A MONOGRAPH ON MAGNETIC FIELDS
FOR LIFE SCIENTISTS

By

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The Franklin Institute Research Laboratories
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THE FRANKLIN INSTITUTE RESEARCH LABORATORIES
BENJAMIN FRANKLIN PARKWAY • PHILADELPHIA, PENNA. 19103

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INTRODUCTION

This monograph was written for two reasons. One, it was felt that workers in magnetobiology would welcome a compilation of material which at present is scattered widely and sometimes difficult to obtain. And two, that it would encourage or at least be helpful to those who wish to become involved in the intricacies of magnetobiology. It is not a text on biology and, in fact, mentions little about the biological effects of magnetic fields. This material is also voluminous and scattered, but can be found rather readily and several bibliographies exist. This work will attempt to present a good deal about magnetic fields (or lack of them) necessary to perform experiments in magnetobiology, or to provide a basis necessary to interpret those which have been done and reported.

Frankly, it is a bit uneven in its treatment of the various aspects of magnetism. This has happened for several reasons. First, it is expected that those who will use the monograph will represent a wide range of sophistication and experience, hence it is hoped both the beginner and the advanced user will find something useful. Second, some aspects required a deeper discussion than others in order to reach a useful point. Thirdly, more material was available for some topics than for others. And lastly, the particular interests of the author, it must be admitted, biased the depth of treatment. It was not meant to be a text but rather a combination of introduction, guide, reference and handbook.

It is suggested that it not be read through from beginning to end, but rather used as needed. If the equations get sticky in some parts skip them and see what happens when they are used without understanding them precisely. Some calculations have been recorded in great detail because in order to be useful we have to end up with a number which we can apply to real life. It is believed that concrete examples are the best way to bridge the gap from theory to practice and concrete examples are thus given. For those who wish to go further and branch out in this field, sufficient references have been given to get onto the main branches of the reference trees and from there to the fine twigs. No claim is made of exhaustive treatment or references in this work. Neither is originally claimed. Rather it is hoped a service has been performed and a need met in bringing together a wide diversity of material and explaining it sufficiently to allow it to be put to immediate and practical use.

R. J. Gibson August 1969

CHAPTER 1
GENERAL DISCUSSION OF MAGNETISM

Magnetism has been known to man for over 2000 years and the lodestone which attracted iron was described by Lucretius who died in 55 B.C. He said "--I will proceed to discuss by what law of nature it comes to pass that iron can be attracted by that stone which the Greeks call the Magnet (Magnes Lopis) from the name of its native place -- the country of the Magnesians. This stone men wonder at--"[1]. These natural magnets, the lodestone, are a variety of naturally occurring iron oxide, magnetite (Fe_2O_3). The Chinese had by 121 A.D.* learned that an iron rod magnetized by a piece of lodestone and hung on a thread would align itself approximately with the earth's north-south direction. Little progress was made in the study of this phenomena until 1600 when William Gilbert published his treatise *De Magnete*. Among other interesting experiments he reports on increasing the holding power of a lodestone by a factor of five by providing it with pole pieces. It was not until 1820, however, that the connection between electric current and magnetism was discovered by Hans Christian Oersted. Immediately thereafter great progress in theory and application was made. Many men whose names are now preserved in the names of various electric and magnetic units investigated and contributed to the knowledge about magnetism. Among these names besides Gilbert and Oersted we find Michael Faraday, Joseph Henry, H. F. E. Lenz, Madame and Pierre Curie, Andre' Ampere' and later James Clark Maxwell. They showed and put on a quantitative basis that one, magnetic effects would be produced by *moving* electric charges and two, electric charges would be induced to move by moving magnets or moving magnetic fields and hence moving charges exert forces on one another entirely independent of the usual electristatic forces. We can say, as a definition of a magnetic field, that if a conductor

*Some legends tell of a crude Chinese compass in use around 2700 B.C.

carrying a current experiences a "magnetic" force then a magnetic field is said to exist in the region of that conductor. Another way of saying this is: a magnetic field is said to exist at a point if a force, in addition to any electrostatic force, is exerted on a moving charge at that point. This puts the phenomenon of a magnetic field on a simple and fundamental basis. These statements in the form of physical equations can be utilized to determine (although sometimes with considerable mathematical complication) both steady and fluctuating magnetic fields in any region due to currents in any configuration of conductors. Further, it can be used to determine the forces and actions of charged moving particles in matter whether biological or nonbiological to a degree limited only by our ability to model satisfactorily these systems and to obtain solutions to the equations describing these models.

We will not go into the latter aspects of this pair of general problems to any degree. Our attention will be focused on the practical aspects of generating magnetic fields of the desired configuration for carrying out experiments with biological systems.

We will attempt in this discussion to write *for* the biologist, zoologist, protozologist, botanist, chemist, ecologist, space flight experimenter, physiologist, bacteriologist, etc. In short, for those *life scientists* not now familiar with the production and use of magnetic fields, but who wish to use them to perform experiments on living systems.

At present, a considerable literature has accumulated and is expanding on the biological effects of magnetic fields. Experiments have been reported on the change in growth rate of bacteria, protozoa, enzymes and seeds, through the orientation of *Paramecium*, planaria and *Volvox* and on up to the scoptic flicker fusion rate change in man as a function of the strength or space rate of change of the magnetic field. And these are only a few of the many and diverse experiments ranging over the whole field of biology and physiology which have been reported. They are mentioned to give some idea of the diversity which, in part, leads to the complexity of the field of biomagnetics or magnetobiology.

Good experiments in biomagnetics are not easy to do and much less easy to interpret. Many experiments have been rather carelessly done or at least poorly described. However, it is not our purpose here to criticize, but rather to make available a body of information that will help the life scientist to set up and perform and report a biomagnetics experiment better and more easily, not from the standpoint of biology but from the standpoint of the physical magnetic field used in his experiment.

Hopefully, we will introduce the proper terminology, make clear the "units", simplify the design procedures and suggest means of getting what is wanted in the way of apparatus. Further, we hope that a better understanding of what a magnetic field is, what it can do and how it does it will occur as a result of the use of this work. We will not attempt to make the life scientist an *expert* in magnetics, there are very few in the physical sciences. We will try to provide a good number of "cookbook" design procedures with sufficient basic theory to fulfill the requirements of many biomagnetics experimenters. There is a large gap between the simple theory in elementary physics texts and that found in advanced texts on electricity and magnetism. The one simply does not tell enough and the other requires some rather sophisticated derivations starting from the most general theories given. Neither usually has much to say about practical design of coils and fields. It is in this immediate area that we hope this exposition will be useful.

The specific material which this monograph will try to cover will be divided into three main categories. The first will be the *fundamentals of magnetism*, the second, the *design of physical apparatus* for producing magnetic fields (and removing them) of the desired size, shape and uniformity and the third, on *how to measure* these fields once they have been produced. The remaining material will be to supplement these three basic areas.

The requirements of life scientists for a magnetic flux density ranging from "zero" (or as small as possible) say 10^{-4} gauss (10^{-8} weber/sqm) to 100,000 gauss (10 weber/sqm) is a very wide range which is not simply handled by any one procedure for producing such flux densities. This problem is further complicated by the fact that the volume in which this flux density may be required may range from a few cubic millimeters to a few cubic meters. Fortunately, the highest fluxes are not always needed in the largest volumes, certainly they cannot be obtained easily. (See Busby[2] for a good listing of high magnetic field experiments.) In this paper we will not attempt to cover either the very smallest or the very largest fields. These are best left to the specialist in shielding or magnetic machinery, but rather to provide an understanding of the intermediate flux densities from say .005 gauss (5×10^{-7} webers/sq meter) to say 1000 gauss (0.1 webers/sq meter). One reason for this is given above, another reason is that it is believed by this writer that in this intermediate region there is still a vast number of experiments which can be performed, many of which will lead to useful information. There is no question that biological effects exist. In extremely high flux densities many of the observed effects can be attributed simply to the relatively large potentials generated by the motion of conducting material emitting lines of flux. As an example of this let us suppose we have a flux density of 50,000 gauss (5 webers/m^2) in which a small animal is moving at a velocity of 2 cm/sec (0.02 m/sec) so that a portion of its body of say 5 cm length (0.05 m) is cutting the flux lines at right angles. The voltage generated from end to end of this length would then be:

$$
\begin{aligned}
E &= Blv \text{ (mks)} \\
&= (5)(0.05)(0.02) = 5 \times 10^{-3} \text{ volts} \\
&= 5 \text{ millivolts}
\end{aligned}
$$

A voltage of this magnitude is on the same order as many normally occurring biological potentials and could be expected to have a very significant effect on the behavior of the animal and if continued over a period of time a significant lasting physiological effect. Thus, some of the effects due to high flux densities can readily be explained, whereas some of the reported effects on, for example, the reduction of sodium influx by some 20% in frog skin at flux densities on the order of 500 gauss (.05 webers/m^2) cannot be explained by this mechanism. Further, a number of interesting effects have been reported due to the reduction of the flux density by several orders of magnitude below that of the earth's normal field. This question, whether the removal of the normal earth's field has a significant biological effect, remains open to further experimentation however. In fact, there remains much to be done in the field of biomagnetics. At present, many effects have been observed, but very few explanations for these effects have been forthcoming. Most biomagnetic phenomena seem unrelated to each other except that they are all the result of an applied magnetic field. Some effects seem not to be proportional to the field strength, others do, some proportional to the space rate of change of the field strength, others more or less independent of that gradient. It seems imperative that carefully thought out and meticulously designed experiments should now be performed specifically aimed at revealing the nature and/or basic mechanism behind the external effect. This will not be easy since there are probably several basic mechanisms which account for the wide range of phenomena observed to date.

Chapter 1 - References

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CHAPTER 2

A DISCUSSION OF THE DIMENSIONS AND UNITS OF THE MAGNETIC QUANTITIES

In order to use magnetic fields and fluxes it is necessary to have names for them and to have measures for them. Further, it is necessary to understand their relationships to each other and to other physical quantities, such as the electrical and mechanical quantities. Because of the many systems of units and dimensions (and hence forms of equations) employed in discussing magnetic quantities it was felt that a discussion of the basic ideas of dimensions and unit systems would be useful. The following is an abbreviated explanation directed towards the development and usage of systems and units of the magnetic quantities. It is intimately connected with the history and fundamental concepts of the magnetic field.

It will be noted that almost every text and paper discussing magnetic fields and utilizing equations to compute the various magnetic quantities will present the equations and/or quantities in slightly different form and use different names for the units involved. This is the result of the many justifiable possibilities in the choice of *dimensions* or *dimension* systems and in the further choice of *unit systems*. It is hoped that in this discussion we may be able to lessen somewhat the aura of confusion and perhaps explain why much of it has occurred. The question of the *proper* dimensions to use for electromagnetic quantities has raised more discussion than any other single question in dimension-system study. It is not surprising, therefore, for the newcomer (or even the experienced) to be confused or to make mistakes in calculations and/or nomenclature.

Mathematics is a symbolic description of the functional relationships between numerical quantities either constant or varying. The numerical values are unique and absolute and do not depend on measurements of the world, but rather on conceptions of the mind. For all practical purposes, two is two in any language anywhere in the world. On the other hand, *physics* is concerned with the *symbolic* relationships of various *measured* quantities. These *quantities* having been *created* or *defined* are subject to various logical or convenient choices in their dimensions and sizes. Physical quantities are determined by their relationship to a *standard*. Thus, a particular rod may be said to be of length L, where L = N x U. N is a pure number and denotes the number of times the standard unit U is contained in the particular length L. U may be meters and N = 10 so that L = 10 meters and the *standard* fits into the particular rod 10 times.

A discussion of units and their relationships will be treated more completely later on in this discussion.

Dimensions and Dimension Systems

The important thing to understand here is that the dimensions of a physical quantity *has nothing to do with the units* used to describe that quantity. The dimension is a common property or description of the class to which the physical quantity belongs, whereas the unit in which it is measured is referred to a physical standard and is the *physical* quantifier, the numerical relation to that standard. Thus, we find that any linear extension in real space has the *dimension* length [L] and may be related to a standard in feet, meters, centimeters, miles or any other conventional or unconventional *standard* unit or measure. Fourier first introduced this concept in the early 1800's. Its use clearly distinguishes physical quantities from mathematical numbers. No physical quantity may be designated as absolutely fundamental although it may be convenient to consider them to be so. Thus, length [L], time [T] and mass [M] are chosen in many dimension systems as fundamental *dimensions* with other physical quantities derived in terms of these three dimensions. In a mechanical system only three dimensions are required to form a consistent and complete set for that field of science. Force [F] or energy [E] could have been chosen as a fundamental dimension along with [L] and [T] and a consistant system developed. This has in fact been done and a comparison of these systems is given.

Physical Quantity and Symbol	Dimensions: Physical System	Gravitational System	Energetical System
length, l	$[L]$	$[L]$	$[L]$
time, t	$[T]$	$[T]$	$[T]$
velocity, v	$[L]\,[T]^{-1}$	$[L]\,[T]^{-1}$	$[L]\,[T]^{-1}$
mass, m	$[M]$	$[F]\,[L]^{-1}\,[T]^{2}$	$[E]\,[L]^{-2}\,[T]^{2}$
force, f	$[M]\,[L]\,[T]^{-2}$	$[F]$	$[E]\,[L]^{-1}$
pressure, p	$[M]\,[L]^{-1}\,[T]^{-2}$	$[F]\,[L]^{-2}$	$[E]\,[L]^{-3}$
energy, E	$[M]\,[L]^{2}\,[T]^{-2}$	$[F]\,[L]$	$[E]$
power, P	$[M]\,[L]^{2}\,[T]^{-3}$	$[F]\,[L]\,[T]^{-1}$	$[E]\,[T]^{-1}$
grav. const, k	$[M]^{-1}\,[L]^{3}\,[T]^{-2}$	$[F]^{-1}\,[L]^{4}\,[T]^{-4}$	$[E]^{-1}\,[L]^{5}\,[T]^{-4}$
etc.			

Note that a perfectly consistent system is developed in each of these systems. Those three and one other, The Astrophysical System are the four which have come into general use. The Astrophysical System defines the gravitational constant as a fundamental unit [k]. The Physical System is generally used in the various branches of science. The Gravitational System is widely used in engineering especially where combined with English units and the Energetical System has been useful in heat engineering.

However, for a heat dimension system an additional dimension is required. To complete this system, the fourth dimension temperature $[\theta]$ is added. This has been done to the three systems shown in the table and all are in use, especially the first and second. So far the complications are not so great. However, for a system involving electrical physical quantitites, another dimension must be added. Here we have a wide choice. *Any*, I repeat, *any* electromagnetic (or electrostatic) physical quantity may be chosen since no philosophical argument is more valid for one than another. However, there are three fundamental electromagnetic *experiments* which involve the "basic" quantities of electromagnetism. These are

1) $f_e = k_e \frac{Q_1 Q_2}{r^2}$ Coulomb's Law Experiment

2) $f_m = k_m \frac{I_1 I_2}{r} l$ Ampere's Law Experiment

(by definition $I = dQ/dt$)

3) $V = -k_i \frac{d\phi}{dt}$ voltage generation experiment.

f_e = the electrostatic force between charges
f_m = the electromagnetic force between current loops
Q = quantity of electrostatic charge
I = electric current
r = center to center distance
l = length of the current carrying conductors
V = the induced voltage or electromotive force
ϕ = the magnetic flux linking the conductor
k_j = various proportionality constants

Depending on how these are chosen a vast multitude of dimensional systems can be developed. There are at least 8 systems in popular use, not including the possible *unit* choices.

System	Quantities Chosen as Fundamental
Electrophysical	L, T, M, Q
Electrogravitational	L, T, F, Q
Definitive	L, T, P, Q
"Practical"	L, T, I, R
Energetical	L, T, E, V
Electrostatic	L, T, M, k_e^{-1}
Electromagnetic	L, T, M, k_m
Gaussian	L, T, M, k_m and $k_e = k_m c^2$

where

L = length
T = time
M = mass

Q = electrostatic charge

F = force

P = power

I = current

E = energy

V = voltage

k_e = proportionality constant in Coulomb's Law and is taken as the absolute dielectric constant.

k_m = proportionality constant in Ampere's Law and is taken as the absolute permeability.

c = a constant of dimensions of velocity and numerically equal to the velocity of light in free space but not uniquely identified with it.

This confusion may be further complicated by choosing in the Electrostatic system $[k_e] = 1$, in the Electromagnetic system $[k_m] = 1$ and in the Gaussian system $[k_e] = [k_m] = 1$, producing the so called "absolute" systems. This essentially reduces the number of fundamental quantities by one, resulting in incomplete systems and adding confusion in checking equations for dimensional homogenity. The present trend is away from incomplete systems and toward the *Electrophysical* or *Practical* systems.

The Practical systems are by far the most convenient for electromagnetic calculations, but are difficult to combine with the popular mechanical dimensional systems. They are used extensively in the engineering literature.

A table showing the dimensions of the various electromagnetic quantities in the Practical and the Electromagnetic systems is given.

The rest of the discussion will be limited to these two systems of dimensions, the Electromagnetic and the Practical systems. The Practical system will be developed as a comprehensive unit system using L, T, M and R. Actually, instead of R, which is difficult to standardize experimentally it has been suggested and adopted to set the value of absolute permeability to exactly 10^{-7} henry/meter (in the MKS unit

Physical Quantity	Symbol	Dimensions: Practical	Dimensions: Electromagnetic
length	l	L	L
time	t	T	T
mass	m	M	M
force	f	$M\ L\ T^{-2}$	$M\ L\ T^{-2}$
energy	E	$M\ L^{2}\ T^{-2}$	$M\ L^{2}\ T^{-2}$
electric charge	Q	Q	$M^{1/2}\ L^{1/2}\ T^{-1}\ k_m^{-1/2}$
current	I	$Q\ T^{-1}$	$M^{1/2}\ L^{1/2}\ T^{-1}\ k_m^{-1/2}$
voltage	V	$I\ R$	$M^{1/2}\ L^{3/2}\ T^{-2}\ k_m^{1/2}$
resistance	R	R	$L^{1/2}\ T^{-1}\ k_m$
magnetic flux	ϕ	$I\ R\ T$	$M^{1/2}\ L^{3/2}\ T^{-1}\ k_m^{1/2}$
induction	B	$I\ R\ L^{-2}\ T$	$M^{1/2}\ L^{-1/2}\ T^{-1}\ k_m^{1/2}$
magnetizing force	H	$I\ L$	$M^{1/2}\ L^{-1/2}\ T^{-1}\ k_m^{-1/2}$
magnetomotive force	F	I	$M^{1/2}\ L^{1/2}\ T^{-1}\ k_m^{-1/2}$
reluctance	R_m	$R^{-1}\ T^{-1}$	$L^{-1}\ k_m^{-1}$
abso. diel const	Δ	$R^{-1}\ L^{-1}\ T$	$L^{-2}\ T^{2}\ k_m^{-1}$
absolute permeability	II	$R\ L^{-1}\ T$	k_m

system). In the rationalized form of this system this becomes $4\pi \times 10^{-7}$ henry/meter. The factor 4π is here (i.e., the system is rationalized) in order to simplify the electromagnetic equations rather than simplifying the Coulomb's law of attraction. The term "Practical" is used here in the sense that the engineering quantities volts, amperes, ohms, henries, farads, etc., become the units in the system and have their usual magnitudes. Some other familiar quantities such as density of water (1 g/cc) however become 10^3 kilograms/m^3. The rationalized system further requires the use of μ_o the permeability of free space to have a value different from 1, that is, equal to $4\pi \times 10^{-7}$ henry/meter and requires its presence in the electromagnetic equations. This can be looked upon as an advantage, however, in that it clearly points out the difference between B and H. Thus in a vacuum $B = \mu_{o}h$. The units of B being webers/m^2 and those of H amperes/meter (or ampere/meter) and the numerical values of B and H being different by the factor $4\pi \times 10^{-7}$.

Units and Unit Systems

In the foregoing section the use of units unavoidably crept into the discussion in places. Direct discussion of *units* will now be undertaken. In order to *measure* a physical quantity it is necessary to compare it with a like physical quantity. The reference physical quantity is referred to as the "unit" of that physical quantity and the "measure" is then a ratio (a purely numerical factor) of the unknown amount to the "unit" measure. Since the "unit" chosen may be arbitrary, an infinity of possibilities exists for choosing the unit. Now since like physical quantities are related only by purely numerical factors there is only one *dimension* for each physical quantity but many units. A dimension system is chosen and then for each dimension in that system a specific unit is chosen. These units become the *fundamental units* and by means of the relationships in the dimension system every physical quantity can then be expressed in these units. For each dimension system an infinity of unit systems can exist but in order to achieve widespread understanding these have been limited to a few.

As was pointed out previously, three dimensions and hence three units are required for a mechanical system; plus one for heat or temperature and one more for electricity. A dimension system and unit system comprised of 5 fundamental units which can be used throughout the whole field of physics is termed a comprehensive unit system. If the units are metric it is a metric comprehensive unit system. Even then we find *eleven* comprehensive metric systems in general use and some of these have modifications. In addition to the modifications in the units, we can have a unrationalized system, a rationalized system, a partially rationalized system or a symmetric system. The Gaussian (cgs) system is a symmetric system and here the dielectric constant of free space is chosen as exactly 1 stat-fard per centimeter and the permeability of free space is chosen as 1-abhenry per meter. This factor is numerically the same as the velocity of light in cgs units but is a numerical factor only. In that case all electrostatic and current quantities appear in electrostatic units and all magnetic quantities appear in electromagnetic units. This system is widely used in European publications. Below is a table showing some relationships between a selected few of the comprehensive systems.

Unit System		CGS	CGS Electrostatic	CGS Electromagnetic	MKS	MKS
Author		Gauss	Maxwell		Giorgi	
Dimension System		Symmetric	Electrostatic	Electromagnetic	Practical	(Rationalized) Practical
Quantities (sym)						
Length	L	cm	cm	cm	meter	meter
time	t	sec	sec	sec	sec	sec
mass	m	gram	gram	kilogram	kilogram	kilogram
force	f	dyne	dyne	joule/m		newton = j/m
charge	Q	statcoulombs	statcoulomb	abcoulomb	coulomb	coulomb
current	I	statampere	statampere	abampere	ampere	ampere
resistance	R	statohm	statohm	abohm	ohm	ohm
voltage	V	statvolt	statvolt	abvolt	volt	volt
magnetic flux	ϕ	statweber	statweber	maxwell	weber	weber
m. flux density	B	gauss	statweber/cm^2	gauss	weber/m^2	weber/m^2
magnetizing force	H	oersted	--	oersted	amp/m	amp-turn/m
magnetomotive force	F	gilbert	--	abhenry	amp-turn	amp-turn
permeability	μ	abhenry/cm	stathenry/cm	abhenry/cm	henry/m	henry/m

The dimension systems and units systems which will be used in this paper are presented on the following page giving the conversion factors from the one system to the other. Formulas presented in this paper will be given in both systems. *In the following table, quantities in the same row are equal.* This will be our working table.

Physical Quantity	Symbol	Absolute Practical or Rationalized Practical MKS Units	cgs Electromagnetic Nonrationalized Units
length	l	1 meter	100 centimeter
mass	m	1 kilogram	1000 grams
time	t	1 sec	1 sec
force	f	1 newton	10^5 dynes
work	W	1 joule	10^7 ergs
energy	U	1 joule	10^7 ergs
power	P	1 watt	10^7 ergs/sec
charge	q	1 coulomb	10^{-1} abcoulomb
current	i or I	1 ampere	10^{-1} abampere
voltage	V	1 volt	10^8 abvolts
resistance	R	1 ohm	10^9 abohms
capacity	C	1 farad	10^{-9} abfarads
magnetic flux	ϕ	1 weber	10^8 maxwells
magnetic induction (flux density)	B	1 weber/m^2	10^4 gauss
magnetizing force (magnetic field)	H	1 amp-turn/m	$4\pi \times 10^{-3}$ oersted
magnetomotive force	F	1 amp-turn	$4\pi/10$ oersted
reluctance	R_m	1 amp-turn/weber	$4\pi \times 10^{-9}$
pole strength	p_m	1 weber	$10^8/4\pi$ maxwells
inductance	L	1 henry	10^9 abhenries
permeability of free space	μ_o	$4\pi \times 10^{-7}$ henry/m	unity
permeability	μ	dimensionless constant	gauss/oersted

Notes: a) 1 volt-sec = 1 weber

b) 1 henry = 1 $\dfrac{\text{weber}}{\text{amp-turn}}$

c) quantities in same row are equal, i.e., 1 m = 100 cm.

These two systems of unity will be abbreviated as (MKS) and (CGS) respectively and placed in Brackets next to the formulas used where necessary to identify the system of units which must be used in the equation.

In order to convert from units from one system to another it is necessary to multiply by the appropriate factor. This factor is obtained from the table. Since quantities in the same row are equal their dimensional ratio is unity but their unit ratio is the value in one column divided by that in the other. The *conversion factor* is this *ratio* and is *dimensionless* and equal to 1 (unity) dimensionally. Thus, we see

$$\frac{1 \text{ meter}}{100 \text{ centimeters}} = [1] = \frac{100 \text{ centimeters}}{1 \text{ meter}}$$

$$\frac{1 \text{ weber/m}^2}{10^4 \text{ gauss}} = [1] = \frac{10^4 \text{ gauss}}{1 \text{ weber/m}^2}$$

Thus to convert a value of B (magnetic induction) in (MKS) of 10 weber/m^2 to the appropriate number of units (CGS) gauss we multiply by the unit factor which will cancel the unwanted units leaving the wanted units and having the proper numerical factor. Note the unity factor can always be inverted.

$$B = 10 \frac{\text{webers}}{m^2} \times [1] = 10 \cancel{\text{weber/m}^2} \times \frac{10^4 \text{ gauss}}{1 \cancel{\text{weber/m}^2}} = 10 \times 10^4 \text{ gauss}$$

$$= 10^5 \text{ gauss (the MKS units cancelling)}$$

$$\text{or } B = 10^5 \text{ gauss} \times [1] = 10^5 \cancel{\text{gauss}} \times \frac{1 \text{ weber/m}^2}{10^4 \cancel{\text{gauss}}} = \frac{10^5}{10^4} \text{ weber/m}^2$$

$$= 10 \text{ webers/m}^2 \text{ (the CGS units cancelling)}$$

This method of converting from units in one system to those in another is the most simple and foolproof known. It takes care of numerical factors and units at the same time. It is only necessary to have a table with equality between units. Any number of [1] (unity) factors may be used in a string without difficulty. The simple example below should suffice to illustrate the method.

Thus to convert 1 yard to cm.

unity factors: $[1] = \frac{3 \text{ feet}}{1 \text{ yard}}$, $[1] = \frac{12 \text{ inches}}{1 \text{ foot}}$, $[1] = \frac{2.54 \text{ cm}}{1 \text{ inch}}$

$$L = 1 \text{ yard} \times [1] \times [1] \times [1]$$

$$L = 1 \cancel{\text{yard}} \times \frac{3 \cancel{\text{feet}}}{1 \cancel{\text{yard}}} \times \frac{12 \cancel{\text{inches}}}{1 \cancel{\text{foot}}} \times \frac{2.54 \text{ cm}}{1 \cancel{\text{inch}}}$$

$$= 3 \times 12 \times 2.54 \text{ cm}$$

$$L = 91.44 \text{ cm}$$

all units cancelling except cm. leaving the proper numerical multiplier.

As was mentioned previously a formula or equation in a rationalized system will be different from one in an unrationalized system even when the same units are used in both systems.

A tabulation of a few frequently used magnetic equations in the MKS rationalized and the CGS EMU unrationalized systems are given below.

Physical Situation	Equation
Biot-Savart Law Field due to element of current	$H = \frac{\mu_o}{4\pi} \frac{\bar{I} \times d\bar{l}}{r^2} \frac{\text{amp-t}}{\text{m}}$ (MKS) $H = \frac{\bar{I} \times d\bar{l}}{r^2}$ oersted (CGS)
Normal force per unit length between infinite parallel currents	$F_u = \frac{\mu_o}{2\pi} \frac{I_1 I_2}{r}$ newtons/m (MKS) $F_u = 2 \frac{I_1 I_2}{r}$ dynes/cm (CGS)
Flux density at center of n turn circular coil in air	$B = \frac{\mu_o}{2} \frac{nI}{r}$ webers (MKS) $B = 2\pi \frac{nI}{r}$ gauss (CGS)
Flux density inside infinite solenoid n turns/unit length	$B = \mu_o nI$ webers (MKS) $B = 4\pi nI$ gauss (CGS)
EMF induced in conductor by rate of flux change	$V = -\frac{d\phi}{dt}$ volts (MKS) $V = -\frac{d\phi}{dt}$ ab volts (CGS)
"ohms" law for magnetic circuits	$F = \phi R_m$ (MKS & CGS)

Chapter 2 - References

Various sources have been used in the development of the discussion of dimensions and unit systems. Among them were the following:

Handbook of Engineering Fundamentals, O. W. Eshbach, John Wiley and Sons, New York, 1936.

Static and Dynamic Electricity, W. R. Smythe, McGraw-Hill Book Co., New York, 1939.

Electricity and Magnetism, N. R. Gilbert, Macmillan Co., New York, 1941.

Principals of Electricity and Electromagnetism, G. P. Harnwell, McGraw-Hill, New York, 1938.

Handbook of Chemistry and Physics, 34th and 47th Ed. Chemical Rubber Publishing Co.

CHAPTER 3
MAGNETIC FIELDS AND MAGNETIC FLUX

We are now in a position to discuss the production of a magnetic field.

Let us first note that we can make some analogies of magnetic fields with simple electric circuits with which many are familiar.

In an electric circuit we have a voltage or electromotive force (E) producing a current (I) through a resistance (R). Similarly, in a magnetic circuit we have a magnetomotive force (F) producing a flux (ϕ) through a reluctance (R_m).

$$E = RI \qquad\qquad F = R_m\phi$$

In the electric circuit the current is generally confined inside the conductor, in the magnetic circuit this is in general, not true, and we have to carefully define the region in which the flux occurs.

The specific magnetomotive force per unit length is called the *magnetic field* or *magnetic field strength* and sometimes just *field*.

It is denoted by the letter H. The specific magnetic flux or flux density is called the *magnetic induction* and is denoted by B. The formal definitions are as follows. The magnetomotive force F is:

$$F = \oint \overline{H}\cdot\overline{dl}$$

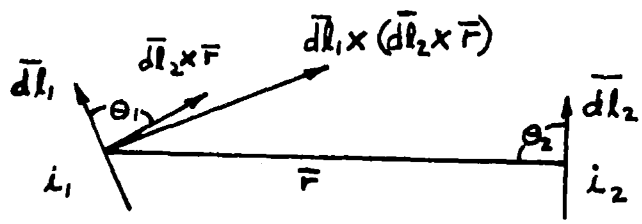
and hence H is a vector with direction and magnitude and units of magnetomotive force per unit length.

We associate H with a current i in such a manner that the following is true:

$$\oint \overline{H}\cdot\overline{dl} = i \text{ (MKS)} \qquad \oint \overline{H}\cdot\overline{dl} = 4\pi i \text{ (CGS)}$$

As a different way of looking at the magnetic field which is exactly equivalent to the above, we can argue in this way: *Ampere performed experiments which showed that the force between current carrying segments of wire varied inversely as the square of their distance apart and directly as the currents in the segments. He further determined by experiment, that the force experienced by a wire carrying a current is always normal to the wire.*

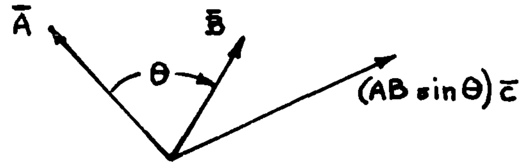
In the figure below these relations are diagrammed.



current element 1 is written $i_1\overline{dl}_1$
current element 2 is written $i_2\overline{dl}_2$

The magnitude and direction of the force is described succinctly as the vector product $\overline{A} \times \overline{B} = (AB\sin\theta)\ \overline{c} = \overline{C}$, where θ is the angle between $\overline{A}$ and $\overline{B}$, $(AB\sin\theta)$ is the magnitude of the vector product and $\overline{c}$ is a unit vector perpendicular to both $\overline{A}$ and $\overline{B}$ and so directed that a right-hand screw advancing in the direction $\overline{c}$ would rotate $\overline{A}$ through the smaller angle into the position of $\overline{B}$.

Thus,



We now can write the equation which describes the elementary force of $i_2\overline{dl}_2$ on $i_1\overline{dl}_1$

$$\overline{df}_1 = K\, i_1 i_2 \frac{\overline{dl}_1 \times (\overline{dl}_2 \times \bar{r}_1)}{r^2}$$

where

$\bar{r}_1$ is the unit vector in the direction element 2 to element 1, and

K is a constant of proportionality.

