

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1360

*Periodic Orbits in the Elliptic Restricted
Three-Body Problem*

R. A. Broucke

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**JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA**

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Preface

The work described in this report was performed by the Mission Analysis Division of the Jet Propulsion Laboratory during the period from July 1, 1967 to June 30, 1968.

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Contents

I. Introduction	1
II. Equations of Motion	3
A. The Underlying Two-Body Problem	3
B. Inertial Barycentric Equations of Motion of the Satellite	4
C. Equations of Motion With Rotating Coordinates	5
D. Equations of Motion for the Rectilinear Elliptic Restricted Three-Body Problem	7
E. An Alternate Form of Equations of Motion for the Elliptic Restricted Three-Body Problem	8
F. Some Formulas for Coordinate Changes	9
1. Inertial coordinates (ξ, η) to rotating coordinates $(\bar{x}, \bar{y})$	9
2. Rotating coordinates $(\bar{x}, \bar{y})$ to inertial coordinates (ξ, η)	10
3. Rotating coordinates $(\bar{x}, \bar{y})$ to pulsating coordinates (x, y)	10
4. Pulsating coordinates (x, y) to rotating coordinates $(\bar{x}, \bar{y})$	10
5. Inertial barycentric coordinates (ξ, η) to inertial geocentric coordinates (ξ_G, η_G)	10
6. Inertial barycentric coordinates (ξ, η) to inertial selenocentric coordinates (ξ_S, η_S)	10
G. The Five Equilibrium Points of the Elliptic Problem	10
H. The Energy Equation of the Elliptic Restricted Three-Body Problem	13
I. Regularization With Birkhoff Coordinates	13
J. Equations of Motion With Birkhoff Coordinates	15
K. Initial Conditions for Ejection Orbits	17
III. Computation of Periodic Orbits	18
A. Integration of the Equations of Motion With Recurrent Power Series	18
B. Integration of the Variational Equations With Recurrent Power Series	20
C. Periodic Orbits in the Elliptic Restricted Three-Body Problem	23
D. Families of Periodic Orbits in the Elliptic Restricted Three-Body Problem	24
E. Differential Corrections for Periodic Orbits	25
F. Properties of the Eigenvalues of the Fundamental Matrix	26
G. Solution of the Characteristic Equation of the Elliptic Problem	27
H. The Seven Types of Stability Characteristics	28

Contents (contd)

IV. Families of Periodic Orbits	30
A. Families 12A and 6A of Periodic Orbits	30
B. Periodic Orbits in the Rectilinear Restricted Three-Body Problem	32
C. Family 7P	39
D. Family 7A	51
E. Family 8P	51
F. Family 8A	51
G. Family 11P	66
H. Family 11A	66
I. Family 10P	66
J. A Family of Periodic Collision Orbits	81
V. Computer Programs	89
A. Introduction	89
B. Derivative Subroutines	89
1. Subroutine DERIV with inertial coordinates	89
2. Subroutine DERIV with rotating-pulsating coordinates	90
3. Subroutine DERIV with regularized Birkhoff coordinates	90
4. Subroutine DERIV with rotating-pulsating variables and with variational equations	90
5. Subroutine ERPSV for recurrent power series	90
C. Conversion Subroutines	91
D. Differential Correction Subroutines	92
E. Main Program With Variational Equations and Recurrent Power Series	93
1. $JN = 1$	94
2. $JN = 2$	94
3. Computing the variable-integration step	94
4. Logic of the differential correction process	94
5. Additional operations for periodic orbits	94
6. Computer time	94
F. A Regularized Main Program	95
References	124

Contents (contd)

Tables

1. Orbit variables and defining equations	19
2. Recurrence relations for coefficients of power series	20
3. Properties of the seven stability regions	29
4. Initial conditions for family 6A	33
5. Initial conditions for family 12A	36
6. Initial conditions for 13 periodic orbits	38
7. Initial and final conditions for 13 periodic orbits	42
8. Initial and final conditions for family 5P	42
9. Initial and final conditions for family 9P	46
10. Initial and final conditions for family 4P	46
11. Initial and final conditions for family 3P	47
12. Initial and final conditions for family 7P	49
13. Initial and final conditions for family 7A	54
14. Initial and final conditions for family 8P	61
15. Initial and final conditions for family 8A	67
16. Initial and final conditions for family 11P	71
17. Initial and final conditions for family 11A	76
18. Initial and final conditions for family 10P	80
19. Final conditions for periodic collision orbits	88
20. Families of periodic orbits	95
21. List of subroutines	95
22. Computer program	96

Figures

1. Configuration of masses M_1 , M_2 and libration point L_4	12
2. Seven stability regions	29
3. Root configuration for stability regions	29
4. Circular restricted three-body problem: Strömgren's Class C of periodic orbits around L_1 (equal masses; $e = 0$; rotating axes)	30
5. Elliptic restricted three-body problem: periodic orbit for mass ratio $\mu = 0.5$ (equal masses; $e = 0.74$; rotating axes)	31

Contents (contd)

Figures (contd)

6. Elliptic restricted three-body problem: periodic orbits for constant mass ratio $\mu = 0.5$ (equal masses) and variable eccentricity ($e = 0.0$ to 1.0 ; inertial axes)	31
7. Family 6A of periodic orbits—rectilinear elliptic restricted three-body problem: periodic orbits for constant eccentricity ($e = 1.0$) and variable mass ratio ($\mu = 0.50$ to 0.166 ; inertial axes)	32
8. Family 12A of periodic orbits	35
9. Diagram of initial conditions for periodic orbits	39
10. The 13 periodic orbits in the barycentric (inertial) coordinate system	40
11. The first six periodic orbits in the geocentric and the selenocentric coordinate systems	41
12. Families 5P, 9P, 4P, and 3P of periodic orbits	43
13. Family 7P of periodic orbits	48
14. Family 7A of periodic orbits	52
15. Family 8 of periodic orbits	56
16. Family 8P of periodic orbits	57
17. Family 8A of periodic orbits	63
18. Family 11 of periodic orbits	68
19. Family 11P of periodic orbits	69
20. Family 11A of periodic orbits	72
21. Family 10P of periodic orbits	78
22. Ejection orbits (from $m_2 = \mu$) with $\theta = 0.0$, $\mu = 0.5$	81
23. Family of ejection orbits (e vs μ)	82
24. Family of ejection orbits (e vs E)	82
25. An extreme orbit of the family of periodic collision orbits ($e = 0$)	83
26. An extreme orbit of the family of periodic collision orbits ($e = 0.98$)	83
27. Periodic collision orbits in different coordinate systems	84

Abstract

Results of a detailed study of the two-dimensional, elliptic restricted three-body problem are presented. The equations of motion and the variational equations have been solved with recurrent power series. Several computer programs have been prepared to use the recurrent power series for the generation of families of periodic orbits. Some of these programs have been regularized with the well-known Birkhoff regularization. A total of 15 families of periodic orbits (1127 orbits) are described. The linear stability of several of these orbits has been computed; that of periodic orbits has been studied theoretically for the elliptic problem and for any nonintegrable, nonconservative, dynamical system with two degrees of freedom, and it has been found that seven types of stability or instability exist. Periodic orbits have been obtained for both the earth-moon mass ratio and the Strömgren mass ratio (equal masses). The whole range of eccentricities e from 0 to 1 (including 1) has also been studied, and it is shown that some periodic orbits exist for all of these eccentricities. A family of periodic collision orbits is also described.

Periodic Orbits in the Elliptic Restricted Three-Body Problem

I. Introduction

The elliptic restricted three-body problem is a particular case of the well-known three-body problem. "Restricted" means that one of the three particles has a zero mass; "elliptic" means that the two other particles move in Keplerian elliptic orbits. The problem, then, is to study the motion of the massless particle or satellite under the influence of the two other massive bodies. In this report, the problem is two-dimensional.

It can be seen in the literature that the particular case of the three-body problem that has been the most extensively studied is the so-called "circular restricted three-body problem," and that most other special cases have been investigated much less. For this reason, it was considered useful to begin a numerical study of the particular case of the three-body problem that comes next to the circular problem—the elliptic problem. The elliptic problem may be considered as a remarkably simple dynamical system that is slightly more complicated than the circular one. In fact, however, the elliptic problem is considerably different from the circular problem. This is because the elliptic problem is nonconservative.

The elliptic problem may be considered as the prototype of all *nonconservative*, nonintegrable dynamical systems with two degrees of freedom. The elliptic three-body problem is probably one of the simplest examples of such a system, the circular problem being the typical example of a *conservative*, nonintegrable system with two degrees of freedom. It has been demonstrated that the fact of being conservative or nonconservative is extremely important, and that the absence of the energy integral has far-reaching consequences.

Another reason for the interest in the elliptic problem is that there are several situations in the solar system for which this configuration is a better approximation and a better model than the circular model. In the study of the motion of the moon, for example, it is better to use an elliptic orbit for the earth than a circular one. When the motion of an artificial earth-moon spacecraft or of an artificial lunar satellite is studied, it is also better to use an elliptic orbit rather than a circular one for the motion of the moon. The literature reveals that only a few authors have studied the influence of nonzero eccentricities for the motion of the primaries of a three-body problem in astronautical applications; among these are

Colombo, Lautman, and Munford (Refs. 1 and 2), Herring (Ref. 3), and Richards (Ref. 4). As for the natural satellites and asteroids, such a study has also been made by Hunter (Ref. 5).

The present report concentrates on the numerical generation of periodic orbits in the elliptic problem. In this area, it must also be said, the circular problem has been studied more extensively than the elliptic problem. In fact, this work is a natural follow-up of the author's previous study of the circular problem (Ref. 6). This work attempts to generalize several concepts and methods that are well established for the circular problem. The recurrent power-series solutions and the computation of the characteristic exponents by the methods of Deprit and Price (Ref. 7), in particular, have been adapted here to the elliptic problem. To the author's knowledge, this has not been done before.

In contrast with the abundance of research on periodic orbits in the circular problem, only a few papers have been published that deal with the numerical computation of periodic orbits in the elliptic problem, e.g., that of Schubart (Ref. 8). This is another reason to undertake such a study.

A search of the literature on the elliptic restricted three-body problem found that the only aspect studied in some detail is the problem of the stability of motion in the neighborhood of the triangular libration points. This aspect of the problem has not been considered in the present work, therefore, and the works of the following authors can be consulted: Bennett (Ref. 9), Danby (Ref. 10), Grebenikov (Ref. 11), Lanzano (Ref. 12), Moulton (Ref. 13), Rabe (Ref. 14), and Szebehely (Ref. 15).

As regards stability, some new theoretical and numerical results concerning the stability of periodic orbits are given herein. Probably the first discussion of the characteristic exponents of a nonconservative dynamical system with two-degrees-of-freedom is presented in this report. It is shown that seven classes of orbits exist. Some preliminary results were presented at the August 1968 meeting of the American Institute of Aeronautics and Astronautics/American Astronautical Society (AIAA/AAS) (Ref. 16). A short, related study of the characteristic exponents of a dynamical system was also published by Broucke and Lass (Ref. 17). Much numerical work has also been accumulated to illustrate numerically the different properties of the characteristic exponents of the orbits; in fact, periodic orbits have been found that belong to all stability classes. These orbits have been

grouped in 15 families. Altogether, more than 1100 periodic orbits are described in the present report. To find these periodic orbits, the circular restricted three-body problem has generally been used as a starting point, and the eccentricity has then been increased. For instance, some starting orbits have been taken from Broucke's report (see Ref. 6) and from Bartlett's publication (Ref. 18).

Another remarkable property of the elliptic three-body problem, which has been the origin of several publications, has to do with the so-called Nechville transformation of the equations of motion. Nechville's paper was published in 1926 (Ref. 19), and his result has been discussed recently by Szebehely (Ref. 20), Kopal and Lyttleton (Ref. 21), Herring (see Ref. 3), and Broucke (Ref. 22). In the present report, an attempt is made to remove the singularity $e = 1$ from the Nechville transformation.

The Nechville transformation has also originated work on regularization of the elliptic problem, as can be seen in the works of Szebehely and Broucke (see Refs. 20 and 22). Earlier work on regularization of the three-body problem has been published by the present author in *Icarus* (Refs. 23 and 24), where several regularizing transformations are discussed; only the Birkhoff transformation, however, has been used herein. Practically no previous numerical work has apparently been done on the elliptic three-body problem in regularized coordinates. For this reason, an extensive study has been made of the behavior of the integration of the regularized equations of motion. A regularization of the elliptic three-body problem that is independent of the Nechville transformation has been published by Schubart (Ref. 25). A review of the equations of motion and the regularization of the elliptic restricted three-body problem is contained in Szebehely's work on the theory of orbits (Ref. 26), which is probably the first description of a family of periodic collision orbits in the elliptic problem that has been published.

The present report is subdivided into five sections. Section II describes the equations of motion relative to different coordinate systems, including the regularized equations. In the equations of motion, three differences between the circular and elliptic three-body problems are apparent, as follows:

- (1) The elliptic problem contains two parameters rather than one—the eccentricity e and the mass ratio μ .
- (2) The elliptic problem has no energy (or Jacobi) integral.

- (3) The independent variable is explicitly present in the equations of motion, even when rotating axes are used.

Section III describes the solution of the equations of motion and the variational equations with recurrent power series, and the properties of periodic orbits. It is shown that, in the elliptic problem, the period of a periodic orbit is always a multiple of 2π . For this reason, when e and μ are kept fixed, only isolated periodic orbits exist in the elliptic problem; however, families of periodic orbits may be generated with e or μ (or both) as variable parameters. The present work is relative to symmetric periodic orbits only, although nonsymmetric periodic orbits may also exist. Another characteristic of the elliptic problem is that periodicity can be obtained in the inertial axes as well as in the rotating system (whereas, in the circular problem, the orbits are generally periodic only relative to the rotating coordinate system). It is also shown in Section III that the linear stability in the elliptic problem—and in any nonconservative, nonintegrable dynamical system with two degrees of freedom—depends upon two independent real numbers rather than upon one, as in the circular problem. As a consequence of the existence of the two independent real stability coefficients, it is shown that seven types of stability or instability exist in the elliptic problem.

Sections IV and V contain descriptions of the families of periodic orbits and the computer programs. The computer programs that have been prepared are too long to be described completely herein, but they can be obtained from the author upon request.

II. Equations of Motion

A. The Underlying Two-Body Problem

In this report, the results are given of a numerical study of the two-dimensional, elliptic restricted three-body problem.

A particle is supposedly moving under the effect of the Newtonian attraction of two massive bodies called the *primaries*. In the present study, the particle or satellite and the two primaries (all three) move in the same plane. The problem is called *restricted* because the two primaries influence the satellite without themselves being influenced by it. In other words, the satellite is small and supposedly “massless” with respect to the primaries. This allows the two primaries to move in Keplerian orbits

relative to each other or relative to their center of mass. These Keplerian orbits will always be elliptic, and no restriction will be made on the eccentricity e . Thus, the case with eccentricity $e = 1$ will be considered herein as well as the other values of e from 0 to 1.

Basically, an inertial barycentric frame of reference will be used, although several rotating systems of axes will also be used. The convention is that, at the initial value of the independent variable (at $t = 0$), the two primaries are always located on the x -axis, and are at an apse—either at minimum elongation (periapsis) or maximum elongation (apoapsis). On the other hand, a system of canonical units will be used in such a way that the semimajor axis a and the mean motion n of the two-body motion of the primaries are unity. It is then possible to designate the two masses of the primaries (including the gravitation constant) by $m_1 = 1 - \mu$ and $m_2 = \mu \leq \frac{1}{2}$. The distance r between the two primaries is then

$$r = (1 - e \cos E) = \frac{p}{1 + e \cos v} \quad (1)$$

where E is the eccentric anomaly, v is the true anomaly, and the semilatus rectum $(1 - e^2)$ has been called p . With respect to the barycentric inertial frame of reference, the coordinates of m_1 and m_2 are taken to be

$$\xi_1 = -\mu r \cos v = -\mu (\cos E - e) \quad (2a)$$

$$\eta_1 = -\mu r \sin v = -\mu (1 - e^2)^{1/2} \sin E \quad (2b)$$

$$\xi_2 = (1 - \mu) r \cos v = (1 - \mu) (\cos E - e) \quad (2c)$$

$$\eta_2 = (1 - \mu) r \sin v = (1 - \mu) (1 - e^2)^{1/2} \sin E \quad (2d)$$

The eccentric anomaly E will be related to the time t through Kepler's equation

$$t + x = E - e \sin E \quad (3)$$

In the present study, the phase constant x is always taken as equal to 0 or π . The case $x = 0$ corresponds to a minimum elongation (periapsis) of the two primaries at $t = 0$, whereas the case $x = \pi$ corresponds to maximum elongation (apoapsis).

A few other formulas relating to the motion of the two primaries will be needed. For instance, the first and

second time derivatives of r , v , and E are

$$\begin{aligned} r' &= \frac{e \sin v}{(p)^{1/2}} = \frac{e \sin E}{r}, & r'' &= \frac{e \cos v}{r^2} = \frac{p - r}{r^3} \\ v' &= \frac{(p)^{1/2}}{r^2}, & v'' &= \frac{-2e \sin v}{r^3} \\ E' &= \frac{1}{r}, & E'' &= \frac{-e \sin v}{r^2 (p)^{1/2}} \end{aligned} \quad (4)$$

Using the above relations, the energy integral of the two-body problem is easily verified:

$$\frac{1}{2} (r'^2 + r^2 v'^2) - \frac{1}{r} = -\frac{1}{2} \quad (5)$$

In the above equations, a prime is used for the derivatives with respect to the time t , but later the true anomaly v will be used as independent variable, and the derivatives with respect to the true anomaly v will be indicated by dots. The derivatives of r are then

$$\begin{aligned} \dot{r} &= \frac{er^2 \sin v}{p} \\ \ddot{r} &= 2 \frac{\dot{r}^2}{r} + r \left(1 - \frac{r}{p} \right) \end{aligned} \quad (6)$$

Energy Eq. (5) may also be written as

$$\frac{p}{2} \left(\frac{\dot{r}^2}{r} + r \right) - \frac{1}{r} = -\frac{1}{2} \quad (7)$$

Through the use of energy Eq. (7), $\dot{r}$ may be eliminated from the second equation, Eq. (6), so that the following differential equation for r is obtained:

$$\ddot{r} = \frac{-2}{p} r^3 + \frac{3}{p} r^2 - r \quad (8)$$

Differential Eq. (8) will be used rather than the explicit solution of Eq. (1) because this will facilitate the work with recurrent power series. This equation is not valid for $\rho = 0$ or $e = 1$. For this reason, in cases with high values of e , Eq. (1) will be used rather than Eq. (8). In this case, Kepler's equation, Eq. (3), has to be solved; there is no difficulty in solving Eq. (3) numerically by successive approximations, however, even for high eccentricities, although some precautions may be necessary when $e = 1$. In what follows, frequent changes of the independent variable from t to v will be made. For any quantity F , the relations between the t -derivatives and v -derivatives according to the equation for v' in Eq. (4) are

$$\begin{aligned} F' &= \frac{(p)^{1/2}}{r^2} \dot{F} \\ F'' &= \frac{p(r\ddot{F} - 2\dot{r}\dot{F})}{r^5} \end{aligned} \quad (9)$$

In the sections that follow, a special set of coordinates $(\bar{\xi}, \bar{\eta})$ will be used. They will be called "reduced" or pulsating coordinates, and will introduce a radial change of scale in such a way that the elliptic motion of the primaries in the system (ξ, η) transforms in a circular motion in the system $(\bar{\xi}, \bar{\eta})$. The change of scale must thus be proportional to r :

$$\xi = r\bar{\xi}, \quad \eta = r\bar{\eta} \quad (10)$$

The reduced coordinates of the primaries m_1 and m_2 are then

$$\begin{aligned} \bar{\xi}_1 &= -\mu \cos v, & \bar{\xi}_2 &= (1-\mu) \cos v \\ \bar{\eta}_1 &= -\mu \sin v, & \bar{\eta}_2 &= (1-\mu) \sin v \end{aligned} \quad (11)$$

and represent circular motion (with nonconstant angular velocity). It is clear that the transformation of Eq. (10) cannot be used when $r = 0$ or when $e = 1$.

B. Inertial Barycentric Equations of Motion of the Satellite

In the present study of the elliptic restricted three-body problem, it has been necessary to use different coordinate systems. The equations of motion of the satellite relative to these coordinate systems will now be given, starting with the equations relative to the barycentric inertial system. Later, two other important barycentric systems will be used; they are called the "rotating" and "pulsating" systems. Finally, some other derived coordinate systems must be used for special purposes, such as regularization. In other cases, coordinate systems centered at m_1 or at m_2 will be used. These systems will be called "geocentric" (at the "earth" m_1) or "selenocentric" (at the "moon" m_2) because, in many computations, the earth-moon mass ratio has been used. Their function is mainly to show the forms of the orbits with respect to these masses (m_1 or m_2).

In the inertial barycentric frame of reference, the equations of motion of the satellite derive from the Lagrangian equation

$$L = \frac{1}{2} (\xi'^2 + \eta'^2) + \frac{1-\mu}{s_1} + \frac{\mu}{s_2} \quad (12)$$

They are thus

$$\xi'' = -(1-\mu) \frac{\xi - \xi_1}{s_1^3} - \mu \frac{\xi - \xi_2}{s_2^3} \quad (13)$$

$$\eta'' = -(1-\mu) \frac{\eta - \eta_1}{s_1^3} - \mu \frac{\eta - \eta_2}{s_2^3}$$

The distances between the satellite and m_1 and m_2 are obtained by

$$s_1^2 = (\xi - \xi_1)^2 + (\eta - \eta_1)^2 \quad (14)$$

$$s_2^2 = (\xi - \xi_2)^2 + (\eta - \eta_2)^2$$

Lagrangian Eq. (12) is not conservative because it explicitly contains the independent variable through s_1 and s_2 .

If the true anomaly v were used as an independent variable, the equations of motion would be derived from the Lagrangian equation (see Ref. 22):

$$L = \frac{r^2}{(p)^{1/2}} L = \frac{(p)^{1/2}}{2r^2} (\dot{\xi}^2 + \dot{\eta}^2) + \frac{r^2}{(p)^{1/2}} \left(\frac{1-\mu}{s_1} + \frac{\mu}{s_2} \right) \quad (15)$$

If the "reduced" or "pulsating" coordinates $(\bar{\xi}, \bar{\eta})$ are used, according to Eq. (10), and

$$\begin{aligned} \xi' &= r' \bar{\xi} + r \bar{\xi}', & \dot{\xi} &= \dot{r} \bar{\xi} + r \dot{\bar{\xi}} \\ \eta' &= r' \bar{\eta} + r \bar{\eta}', & \dot{\eta} &= \dot{r} \bar{\eta} + r \dot{\bar{\eta}} \end{aligned} \quad (16)$$

Lagrangian Eqs. (12) and (15) may be transformed accordingly, to give two new Lagrangian equations:

$$L = \frac{r^2}{2} (\bar{\xi}'^2 + \bar{\eta}'^2) + r r' (\bar{\xi} \bar{\xi}' + \bar{\eta} \bar{\eta}') + \frac{r^2}{2} (\bar{\xi}^2 + \bar{\eta}^2) + \frac{1}{r'} \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right) \quad (17)$$

$$L = \frac{(p)^{1/2}}{2} (\dot{\bar{\xi}}^2 + \dot{\bar{\eta}}^2) + (p)^{1/2} \frac{\dot{r}}{r} (\bar{\xi} \dot{\bar{\xi}} + \bar{\eta} \dot{\bar{\eta}}) + \frac{(p)^{1/2}}{2} \frac{\dot{r}^2}{r^2} (\bar{\xi}^2 + \bar{\eta}^2) + \frac{r}{(p)^{1/2}} \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right) \quad (18)$$

where

$$\begin{aligned} r_1^2 &= (\bar{\xi} - \bar{\xi}_1)^2 + (\bar{\eta} - \bar{\eta}_1)^2 = \frac{s_1^2}{r^2} \\ r_2^2 &= (\bar{\xi} - \bar{\xi}_2)^2 + (\bar{\eta} - \bar{\eta}_2)^2 = \frac{s_2^2}{r^2} \end{aligned} \quad (19)$$

Dividing the Lagrangian L in Eq. (18) by $(p)^{1/2}$ and replacing $\dot{r}$ by its value taken from Eq. (6), the following expression is obtained:

$$L = \frac{1}{2} (\dot{\bar{\xi}}^2 + \dot{\bar{\eta}}^2) + \frac{e r \sin v}{p} (\bar{\xi} \dot{\bar{\xi}} + \bar{\eta} \dot{\bar{\eta}}) + \frac{e^2 r^2 \sin^2 v}{2p^2} (\bar{\xi}^2 + \bar{\eta}^2) + \frac{r}{p} \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right) \quad (20)$$

This Lagrangian equation may finally be replaced by a more simple one,

$$L = \frac{1}{2} (\dot{\bar{\xi}}^2 + \dot{\bar{\eta}}^2) + \frac{1}{2} \left(\frac{r}{p} - 1 \right) (\bar{\xi}^2 + \bar{\eta}^2) + \frac{r}{p} \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right) \quad (21)$$

which is obtained by subtracting from Eq. (20) the following exact differential:

$$\frac{d}{dv} \left[\frac{e r \sin v}{2p} (\bar{\xi}^2 + \bar{\eta}^2) \right] \quad (22)$$

The equations of motion of the satellite, in "inertial barycentric pulsating" coordinates, with the true anomaly v as independent variable, are thus

$$\begin{aligned} \ddot{\bar{\xi}} &= \left(\frac{r}{p} - 1 \right) \bar{\xi} - \frac{r}{p} \left(m_1 \frac{\bar{\xi} - \bar{\xi}_1}{r_1^3} + m_2 \frac{\bar{\xi} - \bar{\xi}_2}{r_2^3} \right) \\ \ddot{\bar{\eta}} &= \left(\frac{r}{p} - 1 \right) \bar{\eta} - \frac{r}{p} \left(m_1 \frac{\bar{\eta} - \bar{\eta}_1}{r_1^3} + m_2 \frac{\bar{\eta} - \bar{\eta}_2}{r_2^3} \right) \end{aligned} \quad (23)$$

The forces present in the last equations of motion, Eq. (23), contain three terms: an apparent radial force that comes only from the radial scale change of the coordinate system, and m_1 and m_2 , which are, of course, the Newtonian attraction potential from the two primaries.

C. Equations of Motion With Rotating Coordinates

Rotating barycentric coordinates are now introduced. The rotation angle is the true anomaly v . Except when the eccentricity e is zero, the angular velocity of the axes will thus be time-dependent. The equations of motion in rotating axes will be written here in different forms—with time and with the true anomaly as independent variable, and with ordinary and reduced pulsating coordinates. The most important set of equations of motion, however, is the set wherein the true anomaly is the independent variable and where pulsating coordinates are used.