In practical rationalized MKS units, K is chosen as $\mu_o/4\pi$ and in CGS electromagnetic non-rationalized units K is chosen as 1. The above equation can be written as:

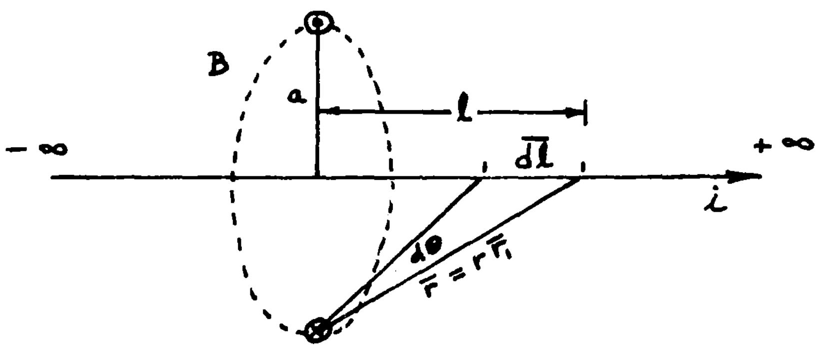
$$\overline{df} = i_1\overline{dl}_1 \times \frac{\mu o}{4\pi} i_2\overline{dl}_2 \times \bar{r}_1 \qquad \text{(MKS)}$$

From this we now define $\overline{dB}_2 = \frac{\mu_o}{4\pi} i_2\overline{dl}_2 \times \bar{r}_1$ and $\overline{dB}_2$ is the magnetic flux produced by $i_2\overline{dl}_2$ to produce the force $\overline{df}$. If we are in a material of permeability μ then:

$$\overline{dB} = \frac{\mu\mu_o}{4\pi} i\overline{dl} \times \bar{r}_1 \quad \text{(MKS)}$$

$$dB = \mu\, i\overline{dl} \times \bar{r}_1 \quad \text{(CGS)}$$

If we have a single current carrying conductor stretching from $-\infty$ to $+\infty$ we find a $d\bar{B}$ surrounding the wire coming out of the plane $\odot$ above the wire and entering the plane below the wire $\otimes$ and at a distance a from the wire.



If we now integrate $d\bar{B}$ for all elements of the wire, or completely around the closed loop (of wire) (closed through ∞) we have for the field around the wire

$$B = \frac{\mu\mu_o i}{4\pi} \oint \frac{\overline{dl} \times \bar{r}_1}{r^2} \text{ (MKS)} \quad \bar{B} = \mu i \oint \frac{\overline{dl} \times \bar{r}_1}{r^2} \text{ (CGS)}$$

where

$$\overline{dl} \times \bar{r}_1 = dl \sin\theta$$
$$a = r \sin\theta$$
$$r^2 = a^2 + l^2$$
$$dr = dl$$

so

$$\overline{B} = \frac{\mu\mu_o i}{4\pi} \int_{-\infty}^{\infty} \frac{a dr}{(a^2 + 1^2)^{3/2}} = \frac{2\mu i}{4\pi a} \left[\frac{1}{(a^2 + 1^2)^{1/2}} \right]_o^{\infty} \quad \text{(MKS)}$$

$$\overline{B} = \frac{2\mu\mu_o i}{4\pi a} \quad \text{(MKS)} \quad \overline{B} = \frac{2\mu i}{a} \quad \text{(CGS)}$$

we may also write

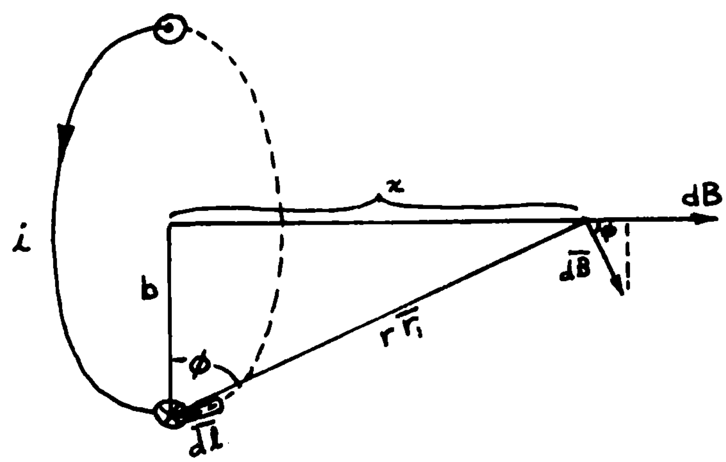
$$\overline{H} = \frac{2i}{4\pi a} \quad \text{(MKS) and } \overline{H} = \frac{2i}{a} \quad \text{(CGS)}$$

since

$$\overline{H} = B/\mu\mu_o \quad \text{(MKS) and } \overline{H} = \overline{B}/\mu \quad \text{(CGS)}$$

This result (or the integral form above) is known as the *Biot-Savart* law and is of great practical importance. It is from this and the following calculation that practically all useful magnetic fields are determined.

A special case, essentially the inverse of the above is now described. Here the current-carrying conductor is bent in the form of a circle (a loop) and the magnetic field is calculated along the axis of the loop. Note, however, that it is not easy to determine the magnetic field *off* the *axis*. This problem will be discussed later.



Now by symmetry each $i\overline{dl}$ will contribute an equal and symmetric amount of $\overline{B}$ to the axial $\overline{B}$ hence the component of $\overline{B}$ perpendicular to the axis vanishes and

$$\overline{B} = \frac{\mu\mu_o i}{4\pi} \oint \frac{\overline{dl} \times \overline{r}_1}{r^2} \quad \text{(MKS)} \qquad \overline{B} = \mu i \oint \frac{\overline{dl} \times \overline{r}_1}{r^2} \quad \text{(CGS)}$$

now $\overline{dl} \times \overline{r}_1 = dl \sin\theta$, but $\sin\theta = \sin \pi/2 = 1$, since $\overline{dl}$ is everywhere perpendicular to $\overline{r}_1$, so $\overline{dl} \times \overline{r}_1 = dl$ and $dB \cos\phi$ is the component of $\overline{dB}$ along the axis hence:

$$B_{axis} = \oint \cos\phi dB = \frac{\mu\mu_o i}{4\pi} \int_o^{l=2\pi b} \frac{\cos\phi dl}{r^2} \qquad \begin{array}{l} \cos\phi = b/r \\ \text{and } b^2 + x^2 = r^2 \end{array}$$

so

$$B_{axis} = \frac{\mu\mu_o i}{4\pi} \frac{2\pi b^2}{(b^2 + x^2)^{3/2}}$$

and

$$B_{axis} = \frac{\mu\mu_o i}{2} \frac{b^2}{(b^2 + x^2)^{3/2}} \text{ (MKS)} \quad B_{axis} = 2\pi\mu i \frac{b^2}{(b^2 + x^2)^{3/2}} \text{ (CGS)}$$

Again we have a very important practical result. This result can be used immediately. We may rewrite the equation as follows:

Since

$$B = \mu\mu_o H \quad \text{(MKS)} \qquad B = \mu H \quad \text{(CGS)}$$

then

$$H_{axis} = \frac{B}{\mu\mu_o} = \frac{i}{2} \frac{b^2}{(b^2 + x^2)^{3/2}} \text{ (MKS)} \quad H_{axis} = \frac{B}{\mu} = 2\pi i \frac{b^2}{(b^2 + x^2)^{3/2}} \text{ (CGS)}$$

Now let $x = o$ and call the field along the axis at the center of the coil H_o

$$H_o = \frac{i}{2b} \text{ (MKS)} \qquad H_o = \frac{2\pi i}{b} \text{ (CGS)}$$

and form the dimensionless quantity H/H_o

$$\frac{H}{H_o} = \frac{b^3}{(x^2 + b^2)^{3/2}} \qquad \frac{H}{H_o} = \frac{b^3}{(x^2 + b^2)^{3/2}}$$

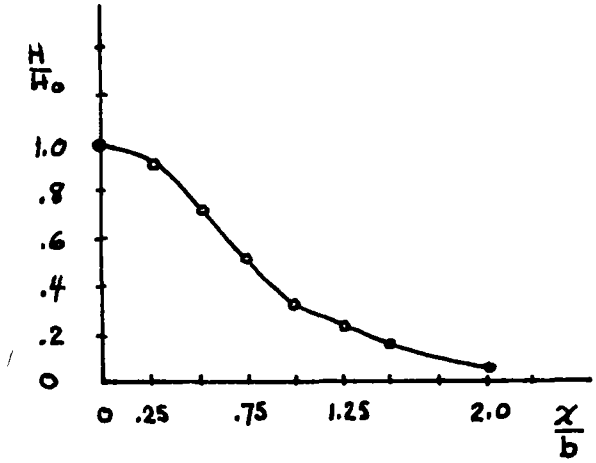
which are the same and dimensionless in both systems

and finally

$$\frac{H}{H_o} = \frac{1}{\left(\left(\frac{x}{b}\right)^2 + 1\right)^{3/2}}$$

A plot of H/H_o vs x/b (also dimensionless) gives us a universal curve for the field on the *axis* of a single loop of wire. If many turns of wire are used in the loop, it is still valid, the only assumption being that the cross sectional dimensions of the loop are small compared to the radius of the loop.

x/b	H/H_o
0	1.000
.25	.915
.50	.716
.75	.512
1.00	.354
1.25	.244
1.50	.171
2.00	.089



This example is being explained in considerable detail since the general principles involved are similar to those which will be encountered in more complicated coil arrangements.

We can see that the slope is practically constant in the region around x/b = 1/2. We can make use of this fact to produce a uniform or nearly uniform H.

Now, if the first derivative of a function can be found to have a constant value its second derivative will be zero and the value of the argument (x/b) can be found which will make the slope constant.

Let us now find dH'/dx', d^2H'/dx'^2, where $H' = H/H_o$ and $x' = x/b$

$$H/H_o = \frac{1}{\left(\left(\frac{x}{b}\right)^2 + 1\right)^{3/2}} \quad \text{then } H' = \frac{1}{(x'^2 + 1)^{3/2}}$$

$$\frac{dH'}{dx'} = -3/2 \ (x'^2 + 1)^{-5/2} \ 2x'$$

and

$$\frac{d^2H'}{dx'^2} = -3/2 \left[-5/2 \ (x'^2 + 1)^{-7/2} \cdot 2x' \cdot x' + (x'^2 + 1)^{-5/2} \cdot 1 \right]$$

Setting this equal to zero and factoring out the common factor $-3/2$ $(x'^2 + 1)^{-5/2}$ we find

$$\frac{5x'^2}{x'^2 + 1} = 1$$

and

$$x'^2 = 1/4 \text{ or } x' = 1/2$$

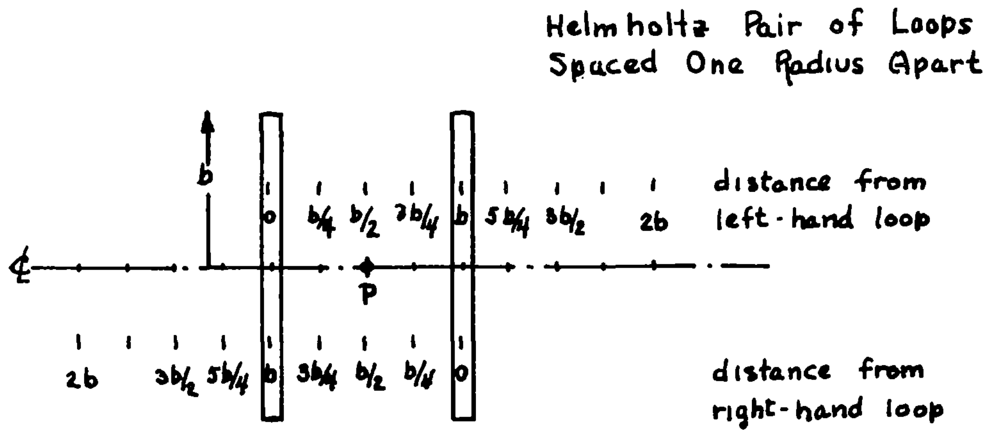
This gives us

$$x/b = 1/2$$

At this point, along the axis, the field is decreasing at a constant rate.

If we now position two coils one to the left of point P a distance of $x/b = 1/2$ and the other to the right of point P a distance $x/b = 1/2$ we will obtain a field which is the sum of the fields of two coils. This arrangement of loops is called a Helmholtz pair. It is one of the first and relatively common and simple ways to produce a uniform field of a reasonable extent. It produces the largest uniform field if the two loops are of equal size. Other configurations using more than two loops of the same or different sizes can be more efficient, that is, give a greater volume of uniform field, for the same exterior dimension of the loops.

At the center P (and approximately so to each side of center) the decrease in the field of the one loop is exactly compensated by the increase of the field of the other loop leading to a uniform field between the loops and becoming less uniform as we move farther from the center P. The slope of the field at the center of a single loop is also zero but as can be seen from the previous figure changes rather rapidly as we move away from center. The field for a Helmholtz using the data just calculated for a single loop is shown in the next example.

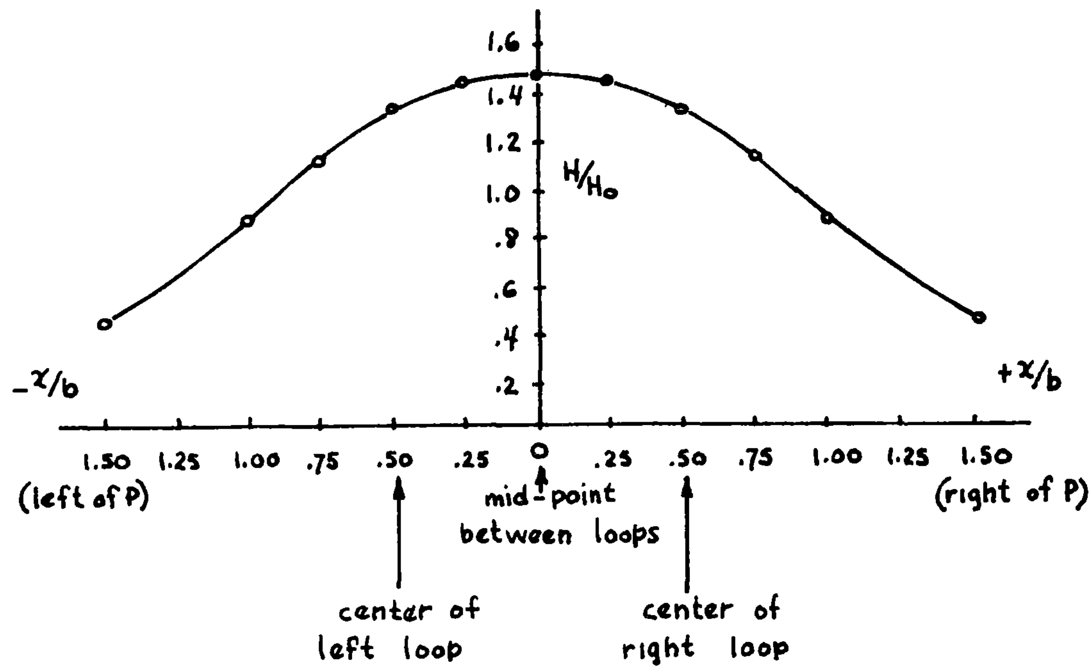


The field from the left hand loop will decrease to the right and the field from the right hand loop will decrease to the left. The two fields will add to give a nearly uniform field. Thus, from our table we have

Distance from P (x/b)	Distance from Left Hand Loop	H/H_0 LH Loop	Distance from Right Hand Loop	H/H_0 RH Loop	Sum of H/H_0 Left and Right
-1.5	(1.00)	(.354)	2.00	.089	.443
-1.00	(0.50)	(.716)	1.50	.171	.887
-0.75	(0.25)	(.915)	1.25	.244	1.159
-0.50	0.00	1.000	1.00	.354	1.354
-0.25	0.25	.915	0.75	.512	1.427
0.00	0.50	.716	0.50	.716	1.432
0.25	0.75	.512	0.25	.915	1.427
0.50	1.00	.354	0.00	1.000	1.354
0.75	1.25	.244	(0.25)	(.915)	1.159
1.00	1.50	.171	(0.50)	(.716)	.887
1.50	2.00	.089	(1.00)	(.354)	.443

where the values in brackets have been filled in by symmetry and the columns have been adjusted vertically so that the field from each coil falls at the point P.

A plot of the first and last columns shows the uniformity of the field on the axis of the loops.



It can be seen that in the region from -0.25 to +0.25 the field along the axis is quite uniform.

To find the point where the combined field falls to 99% of the field at point P we let:

$$\frac{H_L' + H_R'}{2H_o'} = 0.99$$ subscripts L and R stand for "left" and "right"

$$H_L' + H_R' = (0.99)\ (2)\ (.716)$$
$$= 1.417$$

where

$$H_L' = \frac{1}{(x'^2 + 1)^{3/2}} \qquad H_R' = \frac{1}{((1 - x')^2 + 1)^{3/2}}$$

then

$$\frac{1}{\left(\left(1 - \frac{x}{b}\right)^2 + 1\right)^{3/2}} = 1.417 - \frac{1}{\left(\left(\frac{x}{b}\right)^2 + 1\right)^{3/2}}$$

Since this is a bit difficult to solve algebraically we can tabulate a few values in the vicinity of x/b = 0.25 and plot them on an expanded scale we can find where x/b makes this true. This will then be the extent of the 1% region.

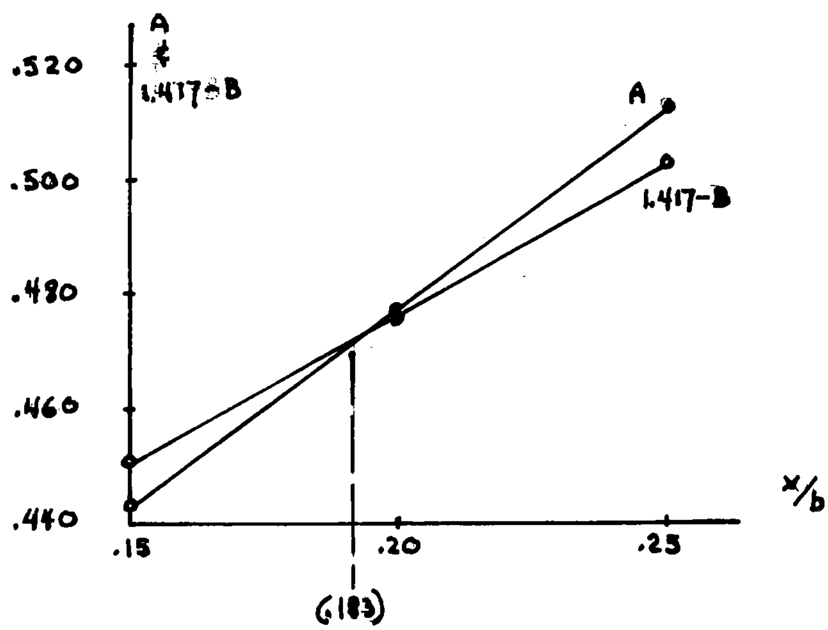
Let

$$\frac{1}{((1 - x/b)^2 + 1)^{3/2}} = A$$

$$\frac{1}{((x/b)^2 + 1)^{3/2}} = B$$

Then

x/b	A	B	(1.417-B)
.15	.443	.966	0.451
.20	.476	.943	0.474
.25	.512	.914	0.503



This gives x/b = 0.183 where the field is 99% of center field.

Putting this value of x/b in our equations for the combined field we find:

$$H'_L + H'_R = \frac{1}{(x'^2 + 1)^{3/2}} + \frac{1}{(1-x')^2 + 1)^{3/2}}$$

$$= \frac{1}{(.183^2 + 1)^{3/2}} + \frac{1}{(.817^2 + 1)^{3/2}}$$

$$= .951 + .461$$

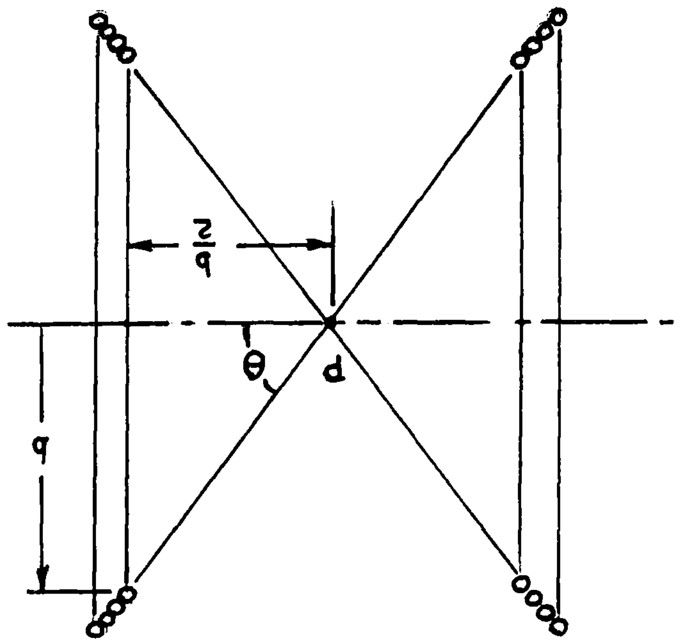
$$= 1.412$$

which is $\frac{1.412}{1.432} = 0.986$ or 99% of field at center as required.

If we had a pair of loops each with a one meter radius (b = 1m), spaced 1m apart, a uniform field (within 1%) would exist on the *axis* from .317m left of center to .317m right of center for a total length of .634 meter.

For a field uniform to say 0.1% we would naturally have a much smaller distance within this uniformity. But note that the length of the useful field for a given uniformity is proportional to the coil radius. Thus a pair of coils 10 cm radius would have a useful length of 6.34 cm.

In the illustration above we made the assumption that the cross section of the loop winding had dimensions that were very small compared to the loop radius. That is the current that was essentially a line. In some cases it may be necessary to have a considerable number of turns taking up a considerable amount of physical space. If these turns are on a cylindrical surface only one of the turns can have the proper spacing from the center. In order to compensate for this, the winding should be on a conical surface so that the radius of each loop will be twice its distance from the center of the pair of coils. This is, the distance between corresponding turns on the two coils should equal the radius of these turns. This is illustrated in the following figure. Here the radius of each turn as it is farther removed from point P is increased in radius sufficiently to keep the ratio $b/b/2 = 2$. The uniformity is then maintained as well as it would be if only the inner turns were considered.



The turns will be in the proper proportion if the bobbin (on which they are wound) is as shown with $\tan\theta = 2$, $\theta = 63.43°$.

Calculations of Field *Off-Axis*

Note that all the foregoing calculations of the field from a circular loop were restricted to the field *on the axis* of the loop and the uniformity was considered *along the axis only*. It is now necessary to consider how the field varies *off the axis*. This cannot be done in any simple way and for this reason is usually left out of most elementary texts and when found in advanced texts is obscured by the derivations. In general, the *off-axis* field is given in terms of complete elliptic integrals of the first and second kinds or of Legendre functions. A complete discussion may be found in Smythe[1] on the use of Legendre functions.

An article by Blewett[2] develops the off-axis field in terms of elliptic integrals and contains a set of tables useful radially out to twice the radius of the loop but extending along the axis only to 0.36 times the radius, which limits the usefulness. (Note: formula 5 of this paper is in error and should have $(1/2)^2$ as coef. of k^2 rather than (1/2)). Dwight[3] develops several formulas for both loops and extended solenoids in terms of Legendre functions. These formulae give the magnitude of the field components in the x and y directions and hence are quite useful, whereas, Smythe gives the components in terms of the radius vector ($\bar{r}$) and (θ) directions. Formulas are given for the entire plane, both where the point of interest is farther away or closer than the loop radius. A large number of formulas is also given by Dwight[4] in a later article for the off-axis field near cylindrical coils or solenoids. He develops these in Legendre functions, elliptic integral functions and in rapidly converging infinite series with recommendations of those most useful in various special regions around the coils. In the same journal in a discussion of Dwight's paper, Welch[5] presents sets of useful curves for flux density around cylindrical coils. Where great accuracy is not required the information can be obtained quite quickly from these curves.

Probably the best reference for practical calculation of magnetic fields (from tables) is Hart[6]. In the introduction to his volume he states:

> "To evaluate the magnetic field of a circular current at an arbitrary point in space has often been a tedious process. The formulas ... are rather involved expressions ... a separate evaluation of the magnetic field at each desired point in space must be performed ... the task may involve considerable labor. The purpose of the tables ... is to reduce to a bare minimum the time and effort required. ... The procedure consists of adding ... the tables apply to all ... distributions."

These tables are far more extensive and of a finer subdivision than any tables published previously. Any circular coil (or number of coils) can be easily handled. For regions where the field is nearly uniform an accuracy of better than 0.01% is possible. Overall the accuracy is better than the accuracy to which the coil dimensions can be measured. The fact that distributed currents are included makes possible the accurate calculation of the fields from coils of finite (and practical) and even large winding cross section size. The calculations for eal coils with eal dimensions can thus be done with greater ease and accuracy than could previously be done for idealized coils. In addition, both the radial and axial fields are given for both on- and off-axis, thus making possible a true knowledge of the uniformity of the field in a real, practical region of interest to the experimenter.

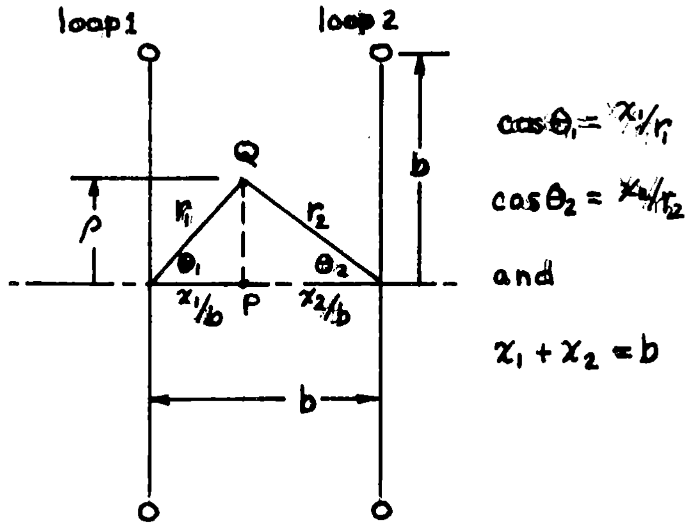
A typical calculation to illustrate the complexities (without tables) of an off-axis calculation is given below. It will use the same coils as for the Helmholtz coil calculation previously given.

In general we wish to find the off axis field at the point Q. This will consist of the field from loop 1 plus the field from loop 2. Point Q is distant from loop 1 by the radius vector r_1 at angle θ_1 similarly r_2 and θ_2 for loop 2. Since loop 1 and 2 are at a fixed distance apart (b in this case) we have:

$$\cos\theta_1 = x_1/r_1 \qquad \cos\theta_2 = x_2/r_2$$

and

$$x_1 + x_2 = b \ .$$



Adapting formulas (5) and (6) from Smythe pg. 271, we obtain:

$$\frac{H_r}{H_{or}} = \frac{1}{.7155} \sum_{n\ \mathrm{odd}}^{1\ \mathrm{to}\ \infty} (-1)^{\frac{n+1}{2}} \frac{1.3.5...n}{2.4.6...(n-1)} \left(\frac{r}{b}\right)^{n-1} P_n\ (\cos\theta)$$

$$\frac{H_\theta}{H_{or}} = \frac{1}{.7155} \sum_{n\ \mathrm{odd}}^{1\ \mathrm{to}\ \infty} (-1)^{\frac{n+1}{2}} \frac{1.3.5...n}{n.2.4.6...(n-1)} \left(\frac{r}{a}\right)^{n-1} P_n'\ (\cos\theta)$$

where H_o = field on axis at r = b/2.

The field components have to be calculated for each loop at the point Q, the x and ρ components calculated and the two added, giving an:

H/H_{or} along x and a H/H_{or} along ρ .

The field from loop 1 will have components in the r_1 and θ_1 direction; loop 2 will have components in the r_2 and θ_2 direction. These will have to be summed along the x and radial (ρ) direction.

If for convenience of calculation we choose the point Q above the center point ρ by the amount $x/b = 0.5$, we will have $\cos\theta_1 = \cos\theta_2 = \cos\left(\frac{\pi}{4}\right) = \sqrt{2}/2$ i.e., $\theta = 45°$ and the ρ components will cancel by symmetry.

The factor .7155 having been obtained by a previous calculation and is the value of the field from one loop on the axis at a distance equal to b/2, we now need to know the values for P_n (cos θ) and P'_n (cos θ). P_n (cos θ) and P'_n (cos θ) are defined as follows: let $\cos\theta = \mu$

$$P_1(\mu) = \mu \qquad P'_1(\mu) = (1-\mu^2)^{1/2}$$

$$P_2(\mu) = 1/2\ (3\mu^2-1) \qquad P'_2(\mu) = 3\ (1-\mu^2)^{1/2}\ \mu$$

$$P_3(\mu) = 1/2\ (5\mu^3-3\mu) \qquad P'_3(\mu) = 3/2\ (1-\mu^2)^{1/2}\ (5\mu^2-1)$$

$$P_4(\mu) = \left(\frac{35\mu^4 - 30\mu^2 + 3}{8}\right) \qquad P'_4(\mu) = 5/2\ (1-\mu^2)^{1/2}\ (7\mu^3-3\mu)$$

$$P_5(\mu) = \left(\frac{63\mu^5 - 70\mu^3 + 15\mu}{8}\right) \qquad P'_5(\mu) = \frac{15}{8}\ (1-\mu^2)^{1/2}\ (21\mu^4 - 14\mu^2 + 1)$$

and in general $P'_n(\mu) = (1-\mu^2)^{1/2} \frac{d}{d\mu} P_n(\mu)$, for $-1 < \mu < +1$.

The recurrence formula for $P_n(\mu)$ (see (8) pg. 334) or (Smythe (1) pg. 150) is

$$-(m - n - 1)\ P^m_{n+1} = (2n+1)\mu\ P^m_n - (m + n)\ P^m_{n-1}$$

The recurrence formula may be used to obtain numerical values of P_n and P'_n quite accurately (except near zeros of P_n and P'_n) as well as to generate the analytic form of P_n and P'_n for higher values of n.

Thus

$$P_{n+1} = \frac{2n+1}{n+1}\ \mu\ P_n - \frac{n}{n+1}\ P_{n-1}$$

and

$$P'_{n+1} = \frac{2n+1}{n}\ \mu\ P'_n = \frac{n+1}{n}\ P'_{n-1}$$

The values of P_n and P'_n are not particularly easy to come by as tables do not generally include both P_n and P'_n nor do they usually give values for n over say 5. It was necessary to calculate most of the P_n and P'_n values shown in the table below directly from the recurrence formulas. No guarantee of accuracy is claimed. For 1% accuracy probably at least two more terms should be computed but these values will illustrate the procedure.

Table of Values for $P_n(\mu)$, $P'_n(\mu)$ $(r/b)^{n-1}$ and Numerical Factors for $r/b = \sqrt{2}/2$ $\mu = \sqrt{2}/2$

n	P_n	$(r/b)^{n-1}$	factor	$(-1)^{\frac{n+1}{2}}$	Product of term.	Sum
1	.70711	1.00000	1	-1	-.70711	
3	-.17678	0.50000	3/2	+1	-.13258	
5	-.37565	0.25000	15.8	-1	+.17608	
7	+.12681	0.12500	35/16	+1	+.03467	
9	+.28511	0.06250	315/128	-1	-.04385	
11	-.10394	0.03125	693/256	+1	-.00879	-.68158

n	P'_n	$(r/b)^{n-1}$	factor	$(-1)^{\frac{n+1}{2}}$	Product of term.	Sum
1	.70711	1.00000	1	-1	-.70711	
3	1.59099	0.50000	1/2	+1	+.39775	
5	-0.94437	0.25000	3/8	-1	+.09322	
7	-2.35888	0.12500	5/16	+1	-.09214	
9	+1.22326	0.06250	35/128	-1	-.02095	
11	+2.93214	0.03125	63/256	+1	+.02255	-.30664

From these values we can obtain

$$H_r/H_{or} = -\frac{1}{.7155}(-.6816) = +.9526$$

$$H_\theta/H_{or} = +\frac{1}{.7155}(-.3066) = -\ .4285$$

These are the values at the point Q in the r and θ directions for one loop (loop 1).

For both loops the x and ρ components are given below:

	loop 1		loop 2	
Components $\rightarrow$	x	ρ	x	ρ
H_r/H_{or}	$.9533 \cos\theta$,	$.9533 \sin\theta$	$(+.9533)\ (+\cos\theta)$,	$+.9533\ (-\sin\theta)$
H_θ/H_{or}	$-.4288\ (-\sin\theta)$,	$-.4288\ (+\cos\theta)$	$(-.4288)\ (-\sin\theta)$,	$-.4288\ (-\cos\theta)$

$$\cos\theta = \sin\theta = \sqrt{2}/2 = .707$$

With the signs chosen for the appropriate directions.