The ordinary rotating coordinates will be designated by $(\bar{x}, \bar{y})$, which are related to the inertial coordinates (ξ, η) by

$$\begin{aligned}\xi &= \bar{x} \cos v - \bar{y} \sin v \\ \eta &= \bar{x} \sin v + \bar{y} \cos v\end{aligned}\quad (24)$$

The rotating coordinates introduce the advantage that the two primaries m_1 and m_2 are now permanently on the $\bar{x}$ -axis, their coordinates becoming

$$\begin{aligned}\bar{x}_1 &= -\mu r, & \bar{x}_2 &= (1-\mu)r \\ \bar{y}_1 &= 0, & \bar{y}_2 &= 0\end{aligned}\quad (25)$$

but these coordinates are not constants; the two primaries m_1 and m_2 are oscillating on the $\bar{x}$ -axis. Lagrangian Eq. (12), under the effect of the rotation of Eq. (24), transforms into

$$\begin{aligned}L &= \frac{1}{2} (\bar{x}'^2 + \bar{y}'^2) + (\bar{x}\bar{y}' - \bar{y}\bar{x}') v' + \frac{1}{2} (x^2 + y^2) v'^2 \\ &+ \left(\frac{m_1}{s_1} + \frac{m_2}{s_2} \right)\end{aligned}\quad (26)$$

The equations of motion corresponding to this Lagrangian equation are

$$\bar{x}'' - 2\bar{y}'v' - \bar{y}v'' - \bar{x}v'^2 = -(1-\mu) \frac{\bar{x} - \bar{x}_1}{s_1^3} - \mu \frac{\bar{x} - \bar{x}_2}{s_2^3} \quad (27a)$$

$$\bar{y}'' + 2\bar{x}'v' + \bar{x}v'' - \bar{y}v'^2 = -(1-\mu) \frac{\bar{y}}{s_1^3} - \mu \frac{\bar{y}}{s_2^3} \quad (27b)$$

Equation (27) will practically never be used, but will be transformed once more in order to use rotating-pulsating coordinates (x, y) . These coordinates are defined in the same way as in Eq. (10):

$$\bar{x} = rx, \quad \bar{y} = ry \quad (28)$$

Lagrangian Eq. (26) then transforms in the new Lagrangian equation

$$\begin{aligned}L &= \frac{r^2}{2} (x'^2 + y'^2) + rr'(xx' + yy') + (p)^{1/2} (xy' - yx') \\ &+ \frac{1}{2} \left(r'^2 + \frac{p}{r^2} \right) (x^2 + y^2) + \frac{1}{r} \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right)\end{aligned}\quad (29)$$

and the corresponding equations of motion may be written as

$$\begin{aligned}r^2 x'' - 2y'(p)^{1/2} + 2rr'x' - \frac{1}{r}x &= \\ - \frac{1}{r} \left(m_1 \frac{x - x_1}{r_1^3} + m_2 \frac{x - x_2}{r_2^3} \right)\end{aligned}\quad (30a)$$

$$\begin{aligned}r^2 y'' + 2x'(p)^{1/2} + 2rr'y' - \frac{1}{r}y &= \\ - \frac{1}{r} \left(m_1 \frac{y}{r_1^3} + m_2 \frac{y}{r_2^3} \right)\end{aligned}\quad (30b)$$

The equations of motion take a more simple form if the true anomaly v is used as an independent variable. This change of variable is done with the use of the expression for v' , given in Eq. (4), in the same way as it was used to obtain Lagrangian Eq. (15). If one also subtracts the exact derivative, which is similar to Eq. (22),

$$\frac{d}{dv} \left[\frac{1}{2} \frac{\dot{r}}{r} (x^2 + y^2) \right] \quad (31)$$

after division of the Lagrangian equation by $(p)^{1/2}$, then Eq. (29) finally transforms into

$$\begin{aligned}L &= \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + (x\dot{y} - y\dot{x}) \\ &+ \frac{r}{p} \left[\frac{1}{2} (x^2 + y^2) + \frac{m_1}{r_1} + \frac{m_2}{r_2} \right]\end{aligned}\quad (32)$$

The equations of motion derived from this Lagrangian equation are

$$\ddot{x} - 2\dot{y} = \frac{r}{p} \left(x - m_1 \frac{x - x_1}{r_1^3} - m_2 \frac{x - x_2}{r_2^3} \right) \quad (33)$$

$$\ddot{y} + 2\dot{x} = \frac{r}{p} \left(y - m_1 \frac{y}{r_1^3} - m_2 \frac{y}{r_2^3} \right)$$

The above Lagrangian and equations of motion, Eqs. (32) and (33), have the remarkable property that they differ only by the factor $r/p = 1/(1 + e \cos v)$ from the Lagrangian and classical equations of motion of the circular restricted three-body problem. For this reason, the equations of motion, Eq. (33), with rotating-pulsating coordinates and with the true anomaly have often been used by different authors. They have also been used as

the basic equations for the study of the orbits described in this report.

It should also be noted that Lagrangian Eq. (32) can be obtained directly from Lagrangian Eq. (21) by the simple application of a rotation similar to Eq. (24). In the different numerical integrations that have been done, the differential equations, Eq. (33), have mostly been treated with the recurrent power series method. For this purpose, the second differential equation, Eq. (6), has been joined to the system by Eq. (33). A system of the sixth order is then obtained that does not explicitly contain the independent variable. When this system is solved, the time t can be obtained by a quadrature. However, it is simpler to obtain the time t by numerical integration of the differential equation

$$\dot{t} = \frac{r^2}{(p)^{1/2}} \quad (34)$$

simultaneously with the equations for x , y , and r . The computer programs to integrate numerically Eqs. (33) and (34) are described in Section V.

D. Equations of Motion for the Rectilinear Elliptic Restricted Three-Body Problem

The equations of motion for the two primaries given in Eq. (2) are still valid for $e = 1$ when the eccentric anomaly—and the equations of motion of the satellite given in Eq. (13), which are relative to the inertial axes—are also valid when the eccentricity e of the primaries is one. The motion of the primaries being on a straight line (on the x -axis), there is not much need for rotating axes in this problem. On the other hand, the “reduced” or “pulsating” coordinates, as used in Eq. (33), cannot be applied when $r = 0$; the rotation of the axes with the true anomaly v as the rotation angle becomes senseless when $r = 0$. For these reasons, it has been necessary to make a separate study of the elliptic problem with $e = 1$.

A different set of computer programs was needed to study the rectilinear problem. It was still possible to introduce a system of reduced coordinates that “fixes” the position of the primaries at the coordinates $+1/2$ and $-1/2$ on the x -axis. This has been necessary mainly for the regularization of this problem with Birkhoff coordinates. However, this transformation is not valid when $r = 0$, and can thus only be used for those arcs of orbits for which r is not too small. Therefore, it is not possible to use this transformation to regularize the simultaneous collision of the two primaries and the satellite (which would be a case of triple collision).

The motion of the satellite when $e = 1$ is still derived from the Lagrangian equation

$$L = \frac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2) + \frac{m_1}{s_1} + \frac{m_2}{s_2} \quad (35)$$

where

$$s_1^2 = (\xi - \xi_1)^2 + \eta^2 \quad (36a)$$

$$s_2^2 = (\xi - \xi_2)^2 + \eta^2 \quad (36b)$$

$$\xi_1 = -\mu (\cos E - 1) = +m_2 r > 0 \quad (36c)$$

$$\xi_2 = (1 - \mu) (\cos E - 1) = -m_1 r < 0 \quad (36d)$$

As before, the masses of the two primaries are $m_1 = 1 - \mu$ and $m_2 = \mu$.

The time t is related to the eccentric anomaly E by Kepler's equation

$$t + \chi = E - \sin E \quad (37)$$

and the distance r between the two primaries is

$$r = 1 - \cos E \quad (38)$$

Lagrangian Eq. (35) will now be transformed by a change of coordinates and a change of independent variable. The time t will be replaced by a new independent variable s , which is defined by the differential relation

$$dt = r^2 ds \quad (39)$$

The derivatives with respect to s will here be designated by dots. With the variable s , the differential equation for r , corresponding to Eq. (6), becomes

$$\ddot{r} = \frac{2}{r} \dot{r}^2 = r^2 \quad (40)$$

whereas the energy integral of Eq. (5) becomes

$$\frac{2}{r} \dot{r}^2 - 4r^2 = -2r^3 \quad (41)$$

Using this energy integral, Eq. (40) may be written as

$$\ddot{r} = -2r^3 + 3r^2 \quad (42)$$

A change of coordinates from (ξ, η) to (x, y) will now also be introduced by

$$\begin{aligned} x &= \frac{\xi}{r} + x_0 \\ y &= \frac{\eta}{r} \end{aligned} \quad (43)$$

where x_0 is defined by

$$x_0 = \frac{1}{2} (m_1 - m_2) = \frac{1}{2} (1 - 2\mu) \quad (44)$$

Solving Eq. (43) for ξ and η then gives

$$\begin{aligned} \xi &= r(x - x_0) \\ \eta &= ry \end{aligned} \quad (45)$$

Using Eq. (36) and the definitions of Eq. (43), it can now be seen that the coordinates of the primaries relative to the new coordinate system are

$$x_1 = +\frac{1}{2}, \quad x_2 = -\frac{1}{2} \quad (46)$$

From Eqs. (36a) and (36b), it is also seen now that

$$\begin{aligned} s_1^2 &= r^2 \left[\left(x - \frac{1}{2} \right)^2 + y^2 \right] = r^2 r_1^2 \\ s_2^2 &= r^2 \left[\left(x + \frac{1}{2} \right)^2 + y^2 \right] = r^2 r_2^2 \end{aligned} \quad (47)$$

Applying the coordinate transformation of Eq. (45) and the time transformation of Eq. (39) to Lagrangian Eq. (35) gives the new Lagrangian equation

$$\begin{aligned} L &= \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + [(x - x_0) \dot{x} + y \dot{y}] \sin E \\ &+ [(x - x_0)^2 + y^2] \frac{\sin^2 E}{2} + r \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right) \end{aligned} \quad (48)$$

Equation (48) can be greatly simplified by subtracting from it the exact differential

$$\frac{d}{ds} \left\{ \frac{\sin E}{2} [(x - x_0)^2 + y^2] \right\} = \frac{d}{ds} \left\{ \frac{\dot{r}}{2r} [(x - x_0)^2 + y^2] \right\} \quad (49)$$

The Lagrangian equation then becomes

$$\begin{aligned} L &= \frac{1}{2} (\dot{x}^2 + \dot{y}^2) \\ &+ r \left\{ \frac{1}{2} [(x - x_0)^2 + y^2] + \frac{m_1}{r_1} + \frac{m_2}{r_2} \right\} \end{aligned} \quad (50)$$

The equations of motion that correspond to this Lagrangian equation are

$$\begin{aligned} \ddot{x} &= r \left[(x - x_0) - m_1 \frac{x - x_1}{r_1^3} - m_2 \frac{x - x_2}{r_2^3} \right] \\ \ddot{y} &= r \left[y - m_1 \frac{y}{r_1^3} - m_2 \frac{y}{r_2^3} \right] \end{aligned} \quad (51)$$

Again, the first of the three terms on the right side gives the artificial centrifugal force coming from the radial change of scale of the coordinates, and the two last terms are the ordinary Newtonian attraction terms.

Because of the straight-line motion of the two primaries, the problem treated here presents some similarity with the well-known two-fixed-center problem, of which the present problem may be considered a generalization (the two "fixed centers" now move on a straight line in accordance with Keplerian laws of motion). Because of this fact, this problem seems more interesting than the two-fixed-center problem. The problem studied here is gravitationally consistent, whereas the definition of the two-fixed-center problem is somewhat arbitrary (the primaries attract the satellite but do not attract each other!). Among the similarities with the two-fixed-center problem, one can also mention the absence of Coriolis force and the characteristic loops in the orbits that are caused by this force.

E. An Alternate Form of Equations of Motion for the Elliptic Restricted Three-Body Problem

As aforementioned, the equations of motion, Eq. (33), with rotating-pulsating coordinates and with the true anomaly as independent variable are not valid when $p = 0$ and $e = 1$, nor when $r = 0$. Attempts have been made to arrive at a corresponding system of equations of motion that would be valid for all values of e (including $e = 1$) and for all values of r (including $r = 0$). Such a set of equations has not been found, but a set is derived below wherein the singularities $r = 0$ and $e = 1$ are removed. However, this set is unsatisfactory because the situation $r = 0$ is now an equilibrium solution of the new equations of motion. Thus, these equations of motion are adequate for all values of e ; when $e = 1$, however, those parts of

the trajectory that are close to $r = 0$ must be avoided, and have to be integrated in a different way, e.g., in ordinary inertial coordinates. These equations of motion will now be derived.

The rotating-pulsating coordinates are used as before, but the independent variable will no longer be the true anomaly. The new independent variable s will be related to t as in Eq. (39), but not in Eq. (34):

$$dt = r^2 ds \quad (52)$$

The fact that the denominator $(p)^{1/2}$ is not present here as in Eq. (34) will make the new time transformation of Eq. (52) valid even when $e = 1$ and $p = 0$.

Using the reduced inertial coordinates $(\bar{\xi}, \bar{\eta})$ as before (but with no rotation of axes), the following Lagrangian equation corresponding to Eq. (21) is obtained:

$$L = \frac{1}{2} (\dot{\bar{\xi}}^2 + \dot{\bar{\eta}}^2) + \frac{(r-p)}{2} (\bar{\xi}^2 + \bar{\eta}^2) + r \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right) \quad (53)$$

The equations of motion derived from this Lagrangian equation are

$$\begin{aligned} \ddot{\bar{\xi}} &= (r-p) \bar{\xi} - r \left(m_1 \frac{\bar{\xi} - \bar{\xi}_1}{r_1^3} + m_2 \frac{\bar{\xi} - \bar{\xi}_2}{r_2^3} \right) \\ \ddot{\bar{\eta}} &= (r-p) \bar{\eta} - r \left(m_1 \frac{\bar{\eta} - \bar{\eta}_1}{r_1^3} + m_2 \frac{\bar{\eta} - \bar{\eta}_2}{r_2^3} \right) \end{aligned} \quad (54)$$

The rotating coordinates (x, y) are now introduced by

$$\begin{aligned} \bar{\xi} &= x \cos v - y \sin v \\ \bar{\eta} &= x \sin v + y \cos v \end{aligned} \quad (55)$$

where v is the true anomaly, related to the new independent variable s by the relation

$$\frac{dv}{ds} = \dot{v} = (p)^{1/2} \quad (56)$$

Lagrangian Eq. (53) now transforms into

$$L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + (p)^{1/2} (x\dot{y} - y\dot{x}) + r \left[\frac{1}{2} (x^2 + y^2) + \frac{m_1}{r_1} + \frac{m_2}{r_2} \right] \quad (57)$$

which gives the associated equations of motion

$$\begin{aligned} \ddot{x} - 2(p)^{1/2} \dot{y} &= r \left(x - m_1 \frac{x - x_1}{r_1^3} - m_2 \frac{y - y_1}{r_2^3} \right) \\ \ddot{y} + 2(p)^{1/2} \dot{x} &= r \left(y - m_1 \frac{y}{r_1^3} - m_2 \frac{y}{r_2^3} \right) \end{aligned} \quad (58)$$

The value of r in these equations of motion can be obtained by simultaneous integration of a differential equation similar to Eq. (6) or (40). If the independent variable s defined in Eq. (52) is used, the second-order differential for r may be written as

$$\ddot{r} = \frac{2}{r^2} + r(p-r) \quad (59)$$

If the energy integral

$$\frac{2}{r} \dot{r}^2 + 2pr - 4r^2 = -2r^3 \quad (60)$$

is used to eliminate $\dot{r}$, the differential equation for $\ddot{r}$ becomes

$$\ddot{r} = -2r^3 + 3r^2 - pr \quad (61)$$

As stated above, however, the differential equation for r and the energy integral show that, when $r = 0$, then also $\dot{r} = \ddot{r} = 0$; thus, this becomes an equilibrium solution.

F. Some Formulas for Coordinate Changes

In the numerical study of periodic orbits, it will frequently be necessary to make changes of coordinates during the integration or for the input of initial conditions. This is also necessary for the output of the orbits, either in printed or in graphical form. The computer programs that have been prepared are all greatly flexible as to what concerns the different types of input and output. Several subroutines have thus had to be prepared to make the changes of coordinates. For this reason, some of the formulas are recapitulated below.

1. Inertial coordinates $(\bar{\xi}, \bar{\eta})$ to rotating coordinates $(\bar{x}, \bar{y})$.

$$\bar{x} = +\bar{\xi} \cos v + \bar{\eta} \sin v \quad (62a)$$

$$\bar{y} = -\bar{\xi} \sin v + \bar{\eta} \cos v \quad (62b)$$

$$\bar{x}' = + \frac{\bar{y}(p)^{1/2}}{r^2} + \xi' \cos v + \eta' \sin v \quad (62c)$$

$$\bar{y}' = - \frac{\bar{x}(p)^{1/2}}{r^2} - \xi' \sin v + \eta' \cos v \quad (62d)$$

2. Rotating coordinates $(\bar{x}, \bar{y})$ to inertial coordinates (ξ, η) .

$$\xi = \bar{x} \cos v - \bar{y} \sin v \quad (63a)$$

$$\eta = \bar{x} \sin v + \bar{y} \cos v \quad (63b)$$

$$\xi' = \left(\bar{x}' - \frac{\bar{y}(p)^{1/2}}{r^2} \right) \cos v - \left(\bar{y}' + \frac{\bar{x}(p)^{1/2}}{r^2} \right) \sin v \quad (63c)$$

$$\eta' = \left(\bar{x}' - \frac{\bar{y}(p)^{1/2}}{r^2} \right) \sin v + \left(\bar{y}' + \frac{\bar{x}(p)^{1/2}}{r^2} \right) \cos v \quad (63d)$$

3. Rotating coordinates $(\bar{x}, \bar{y})$ to pulsating coordinates (x, y) .

$$x = \frac{\bar{x}}{r} \quad (64a)$$

$$y = \frac{\bar{y}}{r} \quad (64b)$$

$$\dot{x} = \frac{(r\bar{x}' - r'\bar{x})}{(p)^{1/2}} \quad (64c)$$

$$\dot{y} = \frac{(r\bar{y}' - r'\bar{y})}{(p)^{1/2}} \quad (64d)$$

The dots here indicate derivatives with respect to the true anomaly v .

4. Pulsating coordinates (x, y) to rotating coordinates $(\bar{x}, \bar{y})$.

$$\bar{x} = rx \quad (65a)$$

$$\bar{y} = ry \quad (65b)$$

$$\bar{x}' = \frac{(p)^{1/2}}{r^2} (r\dot{x} + \dot{r}x) \quad (65c)$$

$$\bar{y}' = \frac{(p)^{1/2}}{r^2} (r\dot{y} + \dot{r}y) \quad (65d)$$

5. Inertial barycentric coordinates (ξ, η) to inertial geocentric coordinates (ξ_G, η_G) .

$$\xi_G = \xi + \mu r \cos v \quad (66)$$

$$\eta_G = \eta + \mu r \sin v$$

6. Inertial barycentric coordinates (ξ, η) to inertial selenocentric coordinates (ξ_s, η_s) .

$$\xi_s = \xi - (1 - \mu)r \cos v \quad (67)$$

$$\eta_s = \eta - (1 - \mu)r \sin v$$

The geocentric coordinates are relative to the larger mass $m_1 = 1 - \mu$, whereas the selenocentric coordinates are relative to the smaller primary $m_2 = \mu$. The last two transformations have been used only for graphical display of the orbits; therefore, the corresponding velocity transformations were not necessary, and are not reproduced here.

G. The Five Equilibrium Points of the Elliptic Problem

It is well known that the five Lagrange points (or equilibrium points) of the circular restricted three-body problem still exist in the elliptic three-body problem. At these five points, the satellite always remains in the same position relative to the two primaries. It can be shown in different ways that five such points exist. This will be proven herein in two ways: (1) with the use of rotating-pulsating coordinates (x, y) and (2) with inertial coordinates (ξ, η) .

In the system of rotating-pulsating coordinates, the five equilibrium points are fixed. It can be seen by consideration of the equations of motion, Eq. (33), that there are five particular solutions with constant coordinates and with $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0$. These constant coordinates must thus be a solution to the equations obtained by setting the right side of Eq. (33) equal to zero.

$$x - m_1 \frac{x - x_1}{r_1^3} - m_2 \frac{x - x_2}{r_2^3} = x \left(1 - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} \right) + \mu(1 - \mu) \left(\frac{1}{r_2^3} - \frac{1}{r_1^3} \right) = 0 \quad (68a)$$

$$y - m_1 \frac{y}{r_1^3} - m_2 \frac{y}{r_2^3} = y \left(1 - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} \right) = 0 \quad (68b)$$

It has been presumed here that r is always different from zero; the case $e = 1$ of the rectilinear elliptic problem is thus excluded in this study of the equilibrium points. Equation (68) is identical to those equations obtained in the treatment of the classical circular three-body problem. The conclusions thus usually derived from Eq. (68) are still valid for the elliptic problem. The two equilibrium solutions that are the easiest to find correspond to $r_1 = r_2 = 1$, as can be seen from Eq. (68); they correspond

to equilateral triangle configurations with the two primaries m_1 and m_2 . These solutions are generally called L_4 and L_5 , and their coordinates are

$$\begin{aligned} L_4 : x_4 &= \frac{1}{2}(1 - 2\mu), & y_4 &= \frac{+(3)^{1/2}}{2} \\ L_5 : x_5 &= \frac{1}{2}(1 - 2\mu), & y_5 &= \frac{-(3)^{1/2}}{2} \end{aligned} \quad (69)$$

The other three solutions of Eq. (68) are L_1 , L_2 , and L_3 . These are called the collinear equilibrium points because they correspond to $y = 0$, and are thus on the *syzygy*-axis (the line of the two primaries).

The abscissa x must thus be a root of the equation

$$f(x) \equiv -x + (1 - \mu) \frac{x - x_1}{r_1^3} + \mu \frac{x - x_2}{r_2^3} = 0 \quad (70)$$

where now

$$r_1^3 = |x - x_1|^3, \quad r_2^3 = |x - x_2|^3 \quad (71)$$

Equation (70) has one root in each of the three intervals on both sides of m_1 and m_2 and between m_1 and m_2 . This is a consequence of the fact that, in each interval, $f(x)$ is monotonously decreasing from $+\infty$ to $-\infty$, the derivative of $f(x)$ being negative in each of the three intervals

$$\frac{df(x)}{dx} = -1 - 2 \frac{(1-\mu)}{|x - x_1|^3} - 2 \frac{\mu}{|x - x_2|^3} < 0 \quad (72)$$

The abscissae of L_1 , L_2 , and L_3 can be obtained by numerically solving Eq. (70), which can be reduced to a fifth-degree equation in x . These coordinates have been tabulated by a large number of authors.

It is of great importance to study the neighborhood of the equilibrium points, mainly to determine the stability of these points. For this purpose, it is interesting to linearize the equations of motion, Eq. (33), in the neighborhood of the equilibrium points to obtain the so-called first-order variational equations. To accomplish this, Eq. (33) may be written in the form

$$\begin{aligned} \ddot{x} - 2\dot{y} - \frac{r}{p} x &= \frac{r}{p} U_x \\ \ddot{y} + 2\dot{x} - \frac{r}{p} y &= \frac{r}{p} U_y \end{aligned} \quad (73)$$

where the potential function U is defined by

$$U = \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \quad (74)$$

The subscripts x and y are used to represent partial derivatives of U .

The first-order variational equations corresponding to Eq. (73) may then be written as

$$\begin{aligned} \delta\ddot{x} - 2\delta\dot{y} - \frac{r}{p} \delta x &= \frac{r}{p} (U_{xx} \delta x + U_{xy} \delta y) \\ \delta\ddot{y} + 2\delta\dot{x} - \frac{r}{p} \delta y &= \frac{r}{p} (U_{yx} \delta x + U_{yy} \delta y) \end{aligned} \quad (75)$$

Equation (75) forms a system of linear differential equations with nonconstant periodic coefficients because of the factor r , which depends upon the cosine of the true anomaly. It is well known that, in the circular three-body problem, a system with constant coefficients would be obtained (r being constant). In this fact resides a principal difference between the elliptic and the circular restricted problems. In the elliptic problem, a more difficult technique is necessary to study the variational equations than in the circular problem. The Floquet theory can be used, for instance, as has been shown by several authors. The characteristic equation of the system formed by Eq. (75) would be

$$\begin{aligned} \lambda^4 + \lambda^2 \left[4 - \frac{r}{p} (2 + U_{xx} + U_{yy}) \right] \\ + \frac{r^2}{p^2} [(1 + U_{xx})(1 + U_{yy}) - U_{xy}^2] = 0 \end{aligned} \quad (76)$$

but again the coefficients of this equation are periodic functions of the true anomaly v .

It is also possible to show the existence of the five equilibrium points without using rotating coordinates. This will be done herein with the use of inertial barycentric coordinates. This demonstration will be done separately, again, for the two equilateral points L_4 and L_5 and for the collinear points L_1 , L_2 , and L_3 .

For the triangular points, it can be concluded that the inertial equations of motion, Eq. (13), have the corresponding two particular solutions by simple geometrical considerations (Fig. 1). The two primaries are supposed to be at points M_1 and M_2 , with the center of mass at the

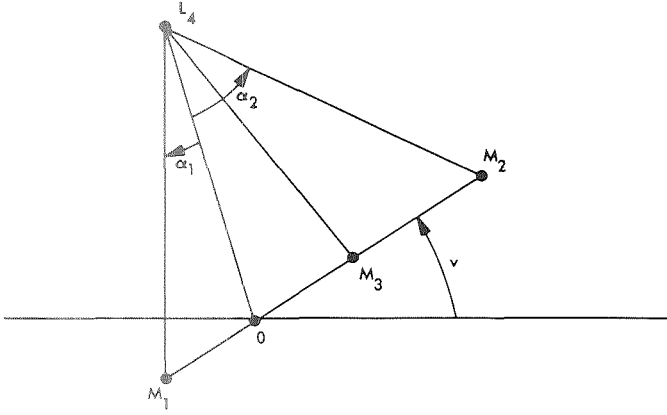


Fig. 1. Configuration of masses M_1 , M_2 and libration point L_4

origin O of the coordinate system. Each side of the triangle $L_4M_1M_2$ has the length r , as given by Eq. (1). The coordinates of M_1 and M_2 are given in Eq. (2). The coordinates of the midpoint M_3 of the segment M_1M_2 are then

$$\xi_3 = \xi_0 r \cos v, \quad \eta_3 = \xi_0 r \sin v \quad (77)$$

where ξ_0 is the same as the quantity x_0 already defined in Eq. (44): $\xi_0 = (1/2) - \mu = (1 - 2\mu)/2$. The equilibrium point L_4 then has the coordinates

$$\begin{aligned} \xi_4 &= \xi_3 + \frac{(3)^{1/2}}{2} r \cos (90 \text{ deg} + v) \\ &= r \left[\xi_0 \cos v - \frac{(3)^{1/2}}{2} \sin v \right] \end{aligned} \quad (78a)$$

$$\begin{aligned} \eta_4 &= \eta_3 + \frac{(3)^{1/2}}{2} r \sin (90 \text{ deg} + v) \\ &= r \left[\xi_0 \sin v + \frac{(3)^{1/2}}{2} \cos v \right] \end{aligned} \quad (78b)$$

the length of the segment M_3L_4 being $r(3/2)^{1/2}$. It is now easy to verify that the point L_4 with the coordinates of Eq. (78) describes a Keplerian ellipse that is a solution of the equations of motion, Eq. (13), of the particle. The second derivatives of Eq. (78) are

$$\xi_4'' = -\frac{\xi_4}{r^3}, \quad \eta_4'' = -\frac{\eta_4}{r^3} \quad (79)$$

Substitution of the coordinates of Eq. (78) and the accelerations of Eq. (79) verifies Eq. (13), which thus proves that Eq. (78) is a particular solution.

It is remarkable that the geometry of the triangle $L_4M_1M_2$ depends upon the eccentricity e through the

distance r only. In other words, the shape of the triangle depends upon e only, whereas angular quantities in this triangle depend upon the mass-ratio μ only. For instance,

$$\cos \alpha_1 = \frac{(2 - \mu)}{2(1 - \mu + \mu^2)^{1/2}} \quad (80a)$$

$$\cos \alpha_2 = \frac{(1 - \mu)}{2(1 - \mu + \mu^2)^{1/2}} \quad (80b)$$

$$\sin \alpha_1 = \frac{(3)^{1/2}}{2} \frac{\mu}{(1 - \mu + \mu^2)^{1/2}} \quad (80c)$$

$$\sin \alpha_2 = \frac{(3)^{1/2}}{2} \frac{(1 - \mu)}{(1 - \mu + \mu^2)^{1/2}} \quad (80d)$$

The length of the segment OL_4 is $r(1 - \mu + \mu^2)^{1/2}$. The three forces that act on the point L_4 are the attractive forces F_1 and F_2 from m_1 and m_2 , together with the centrifugal force F_3 :

$$\begin{aligned} F_1 &= -\frac{1 - \mu}{r^2}, \quad F_2 = -\frac{\mu}{r^2} \\ F_3 &= \frac{p(1 - \mu + \mu^2)^{1/2}}{r^3} \end{aligned} \quad (81)$$

The resultant of the three forces is along the direction OL_4 with magnitude

$$\begin{aligned} F &= F_1 \cos \alpha + F_2 \cos \alpha_2 + F_3 \\ &= \frac{(1 - \mu + \mu^2)^{1/2}}{r^2} \left(\frac{p}{r} - 1 \right) = \frac{e(1 - \mu + \mu^2)^{1/2}}{r^2} \cos v \end{aligned} \quad (82)$$

The existence of the collinear equilibrium solutions can be shown in a way similar to the above demonstration for L_4 and L_5 . The coordinates of any point that is collinear with m_1 and m_2 may be expressed as

$$\begin{aligned} \xi &= (1 - \alpha)\xi_1 + \alpha\xi_2 = (\alpha - \mu)r \cos v \\ \eta &= (1 - \alpha)\eta_1 + \alpha\eta_2 = (\alpha - \mu)r \sin v \end{aligned} \quad (83)$$

The constant parameter α is between 0 and 1 for points that are between m_1 and m_2 , and outside of these two limits for points that are outside m_1 and m_2 . Substitution in the inertial equations of motion, Eq. (13), shows that Eq. (83) is a particular solution for the values of α that satisfy the condition

$$\frac{(1 - \alpha)}{\mu^2} - \frac{\alpha}{(1 - \mu)^2} = \frac{-\mu\alpha}{|\alpha|^3} - \frac{(1 - \mu)(\alpha - 1)}{|\alpha - 1|^3} \quad (84)$$

For each of the three intervals, inside and outside the limits 0 and +1, Eq. (84) becomes a fifth-degree equation in α , and each time has one single real root. In this way, the three collinear points L_1 , L_2 , and L_3 are thus obtained. In fact, the fifth-degree equation obtained from Eq. (84) is fundamentally equivalent to Lagrange's well-known fifth-degree equation in the masses for the collinear solutions of the general three-body problem.