Adding components in the x and ρ directions

total

$H_x/H_{or} = 1.9532$

$H_\rho/H_{or} = 0$, that is, at this point the field is horizontal or parallel to the x-axis.

This compares to the value of H_x/H_{or} at P (i.e., on the axis below Q) which is:

$$H_x/H_{or} = 1.432$$

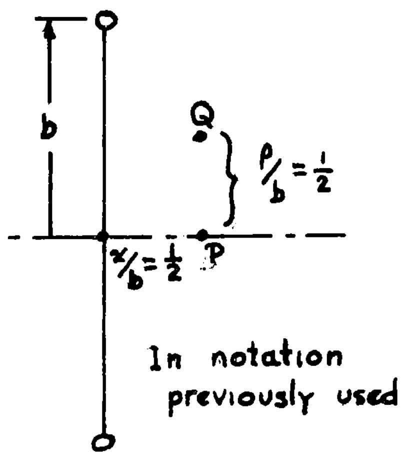
or

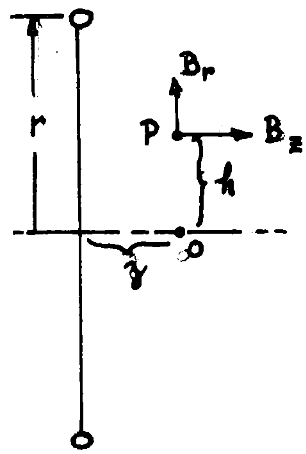
$$100 \times \frac{H_Q - H_P}{H_P} = 26.7\%$$

H_Q is 26.7% greater than H_P.

If we do this same calculation using the Hart tables the work is *greatly simplified* and the chance of error is greatly reduced because fewer calculations are required.

Thus, for one loop,





then form: $Z = z/h$

$R = r/h$

table is in Z and R coordinates.

Thus $z = 1/2b$

$h = 1/2b$

$r = b$

$Z = 1 \quad R = 2$ Enter table III (Hart).

and obtain $F_Z = 217.29 \qquad B_Z = KI/h\ F_Z$

$F_R = 80.84 \qquad B_R = KI/h\ F_R$

We must also obtain the field at P(ours), 0(Hart) here.

$z = 1/2 \quad h = 0$. This requires the use of Hart's table IV in terms of W.

$r = b$

$z = 1/2b$

$W = r/z = 2$

$$G = 449.59$$

$$B_Z = KIG/r = KIG/b \quad \text{where } I = \text{current}$$
$$K = \text{constant factor}$$

Forming the ratios as before

$$\frac{B_Z}{B_{ZO}} = \frac{H_Z}{H_D} = \frac{H_r}{H_{or}} = \frac{KI/h}{KI/b}\frac{F_Z}{G} = \frac{b}{h}\frac{F_Z}{G} = \frac{b}{b/2}\frac{F_Z}{G} = 2\,\frac{F_Z}{G}$$

$$\frac{H_r}{H_{or}} = 2\,\frac{F_Z}{G} \quad \text{for one loop.}$$

or

$$\frac{H_r}{H_{or}} = 4\,\frac{F_Z}{G} \quad \text{for two loops.}$$

Similarly, $\frac{H_\theta}{H_{or}} = 2\,\frac{F_R}{G}$ for one loop and because of symmetry $\frac{H_\theta}{H_{or}} = 0$ for two loops.

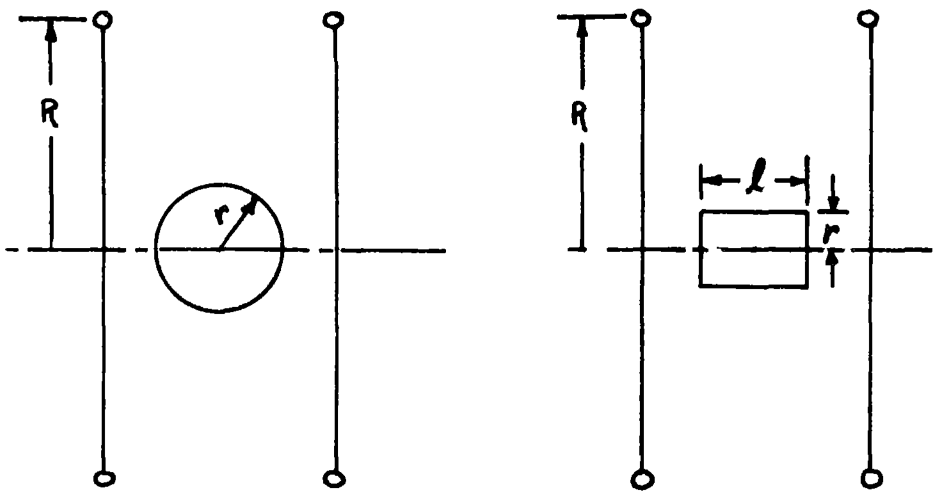
$$H_r/H_{or} = 4\,\frac{217.29}{449.59} = 1.9332$$

This compares to 1.9532 the value obtained by the summation of the two series in Legendre polynomials the small difference being due to the fact that only 6 terms (odd, 1 thru 11) were summed. It can be seen clearly that the amount of time required using the Hart tables is only a fraction of that required in finding the series sum.

The Hart tables can be used for a distributed winding of rectangular cross section and circular symmetry (cylindrical) of any relative size in a similar fashion. The only complexity added is that four values have to be determined (essentially one for each corner of the coil) and combined by well defined simple rules in order to determine the radial and axial field components.

Using the Hart tables, a plot of values of the field for a series of points can be made and the region of desired field uniformity can be determined to a first approximation by inspection. This can then be improved by the calculation of a series of additional points in the region of interest.

Some useful information on Helmholtz coils has been reported by Wolff[9]. He gives the following regions of uniformity.



Spherical region

r = 0.1R 0.02% uniformity

r = 0.2R 0.2% uniformity

Cylindrical region

1% uniformity

1 = 0.6r r = .25R

or

1 = 0.4r r = 0.35R

5% uniformity

1 = 1.0r r = 0.4R

or

1 = 0.5r r = 0.5R

0.1% uniformity

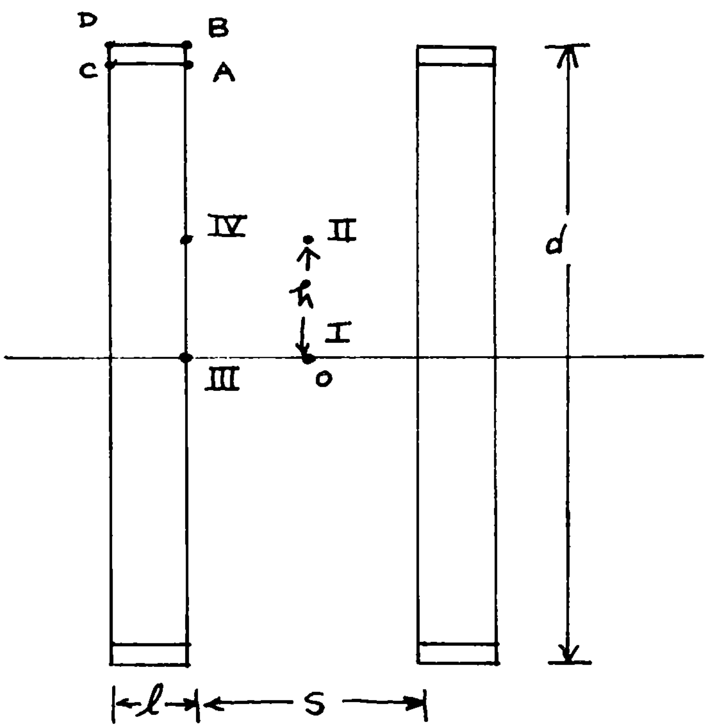
1 = 0.2r r = 0.2R

or

1 = 0.3r r = 0.15R

The assumption is made here that the coil cross section is very small compared to the coil diameter. This is *not* a restriction where utilizing the Hart tables.

A calculation for a practical size Helmholtz pair of coils, circular and spaced one radius apart, of *finite size coil cross section* is given below. The calculations are done using the Hart tables and the details of the calculation are not shown. The resulting field at four points is given. The region chosen is much larger than is generally chosen for a Helmholtz pair and therefore may prove useful.



d = 50 cm (coil dia)

s = 20 cm (coil spacing-inside)

1 = 5 cm (coil length)

t = 0.90 cm (coil thickness)

h = 10 cm

I = 1.0 amp (current in coil)

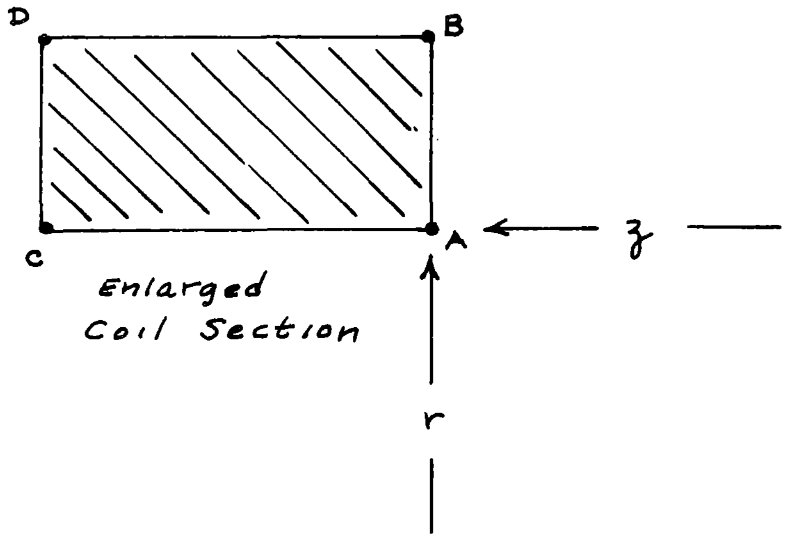
N = 50 each (number of turns)

wire size #19 enameled.

J = current density in coil = $N \frac{I}{A}$ = 109.65 amps/cm^2.

Coordinates of points where field is calculated (origin at center of coil pair).

Point	Z	r = h
I	0	0
II	0	10 cm
III	10 cm	0
IV	10 cm	10 cm



Z measured from points I through IV to corner points A through D.
r measured from coil axis to points A through D.

The flux density calculated at the points I, II, III, and IV is given below. B_Z is in the axial direction and B_r is in the radial direction.

	B_Z (gauss)	B_r (gauss)
Point I	1.788	0
II	1.757	0
III	1.739	0
IV	1.834	.056

If we consider point I (at the center of the pair) as the point from which we compute deviations we find,

Point II	1.7% low
III	2.7% low
IV	2.6% high

and a radial field at point IV amounting to less than 1/30 of the center field.

This shows that if an inhomogeneity of about 3% can be tolerated a very large volume is usable inside a Helmholtz pair. The cylinder length extending from the face of one coil to the other in this case is equal to 40% of the coil radius and with a radius equal to 20% of the coil radius.

For better than 3% homogeneity we have a cylinder; $l = 0.8R$, $r = 0.2R$.

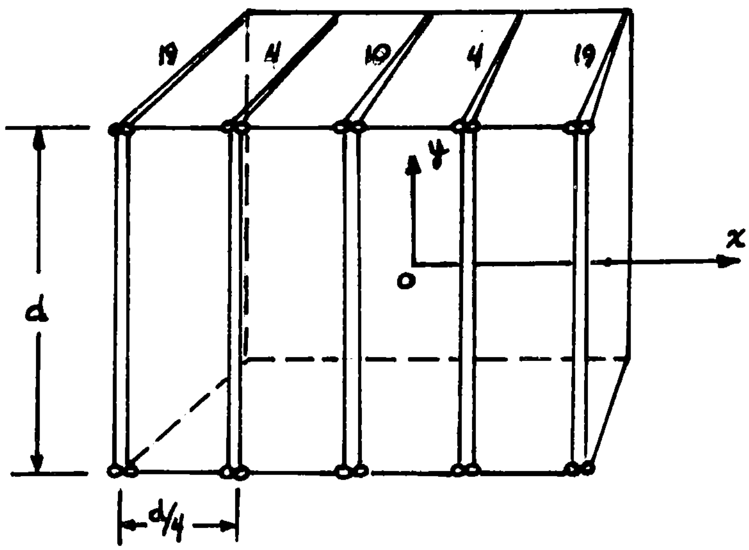
The absolute value of the flux density can be adjusted to any desired value by adjusting the current in the coils, subject only to overheating of the coils by I^2R losses.

Other Coil Configurations for Uniform Fields (more than two loops)

So far we have discussed only the very simple configuration of a single loop or two loops placed with their centers one loop-radius apart, the so-called Helmholtz configuration.

This configuration produces a region of relatively small volume of uniform field compared to the total region occupied by the two coils. By the addition of more loops it was realized long ago that a greater relative volume of uniform field could be produced. This comes about by the fact that more parameters are available for adjustment (such as loop radius, loop spacing, loop current and number of loops) to obtain the desired uniformity.

Many different configurations are possible once we allow multiple coils. Most of these utilize circular coils of varying spacings and numbers of turns. One interesting configuration which is non-circular is the square five coil arrangement known as the Rubens coil, Rubens[10]. This configuration has five equally spaced square coils in line, forming a cube. The number of turns on each of the coils is then selected to give the largest possible region of uniform field in the interior. A relatively simple derivation is employed to obtain the required number of ampere-turns for each coil. By choosing different criteria for uniformity many different sets of values for the current-turns ratio may be found. The particular set chosen for computation in this paper is 19:4:10:4:19. If the coils are series connected the number of turns in each coil is then made proportional to these numbers.



Schematic arrangement of Ruben's Coil.

Integers are proportional to the ampere-turns in each loop.

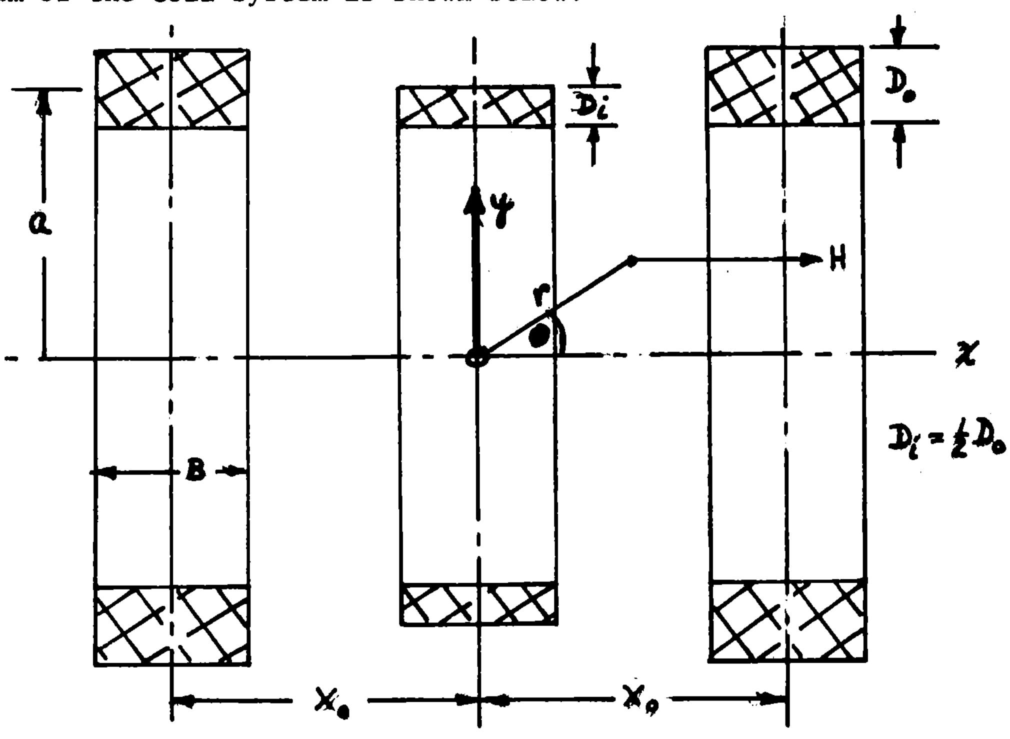
If x and y are measured as shown, the ratio of the field at x and y to that at the center is as shown in the table abbreviated from Rubens.

Table of H_{xy}/H_{oo}

	y/(d/2)→ 0	0.2	0.5
x/(d/2) 0	1.000	1.003	1.009
0.2	.9997	.9995	.997
0.5	.9990	1.0005	.996

As can be seen over this whole region, the field does not deviate from that at the center by more than 4 parts per 1000 or 0.4%. For a 1% deviation he shows that the volume of homogeneity can be about seven times greater for the cube coil than for a Helmholtz pair of the same diameters as the cube coil edge (d). Similar comparisons are made for other percent deviations. The analysis given by Rubens does not take into account the finite size of the coils (distributed windings). Some reasonable approximations can be made for the practical case which should permit results very close to those given by Rubens. Discussion of approximate methods will be treated in later sections.

Another interesting system using three circular coils having the same coil-forms (bobbins) and of finite winding cross section is that given by Barker[11]. This paper is especially valuable because it considers carefully, very practical and significant problems such as the wire size, the insulation size, the similarity of coils forms and other fine points of coil winding and construction. A complete set of formulae are developed taking into consideration just these points and examples of their use in the construction of a real system is given. The starting point of his system is given. The starting point of his derivation is the standard expansion of the field in a series of Legendre polynomials in which the coefficients of two of these terms is made equal to zero (this will be discussed more fully later). A diagram of the coil system is shown below.



All three *coil-forms* or *bobbins* are exactly the same. The center with one-half the number of turns of the outer coils.

The uniform field region is roughly twice the linear dimensions (approx. 8 times the volume) of a Helmholtz pair using the two outer coils.

Typical values for the extent of the field uniformity are given in the table from Barker:

	median plane (thru 0)		along axis		
uniformity	0.1%	1%	uniformity	0.1%	1%
y_{max}	0.411a	0.603a	x_{max}	0.338a	0.496a

Approximate coil separation: $x_o = 0.76a$.
Adjusted by calculation to give desired uniformity.

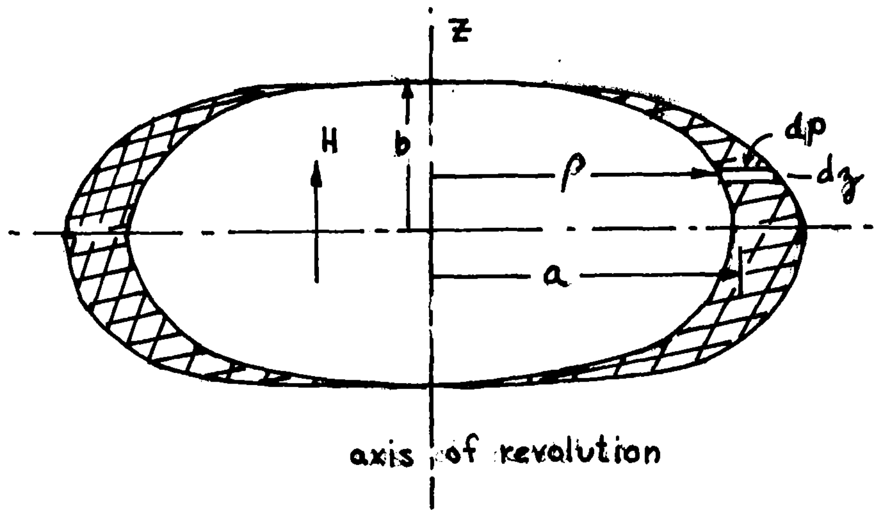
This is one of the very few articles which takes into account the practical construction problems of a coil with finite winding depth and length. It should be studied for this information even if a different configuration is ultimately chosen.

Ellipsoidal Coil

A coil can be constructed in the shape of an ellipsoid of revolution in which the magnetic field is uniform in the direction of the axis of symmetry throughout the *whole internal volume*, see Blewett[(2)].

In cylindrical coordinates the equation of the ellipsoid is:

$$\frac{\rho^2}{a^2} + \frac{z^2}{b^2} = 1$$



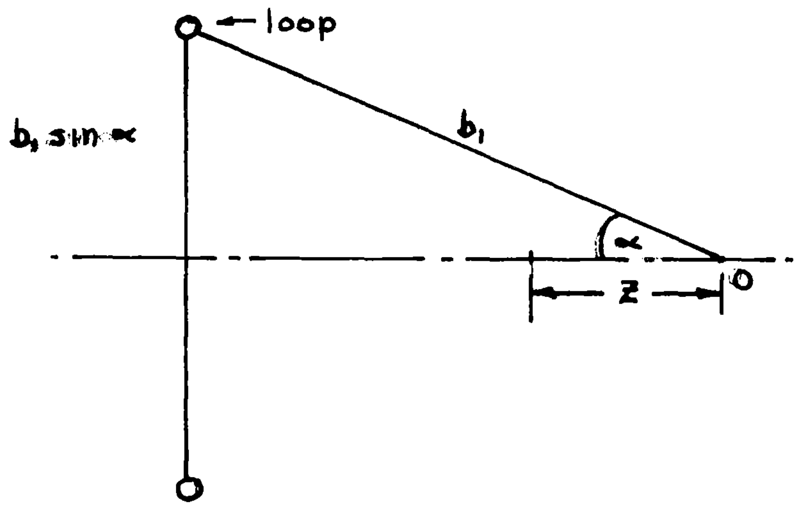
If the winding has constant current density and the wall thickness in the radial direction is held constant ($d\rho = c$) by varying the wall thickness the field inside will be uniform in the z direction. In a real coil $(d\rho)_{max}$ must be very small with respect to a or b, whichever is smaller for this simple condition to hold. A coil of this configuration would be very difficult to build and since it completely envelops the internal volume prohibits access to that volume. While two coils (above and below $z = 0$) could be made so that the halves could be separated the coil form along that surface would have to be of negligible thickness. Blewett discusses configurations derived from this shape (pancake, sphere and infinite solenoid), their energy storage and power consumption.

A comparison of circular coil systems with from one to 8 loops is given by Pittman and Waidelich[12]. In this paper they also develop the equations for coil size, spacing and current to achieve the maximum uniform region in a four loop coil arrangement.

It is pertinent *here* to give a rather full review of the numerous coil arrangements which they discuss because as far as I know no other single paper discusses so many different arrangements.

Fifteen different systems are described; from a single loop (Ampère), two loops (Helmholtz), three loops (Barker, Maxwell and their own "minimum" solution, 4 loops (5 varieties), 6 loops (three varieties) and 8 loops (2 varieties).

Their notation is somewhat different from that used elsewhere, but involves the use of a Lengendre polynominal expansion of the magnetic field from a loop.



If the origin is at "O", radius of the loop $b_1 \sin \alpha$, the distance from the origin to the center of the loop $\cos \alpha = X$, then the magnetic field on the axis H is:

$$H = \sum_{n=0}^{\infty} a_n Z^n$$ where Z is the distance along the axis from the origin.

where

$$a_n = \frac{NI\ (1-x^2)}{2b^{n+1}}\ P'_{n+1}\ (X)$$

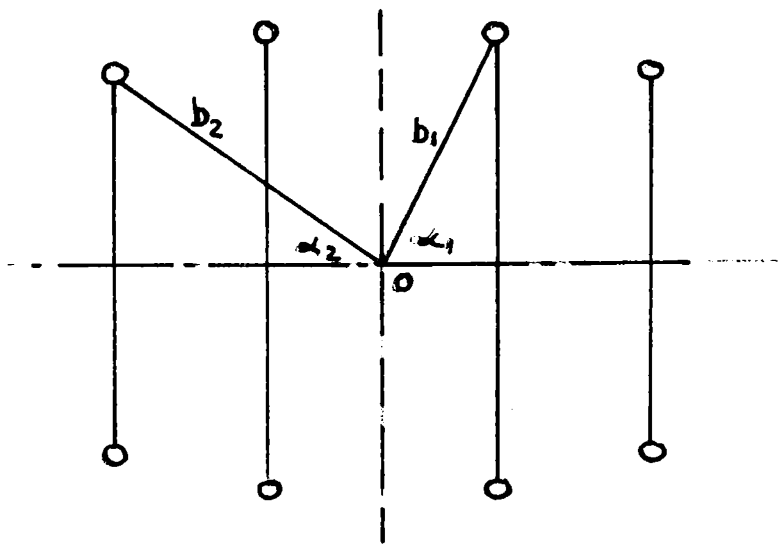
and

$$P'_{n+1}(X) = \frac{dP_{n+1}(X)}{dX}$$

and $P_n(X)$ is the n^{th} order Legendre polynomial with argument $X = \cos\alpha$ and P'_{n+1} is the derivative.

Two or more coils on a common axis may be used to produce a more uniform or homogeneous magnetic field than is possible with one coil.

For a symmetric four-coil system as shown in the figure.



we have

$$H = \sum_{\substack{n=0,2\\ \text{even}}}^{\infty} a_n Z^n$$

where

$$a_n = \frac{N_1 I_1 (1-X_2^2)}{b_1^{n+1}} P'_{n+1}(X) + \frac{N_2 I_2 (1-X_2^2)}{b_2^{n+1}} P'_{n+1}(X_2)$$

These equations are shown, not because they will be used by us for calculation but rather to show the method of obtaining a uniform field.

If we will write the equation for the magnetic field for 4 coils as below.

$$\frac{H}{a_o} = [1 + a_2Z^2 + a_4Z^4 + a_6Z^6 + a_8Z^8 + a_{10}Z^{10} \ \ldots]$$

We obtain the ratio of the field at Z in terms of the field at Z = 0 i.e., the origin. At the origin $H = a_o$. What we want then is for H/a_o to be *independent* of Z, i.e., $H/a_o = 1$.

This can be accomplished to a very high degree of accuracy, because of two factors operating in our favor. First, the series is convergent, each term is successively smaller than the one preceding it. Second, several parameters are available which can be adjusted, namely, the radius of the coils $r_1 = b_1 \sin \alpha_1$ and $r_2 = b_2 \sin \alpha_2$ by adjusting α; the separation of the coils $x_1 = \cos \alpha_1$ and $x_2 = \cos \alpha_2$ and finally, N_1I_1 and N_2I_2 the turns current product in the two pairs of coils. It turns out that this number of parameters is just sufficient (for particular values of these parameters) as a_2, a_4 and a_6 can be made zero and even make some adjustment in a_8 to make it a minimum. Hence, we have

$$\frac{H}{a_o} = [1 + 0 + 0 + 0 + a_8Z^8 + a_{10}Z^{10} + \ldots]$$

or

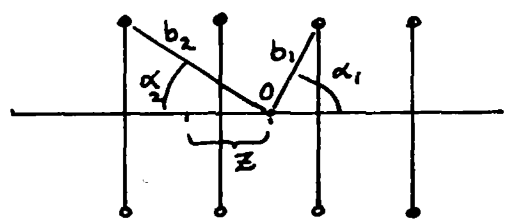
$$\frac{H}{a_o} = [1 + a_8Z^4] \quad \text{approximately since } a_{10}Z^{10} \text{ etc. are very small.}$$

A similar situation exists for three coils. Here the two inner coils are brought together to form a single center coil plus an outer pair thus becoming effectively three loops.

A modified portion of the table given in Pittman and Waidelich is shown below. The Barker 3 and 4 coil configurations are given in preference to several others because of the desirability of equal loop diameters. Some of the other configurations have the loops on the surface of a sphere or other non-equal diameters with little or no advantage in the volume of uniformity. Very little advantage is gained by the *minimum* configurations also given, compared to the Barker configurations while complexity of construction is increased. The Barker configurations, of 3 and 4 loops are not as good as the Pittman and Waidelich configuration since the a_6 and a_8 terms are not minimum (in the 3 and 4 coil configurations respectively) but the a_2, a_4 and a_2, a_4 and a_6 terms are made zero. By requiring equal diameter we do not have freedom to minimize the a_6 or a_8 terms but we have the space and construction advantages of equal diameters.

Number of Coils	2	3	3	4	4
Name Assumptions	Helmholtz none	Barker Equal dia. coils	Min a_6 none	Barker Equal dia. coils	Min a_8 none
a's = 0	a_2	a_2, a_4	a_2, a_4	a_2, a_4, a_6	a_2, a_4, a_6
coef. of next An term	-1.8	-2.037	-2.036	-2.471	-2.222
coef of next Bm term	—	-3.214	-2.925	-7.828	-2.758
$X_1 = \cos \alpha_1 = 1/2$ spacing of pair $1/b_1$	0.4472	0.0	0.0	0.2363	0.2724
$X_2 = \cos \alpha_2 = 1/2$ spacing of pair $2/b_2$	—	0.6051	0.6163	0.6852	0.7484
b_1/b_2	—	1.2561	1.1970	1.3341	1.0713
N_1I_1/N_2I_2	—	3.7632	3.0758	2.2606	.9041
Z/b_m for $\Delta H/H \leq 10^{-5}$	.0486	0.1208	0.1228	0.1858	.2094
Z/b_m for $\Delta H/H \leq .01$	0.164	0.382	—	0.435	—

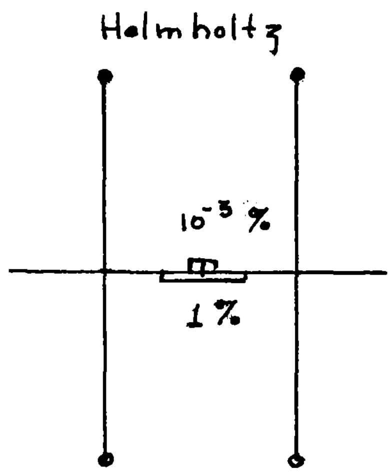
b_m = the maximum b.

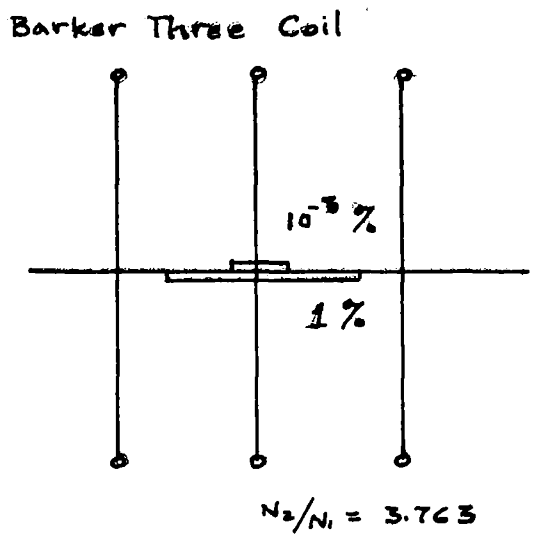


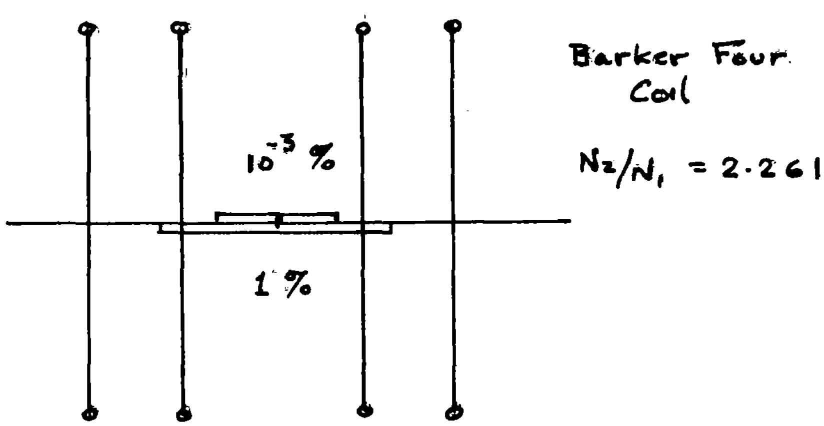
Note: Since in the three loop systems, the center loop is really two loops brought together the number of ampere turns of the center loop becomes $2N_1I_1$.

Three of these coil configurations are shown in the same scale with their approximate dimensions, spacing, and the extent of uniform field for 10^{-5} nonuniformity and 10^{-2} nonuniformity. For series connection $I_1 = I_2$ and turns ratio is shown.

The radius of all coils is chosen to be the same in the figures.



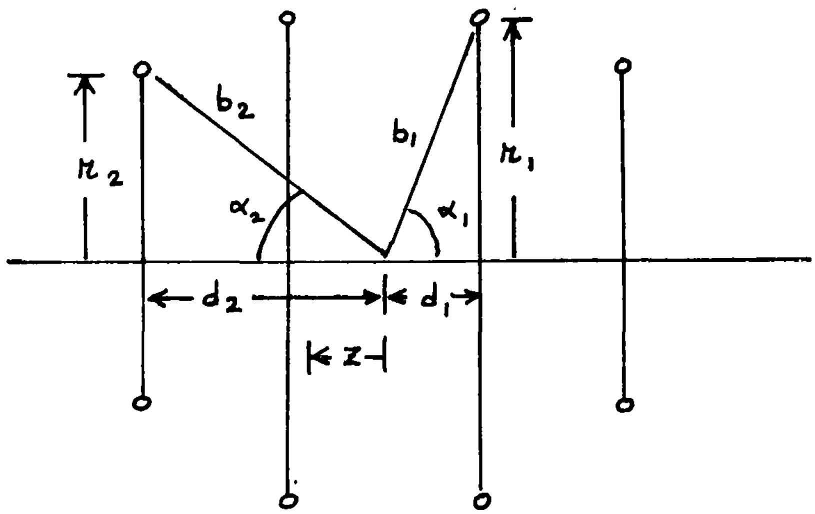




The extent of the uniform field for the three configurations are, at 1%, very roughly in the ratio 3/8/10 in the order shown for same loop diameters.

As can be seen from the table the Pittman & Waidelich configurations for 3 and 4 coils are very smiilar to the Barker coils but the coil pairs are slightly different in diameter.

Now there are various relationships between the quantities shown and those useful in constructing or designing a set of coils, these are related to the figure shown.



Then:

$$x_1 = \cos\alpha_1 \qquad x_2 = \cos\alpha_2$$
$$r_1 = b_1 \sin\alpha_1 \qquad r_2 = b_2 \sin\alpha_2$$
$$d_1 = b_1 \cos\alpha_1 \qquad d_2 = b_2 \cos\alpha_2$$

$$Z = b_2 \text{ (constant for coils)}$$

and from this

$$d_1 = r_1/\tan\alpha_1 \qquad d_2 = r_2/\tan\alpha_2$$

and $$Z = (r_2/\sin\alpha_2) \text{ (constant for coils)}$$

The constant for the coils has to be determined from the Pittman and Waidelich computation for the degree of inhomogeneity permitted and for this equal to 10^{-5} the constant is given in the table. For other values we must calculate the constant from the B_m values given in the table.

The B's (B_6 and B_8) for the three and four coil configurations can be used to calculate the extent of field (Z/b_m) for any desired homogeneity $\Delta H/H_o$ as follows:

$$\frac{\Delta H}{H} = \left| B_n \left(\frac{Z}{b_m}\right)^n \right|$$

In the case of the Barker 4 loop configuration let $\Delta H/H = .01$ (1%), then $B_8 = 7.828$.

Then

$$\left(\frac{Z}{b_m}\right)^8 = \frac{.01}{7.828}$$

and

$$(Z/b_m) = (Z/b_2) = \left(\frac{.01}{7.828}\right)^{1/8}$$

$$(Z/b_2) = 0.435 \quad \text{for 1\% homogeneity.}$$

For the Barker 3 loop configuration

$$B_6 = -3.214 \qquad \Delta H/H = .01$$

and

$$(Z/b_m) = \left(\frac{.01}{3.214}\right)^{1/6}$$

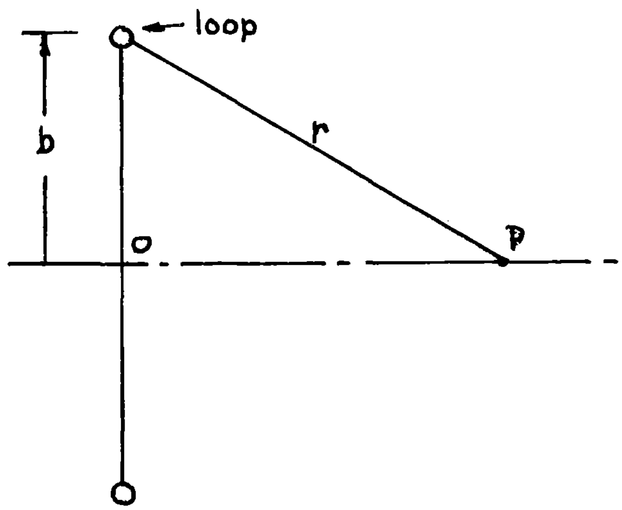
$$= 0.382 \quad \text{for 1\% homogeneity}$$

The 1% and 10^{-3}% regions are shown in the figure for Helmholtz and Barker 3 and 4 coil configuration and are indicated by the dotted and solid lines respectively.

Using these formulas and the abbreviated table given here three or four coil configurations can be readily designed to obtain any size field with the desired degree of homogeneity.

Loops of Finite Cross Section (Distributed Currents and Approximations)

For the most part, the discussion of various loop configurations has considered the windings to be of infinitely small cross sectional area with the exception of the Barker Coils and the Hart tables. In real life, coils windings have finite cross sections and while this can be accurately taken into account by integrating over the area, assuming a uniform current density, it is often useful to have an approximation to simplify calculations instead of an integration to complicate them. Let us make some simple assumptions and see what approximations will be useful.



The magnetic field at point P on the axis of a loop is given by

$$H_{axis} = \frac{\mu_o i}{2} \frac{b^2}{r^3} \text{ (MKS)} \qquad H_{axis} = 2\pi i \frac{b^2}{r^3} \text{ (CGS)}$$

In either case, this may be written as

$$H = K \frac{b^2}{r^3}$$

To illustrate how we can make an approximation for finite dimensions of a coil winding we will make the following calculation. This calculation shows that under certain conditions, by using the average dimensions of a finite winding good accuracy can still be obtained.

In the figure below we have a coil of dimensions as shown. b_1 is the outer radius, b_2 the inner radius, b_a the average radius = $(b_1 + b_2)/2$, the radial width of the coil is $(b_1 - b_2)$ and the length of the coil is $[(r_1^2 - b_1^2)^{1/2} - (r_2^2 - b_2^2)^{1/2}]$.

The value of the field at the point P from loop, is

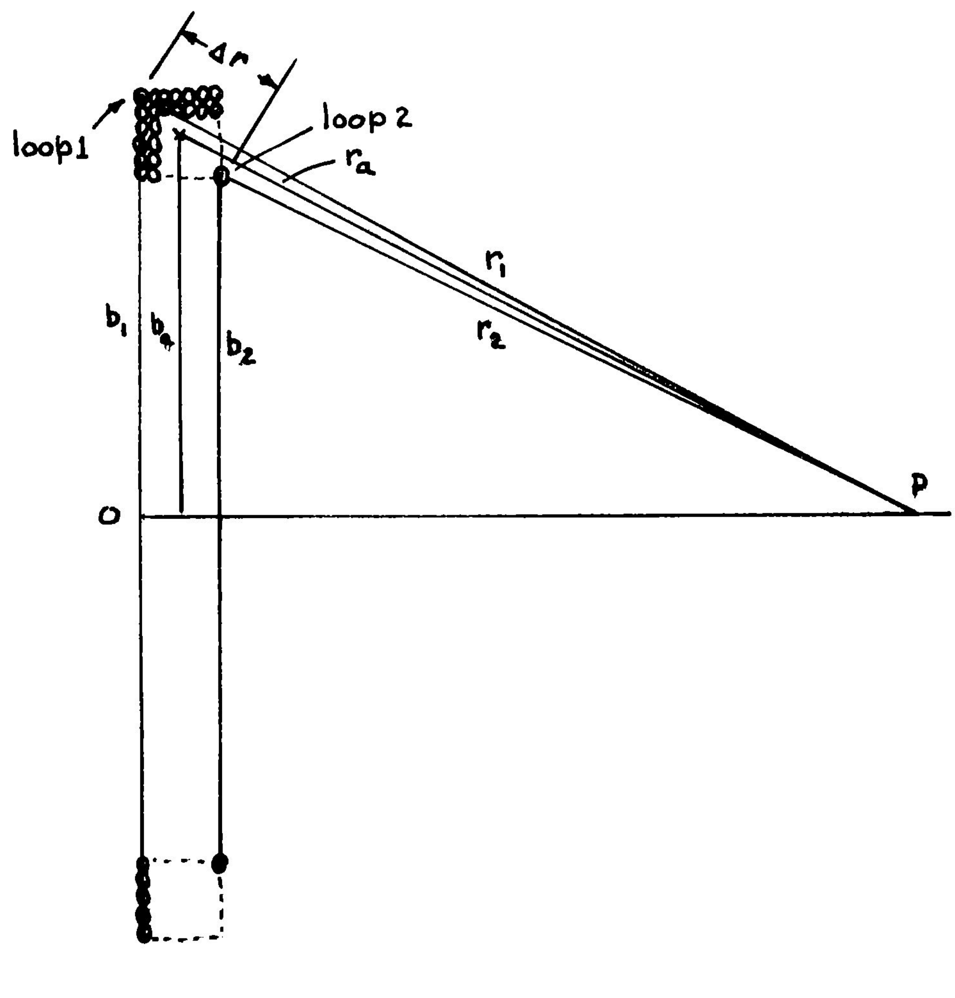
$$H_1 = K \frac{b_1^2}{r_1^3}$$

and for loop 2 is

$$H_2 = K \frac{b_2^2}{r_2^3}$$

while that due to both is

$$H = H_1 + H_2 = K \left[\frac{b_1^2}{r_1^3} + \frac{b_2^2}{r_2^3} \right]$$



Whereas the value of H due to two loops of average r and average b is:

$$H_a = K \frac{2\,b_a^2}{r_a^3}$$

and the fractional error in H made by this approximation is

$$\text{fract. error} = \frac{H - H_a}{H} = 1 - \frac{H_a}{H}$$

We then have to evaluate H_a/H, let

$$b_a = (b_1 + b_2)/2 \qquad r_a = (r_1 + r_2)/2$$

$$\Delta b = b_1 - b_2 \qquad \Delta r = r_1 - r_2$$

and

$$H_a/H = \frac{2\,b_a^2}{r_a^3} \bigg/ \frac{b_1^2}{r_1^3} + \frac{b_2^2}{r_2^3} \qquad \text{is}$$

the fractional change in H due to approximation of average b and r, the K's cancelling, then for

$$2b_a = b_1 + b_2$$

$$2b_a = b_1 - b_2$$

$$b_1 = b_a + \frac{\Delta b}{2} \qquad r_1 = r_a + \frac{\Delta r}{2}$$

$$b_2 = b_a - \frac{\Delta b}{2} \qquad r_2 = r_a - \frac{\Delta r}{2}$$

This becomes

$$\frac{H_a}{H} = \frac{2b_a^2}{r_a^3} \; \frac{r_1^3\,r_2^3}{b_1^2 r_2^3 + b_2^2 r_1^3}$$

substituting for r_1 and r_2, b_1 and b_2 their values in terms of b_a and Δb and r_a and Δr we obtain,

$$\frac{H_a}{H} = \frac{2b_a^2}{r_a^3} \cdot \frac{(r_a + \frac{\Delta r}{2})^3 \; (r_a - \frac{\Delta r}{2})^3}{(b_a + \frac{\Delta b}{2})^2 \; (r_a - \frac{\Delta r}{2})^3 + (b_a - \frac{\Delta b}{2})^2 \; (r_a + \frac{\Delta r}{2})^3}$$

Carrying out some algebra and assuming that

$$\frac{\Delta b}{b_a} \simeq \frac{\Delta r}{r_a}$$

we have:

$$\frac{H_a}{H} = \frac{2\;(1 - (\frac{\Delta b}{2b_a})^2)^3}{(1 + \frac{\Delta b}{2b_a})^2 \; (1 - \frac{\Delta b}{2b_a})^3 + (1 - \frac{\Delta b}{2b_a})^2 \; (1 + \frac{\Delta b}{2b_a})^3}$$

which reduces to

$$\frac{H_a}{H} = (1 - (\frac{\Delta b}{2b_a})^2) \quad \text{exactly}$$

The fractional error $= 1 - \frac{H_a}{H} = (\frac{\Delta b}{2b_a})^2 = \frac{1}{4}\,(\frac{\Delta b}{b_a})^2$

For a fractional error of 1% = .01

$\frac{\Delta b}{b_a}$ must be $\leq 0.2 \leq \frac{1}{5}$ which is quite generous

and for a fractional error of 0.1% = .001

$\frac{b}{b_a}$ must be $\leq .0632 \lesssim \frac{1}{16}$ which is not bad.

This says that for all practical purposes if the coil cross section dimensions are less than 1/16 of b or r we may substitute for the coil cross section the dimensions of its *center* with very little loss in accuracy. Calculations may similarly be made for off-axis fields if the distance from the point in question to the coil is small compared to the extent of the coil cross section.

As with all approximations, care must be observed and an occasional check made of the error occurring in making the approximation.

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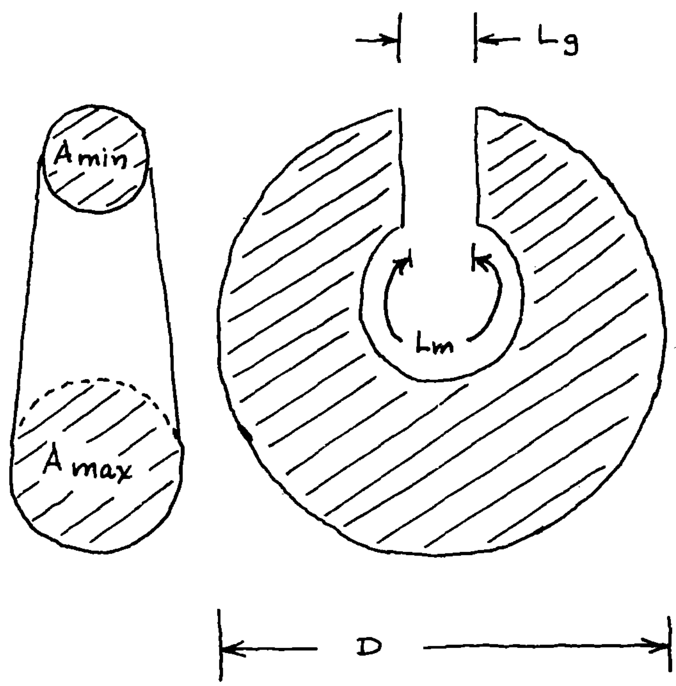
CHAPTER 4
PERMANENT MAGNETS

Permanent magnets as mentioned in the first chapter were known as exciting curiosities long before magnetic fields were associated with currents or moving charges. The early permanent magnets were relatively weak. Today, with modern materials, flux densities to 10,000 gauss (1 weber/m^2) (being chiefly limited by the saturation of soft iron pole pieces) and filling air gap volumes to many cubic centimeters are possible. A preliminary example to obtain some idea of the size and weight of magnets required to produce a given flux in a specific air gap is given below.[1]



The dimensions are:

B_g = flux density in gap = 4000 gauss

L_g = length of working gap = 2 cm

L_m = length of magnet, min = 14.5 cm

A_{max} = max cross section area = 16.3 cm^2

D = external diameter magnet = 12 cm

V_m = vol magnet $\approx$ 300 cm^3

Weight of magnet (Alcomax III) equals approximately 2.2 Kg (4.8 lbs).

This magnet produces a flux density in the gap, 2 centimeters long by about 2 centimeters diameter of 4000 gauss (0.4 webers/m^2). This is a relatively large flux density in a volume of about 6.3 cm^3 from a magnet weighing less than 5 pounds. Rule-of-thumb formulas for proportioning magnets to other sizes show for a given flux density in the gap that

length of magnet $\propto$ to length of gap

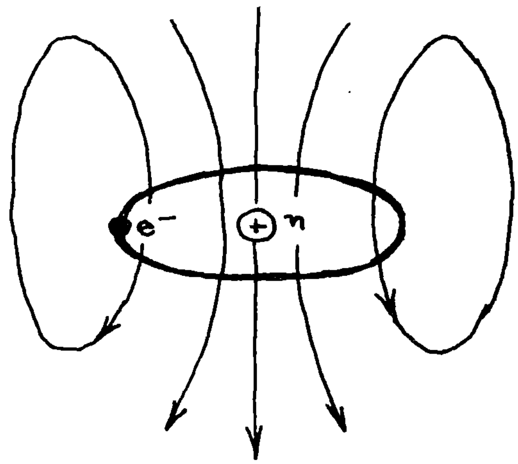
area of magnet $\propto$ to (length of gap)^2

hence, vol. of magnet $\propto$ to (length of gap)^3

Thus, to double the gap length would increase the magnet length by a factor of 2, the cross sectional area by 4 and the volume and weight by 8.

As can be seen, for very large flux densities and working gap volumes, the magnet volume increases very rapidly because of the cubic relationship. This is expected however, because it is the gap *volume* which must be filled with the flux.

Two questions now arise. One, is the magnetic field produced by a permanent magnet any different from that produced by dc current carrying coils? And two, where does the magnetic field come from in a permanent magnet? The first question can be answered categorically *no*. There is no way to differentiate one kind of magnetic field from the other. The method of producing the field gives no properties to the field which can be attributed to the way in which it was produced although it may be more convenient to produce a field by one means rather than another. Fields produced by permanent magnets require no power supply, they are compact, they are stable. If a coil alone (with no iron) was used to produce a field of the flux density in the example above very high currents would be required and a great amount of heat would be generated in the coil. This would lead to complications in the apparatus to remove the heat so that it would not effect any biological materials in the region. However, the magnetic field produced would have the same properties as the field from the permanent magnet. The design of the magnet and its pole pieces determine the flux density and the uniformity of the field, but the effect of *any* flux density on any material whatever has been shown to be the same, however that field was produced. Suppose we wind a coil of wire around a wooden cylinder. Then, except very close to the wires, the external magnetic field will look like that of a permanent magnet of the same size. Two such coils will produce forces on each other which are the same as the force produced by two bar magnets. The coil of wire will behave as if it had two poles near the ends just as the bar magnet behaves. If the coils and bars were carefully proportioned and camoulflaged they could not be distinguishable from each other by their actions or the fields they produced. How then does the magnetic field of the permanent magnet come about? In a very simplified way let us look at the structure of matter. All matter consists of atoms, each atom consisting of a nucleous surrounded by a cloud of electrons of negative charge. These electrons can be considered to be rotating in orbits at very high velocities both clockwise and counterclockwise. Each electron, although of a very small charge (the smallest known, a unit charge) will, with its high velocity, produce a magnetic field similar to that shown in the figure on the next page. In some materials, iron and other ferromagnetic materials, many of these atoms will be constrained by crystal lattice forces to have the resultant of the fields from their electrons pointing in the same direction. These groups of atoms with a net magnetic field are designated magnetic domains and are large enough to be seen with a microscope. A rather thorough and mathematical treatment is given in Hadfield(1). The direction of the fields of these domains are randomly oriented.



But the resultant magnetic fields of these domains can be aligned by the application of an external magnet field. When the external field is removed these domains remain aligned and their magnetic fields add up to produce a large resultant magnetic field. The material has become a permanent magnet. Essentially what we have done is to align electrons moving in orbits producing what can be considered as a net current sheet flowing on the surface of the material. Thus, if we consider a cylindrical bar magnet we visualize a cylindrical current sheet on the surface of the magnet. Hence, we should not be astonished to find that a coil of wire wound on a wooden rod will produce the same external field as a permanent bar magnet. This explanation is admittedly oversimplified, but even this simple description will account for the fact that when a permanent magnet is broken in half two permanent magnets are produced and not separate poles. Magnetic poles are a fiction, but for some purposes a useful fiction. Even the most modern theories of permanent magnetism are not entirely satisfactory and do not approach the rigor and completeness obtained with the theory of magnetic fields from currents in conductors. However, it is not necessary to know or understand the origin of the magnetic field of a permanent magnet to be able to use it or even to design it.

Permanent magnet materials most suitable for constructing laboratory magnets are those known as the alnicos. Alino magnets are composed of an alloy of iron, nickel, aluminum and cobalt. The percentages being in the range 10-12% Al, 15-20% Ni, 5-14% Co, 0-6% Cu and the remainder iron. After melting and casting a heat treatment sequence is followed with reheating and controlled cooling rates to develope optimum and uniform magnetic properties.

The magnetic properties obtained in magnets made according to this procedure are in the range,

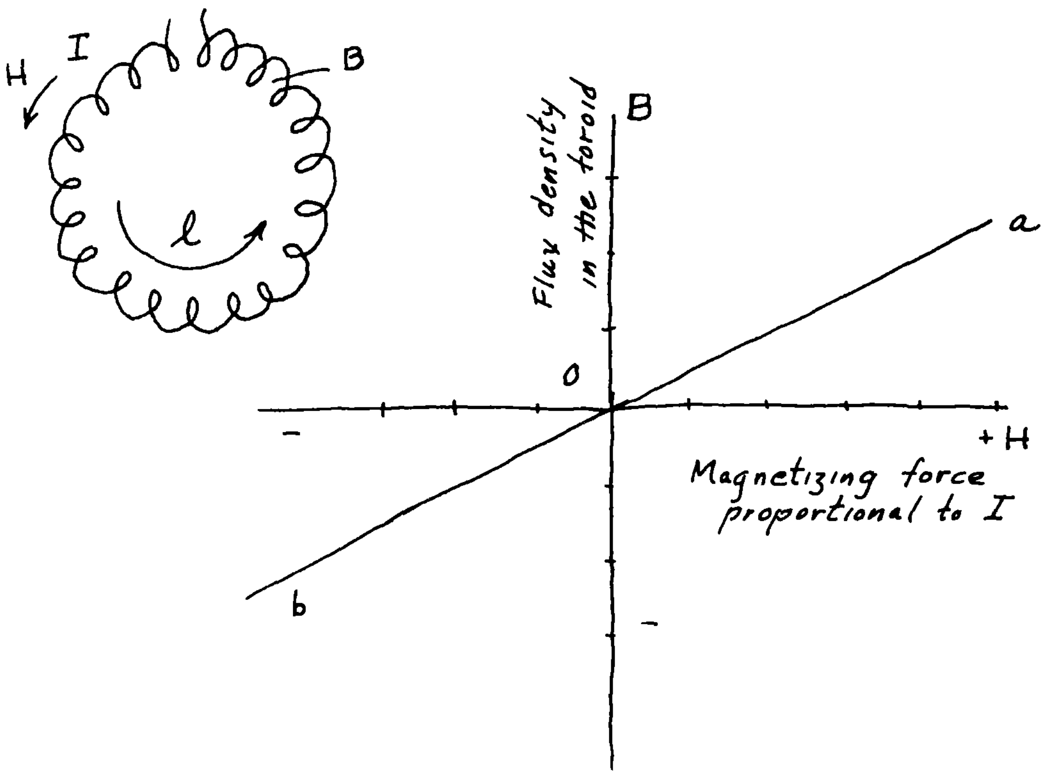
B_r (remanence) = 6000-8000 (0.6-0.8 w/m^2)

H_c (coercivity) = 450-650 oersted (36000-52000 ampere turns/meter)

B_{max} (max. energy product) = 1.5-1.7 MGO (Million-Gauss-Oersted) = (12000, 13600 joules/m^3)

These terms will be explained shortly. Modern alloys of this type are far superior to carbon steel and about twice as good as the best cobalt steel in the amount of magnetic field energy available per pound. They are far less expensive and more available than any of the exotic materials which may possess ten times the magnetic energy per unit volume.

To understand the terminology of permanent magnets let us consider a toroidal winding with n-turns of wire. The ring has central circumferential length 1 and cross sectional area A. As current is increased through the wire we find that if we plot B vs. H a curve similar to the following figure is described. We obtain a straight line oa, reducing the current we return to o, reversing the current we proceed to b and reducing -I to o we again return to o.



The magnetizing force or field intensity is given for this special case as,

$$H = \frac{NI}{l} \text{ (MKS)} \qquad H = \frac{4\pi NI}{l} \text{ (CGS)}$$

and the flux density inside the toroidal coil of the area A is on the average for $\mu = 1$

$$B = \mu_o H \text{ (MKS)} \qquad B = H \text{ (CGS)}$$

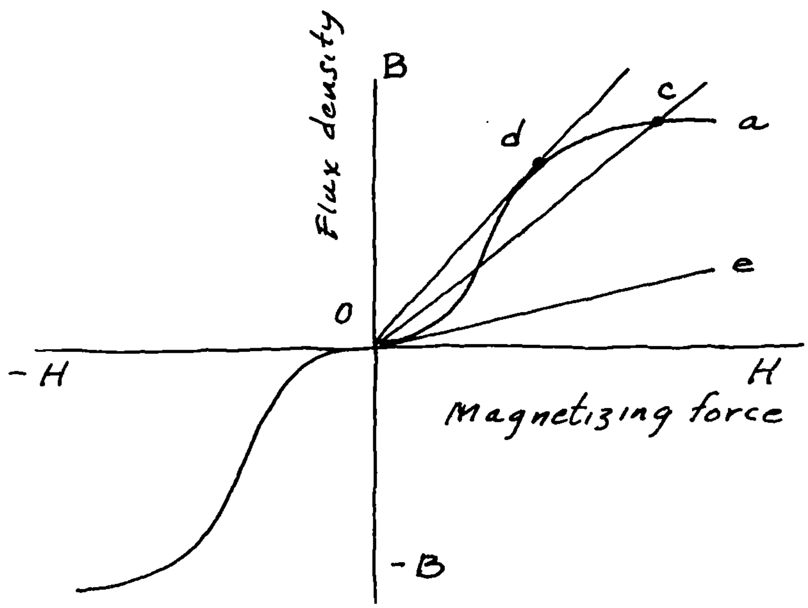
The flux in the area A will then be

$$\phi = BA \text{ (MKS)} \qquad \phi = BA \text{ (CGS)}$$

or

$$\phi = \mu_o \frac{NIA}{l} \text{ (MKS)} \qquad \phi = \frac{4\pi NIA}{l} \text{ (CGS)}$$

If we could now fill the region inside the windings with a ferromagnetic material, such as "soft iron," a material which would retain no magnetic field after the current is removed, we would find a different curve of B vs. H as shown in the figure below.

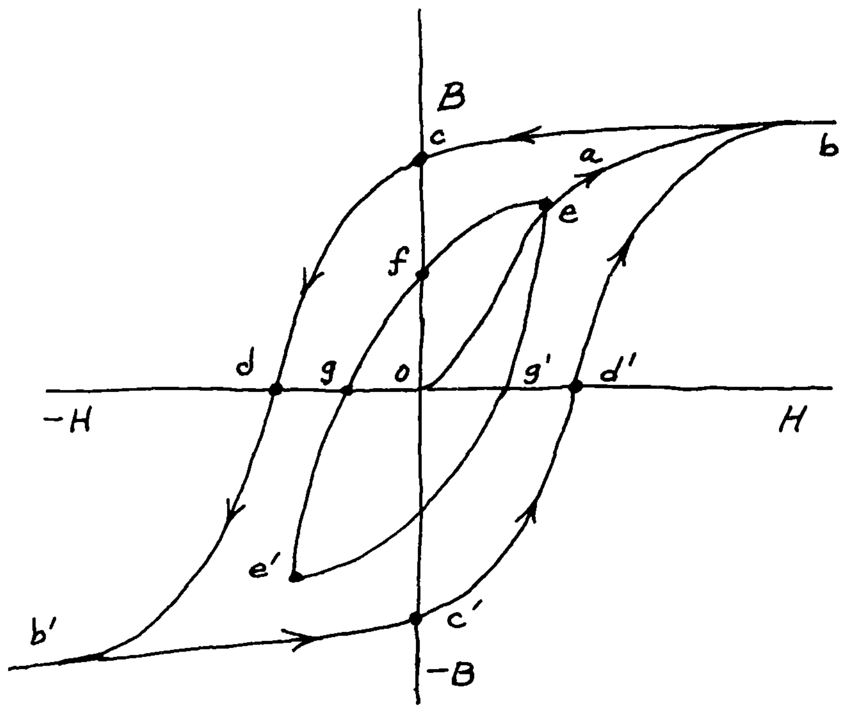


The third quadrant would be an inverted mirror image of the first quadrant. The permeability, μ, is defined as the slope of the line oc, the *maximum* permeability, μ_{max}, as the slope of the line od and the *initial* permeability, μ_i, as the slope of the line oe as the curve approaches the origin. Note that the permeability μ is not the slope of the B-H curve, this slope is called the *differential* permeability and is defined by

$$\mu_d = \frac{1}{\mu_o} \frac{dB}{dH} \text{ (MKS)} \qquad \mu_d = \frac{dB}{dH} \text{ (CGS)}$$

only at very small fields wil $\mu = \mu_d$ which is then μ_i.

If we now replace the "soft iron" with any real ferromagnetic material and especially with one which would be called a permanent magnet material, we find an unexpected development -- hysteresis. That is, as we traverse the B-H curve first increasing, then decreasing the current as before, we do not retrace the same curve but rather find we may be on any of an infinite number of curves as shown in the figure.



where

oeab   initial magnetization curve

efge'f'g'   minor hysteresis loop

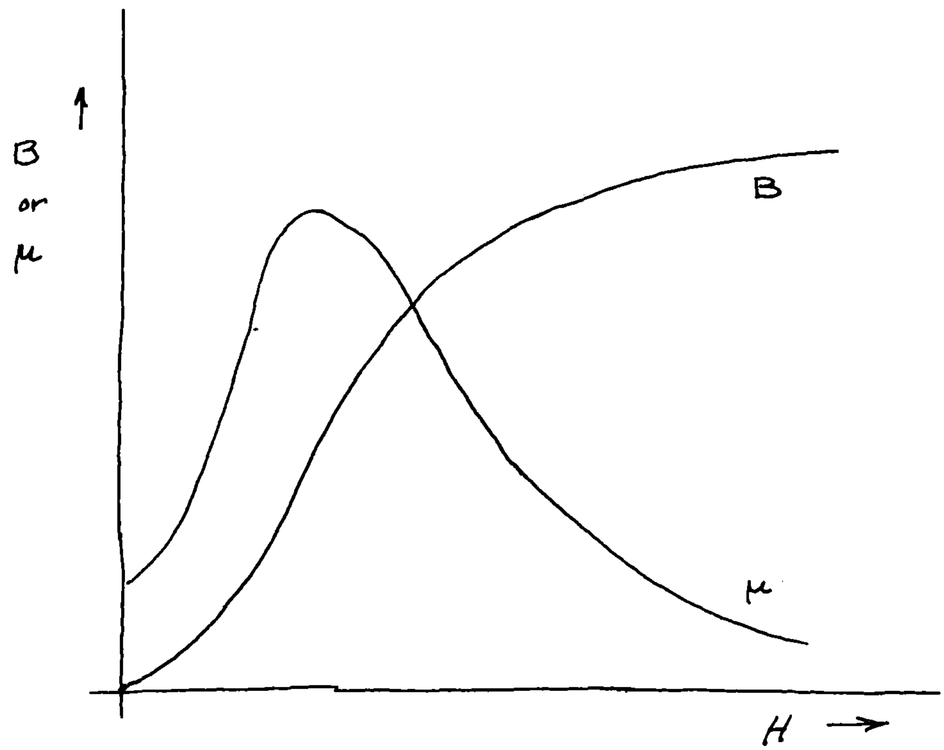
B_r = oc   residual or remanent magnetization of material

$-H_c$ = od   coercive force of material

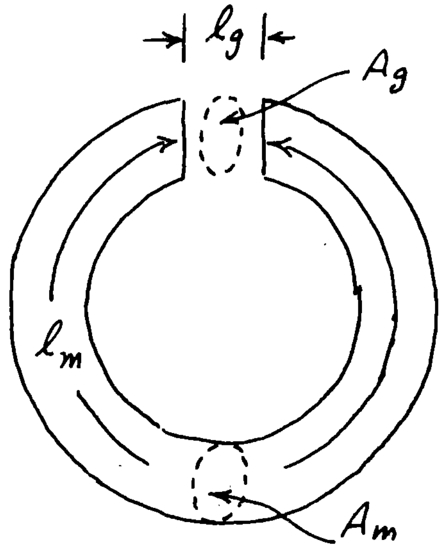
bcdb'c'd'   the hysteresis loop of the material

If the material has no initial magnetization and we apply a current to the surrounding coil, increasing it steadily we traverse the curve oeaf, the initial magnetization curve, similar to that shown in the preceeding figure. If we now reduce the current (and for a short way retrace our path) we are on the major, or simply the hysteresis loop of the material--the loop which contains the largest possible area of all, the hysteresis loops. Returning to zero current and hence zero H we find we are at point c. The flux density corresponding to this point is called the residual magnetism, retentivity or remanence and is denoted by B_r. At this point the material has become a *permanent magnet*. If we now reverse the direction of the current and increase its magnitude, we continue on the curve from c and eventually reach d. At point d the flux density in the material has been reduced to zero but has required a field intensity H_c to accomplish this. This value of $H = H_c$ is the reversed magnetic intensity required to produce zero flux density after the material has been magnitized to the point b or to saturation. H_c is the coercive force or coercivity of the magnetic material. The third quadrant is identical to the first and the second is identical to the fourth except for inversion and reversal. This general curve shape is typical of all permanent magnet materials; only the slopes, magnitudes and width of the loop will be different depending on the material involved.