H. The Energy Equation of the Elliptic Restricted Three-Body Problem

The classical energy or Jacobi integral of the circular restricted three-body problem has no analog in the elliptic problem. This is a well-known result and, because of this fact, the elliptic problem is fundamentally different from the circular problem. Even when rotating or pulsating axes are used, the independent variable (e.g., time or true anomaly) is still present in the Lagrangian equations and in the equations of motion; in the circular problem, the independent variable disappears by a simple rotation of axes. For this reason, the elliptic problem must be considered *nonconservative* as opposed to the circular problem, which is *conservative*.

Although there is no known energy integral in the elliptic restricted three-body problem, it is still possible and even useful to define a quantity that can be called energy. This energy, however, will not be constant along any given trajectory. In this case, it will be possible to establish the differential equation for the variation of the energy.

There are different ways of defining the energy and its differential equation, but the classical definition—with the Lagrangian or Hamiltonian equations—will probably be the most useful. If canonical variables and canonical equations of motion are used, the Hamiltonian H may be considered as the energy of the satellite; if Lagrangian equations of motion are used, the energy E is the difference between the second- and zero-degree terms in the velocity components of the Lagrangian equation, $E = L_2 - L_0$. In both cases, the derivative of the energy has a very simple expression:

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial t} \\ \frac{dE}{dt} &= -\frac{\partial L}{\partial t}\end{aligned}\quad (85)$$

Similar expressions for the derivative may be written when some other independent variable is used. Proceeding in this way, the different coordinate systems that have been introduced above could all be used. The most elegant way of introducing the energy is probably to use the rotating coordinates and the true anomaly, as in Eqs. (32) and (33):

$$\begin{aligned}E &= \frac{1}{2} (\dot{x}^2 + \dot{y}^2) \frac{r}{p} \left[\frac{1}{2} (x^2 + y^2) + \frac{m_1}{r_1} + \frac{m_2}{r_2} \right] \\ &= \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \frac{r}{p} U\end{aligned}\quad (86)$$

where U represents the negative of the quantity in brackets in the first expression. The derivative of E with respect to the time t or the true anomaly v can be obtained by using the expressions for the derivatives given in Eqs. (4) and (6) for the variable r :

$$\begin{aligned}\frac{dE}{dt} &= -\frac{r'}{p} U = -\frac{e \sin v}{(p^3)^{1/2}} U \\ \frac{dE}{dv} &= -\frac{\dot{r}}{p} U = -\frac{e r^2 \sin v}{p^2} U\end{aligned}\quad (87)$$

It will be seen in the following sections that the "energy-differential equation," Eq. (87), plays an important role in the regularization of the equations of motion. These differential equations have also been used for controlling the precision of the numerical integration of the equations of motion. One of the equations, Eq. (87), is integrated along with the equations of motion, and the result of the integration is continuously compared with the value of E computed directly with Eq. (86).

I. Regularization With Birkhoff Coordinates

For the computation of close approaches and collision trajectories, it has been necessary to use a system of regularized equations of motion. Only the Birkhoff coordinates have been used herein for the elliptic restricted three-body problem. The Birkhoff coordinates are known to regularize simultaneously both singularities of the problem; they also lead to relatively simple equations in both the Lagrangian and Hamiltonian forms.

The Birkhoff coordinates will be introduced in relation with the rotating-pulsating coordinates (x, y) defined in Eq. (28). They will be called ξ, η . No confusion with the inertial coordinates (ξ, η) used in Section II-B will result because these inertial coordinates will not be used from

this point on. In fact, a coordinate system (X,Y) will also be used that is intermediate between the rotating coordinates (x,y) and the Birkhoff coordinates (ξ,η) . The coordinates (X,Y) are obtained from the coordinates (x,y) by a simple translation along the x -axis, which is made in such a way that the new origin becomes the midpoint of the two primaries m_1 and m_2 :

$$\begin{aligned} x &= X + x_0 \\ y &= Y \end{aligned} \quad (88)$$

where x_0 is the quantity

$$x_0 = \frac{1}{2} - \mu = \frac{1}{2} - m_2 = \frac{1}{2} (1 - 2\mu) \quad (89)$$

which has already been used in Eq. (44). The coordinates (X,Y) will be called *median* coordinates. The coordinates of the primaries m_1 and m_2 are now

$$\begin{aligned} X_1 &= -\frac{1}{2}, & Y_1 &= 0 \\ X_2 &= +\frac{1}{2}, & Y_2 &= 0 \end{aligned} \quad (90)$$

and the distances from the satellite to the primaries are obtained by

$$\begin{aligned} r_1^2 &= \left(X + \frac{1}{2}\right)^2 + Y^2 \\ r_2^2 &= \left(X - \frac{1}{2}\right)^2 + Y^2 \end{aligned} \quad (91)$$

The Birkhoff coordinates $\zeta = \xi + i\eta$ are related to the median coordinates $Z = X + iY$ by

$$Z = \frac{1}{4} \left(\zeta + \frac{1}{\zeta} \right) \quad (92)$$

The function Z of ζ is thus analytic everywhere except at $\zeta = 0$ and $\zeta = \infty$.

Separating in Eq. (92) the real and imaginary parts gives for X and Y the expressions

$$\begin{aligned} X &= \frac{1}{4S} (S+1) \xi = \frac{\xi}{4} \left(1 + \frac{1}{S} \right) \\ Y &= \frac{1}{4S} (S-1) \eta = \frac{\eta}{4} \left(1 - \frac{1}{S} \right) \end{aligned} \quad (93)$$

where the following abbreviation S is used:

$$S = \xi^2 + \eta^2 \quad (94)$$

Solving Eq. (92) for ζ gives

$$\zeta = 2 \left[Z \pm \left(Z^2 - \frac{1}{4} \right)^{1/2} \right] \quad (95)$$

and this shows that Eq. (92) is a one-to-two mapping between the Z -plane and the ζ -plane except at the two branch points $Z = \pm 1/2$. These two branch points thus correspond with the two singularities of the problem, i.e., with the location of the two primaries m_1 and m_2 . The derivative Z' of Eq. (92) is

$$\frac{dZ}{d\zeta} = Z' = \frac{\zeta^2 - 1}{4\zeta} \quad (96)$$

and the value of Z' is thus zero at the two critical points $\zeta = +1$ and $\zeta = -1$, which are also the branch points $Z = +1/2$ and $Z = -1/2$ described above. Except at these two points, Eq. (92) defines a conformed mapping between the Z -plane and the ζ -plane. The absolute value of Z' in Eq. (96) is given by

$$\left| \frac{dZ}{d\zeta} \right|^2 = J = \frac{\partial(X,Y)}{\partial(\xi,\eta)} = \frac{1}{16S^2} \left[(S+1)^2 - 4\xi^2 \right] \quad (97)$$

The symbol J is used here to represent the Jacobian or the functional determinant of X,Y with respect to ξ,η .

In what follows, it will be necessary to express r_1, r_2 and $r_1 r_2$ with the Birkhoff coordinates. This can be done with the use of Eqs. (91) and (93):

$$\begin{aligned} r_1 &= \frac{1}{4(S)^{1/2}} [(S+1) + 2\xi] \\ r_2 &= \frac{1}{4(S)^{1/2}} [(S+1) - 2\xi] \\ r_1 r_2 &= \frac{1}{16S} [(S+1)^2 - 4\xi^2] \end{aligned} \quad (98)$$

Comparing the last relation of Eq. (98) with the expression of Eq. (97) for the Jacobian determinant gives the new formula

$$\left| \frac{dZ}{d\zeta} \right|^2 = J = \frac{r_1 r_2}{S} \quad (99)$$

This last expression will be used to define a new independent variable s , which will be used instead of the true anomaly v . In what follows, dots are used to designate derivatives with respect to the true anomaly v (the “old” independent variable) and the symbol $\circ$ for derivatives with respect to s (the “new” independent variable). Thus, s will be related to v by

$$\frac{dv}{ds} = \circ v = J = \frac{r_1 r_2}{S} \quad (100)$$

From the transformation in Eq. (93) of the coordinates, the corresponding transformation of the velocity components is obtained by

$$\begin{aligned} \dot{X} &= \frac{1}{J} \left(\frac{\partial X}{\partial \xi} \dot{\xi} + \frac{\partial X}{\partial \eta} \circ \eta \right) \\ \dot{Y} &= \frac{1}{J} \left(\frac{\partial Y}{\partial \xi} \dot{\xi} + \frac{\partial Y}{\partial \eta} \circ \eta \right) \end{aligned} \quad (101)$$

where the partial derivatives have the following values:

$$\begin{aligned} \frac{\partial X}{\partial \xi} &= \frac{\partial Y}{\partial \eta} = \frac{1}{4} \left(1 - \frac{D}{S^2} \right) \\ \frac{\partial X}{\partial \eta} &= -\frac{\partial Y}{\partial \xi} = -\frac{\xi \eta}{2S^2} \end{aligned} \quad (102)$$

and where D is the difference

$$D = \xi^2 - \eta^2 \quad (103)$$

To Eq. (101) corresponds the inverse transformation of the derivatives (velocity components):

$$\begin{aligned} \dot{\xi} &= + \frac{\partial Y}{\partial \eta} \dot{X} - \frac{\partial X}{\partial \eta} \dot{Y} \\ \circ \eta &= - \frac{\partial Y}{\partial \xi} \dot{X} + \frac{\partial X}{\partial \xi} \dot{Y} \end{aligned} \quad (104)$$

The coordinates (ξ, η) themselves are easily derived from X and Y by taking the real and imaginary parts of Eq. (95).

J. Equations of Motion With Birkhoff Coordinates

Lagrangian Eq. (32) in rotating-pulsating coordinates (x, y) will now be transformed to the Birkhoff coordinates (ξ, η) and to the new independent variable s , and the corresponding equations of motion will be derived. If

the median coordinates (X, Y) are used instead of (x, y) , Lagrangian Eq. (32) becomes

$$\begin{aligned} L &= \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) + (X\dot{Y} - Y\dot{X}) \\ &\quad + \frac{r}{p} \left[\frac{1}{2} (X^2 + Y^2) + x_0 X + \frac{m_1}{r_1} + \frac{m_2}{r_2} \right] \\ &= \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) + (X\dot{Y} - Y\dot{X}) + \frac{r}{p} U \end{aligned} \quad (105)$$

The different terms of this Lagrangian equation transform in the following way:

$$\dot{X}^2 + \dot{Y}^2 = \frac{1}{J} (\dot{\xi}^2 + \dot{\eta}^2) \quad (106a)$$

$$X\dot{Y} - Y\dot{X} = \frac{1}{2J} \left[\eta \frac{\partial}{\partial \xi} (X^2 + Y^2) - \xi \frac{\partial}{\partial \eta} (X^2 + Y^2) \right] \quad (106b)$$

$$X^2 + Y^2 = \frac{1}{16} \left[\left(S + \frac{1}{S} \right) + \frac{2D}{S} \right] \quad (106c)$$

The second of the three preceding expressions may be written as

$$X\dot{Y} - Y\dot{X} = \frac{1}{J} (A_\xi \dot{\xi} + A_\eta \circ \eta) \quad (107)$$

where A_ξ and A_η are defined by

$$\begin{aligned} A_\xi &= -\frac{1}{2} \frac{\partial}{\partial \eta} (X^2 + Y^2) = -X \frac{\partial X}{\partial \eta} - Y \frac{\partial Y}{\partial \eta} \\ &= -\frac{\eta}{16S^2} (S^2 - 4\xi^2 - 1) \end{aligned} \quad (108a)$$

$$\begin{aligned} A_\eta &= +\frac{1}{2} \frac{\partial}{\partial \xi} (X^2 + Y^2) = +X \frac{\partial X}{\partial \xi} + Y \frac{\partial Y}{\partial \xi} \\ &= +\frac{\xi}{16S^2} (S^2 + 4\eta^2 - 1) \end{aligned} \quad (108b)$$

To account for the change of independent variable from v to s according to Eq. (100), Lagrangian Eq. (105) must be multiplied by J . The following new Lagrangian equation is thus obtained:

$$L = \frac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2) + (A_\xi \dot{\xi} + A_\eta \circ \eta) + \frac{r}{p} JU \quad (109)$$

The equations of motion derived from this Lagrangian equation take the simple form

$$\ddot{\xi} - 2J\dot{\eta} = \frac{r}{p} \frac{\partial}{\partial \xi} (JU) + E_m \frac{\partial J}{\partial \xi} \quad (110)$$

$$\ddot{\eta} + 2J\dot{\xi} = \frac{r}{p} \frac{\partial}{\partial \eta} (JU) + E_m \frac{\partial J}{\partial \eta}$$

The last term in these equations, proportional to the energy E_m , is due to the change of independent variable (see Ref. 22).

To be able to apply the above equations of motion to a numerical integration on a digital computer, all of the explicit expressions for the partial derivatives must be developed. These are given below. The four partials of A_ξ , A_η are

$$\frac{\partial A_\xi}{\partial \xi} = -\frac{\xi\eta}{4S^3} (2D + 1) \quad (111a)$$

$$\frac{\partial A_\eta}{\partial \eta} = +\frac{\xi\eta}{4S^3} (2D + 1) \quad (111b)$$

$$\frac{\partial A_\xi}{\partial \eta} = \frac{-1}{16S^3} (S^3 - S - 4S\xi^2 + 4\eta^2 + 16\xi^2\eta^2) \quad (111c)$$

$$\frac{\partial A_\eta}{\partial \xi} = \frac{+1}{16S^3} (S^3 - S + 4S\eta^2 + 4\xi^2 - 16\xi^2\eta^2) \quad (111d)$$

It can thus be seen that these partial derivatives satisfy the relations that have been used to simplify the equations of motion, Eq. (110):

$$\begin{aligned} \frac{\partial A_\eta}{\partial \xi} - \frac{\partial A_\xi}{\partial \eta} &= 2J \\ \frac{\partial A_\xi}{\partial \xi} + \frac{\partial A_\eta}{\partial \eta} &= 0 \end{aligned} \quad (112)$$

The partial derivatives of J , r_1 , r_2 , and r_1r_2 are

$$\frac{\partial J}{\partial \xi} = \frac{\xi}{4S^3} (2D - S - 1) \quad (113a)$$

$$\frac{\partial J}{\partial \eta} = \frac{\eta}{4S^3} (2D + S - 1)$$

$$\frac{\partial r_1}{\partial \xi} = \frac{1}{4(S^3)^{1/2}} [(S-1)\xi + 2\eta^2] \quad (113b)$$

$$\frac{\partial r_1}{\partial \eta} = \frac{\eta}{4(S^3)^{1/2}} [(S-1) - 2\xi]$$

$$\frac{\partial r_2}{\partial \xi} = \frac{1}{4(S^3)^{1/2}} [(S-1)\xi - 2\eta^2] \quad (113c)$$

$$\frac{\partial r_2}{\partial \eta} = \frac{\eta}{4(S^3)^{1/2}} [(S-1) + 2\xi]$$

$$\frac{\partial(r_1r_2)}{\partial \xi} = \frac{\xi}{8S^2} (S^2 - 4\eta^2 - 1) \quad (113d)$$

$$\frac{\partial(r_1r_2)}{\partial \eta} = \frac{\eta}{8S^2} (S^2 + 4\xi^2 - 1)$$

The above expressions will now be applied to obtain the partial derivatives of (JU) , which are present in the equations of motion, Eq. (110), the expression for (JU) being

$$JU = \frac{1}{S} \left\{ m_1r_1 + m_2r_2 + r_1r_2 \left[\frac{1}{2} (X^2 + Y^2) + x_0X \right] \right\} \quad (114)$$

The expression E_m for the energy used in the equations of motion, Eq. (110), is the "median" energy in opposition to the barycentric energy E defined in Eq. (86). The difference between the two values is given by

$$E_m = E + \frac{r}{p} x_0^2 \quad (115)$$

This difference comes from the fact that the exact differential rx_0^2/p has been neglected in deriving Lagrangian Eq. (105) from Lagrangian Eq. (32).

For practical computations, it is important to note that the expressions of Eq. (86) or (115) for energy should not be used in the equations of motion, Eq. (110), for the numerical integrations because the benefit of the regularization would then be lost; the denominators r_1 and r_2 are both present in the expression of Eq. (86) for the energy E . Instead of this formula for the energy, the second differential equation, Eq. (87), should be regularized and integrated numerically, together with the equations of motion, Eq. (110):

$$\dot{E} = \frac{dE}{ds} = -\frac{er^2 \sin v}{p^2} \left(\frac{r_1r_2U}{S} \right) \quad (116)$$

To obtain the true anomaly v and the time t as a function of the independent variable s , the following differential equations must be integrated simultaneously with Eqs. (110) and (116):

$$\begin{aligned}\dot{v} &= \frac{dv}{ds} = \frac{r_1 r_2}{S} \\ \dot{t} &= \frac{dt}{ds} = \frac{r_1 r_2 r^2}{(p)^{3/2} S}\end{aligned}\quad (117)$$

Equations (110), (116), and (117) thus form a seventh-order system. It is this system that has been used extensively for the integration of orbits. The computer programs are described in Section V.

It is just as easy to obtain a system of Hamiltonian rather than Lagrangian equations of motion; however, most of the numerical work herein has been done with the Lagrangian equations. With the use of rotating-pulsating coordinates (x, y) , the Hamiltonian equation of the problem is

$$\begin{aligned}H &= \frac{1}{2} (p_x^2 + p_y^2) + (yp_x - xp_y) \\ &\quad - \frac{r}{p} \left[\frac{1}{2} (x^2 + y^2) + \frac{m_1}{r_1} + \frac{m_2}{r_2} \right]\end{aligned}\quad (118)$$

whereas, with the median coordinates,

$$H = \frac{1}{2} (p_x^2 + p_y^2) + (Yp_x - Xp_y) - \frac{r}{p} U \quad (119)$$

With the Birkhoff coordinates (ξ, η) , the Hamiltonian equation is

$$H = \frac{1}{J} \left[\frac{1}{2} (p_\xi^2 + p_\eta^2) - (A_\xi p_\xi + A_\eta p_\eta) \right] - \frac{r}{p} U \quad (120)$$

This Hamiltonian equation must be multiplied by the function J to take account of the change of independent variable from v to s :

$$H = \frac{1}{2} (p_\xi^2 + p_\eta^2) - (A_\xi p_\xi + A_\eta p_\eta) - \frac{r}{p} (JU) \quad (121)$$

The momenta (p_ξ, p_η) are related to the velocity components $(\dot{\xi}, \dot{\eta})$ by

$$p_\xi = \dot{\xi} + A_\xi \quad (122)$$

$$p_\eta = \dot{\eta} + A_\eta$$

The Hamiltonian equations of motion that are derived from Eq. (121) are

$$\dot{\xi} = p_\xi - A_\xi \quad (123a)$$

$$\dot{\eta} = p_\eta - A_\eta \quad (123b)$$

$$\dot{p}_\xi = p_\xi \frac{\partial A_\xi}{\partial \xi} + p_\eta \frac{\partial A_\eta}{\partial \xi} + \frac{r}{p} \frac{\partial}{\partial \xi} (JU) + E_m \frac{\partial J}{\partial \xi} \quad (123c)$$

$$\dot{p}_\eta = p_\xi \frac{\partial A_\xi}{\partial \eta} + p_\eta \frac{\partial A_\eta}{\partial \eta} + \frac{r}{p} \frac{\partial}{\partial \eta} (JU) + E_m \frac{\partial J}{\partial \eta} \quad (123d)$$

The differential equations for E , v , and t should again be joined to Eq. (123) so that a seventh-order system is again obtained. On the other hand, if the value of t is not wanted, the differential equation for this variable may be neglected, t not being present in the other equations. The value of the energy used in Eq. (123) is again the "median" value because, in deriving Hamiltonian Eq. (119) from Hamiltonian Eq. (118), a term rx_0^2/p has been suppressed. The value of r was obtained by the known two-body formula of Eq. (1) instead of by integrating the differential equation for r (the differential equation for r has been integrated by recurrent power series in another nonregularized program).

K. Initial Conditions for Ejection Orbits

If regularized variables and equations of motion are used, it is possible to compute orbits that go through the singularities at m_1 ($r_1 = 0$, $\xi = -1$, $\eta = 0$) or at m_2 ($r_2 = 0$, $\xi = +1$, $\eta = 0$). It is also possible to compute orbits that initially start from one of the primaries, and these orbits may then be called *ejection orbits*. It is well known that, in the study of periodic orbits in the three-body problem, collision orbits and ejection orbits play an important role.

It is shown herein how the initial conditions of the equations of motion, Eq. (110), can be determined for the case of ejection orbits. Ejection orbits have the remarkable property of depending upon only two variable parameters: the energy E and the angle of ejection θ .

The regularized velocity components $(\dot{\xi}, \dot{\eta})$ will then depend upon the angle θ , but not upon the value of E . In other words, the magnitude of the regularized velocity does not depend upon the energy E , as will be shown below. The energy equation may be written as

$$r_1 r_2 E = \frac{S}{2} (\dot{\xi}^2 + \dot{\eta}^2) - \frac{r}{p} \left\{ r_1 r_2 \left[\frac{1}{2} (X^2 + Y^2) + x_0 X + \frac{x_0^2}{2} \right] + [m_1 r_2 + m_2 r_1] \right\} \quad (124)$$

Equation (124) has been written in such a form that it is still valid when $r_1 = 0$ or $r_2 = 0$. The case of $r_1 = 0$ will be considered first. This corresponds to a collision with (or ejection from) the larger primary m_1 (earth), upon which the following special values hold:

$$\begin{aligned} r_1 &= 0, & r_2 &= +1, & \xi &= -1, & \eta &= 0 \\ S &= +1, & D &= +1 \end{aligned} \quad (125)$$

Energy Eq. (124) then gives

$$\frac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2) - \frac{m_1 r}{p} = 0 \quad (126)$$

and, according to this equation, $\dot{\xi}$ and $\dot{\eta}$ may be related to the collision angle θ (in the Birkhoff plane) by

$$\begin{aligned} \dot{\xi} &= \left(\frac{2m_1 r}{p} \right)^{1/2} \cos \theta \\ \dot{\eta} &= \left(\frac{2m_1 r}{p} \right)^{1/2} \sin \theta \end{aligned} \quad (127)$$

The ejection orbits from m_1 may thus be integrated with the initial values for $(\xi, \eta, \dot{\xi}, \dot{\eta})$ given by Eqs. (125) and (127), E and θ being two arbitrary parameters that determine the particular ejection orbits. The collision with the smaller primary m_2 (moon) must be treated in the same way. This corresponds with

$$r_1 = 1, \quad r_2 = 0, \quad \xi = +1, \quad \eta = 0, \quad S = +1, \quad D = +1 \quad (128)$$

Energy Eq. (124) becomes here

$$\frac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2) - \frac{m_2 r}{p} = 0 \quad (129)$$

and the velocity components may be expressed by

$$\begin{aligned} \dot{\xi} &= \left(\frac{2m_2 r}{p} \right)^{1/2} \cos \theta \\ \dot{\eta} &= \left(\frac{2m_2 r}{p} \right)^{1/2} \sin \theta \end{aligned} \quad (130)$$

III. Computation of Periodic Orbits

A. Integration of the Equations of Motion With Recurrent Power Series

The well-known technique for integration with recurrent power series has been used to integrate the equations of motion, Eq. (33), in rotating-pulsating coordinates. This method is usually referred to as the Steffensen method, and it has been extensively used by the present author for the circular problem. The ideas of A. Deprit (see Ref. 7) have also been closely followed in the preparation of the recurrence relations. Both two- and three-dimensional programs have been prepared; however, only the two-dimensional case will be described herein. For the two-dimensional case, the system of simultaneous differential equations is of order seven. It contains the two second-order differential equations for x and y given in Eq. (33), the first-order equation for t given in Eq. (34), and the second-order equation for r given in Eq. (6) (second equation).

To make this system suitable for the recurrent power series approach, a few transformations have to be made. The recurrence relations will be simplified if some auxiliary variables are introduced. Redundant variables are generally introduced to enable the writing of the transformed system of equations in such a way that no more than two variables (two power series) ever have to be multiplied together. The system then obtained is sometimes called *second degree*.

In the present problem, 18 variables (parameters) P_i have been introduced for the three-dimensional case. For the two-dimensional case, three parameters (P_3 , P_6 , and P_{17}) are set equal to zero. For the first 13 parameters, there are 13 "first-order/second-degree" differential equations. For the last five parameters (P_{14} to P_{18}), there are five second-degree equations that relate them directly to the preceding 13 parameters without using more differential equations. The 18 parameters P_i are given in Table 1, together with their defining equations.

Next, power series must be written for each of the 18 variables, with the true anomaly v as the independent variable. These power series should be of the following form:

$$P_i = P_i(1) + P_i(2)v + P_i(3)v^2 + P_i(4)v^3 + \cdots P_i(18)v^{17} \quad (131)$$

where $i = 1, 2, \dots, 18$.

By substitution of these expressions in Table 1, 18 recurrent relations are obtained for the different coefficients $P_i(n)$ of the power series. The zero-order terms $P_i(1)$ are determined by means of the initial conditions. The recurrent relations then allow one to obtain all of the coefficients $P_i(n+1)$ of order n as a function of the coefficients $P_i(1)$ to $P_i(n)$ of lower orders. The different recurrence relations are collected in Table 2.

It is important to note that these relations are really recurrent only if they are used in the proper order (1 to 18) because it can be seen, e.g., that to obtain the coefficient $P_{14}(n+1)$, the coefficients $P_{12}(n+1)$ and $P_{13}(n+1)$ of the same order are needed. In the recurrence relations given in Table 2, a capital P is used for the semilatus rectum (instead of the usual lower-case p), and a capital Q is used for the square root of the semilatus rectum. Three different kinds of summation symbols Σ have also been used to designate sums with different limits in the indices p and q . These have the following meaning:

$$\Sigma_1 = \text{sum for } \begin{bmatrix} p = 1, 2, \dots, n \\ q = n, n-1, \dots, 1 \\ p+q = 1+n \end{bmatrix} \quad (132a)$$

$$\Sigma_2 = \text{sum for } \begin{bmatrix} p = 2, 3, \dots, n \\ q = n-1, n-2, \dots, 1 \\ p+2 = n+1 \end{bmatrix} \quad (132b)$$

$$\Sigma_3 = \text{sum for } \begin{bmatrix} p = 1, 2, \dots, n+1 \\ q = n+1, n, \dots, 1 \\ p+q = n+2 \end{bmatrix} \quad (132c)$$

These equations have been programmed for the two- and three-dimensional problems in two separate programs, but no research has been done on the three-dimensional case.