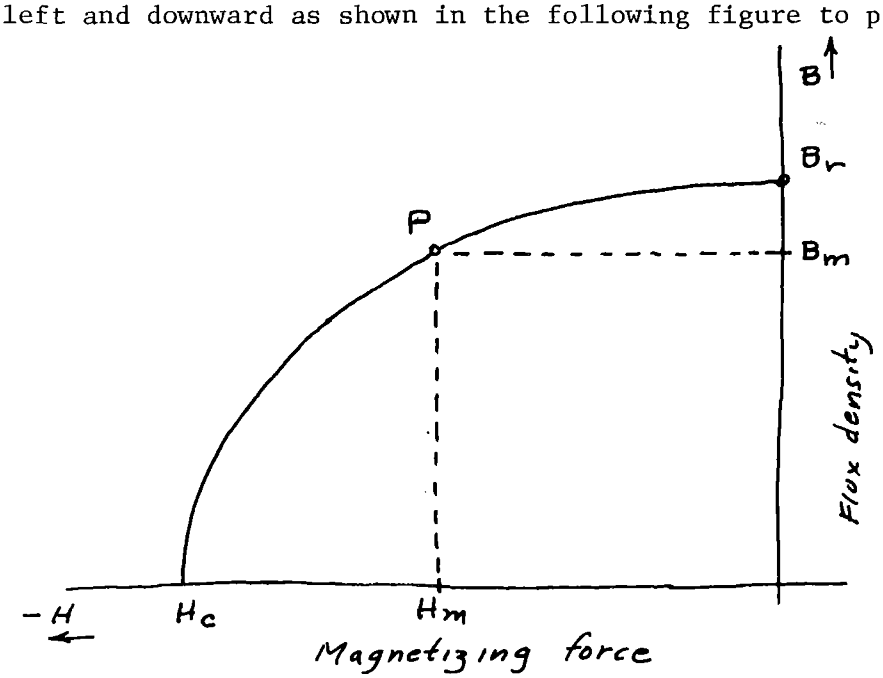
Note that the permeability μ is *not* a constant, but first increases to a maximum and then decreases as shown in the following figure.



So far we have been discussing a closed ring of magnetic material. No external field or flux density is available for use,therefore let us now consider a toroidal ring which has been magnetized to saturation, the coil removed and a small section of the material removed as in the figure.



After magnetizing to saturation and removing the current we arrive at some point on the magnetization curve. The point we arrive at if the ring were closed is B_r. When the ring has a gap we move to the left and downward as shown in the following figure to point P.



The equations governing the point at which we stop are as follows:

In a permanent magnet system, with no actual current in a coil, the total magnetizing force or magnetomotive force in the *complete* circuit is zero, just as the voltage drops through a complete circuit of a simple battery-resistance circuit add to zero. The magnetomotive force in a magnetic circuit is just the magnetizing force per unit length times the length or

$$F = \int H dl$$

therefore

$$\int H_{gap} dl_{gap} + \int H_{magnet} dl_{magnet} = 0$$

this becomes

$$H_g l_g + H_m l_m = 0$$

and magnetomotive forces are

$$F_{gap} + F_{magnet} = 0$$

Further, flux lines are continuous hence none are lost when leaving the magnet and going into the air gap and so we have for flux:

$$\phi_{gap} = \phi_{magnet}$$

but

$$B_{gap} A_{gap} = B_{mag} A_{mag}$$

or

$$B_g A_S = B_m A_m$$

These two equations

$$H_g l_g = -H_m l_m$$

$$B_g A_g = B_m A_m$$

are essentially the fundamental equations of the permanent magnet circuit.

If one is multiplied by the other we have

$$B_g H_g l_g A_g = - B_m H_m l_m A_m$$

but

$$l_m A_m = V_m$$

gap or magnet volume, hence,

$$B_g H s_g V_g = -B_m H_m V_m$$

but

$$B_g = \mu_o H_g \text{ (MKS)} \qquad B_g = H_g \text{ (CGS)}$$

hence,

$$\frac{B_g^2}{\mu_o} V_g = -(B_m H_m) V_m \text{ (MKS)}$$

or

$$B_g^2 V_s = -(B_m H_m) V_m \text{ (CGS)}$$

This shows that to produce a flux density B_g in a gap volume V_g, we need a magnet volume V_m with a certain BH product. If this product can be maximized we will have the *minimum* magnet volume. If we now plot B vs. BH for our magnetization curve we can find this maximum BH = $(BH)_{max}$. This is done in the figure on the next page. The regular demagnetization curve is on the left and with the same ordinate the BH product curve is in the right.

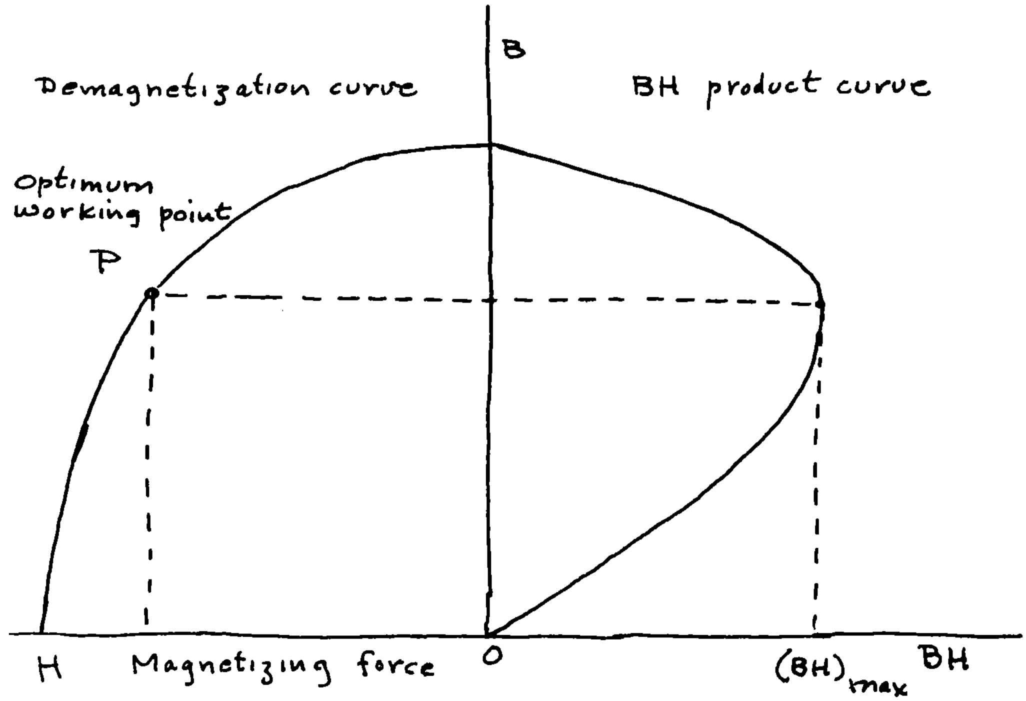
If, for any given material (demagnetization curve given) we operate at a B/H = μ ratio such that (BH) = $(BH)_{max}$ is a maximum, we will than be obtaining the maximum B in the gap.

Often the BH product is referred to as the energy product of the flux in the magnet or gap and $(BH)_{max}$ is the maximum energy product.

Strictly speaking

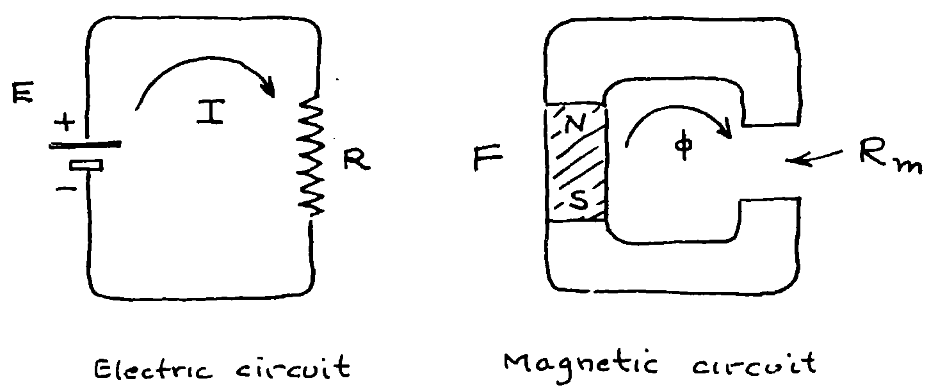
$$\frac{BH}{2} \text{ (joules/meter}^3\text{) (MKS)}$$

$$\frac{BH}{8\pi} \text{ (ergs/cm}^3\text{) (CGS)}$$



is the magnetic field space-energy density, but generally the numerical factors (1/2 an 1/8π) are omitted.

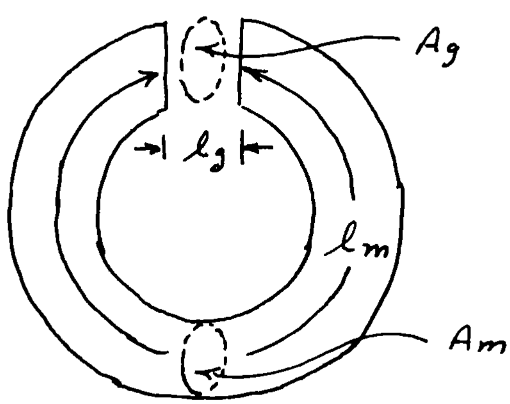
The magnetic circuit can be considered analogous to the direct current electric circuit (see figures below).



with one main reservation: *energy is not dissipated in the permanent magnet circuit.*

Electric Circuit	Magnetic Circuit
$E = RI$	$F = R_m\phi$
E = electromotive force	F = magnetomotive force
R = resistance to current flow	R_m = reluctance to flux
I = current	ϕ = flux

The second reservation is that the electrical circuit equation is usually a *very good* approximation to the physical battery-wire-resistor circuit because the resistance is linear with current magnitude and the current is constrained to remain in the wire and resistor. The magnetic circuit equation is at *best* only a good approximation. Often it is only a fair approximation and occasionally a poor one. The reluctance (depending on μ) is hardly ever a constant and depends on the magnetic history of the magnetic material. Further, the flux is only semiconstrained to lie in the magnetic material and spreads out from it. Also, in the *gap* it spreads greatly and not being uniform must be accounted for. Nevertheless, the approximation is very useful and corrections can be made for the various deviations from the simple picture. Rules for the addition of magnetomotive forces are similar to those for electromotive forces and rules for the addition of reluctances in series and in parallel are similar to those for resistances in series and parallel. With this in mind let us again consider the magnetized toroidal ring:



$$H_g l_g = -H_m l_m$$

$$B_g A_g = B_m A_m$$

Then magnetomotive force of the magnet is

$$F_m = H_m l_m$$

and this F_m appears across the gap as $-F_g$ which contains the flux ϕ_g.

The gap reluctance is

$$R_g = \frac{l_g}{\mu_o A_S} \text{ (MKS)} \qquad R_g = \frac{l_g}{A_g} \text{ (CGS)}$$

The magnet reluctance is

$$R_m = \frac{l_m}{\mu\mu_o A_m} \text{ (MKS)} \qquad R_g = \frac{l_m}{\mu A_m} \text{ (CGS)}$$

Hence,

$$F_g = R_g \phi_g \text{ and } F_m = R_m \phi_m$$

or

$$F_g = \frac{l_g}{\mu_o A_g} \phi_g \text{ (MKS)} \qquad F_g = \frac{l_g}{A_g} \phi_g \text{ (CGS)}$$

and

$$F_m = \frac{l_m}{\mu\mu_o A_m} \phi_m \text{ (MKS)} \qquad F_m = \frac{l_m}{\mu A_m} \phi_m \text{ (CGS)}$$

Now, in order to use these equations it is necessary to find the working point P of the magnet with given gap and magnet dimensions. To do this it is necessary to find B_m/H_m.

Since

$$B_m A_m = B_g A_g$$

and

$$H_m l_m = -H_g l_g$$

we take their ratio and solve for B_m/H_m

$$\frac{B_m}{H_m} = -\frac{B_g}{H_g}\frac{(A_g/l_g)}{(A_m/l_m)}$$

Since in air

$$B_g = \mu_o H_g \text{ (MKS)} \qquad B_g = H_g \text{ (CGS)}$$

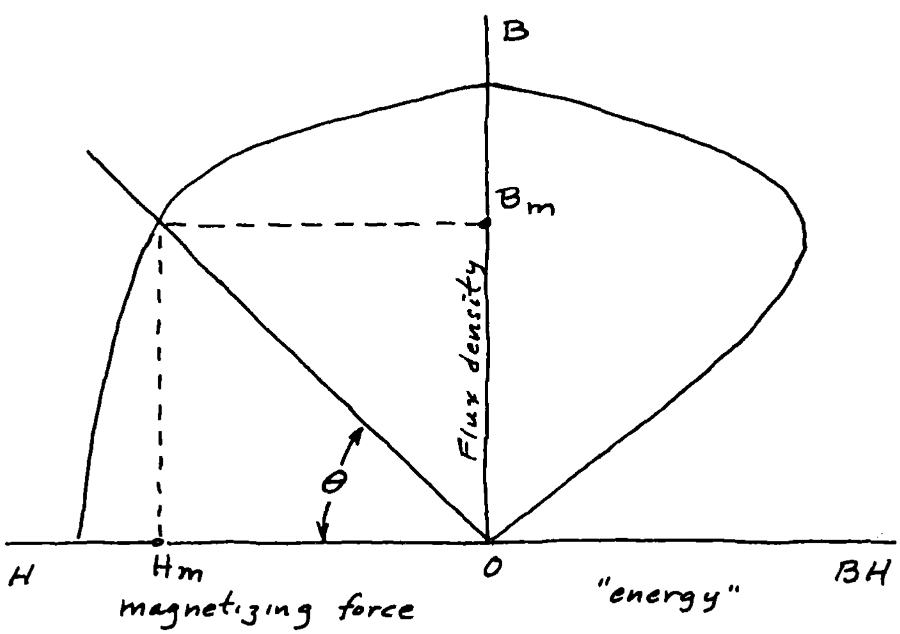
we have

$$\frac{B_m}{H_m} = -\frac{\mu_o (A_g/l_g)}{A_m/l_m} \frac{\text{henries}}{\text{meter}} \text{ (MKS)} \qquad \frac{B_m}{H_m} = \frac{A_g/l_g}{(A_m/l_m)} \text{ (gauss/oersted) (CGS)}$$

Now from the figure we see that

$$\frac{B_m}{H_m} = \tan\theta = -\frac{\mu_o (A_g/l_g)}{A_m/l_m} \quad \text{(MKS)}$$

$$\frac{B_m}{H_m} = \tan\theta = -\frac{A_g/l_g}{A_m/l_m} \quad \text{(CGS)}$$



This gives us the working point P which may or may not be at $(BH)_{max}$, the optimum working point for the particular magnetic material we have chosen. Generally, it is best to work backwards, choosing the working point (hence $\tan\theta$) and the gap dimensions and then calculating the magnet dimensions. Actually, this procedure is a bit crude (neglecting many things) and it is necessary for useful design work to include two correction factors in the equations. These factors account for several things: leakage flux from the magnet, a fringing in the working gap, and a magnetomotive force drop in soft iron pole pieces and the fact that unless the magnet is ellipsoidal in shape, it is *not* magnetized uniformly throughout its volume.

The correction factors are applied in this way

$$B_m A_m = f_B B_g A_g \qquad H_m l_m = - f_H H_g l_g$$

and

$$\tan\theta = - \frac{f_B}{f_H} \mu_o \frac{(A_g/l_g)}{(A_m/l_m)} \text{ (MKS)}$$

$$= - \frac{f_B}{f_H} \frac{(A_g/l_g)}{(A_m/l_m)} \text{ (CGS)}$$

Many empirical tables and formulas have been worked out from measurements of actual designs. Alternatively, the actual magnetomotive force drops in the soft iron and the fringing flux around the magnet and in the working gap can be calculated. The various parallel reluctances in the gap and around the magnet can be calculated separately and included in the calculations. The factors f_B and f_H are dimensionless, f_H is rarely less than 1.1 or greater than 1.5, 1.35 has been suggested as a good engineering approximation. Factor f_B may be approximated for many purposes as:

$$f_B = 1 + 7 \left(\frac{l_g}{d_g}\right)$$

where

l_g = length gap

d_g = diameter gap

The factor f_H may be defined in terms of the magnetomotive force drops in soft iron pole pieces and the gaps between these pieces and the working gap. Thus,

$$f_H = 1 + \frac{F_i}{F_g}$$

where

F_i = mmf drop in the iron and joints

F_g = mmf drop in the working gap

The factor f_B may be defined in terms of the fringing flux and the flux in the working gap. Assuming the same magnetomotive force across all gaps, these fluxes may be described in terms of the reluctances of these gaps. Thus,

$$f_B = 1 + \frac{R_g}{R_1}$$

where

R_g = reluctance of useful gap

R_1 = total reluctance of leakage gap

A number of formulas have been derived to evaluate the reluctance of many different geometrical shapes and a few are given in Hadfield(1) with procedures for their application, while a great number are given in Rotors(2). A good, short design manual has been published by the Thomas and Skinner Co.(3).

Very good estimates can be made of the reluctance of unusual geometries by means of electrical circuit analogs. These may take the form of capacitance or resistance models in two or three dimensions using electrolytic tanks, conducting paper or physical capacitances of sheet metal. These are usually resorted to only when an estimate cannot be made by the Rotors formulas.

Further details of the design of permanent magnets will not be gone into here as the references given are rather complete and with worked out examples.

It will be necessary in the design of permanent magnet assemblies to have available the B-H curves of several magnet materials. These are readily available from the manufacturers of magnet alloys. A few are listed in the references (4 through 11). Generally, except for special applications, one of the readily available Alnicos (such as Al-5) is best. Many manufacturers are willing to supply engineering and design advice and it is wise to seek and use it.

One problem which has not been mentioned is that of magnetizing the magnetic material. In general, it is necessary to magnetize the material *after* it has been assembled into its final configuration in order to achieve the optimum properties of the magnet material and configuration. Except in small assemblies this is difficult to achieve without expensive equipment. The manufacturers of magnet materials can often be persuaded to do this, however, for a nominal fee. Once a magnet structure has been assembled and magnetized it should *not* thereafter be disassembled for its properties will not remain same in consequence of the motion of the working point on the B-H curve. If it is necessary to have a magnet assembly which must be taken apart or which must have a variety of pole pieces fitted for different experiments, then this *must be known and allowed for in the design beforehand.* This will result in a magnet which may require several times the volume of magnetic material than that which would be required for the most economical single design (optimum). This will, however, permit a wide variety of pole piece configurations if this is what is desired. Except in very large magnets this is *not* an expensive solution to the problem. The design of a magnet which will work as required under a large variety of magnetic circuit conditions is considerably more complicated than a simple circuit design and it would be best to seek experienced aid for its design.

Chapter 4 - References

1. *Permanent Magnets and Magnetism,* D. Hadfield, Ed., John Wiley and Sons, New York, 1962.

2. *Electromagnetic Devices,* H. C. Rotors, Chapman and Hall, London, 1941.

3. Permanent Magnet Design, Thomas and Skinner, Bulletin M303, 1120 E. 23rd Street, Indianapolis, Indiana, (also ask for data bulletins on permanent magnets.)

4. U. S. Magnet and Alloy Corporation, 226 Glenwood Avenue, Bloomfield, New Jersey.

5. Arnold Engineering Company, Div. of Allegheny Ludlum Corp., Marengo, Illinois.

6. General Electric Co., Magnet Materials Business Section, Edmore, Mich.

7. Indiana General Corp., Magnet Division, Valparaizo, Indiana.

8. Permanent Magnet Co., 4437 Bragdon Stree, Indianapolis (Lawrence), Indiana.

9. Thomas and Skinner, Inc., 1110 E. 23rd Street, Indianapolis, Indiana.

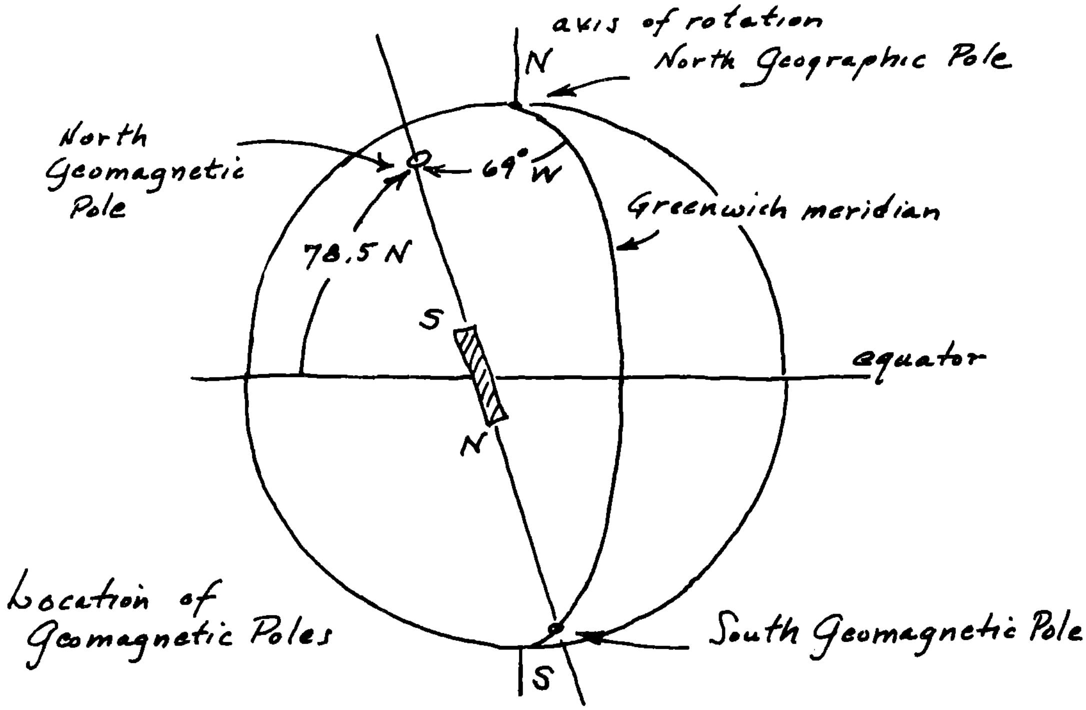
10. Westinghouse Electric Co., Metals Div., Blairsville, Pa.

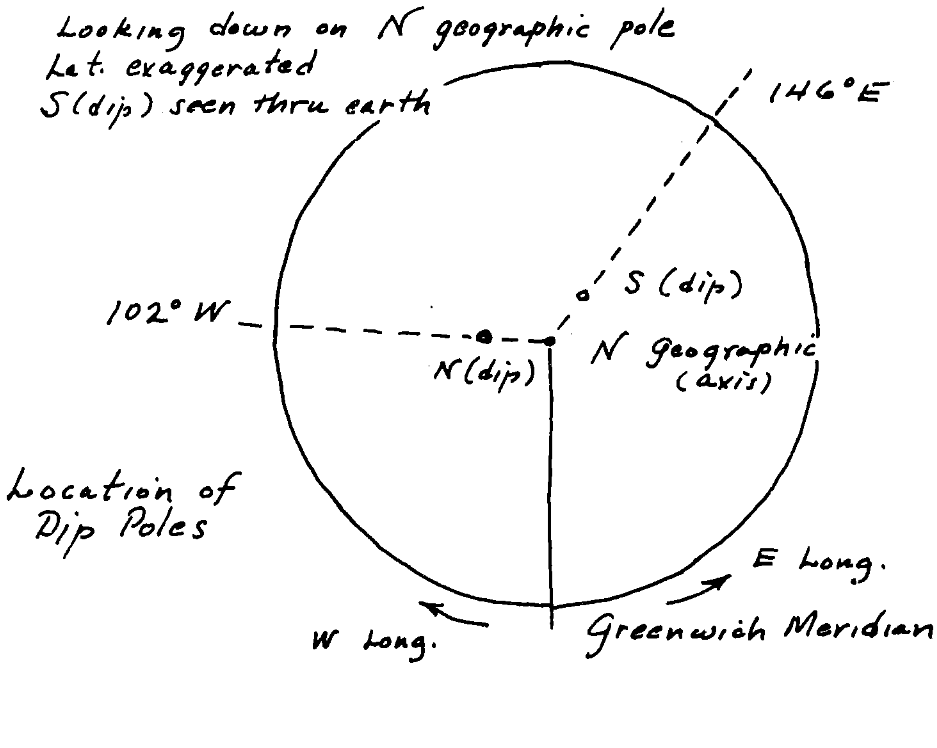
11. Crucible Steel Co. (Sub. Colt Industries) Magnet Div., Elizabethtown, Kentucky.

CHAPTER 5
THE EARTH'S MAGNETIC FIELD

For many centuries it has been known that a magnetized rod, when freely suspended at its center, would align itself in roughly a north-south direction, the earth acting like a huge magnet itself.

The magnetic directions on the earth were known not to be completely regular nor exactly geographically north-south. Around the year 1600 William Gilbert, in attempting to explain the magnetic lines of the earth, constructed a small sphere from the only magnetic material available to him, magnetite or lodestone and plotted the directions of the field outside the model. The magnetic field of the model closely resembled the magnetic field known on the earth. Later Gauss in 1840 showed by calculation that the earth's field could be described approximately by assuming that the earth was a uniformly magnetized sphere. He further showed that the same field could be obtained external to the earth by assuming a bar magnet, small compared to the earth, placed near the center of the earth. To account for the fact that the lines of flux are not in a geographic north-south direction, this magnet must have its axis at about 11.5° from the earth's axis of rotation and be displaced to one side of that axis. This is shown in the figure. The extended axis of this imaginary dipole intersects the earth at about 78.5°N Lat, 69°W Long and 78.5°S Lat, 111°E Long, these points are called the *geomagnetic poles*. On the other hand, the directly determined "apparent" North and South magnetic poles are those *main* north and south points on the earth's surface where a dip needle aligning itself with the lines of magnetic flux point downward 90° from the horizontal. These apparent, or dip poles, are located at 70.8°N Lat, 96.0°W Long and 71.2°S Lat, 150.8°E Long (about 1936). As can be seen, these do not correspond to the dipole axis poles and further are not symmetrically placed with respect to the geographic poles. Actually, there


axis of rotation
N
North Geographic Pole
North Geomagnetic Pole
69° W
Greenwich meridian
78.5 N
S
equator
N
Location of Geomagnetic Poles
South Geomagnetic Pole
S


Looking down on N geographic pole
Lat. exaggerated
S(dip) seen thru earth
146°E
S (dip)
102° W
N(dip)
N Geographic (axis)
Location of Dip Poles
E Long.
W Long.
Greenwich Meridian

are many such points scattered over the earth due to local anomolies. These poles move irregularly with time a few degrees per hundred years (mainly a rotation back and forth around the geographic pole). In addition, the magnetization of the earth has been gradually decreasing by about 1 part in 1000 to 1:1500 per year for about 100 years. It would be extremely risky to extrapolate this figure backward, however, as it would yield a very high intensity thousands of years ago which is not confirmed by paleomagnetic observations. Much of the published data on the magnitude and direction of the earth's field is quite old and sometimes inconsistant. The average value of the intensity of magnetization of the earth per unit volume (assuming uniform magnetization) is 0.08 cgs units/cm^3 which is about 100 to 1000 times the average value for ordinary rocks on the earth's surface. A thorough review of measurements of the earth's field over the centuries is given by Chapman(1) along with the various historical and modern techniques used.

In addition to the non-symmetry because of this general off-centerness there are local variations due possibly to large deposits of iron ore within the earth. Superimposed on the steady field there are daily and yearly cycles and an 11 year cycle. The yearly cyclic variation is quite small amounting to about 2.5 minutes of angle (opposite in northern and southern hemispheres). The daily variation is on the same order varying with location on the earth's surface. The 11 year cycle is probably associated with the 11 year sunspot cycle. Electric ion currents in the atmosphere probably account for irregular variations in the earth's magnetic field sometimes causing much larger variations than those mentioned above. These are known as magnetic storms and may also be associated with sunspot activity. All these variations can be in both direction and magnitude of the earth's field.

There are three aspects of the earths magnetic field regularly measured and recorded. These are:

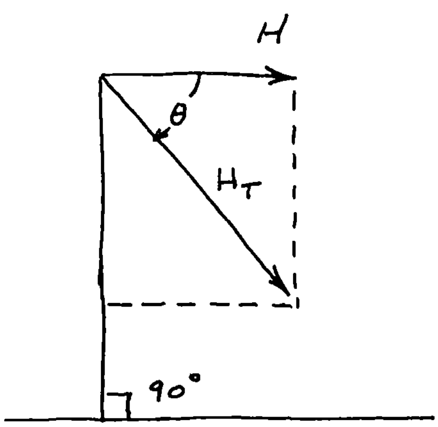
1. Declination - the angle between the resultant magnetic direction and the geographic direction.

2. The angle of dip - the angle measured between the direction of the earths magnetic field and the horizontal plane at that point.
3. The horizontal component of the earth's magnetic field intensity.

The declination varies from point-to-point on the earth's surface in a fairly regular manner. It is given in degrees deviation east or west from the true north-south direction. The line of zero declination is known as the *agonic* line and crosses the U. S. roughly from Chicago to Savannah. This agonic line is gradually moving westward as are all the lines of declination. To the east of this line the declination is west and is increasing and to the west of this line it is east and is decreasing. This amounts to about .05° per year in the U.S.

The dip angle varies from approximately zero at the equator to 90° at the "apparent north and south magnetic poles." It is measured in the vertical plane which intersects a compass needle at that point on the earths surface. There are many local anomolies.

The horizontal intensity of the earths field is related to the total earth's field by the dip angle.



Earth's surface (horizontal)

$$H = H_T \cos\theta$$

H = horizontal intensity earth's field

H_T = total intensity earth's field

θ = dip angle

$$H = H_T \cos\theta$$

where

H = Horizontal intensity earth's field

H_T = Total intensity earth's field

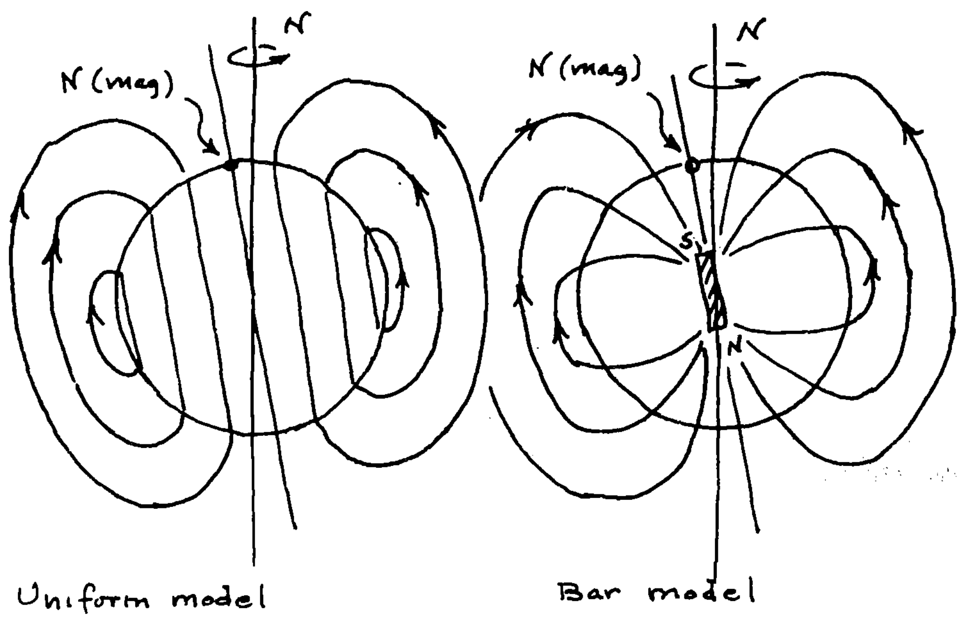
θ = dip angle

Knowing the horizontal component and the dip angle the total intensity of the earth's field can be determined.

A short table of dip angle, declination and horizontal intensity in the U.S. is given below (about 1920):

	Dip Angle (deg)	Declination (deg)	H (oersted)
Maine	74 - 76	16 - 21 W	.14 - .16
New York	74	4 - 12 W	.16 - .17
Florida	57 - 58	1 - 2.5 E	.27 - .29
Michigan	73 - 76	2 E - 3 W	.15 - .18
Kansas	67 - 69	10 - 12 E	.21 - .23
Mississippi	61 - 66	5 - 7 E	.24 - .26
Washington State	71	23 - 24 E	.19
California	58 - 62	14 - 19 E	.25 - .27

The pattern of the lines of flux as they are distributed over the earth is shown approximately in the figure below for both a uniformly magnetized sphere and for a bar near the center of the earth.



Note that the flux lines inside the earth's surface are quite different (the field is uniform) for the uniformly magnetized sphere model as compared to those for the bar magnet model. There is no way to determine exactly what the pattern is inside the earth so we cannot, of present, tell which model is closer to reality since the external pattern of flux lines is identical for both cases. Since it is of no practical consequence to know which model is more correct as both give the same external field, it is unnecessary here to pursue the subject further. However, an excellent article on the history and measurements of the earth's magnetic field has recently appeared in the Journal of Applied Physics[2]. It discusses the change in magnetism of the earth through the geologic ages and the paleomagnetic techniques for determining these changes. Included are many references to other studies and theories.