Table 1. Orbit variables and defining equations

Parameter	Equation	Parameter	Equation
$P_1 = x$	$\frac{dx}{dv} = \dot{x}$	$P_9 = t$	$\frac{dt}{dv} = \frac{1}{p^{1/2}} r^2$
$P_2 = y$	$\frac{dy}{dv} = \dot{y}$	$P_{10} = x_1$	$x_1 \frac{dx_1}{dv} = x\dot{x} + y\dot{y} + z\dot{z} + m_2\dot{x}$
$P_3 = z$	$\frac{dz}{dv} = \dot{z}$	$P_{11} = x_2$	$x_2 \frac{dx_2}{dv} = x\dot{x} + y\dot{y} + z\dot{z} - m_1\dot{x}$
$P_4 = \dot{x}$	$\frac{d\dot{x}}{dv} = +2\dot{y} + \frac{1}{p} r(xA + B)$	$P_{12} = \sigma_1 = x_1^{-3}$	$x_1 \frac{d\sigma_1}{dv} = -3\sigma_1 \frac{dx_1}{dv}$
$P_5 = \dot{y}$	$\frac{d\dot{y}}{dv} = -2\dot{x} + \frac{1}{p} r(yA)$	$P_{13} = \sigma_2 = x_2^{-3}$	$x_2 \frac{d\sigma_2}{dv} = -3\sigma_2 \frac{dx_2}{dv}$
$P_6 = \dot{z}$	$\frac{d\dot{z}}{dv} = -z + \frac{1}{p} r(zA)$	$P_{14} = A$	$= 1 - m_1\sigma_1 - m_2\sigma_2$
$P_7 = r$	$\frac{dr}{dv} = \dot{r}$	$P_{15} = xA + B$	$= xA + m_1m_2(\sigma_2 - \sigma_1)$
$P_8 = \dot{r}$	$\dot{r} \frac{dr}{dv} = 2\dot{r}^2 + r^2 - \frac{1}{p} rr^2$	$P_{16} = yA$	$= yA$
		$P_{17} = zA$	$= zA$
		$P_{18} = r^2$	$= rr$

Table 2. Recurrence relations for coefficients of power series

Coefficient	Recurrence relation	Coefficient	Recurrence relation
$nP_1(n+1)$	$P_1(n)$	$nP_{10}(1)P_{10}(n+1)$	$-\sum_2 qP_{10}(p)P_{10}(q+1) + m_2P_1(n)$
$nP_2(n+1)$	$P_2(n)$		$+\sum_1 [P_1(p)P_2(q) + P_2(p)P_5(q)$
$nP_3(n+1)$	$P_3(n)$		$+ P_3(p)P_6(q)]$
$nP_4(n+1)$	$+2P_3(n) + \frac{1}{p}\sum_1 P_7(p)P_{15}(q)$	$nP_{11}(1)P_{11}(n+1)$	$-\sum_2 qP_{11}(p)P_{11}(q+1) - m_1P_4(n)$
$nP_5(n+1)$	$-2P_4(n) + \frac{1}{p}\sum_1 P_7(p)P_{16}(q)$		$+\sum_1 [P_1(p)P_5(q) + P_2(p)P_5(q)$
$nP_6(n+1)$	$-P_3(n) + \frac{1}{p}\sum_1 P_7(p)P_{17}(q)$		$+ P_3(p)P_6(q)]$
$nP_7(n+1)$	$P_3(n)$	$nP_{10}(1)P_{12}(n+1)$	$-\sum_2 qP_{10}(p)P_{12}(q+1) - 3\sum_1 qP_{12}(p)P_{10}(q+1)$
$nP_8(1)P_8(n+1)$	$-\sum_2 qP_7(p)P_8(q+1) + P_{18}(n)$	$nP_{11}(1)P_{13}(n+1)$	$-\sum_2 qP_{11}(p)P_{13}(q+1) - 3\sum_1 qP_{13}(p)P_{11}(q+1)$
	$+ \sum_1 [2P_8(p)P_8(q) - \frac{1}{p}P_7(p)P_{18}(q)]$	$P_{14}(n+1)$	$-m_1P_{12}(n+1) - m_2P_{13}(n+1)$
$nP_9(n+1)$	$\frac{1}{Q}P_{18}(n)$	$P_{15}(n+1)$	$\sum_2 P_5(p)P_{14}(q) + m_2m_1[P_{13}(n+1) - P_{12}(n+1)]$
		$P_{16}(n+1)$	$\sum_2 P_5(p)P_{14}(q)$
		$P_{17}(n+1)$	$\sum_2 P_5(p)P_{14}(q)$
		$P_{18}(n+1)$	$\sum_2 P_7(p)P_7(q)$

The notation used herein for the coefficients $P_i(n)$ of the power series expansions for the parameters P_i was derived from the FORTRAN 4 programming language. Also, because of some characteristics of FORTRAN 4, the indices n of the coefficients $P_i(n)$ start at one rather than at zero. Although these notations are somewhat unusual, they greatly facilitate programming of the equations for the computer.

The main disadvantage of the recurrent power series method is that it requires a complete transformation of the differential equations and the introduction of many auxiliary variables. On the other hand, the redundancy of the variables can be used for checking the precision; in fact, each auxiliary variable may be used for a check in the same way as a first integral of the system. In the numerical calculations, e.g., the variables P_{10} , P_{11} , P_{12} , and P_{13} have been used for checking purposes. This is done in the following way: Using P_1 and P_2 , the values of r_1 and r_2 are computed:

$$\begin{aligned} r_1 &= [(P_1 - x_1)^2 + (P_2)^2]^{1/2} \\ r_2 &= [(P_1 - x_2)^2 + (P_2)^2]^{1/2} \end{aligned} \quad (133)$$

The following four absolute values give a measure of the local integration error:

$$|P_{10} - r_1|, |P_{11} - r_2|, |P_{12} - r_1^{-3}|, |P_{13} - r_2^{-3}| \quad (134)$$

In the computer programs, the four quantities of Eq. (134) are computed at each integration step; the average of all four is then printed out.

B. Integration of the Variational Equations With Recurrent Power Series

The variational equations play an important role in the study of periodic orbits because, by their solution, the partial derivatives of the variables $(x, y, \dot{x}, \dot{y})$ with respect to the initial conditions $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ are obtained. These partial derivatives are needed for two important reasons: (1) for the differential correction process to find the periodic orbits and (2) for the construction of the fundamental matrix (by examination of the eigenvalues of this matrix, information on the stability of the orbits is obtained).

The 16 partial derivatives are obtained by numerically integrating the variational equations four times. These four independent solutions can be considered as the four columns of a four-by-four matrix, which is called the fundamental matrix. In the most frequent applications, the initial conditions form a unit matrix. The particular fundamental matrix, which is the unit matrix at $t = 0$, will here be called the *principal fundamental matrix*.

The equations of motion may be written in the following form:

$$\frac{dx}{dv} = \dot{x} \quad (135a)$$

$$\frac{dy}{dv} = \dot{y} \quad (135b)$$

$$\frac{d\dot{x}}{dv} = f_x(x, y, \dot{y}) = +2\dot{y} + \frac{r}{p} \frac{\partial U}{\partial x} \quad (135c)$$

$$\frac{d\dot{y}}{dv} = f_y(x, y, \dot{x}) = -2\dot{x} + \frac{r}{p} \frac{\partial U}{\partial y} \quad (135d)$$

The potential function U was defined previously in Eq. (86). The equations of motion are the same as Eq. (33), but are written here in the form of first-order equations. The variational equations corresponding to Eq. (135) are

$$\frac{d \delta x}{dv} = \delta \dot{x} \quad (136a)$$

$$\frac{d \delta y}{dv} = \delta \dot{y} \quad (136b)$$

$$\begin{aligned} \frac{d \delta \dot{x}}{dv} &= \frac{\partial f_x}{\partial x} \delta x + \frac{\partial f_x}{\partial y} \delta y + 2\delta \dot{y} \\ &= \frac{r}{p} \left(\frac{\partial^2 U}{\partial x \partial x} \delta x + \frac{\partial^2 U}{\partial x \partial y} \delta y \right) + 2\delta \dot{y} \end{aligned} \quad (136c)$$

$$\begin{aligned} \frac{d \delta \dot{y}}{dv} &= \frac{\partial f_y}{\partial x} \delta x + \frac{\partial f_y}{\partial y} \delta y + 2\delta \dot{x} \\ &= \frac{r}{p} \left(\frac{\partial^2 U}{\partial y \partial x} \delta x + \frac{\partial^2 U}{\partial y \partial y} \delta y \right) - 2\delta \dot{x} \end{aligned} \quad (136d)$$

The second partial derivatives of U have the value

$$\frac{\partial^2 U}{\partial x \partial x} = 1 - V_3 + 3 \left[m_1 \frac{(x-x_1)^2}{r_1^5} + m_2 \frac{(x-x_2)^2}{r_2^5} \right] \quad (137a)$$

$$\frac{\partial^2 U}{\partial x \partial y} = 3y \left[m_1 \frac{(x-x_1)}{r_1^5} + m_2 \frac{(x-x_2)}{r_2^5} \right] \quad (137b)$$

$$\frac{\partial^2 U}{\partial y \partial y} = 1 - V_3 - 3y^2 \left[m_1 \frac{1}{r_1^5} + m_2 \frac{1}{r_2^5} \right] \quad (137c)$$

where V_3 is the auxiliary expression

$$V_3 = \frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} \quad (138)$$

To obtain the four independent solutions of the variational equations, Eq. (136), to form the fundamental matrix, different computer programs have been prepared. In one program, the numerical integration is performed by a classical predictor-corrector method by which 21 first-order differential equations must be solved simultaneously. Of these, the first to fourth equations are the equations of motion, Eq. (135), and give the values of $x, y, \dot{x}, \dot{y}$, whereas the fifth to twentieth equations give the matrix with four solutions of the variational equations, Eq. (136). Finally, the last equation gives the time t as a function of the true anomaly v .

However, the best results have been obtained by solving the variational equations, Eq. (136), with the recurrent power series method, together with the equations of motion, Eq. (135). The formulation described in the preceding section has been maintained to solve the equations of motion; besides the 18 parameters P_i used in the preceding section, however, 32 new parameters P_i ($i = 19$ to 50) have been introduced. The 16 parameters P_{19} to P_{34} give the four required solutions of the variational equations, and the last 16 parameters P_{35} to P_{50} are auxiliary quantities that are used to obtain simple recurrence relations for the coefficients of the series.

With the use of a large number of auxiliary variables, the chain of computations can be set up in such a way that more than two power series never have to be multiplied together. Very simple recurrence relations for the coefficients are then obtained. The formulation that has been used here is certainly not unique; in fact, a formulation with fewer auxiliary variables is possible.

All 32 new parameters P_{19} to P_{50} are represented by a power series in the true anomaly v , with coefficients $[P_i(n) (n = 1, 2, \dots)]$. For parameters P_{19} to P_{34} , the corresponding differential equations are used, but no auxiliary differential equations are introduced for the other 16 parameters P_{35} to P_{50} . For the variational equations, these last 16 auxiliary quantities play the same role as the variables P_{14} to P_{18} play for the equations of motion. The definitions of the variables P_{35} to P_{50} are given in the following:

$$P_{35} = \frac{x-x_1}{r_1} \quad (139a)$$

$$P_{36} = \frac{x-x_2}{r_2} \quad (139b)$$

$$P_{37} = \frac{y}{r_1} \quad (139c)$$

$$P_{38} = \frac{y}{r_2} \quad (139d)$$

$$P_{39} = -1 + 3 \left(\frac{x-x_1}{r_1} \right)^2 \quad (139e)$$

$$P_{40} = -1 + 3 \left(\frac{x-x_2}{r_2} \right)^2 \quad (139f)$$

$$P_{41} = -1 + 3 \left(\frac{y}{r_1} \right)^2 \quad (139g)$$

$$P_{42} = -1 + 3 \left(\frac{y}{r_2} \right)^2 \quad (139h)$$

$$P_{43} = 3 \frac{y}{r_1} \frac{(x-x_1)}{r_1} \quad (139i)$$

$$P_{44} = 3 \frac{y}{r_2} \left(\frac{x-x_2}{r_2} \right) \quad (139j)$$

$$P_{45} = 1 + m_1 P_{12} P_{39} + m_2 P_{13} P_{40} \quad (139k)$$

$$P_{46} = m_1 P_{12} P_{43} + m_2 P_{13} P_{44} \quad (139l)$$

$$P_{47} = 1 + m_1 P_{12} P_{41} + m_2 P_{13} P_{42} \quad (139m)$$

$$P_{48} = \frac{r}{p} P_{45} \quad (139n)$$

$$P_{49} = \frac{r}{p} P_{46} \quad (139o)$$

$$P_{50} = \frac{r}{p} P_{48} \quad (139p)$$

The last two of the four differential equations, Eqs. (136c) and (136d), may then be written as

$$\frac{d\delta\dot{x}}{dv} = P_{48} \delta x + P_{49} \delta y + 2 \delta \dot{y} \quad (140)$$

$$\frac{d\delta\dot{y}}{dv} = P_{49} \delta x + P_{50} \delta y - 2 \delta \dot{x}$$

The recurrence relations for the coefficients of the quantities P_{19} to P_{50} are given in the following:

$$n P_{19}(n+1) = P_{21}(n) \quad (141a)$$

$$n P_{20}(n+1) = P_{22}(n) \quad (141b)$$

$$\begin{aligned} n P_{21}(n+1) &= +2 P_{22}(n) \\ &+ \Sigma_1 [P_{48}(p) P_{19}(9) + P_{49}(p) P_{20}(9)] \end{aligned} \quad (141c)$$

$$\begin{aligned} n P_{22}(n+1) &= -2 P_{21}(n) \\ &+ \Sigma_1 [P_{49}(p) P_{19}(9) + P_{50}(p) P_{20}(9)] \end{aligned} \quad (141d)$$

$$P_{10}(1) P_{35}(n+1) = P_1(n+1) - \Sigma_4 P_{10}(p) P_{35}(9) \quad (141e)$$

$$P_{11}(1) P_{36}(n+1) = P_1(n+1) - \Sigma_4 P_{11}(p) P_{36}(9) \quad (141f)$$

$$P_{10}(1) P_{37}(n+1) = P_2(n+1) - \Sigma_4 P_{10}(p) P_{37}(9) \quad (141g)$$

$$P_{11}(1) P_{38}(n+1) = P_2(n+1) - \Sigma_4 P_{11}(p) P_{38}(9) \quad (141h)$$

$$P_{39}(n+1) = 3 \Sigma_3 P_{35}(p) P_{35}(9) \quad (141i)$$

$$P_{40}(n+1) = 3 \Sigma_3 P_{36}(p) P_{36}(9) \quad (141j)$$

$$P_{41}(n+1) = 3 \Sigma_3 P_{37}(p) P_{37}(9) \quad (141k)$$

$$P_{42}(n+1) = 3 \Sigma_3 P_{38}(p) P_{38}(9) \quad (141l)$$

$$P_{43}(n+1) = 3 \Sigma_3 P_{37}(p) P_{35}(9) \quad (141m)$$

$$P_{44}(n+1) = 3 \Sigma_3 P_{38}(p) P_{36}(9) \quad (141n)$$

$$\begin{aligned} P_{45}(n+1) &= \Sigma_3 [m_1 P_{12}(p) P_{39}(9) \\ &+ m_2 P_{13}(p) P_{40}(9)] \end{aligned} \quad (141o)$$

$$\begin{aligned} P_{46}(n+1) &= \Sigma_3 [m_1 P_{12}(p) P_{43}(9) \\ &+ m_2 P_{13}(p) P_{44}(9)] \end{aligned} \quad (141p)$$

$$\begin{aligned} P_{47}(n+1) &= \Sigma_3 [m_1 P_{12}(p) P_{41}(9) \\ &+ m_2 P_{13}(p) P_{42}(9)] \end{aligned} \quad (141q)$$

$$P_{48}(n+1) = \frac{\Sigma_3 P_7(p) P_{45}(9)}{(1-e^2)} \quad (141r)$$

$$P_{49}(n+1) = \frac{\Sigma_3 P_7(p) P_{46}(9)}{(1-e^2)} \quad (141s)$$

$$P_{50}(n+1) = \frac{\Sigma_3 P_7(p) P_{47}(9)}{(1-e^2)} \quad (141t)$$

The recurrence equations for the coefficients of P_{23} to P_{34} are not reproduced because they are identical to those of P_{19} to P_{22} . The meaning of the different summation

signs is the same as is indicated in Eq. (132), except that the following new symbol is used:

$$\Sigma_4 = \text{sum for } \begin{bmatrix} p = 2, \dots, n+1 \\ q = n, \dots, 1 \\ p+q = n+2 \end{bmatrix} \quad (142)$$

As aforesaid, the recurrence relations are recurrent only if they are used in the same order as they are given here (from P_1 to P_{50}). More precisely, they should be used in seven consecutive groups in the following order: (P_1 to P_{13}), (P_{14} to P_{18}), (P_{19} to P_{34}), (P_{35} to P_{38}), (P_{39} to P_{44}), (P_{45} to P_{47}), and (P_{48} to P_{50}).

C. Periodic Orbits in the Elliptic Restricted Three-Body Problem

The elliptic restricted three-body problem has some fundamental differences from the circular problem, the most important of which is that the elliptic problem has no Jacobi integral or energy integral, as does the circular problem. The equations of motion, even in rotating or rotating-pulsating axes, explicitly contain the independent variable, e.g., the time, or the true or eccentric anomalies ($\cos v$ and $\sin v$ or $\cos E$ and $\sin E$). This problem is thus essentially nonconservative (nonautonomous according to another terminology).

A consequence of this fact is that any periodic solution must have a period that is an integer multiple of the period of the periodic functions $\cos v$ and $\sin v$. In other words, the period of any periodic orbit in the elliptic restricted three-body problem must be 2π or a multiple of this number. The period of rotation of the rotating axes being also 2π , a periodic orbit of the elliptic problem will be periodic with respect to the inertial axes as well as the rotating axes. Except for some details related, e.g., to the convenience of numerical integration, it is thus not essential to use rotating axes in this problem.

Periodic orbits can also be studied with respect to the inertial axes. Because of the fixed value of the periods of periodic orbits in the elliptic problem, there are no families of periodic orbits similar to the families with continuously varying periods that exist in the circular problem. Only isolated periodic orbits can exist in the elliptic problem, with period $2k\pi$ ($k = 1, 2, \dots$). These periodic orbits are thus all "commensurable" or "synchronous" or "in resonance" with the primaries.

The question of symmetric orbits with respect to the rotating x -axis (or *syzygy*-axis), which is so important in

the circular problem, retains all of its value in the elliptic problem. Symmetric periodic orbits still exist in the elliptic problem, but the criteria or sufficient conditions for symmetric periodic orbits in the elliptic problem are slightly different, these criteria being more strict in the elliptic problem. In the circular problem, what may be called a "weak periodicity criterion" exists, unlike the elliptic problem, where there is a "strong periodicity criterion." In the circular problem, the weak criterion for symmetry could be stated as follows:

If the satellite crosses the *syzygy*-axis perpendicularly, its motion is symmetric with respect to this axis.

In the elliptic problem, the corresponding strong criterion for symmetry could be stated in the form given by Moulton (see Ref. 13):

If the satellite crosses the *syzygy*-axis perpendicularly, at a time when the primaries are at an apse, its motion is symmetrical with this axis.

The time of perpendicular crossing is important, therefore, in the elliptic problem. The crossing must be at the moment of an apse (maximum or minimum elongation of the primaries). In the circular problem, the time of perpendicular crossing of the *syzygy*-axis is irrelevant.

The weak criterion for symmetric periodic orbits in the circular problem can be stated as follows:

If an orbit has two perpendicular crossings with the *syzygy*-axis, it is periodic.

It is because the times of crossing are unimportant that families of periodic orbits with continuously varying periods exist. In the elliptic problem, the perpendicular crossings must be such that they assure a period $2k\pi$; the strong criterion for symmetric periodic orbits may then be expressed in the following form:

An orbit is periodic if it has two perpendicular crossings with the *syzygy*-axis, and if the crossings are at moments when the primaries are at an apse.

The time between the two apsides of the primaries in Keplerian motion being a multiple of π (say, $k\pi$), the period will then be $2k\pi$. The above periodicity criterion gives a sufficient but not necessary condition for periodicity, of course, as nonsymmetric periodic orbits may also exist. In the present study of the elliptic restricted three-body problem, some cases corresponding to the

integers $k = 1, 2, 3, 4, 5$ have been investigated. Only symmetric periodic orbits have been studied.

D. Families of Periodic Orbits in the Elliptic Restricted Three-Body Problem

As was stated in the preceding section, in the sense of what is generally done in the circular restricted three-body problem, no families of periodic orbits exist. However, the elliptic three-body problem has another important characteristic that will make possible the generation of continuous families of periodic orbits.

This problem contains two parameters—the eccentricity e and the mass-ratio μ of the primaries—in contrast with the single parameter μ of the circular problem. The extra parameter e , in fact, adds one degree of freedom to the problem of finding periodic orbits. When e and μ are fixed, only isolated symmetric periodic orbits eventually exist in the elliptic problem. However, if such a periodic orbit has been found, and if e and μ are allowed to vary, this periodic orbit may be prolonged in a double infinity of periodic orbits; in other words, in a family depending upon the two parameters e and μ . Numerical experiments have disclosed that several such families actually exist. It has even been found that some families exist for all values of e and μ : $0 \leq e \leq 1$ and $0 \leq \mu \leq 1$. Most of the families described below have one parameter that has been fixed (μ) and the other parameter variable (e). The computer programs are capable of generating periodic orbits by perturbing and then differentially correcting both e and μ .

Most of the numerical integrations have been done with constant μ and increasing e , starting from the circular problem, at $e = 0$. However, it can be seen that, for every symmetric periodic orbit with $e = 0$, there are two ways of prolongating it to the elliptic problem. This is because the symmetric orbits have two perpendicular intersections with the *syzygy*-axis, and each intersection can be used as a starting point for the integration.

If one point is used, it is considered that at $t = 0$, then $v = 0$; i.e., at $t = 0$ the two primaries are at periapsis (or minimum elongation). Taking the other intersection point is equivalent to taking $v = \pi$ when $t = 0$, and it is then supposed that the two primaries are at apoapsis at the moment of the first crossing of the *syzygy*-axis. When $e = 0$, the forms of the orbits are identical, but there is a phase delay of a half revolution between the two primaries and the satellite. When the eccentricity is increased to nonzero values, the forms of the orbits are different, and different families are thus generated. These

two families join at $e = 0$. Several illustrations of this situation have been found numerically.

In all of the computations, two types of initial conditions have thus been used. The orbits that start with $t = 0, v = 0$ are called *periapsis orbits*, and will be designated with the letter P . The orbits that start with $t = 0, v = \pi$ are called *apoapsis orbits*, and will be designated with the letter A joined to the family number.

To use the circular restricted three-body problem as a first approximation to the elliptic problem, the procedure described below has been followed.

In a previous work on the circular problem, some 4000 periodic orbits have been classified, all corresponding to the earth-moon mass ratio. The initial conditions of the periodic orbits published by Bartlett (see Ref. 18) and Henon (Ref. 27) have also been collected, and these all correspond to $\mu = 1/2$ (equal masses). The initial conditions for these periodic orbits have been punched on cards. Out of this large collection of orbits, the isolated periodic orbits with period $2k\pi$ ($k = 1, 2, 3, 4, 5$) have then been extracted by automatic interpolation with the use of a computer program. In this way, a total of 150 resonant periodic orbits have thus been found.

Some of these orbits have then been used for a starting point by an analytical continuation along increasing eccentricities e (and constant period) implemented in different programs. In a few other cases, the eccentricity has been kept constant and the mass-ratio has been varied.

The computer programs are capable of varying any of the initial parameters of the problem and generating a family in this parameter. More precisely, only symmetric orbits have been studied, and periodic orbits have then been obtained by differential correction of two of the initial parameters. The symmetric periodic orbits are defined by two end conditions ($y = 0, \dot{x} = 0$); two initial conditions are thus necessary to meet the requirements at the end of a half orbit. Altogether, three parameters are thus continuously varied by the computer programs. The first two parameters are varied by differential corrections to satisfy the orthogonal crossing condition $y = \dot{x} = 0$ at the end; the third parameter is varied to generate the family of orbits. For the ordinary symmetric periodic orbits, the first two initial parameters have always been $(x_0, \dot{y}_0)$ and the third parameter e or μ . For periodic symmetric collision orbits, the two initial conditions have been taken as the energy E and the eccentricity e , the final conditions remaining $y = \dot{x} = 0$; the family parameter has been taken as μ .

It is noteworthy that, in the case of periodic symmetric collision orbits, only one-parameter families of periodic orbits exist. These orbits are determined by only two initial conditions—the energy E and the collision angle θ . To fulfill the symmetry conditions, θ must be kept constant and only E can be varied. Thus, for e and μ constants, there will generally be no periodic symmetric collision orbits. If E and e are varied simultaneously by differential corrections, periodic orbits may then be found; these orbits form a one-parameter family in μ .

E. Differential Corrections for Periodic Orbits

To find periodic orbits in general, for the planar restricted problem, differential corrections must be made in four variables; for symmetric periodic orbits, differential corrections in only two variables are necessary. Computer programs have been written to handle both cases.

In the general case, the vector with the four initial conditions $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ of an orbit may be designated by x_0 , and the orbit with these initial conditions is designated by $x(t, x_0)$. At the end of this orbit, at $t = T = 2k\pi$, the values of the coordinates are thus $x(T, x_0)$. If the initial conditions of this orbit must be corrected by Δx_0 in such a way as to make the orbit periodic, the initial coordinates of the new orbit must match its final coordinates:

$$x_0 + \Delta x_0 = x(T, x_0 + \Delta x_0) \quad (143)$$

If the correction Δx_0 is assumed to be small, the right side of Eq. (143) may be expanded in a Taylor series truncated at the first-order terms:

$$x_0 + \Delta x_0 = x(T, x_0) + \frac{\partial x(T, x_0)}{\partial x_0} \Delta x_0 \quad (144)$$

or

$$\left[\frac{\partial x(T, x_0)}{\partial x_0} - I_4 \right] \Delta x_0 = x_0 - x(T, x_0) \quad (145)$$

The corrections Δx_0 are thus obtained by solving the linear system of Eq. (145), whereas the partial derivatives are obtained by solving the variational equations. This system is the basis of the differential corrections to obtain periodic orbits. Because the problem is now linear and Eq. (145) has been made linear by making a truncation of the Taylor series, the corrections of Eq. (145) must be computed and applied in an iterative way. If a sufficiently good approximation is known, the differential corrections converge in about four to five iterations.

In the present work on the elliptic problem, it has always been possible to begin with the fairly good initial conditions given by the circular problem. On the other hand, in the generation of families of periodic orbits, good initial approximations have always been obtained by extrapolation from the previously computed orbits in the family. For the purpose of extrapolations, the programs save the last 10 orbits or so in the memory of the computer. Using this available information has allowed considerable savings of iterations and computer time.

It is known that, in the circular restricted problem, the matrix of Eq. (145) becomes singular because of the presence of an eigenvalue $+1$ (of multiplicity two) of the fundamental matrix. This difficulty can easily be avoided in the circular problem by keeping one of the four variables constant and allowing no variation Δx_0 for it. In the elliptic problem, however, this difficulty does not occur because of the nonexistence of the unit eigenvalue. In many cases, the matrix has been found to be nearly singular for other reasons, mainly because of the existence of several solutions. It has thus been necessary to treat the near-singular case in a special way. This has been done with some programs provided by Dr. C. Lawson. In cases where the matrix of the system is nearly singular, a least-squares approach is adopted for the solution of the system, Eq. (145). In this case, a correction Δx_0 with minimum length is used. This type of solution for the near-singular situation has given highly satisfactory results for the differential corrections.