The experimenter who wishes to reduce the earth's magnetic field (in an experimental region) to a very low value is concerned with the earth's external magnetic field. If he uses a coil assembly to produce an equal and opposite field in that of the earth, so that the earth's field is cancelled by the generated field, he must know approximately the magnitude and direction of the total field in the region of his experiment. The figures given in the table are only approximate and in addition they were taken in regions far removed from civilization. The local field in a laboratory is often considerably different from that given for the local field in tables. In fact, at different locations in the laboratory it may be different by a factor of two in magnitude and the dip may vary by $\pm$ 20 degrees or more. This is generally due to the iron structure in the building, but also may be caused by iron bench frames or other iron work in the laboratory. Occasionally, the iron work in the laboratory may be permanently magnetized producing very strange patterns of flux. Often this permanent magnetism can be removed from the iron work in benches by passing a large coil, of many turns, energized by ordinary 60 cycle mains current, over and around the bench structure and other permanent iron structures in the room.

Naturally, it is best if these iron frame works can be removed from the magnetobiology laboratory and use only wood or aluminum or other non-magnetic materials. Even aluminum and brass are often slightly magnetic due probably to impurities of iron in the alloys; generally this is of little consequence except in the most precise experiments. In the construction of wooden frames for coils or the support of specimens it is best to avoid the use of iron screws or nails which will cause small but definite distortions in the flux distribution. It is also necessary to know the direction and magnitude of the earth's field when shielding boxes or cylinders are used to reduce the flux in the experimental region. A long shielding cylinder must be very carefully oriented with its axis perpendicular to the direction of the earth's field at that point in space. This problem is discussed elsewhere in the section on the design of magnetic shields, but it is pointed out here that a change in angle by *just a few degrees* may mean the difference of *a factor of 10 or more* in the attenuation achieved with such a shield.

The fluctuations of the earth's field must also be considered in relation to the method used to reduce or cancel the earth's field. A coil assembly will subtract the steady portion of the earth's field leaving virtually undisturbed the fluctuating component, whereas a shield will attenuate both the steady portion and the fluctuating magnitude of the earth's field to about the same extent. A fluctuation of the *direction* of the magnetic field, if appreciable, may produce a greater change in the attenuated field inside a shield than a change in magnitude only. For these reasons, it is essential, to be able to measure the flux density in direction, magnitude and time, inside a coil or shield assembly used to reduce the earth's magnetic field. This will then give an indication of the kind and percentage variation in the field in the region of the experiment and enable the experimenter to determine whether this is tolerable. Low frequency fields of 60 cycle frequency from the mains may also be present and are likely to fluctuate with the power demand. These can often be checked by means of a many-turn coil of wire placed in the experiment region and observed on an oscilloscope -- great care must be taken to ensure that the signal observed is due to the magnetic field and not pick up in the wiring. This can be partially checked by enclosing the coil in a grounded electrostatic shield without changing anything else in the circuit. Another test for pick up is to substitute a resistance of the same value as that of the coil and observe if any signal is present.

Magnetic fields in regions other than the earth, such as in far space or on the moon are quite low compared to those on the earth, although the magnetic field inside a space craft may be almost any value depending on the equipment carried. Any biological experiments carried out in a space craft will be subject to this ambient field and it *should be known.* The biological experimenter should request this information from those making the final qualification tests on the space craft even if this experiment *apparently* is not susceptible to magnetic fields.

While the earth's magnetic field is normally 0.4 to 0.8 gauss, the magnetic field of the moon has a field of about $40\text{-}100 \times 10^{-5}$ gauss ($40\text{-}100 \times 10^{-9}$ weber/m^2) or about 1/500 that of the earth. Interplanetary space contains fields on the order of about 1/2 of that of the moon. G. E. Hale in 1913 concluded that the Sun had a magnetic field similar to that of the earth but about 100 times greater. Local fields on the order of 4000 gauss have been estimated in the region of sunspots. Various hypotheses as to the origin of the magnetic field on both the Earth and the Sun are discussed in Fleming[(3)].

The subject of the earth's magnetic field and its possible biological effects is a complicated and fascinating one. Decreases and reversals of the magnetic field in the distant past[(2)] (to 5 million years ago) as revealed by paleomagnetic studies may have influenced the rise and fall of several ancient species. This may have been a direct consequence of the flux density on the organism or indirectly through the changes in shielding of the earth against energetic particles from space. Correlations between various fossil abundances and the strength and direction of the earth's magnetic field raise new questions and suggest further experimentation in magnetobiology.

Chapter 5 - References

1. *Geomagnetism*, Sydney Chapman and Julius Bartels, Vol. 1, Oxford, 1940.

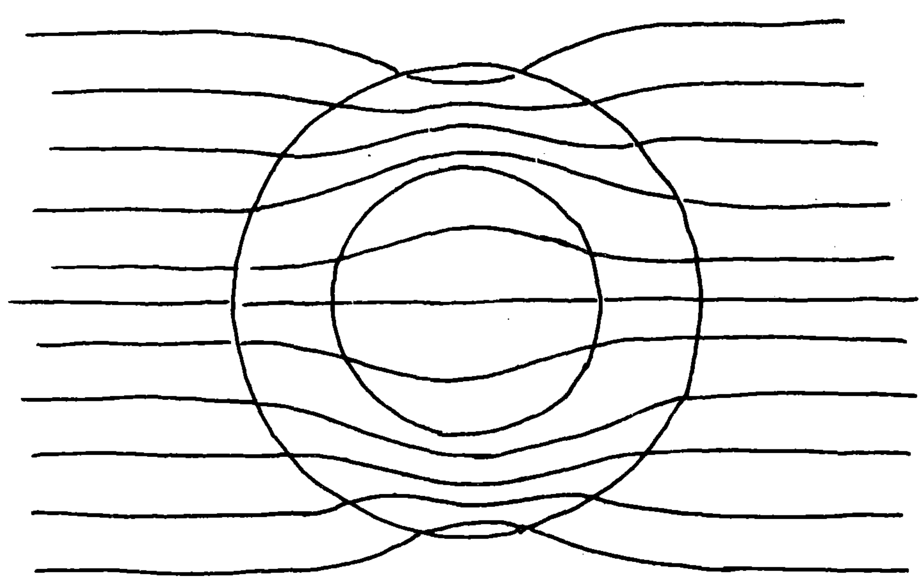
2. History of the Geomagnetic Field, Doell, Richard R., J. Appl. Physics, Vol. 40, No. 3, March 1, 1969, pg. 945-954.

3. *Terrestrial Magnetism and Electricity*, Physics of the Earth, Vol. VIII, J. A. Fleming, Ed., McGraw-Hill, 1939.

CHAPTER 6

DESIGN OF CYLINDRICAL SHIELDS TO PRODUCE "NULL" OR GREATLY REDUCED MAGNETIC FIELDS

In studying biomagnetic phenomena we have two choices with respect to the magnetic field of the earth. We can use fields higher than the earth's field or we may use fields lower than the earth's field. Various coil assemblies discussed previously are one means of reducing or cancelling the field of the earth. Another and in some ways, more convenient method is to shield the biological material from the field of the earth. This is possible because materials of high magnetic permeability can be used to form a diverting pathway for the lines of magnetic flux around the specimen. The resistance (more correctly reluctance) of a path through iron or iron alloys of high permeability is many times less than the reluctance through air so that a shield of this material acts like a shunt or "short circuit" of the magnetic flux. The flux is thus conducted *around* the internal region of the shield. This shunting is not perfect and some of the flux will enter the region inside the shield (see figure below).



For this reason it is necessary to have a shield of sufficiently low reluctance, i.e., (1) thick enough or (2) of high enough permeability or (3) multiple shields. It has been shown by Rücker[1] that is is profitable to increase the shield thickness up to the point where

$$t \leq 3a/2\mu$$

t = thickness of shell

a = inner radius

μ = permeability of material

When this point is reached it is magnetically worthless to increase the thickness further. To achieve greater shielding it is necessary to use multiple concentric shields. Each shield then attenuating the residual field remaining from the previous one. The shielding effectiveness is usually defined as the ratio g or F of the field or flux density existing before the shield is put into place (H_o or B_o) to that remaining inside the shield after it is put into place (H_i or B_i):

$$g = F = \frac{H_o}{H_i} = \frac{B_o}{B_i}$$

As can be seen from the maximum thickness formula given above, if we use modern materials of very high permeability the thickness is very small

μ = 20,000 (Mu-metal initial permeability)

a = 0.5 meter radius

then

$$t = \frac{3(0.5)}{2 \times 20 \times 10^3} = 0.375 \times 10^{-4} \mathrm{m} = .0375 \text{ millimeters}$$

Usually a much thicker material is used for *mechanical* reasons. Some shield designs utilize foil which can be wound in a spiral cylinder interleaved with a non-magnetic separator to provide separation of the layers. For a self supporting shield, however, a material of .020" (1/2 mm) to .060 (1 1/2 mm) thickness is suitable. Suppose then we wish to design a shield. It is necessary to pick a shielding material. For shielding against the earth's magnetic field Mu-metal (or molypermalloy which has very similar properties) is quite suitable. Next, it is necessary to determine the dimensions desired. We shall choose a cylindrical geometry and specify a length three or four times the internal diameter. (This length is not considered in the shielding factor calculation since this factor is calculated assuming an infinite length shield.) This length, however, will be discussed in more detail later. The material thickness is chosen generally for mechanical stability. Stern[2] has theoretically developed a recursion procedure for calculating the shielding factor for *any* number of shields. The formulas and definitions for this procedure are given below with two examples along with a discussion of the choice of the value of permeability to be used for each cylinder. Since permeability of magnetic materials is not constant, but depends upon the flux density in it, the permeability to be used in each cylinder has to be determined.

Formulas for recursion solution of shielding factor for multiple cylindrical shields after Stern[2]

Shielding factor $B_o/B_i = F$ (theoretical for infinite shield length)

$$F_{m+1} = 1/2\ (u_{n+1} + v_{n+1})$$

$$u_1 = v_1 = 1$$

$$u_{i+1} = \alpha_i u_i + \beta_i v_i$$

$$v_{i+1} = \alpha_i u_i + \delta_i v_i$$

$$\alpha_i = 1 - \varepsilon_i - \varepsilon_{i,i+1} + (\mu+1)\ \varepsilon_i \varepsilon_{i+1}$$

$$\beta_i = \varepsilon_{i,i+1}\ (1-\varepsilon_i) + \varepsilon_i/\mu\ (1-\varepsilon_{i,i+1})$$

$$\alpha_i = \mu\varepsilon_i + \varepsilon_{i,i+1} - (\mu+1)\ (\varepsilon_i \varepsilon_{1,i+1})$$

$$\delta_i = 1 - \varepsilon_i - \varepsilon_{i,i+1} + \varepsilon_i \varepsilon_{i+1}\ [(\mu+1)/\mu]$$

$$\varepsilon_i = [(b_i-a_i)/b_i] - 1/2\ [(b_i-a_i)/b_i]^2$$

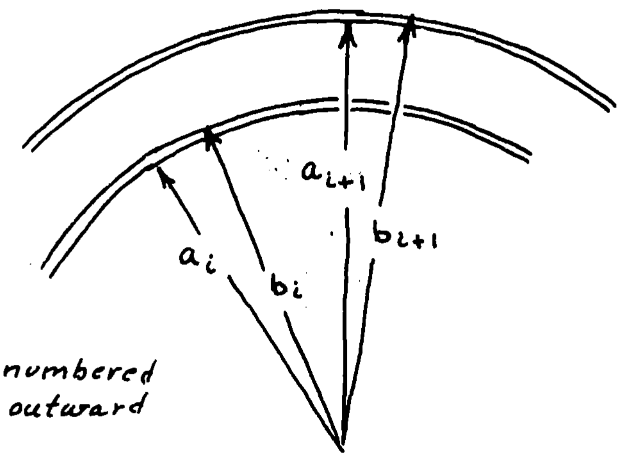
$$\varepsilon_{i,i+1} = [(a_{i+1}-b_i)/a_{i+1}] - 1/2\ [(a_{i+1}-b_i)/a_{i+1}]^2$$

and

$$\varepsilon_{n,n+1} = 0$$

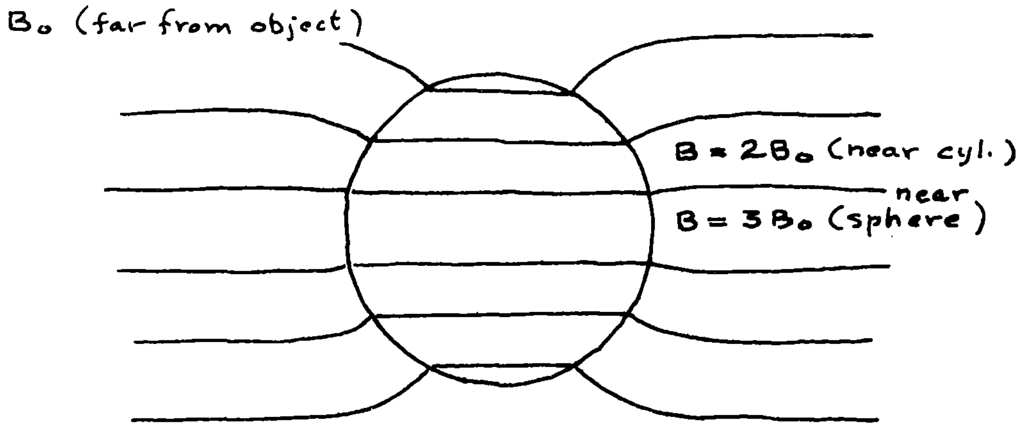
where

a_i, b_i are the inner and outer radii of the i^{th} shield respectively.

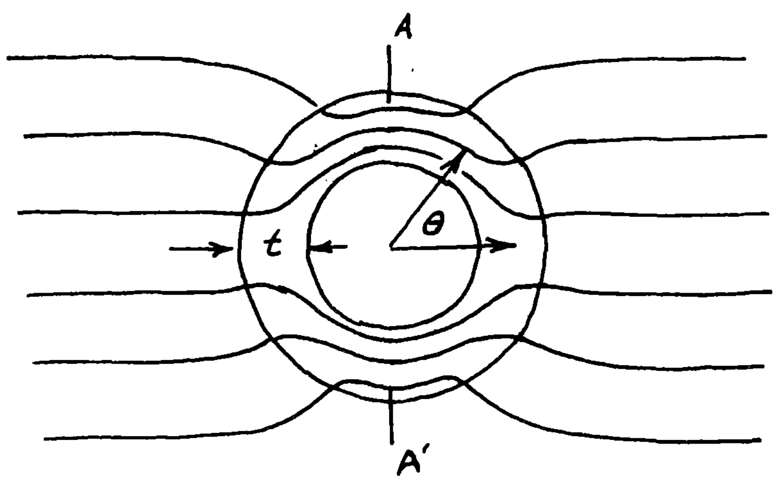


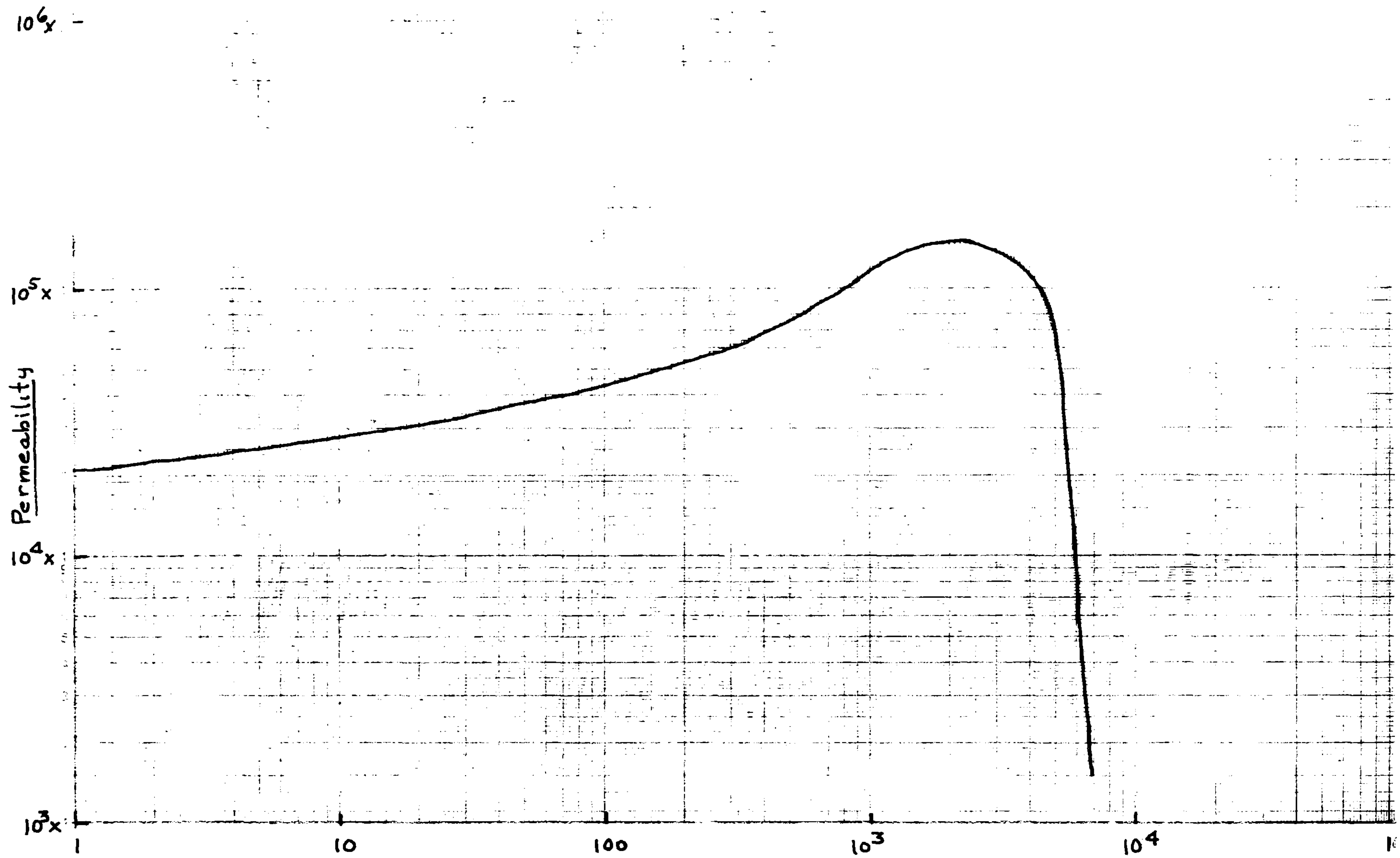
Shields are numbered from inside outward

It is only necessary to pick the proper value of the permeability μ for each shield. Since the permeability μ_i is a function of the flux density B_i in the i^{th} shield and since this increases in each shield as we move from the center outwards μ_i will be somewhat different for each shield, see graph (μ vs B). If the shield is to be used in high fields or is very large and of very thin material, it is necessary to guard against saturation of the shielding material. If the shield is to be used in the earth's field, however, it is necessary to take saturation into account only when the shield is very thin or very large in diameter. Usually, the wall thickness is made many times thicker than necessary because of the structural strength required. In case of doubt this may be estimated as follows. First, any sphere or cylinder (hollow or solid) of high permeability will, when placed in a uniform field of flux density B, concentrate that field at its surface as shown in the figure.



The flux density before the cylinder or sphere is put into it is B, after putting the cylinder or sphere into the field the flux density just outside the cylinder or sphere along the diameter in the field direction will be $2B_o$ for the cylinder and $3B_o$ for a sphere. If the cylinder or sphere is hollow and of high permeability, the flux will be concentrated in the shell as shown in the figure below, increasing the flux density





Mu-metal Permeability .014" sheet

derived from Allegheny Ludlum EM-12 Ed 9 data sheet in the shell at the diameter A A' by approximately the ratio a/t. In an unpublished paper by W. G. Wadey[4] the flux density in the shell (cylindrical) is given by

$$B(\theta) = 2B_o \left[1 + \frac{a}{t} \sin\theta\right]$$

as a function of the angle θ. When θ is 90° the flux density in the shell is a maximum and is

$$B_{max} = 2B_o \left[1 + \frac{a}{t}\right]$$

Wadey further suggests that since the flux density varies from point to point and from comparison with measured values that an effective B_{eff} should be used which is 0.6 B_{max}. Then assuming a/t >> 1 we have:

$$B_{eff} = 0.6\ (a/t)\ 2B_o$$

or

$$B_{eff} = 1.2\ (a/t)\ B_o$$

This B_{eff} can then be entered into a B vs. μ curve for the material used, shield and an approximate value of μ obtained.

For shields used in the earth's field this calculation is usually only necessary for the outermost shield. As an example let B_o = 0.7 gauss, shield radius = 0.5 meters, material thickness = 1 mm = 10^{-3} meters, material mu-metal, then:

$$B_{eff} = 1.2\ \left(\frac{0.5}{10^{-3}}\right)\ (0.7) = 1680 \text{ gauss}$$

from the curve we obtain μ = 140,000 which is very close to the maximum permeability and not far below saturation.

The exact shielding factor for a *single* shield cylinder of infinite length given by Stern[2] is

$$\frac{B_o}{B_i} = F = 1 + \frac{1}{4}\ \left(1 - \left(\frac{a}{b}\right)^2\right)\ \left(\mu + \frac{1}{\mu} - 2\right)$$

if $\mu >> 2$ and $(a/b)^2 >> 1$ this reduces to

$$F = \frac{\mu}{4} [1 - (\frac{a}{b})^2]$$

this is also given by Wills[3] in a much earlier paper.

Using this formula and the same figures as above where:

$a = 1.000$ m
$b = 1.001$ m

$$F \simeq \frac{1.4 \times 10^5}{4} \left[1 - \left(\frac{1.000}{1.001}\right)^2 \right]$$

$$= 35{,}000\ [.002] = 70$$

Then

$$B_i = 0.7/70 = 0.01 \text{ gauss}$$

and if this value were applied to a second shell inside the first of approximately similar concentration

$$B_{eff_2} = 24 \text{ (in the second shell)}$$

and

$$\mu_2 = 32{,}000$$

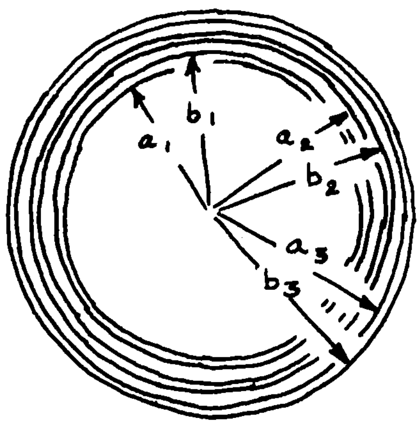
Note that in this case of large diameter and small thickness we come very close to the maximum permeability. One half the thickness of material or twice the diameter would have pushed us just over the hump into saturation. If we had a flux density of around 7000 gauss we would have had both a drastically reduced permeability and shielding factor. In this case it was prudent to make this calculation and also to make the calculation for the second shield since we now have a permeability of 32,000 well above the initial permeability of 20,000. For any additional shields we would use μ = 20,000, the initial permeability of mu-metal.

We would now be in a position to use Stern's iterative formulas given previously, utilizing the just calculated values of μ;

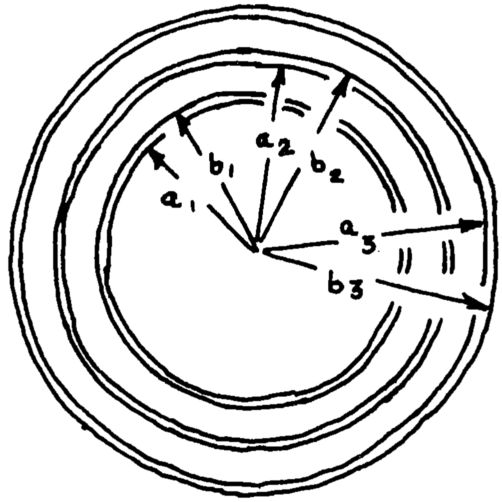
20,000 for all inner cylinders, 32,000 for next to the outer and 140,000 for the outermost cylinder. While the estimates of these values of permeability were made from the outside in, *it must be remembered that the shielding factor calculations are made from the inside out.* Wadey, in the previously mentioned unpublished paper, describes a method whereby a value of flux density is assumed inside the innermost cylinder and by a series of iterations the proper permeability is found for each concentric shell, one at a time, while using the Stern equations. This is an excellent procedure, but is usually not necessary if the external flux density is no greater than the earth's field and it does involve a considerable additional amount of calculation. It can be used to estimate the total number of shields required, but the standard Sterne procedure can be run through for a few sample shields of different n and estimates of the total shielding factor F can be made after any number of shields by setting $\varepsilon_{n,n+1} = 0$ at that point to see if the desired field reduction has been obtained.

We will now calculate the shielding factors for two similar, but different triple shields utilizing values of permeability obtained by the method above, but for new dimensions. There will be, three nested shields, each of one millimeter material thickness of mu-metal. The shield nest 1 will have an inner diameter of 10 cm and a spacing between cylinders of 1 millimeter. The second shield nest will also have an internal diameter of 10 cm but will have a spacing between cylinders of 1 cm. This greater spacing will yield a greater shielding factor, but will increase the external diameter considerably.

The calculation will be carried out in some detail so that anyone wishing to make a similar calculation may readily do so.



Shield nest 1



shield nest 2

For Shield Nest 1

Permeabilities estimated by concentration formula for mu-metal.

$\mu_1 = 20 \times 10^3$ (innermost cylinder)

$\mu_2 = 20 \times 10^3$ (middle cylinder)

$\mu_3 = 42 \times 10^3$ (outer cylinder)

Dimensions

Spacing of shields 0.1 cm

Thickness of shield material 0.1 cm

inside dia. of cylinders	outside dia. of cylinders
$a_1 = 10.0$	$b_1 = 10.1$ (inner cylinder)
$a_2 = 10.2$	$b_2 = 10.3$ (middle cylinder)
$a_3 = 10.4$	$b_3 = 10.5$ (outer cylinder

From these values of a and b the ε_i and $\varepsilon_{i,i+1}$ values are calculated.

$$\varepsilon_1 = (b_1-a_1)/b_1 - 1/2\ [(b_1-a_1/b_1]^2$$

$$= (10.1-10)/10.1 - 1/2\ [(10.1-10)/10.1]^2$$

$$= 0.1/10.1 - 1/2\ [0.1/10.1]^2$$

$$= .009852$$

$$\varepsilon_2 = (10.3-10.2)/10.3 - 1/2 \left(\frac{0.1}{10.3}\right)^2$$

$$= .009662$$

$$\varepsilon_3 = 0.1/10.5 - 1/2 \left(\frac{0.1}{10.5}\right)^2$$

$$= .009478$$

Similarly the $\varepsilon_{i,i+1}$ are calculated. However, $\varepsilon_{n,n+1}$, the outermost value, cannot be calculated and most have some value assigned. Since F is not affected by this choice it may be made zero, hence $\varepsilon_{34} = 0$.

$$\varepsilon_{12} = (a_2-b_1)/a_2 - 1/2\ [(a_2-b_1)a_2]^2$$

$$= (10.2-10.1)/10.2 - 1/2\ [0.1/10.2]^2$$

$$= .009756$$

$$\varepsilon_{23} = (10.4-10.3/10.4 - 1/2\ [0.1/10.4]^2$$

$$= .009569$$

$$\varepsilon_{34} = 0$$

$\varepsilon_1 = .009852$	$\varepsilon_{12} = .009756$
$\varepsilon_2 = .009662$	$\varepsilon_{23} = .009569$
$\varepsilon_3 = .009478$	$\varepsilon_{34} = 0$

The values of α, β, γ, δ are next computed.

$$\alpha_1 = 1 - \varepsilon_1 - \varepsilon_{12} + (\mu_1 + 1)\ \varepsilon_1 \varepsilon_{12}$$

$$= 1 - .009852 - .009756 + 20 \times 10^3\ (.009852)\ (.009756)$$

$$= 1 - .009852 - .009756 + 1.922322$$

$$= 2.902712$$

$$\alpha_2 = 1 - \varepsilon_2 - \varepsilon_{23} + (\mu_2 + 1)\ \varepsilon_2 \varepsilon_{23}$$

$$= 1 - .009662 - .009569 + 20 \times 10^3\ (.009662)\ (.009569)$$

$$= 1 - .009662 - .009569 + 1.849114$$

$$= 2.829883$$

$$\alpha_3 = 1 - \varepsilon_3 - \varepsilon_{34} + (\mu + 1)\ \varepsilon_3 \varepsilon_{34}$$

$$= 1 - .009478 - 0 + 0$$

$$= 0.990522$$

In a like manner the β's are computed

$$\beta_1 = \varepsilon_{12}\ (1-\varepsilon_1) + \frac{\mu_1}{\mu_1}\ (1-\varepsilon_{12})$$

$$= (.009756)\ (1-.009852) + \frac{.009852}{20 \times 10^3}\ (1-.009756)$$

$$= .009660 + 0$$

$$\beta_2 = \varepsilon_{23}\ (1-\varepsilon_2) + \frac{\mu_2}{\mu_2}\ (1-\varepsilon_{23})$$

$$= .009569\ (1-.009662) + \frac{.009662}{20 \times 10^3}\ (1-.009569)$$

$$= .009477 + 0$$

$$\beta_3 = \varepsilon_{34}\ (1-\varepsilon_3) + \frac{\mu_3}{\mu_3}\ (1-\varepsilon_{34})$$

For γ's we have

$$\gamma_1 = \mu_1\varepsilon_1 + \varepsilon_{12} - (\mu_1+1)\ \varepsilon_1\varepsilon_{12}$$

$$= 30 \times 10^3\ (.009852) + .009756 - 20 \times 10^3\ (.009852)\ (.009756)$$

$$= 197.04 + .009756 - 1.922322$$

$$= 195.127$$

$$\gamma_2 = \mu_2\varepsilon_2 + \varepsilon_{23} - (\mu_2+1)\ \varepsilon_2\varepsilon_{23}$$

$$= 20 \times 10^3\ (.009662) + .009569 - 20 \times 10^3\ (.009662)\ (.009569)$$

$$= 191.400$$

$$\gamma_3 = \mu_3\varepsilon_3 + \varepsilon_{34} + (\mu_3+1)\ \varepsilon_3\varepsilon_{34}$$

$$= 42 \times 10^3\ (.009478) + 0 + 0$$

$$= 398.076$$

and finally the δ's

$$\delta_1 = 1 - \varepsilon_1 - \varepsilon_{12} + \varepsilon_1\varepsilon_{12} \left[\frac{\mu_1+1}{\mu_1}\right]$$

$$= 1 - .009852 - .009756 + (.009852)\ (.009756)\ [1]$$

$$= .980488$$

$$\delta_2 = 1 - \varepsilon_2 - \varepsilon_{23} + \varepsilon_2\varepsilon_{23}\ [1]$$

$$= 1 - .009662 - .009569 + (.009662)\ (.009569)$$

$$= 0.980861$$

$$\delta_3 = 1 - \varepsilon_3 - \varepsilon_{34} + \varepsilon_3\varepsilon_{34}\ [1]$$

$$= 1 - .009478 - 0 + 0$$

$$= 0.990522$$

Summarizing we have

$\alpha_1 = 2.9027$	$\beta_1 = .00966$	$\gamma_1 = 195.127$	$\delta_1 = 0.9805$
$\alpha_2 = 2.8299$	$\beta_2 = .00948$	$\gamma_2 = 191.400$	$\delta_2 = 0.9809$
$\alpha_3 = 0.9905$	$\beta_3 = 0$	$\gamma_3 = 398.076$	$\delta_3 = 0.9905$

From these we calculate the values of u and v starting with $u_1 = 1$ and $v_1 = 1$ and each from the preceeding. Then

$$u_2 = \alpha_1 u_1 + \beta_1 v_1 = 2.90271 + .00966 = 2.91237$$

$$v_2 = \gamma_1 u_1 + \delta_1 v_1 = 195.127 + 0.9805 = 196.1075$$

$$u_3 = \alpha_2 u_2 + \beta_2 v_2$$

$$u_3 = (2.8299)(2.91237) + (.00948)(196.1075) = 10.1008$$

$$v_3 = \gamma_2 u_2 + \delta_2 v_2$$

$$v_3 = (191.400)(2.91237) + (0.9809)(196.1075) = 749.789$$

$$u_4 = \alpha_3 u_3 + \beta_3 v_3 = (.9905)(10.1008) + 0 = 10.005$$

$$v_4 = \gamma_3 u_3 + \delta_3 v_3 = (398.076)(10.1008) + (.9905)(749.789) = 4763.552$$

$u_1 = 1.0000$	$v_1 = 1.000$
$u_2 = 2.9124$	$v_2 = 196.1075$
$u_3 = 10.1008$	$v_3 = 749.789$
$u_4 = 10.005$	$v_4 = 4763.552$

and finally the value of the shielding factor is determined as

$$F = 1/2\ (u_4+v_4) = 1/2\ (10.01+4763.55)$$

$$F = 2386.8$$

Notice that it is difficult to assign ahead of time the number of decimal places to be carried. Rather, it is best to carry along at least 3 and possibly 4 significant figures and round off the final value of F.

We now make a similar calculation for the shield nest 2, in which the inner shield is identical to that in nest 1, but the next two shields have been increased in diameter so that the spacing between the shields is 1.0 centimeter instead of 1 millimeter. The shielding factor will be increased at the expense of more volume required for the shield and in addition will use somewhat more shielding material due to the larger diameters of the two outer shields.

The calculations will not be shown in detail but a table of the ε's and other parameters is given below for comparison with those obtained for the previous calculation.

The dimensions of the cylinder for this nest are:

$a_1 = 10.0$	$b_1 = 10.1$	inner cyl.
$a_2 = 11.1$	$b_2 = 11.2$	middle cyl.
$a_3 = 12.2$	$b_3 = 12.3$	outer cyl.

and $\mu_1 = \mu_2 = 20 \times 10^3$ but now $\mu_3 = 47 \times 10^3$ due to the increased diameter and hence increased saturation of the mu-metal. This in itself will lead to an increased shielding factor.

Then:

$\varepsilon_1 = .009852$	$\varepsilon_{12} = 0.086032$
$\varepsilon_2 = .008889$	$\varepsilon_{23} = 0.078608$
$\varepsilon_3 = .008097$	$\varepsilon_{34} = 0$

$\alpha_1 = 17.855871$	$\beta_1 = .0851844$	$\gamma_1 = 180.174$	$\delta_1 = .904963$
$\alpha_2 = 14.887433$	$\beta_2 = .077909$	$\gamma_2 = 163.884$	$\delta_2 = .913202$
$\alpha_3 = 0.991903$	$\beta_3 = 0$	$\gamma_3 = 380.559$	$\delta_3 = .991903$

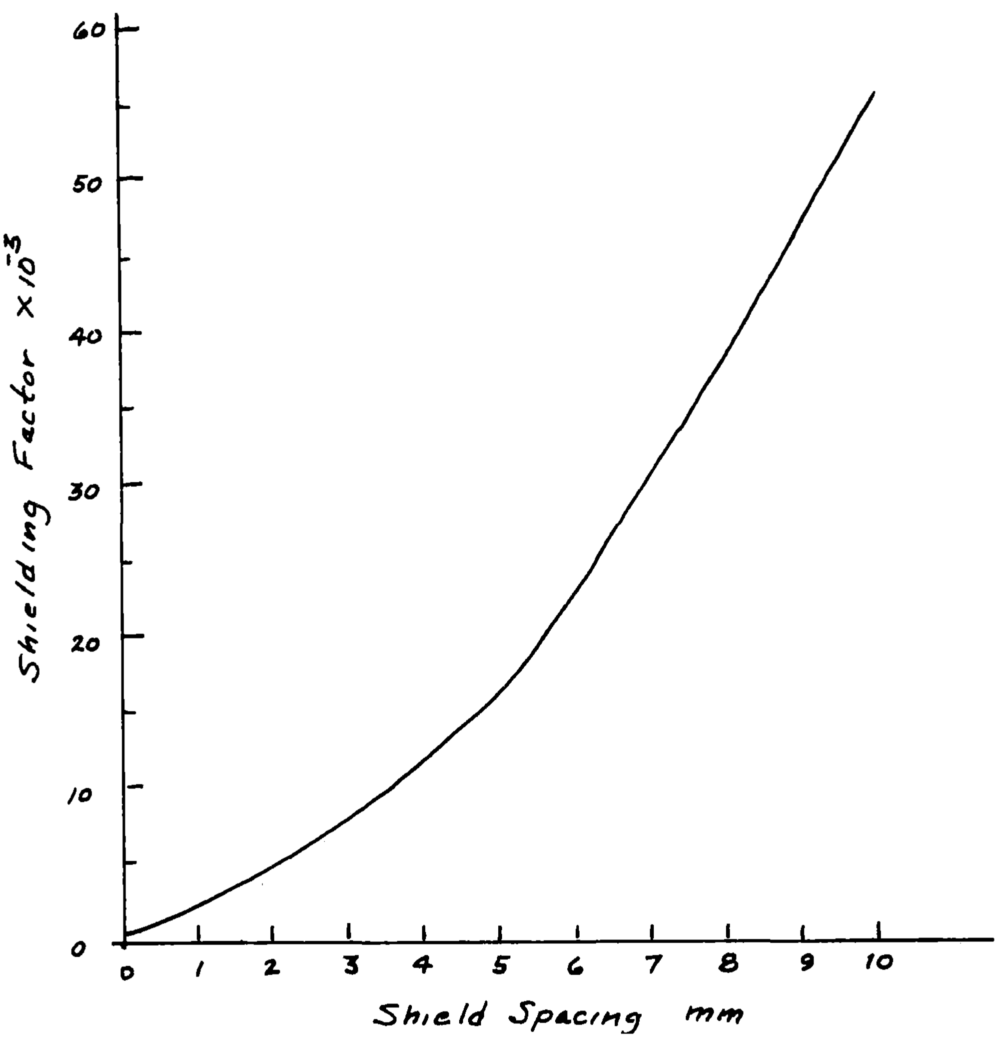
$u_1 = 1$	$v_1 = 1$
$u_2 = 17.941$	$v_2 = 181.079$
$u_3 = 282.519$	$v_3 = 3105.605$
$u_4 = 280.231$	$v_4 = 110{,}595.6$

The shielding factor is calculated as

$$F = 1/2\ (u_4+v_4) = 1/2\ (280.2+110{,}595.6)$$
$$F = 55{,}438$$

The shielding factor for several other spacings has also been computed (with the proper value of μ) and has been plotted (see graph) to show how an improvement in the shielding factor can be obtained by simply increasing only this spacing.

Spacing mm	F
0	474
1	2,386
5	16,328
8	38,383
10	55,438



As can be seen the shielding factor of the second nest is over 20 times that of the first. The increased spacing increased the $\varepsilon_{i,i+1}$ values which in turn increased the α and β values which in turn increased the u and v values. This is an appreciable gain in shielding factor obtained at little expense and as far as this writer knows has not been mentioned in other articles relating to the subject of shielding.

It might be mentioned here that because of the large value of the shielding factor it is sometimes given in terms of db, in which case;

$$F\ (db) = 20 \log_{10} F$$

and so for F = 2,386

$$F\ (db) = 20 \log_{10}\ (2,386) = 20\ (3.3777)$$

$$F\ (db) = 68\ db \qquad (\text{nest } 1)$$

and for F = 55,438

$$F\ (db) = 20 \log\ (55,438) = 20\ (4.7438)$$

$$F\ (db) = 95\ db \qquad (\text{nest } 2)$$

Each 6 db increase corresponds very closely to a factor of 2 improvement in the shielding factor (each 3 db, a factor of 1.4 and each 20 db, a factor of 10). It might be interesting to see what the shielding factor would be for a nest of cylinders which has the same outer radius (12.3 centimeters) as the outside cylinder of nest 2 but has the same spacing (1 millimeter) as nest 1. This would give an inner radius and hence working volume considerably larger but would have a shielding factor more like that of nest 1.

Let

Shield Nest 3

$a_1 = 11.8$ $\qquad b_1 = 11.9$ (inner cyl)

$a_2 = 12.0$ $\qquad b_2 = 12.1$ (middle cyl)

$a_3 = 12.2$ $\qquad b_3 = 12.3$ (outer cyl)

$\mu_1 = \mu_2 = 20 \times 10^3$ $\qquad$ inner and middle cyl.

$\mu_3 = 47 \times 10^3$ $\qquad$ outer cyl.

Now, as pointed out by Wadey[5] if all dimensions are changed by the same factor the concentration ratio of each shell stays the same, hence permeability stays the same and since the shielding factor F depends only to the *ratio* of the dimensions, the factor F remains unchanged. Hence scaling all dimensions (thicknesses and spacings) by the same factor will give the same value of F.

If we apply this idea to cylinder nest 1 multiplying each radius by 1.1714 to scale it up to the size of shield nest 3, we find

$$
\begin{array}{lll}
a_1 = 11.71 & b_1 = 11.83 & \text{(inner cyl.)} \\
a_2 = 11.95 & b_2 = 12.07 & \text{(middle cyl.)} \\
a_3 = 12.18 & b_3 = 12.30 & \text{(outer cyl.)}
\end{array}
$$

A nest of cylinders with these dimensions would have exactly the same shielding factor (F = 2386) as nest 1 and since these dimensions are very similar to those just suggested we can conclude that the shielding factor would be nearly the same. Hence we have avoided the lengthly direct calculation by utilizing this property of shields to obtain a good approximation to this new configuration.

Some other remarks are pertinent here. In designing a shield against the earth's magnetic flux density ($B \simeq 0.7$ gauss; 0.7×10^{-4} webers/sq. m.) a considerable improvement in shielding factor can be obtained if the shield thickness and diameter are made such that the flux concentration places the material at or near the maximum of the material permeability curve (see curve). If the flux density in the material is kept above 800 gauss and below 4000 gauss the permeability will be greater than 100,000 for mu-metal. For small diameter shields this may require a very thin material.

If we assume $B_o = 0.7$ gauss, $B_{eff} = 2000$ gauss (max μ)

$$B_{eff} = 1.2\ (\frac{a}{t})\ B_o$$

Then

$$(\frac{a}{t}) = .83\ \frac{B_{eff}}{B_o} = .83\ \frac{2000}{0.7}$$

$$(\frac{a}{t}) = 2380$$

If $a = 10$ cm, $t = 4.2 \times 10^{-3}$ cm or .0017 inches

In this case the cylinder would not be self supporting and would be difficult to handle. On the other hand, 0.05 cm (about .020 inches) material can be made into cylinders up to 10 cm radius rather easily and a nest can be separated with styrofoam or balsa wood wedges to form a sturdy configuration. A 10 cm radius with t = 0.05 cm gives

$$B_{eff} = (1.2)\left(\frac{10}{0.05}\right)(0.7)$$
$$= 168$$

with a μ = 51,000 for the outermost cylinder.

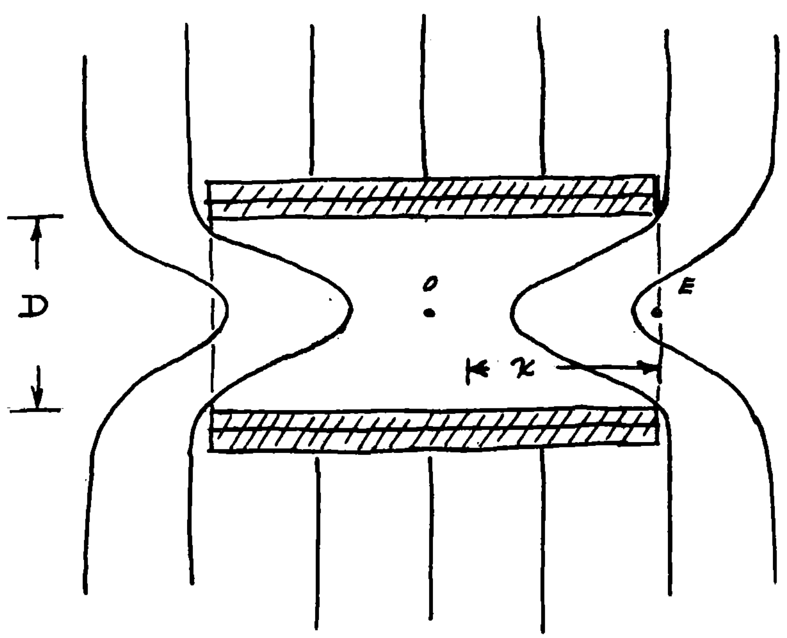
The flux concentration must then, as mentioned previously, be calculated for each succeeding cylinder going inwards until the initial permeability of μ = 20,000 is reached.

As was mentioned previously, shields can be wound in spiral form with an interleaving separator of cardboard or copper sheet. This would allow us to have a good mechanical configuration with a very thin shielding material. Several difficulties here are that the permeability of thin sheets is not the same as that of thicker material, data is not as readily available and the shield cannot be heat treated after construction. Heat treating (annealing) in a hydrogen atmosphere is necessary, *after* mechanical working, to obtain the optimum permeability of mu-metal. The advantages and disadvantages of spiral construction should be thoroughly considered before a decision to use this type construction is made.

End Effects

The calculations above are based on the assumption that the shields are infinite length cylinders with the field transverse or perpendicular to the cylinder axis. In practice the shield cannot be infinite in length. An alternative is to provide a tight fitting overlapping cap for each of the cylinders in the nest on both ends of a finite length cylinder. In the practical case where the shield length is a few diameters this approximates the infinite length cylinder sufficiently well for all cases except the most precise. In many cases, however, this is not possible. At least one end of the cylinder must be open. Furthermore, such caps are quite expensive to have fabricated. It has been shown that if a semi-infinite cylinder is considered, that the field at the open end is just one-half the original external field. Measurements on even short cylinders have shown that this is in fact true and may be even less and nearer to 1/3 the original field. Teasdale[6] in a very practical paper discusses the field penetration into very short cylinders and cones and presents data for a number of sizes. *For an axial field this is not true* and the field at the mouth of the cylinder is approximately equal to the original external field. It is reiterated here that to obtain maximum attenuation of the field it is necessary to orient the shield so that the axis is perpendicular to the original external field direction. This is usually not difficult to do. The shield may be placed with its axis horizontal and this axis rotated in the horizontal plane until the above condition obtains.

If one or both ends of the shielding cylinder is open, the flux will penetrate into the ends and reduce the shielding there as shown in the figure. If one goes into the cylinder far enough the maximum



MAX. Shielding at O
Shielding 1/2 at E

Flux penetration into shield nest

attenuation is reached. As a practical measure the point where the shielding factor is 68% of maximum F is a function of the diameter D and also of F. Wadey[5] using Esmarchs[6] data has proposed a function illustrated in the logarithmic graph. Points from Teasdales paper and data which we have obtained fall close to this line. It is suggested that this relationship be taken as a good general guide as to how far one must go into an open-ended cylindrical shield to obtain approximately 2/3 of the maximum shielding possible. The data can be interpreted as giving a function as follows

$$\log_{10}F = 8.83 \left(\frac{x}{D}\right) - 1.826$$

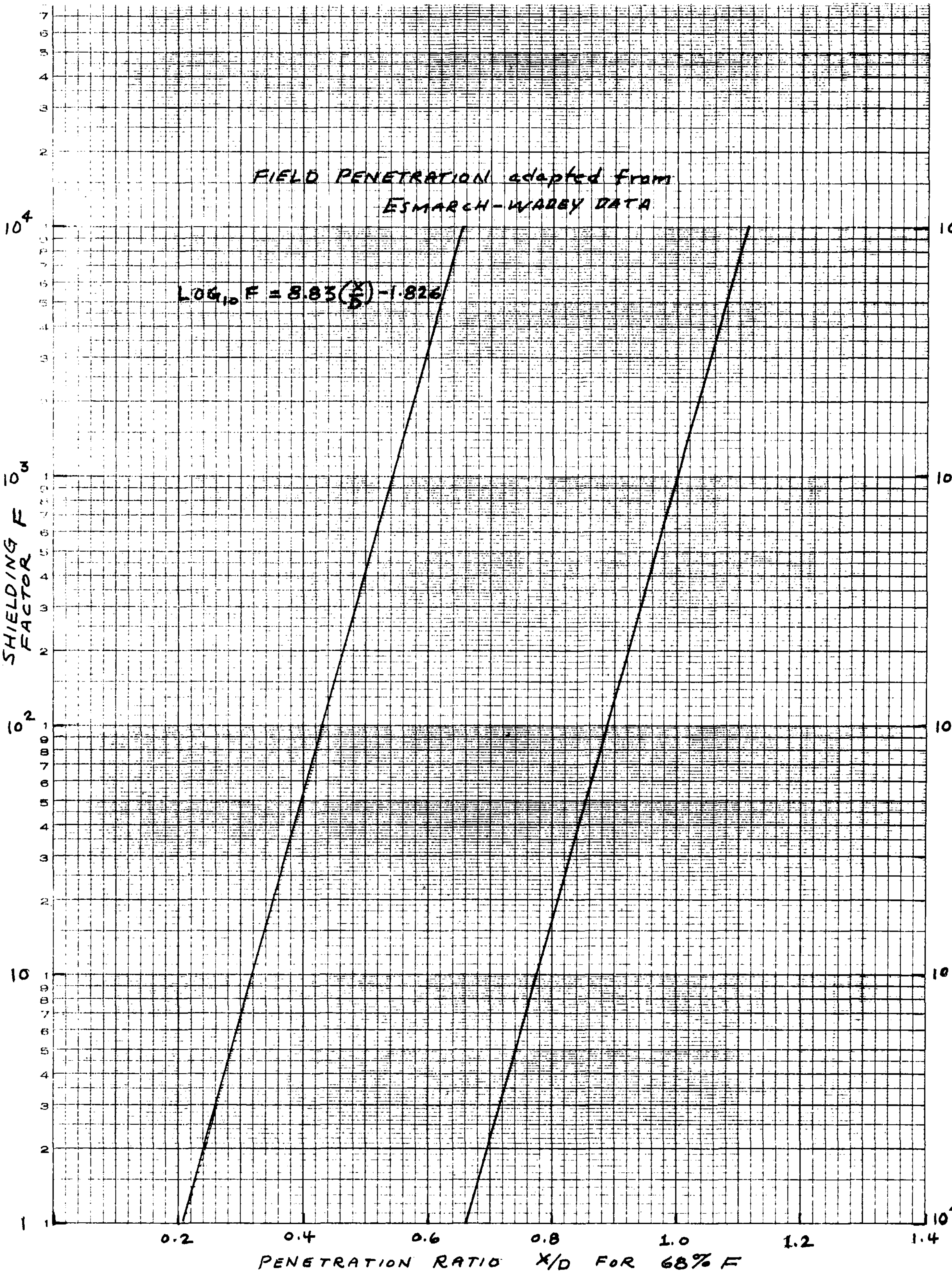
This is an empirical formula, but it covers data from three sources relatively well over a wide range of cylinder length to diameter ratios and from one to many shells in a nest. As can be seen from the curve, the higher the shielding factor the farther one must go into the cylinder to achieve 68% of that factor. It also shows that if one goes into the cylinder at least one diameter (for a nest, the inside diameter) the 68% point is reached even for $F = 10^7$. For a nest of shells of length three times its inner diameter, the middle third should then have a shielding factor greater than that 68% of F, even for high values of F. For this reason it was mentioned earlier that the L/D ratio of open ended shields should be around 3 or 4. If one end of the shield nest is tightly closed with well-fitting caps on each of the shields in the nest, this capped end should look like an infinite length end except possibly very close to the end surface. Approximately two-thirds of this nest (L/D = 3) should then be usable.

As a practical rule of thumb we give the following for usable length (L_u) (where effective shielding ≥ .68 F)

Both ends open	$L_u = (L - 2D)$
One end open	$L_u = (L - D)$
Both ends capped	$L_u = L$

where D = dia. inner cylinder

L = length of cylinder.


FIELD PENETRATION adapted from
ESMARCH-WADEY DATA
LOG10 F = 8.83(X/D) - 1.826
SHIELDING FACTOR F
PENETRATION RATIO X/D FOR 68% F
10^4
10^3
10^2
10
1
10^8
10^7
10^6
10^5
10^4
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Subject to the restrictions noted above this should always give a safe shielding length. Care must be taken that the shield axis is perpendicular to the original field direction to achieve maximum shielding.

It may also be desired to shield against alternating fields such as would be produced by transformers, lamps, motors or main wiring at say the usual line frequency of 60 cycles per second. If the estimated or measured peak flux density is approximately the same as the static flux density for which the shield was designed no additional precautions are necessary since the shielding factor for varying fields increases as the frequency increases. Nevertheless, because the orientation of the shield with respect to the A.C. fields may not be optimum it is best that devices operating from the power line should be kept at some distance (at least a few feet) from the shields in which case there should be no problem. It might be desired, however, to have a light inside the shield. In this case, it would be possible to design a special shield just for the light and related wiring, but it would perhaps be better to locate the light at some distance from the shield and focus the light into it by mirrors and lenses. An alternative method would be to lead the light by means of a "light pipe", into the shield. A very simple but effective light pipe consists of a solid lucite rod bent to the desired shape. The ends are cut straight across and polished. To obtain maximum light transmission the "pipe" sides should be polished and supported by narrow edges such as a hole through a thin sheet of metal or a wire loop.

If the varying field is of considerable magnitude and cannot be removed it may be necessary to specifically design a shield for this purpose. Wadey[5] gives formulas for the attenuation of shields against varying fields.

Schweizer[8] derives a general procedure for *concentric spherical shells* for any number and gives specific formulas for one, two and three shells. He further makes various approximations when the thickness of the shells are small and when the permeability is large. He does not take into consideration the fact that the permeability is different for different shells, however, except in the general procedure. His final three approximations appear to have an error in the numerical factor and a further condition must be true, i.e., d must not be too small. Schweizer's earlier approximations which appear to be correct are:

if: $(b_i-a_i)/b_i = t_i \ll 1$ fractional metal thickness

$(a_{i+1}-b_i)/a_{i+1} = d_i \ll 1$ fractional air space thickness

and also

$\mu \gg 1$

Then for one shell

$$F \simeq 1 + 2/3\ \mu\ t_i$$

two shells

$$F \simeq 1 + 2/3\ \mu\ (t_1+t_2) + 4/3\ \mu^2\ t_1 t_2\ d_1$$

three shells

$$F \simeq 1 + 2/3\ \mu\ [t_1+t_2+t_3] + 4/3\ \mu^2\ [t_1t_2d_1+t_2t_3d_2+t_1t_3(d_1+d_2)] + 8/3\ \mu^3\ t_1t_2t_3\ d_1d_2$$

These are useful and realistic approximations and can be used with reservations to estimate the shielding factor of cylindrical shields. For short capped cylindrical shields where L $\sim$ D they are probably quite good. For long cylindrical shields it would be best to develop similar approximate formulas starting from Stern's exact equations or use Will's formulas for one to three shells as given in Wadey(5). As an example we will calculate using the spherical formula above the shielding factor for a triple spherical shield with the same radii as the cylindrical shield nest 1 previously computed using Stern's formulas.

$a_1 = 10.0$	$b_1 = 10.1$	inner sphere radii
$a_2 = 10.2$	$b_2 = 10.3$	middle sphere radii
$a_3 = 10.4$	$b_3 = 10.5$	outer sphere radii

also let

$$\mu = 20 \times 10^3$$

$$\mu^2 = \mu_1\mu_2 = 20 \times 10^3 \times 20 \times 10^3$$

$$\mu^3 = \mu_1\mu_2\mu_3 = 20 \times 10^3 \times 20 \times 10^3 \times 42 \times 10^3$$

as an attempt to compensate for changing permeability.

Then

$$t_1 = (b_1-a_1)/b_i = 0.1/10.1 = .00990$$

$$t_2 = 0.1/10.3 = .00971$$

$$t_3 = 0.1/10.5 = .00952$$

$$d_1 = (a_2-b_1)/a_2 = 0.1/10.2 = .0098$$

$$d_2 = 0.1/10.4 = .00962$$

$$F \simeq 1 + 2/3\ (20\text{x}10^3)\ (.02913) + 4/3\ (20\text{x}10^3)^2\ (3.662\text{x}10^{-6}) + 8/3\ (20\text{x}10^3)^2\ (42\text{x}10^3)\ (86{,}431\text{x}10^{-15})$$

$$F \simeq 1 + 38.8 + 1953 + 3872 = 5865$$

Approximately 3/4 of this should be taken to approximate a cylinder, hence $F_c \approx 4398$ which is about 1.8 times too large as compared to our previous result but it admittedly is only an approximation.

Chapter 6 - References

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CHAPTER 7
HOW TO MEASURE MAGNETIC FIELDS

In order to draw conclusions and make calculations as well as to describe a magnetic field for others it is necessary to give it a number or numbers with appropriate units attached. Sometimes this can be done simply by knowing the configuration of conductors, currents and magnetic materials and making the appropriate calculations. The flux density in the region of interest is generally designed to be a certain value and this is known. But to rely on these calculations alone is most unsatisfactory unless measurements simply cannot be made for some reason. For example, the region of interest might be extremely small and/or inaccessible when all component parts are in place. In this case, we would have to rely on the calculated value alone. In most cases, however, it is possible and *desirable* to measure, by means of a calibrated instrument, the magnitude of the flux in the region of interest.

Most magnetic flux measuring instruments do just that, measure the flux or flux density in the region of interest and not the magnetomotive force or the magnetic field strength or intensity. These instruments may be divided into the following categories based on principle of operation:

moving coils - snatch coils, rotating coils
Hall effect - semiconductor material
flux gate - high permeability materials
miscellaneous - semiconductor material-magnet diodes
attraction of small magnets, rotation of
small magnets, magnetic sensitive resistances

Any of these principles can be incorporated into an instrument of fairly high accuracy and precision which will hold its calibration for long periods of time.

The accuracy required in the description of the magnetic field in biological experiments is not necessarily high. If the absolute value of the flux density is on the order of 5% or even 10% this may be quite satisfactory. As was explained in Chapter 4, if a high degree of uniformity of the flux density is not required then for a given size coil system we have a tremendously increased working volume available. Only a few experiments to date have shown a marked dependence of the biological effect on the magnetic flux density and it is difficult to imagine any biological mechanism which would be more than linearly dependent on small differences in the magnetic flux density. There is, of course, the possibility of a "resonance" mechanism which would be sharply dependant on the flux density, but from the reported results this seems remote. Thus, it is unlikely that an absolute measure of the magnetic flux density of an accuracy better than 10% would be necessary. A precision of measurement (reproduceability) of about 2 or 3% is quite desirable, however. Usually this can be accomplished with most commercial instruments utilizing moving coil, flux gate or Hall effect transducers.

The moving coil flux meters operate on the following principle. A conductor cutting lines of flux will have an electric potential produced at its ends due to the force on the free charges in the conductor which is proportional to the velocity with which the conductor cuts the lines of flux and the flux density. This may be expressed in several ways:

$$V = -\frac{d\phi}{dt} \text{ (MKS and CGS)}$$

where

V = potential at the wire terminals

ϕ = flux

or

$$V = Blv \quad \text{(MKS and CGS)}$$

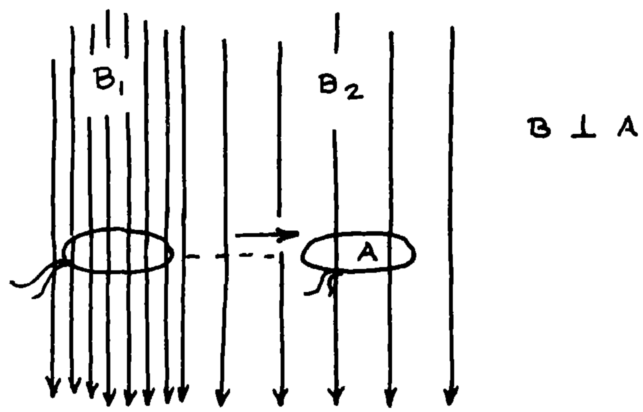
where

B = flux density

l = length of the wire

v = velocity of the wire

One of two mechanical arrangements is usually employed: a small flat coil of wire of a number of turns, is "snatched" or rapidly removed from the region of flux density B_1 to another region of flux density $B_2 \simeq 0$. It is positioned so that the original flux density B_1 is perpendicular to the plane of the coil.



Let

e = the instantaneous voltage produced at the coil terminals

i = the instantaneous current produced by the voltage through the circuit resistance

r = the circuit resistance

N = the number of turns in the coil

ϕ = the flux through the coil where $\phi = BA$

A = the coil area

then

$$e = -N \frac{d\phi}{dt} \quad \text{(MKS and CGS)}$$

and

$$i = e/r = -\frac{N}{r}\frac{d\phi}{dt}$$

if we integrate this over the time t required to move the coil from flux ϕ_1 to ϕ_2 (flux density B_1 to B_2)

then

$$q = \int_{o}^{t} idt = -\frac{N}{r}\int_{\phi_1}^{\phi_2} d\phi$$

$$q = -\frac{N}{r}(\phi_2-\phi_1)$$

where

$$\phi_2 = 0$$

$$q = \frac{N}{r}\phi_1 = \frac{N}{r}B_1A$$

Solving we find

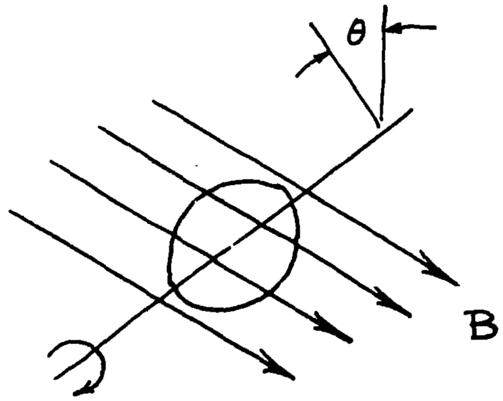
$$\phi = \frac{qr}{N} \text{ and } B = \frac{qr}{NA} \text{ (MKS and CGS)}$$

A ballistic galvanometer will produce a deflection proportional to the charge passed through it in a short time. Thus, the deflection of a ballistic galvanometer attached to a "snatch" coil can be calibrated in terms of either ϕ or B if n, the number of turns in the coil, r the total resistance of the circuit and A the area of the coil are known. Portable ballistic galvanometers calibrated in terms of flux are known as fluxmeters or integrating fluxmeters.* Integrating electronic circuits utilizing operational amplifiers are also used to provide the conversion of q to e and hence B, i.e., to integrate the current produced by the search coil.** Search or "snatch" coils for these instruments are not difficult to make for specific applications and can be readily calibrated in terms of their NA or turns x area by comparison with a coil of known NA using a fixed uniform magnetic flux density.

*A typical instrument of this type is manufactured by the Sensitive Research Instrument Corp., New Rochelle, New York, Model FM, accuracy 0.5 of 1%, 5 inch mirror scale range, 10 maxwells per division (10^{-3} webers) (1000 turn coil) most sensitive scale to 5×10^6 maxwells (50×10^2 webers) full scale (1 turn coil), least sensitive scale, r ranges, 8" x 7 1/2" x 6" portable. Search coils of various areas available.

**A typical instrument of this type is manufactured by Magnetmetrics, O.S. Walker Co., Worcester, Mass., Model MF-1, utilizes a chopper stabilized operational integrator, accuracy $\pm 2\%$, 6 inch scale, range 10^5 maxwells full scale (most sensitive), to 10^7 maxwells 3 ranges can be directly set for various search coil NA.

A second type of flux measuring device using a moving coil is one in which the coil is rapidly rotated along a diameter so that the plane of the coil is continuously changing its angle to the magnetic flux.



axis of rotation
⊥ to B

The flux passing through the coil when its plane is at an angle ϕ to the direction of the flux is:

$$\phi = BA \cos\theta$$

where A is the coil area: differentiating with respect to time

$$\frac{d\phi}{dt} = - BA \sin \frac{d\theta}{dt}$$

where $d\theta/dt$ is the angular velocity of rotation.

For N turns of the coil the induced voltage is then:

$$V = -N \frac{d\phi}{dt} = BNA \sin\theta \frac{d\theta}{dt}$$

if we let $\omega = \frac{d\theta}{dt}$ $\theta = \omega t$ and $V = BNA\ \omega \sin \omega t$

The angular velocity ω is known and fixed as is the product NA, hence

$$V = Bk \sin \omega t$$

where k is a constant of the instrument and the voltage generated is an alternating sinusoidal voltage whose magnitude is proportional to the flux density B. This voltage may be measured by any number of kinds of sensitive A.C. voltmeters* or its amplitude observed on an oscilloscope.

To show the magnitude of the voltage generated we give the following example:

let

$N = 100$ turns

$A = 1\ cm^2 = 10^{-4}\ m^2$

Speed of rotation 1200 rpm = 20 rev/sec

$\omega = 2\pi/60 \times 1200 = 125.7$ rad/sec

$B = 1000$ gauss $= 0.1$ webers/m^2

then

$$V = BNA\ \omega \sin \omega t$$

$$V = (1000)\ (100)\ (1)\ (125.7) \sin (125.7t)$$
$$= (0.1257 \times 10^8) \sin \omega t \quad \text{ab-volts (CGS,EMU)}$$

or

$$V = (0.1)\ (100)\ (10^{-4})\ (125.7) \sin (125.7t)$$
$$= 0.1257 \sin (125.7t) \quad \text{volts} \quad \text{(MKS practical)}$$

or a 20 cycle wave of approximately 1/8 volt peak amplitude or .0889 volts RMS.

Various modifications of these basic instruments are possible and available, such as differential or flux-gradient utilizing two similar coils spaced a known distance apart and an axial field design having two coils rotating at right angles to each other for measuring two right angle flux components.