In most cases, the differential correction process described above can be slightly simplified in the case of symmetric periodic orbits. There are only two variables and, instead of a fourth-order matrix, only a second-order matrix must be inverted. Here the initial conditions x_0 must be corrected by Δx_0 in such a way as to obtain a given preassigned final value $\mathbf{x}_F$ for the state vector. The equations of the periodicity conditions are thus

$$x(T, x_0 + \Delta x_0) = \mathbf{x}_F \quad (146)$$

Again making a Taylor series expansion limited to the first-order terms gives

$$x(T, x_0) + \frac{\partial x(T, x_0)}{\partial x_0} \Delta x_0 = \mathbf{x}_F \quad (147)$$

or, finally, the following linear system in Δx_0 :

$$\frac{\partial x(T, x_0)}{\partial x_0} \Delta x_0 = \mathbf{x}_F - x(T, x_0) \quad (148)$$

In the case of only two variables $(x_0, \dot{y}_0)$ to be corrected, this system (Eq. 148) may be written explicitly as follows:

$$\begin{aligned} \frac{\partial y}{\partial x_0} \Delta x_0 + \frac{\partial y}{\partial \dot{y}_0} \Delta \dot{y}_0 &= -y(x_0, \dot{y}_0, T) \\ \frac{\partial \dot{x}}{\partial x_0} \Delta x_0 + \frac{\partial \dot{x}}{\partial \dot{y}_0} \Delta \dot{y}_0 &= -\dot{x}(x_0, \dot{y}_0, T) \end{aligned} \quad (149)$$

The solution of this system is

$$\begin{aligned} \Delta x_0 &= \frac{1}{D} \left[-\frac{\partial \dot{x}}{\partial \dot{y}_0} y(x_0, \dot{y}_0, T) + \frac{\partial y}{\partial \dot{y}_0} \dot{x}(x_0, \dot{y}_0, T) \right] \\ \Delta \dot{y}_0 &= \frac{1}{D} \left[+\frac{\partial \dot{x}}{\partial x_0} y(x_0, \dot{y}_0, T) - \frac{\partial y}{\partial x_0} \dot{x}(x_0, \dot{y}_0, T) \right] \\ D &= \frac{\partial y}{\partial x_0} \frac{\partial \dot{x}}{\partial \dot{y}_0} - \frac{\partial \dot{x}}{\partial x_0} \frac{\partial y}{\partial \dot{y}_0} \end{aligned} \quad (150)$$

It has been seen that, even with the simple system of Eq. (149), difficulties may occur because of indetermina-tion of the solution, and that the matrix may be nearly of rank one. In this case, the least-squares approach has again been adopted. For the problem of finding periodic collision orbits, the differential corrections have been made in the energy E and the eccentricity e by the fol-lowing system, which is the analog of Eq. (149):

$$\begin{aligned} \frac{\partial y}{\partial E} \Delta E - \frac{\partial y}{\partial e} \Delta e &= -y(E, e, T) \\ \frac{\partial \dot{x}}{\partial E} \Delta E + \frac{\partial \dot{x}}{\partial e} \Delta e &= -\dot{x}(E, e, T) \end{aligned} \quad (151)$$

As was stated above, the periodic collision orbits ob-tained in this way form a one-parameter family, with μ as the variable parameter. For the collision orbits, the problem has been integrated with regularized Birkhoff coordinates. Because no program has yet been written to solve the variational equations in Birkhoff coordinates, the partial derivatives in this case have been obtained by numerical differencing of perturbed solutions.

F. Properties of the Eigenvalues of the Fundamental Matrix

In what follows, the stability of periodic orbits in the elliptic restricted three-body problem will be examined. Only first-order or linear stability will be considered here. This stability will depend exclusively upon the behavior of the eigenvalues of the principal fundamental matrix $\mathbf{R}$. As aforesaid, the principal fundamental matrix

(matrix of partial derivatives) is obtained by solving the variational equations. Before the stability itself is dis-cussed, some important properties of the fundamental matrix $\mathbf{R}$ will be described herein.

The variational equations, Eq. (136), may be written in a matrix form,

$$\frac{d \delta \mathbf{x}}{dv} = \mathbf{A} \delta \mathbf{x} \quad (152)$$

where $\mathbf{A}$ is the fourth-order square matrix of Eq. (136). Matrix $\mathbf{A}$ may be written in the form

$$\mathbf{A} = \begin{vmatrix} 0 & \mathbf{I} \\ \mathbf{a} & 2\mathbf{J} \end{vmatrix} \quad (153)$$

where $\mathbf{I}$ is the second-order unit matrix and $\mathbf{a}$ is the second-order symmetric nonconstant matrix of the second-order partial derivatives of U given in Eq. (136), and

$$\mathbf{J} = \begin{vmatrix} 0 & +\mathbf{I} \\ -\mathbf{I} & 0 \end{vmatrix} \quad (154)$$

A new fourth-order constant matrix $\mathbf{S}$ will now be defined as

$$\mathbf{S} = \begin{vmatrix} 0 & +\mathbf{I} \\ -\mathbf{I} & 2\mathbf{J} \end{vmatrix} \quad (155)$$

It can be verified directly that matrix $\mathbf{A}$ of the varia-tional equations given in Eq. (153) satisfies the following property:

$$\mathbf{S} \mathbf{A}^T \mathbf{S}^{-1} = -\mathbf{A} \quad (156)$$

The superscript T indicates the transpose operation. Matrices $\mathbf{A}$ that satisfy this relation (Eq. 156) will be called *skew-symplectic* relative to the matrix $\mathbf{S}$. The important relation of Eq. (156) will be needed to estab-lish a similar important relation for the principal funda-mental matrix $\mathbf{R}$. In fact, a very simple expression will be obtained for the inverse $\mathbf{R}^{-1}$ of matrix $\mathbf{R}$. Matrix $\mathbf{R}$ satisfies differential Eq. (152):

$$\frac{d\mathbf{R}}{dv} = \mathbf{A}\mathbf{R} \quad (157)$$

and the inverse $\mathbf{R}^{-1}$ of $\mathbf{R}$ is a solution of the adjoining system

$$\frac{d\mathbf{R}^{-1}}{dv} = -\mathbf{R}^{-1}\mathbf{A} \quad (158)$$

Using the property (Eq. 156) of matrix $\mathbf{A}$, it can now be verified that inverse matrix $\mathbf{R}^{-1}$, which is a solution of Eq. (158), is obtained by

$$\mathbf{S}\mathbf{R}^T\mathbf{S}^{-1} = \mathbf{R}^{-1} \quad (159)$$

Matrices $\mathbf{R}$ that satisfy a relation of the form of Eq. (159) may be called *symplectic* with respect to the given matrix $\mathbf{S}$. An important consequence of the relation of Eq. (159) is that matrices $\mathbf{R}$, $\mathbf{R}^T$, and $\mathbf{R}^{-1}$ all have the same eigenvalues. Thus, whenever λ is an eigenvalue of $\mathbf{R}$, $1/\lambda$ is also an eigenvalue. Zero eigenvalues are excluded, of course, because the determinant of both matrices $\mathbf{R}$ and $\mathbf{R}^{-1}$ is $+1$. The four eigenvalues of $\mathbf{R}$ may thus be written in the form

$$\lambda, \frac{1}{\lambda}, \mu, \frac{1}{\mu} \quad (160)$$

This property is true for all values of the independent variable and for all solutions, periodic and nonperiodic, as has been shown in more detail in Ref. 17. However, the different properties of the eigenvalues are mostly used for periodic orbits, and only for the value $2k\pi = T$ of the independent variable v . The characteristic exponents related to the above eigenvalues may be designated by $(\alpha, -\alpha, \beta, -\beta)$, and are defined by the relations

$$\lambda = e^{\alpha T}, \quad \mu = e^{\beta T} \quad (161)$$

They are defined, of course, only for modulo $2\pi i/T$.

G. Solution of the Characteristic Equation of the Elliptic Problem

As a consequence of the form of the roots of Eq. (160), the characteristic equation of matrix $\mathbf{R}$ may be written in the following form:

$$s^4 + a_1 s^3 + a_2 s^2 + a_1 s + 1 = 0 \quad (162)$$

The two coefficients of s^4 and s^0 ($+1$), as well as the two coefficients of s^3 and s , are thus equal. These two simple relations between the coefficients of the characteristic equation are used for checking the precision of the numerical computations. In most cases, a precision of at least 10^{-10} has been obtained for these two relations. It will be seen that the stability of the orbits depends

only upon the roots of the characteristic equation; therefore, upon two numbers: (λ, μ) or (a_1, a_2) . In other words, two stability numbers correspond to each periodic orbit. The real numbers (a_1, a_2) may be considered as the coordinates of a point in the (a_1, a_2) plane, and each periodic orbit is thus a point in this plane. The real quantities (a_1, a_2) will be called *stability coefficients* of the orbit. Two other numbers (k_1 and k_2), called *stability indices*, will also be used in the following:

$$k_1 = \lambda + \frac{1}{\lambda}, \quad k_2 = \mu + \frac{1}{\mu} \quad (163)$$

One is reminded here that, in the circular problem, there is only a single nontrivial stability index k because of the presence of two unit eigenvalues, the eigenvalues being of the form $(\lambda, 1/\lambda, 1, 1)$ because of the existence of the energy integral. In the circular problem, stability is discussed on the real k -axis ($k = \lambda + 1/\lambda$); in the elliptic problem, stability is discussed in the real (a_1, a_2) plane.

The characteristic equation, Eq. (162), may be written in the following form:

$$(s-\lambda)\left(s-\frac{1}{\lambda}\right)(s-\mu)\left(s-\frac{1}{\mu}\right) = 0 \quad (164)$$

and this form gives a relation between the stability coefficients (a_1, a_2) and the stability indices (k_1, k_2) :

$$a_1 = -\left(\lambda + \frac{1}{\lambda} + \mu + \frac{1}{\mu}\right) = -(k_1 + k_2) \quad (165)$$

$$a_2 = 2 + \left(\lambda + \frac{1}{\lambda}\right)\left(\mu + \frac{1}{\mu}\right) = 2 + k_1 k_2$$

This shows that k_1 and k_2 are the roots of the second-degree equation

$$X^2 + a_1 X + (a_2 - 2) = 0 \quad (166)$$

The expressions for k_1 and k_2 may thus be written as

$$\begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix} = \frac{-a_1 \pm (a_1^2 - 4a_2 + 8)^{1/2}}{2} \quad (167)$$

When k_1 and k_2 have been found, the roots λ and μ can be found again by solving second-degree equations. The solution of

$$x^2 - k_1 x + 1 = 0 \quad (168)$$

will give λ and $1/\lambda$ by

$$\left. \begin{array}{l} \lambda \\ 1/\lambda \end{array} \right\} = \frac{k_1 \pm (k_1^2 - 4)^{1/2}}{2} \quad (169)$$

whereas the solution of

$$y^2 - k_2 y + 1 = 0 \quad (170)$$

gives μ and $1/\mu$ by

$$\left. \begin{array}{l} \mu \\ 1/\mu \end{array} \right\} = \frac{k_2 \pm (k_2^2 - 4)^{1/2}}{2} \quad (171)$$

The problem of finding the four roots of the characteristic equation (i.e., the four eigenvalues of the fundamental matrix $\mathbf{R}$) is thus reduced to solving three second-degree equations. Coefficients a_1 and a_2 of the characteristic equation are obtained directly from the fundamental matrix $\mathbf{R}$, which is obtained by numerical integration of the variational equations. If the four columns of the matrix $\mathbf{R}$ are designated by x_i , y_i , $\dot{x}_i$, and $\dot{y}_i$ ($i = 1, 2, 3, 4$), then coefficients a_1 and a_2 are obtained by

$$-a_1 = x_1 + y_2 + \dot{x}_3 + \dot{y}_4 \quad (172a)$$

$$+a_2 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_1 & x_3 \\ \dot{x}_1 & \dot{x}_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_4 \\ \dot{y}_1 & \dot{y}_4 \end{vmatrix} \\ + \begin{vmatrix} y_2 & y_3 \\ \dot{x}_2 & \dot{x}_3 \end{vmatrix} + \begin{vmatrix} y_2 & y_4 \\ \dot{y}_2 & \dot{y}_4 \end{vmatrix} + \begin{vmatrix} \dot{x}_3 & \dot{x}_4 \\ \dot{y}_3 & \dot{y}_4 \end{vmatrix} \quad (172b)$$

The characteristics of the roots—and, of course, the stability of the orbits—will depend upon the discriminants of these three second-degree equations. Equation (166) has a zero discriminant when

$$a_2 = \frac{a_1^2}{4} + 2 \quad (173)$$

and this locus is a parabola in the (a_1, a_2) plane. For the other two second-degree equations, the discriminants

are zero when k_1 or k_2 takes on the value ± 2 . In the (a_1, a_2) plane, this gives the two straight lines

$$\begin{aligned} a_2 &= +2a_1 - 2 \\ a_2 &= -2a_1 - 2 \end{aligned} \quad (174)$$

These two lines are tangents of the parabola of Eq. (173) at the points $(a_1 = \pm 4, a_2 = 6)$.

H. The Seven Types of Stability Characteristics

The stability of periodic orbits will be discussed in the (a_1, a_2) plane. This stability thus depends upon stability coefficients (a_1, a_2) or stability indices (k_1, k_2) , or on eigenvalues (λ, μ) and the corresponding characteristic exponents.

A periodic orbit is said to be stable if the solutions of the associated variational equations have no terms with exponentially increasing terms. This happens if (and only if) the characteristic exponents have no positive real parts. However, as was seen above, the eigenvalues of the fundamental matrix $\mathbf{R}$ occur in reciprocal pairs, and the characteristic exponents occur in opposite pairs. Therefore, there is stability if (and only if) the characteristic exponents of the orbit are all purely imaginary.

If the characteristic exponents α and β given by Eq. (161) are purely imaginary, the corresponding eigenvalues λ and μ thus must be on the unit circle. In other words, stability exists only when all four eigenvalues λ , $1/\lambda$, μ , $1/\mu$ are on the unit circle.

The location of the four eigenvalues with respect to the unit circle is discussed below. This can easily be done in the (a_1, a_2) plane. Considering the parabola of Eq. (173) and its two tangents (Eq. 174), the plane will be subdivided into seven distinct regions, in each of which will be different types of stability properties. The properties of the roots in all seven regions are examined below.

There are six unstable regions and one stable region; names are proposed for the six types of instability. The seven regions are shown in Fig. 2, and the corresponding configurations of roots are shown in Fig. 3. It can also be noted on the stability diagram (Fig. 2) that the circular restricted three-body problem—which has two roots $+1$, and thus $-k_1 = +2$ or $k_2 = +2$ —corresponds to the line $a_2 = -2a_1 - 2$.

The properties of the roots in each region are described in Table 3.

Table 3. Properties of the seven stability regions

Region	Properties of roots						Properties of orbits
	$a_1^2 - 4a_2 + 8$	k_1 and k_2	$k_1^2 - 4$	$k_2^2 - 4$	Properties of roots λ and μ and $1/\lambda$, $1/\mu$	Remarks	
1	>0	Real	<0	<0	$\bar{\lambda} = 1/\lambda$, $\bar{\mu} = 1/\mu$ (λ and μ complex)	All four on unit circle	Stability
2	<0	Complex conjugates	Complex	Complex	$\bar{\lambda} = \mu$, $\bar{\mu} = \lambda$ (λ and μ complex)	Not on unit circle	Complex instability
3	>0	Real	>0	>0	λ real, μ real $\lambda\mu < 0$	Two positive and two negative roots	Even-odd instability
4	>0	Real	>0	>0	λ real >0 μ real >0	Four real positive roots	Even-even instability
5	>0	Real	>0	>0	λ real <0 μ real <0	Four real negative roots	Odd-odd instability
6	>0	Real	>0	<0	λ real >0 , μ complex $\bar{\mu} = 1/\mu$	Two real positive roots and two complex roots on unit circle	Even-semi-instability
7	>0	Real	<0	>0	λ complex, $\bar{\lambda} = 1/\lambda$ μ real <0	Two real negative roots and two complex roots on unit circle	Odd-semi-instability

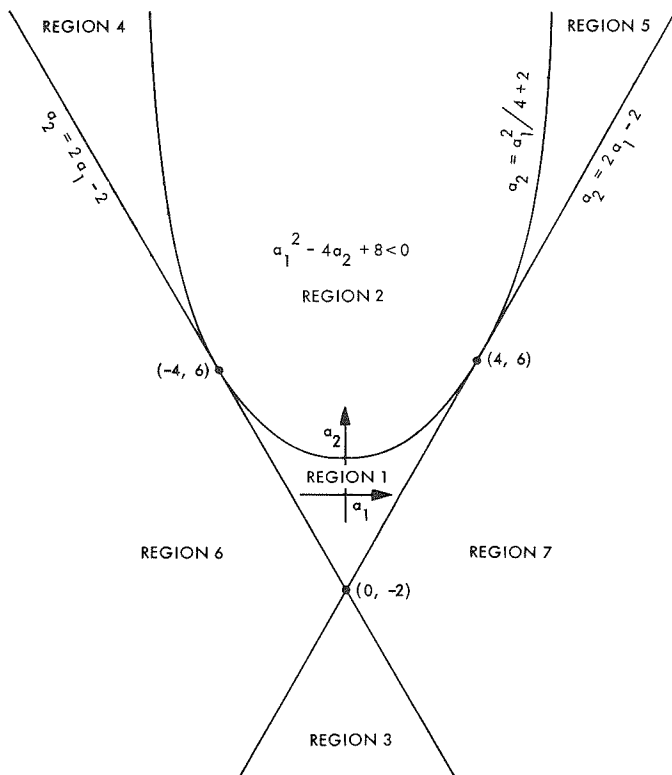


Fig. 2. Seven stability regions

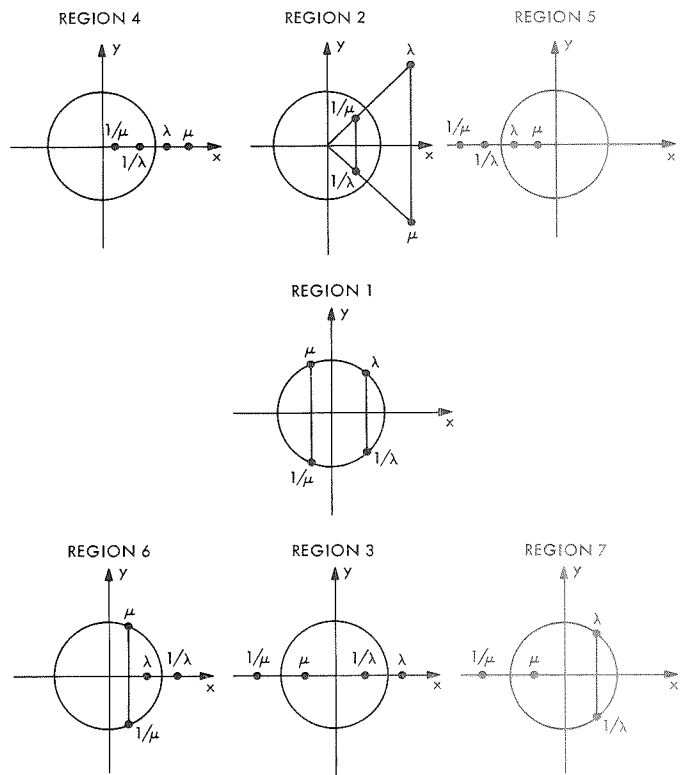


Fig. 3. Root configuration for stability regions

IV. Families of Periodic Orbits

In this section, the actual results of the machine calculations are described. These results consist of 1127 periodic orbits, which are grouped as follows:

- (1) One group of 13 isolated periodic orbits of the rectilinear problem (1P).
- (2) Four short segments of families of periodic orbits with $e = 1.0$ or $e \approx 1.0$ (3P, 4P, 5P, and 9P).
- (3) Two families related to the eccentricity $e = 1$ (6A and 12A).
- (4) Two families with the earth-moon mass-ratio $\mu = 0.012155$ (7P and 7A).
- (5) Five families with equal masses (8P, 8A, 10P, 11P, and 11A).
- (6) One family with periodic collision orbits.

Although much effort was made at the beginning to complete each family, this has turned out to be impossible because, in the investigation of the continuation of one family, several new families are always discovered.

The letters "P" and "A" designate the periapsis- and apoapsis-types of orbits, respectively, as described in Section III. Following the descriptions are tables with initial and final conditions (see Tables 4–19). The column headings at the top of each of these tables briefly describe the family. Columns with suffix 0 (e.g., X0, YD0T0) correspond to the initial conditions; columns with suffix 1 (e.g., X1, YD0T1) correspond to the final conditions. The initial and final conditions are given in barycentric inertial coordinates for families 1P, 3P, 4P, 5P, 9P, 6A, and 12A. For the eight other families, the initial and final conditions are given in barycentric rotating-pulsating coordinates.

One of the computer programs was prepared to automatically give the plots of the orbits on photographic paper and on microfilm, and in five different coordinate systems (the orbits being periodic in all five systems). For this reason, some graphs are included in the text with representations of orbits with respect to these frames of reference. Not much information is gained by seeing the orbits in different frames of reference, but it has been judged useful to give a few figures in different coordinate systems. If one would like to do a Fourier analysis of the orbits, e.g., these figures may help in selecting the most appropriate coordinate system. The following five systems have been used:

- (1) Barycentric inertial.

- (2) Barycentric rotating.

- (3) Barycentric rotating-pulsating.

- (4) Geocentric inertial (centered at $m_1 = 1 - \mu$).

- (5) Selenocentric inertial (centered at $m_2 = \mu$).

A. Families 12A and 6A of Periodic Orbits

The orbits of this family are symmetric with respect to the $0x$ -axis in the rotating coordinate system, i.e., the m_1 - m_2 line or *syzygy*-axis. They all satisfy the strong periodicity criterion—that they cross the *syzygy*-axis twice with a right angle at moments when m_1 and m_2 are at an apse (i.e., at the minimum or maximum elongation of m_1 and m_2).

This family is a continuation of an orbit with period 4π , which belongs to Strömberg's Class C of symmetric periodic orbits around the libration point L_1 . Thus, its mass ratio is $\mu = 0.50$.

Strömberg's Class C of orbits is one of the well-known classes of periodic orbits in the circular restricted three-body problem with equal masses. The orbits are symmetric with respect to both the $0x$ -axis and the $0y$ -axis. This class begins with infinitesimal retrograde periodic orbits around the libration point L_1 , and its end is unknown. It is also the class that contains one of the first double-collision orbits to be discovered. Some of the details of the evolution of Class C are shown in Fig. 4. Orbit 3 is the double-collision orbit. The evolution of

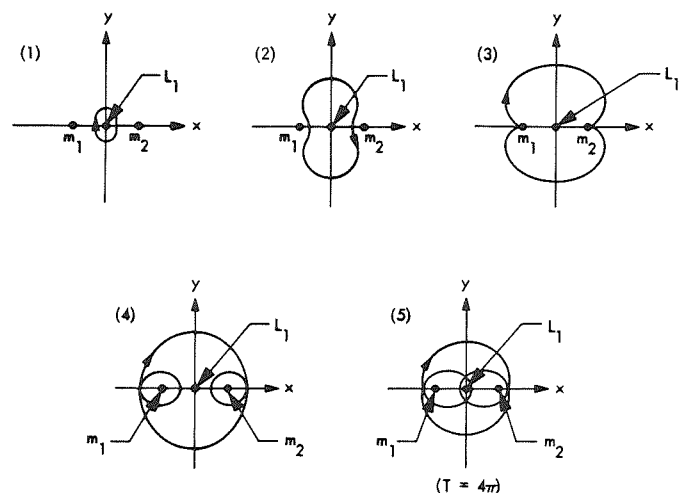


Fig. 4. Circular restricted three-body problem: Strömberg's Class C of periodic orbits around L_1 (equal masses; $e = 0$; rotating axes)

Class C beyond the collision orbit has been studied in detail; it appears, in particular, that there is an orbit with period 4π (orbit 5, Fig. 4).

This particular orbit with period 4π can be generalized in an interesting way in the elliptic problem. More precisely, it is concluded that orbit 5 can be continued for all eccentricities e from 0.0 to 1.0. For $e = 1.0$, orbit 5 seems also to exist for all mass ratios μ from 0.0 to 0.5.

A series of periodic orbits with constant mass ratio $\mu = 0.5$, and with increasing eccentricity from 0.0 to about 0.75, was first computed. These orbits have been integrated with a recurrent power-series solution and with the true anomaly as an independent variable. This system of differential equations, which essentially results from the well-known Nechville transformation, has a singularity for $e = 1.0$; numerical difficulties thus arise for high eccentricities. For this reason, the integration method has been changed at the higher eccentricities. The inertial axes have been used, and the classical Runge-Kutta integration method has been adopted, with which only about five- or six-place accuracy has been desired.

The Runge-Kutta program integrates a system of differential equations that is valid for all eccentricities (including $e = 1.0$) where the two primaries are oscillating (and colliding) on a straight line with a finite amplitude (semimajor axis = 1.0). Using the barycentric inertial formulation of the problem, there is still periodicity of the orbits (the periodicity criterion being unchanged). It then appears that the family of periodic orbits can be continued up to eccentricity 1.0.

In Fig. 5, one orbit is shown in rotating axes. The form is very similar for all eccentricities. Figure 6 shows the complete evolution, with respect to inertial axes, for all eccentricities. The outer orbit is the one that belongs to Strömgen's Class C ($e = 0.0$). It can be seen that the orbits keep shrinking when the eccentricity increases up to a rectilinear orbit corresponding to $e = 1.0$. In this last rectilinear orbit, there is oscillation of the satellite on the Oy -axis in synchronization with the oscillation of the two primaries on the Ox -axis. The satellite reaches maximum elongation (about ± 1.69) when the primaries collide, and it passes between the two primaries when they are at maximum elongation.

It is rather surprising that the rectilinear periodic orbit for $e = 1.0$ was published in 1956 by Schubart. In fact, Schubart has proposed this orbit as a starting point for

the study of the elliptic restricted three-body problem (see Ref. 8).

It has also been seen that this orbit has an interesting development not only along e , but also along μ , when e is kept equal to 1.0. The evolution of this orbit has been

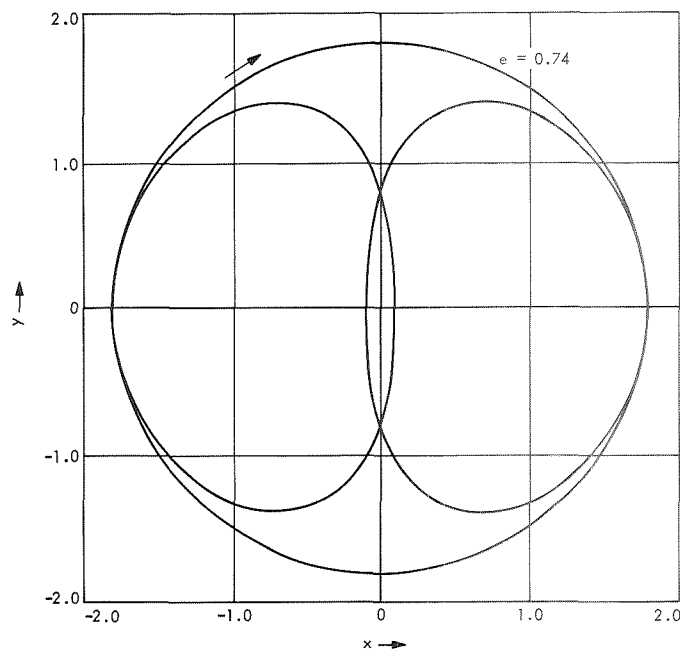


Fig. 5. Elliptic restricted three-body problem: periodic orbit for mass ratio $\mu = 0.5$ (equal masses; $e = 0.74$; rotating axes)

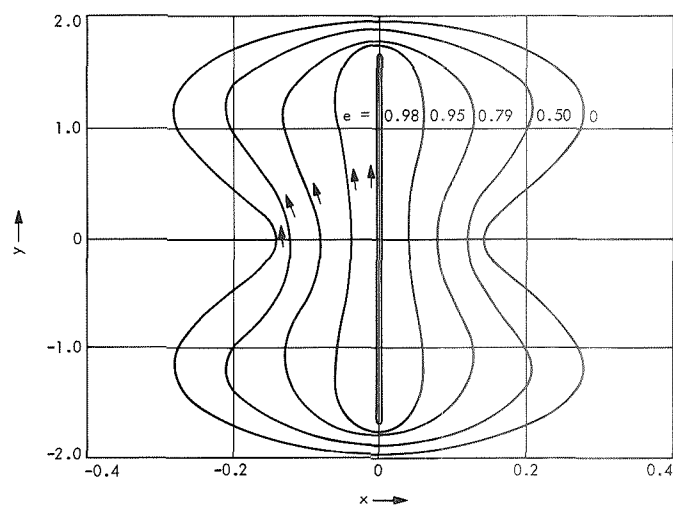


Fig. 6. Elliptic restricted three-body problem: periodic orbits for constant mass ratio $\mu = 0.5$ (equal masses) and variable eccentricity ($e = 0.0$ to 1.0; inertial axes)

followed for variable mass ratios up to $\mu = 0.166$. Judging from the results obtained, the evolution can probably be extrapolated up to $e = 0.0$, although exact computations should be made, and a regularization appears to be necessary in this case. These orbits, which form family 6A, are shown in Fig. 7. The initial conditions for family 6A are listed in Table 4. When μ decreases, a bending of the initial rectilinear orbit is seen. There is thus a back-and-forth oscillation, with period 4π , along a pseudo-parabolic path. These paths are included in an envelope of zero-velocity points, which are shown by dashed lines in Fig. 7.

Of periodic orbits with $\mu = 0.50$ and with e from 0.0 to 1.0, 152 have been computed, and 118 orbits have been computed with $e = 1.0$ and with μ from 0.5 to 0.166.