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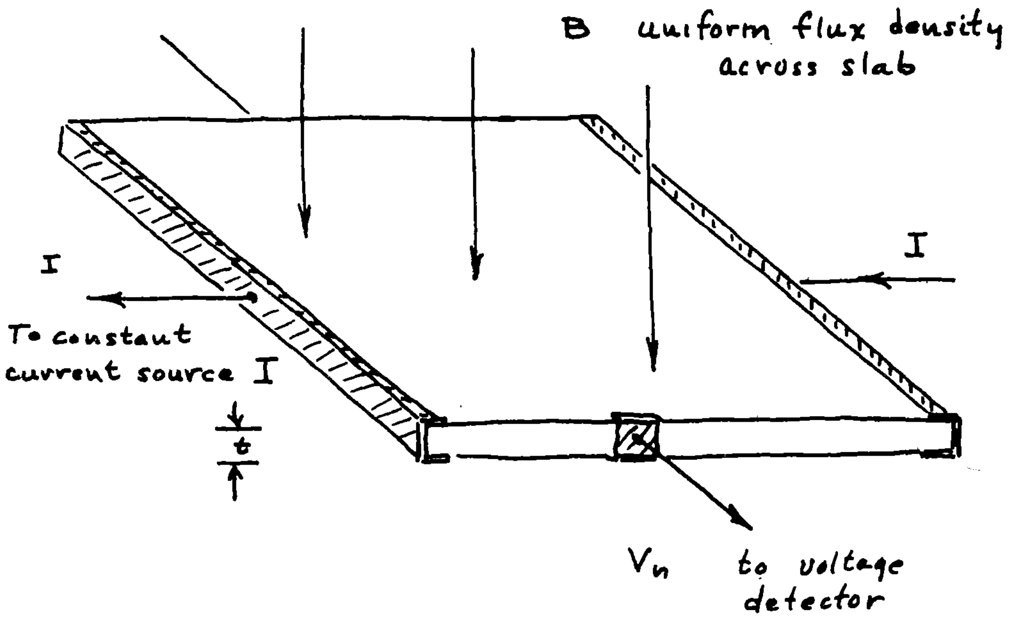
*A typical instrument of this type is manufactured by the Rawson Electrical Instrument Co., Cambridge, Mass., Model 822 (other models avail- for different ranges), accuracy may be adjusted to ± 0.1% or ± 0.05 gausses probe dia. 3/4", length 19", range 0-1000 gausses, sensitivity lowest range 0.1 gauss/division. Instrument can be used as direct reading or as null balance with greater sensitivity. Calibration magnets available.

The Hall effect (discovered by E. W. Hall in 1879) magnetic flux density detector is based on the basic principle that a moving charge in a magnetic field experiences a force perpendicular both to the direction of motion and to the direction of the field.[1] The force is proportional to the charge, the velocity and the flux density. When current flows through a slab of semiconductor material (see figure) (indium arsenide, indium antimonide, bismuth, germanium, etc.) this force on the electrons in the current pushes these electrons towards the edge until an equilibrium condition is obtained. The potential obtained is proportional to the current, thickness (t) and flux density through the slab.



The simple theory relationship found is:

$$V_{Hall} = K \frac{IB}{t}$$

where K is the Hall constant.

Without going into a host of complications we find that for mixed units the order of magnitude of the voltage is about 0.1 volt for a field of 10^4 gauss (1 weber/m^2) for a current of 0.1 ampere and a thickness of slab about one millimeter. In order to make a practical instrument many disturbing factors must be overcome. Temperature coefficient, precision of location of contacts, resistance of the voltage detector, heat loss, size and shape of the slab, internal resistance, flux density which is a function of the magnetic field concentrators and choice of material are some of these factors which have to be successfully chosen to make practical operating instruments.

One great advantage of the Hall probe is the very small size in which the probe can be made. This is extremely useful for obtaining good resolution of the changes in flux density in a small region. Typical probe sizes are in the range of one-half to one and one-half millimeters thick by two to 10 millimeters wide by one to two centimeters long. Special probes can be made even smaller, but are generally less sensitive. Another advantage is that Hall detectors can measure fluctuating magnetic fluxes up to frequencies of the order of one-half megahertz with the proper voltage sensing equipment. Many companies supply either or both Hall probes and auxilliary measuring equipment and many models and ranges are available each as battery operated models and flux gradient probes.*

The Flux-Gate Magnetometer

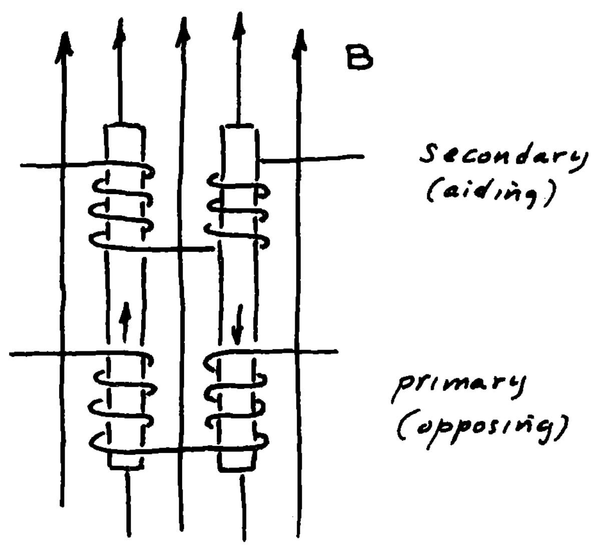
This device for measuring magnetic flux is based on the transformer principle[(2)]. The sensing device consists of two highly permeable cores of magnetic materials, placed parallel to each other, (some units

*Two such typical instruments are:

Bell, Inc., Columbus, Ohio, model 240 Incremental Gaussmeter. Direct reading on 12 ranges from 0.1 gauss (10^{-5} w/m^2) to 30,000 (3 w/m^2) full scale. Accuracy 1% of full scale. Synchronous modulation and demodulation amplifier system. Zero suppression to 14 x 10^3 gauss to enable differential measurements to be made (accuracy 1% to 5%), various typical small probes and calibration magnets available.

Radio Frequency Laboratories (RFL) Boonton, New Jersey, Model 1965 Universal Gaussmeter, Direct reading on 16 ranges from 0.1 gauss (10^{-5} w/m^2) to 10^4 gauss (1 w/m^2), accuracy $\pm$ 3% of full scale direct reading, $\pm$ 1% with calibration. Battery operation option built in, various scale expansion and suppression built in measures static and AC fields 20 Hz to 400 Hz various, small probes and reference magnets available.

use a toroidal ring of material) on each of which is wound two windings, a primary and a secondary. In one configuration, the primaries are connected series-opposing and the secondaries series-aiding.



When an exciting current of some low frequency, say 20 Hz to 20 KHz is fed to the primaries, the voltage induced in the secondaries is zero with zero external B field because of the cancellation due to the opposite phase. When a constant flux from an external field is applied, an unbalance in flux in the cores occurs. The external field produces a flux in the cores which adds to that produced by the primary in one rod and subtracts in the other. Due to the extreme non-linearity of the permeability of these cores with respect to the flux density in them, a voltage is generated in one secondary which is not balanced by that in the other. This difference or unbalanced net voltage from the two secondaries is proportional to the external flux density B and is of a frequency twice that of the driving frequency. The amplitude of this second harmonic is then a measure of the external magnetic flux density.

Suitable electronic circuits are used to amplify, filter and phase defect this signal which can then be displayed as a meter indication.*

In some flux gate transformers, additional windings are placed on the cores. A steady current may be passed through these windings to produce a flux which cancels the flux produced in the rods by the external field. These detectors may then be used as differential-flux measuring devices and/or as a means of expanding the scale in the region of interest.

These instruments are quite stable and can be made highly sensitive. Commercial instruments are available which will read to 10^{-4} gauss (10^{-8} weber/m^2) full scale and special instruments have been made for geomagnetic and space surveys with sensitivities several orders of magnitude greater. In general, the probes are fairly large in size an it is impossible to use a probe two inches long to measure the flux distribution in a region say 4 inches long. For fields of large extent this is no problem, but for many laboratory experiments this drawback may prohibit the use of this type instrument.

The above techniques are the most common and useful methods for determining the magnitude, gradient and direction of magnetic fields which might be used for Biomagnetics experiments.

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*Typical instruments of this type are:

Förster/Hoover Electronics, Inc., Ann Arbor, Michigan, Magnetometer Type MF-5050 ten ranges, 10^{-3} gauss (10^{-7} w/m^2) to 1 gauss (10^{-4} w/m^2) full scale with a factor of 10 increase sensitivity switch. Internal semicalibration and compensation or zero suppression up to 1 gauss (10^{-4} w/m^2) on any range. Calibration accuracy $\pm$ 1% on the 4 decade ranges. Standard probe size approx. 2 1/2" x 1" dia. AC fields can be monitored via scope jack.

Gammatronix Inc., Dublin, Ohio, Model 110 Magnetometer, five ranges 10^{-2} gauss (10^{-6} w/m^2) to 1 gauss (10^{-4} w/m^2), Linearity 2% of full scale, accuracy not stated. Probe 1" dia. x 7 1/2" long, solid state circuits. Low cost, small size.

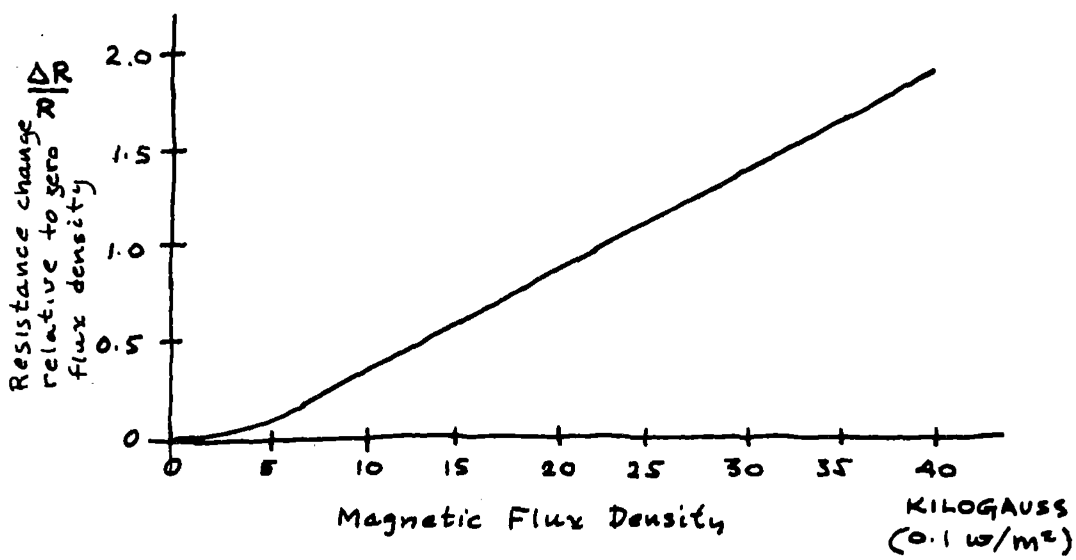
Magnaflux Corp., New York, New York, Model FM-204, ranges 10, 10^{-3} gauss (10^{-7} w/m^2), full scale, most sensitive range. Accuracy $\pm$ 1% fo full scale, long term stability 10^{-5} gauss. Smallest probe 2" long x 1/4" x 1/4". Speed of response .01 seconds. A.C. fields may be monitored via scope jack. Other probes for gradient, etc., available.

Miscellaneous Methods

Movement of Small Magnets

Almost any magnetic phenomena can be used to measure the flux density of magnetic fields. A simple small compass can be used to indicate the field direction, the needle aligning itself with the direction of the field with the geographic north seeking end of the needle pointing toward the south pole of the magnetic field. It is the convention to say that the sense of the direction of a magnetic field is from the north magnetic pole towards the south magnetic pole. If a spring is attached to the compass needle which will restore it to an equilibrium position with respect to a fiducial mark in the absence of any external field, then the angular deflection of the needle when placed perpendicular to an external field can be used as a measure of the magnitude and direction of the flux density of the field. The force of attraction (or repulsion) of a small magnet acting against a spring (or gravity) has also been used to measure the strength of magnets. Simple special purpose instruments of not very great accuracy or sensitivity have been made using these principles.

Other instruments have been based on the change in resistance of certain conductors or semiconductors when placed in a magnetic



field.* The resistance curve is not linear except above about 2 x 10^3 gauss (0.2 webers/m^2) (linearity is approximately 1% above this value). The approximate characteristic is shown on the preceding page.

These units are quite small--on the order of a few millimeters square by a fraction of a millimeter thick and will respond to frequencies as high as one megahertz and can be used from cryogenic temperatures (4.2°K) up to 100°C. A disadvantage is the rather high temperature coefficient of resistance 0.4% per °C. Conventional resistance bridge circuits may be used to measure the resistance or direct reading resistance circuits may provide a readout directly in flux density. Accurate calibration is required at any flux density at which this device is to be used and especially at values below the linear range. No figures are available on repeatability, but this is presumably good.

The Magnetodiode

A new device called the Sony Magnetodiode**(SMD)[(3)] is a magnetosensitive, semiconductor device of germanium or silicon which works on the principle of controlled lifetime of injected carriers by an external magnetic field. It is highly sensitive to magnetic fields, but is limited by noise at about 0.1 gauss (10^{-5} w/m^2). A typical unit is made in the form of a small block of intrinsic germanium 3 x 0.6 x 0.4 mm. The units are diodes and have typical diode characteristics as shown on the next page.

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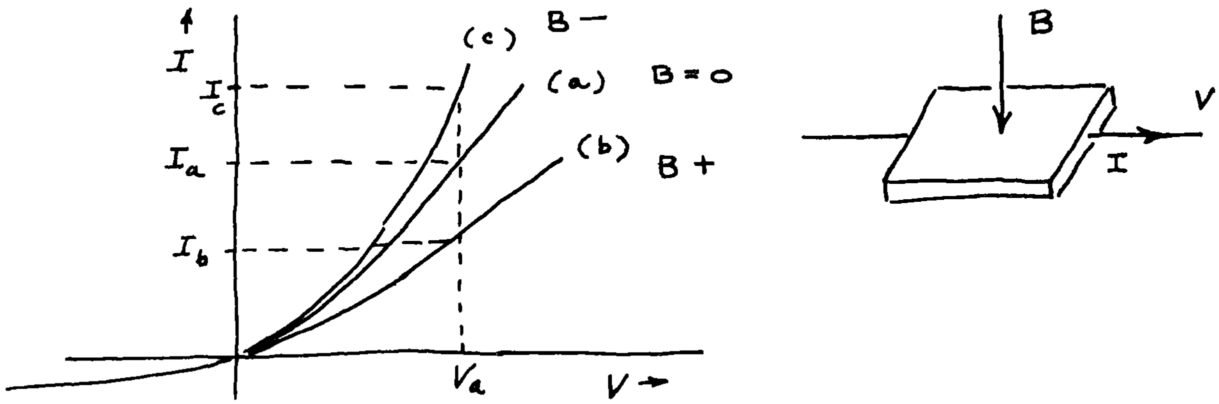
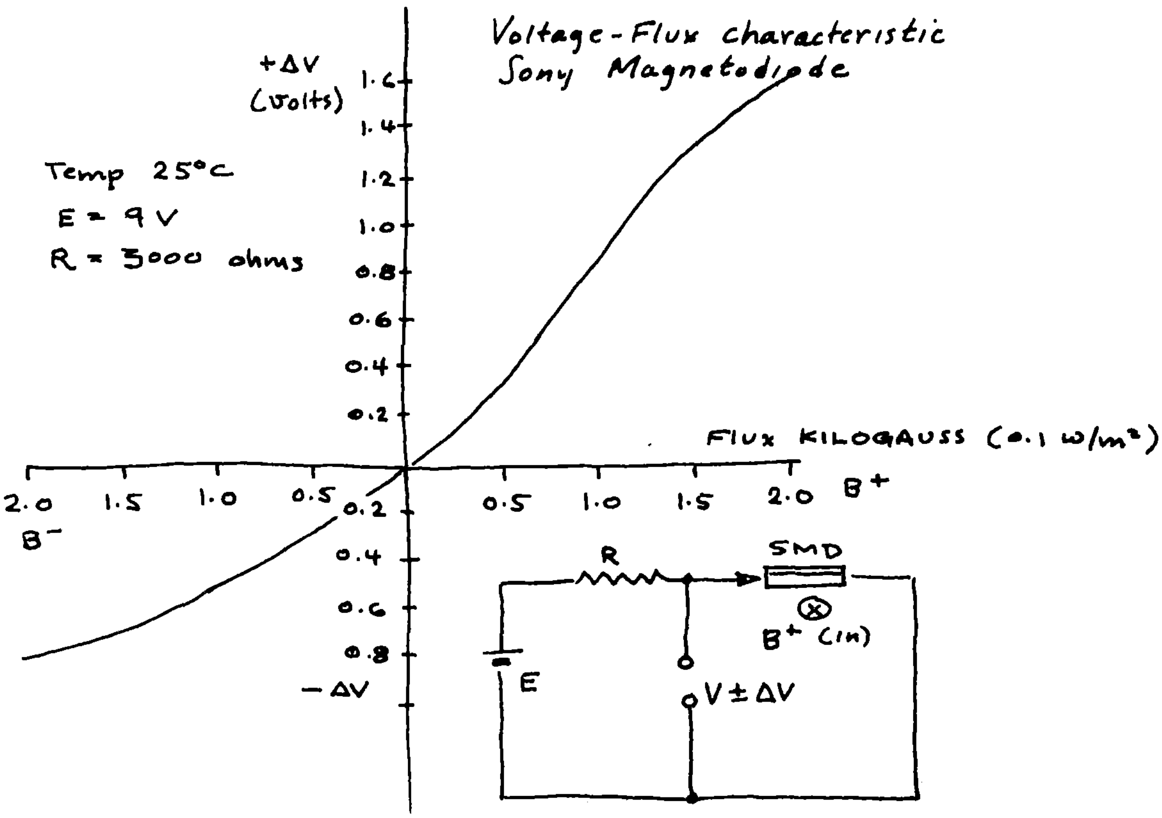
*A typical example of this instrument is a sensor built by American Aerospace Controls Inc., Farmingdale, New York. The MISTOR Model MR-A. Various models ranging in resistance from one to 5000 ohms at zero flux density. Sizes 1 1/2mm square by 0.1mm thick to 1 cm square x 0.2 mm thick. Accuracy figures are not given.

**Invention of Toshiyuki Yamada of SONY Corporation research laboratory. Information on this device from SONY data sheets. Sony corporation Tokyo, Japan.



SMD diode characteristics

Curve (a) shows the current I_a, for a given applied voltage V_a, when the transverse magnetic flux is zero. When a field B is applied in one direction (designated B+) the current will increase to I_c, when the same field is applied in the opposite sense the current will decrease to I_b. The voltage-flux characteristic in a simple circuit is shown in the next figure.



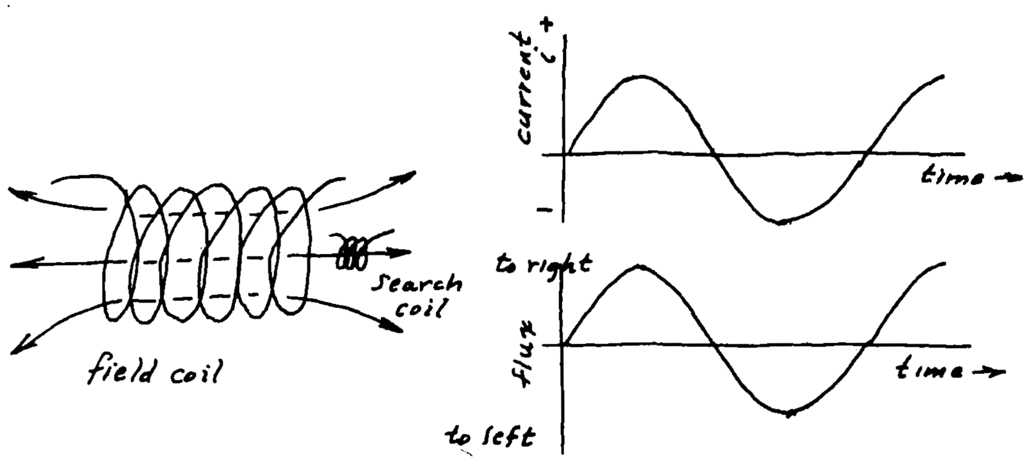
The response of these diodes is linear up to about 500 gauss (.05 w/m^2) flux density and gradually saturates beyond this point. The noise level is equivalent to about 5 x 10^{-2} gauss (5 x 10^{-6} w/m^2) which limits accurate measurement of flux densities to values somewhat higher than this value without the use of flux concentrators. Flux concentrators, essentially pole pieces of high permeability material designed to increase the flux density through the sensitive diode, will allow external fields as low as 10^{-5} gauss (10^{-9} w/m^2) to be measured linearly. Frequency response is excellent there being practically no change from d.c. or static fields up to about 2000 hertz and down by a factor of 2 at about 10,000 hertz. Flat response in special small samples has been obtained up to 100 KHz. Temperature dependence is a function of the material and the dimensions; with a suitable choice it can be made nearly independent (about ± 5%) of temperature from -10° C to 50° C.

This device is useful in applications not requiring the high accuracy and precision of Hall devices nor the high linearity (at high flux densities) of the magnetoresistive devices. It can be operated in very simple circuits. It is available in dual units which are useful in cancelling temperature dependence. In a bridge circuit its sensitivity may be given as about 1 volt per milliamp-kilogauss which is some two to four orders of magnitude more sensitive than Hall devices.

At the present time no instruments are being manufactured utilizing these magnetodiodes. I would expect, however, that in the near future enough experience will have been obtained with them that small relatively low priced instruments will be introduced. They should be quite useful for mapping fields because of the small probe size. Even with reasonable flux concentrators the probe size will probably be less than 1 cm dia x 2 cm long.

Measurement of A.C. Fields

If the magnetic field magnitude we wish to measure is not steady with time but is regularly fluctuating with we have a somewhat easier job. Let us assume we have a cylindrical coil of wire in air energized by a 60 cycle current which reverses its direction 60 times per second.



The flux and flux density magnitude follow the current exactly in phase and are proportional in amplitude. When the current reverses, the magnetic flux direction reverses. When the current goes to zero, the flux density goes to zero. Suppose we now put a small stationary search coil of N turns and area A in a part of the field where the flux density is B. Then with the following equation we find,

$$V = - N \frac{d\phi}{dt} \text{ (CGS and MKS)}$$

Now the current can be represented as

$$I = I_{max} \sin \omega t = I_{max} \sin (2\pi ft)$$

The flux density is proportional to the current so:

$$B = B_{max} \sin (2\pi ft)$$

The flux through the search coil is

$$\phi = AB = AB_{max} \sin (2\pi ft)$$

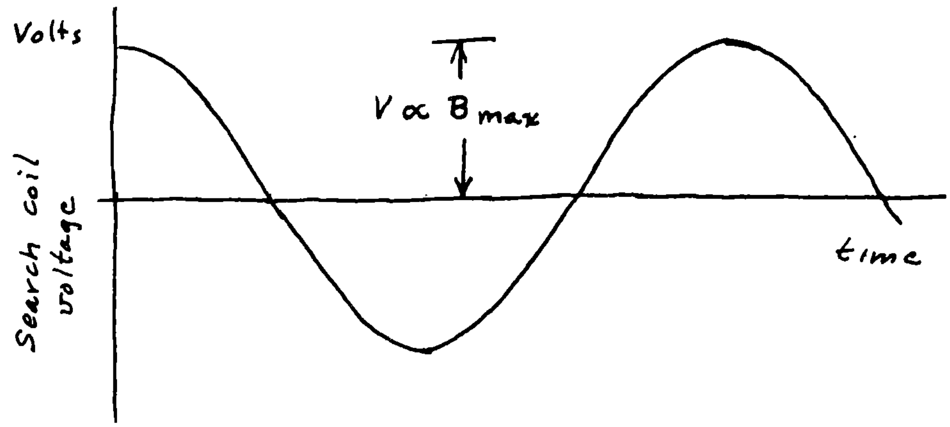
differentiating

$$\frac{d\phi}{dt} = AB_{max} 2\pi f \cos (2\pi ft)$$

and the voltage at the terminals of the search coil will be

$$V = - N \frac{d\phi}{dt} = - NA B_{max} 2\pi f \cos (2\pi ft)$$

If this cosine wave is observed on a calibrated oscilloscope the maximum voltage from zero to peak may be measured and will be proportional to B_{max}.



Let us put in some typical values for the various components:

$A_{coil} = 5\ cm^2 = 5 \times 10^{-4}\ m^2$

$N = 100$ turns

$B_{max} = 500$ gauss $= .05\ w/m^2$

$f = 60$ hertz $\quad 2\pi f \simeq 377$ rad/sec

Then:

$$V = -\ NA\ B_{max}\ 2\pi f \cos (2\pi f t)$$

$$V = -(100)(5)(500)(377) \cos (377t) \quad \text{(CGS)}$$

$$V = -(100)(5\times10^{-4})(.05)(377) \cos (377t) \quad \text{(MKS)}$$

$$V = -\ 0.963 \times 10^8 \cos (377t) \text{ emu volts or ab volts} \quad \text{(CGS)}$$

$$V = 0.963 \cos (377t) \text{ volts} \quad \text{(MKS)}$$

Thus we find that we obtain approximately 1 volt zero to peak deflection on the oscilloscope trace.

If we had measured this voltage with a high impedance AC voltmeter reading rms volts (as most a.c. meters do) we would have obtained $0.963/\sqrt{2}$ rms volts equals 0.682 rms volts. Since for a sine (or cosine) waveform the zero to peak voltage is related to the root mean square (rms) voltage as $\sqrt{2}$ V (rms) = V (zero to peak). This method is an extremely simple and accurate way of measuring alternating magnetic flux densities which are not too small.

The Hall effect device, the magnetoresistor and the magneto-diode (if they are energized by a steady dc current) are suitable for measuring regularly fluctuating magnetic fields. In some instruments these devices are energized by pulsating or high frequency alternating currents, in which case some difficulty may be encountered with them in measuring fluctuating magnetic fields, especially if the frequency of the field is close to the frequency of the energizing current.

The use of low frequency currents to measure and map the fields of coils such as Helmholtz, Reubens, Barker, etc., intended to be used with dc currents is sometimes useful. If the distributed capacity of the coil is small (and this will be true for many field coils - see [4] and [5]) we may energize the coil with a low frequency alternating current whose peak value is on the order of the value of the d.c. current we eventually wish to use. If the field is measured by a small pickup coil as just described we have a measure of the flux density that would be produced by the direct current. For example, let us assume the following values for a field producing coil.

$a = 50$ cm $= 0.5$ m   radius of coil

$l = 10$ cm $= 0.1$ m   length of coil

$f = 1000$ hertz

$n = 100$ turns

$I = 0.5$ ab amperes $= 5$ amperes

$\mu_o = 4\pi \times 10^{-7}$ b/m

Then the flux density at the center of the coil is given approximately by

$$B = \frac{2\pi In}{a} \quad \text{(CGS)} \qquad\qquad B = \frac{\mu_o In}{2a} \quad \text{(MKS)}$$

$$B = \frac{(6.28)(0.5)(100)}{50} \qquad\qquad B = \frac{4\pi \times 10^{-7}\ (5)(100)}{2(0.5)}$$

$$= 6.28 \text{ gauss} \qquad\qquad = 6.28 \times 10^{-4}\ \text{w/m}^2$$

The inductance of a coil of this size is approximately given by[5]

$$L = \frac{0.04\ a^2 n^2}{b} K \quad \text{micro henrys in mixed units}$$

where

a = radius cm

n = number of turns

b = length cm

K = 0.2 approx, for this a/b ratio

$$L = \frac{(.04)(50)^2(100)^2}{(10)} (.2) = 2 \times 10^{-2} \text{ henrys}$$

An estimate of the self capacitance is given by

$$C_o = HD \quad \text{micro microfarads in mixed units}$$

where

$D = 2a$ = dia. coil in cm

H = 0.7 approx. for this b/a ratio

$$C_o = (0.7)(100) = 70\ \mu\mu\text{fds.} = 7 \times 10^{-11} \text{ farads}$$

The frequency required to resonate this combination of L and C is

$$\omega^2 = \frac{1}{LC} = \frac{1}{2 \times 10^{-2} \times 7 \times 10^{-11}} = \frac{1}{14} \times 10^{13}$$

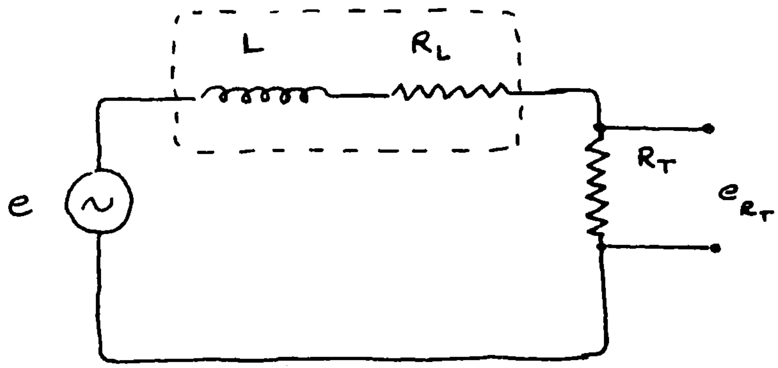
$$\omega^2 = 71 \times 10^{10}$$

$$\omega = 8.4 \times 10^5 \text{ rad/sec}$$

$$f = 134{,}000 \text{ hertz}$$

So that for any frequency say on the order of $\frac{1}{20}$ of this frequency or less, the distributed capacity may be neglected, hence at 1000 cycles we can say the self capacity is negligible.

If we now assume a testing circuit as follows:



Test circuit

where

$e \;= E_{max}$ sin $2\pi ft$ (volts) (low resistance generator)

R_T = test resistance (ohms)

$L \;=$ industance of coil (henry)

R_L = resistance of coil (ohms)

$R \;= R_T + R_L$

$$\bar{e} = \bar{Z}\,\bar{i}$$

$$\bar{i} = \frac{E_{max} \sin 2\pi ft}{\bar{Z}} = \frac{E_{max} \sin 2\pi ft}{(R^2 + \omega^2 L^2)} \quad \text{angle } \phi = \tan^{-1} \left(\frac{\omega L}{R}\right)$$

$$\bar{i} = \frac{E_{max}}{Z} \sin (2\pi ft - \phi) \qquad \text{where } Z^2 = R^2 + \omega^2 L^2$$

The voltage across the test resistor R_T is then:

$$e_{R_T} = \overline{L}R_T = \frac{R_T}{A} E_{max} \sin (2\pi ft - \phi)$$

This voltage is in phase with and proportional to the current through L. It is the current through L which produces the field in the region of L so that the field will then be proportional to this voltage e_{R_T} which in turn is proportional to E_{max}. By measuring the peak value of e_{R_T}

we obtain $\frac{R_T E_{max}}{Z}$ and dividing by R_T we then have a measure of E_{max}/Z which on comparing to $\bar{i}$ above we see is the peak value of $\bar{i}$ or i_{max}. This will give the peak value of B and may be used in,

$$B = \frac{2\pi I_n}{a} \quad \text{(CGS)} \qquad\qquad B = \frac{\mu_o I_n}{2a} \quad \text{(MKS)}$$

$$B_{max} = \frac{2\pi i_{max} N}{a} \qquad\qquad B_{max} = \frac{\mu_o i_{max} N}{2a}$$

Using the same small test coil as in the previous example

$A = 5 \text{ cm}^2 = 5 \times 10^{-4} \text{ m}^2$

$N = 100$ turns

$B_{max} = 6.28$ gauss $= 6.28 \times 10^{-4}$ w/m^2

$f = 1000$ hertz

we find for $V = - NA\, B_{max}\, 2\pi f \cos (2\pi ft-\phi)$

$$V = -(100)(5)(6.28)(6.28\text{x}10^3) \cos (6.28\text{x}10^3-\phi) \quad \text{(CGS)}$$

$$V = -(100)(5\text{x}10^{-4})(6.28\text{x}10^{-4})(6.28\text{x}10^3) \cos (6.28\text{x}10^3-\phi) \quad \text{(MKS)}$$

$V_{max} = 1.97 \times 10^7$ ab-volts (CGS) peak at center of field coil

$V_{max} = 0.197$ volts (MKS) peak at center of field coil.

Thus, we find that by using a thousand cycle energizing current for the field coil we can obtain with a simple test coil a reasonable voltage for even a small flux density in the field coil. Every measurement made of the flux density with the alternating field will be exactly equal to the field which will exist when the peak alternating current is replaced by a steady direct current of the same value. Further, the alternating current peak value can be simply measured by measuring the peak voltage across a known resistance in series with the field coil when the self capacitance of the field coil is small.

Chapter 7 - References

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