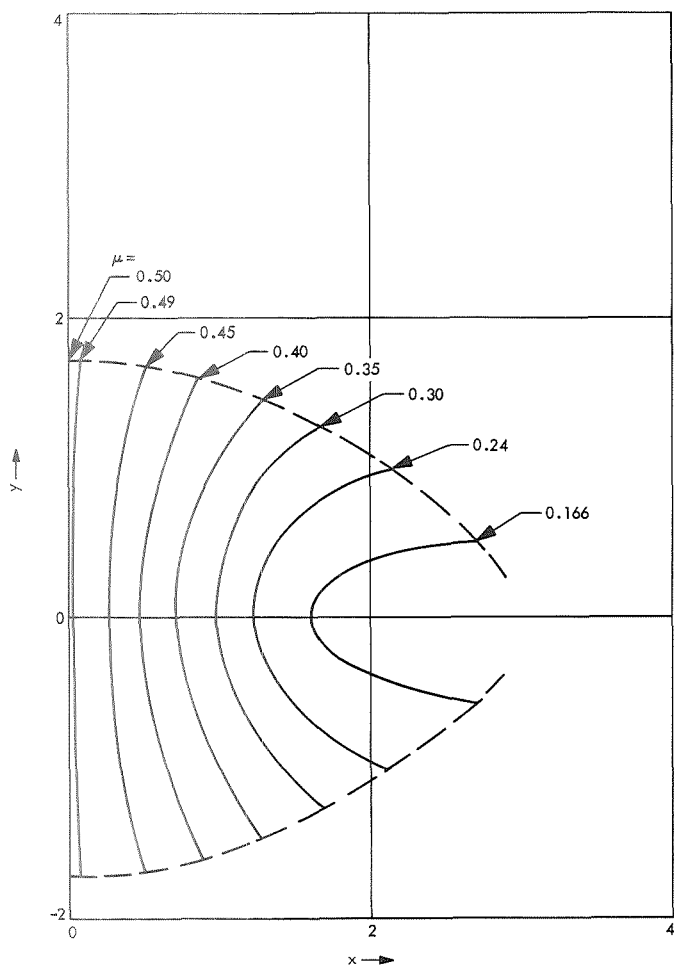


Fig. 7. Family 6A of periodic orbits—rectilinear elliptic restricted three-body problem: periodic orbits for constant eccentricity ($e = 1.0$) and variable mass ratio ($\mu = 0.50$ to 0.166 ; inertial axes)

More orbits could have been computed, e.g., with fixed e (different from 1.0) and variable μ , or with fixed μ (different from 0.5) and variable e . This has not been done because of the extensive computer time involved, but it is possible that this family exists for all values of e and μ . The stability of this family has not been studied.

Orbits with $\mu = 0.5$ form family 12A (Fig. 8), whereas orbits with $e = 1.0$ form family 6A (see Fig. 7 and Table 4). In the integration of these orbits, the two primaries have always been taken at maximum elongation (at apoapsis) at $t = 0$, and so the letter A is used in the symbols 6A and 12A designating these families. The initial conditions for family 12A are listed in Table 5.

B. Periodic Orbits in the Rectilinear Restricted Three-Body Problem

By studying the continuation of one of Strömberg's periodic orbits, it is seen that the eccentricities $e = 1$ cannot be avoided in the natural prolongation of the families, and that the rectilinear restricted three-body problem thus plays a special role. For this reason, a special study has been made of the rectilinear problem, and the most important numerical results are described in this section. Schubart had already proposed to take the eccentricity $e = +1$ as a starting point for a systematic study of the elliptic restricted three-body problem.

Because of the simplicity of the equations of motion, it was decided first to integrate some orbits with regularly spaced initial conditions to see if some periodic orbits would eventually exist. The rather surprising conclusion was soon reached that a large number of periodic orbits exist, although they are all isolated. Thirteen of these periodic orbits have been computed with somewhat higher precision. The classical Runge-Kutta numerical integration procedure has been used with a variable step (equal to $0.005 r_1 r_2$). The objective was not mainly the high precision of the numerical results; e.g., by using full double precision, the end results have probably only five- or six-place accuracy. The periodic orbits were obtained by two-dimensional linear differential corrections when a good approximation had been obtained by the previous computer runs.

The initial conditions that have been explored are of the form

$$(x_0, 0, 0, \dot{y}_0) \quad (175)$$

and have been arbitrarily restricted to the limits

$$0.1 < x_0 < 1.0, \quad 0.2 < \dot{y}_0 < 2.2 \quad (176)$$

Table 4. Initial conditions for family 6A

RECT.ELL.PROBLEM.MU=VARIABLE,E=1.0, (MU=0.5 IS RECTIL.) FAMILY 6				
NR	X0	YDDT0	MASS RATIO	ECC
1	0.0000000	1.0530847	0.5000000	1.0000000
2	-0.0050094	1.0530873	0.4990000	1.0000000
3	-0.0150282	1.0531077	0.4970000	1.0000000
4	-0.0250467	1.0531486	0.4950000	1.0000000
5	-0.0350646	1.0532099	0.4930000	1.0000000
6	-0.0450817	1.0532917	0.4910000	1.0000000
7	-0.0550978	1.0533940	0.4890000	1.0000000
8	-0.0651127	1.0535167	0.4870000	1.0000000
9	-0.0751261	1.0536600	0.4850000	1.0000000
10	-0.0851378	1.0538237	0.4830000	1.0000000
11	-0.0951477	1.0540081	0.4810000	1.0000000
12	-0.1051555	1.0542130	0.4790000	1.0000000
13	-0.1151609	1.0544385	0.4770000	1.0000000
14	-0.1251638	1.0546847	0.4750000	1.0000000
15	-0.1351639	1.0549516	0.4730000	1.0000000
16	-0.1451611	1.0552392	0.4710000	1.0000000
17	-0.1551551	1.0555476	0.4690000	1.0000000
18	-0.1651458	1.0558768	0.4670000	1.0000000
19	-0.1751328	1.0562268	0.4650000	1.0000000
20	-0.1851162	1.0565980	0.4630000	1.0000000
21	-0.1950956	1.0569900	0.4610000	1.0000000
22	-0.2050708	1.0574032	0.4590000	1.0000000
23	-0.2150417	1.0578375	0.4570000	1.0000000
24	-0.2250081	1.0582930	0.4550000	1.0000000
25	-0.2349699	1.0587699	0.4530000	1.0000000
26	-0.2449265	1.0592681	0.4510000	1.0000000
27	-0.2548782	1.0597878	0.4490000	1.0000000
28	-0.2648248	1.0603291	0.4470000	1.0000000
29	-0.2747658	1.0608921	0.4450000	1.0000000
30	-0.2847019	1.0614773	0.4430000	1.0000000
31	-0.2946312	1.0620836	0.4410000	1.0000000
32	-0.3045552	1.0627123	0.4390000	1.0000000
33	-0.3144731	1.0633632	0.4370000	1.0000000
34	-0.3243849	1.0640364	0.4350000	1.0000000
35	-0.3342904	1.0647319	0.4330000	1.0000000
36	-0.3441890	1.0654498	0.4310000	1.0000000
37	-0.3540818	1.0661909	0.4290000	1.0000000
38	-0.3639676	1.0669547	0.4270000	1.0000000
39	-0.3738465	1.0677414	0.4250000	1.0000000
40	-0.3837184	1.0685514	0.4230000	1.0000000
41	-0.3935833	1.0693848	0.4210000	1.0000000
42	-0.4034412	1.0702420	0.4190000	1.0000000
43	-0.4132914	1.0711226	0.4170000	1.0000000
44	-0.4231344	1.0720273	0.4150000	1.0000000
45	-0.4329699	1.0729563	0.4130000	1.0000000
46	-0.4427978	1.0739097	0.4110000	1.0000000
47	-0.4526181	1.0748878	0.4090000	1.0000000
48	-0.4624305	1.0758906	0.4070000	1.0000000
49	-0.4722352	1.0769190	0.4050000	1.0000000
50	-0.4820318	1.0779726	0.4030000	1.0000000
51	-0.4918205	1.0790519	0.4010000	1.0000000
52	-0.5016011	1.0801573	0.3990000	1.0000000
53	-0.5113735	1.0812889	0.3970000	1.0000000
54	-0.5211378	1.0824474	0.3950000	1.0000000
55	-0.5308937	1.0836324	0.3930000	1.0000000
56	-0.5406414	1.0848449	0.3910000	1.0000000
57	-0.5503806	1.0860851	0.3890000	1.0000000
58	-0.5601115	1.0873532	0.3870000	1.0000000
59	-0.5698338	1.0886497	0.3850000	1.0000000
60	-0.5795476	1.0899749	0.3830000	1.0000000

Table 4 (contd)

RECT.ELL.PROBLEM.MU=VARIABLE,E=1.0, (MU=0.5 IS RECTIL.) FAMILY 6				
NR	X0	YDDT0	MASS RATIO	ECC
61	-0.5892529	1.0913295	0.3810000	1.0000000
62	-0.5989496	1.0927137	0.3790000	1.0000000
63	-0.6086376	1.0941278	0.3770000	1.0000000
64	-0.6183170	1.0955725	0.3750000	1.0000000
65	-0.6279877	1.0970482	0.3730000	1.0000000
66	-0.6376498	1.0985555	0.3710000	1.0000000
67	-0.6473030	1.1000946	0.3690000	1.0000000
68	-0.6569476	1.1016664	0.3670000	1.0000000
69	-0.6665833	1.1032712	0.3650000	1.0000000
70	-0.6762103	1.1049098	0.3630000	1.0000000
71	-0.6858286	1.1065826	0.3610000	1.0000000
72	-0.6954380	1.1082903	0.3590000	1.0000000
73	-0.7050386	1.1100337	0.3570000	1.0000000
74	-0.7146305	1.1118133	0.3550000	1.0000000
75	-0.7242135	1.1136298	0.3530000	1.0000000
76	-0.7337878	1.1154840	0.3510000	1.0000000
77	-0.7385716	1.1164254	0.3500000	1.0000000
78	-0.7624579	1.1212802	0.3450000	1.0000000
79	-0.7862893	1.1263905	0.3400000	1.0000000
80	-0.8100663	1.1317705	0.3350000	1.0000000
81	-0.8337889	1.1374358	0.3300000	1.0000000
82	-0.8574577	1.1434038	0.3250000	1.0000000
83	-0.8810729	1.1496934	0.3200000	1.0000000
84	-0.9046351	1.1563259	0.3150000	1.0000000
85	-0.9281448	1.1633246	0.3100000	1.0000000
86	-0.9516025	1.1707156	0.3050000	1.0000000
87	-0.9750087	1.1785277	0.3000000	1.0000000
88	-0.9983642	1.1867933	0.2950000	1.0000000
89	-1.0216696	1.1955485	0.2900000	1.0000000
90	-1.0449255	1.2048340	0.2850000	1.0000000
91	-1.0681326	1.2146951	0.2800000	1.0000000
92	-1.0912918	1.2251836	0.2750000	1.0000000
93	-1.1144036	1.2363577	0.2700000	1.0000000
94	-1.1374689	1.2482838	0.2650000	1.0000000
95	-1.1604884	1.2610378	0.2600000	1.0000000
96	-1.1834629	1.2747065	0.2550000	1.0000000
97	-1.2063931	1.2893905	0.2500000	1.0000000
98	-1.2292798	1.3052059	0.2450000	1.0000000
99	-1.2521239	1.3222887	0.2400000	1.0000000
100	-1.2749260	1.3407982	0.2350000	1.0000000
101	-1.2976870	1.3609228	0.2300000	1.0000000
102	-1.3204076	1.3828872	0.2250000	1.0000000
103	-1.3430885	1.4069615	0.2200000	1.0000000
104	-1.3657305	1.4334735	0.2150000	1.0000000
105	-1.3883343	1.4628257	0.2100000	1.0000000
106	-1.4109007	1.4955178	0.2050000	1.0000000
107	-1.4334302	1.5321793	0.2000000	1.0000000
108	-1.4559237	1.5736156	0.1950000	1.0000000
109	-1.4783821	1.6208768	0.1900000	1.0000000
110	-1.5008043	1.6753471	0.1850000	1.0000000
111	-1.5231925	1.7389361	0.1800000	1.0000000
112	-1.5321383	1.7674878	0.1780000	1.0000000
113	-1.5410783	1.7981249	0.1760000	1.0000000
114	-1.5500127	1.8311020	0.1740000	1.0000000
115	-1.5589428	1.8667278	0.1720000	1.0000000
116	-1.5678681	1.9053503	0.1700000	1.0000000
117	-1.5767853	1.9473569	0.1680000	1.0000000
118	-1.5856986	1.9932978	0.1660000	1.0000000

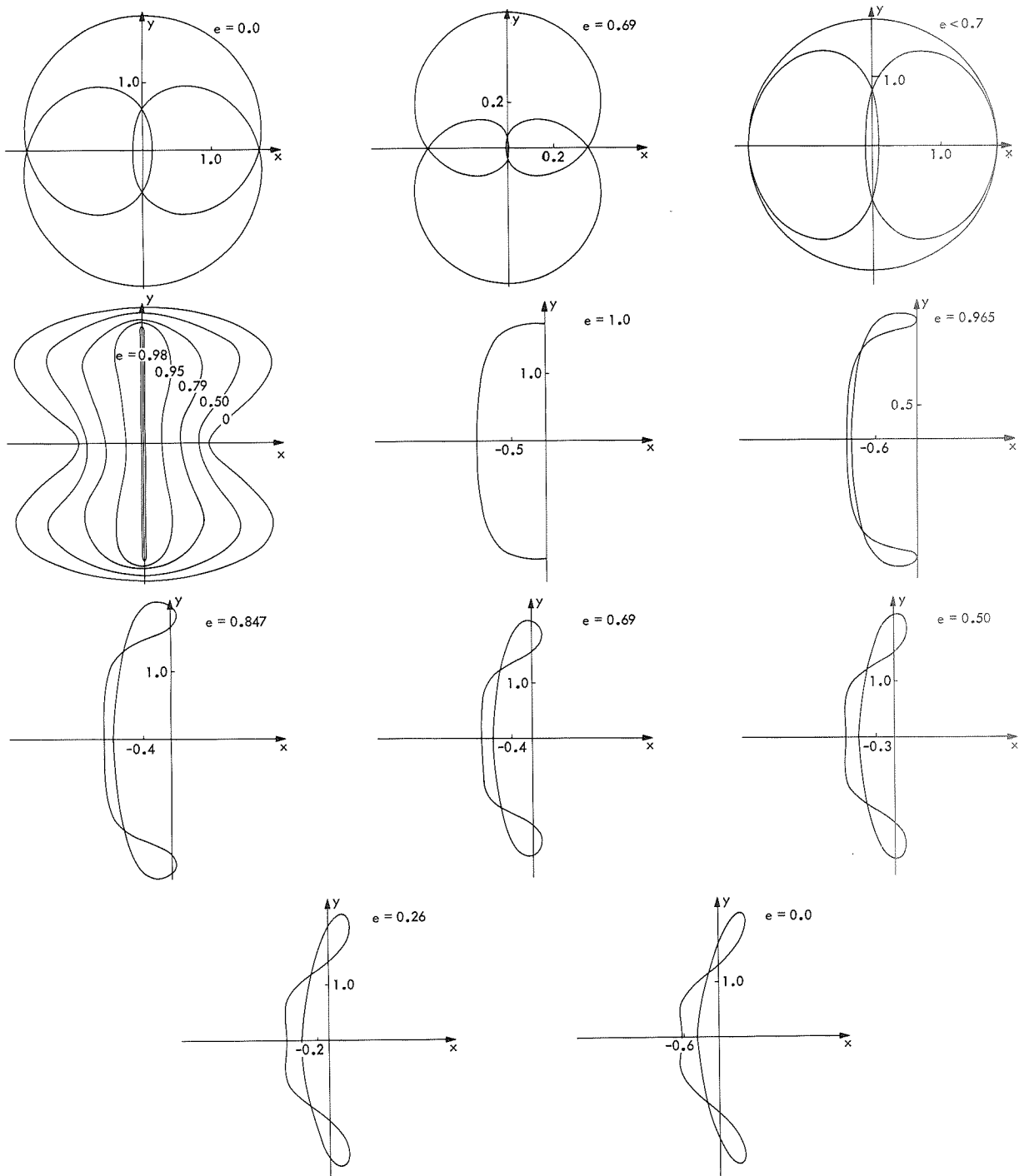


Fig. 8. Family 12A of periodic orbits

Table 5. Initial conditions for family 12A

ELL.PROBLEM.MU=0.5,E=1.0 TO 0.0 (E=1 IS RECT.) FAMILY 12				
NR	X0	YD0T0	MASS RATIO	ECC
1	0.0000000	1.0530847	0.5000000	1.0000000
2	-0.0055055	1.0536413	0.5000000	0.9990000
3	-0.0095377	1.0547550	0.5000000	0.9970000
4	-0.0123140	1.0558693	0.5000000	0.9950000
5	-0.0145724	1.0569843	0.5000000	0.9930000
6	-0.0165242	1.0580999	0.5000000	0.9910000
7	-0.0182709	1.0592159	0.5000000	0.9890000
8	-0.0198647	1.0603329	0.5000000	0.9870000
9	-0.0213403	1.0614508	0.5000000	0.9850000
10	-0.0227208	1.0625691	0.5000000	0.9830000
11	-0.0240226	1.0636881	0.5000000	0.9810000
12	-0.0252577	1.0648078	0.5000000	0.9790000
13	-0.0264356	1.0659282	0.5000000	0.9770000
14	-0.0275644	1.0670495	0.5000000	0.9750000
15	-0.0286476	1.0681712	0.5000000	0.9730000
16	-0.0296924	1.0692937	0.5000000	0.9710000
17	-0.0307019	1.0704170	0.5000000	0.9690000
18	-0.0316795	1.0715410	0.5000000	0.9670000
19	-0.0326282	1.0726657	0.5000000	0.9650000
20	-0.0335496	1.0737910	0.5000000	0.9630000
21	-0.0344479	1.0749174	0.5000000	0.9610000
22	-0.0353229	1.0760444	0.5000000	0.9590000
23	-0.0361770	1.0771721	0.5000000	0.9570000
24	-0.0370116	1.0783006	0.5000000	0.9550000
25	-0.0378280	1.0794298	0.5000000	0.9530000
26	-0.0386280	1.0805601	0.5000000	0.9510000
27	-0.0394105	1.0816907	0.5000000	0.9490000
28	-0.0401786	1.0828223	0.5000000	0.9470000
29	-0.0409324	1.0839547	0.5000000	0.9450000
30	-0.0416727	1.0850879	0.5000000	0.9430000
31	-0.0424003	1.0862219	0.5000000	0.9410000
32	-0.0431149	1.0873565	0.5000000	0.9390000
33	-0.0438194	1.0884923	0.5000000	0.9370000
34	-0.0445121	1.0896288	0.5000000	0.9350000
35	-0.0451943	1.0907661	0.5000000	0.9330000
36	-0.0458664	1.0919042	0.5000000	0.9310000
37	-0.0465289	1.0930431	0.5000000	0.9290000
38	-0.0471828	1.0941831	0.5000000	0.9270000
39	-0.0478264	1.0953236	0.5000000	0.9250000
40	-0.0484623	1.0964651	0.5000000	0.9230000
41	-0.0490899	1.0976075	0.5000000	0.9210000
42	-0.0497096	1.0987507	0.5000000	0.9190000
43	-0.0503217	1.0998948	0.5000000	0.9170000
44	-0.0509258	1.1010396	0.5000000	0.9150000
45	-0.0515242	1.1021857	0.5000000	0.9130000
46	-0.0521150	1.1033324	0.5000000	0.9110000
47	-0.0526992	1.1044801	0.5000000	0.9090000
48	-0.0532770	1.1056286	0.5000000	0.9070000
49	-0.0538486	1.1067781	0.5000000	0.9050000
50	-0.0544148	1.1079287	0.5000000	0.9030000
51	-0.0549738	1.1090798	0.5000000	0.9010000
52	-0.0555278	1.1102320	0.5000000	0.8990000
53	-0.0560763	1.1113852	0.5000000	0.8970000
54	-0.0566195	1.1125393	0.5000000	0.8950000
55	-0.0571574	1.1136944	0.5000000	0.8930000
56	-0.0576902	1.1148503	0.5000000	0.8910000
57	-0.0582182	1.1160073	0.5000000	0.8890000
58	-0.0587413	1.1171652	0.5000000	0.8870000
59	-0.0592598	1.1183241	0.5000000	0.8850000
60	-0.0597738	1.1194839	0.5000000	0.8830000

Table 5 (contd)

ELL.PROBLEM.MU=0.5,E=1.0 TO 0.0 (E=1 IS RECT.) FAMILY 12				
NR	XO	YDOTO	MASS RATIO	ECC
61	-0.0602834	1.1206447	0.5000000	0.8810000
62	-0.0607890	1.1218065	0.5000000	0.8790000
63	-0.0612888	1.1229694	0.5000000	0.8770000
64	-0.0617854	1.1241332	0.5000000	0.8750000
65	-0.0622780	1.1252980	0.5000000	0.8730000
66	-0.0627666	1.1264638	0.5000000	0.8710000
67	-0.0632513	1.1276306	0.5000000	0.8690000
68	-0.0637312	1.1287985	0.5000000	0.8670000
69	-0.0642094	1.1299673	0.5000000	0.8650000
70	-0.0646830	1.1311372	0.5000000	0.8630000
71	-0.0651530	1.1323082	0.5000000	0.8610000
72	-0.0656195	1.1334802	0.5000000	0.8590000
73	-0.0660825	1.1346532	0.5000000	0.8570000
74	-0.0665432	1.1358272	0.5000000	0.8550000
75	-0.0669987	1.1370024	0.5000000	0.8530000
76	-0.0674518	1.1381786	0.5000000	0.8510000
77	-0.0679018	1.1393559	0.5000000	0.8490000
78	-0.0683487	1.1405343	0.5000000	0.8470000
79	-0.0687926	1.1417138	0.5000000	0.8450000
80	-0.0692324	1.1428944	0.5000000	0.8430000
81	-0.0696713	1.1440759	0.5000000	0.8410000
82	-0.0698891	1.1446672	0.5000000	0.8400000
83	-0.0709677	1.1476275	0.5000000	0.8350000
84	-0.0720289	1.1505949	0.5000000	0.8300000
85	-0.0730738	1.1535693	0.5000000	0.8250000
86	-0.0741024	1.1565510	0.5000000	0.8200000
87	-0.0751157	1.1595399	0.5000000	0.8150000
88	-0.0761143	1.1625362	0.5000000	0.8100000
89	-0.0770985	1.1655401	0.5000000	0.8050000
90	-0.0780689	1.1685514	0.5000000	0.8000000
91	-0.0790258	1.1715706	0.5000000	0.7950000
92	-0.0799699	1.1745974	0.5000000	0.7900000
93	-0.0809014	1.1776321	0.5000000	0.7850000
94	-0.0818206	1.1806748	0.5000000	0.7800000
95	-0.0827279	1.1837256	0.5000000	0.7750000
96	-0.0836236	1.1867845	0.5000000	0.7700000
97	-0.0845082	1.1898517	0.5000000	0.7650000
98	-0.0853816	1.1929273	0.5000000	0.7600000
99	-0.0862444	1.1960114	0.5000000	0.7550000
100	-0.0870968	1.1991041	0.5000000	0.7500000
101	-0.0879389	1.2022054	0.5000000	0.7450000
102	-0.0887711	1.2053155	0.5000000	0.7400000
103	-0.0895934	1.2084345	0.5000000	0.7350000
104	-0.0904064	1.2115624	0.5000000	0.7300000
105	-0.0912099	1.2146995	0.5000000	0.7250000
106	-0.0920042	1.2178458	0.5000000	0.7200000
107	-0.0935661	1.2241663	0.5000000	0.7100000
108	-0.0950933	1.2305249	0.5000000	0.7000000
109	-0.0965871	1.2369225	0.5000000	0.6900000
110	-0.0980485	1.2433599	0.5000000	0.6800000
111	-0.0994787	1.2498380	0.5000000	0.6700000
112	-0.1008790	1.2563577	0.5000000	0.6600000
113	-0.1022474	1.2629201	0.5000000	0.6500000
114	-0.1035882	1.2695259	0.5000000	0.6400000
115	-0.1049006	1.2761761	0.5000000	0.6300000
116	-0.1061855	1.2828717	0.5000000	0.6200000
117	-0.1074432	1.2896137	0.5000000	0.6100000
118	-0.1086732	1.2964032	0.5000000	0.6000000
119	-0.1098796	1.3032409	0.5000000	0.5900000
120	-0.1110591	1.3101281	0.5000000	0.5800000

Table 5 (contd)

ELL.PROBLEM.MU=0.5,E=1.0 TO 0.0 (E=1 IS RECT.) FAMILY 12				
NR	X0	YDDTO	MASS RATIO	ECC
121	-0.1122135	1.3170658	0.5000000	0.5700000
122	-0.1133429	1.3240550	0.5000000	0.5600000
123	-0.1144479	1.3310968	0.5000000	0.5500000
124	-0.1155300	1.3381924	0.5000000	0.5400000
125	-0.1165856	1.3453429	0.5000000	0.5300000
126	-0.1176189	1.3525494	0.5000000	0.5200000
127	-0.1196155	1.3671351	0.5000000	0.5000000
128	-0.1215201	1.3819593	0.5000000	0.4800000
129	-0.1233338	1.3970321	0.5000000	0.4600000
130	-0.1250580	1.4123639	0.5000000	0.4400000
131	-0.1266935	1.4279656	0.5000000	0.4200000
132	-0.1282409	1.4438486	0.5000000	0.4000000
133	-0.1297006	1.4600249	0.5000000	0.3800000
134	-0.1310729	1.4765068	0.5000000	0.3600000
135	-0.1323581	1.4933075	0.5000000	0.3400000
136	-0.1335557	1.5104406	0.5000000	0.3200000
137	-0.1346660	1.5279203	0.5000000	0.3000000
138	-0.1356886	1.5457617	0.5000000	0.2800000
139	-0.1366232	1.5639806	0.5000000	0.2600000
140	-0.1374693	1.5825935	0.5000000	0.2400000
141	-0.1382262	1.6016178	0.5000000	0.2200000
142	-0.1388937	1.6210719	0.5000000	0.2000000
143	-0.1394707	1.6409752	0.5000000	0.1800000
144	-0.1399566	1.6613480	0.5000000	0.1600000
145	-0.1403505	1.6822119	0.5000000	0.1400000
146	-0.1406515	1.7035896	0.5000000	0.1200000
147	-0.1408588	1.7255053	0.5000000	0.1000000
148	-0.1409711	1.7479844	0.5000000	0.0800000
149	-0.1409871	1.7710539	0.5000000	0.0600000
150	-0.1409055	1.7947423	0.5000000	0.0400000
151	-0.1407288	1.8190812	0.5000000	0.0200000
152	-0.1404511	1.8441018	0.5000000	0.0000000

The velocity $\dot{y}_0$ has also been restricted to be below some parabolic or approximate escape velocity defined by

$$\dot{y}_0 = \left(\frac{2}{x_0} \right)^{1/2} \quad (177)$$

Figure 9 shows these initial conditions. Imposing the above restrictions, the shaded area is obtained. This area was then swept with regular intervals of 0.05 for x_0 as well as for $\dot{y}_0$. The 13 points on Fig. 9 correspond to the isolated periodic orbits that have been found. Eleven of these orbits (3-13) form a single sequence, all of them having one loop around one primary and an increasing number of loops around the other primary.

The initial conditions of the 13 periodic orbits are given in Table 6. A printout of the initial and final conditions is presented in Table 7. All 13 orbits have the period $T = 2\pi$. In Fig. 10, the actual orbits are shown referred to the barycentric coordinate system. Figure 11 shows the six simplest orbits referred to a coordinate system centered at one primary $m_1 = 1 - \mu$ (geocentric) and

at the other primary $m_2 = \mu$ (selenocentric). In this particular problem, with equal masses $1 - \mu = \mu = 0.5$, the word *geocentric* is used for the coordinate system centered

Table 6. Initial conditions for 13 periodic orbits

Orbit number	x_0	$\dot{y}_0$
1	0.579152	1.680050
2	0.574506	0.346101
3	0.682309	0.981417
4	0.492110	1.192088
5	0.396211	1.347039
6	0.336523	1.473383
7	0.295148	1.581601
8	0.264484	1.677079
9	0.240690	1.763014
10	0.221596	1.841481
11	0.205876	1.913916
12	0.192667	1.981358
13	0.171617	2.104198

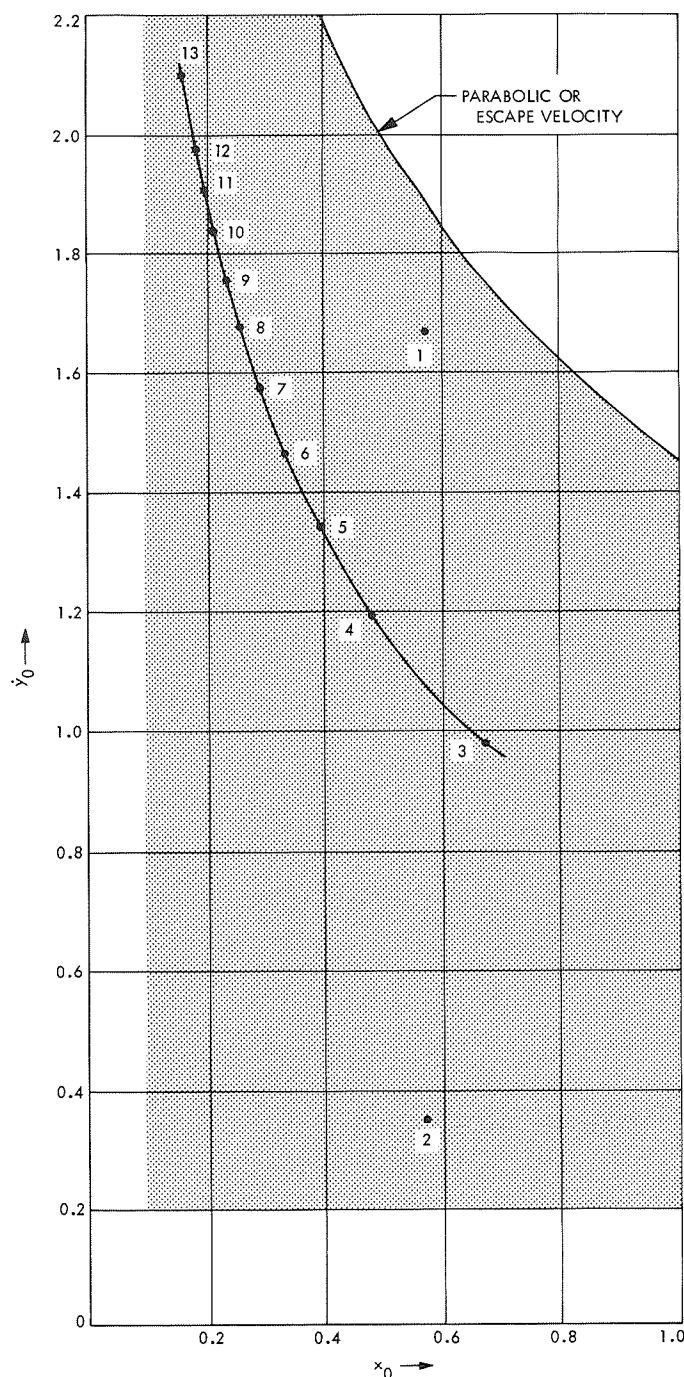


Fig. 9. Diagram of initial conditions for periodic orbits

at the primary shown on the left side of Fig. 11; similarly, *selenocentric* refers to the primary shown on the right side. When geocentric or selenocentric coordinates are used, it can be seen that cusps are present in the form of the orbits. These cusps are due to the accelerated translational motion of the frame of reference rather than the motion of the satellite.

The 13 periodic orbits that have been computed are isolated as far as e and μ are concerned. When e and μ are varied, however, families of periodic orbits are generated in the neighborhood of $e = 1$, $\mu = 0.5$. To verify the existence of these families, the following four families (5P, 9P, 4P, and 3P—Fig. 12) have been initiated:

- (1) 5P—In the neighborhood of orbit 1: 22 periodic orbits with $e = 1.0$ and $\mu = 0.5$ to 0.479 (Table 8).
- (2) 9P—In the neighborhood of orbit 2: 12 periodic orbits with $\mu = 0.5$ and $e = 1.0$ to 0.988 (Table 9).
- (3) 4P—In the neighborhood of orbit 2: 26 periodic orbits with $e = 1.0$ and $\mu = 0.5$ to 0.458 (Table 10).
- (4) 3P—In the neighborhood of orbit 3: 23 periodic orbits with $e = 1.0$ and $\mu = 0.5$ to 0.466 (Table 11).

C. Family 7P

This family (Fig. 13) begins with one of the periodic orbits described in Ref. 6—an orbit of family C of retrograde satellite orbits around the smaller primary m_2 in the circular restricted problem. The orbit that has been chosen is close to orbit 87 in Ref. 6 because this orbit has a period of 2π . The initial conditions, in rotating axes, may be given as follows:

$$x_0 = 0.15212027 \quad (178a)$$

$$\dot{y}_0 = 3.16076559 \quad (178b)$$

$$\mu = 0.012155 \quad (178c)$$

$$e = 0.0 \quad (178d)$$

Family 7P thus belongs to the earth-moon mass ratio, and has the eccentricity e as the variable parameter. The initial and final conditions are listed in Table 12.

About 130 orbits have been computed in this family, with eccentricities e from 0 to 0.50. No orbits with higher eccentricity have been obtained as yet because there is a collision with the larger primary just above this value of e . Although the orbits of this family belong to the class of "satellite orbits" in the circular problem, they are all quite large in shape; in fact, they come closer to the larger primary m_1 than to m_2 .

Good stability information has been obtained for the eccentricities up to 0.35 only. The orbits are all unstable, and belong to region 6 in the stability diagram (see Fig. 2).

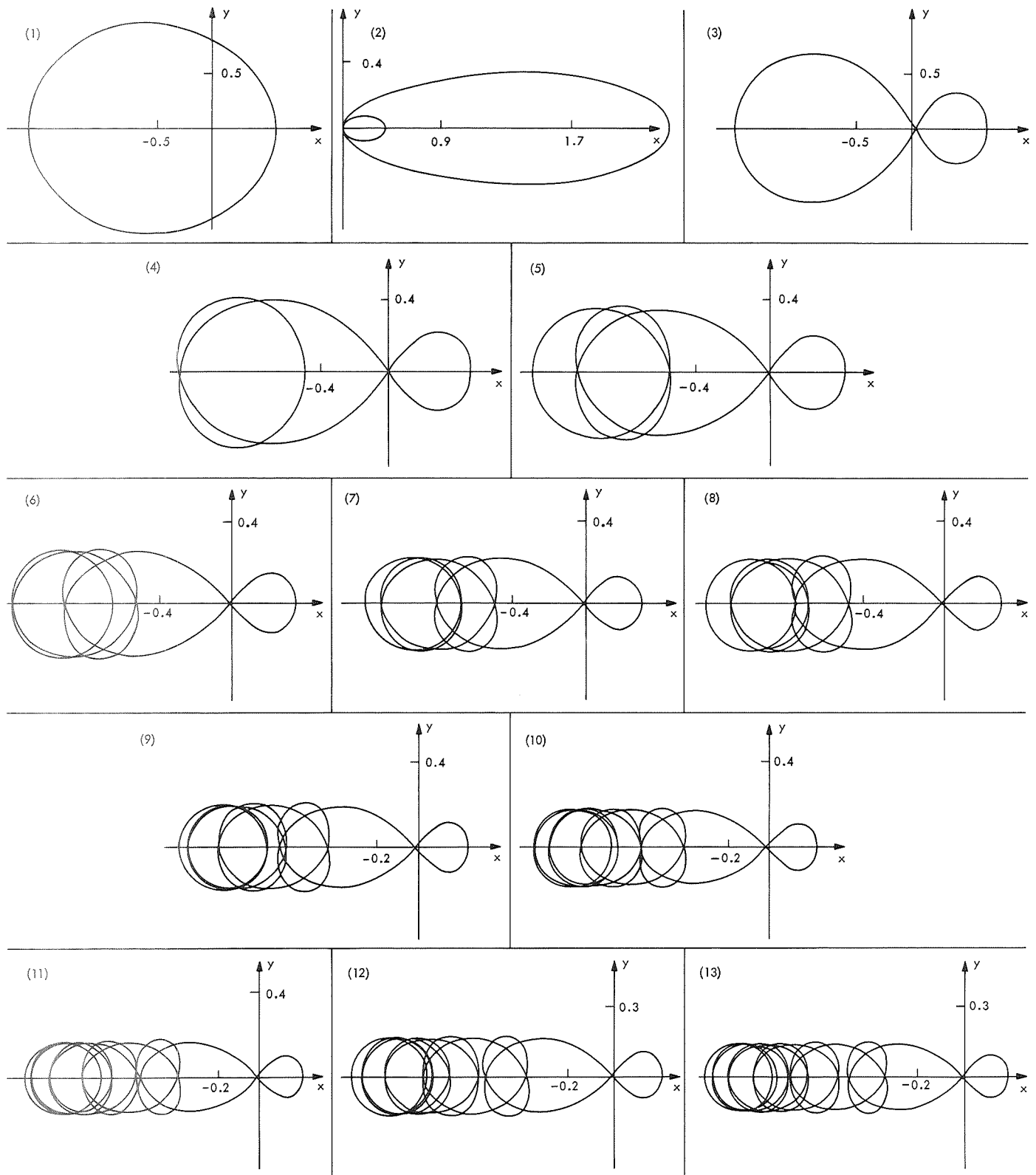


Fig. 10. The 13 periodic orbits in the barycentric (inertial) coordinate system

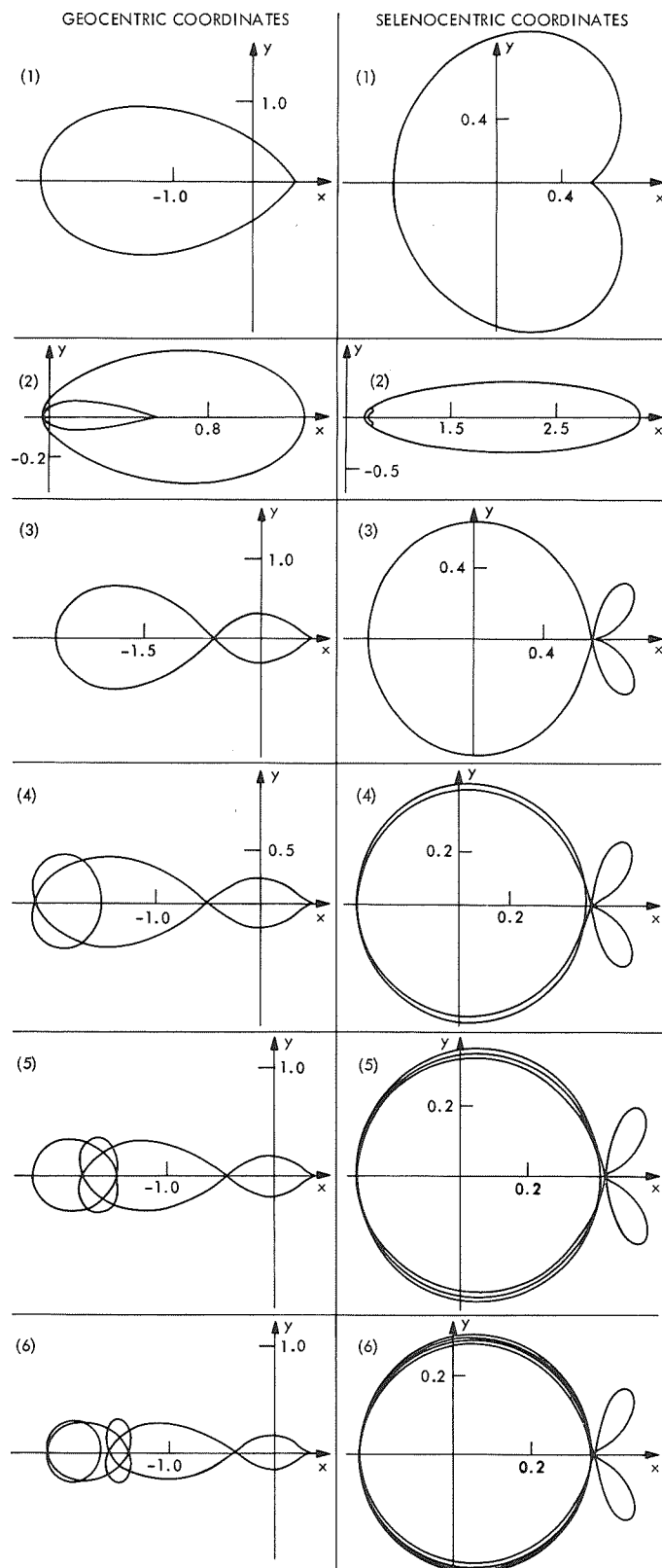


Fig. 11. The first six periodic orbits in the geocentric and the selenocentric coordinate systems

Table 7. Initial and final conditions for 13 periodic orbits

RECTILINEAR ELLIPTIC RESTRICTED THREE-BODY PROBLEM 1 HALF REV. PERI.						
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC
1	0.5745062	0.3461014	2.2888544	0.2107393	0.5000000	1.0000000
2	0.5791522	1.6800505	-1.6589334	-1.0153199	0.5000000	1.0000000
3	0.6823097	0.9814177	-1.6020894	0.9510169	0.5000000	1.0000000
4	0.4921103	1.1920888	-0.5101565	-0.9672414	0.5000000	1.0000000
5	0.3962118	1.3470394	-1.2921322	1.4104449	0.5000000	1.0000000
6	0.3365234	1.4733837	-0.6546279	-1.0962909	0.5000000	1.0000000
7	0.2951483	1.5816018	-1.2049788	1.7100109	0.5000000	1.0000000
8	0.2644841	1.6770793	-0.7239846	-1.2009274	0.5000000	1.0000000
9	0.2406901	1.7630140	-1.1620703	1.9393112	0.5000000	1.0000000
10	0.2215965	1.8414813	-0.7662645	-1.2895975	0.5000000	1.0000000
11	0.2058762	1.9139168	-1.1360074	2.1282105	0.5000000	1.0000000
12	0.1926678	1.9813582	-0.7952567	-1.3671887	0.5000000	1.0000000
13	0.1716180	2.1041985	-0.8166100	-1.4366001	0.5000000	1.0000000

Table 8. Initial and final conditions for family 5P

RECT. ELL. PROBLEM. MU=VARIABLE, E=1.0, FAMILY 5 (PERIAPSIS)						
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC
1	0.5791522	1.6800505	-1.6589334	-1.0153199	0.5000000	1.0000000
2	0.5798281	1.6787450	-1.6577717	-1.0153041	0.4990000	1.0000000
3	0.5805048	1.6774394	-1.6566089	-1.0152883	0.4980000	1.0000000
4	0.5811814	1.6761353	-1.6554461	-1.0152713	0.4970000	1.0000000
5	0.5818578	1.6748327	-1.6542834	-1.0152530	0.4960000	1.0000000
6	0.5825344	1.6735311	-1.6531205	-1.0152337	0.4950000	1.0000000
7	0.5832109	1.6722311	-1.6519577	-1.0152131	0.4940000	1.0000000
8	0.5838874	1.6709325	-1.6507949	-1.0151914	0.4930000	1.0000000
9	0.5845639	1.6696351	-1.6496321	-1.0151686	0.4920000	1.0000000
10	0.5852403	1.6683392	-1.6484693	-1.0151446	0.4910000	1.0000000
11	0.5859166	1.6670446	-1.6473065	-1.0151194	0.4900000	1.0000000
12	0.5865931	1.6657512	-1.6461436	-1.0150931	0.4890000	1.0000000
13	0.5872694	1.6644593	-1.6449808	-1.0150656	0.4880000	1.0000000
14	0.5879457	1.6631688	-1.6438180	-1.0150370	0.4870000	1.0000000
15	0.5886221	1.6618792	-1.6426551	-1.0150074	0.4860000	1.0000000
16	0.5892983	1.6605914	-1.6414924	-1.0149764	0.4850000	1.0000000
17	0.5899745	1.6593047	-1.6403296	-1.0149444	0.4840000	1.0000000
18	0.5906507	1.6580193	-1.6391668	-1.0149113	0.4830000	1.0000000
19	0.5913268	1.6567353	-1.6380040	-1.0148770	0.4820000	1.0000000
20	0.5920030	1.6554525	-1.6368412	-1.0148416	0.4810000	1.0000000
21	0.5926790	1.6541710	-1.6356785	-1.0148051	0.4800000	1.0000000
22	0.5933551	1.6528909	-1.6345158	-1.0147675	0.4790000	1.0000000

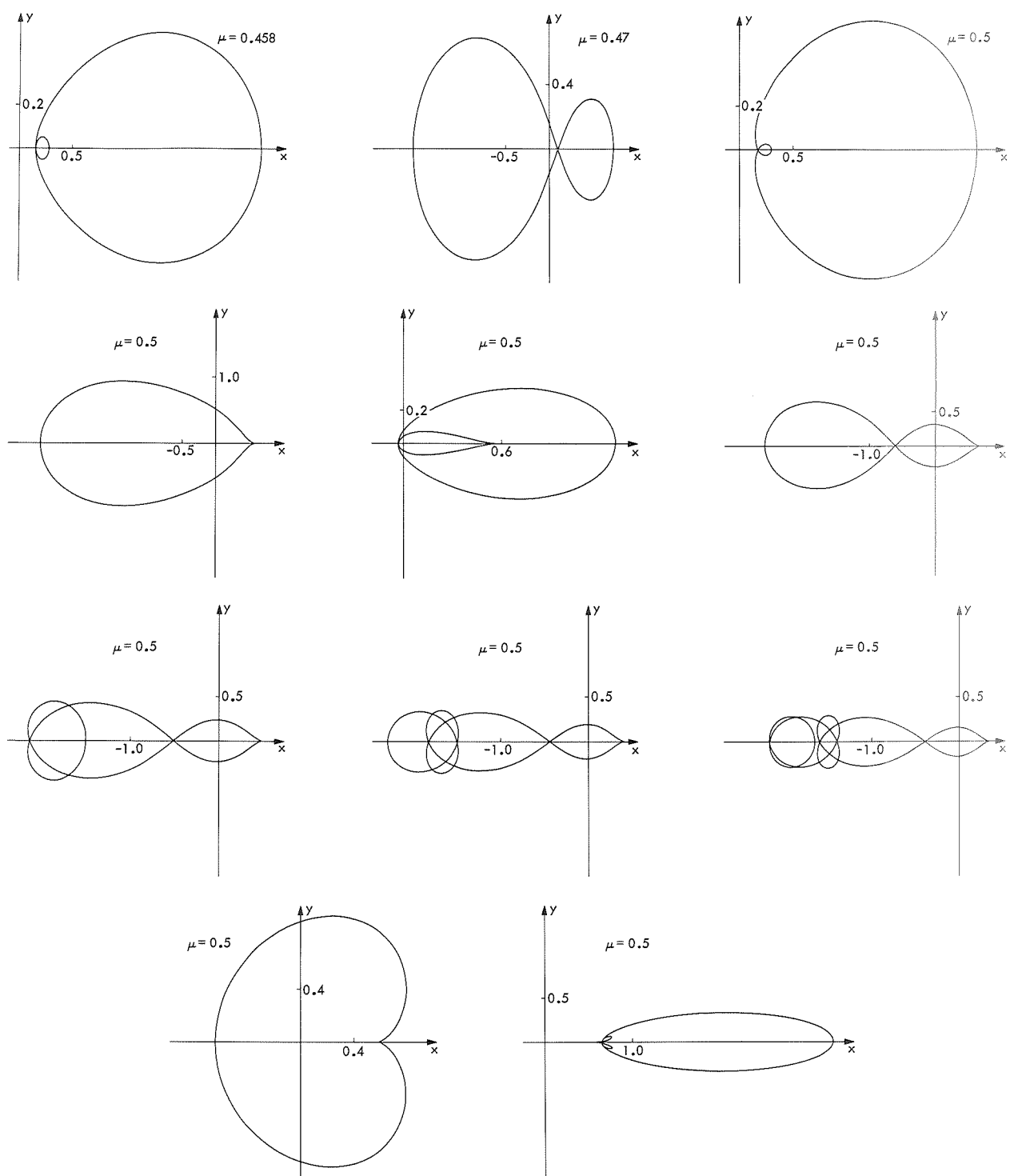


Fig. 12. Families 5P, 9P, 4P, and 3P of periodic orbits

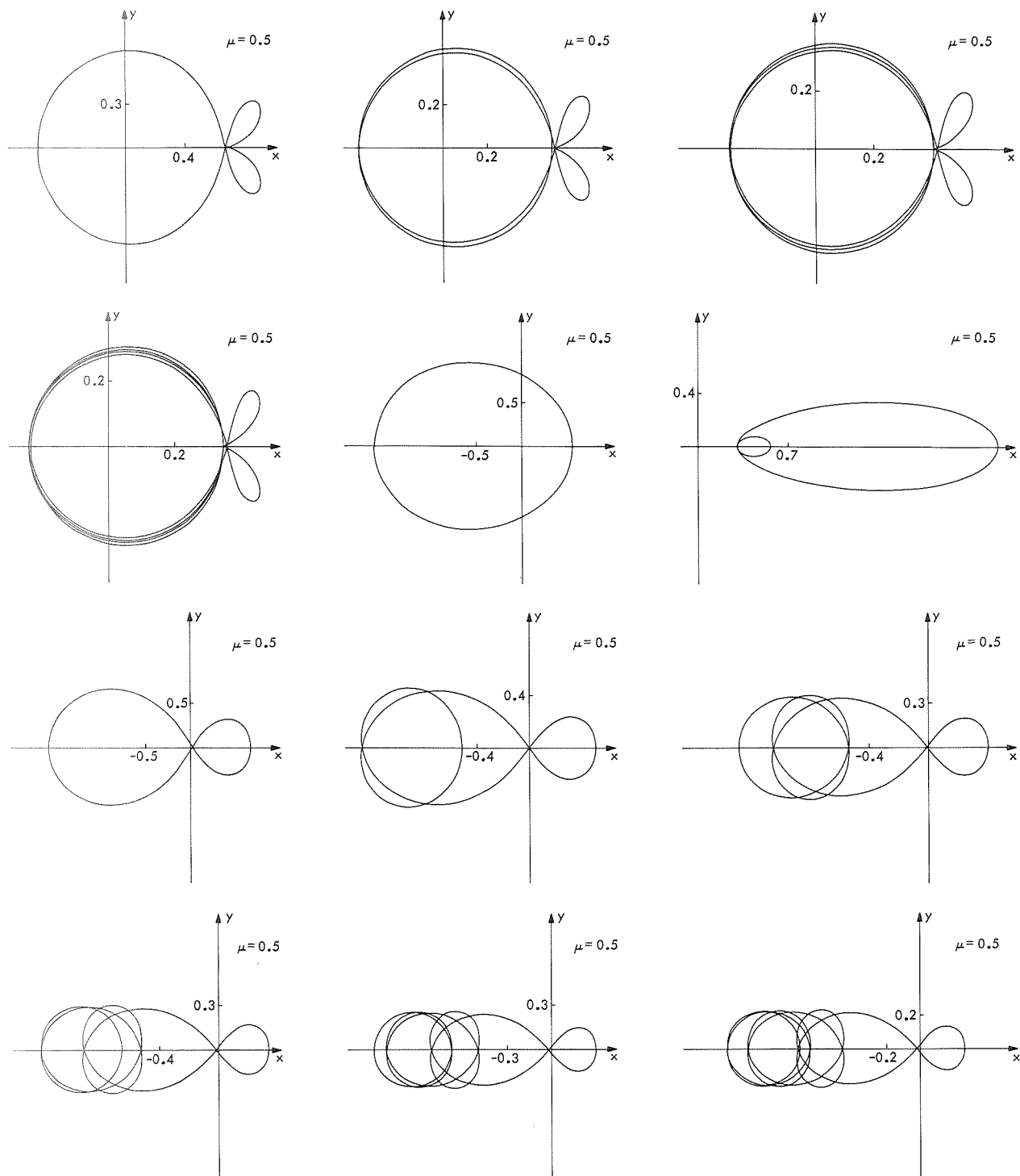


Fig. 12 (contd)

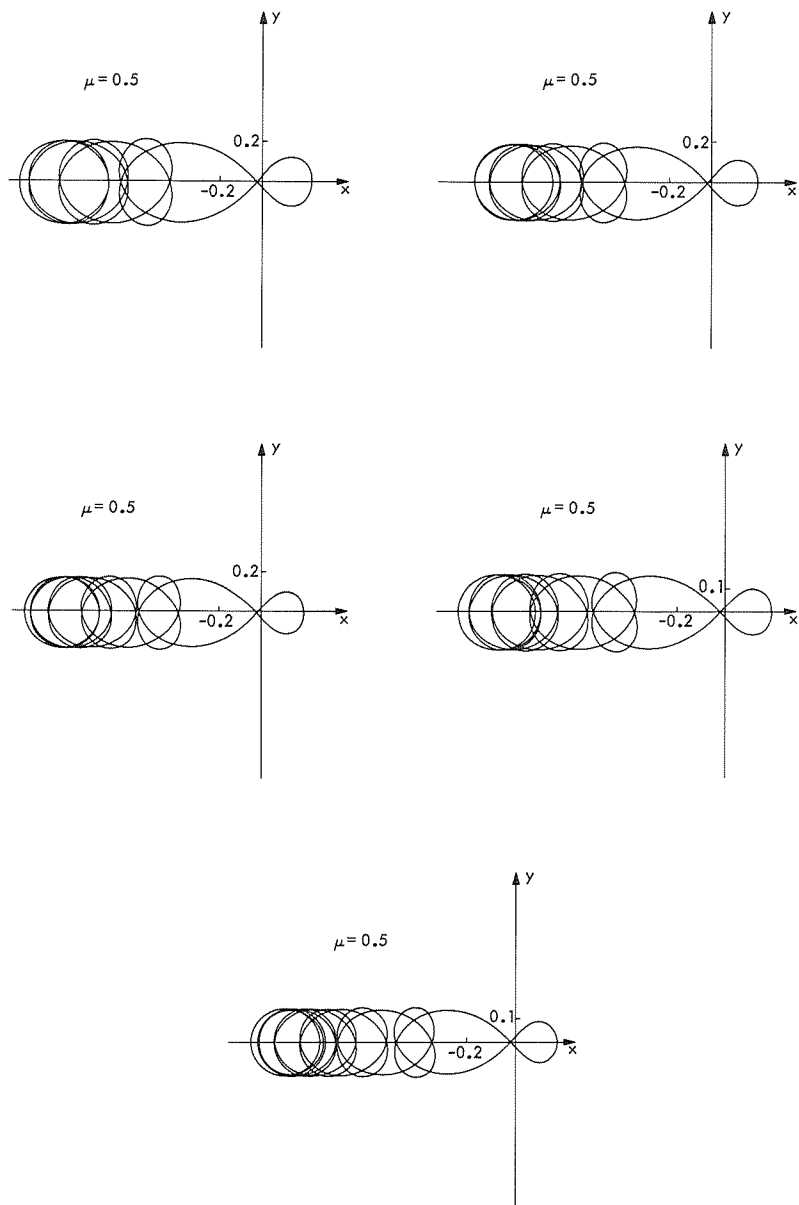


Fig. 12 (contd)

Table 9. Initial and final conditions for family 9P

ELL.RESTR.PROBLEM. C E VARIABLE FROM 1.0 MU=0.5 PERI. 1 HALF REV. FAM 9						
NR	X0	YDOTO	X1	YDOT1	MASS RATIO	ECC
1	0.5745062	0.3461014	2.2888544	0.2107393	0.5000000	1.0000000
2	0.5554261	0.3536558	2.2735915	0.2485901	0.5000000	0.9990000
3	0.5391346	0.3588893	2.2672086	0.2649357	0.5000000	0.9980000
4	0.5226942	0.3634967	2.2623948	0.2776792	0.5000000	0.9970000
5	0.5057545	0.3677720	2.2584234	0.2885861	0.5000000	0.9960000
6	0.4880889	0.3718801	2.2550035	0.2983681	0.5000000	0.9950000
7	0.4694702	0.3759493	2.2519789	0.3074152	0.5000000	0.9940000
8	0.4496181	0.3801073	2.2492501	0.3159891	0.5000000	0.9930000
9	0.4281412	0.3845098	2.2467439	0.3243015	0.5000000	0.9920000
10	0.4044309	0.3893839	2.2443955	0.3325661	0.5000000	0.9910000
11	0.3774115	0.3951273	2.2421329	0.3410634	0.5000000	0.9900000
12	0.2984375	0.4150494	2.2371638	0.3619545	0.5000000	0.9880000

Table 10. Initial and final conditions for family 4P

RECT.ELL.PROBLEM.MU=VARIABLE,E=1.0, FAMILY 4 (PERIAPSIS)						
NR	X0	YDOTO	X1	YDOT1	MASS RATIO	ECC
1	0.5745062	0.3461014	2.2888544	0.2107393	0.5000000	1.0000000
2	0.5674326	0.3542697	2.2889281	0.2144953	0.4990000	1.0000000
3	0.5603530	0.3624437	2.2889941	0.2181885	0.4980000	1.0000000
4	0.5532670	0.3706281	2.2890527	0.2218205	0.4970000	1.0000000
5	0.5461754	0.3788288	2.2891041	0.2253941	0.4960000	1.0000000
6	0.5390791	0.3870518	2.2891482	0.2289118	0.4950000	1.0000000
7	0.5319790	0.3953027	2.2891852	0.2323760	0.4940000	1.0000000
8	0.5248759	0.4035872	2.2892151	0.2357887	0.4930000	1.0000000
9	0.5177707	0.4119105	2.2892379	0.2391520	0.4920000	1.0000000
10	0.5035575	0.4286952	2.2892623	0.2457375	0.4900000	1.0000000
11	0.4893471	0.4456986	2.2892590	0.2521460	0.4880000	1.0000000
12	0.4751470	0.4629622	2.2892282	0.2583889	0.4860000	1.0000000
13	0.4609653	0.4805276	2.2891704	0.2644764	0.4840000	1.0000000
14	0.4468103	0.4984367	2.2890859	0.2704176	0.4820000	1.0000000
15	0.4326905	0.5167328	2.2889751	0.2762204	0.4800000	1.0000000
16	0.4186150	0.5354604	2.2888385	0.2818918	0.4780000	1.0000000
17	0.4045927	0.5546663	2.2886764	0.2874382	0.4760000	1.0000000
18	0.3906333	0.5743994	2.2884891	0.2928654	0.4740000	1.0000000
19	0.3767464	0.5947116	2.2882772	0.2981786	0.4720000	1.0000000
20	0.3629422	0.6156580	2.2880409	0.3033823	0.4700000	1.0000000
21	0.3492310	0.6372978	2.2877808	0.3084808	0.4680000	1.0000000
22	0.3356234	0.6596942	2.2874972	0.3134780	0.4660000	1.0000000
23	0.3221304	0.6829159	2.2871903	0.3183777	0.4640000	1.0000000
24	0.3087633	0.7070368	2.2868607	0.3231828	0.4620000	1.0000000
25	0.2955335	0.7321377	2.2865088	0.3278966	0.4600000	1.0000000
26	0.2824529	0.7583063	2.2861349	0.3325218	0.4580000	1.0000000

Table 11. Initial and final conditions for family 3P

RECT.ELL.PROBLEM.MU=VARIABLE,E=1.0, FAMILY 3 (PERIAPSIS)						
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC
1	0.6823097	0.9814177	-1.6020894	0.9510169	0.5000000	1.0000000
2	0.6836911	0.9788757	-1.6002956	0.9519198	0.4990000	1.0000000
3	0.6850722	0.9763395	-1.5985012	0.9528221	0.4980000	1.0000000
4	0.6864528	0.9738089	-1.5967064	0.9537237	0.4970000	1.0000000
5	0.6878332	0.9712840	-1.5949112	0.9546246	0.4960000	1.0000000
6	0.6892132	0.9687647	-1.5931155	0.9555248	0.4950000	1.0000000
7	0.6905929	0.9662509	-1.5913194	0.9564243	0.4940000	1.0000000
8	0.6919722	0.9637427	-1.5895228	0.9573231	0.4930000	1.0000000
9	0.6947297	0.9587429	-1.5859284	0.9591187	0.4910000	1.0000000
10	0.6933511	0.9612401	-1.5877258	0.9582213	0.4920000	1.0000000
11	0.6961080	0.9562512	-1.5841305	0.9600155	0.4900000	1.0000000
12	0.6988633	0.9512840	-1.5805335	0.9618069	0.4880000	1.0000000
13	0.7016172	0.9463383	-1.5769350	0.9635956	0.4860000	1.0000000
14	0.7043695	0.9414138	-1.5733348	0.9653814	0.4840000	1.0000000
15	0.7071203	0.9365104	-1.5697331	0.9671644	0.4820000	1.0000000
16	0.7098694	0.9316278	-1.5661298	0.9689446	0.4800000	1.0000000
17	0.7126169	0.9267658	-1.5625252	0.9707218	0.4780000	1.0000000
18	0.7153627	0.9219241	-1.5589190	0.9724963	0.4760000	1.0000000
19	0.7181068	0.9171026	-1.5553115	0.9742678	0.4740000	1.0000000
20	0.7208490	0.9123010	-1.5517027	0.9760364	0.4720000	1.0000000
21	0.7235895	0.9075192	-1.5480925	0.9778020	0.4700000	1.0000000
22	0.7263281	0.9027568	-1.5444811	0.9795647	0.4680000	1.0000000
23	0.7290649	0.8980138	-1.5408685	0.9813245	0.4660000	1.0000000

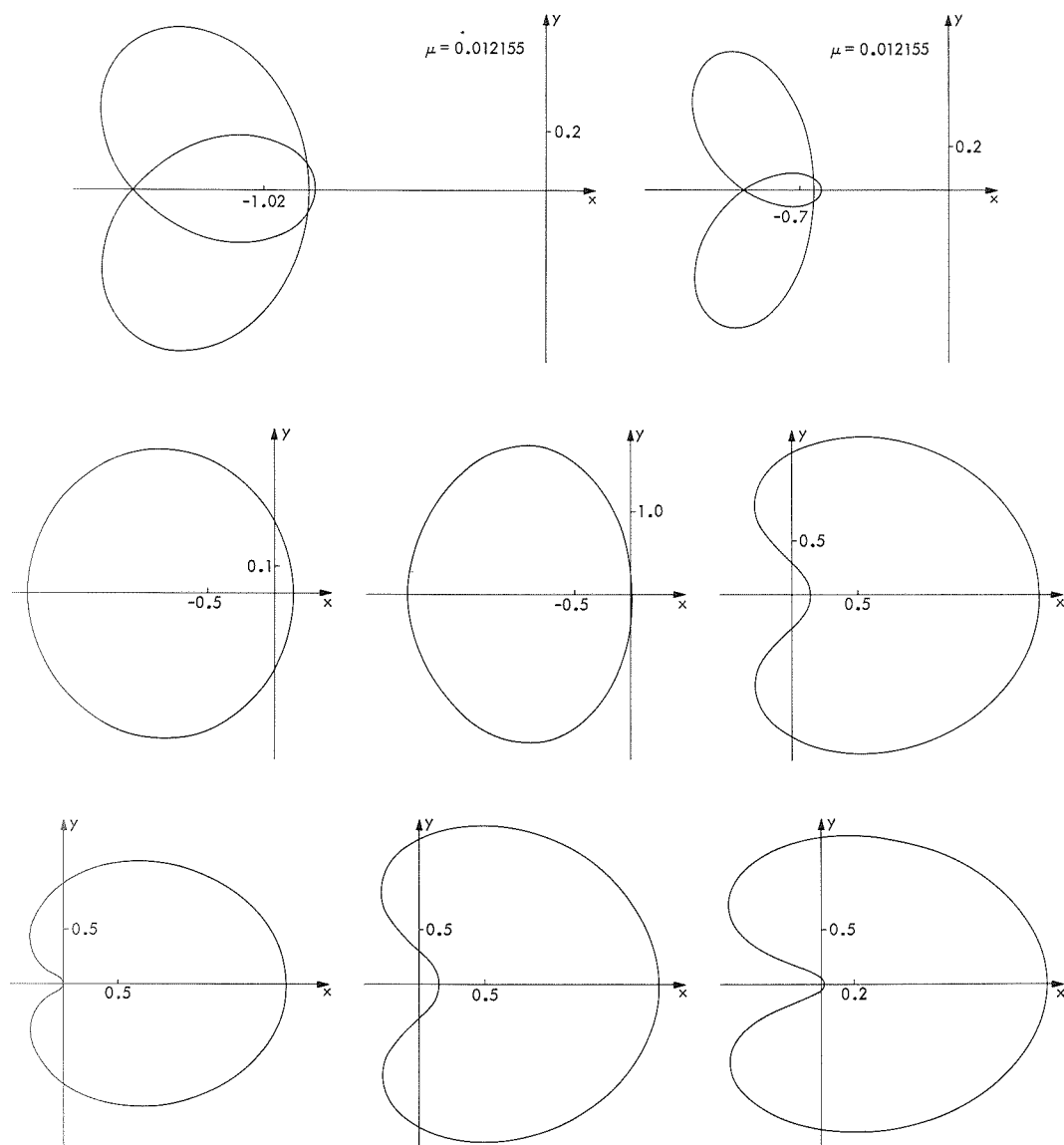


Fig. 13. Family 7P of periodic orbits

Table 12. Initial and final conditions for family 7P

ELLIPTIC PROBLEM MU=0.012155 PS 1 HALF REV PERIAPSIS FAMILY 7						
NR	X0	YD0T0	X1	YD0T1	MASS RATIO	ECC
1	0.1520965	3.1608994	1.8428499	-1.5495568	0.0121550	0.0001000
2	0.1520728	3.1610334	1.8427078	-1.5494262	0.0121550	0.0002000
3	0.1520491	3.1611673	1.8425657	-1.5492956	0.0121550	0.0003000
4	0.1520253	3.1613014	1.8424237	-1.5491650	0.0121550	0.0004000
5	0.1520059	3.1613835	1.8422775	-1.5490262	0.0121550	0.0005000
6	0.1519822	3.1615176	1.8421355	-1.5488957	0.0121550	0.0006000
7	0.1519584	3.1616519	1.8419936	-1.5487653	0.0121550	0.0007000
8	0.1519347	3.1617862	1.8418517	-1.5486348	0.0121550	0.0008000
9	0.1519109	3.1619206	1.8417098	-1.5485044	0.0121550	0.0009000
10	0.1518872	3.1620550	1.8415679	-1.5483741	0.0121550	0.0010000
11	0.1518634	3.1621895	1.8414260	-1.5482437	0.0121550	0.0011000
12	0.1518354	3.1623755	1.8412882	-1.5481215	0.0121550	0.0012000
13	0.1517879	3.1626453	1.8410047	-1.5478611	0.0121550	0.0014000
14	0.1517404	3.1629148	1.8407213	-1.5476007	0.0121550	0.0016000
15	0.1516928	3.1631846	1.8404379	-1.5473404	0.0121550	0.0018000
16	0.1516453	3.1634547	1.8401546	-1.5470802	0.0121550	0.0020000
17	0.1515978	3.1637250	1.8398714	-1.5468201	0.0121550	0.0022000
18	0.1515503	3.1639956	1.8395883	-1.5465602	0.0121550	0.0024000
19	0.1515028	3.1642664	1.8393054	-1.5463004	0.0121550	0.0026000
20	0.1514552	3.1645375	1.8390225	-1.5460407	0.0121550	0.0028000
21	0.1514077	3.1648088	1.8387397	-1.5457812	0.0121550	0.0030000
22	0.1513602	3.1650804	1.8384570	-1.5455218	0.0121550	0.0032000
23	0.1513126	3.1653523	1.8381745	-1.5452625	0.0121550	0.0034000
24	0.1512651	3.1656244	1.8378920	-1.5450033	0.0121550	0.0036000
25	0.1512175	3.1658967	1.8376096	-1.5447442	0.0121550	0.0038000
26	0.1511700	3.1661693	1.8373273	-1.5444853	0.0121550	0.0040000
27	0.1511224	3.1664422	1.8370452	-1.5442264	0.0121550	0.0042000
28	0.1510748	3.1667153	1.8367631	-1.5439678	0.0121550	0.0044000
29	0.1510272	3.1669887	1.8364811	-1.5437092	0.0121550	0.0046000
30	0.1509797	3.1672623	1.8361993	-1.5434507	0.0121550	0.0048000
31	0.1509321	3.1675362	1.8359175	-1.5431924	0.0121550	0.0050000
32	0.1508845	3.1678104	1.8356358	-1.5429342	0.0121550	0.0052000
33	0.1508369	3.1680847	1.8353543	-1.5426761	0.0121550	0.0054000
34	0.1508131	3.1682220	1.8352135	-1.5425471	0.0121550	0.0055000
35	0.1507893	3.1683594	1.8350728	-1.5424181	0.0121550	0.0056000
36	0.1506941	3.1689094	1.8345102	-1.5419026	0.0121550	0.0050000
37	0.1505751	3.1695985	1.8338074	-1.5412588	0.0121550	0.0065000
38	0.1504560	3.1702892	1.8331053	-1.5406158	0.0121550	0.0070000
39	0.1503370	3.1709814	1.8324038	-1.5399736	0.0121550	0.0075000
40	0.1502179	3.1716753	1.8317030	-1.5393321	0.0121550	0.0080000
41	0.1500987	3.1723708	1.8310027	-1.5386913	0.0121550	0.0085000
42	0.1499795	3.1730680	1.8303031	-1.5380514	0.0121550	0.0090000
43	0.1498604	3.1737667	1.8296041	-1.5374122	0.0121550	0.0095000
44	0.1497411	3.1744671	1.8289057	-1.5367737	0.0121550	0.0100000
45	0.1495029	3.1758696	1.8275105	-1.5354985	0.0121550	0.0110000
46	0.1492642	3.1772816	1.8261181	-1.5342269	0.0121550	0.0120000
47	0.1490255	3.1787003	1.8247281	-1.5329582	0.0121550	0.0130000
48	0.1487866	3.1801254	1.8233405	-1.5316926	0.0121550	0.0140000
49	0.1485477	3.1815572	1.8219554	-1.5304299	0.0121550	0.0150000
50	0.1483086	3.1829956	1.8205727	-1.5291703	0.0121550	0.0160000
51	0.1480694	3.1844405	1.8191925	-1.5279136	0.0121550	0.0170000
52	0.1478301	3.1858921	1.8178146	-1.5266599	0.0121550	0.0180000
53	0.1475907	3.1873503	1.8164392	-1.5254092	0.0121550	0.0190000
54	0.1473511	3.1888152	1.8150662	-1.5241614	0.0121550	0.0200000
55	0.1471115	3.1902868	1.8136956	-1.5229166	0.0121550	0.0210000
56	0.1468718	3.1917651	1.8123273	-1.5216747	0.0121550	0.0220000
57	0.1466319	3.1932502	1.8109615	-1.5204358	0.0121550	0.0230000
58	0.1463920	3.1947419	1.8095980	-1.5191998	0.0121550	0.0240000
59	0.1461519	3.1962405	1.8082369	-1.5179667	0.0121550	0.0250000
60	0.1459117	3.1977458	1.8068781	-1.5167366	0.0121550	0.0260000

Table 12 (contd)

ELLIPTIC PROBLEM MU=0.012155 PS 1 HALF REV PERIAPSIS FAMILY 7						
NR	X0	YD0T0	X1	YD0T1	MASS RATIO	ECC
61	0.1456714	3.1992580	1.8055217	-1.5155093	0.0121550	0.0270000
62	0.1454310	3.2007770	1.8041677	-1.5142850	0.0121550	0.0280000
63	0.1451905	3.2023028	1.8028160	-1.5130636	0.0121550	0.0290000
64	0.1449498	3.2038355	1.8014666	-1.5118451	0.0121550	0.0300000
65	0.1437451	3.2116029	1.7947543	-1.5057956	0.0121550	0.0350000
66	0.1425372	3.2195499	1.7881000	-1.4998187	0.0121550	0.0400000
67	0.1413268	3.2276708	1.7815017	-1.4939117	0.0121550	0.0450000
68	0.1401138	3.2359716	1.7749588	-1.4880748	0.0121550	0.0500000
69	0.1388980	3.2444577	1.7684709	-1.4823074	0.0121550	0.0550000
70	0.1376795	3.2531297	1.7620370	-1.4766087	0.0121550	0.0600000
71	0.1364583	3.2619923	1.7556563	-1.4709780	0.0121550	0.0650000
72	0.1352343	3.2710488	1.7493281	-1.4654149	0.0121550	0.0700000
73	0.1340076	3.2803021	1.7430516	-1.4599185	0.0121550	0.0750000
74	0.1327783	3.2897563	1.7368262	-1.4544885	0.0121550	0.0800000
75	0.1315462	3.2994151	1.7306510	-1.4491240	0.0121550	0.0850000
76	0.1303114	3.3092823	1.7245254	-1.4438247	0.0121550	0.0900000
77	0.1290740	3.3193620	1.7184486	-1.4385899	0.0121550	0.0950000
78	0.1278339	3.3296584	1.7124201	-1.4334192	0.0121550	0.1000000
79	0.1265911	3.3401755	1.7064393	-1.4283121	0.0121550	0.1050000
80	0.1253457	3.3509190	1.7005051	-1.4232678	0.0121550	0.1100000
81	0.1240976	3.3618924	1.6946172	-1.4182861	0.0121550	0.1150000
82	0.1228469	3.3731008	1.6887750	-1.4133666	0.0121550	0.1200000
83	0.1215936	3.3845496	1.6829778	-1.4085087	0.0121550	0.1250000
84	0.1203377	3.3962437	1.6772250	-1.4037120	0.0121550	0.1300000
85	0.1190792	3.4081887	1.6715159	-1.3989761	0.0121550	0.1350000
86	0.1178181	3.4203902	1.6658501	-1.3943007	0.0121550	0.1400000
87	0.1165544	3.4328541	1.6602269	-1.3896852	0.0121550	0.1450000
88	0.1152882	3.4455858	1.6546459	-1.3851295	0.0121550	0.1500000
89	0.1140193	3.4585936	1.6491062	-1.3806330	0.0121550	0.1550000
90	0.1127480	3.4718822	1.6436076	-1.3761955	0.0121550	0.1600000
91	0.1114742	3.4854586	1.6381494	-1.3718167	0.0121550	0.1650000
92	0.1101979	3.4993308	1.6327311	-1.3674963	0.0121550	0.1700000
93	0.1089191	3.5135053	1.6273523	-1.3632339	0.0121550	0.1750000
94	0.1076379	3.5279901	1.6220123	-1.3590293	0.0121550	0.1800000
95	0.1063542	3.5427935	1.6167108	-1.3548823	0.0121550	0.1850000
96	0.1050681	3.5579232	1.6114471	-1.3507926	0.0121550	0.1900000
97	0.1037797	3.5733881	1.6062209	-1.3467599	0.0121550	0.1950000
98	0.1024888	3.5891975	1.6010317	-1.3427842	0.0121550	0.2000000
99	0.1011956	3.6053602	1.5958790	-1.3388650	0.0121550	0.2050000
100	0.0999002	3.6218855	1.5907624	-1.3350025	0.0121550	0.2100000
101	0.0986024	3.6387860	1.5856812	-1.3311961	0.0121550	0.2150000
102	0.0973023	3.6560701	1.5806353	-1.3274460	0.0121550	0.2200000
103	0.0960001	3.6737485	1.5756241	-1.3237518	0.0121550	0.2250000
104	0.0946957	3.6918346	1.5706472	-1.3201137	0.0121550	0.2300000
105	0.0920803	3.7292754	1.5607948	-1.3130047	0.0121550	0.2400000
106	0.0894569	3.7684904	1.5510745	-1.3061179	0.0121550	0.2500000
107	0.0868251	3.8096025	1.5414837	-1.2994542	0.0121550	0.2600000
108	0.0841858	3.8527260	1.5320190	-1.2930122	0.0121550	0.2700000
109	0.0815395	3.8979911	1.5226774	-1.2867919	0.0121550	0.2800000
110	0.0788861	3.9455605	1.5134566	-1.2807952	0.0121550	0.2900000
111	0.0762265	3.9955870	1.5043534	-1.2750217	0.0121550	0.3000000
112	0.0735613	4.0482459	1.4953653	-1.2694722	0.0121550	0.3100000
113	0.0708908	4.1037439	1.4864898	-1.2641485	0.0121550	0.3200000
114	0.0682157	4.1622978	1.4777244	-1.2590522	0.0121550	0.3300000
115	0.0655371	4.2241443	1.4690667	-1.2541846	0.0121550	0.3400000
116	0.0628552	4.2895686	1.4605147	-1.2495490	0.0121550	0.3500000
117	0.0601710	4.3588715	1.4520660	-1.2451478	0.0121550	0.3600000
118	0.0574860	4.4323779	1.4437183	-1.2409829	0.0121550	0.3700000
119	0.0548006	4.5104873	1.4354699	-1.2370589	0.0121550	0.3800000
120	0.0521161	4.5936304	1.4273187	-1.2333795	0.0121550	0.3900000

Table 12 (contd)

ELLIPTIC PROBLEM MU=0.012155 PS 1 HALF REV PERIAPSIS FAMILY 7						
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC
121	0.0494341	4.6822788	1.4192626	-1.2299480	0.0121550	0.4000000
122	0.0467555	4.7770100	1.4113001	-1.2267703	0.0121550	0.4100000
123	0.0440821	4.8784508	1.4034294	-1.2238513	0.0121550	0.4200000
124	0.0414156	4.9873240	1.3956485	-1.2211963	0.0121550	0.4300000
125	0.0387577	5.1044787	1.3879562	-1.2188122	0.0121550	0.4400000
126	0.0361103	5.2308875	1.3803507	-1.2167060	0.0121550	0.4500000
127	0.0334761	5.3676731	1.3728305	-1.2148847	0.0121550	0.4600000
128	0.0308572	5.5161770	1.3653942	-1.2133572	0.0121550	0.4700000
129	0.0282564	5.6779674	1.3580405	-1.2121326	0.0121550	0.4800000
130	0.0256766	5.8549100	1.3507681	-1.2112207	0.0121550	0.4900000
131	0.0231214	6.0492417	1.3435756	-1.2106323	0.0121550	0.5000000

However, they are all very close to the line $a_2 = -2a_1 - 2$ in this diagram. This is to be expected because the orbits are essentially two-body orbits around the larger primary m_1 (in the inertial axes, these orbits appear as ellipses), which are only weakly perturbed by the smaller primary. As a consequence, there is still approximately an integral of the motion, and there is a pair of eigenvalues of the fundamental matrix that are close to $+1$. It should be noted here that, in the circular restricted three-body problem, all of the orbits belong to the straight line $a_2 = -2a_1 - 2$.

D. Family 7A

This family (Fig. 14) begins with the same orbit as family 7P (periapsis). About 120 periodic orbits have been computed with eccentricity e from 0.0 to 0.99 (Table 13). Therefore, this is a family that probably exists for all eccentricities. All of the orbits have been computed in rotating (pulsating) axes, using the Nechville transformation. The orbit corresponding to the eccentricity $e=1.0$ would have to be computed in inertial axes.

Good stability information has been obtained only for the orbits with eccentricities below 0.75. All of the orbits are stable, and belong to region 1 (see Fig. 2). They are on a line that is close to the line $a_2 = -2a_1 - 2$. When the eccentricity e increases, the orbits approach the separation point $(-4, 6)$ on the stability diagram.

E. Family 8P

Family 8 of periodic orbits is shown in Fig. 15. Family 8P (Fig. 16) begins at $e = 0.0$ with an orbit that has been taken from Strömberg's problem (circular restricted three-body problem with equal masses). In fact, the starting orbit has been taken from Bartlett's class g (see

Ref. 18). The period 2π and the initial conditions for this orbit may be written as follows (in rotating axes):

$$x_0 = -0.4017933 \quad (179a)$$

$$\dot{y}_0 = +3.1437189 \quad (179b)$$

$$\mu = 0.5 \quad (179c)$$

$$e = 0.0 \quad (179d)$$

About 120 orbits have been obtained in family 8P with eccentricities ranging from 0.0 to 0.075 (Table 14). It is not yet known whether this family can be prolonged all the way to $e = 1.0$. These last orbits, with very high e , would have to be computed in inertial axes; on the other hand, there seems to be an additional difficulty because the satellite has a close approach with one of the primaries when e approaches 0.97.

The stability has been computed only for the orbits with eccentricities up to 0.55. All of these orbits are unstable, and belong to region 6 (even semi-instability) in the stability diagram (see Fig. 2).

F. Family 8A

At $e = 0.0$, this family (Fig. 17) begins at the same orbit as family 8P (periapsis). About 100 periodic orbits have been computed with eccentricities from 0 to 0.85. No orbits with eccentricities higher than 0.85 have yet been obtained because the convergence of the differential correction process was becoming increasingly slow. On the other hand, there seems to be a moderately close approach with one of the primaries for the last orbits that have been computed.

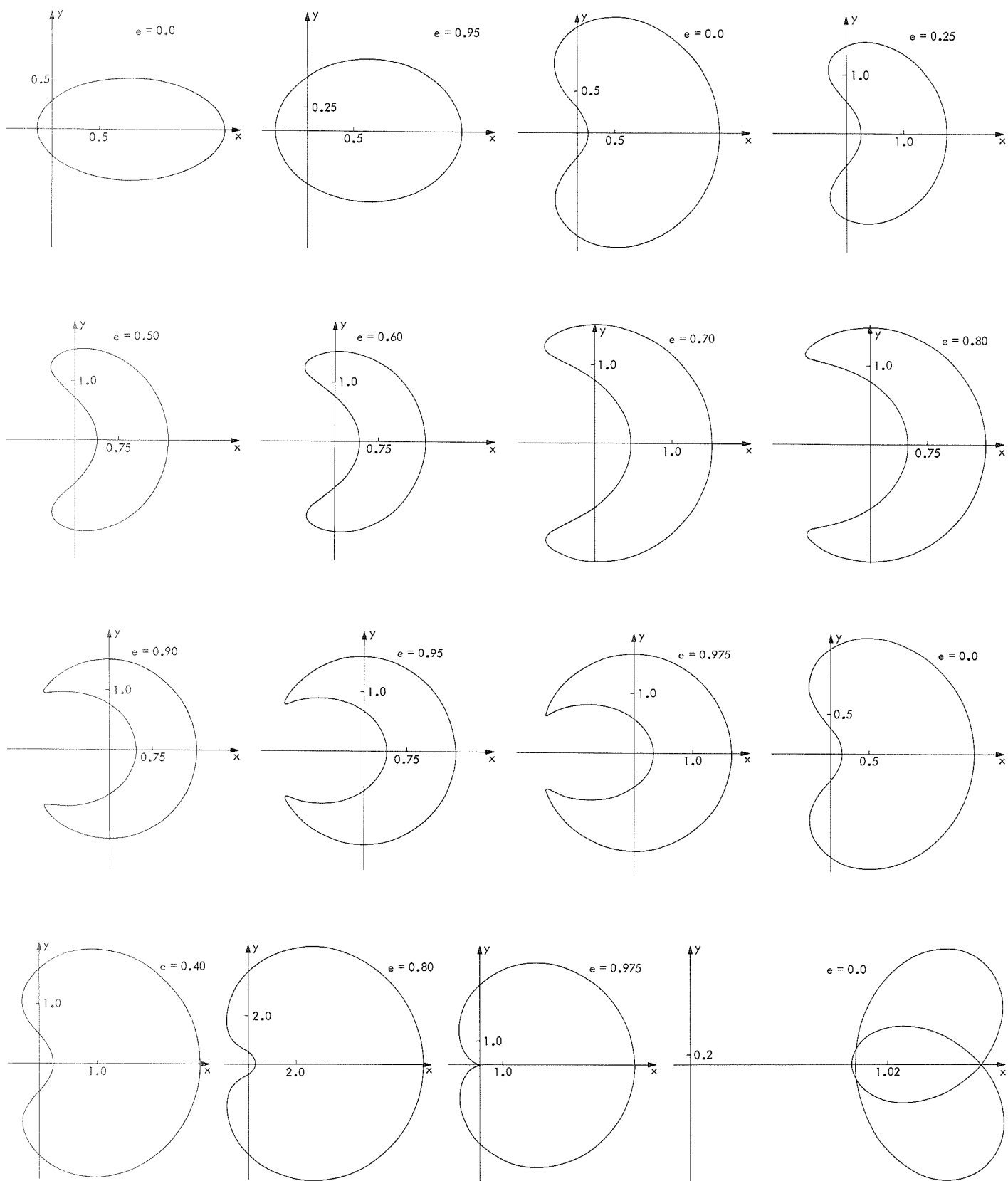


Fig. 14. Family 7A of periodic orbits

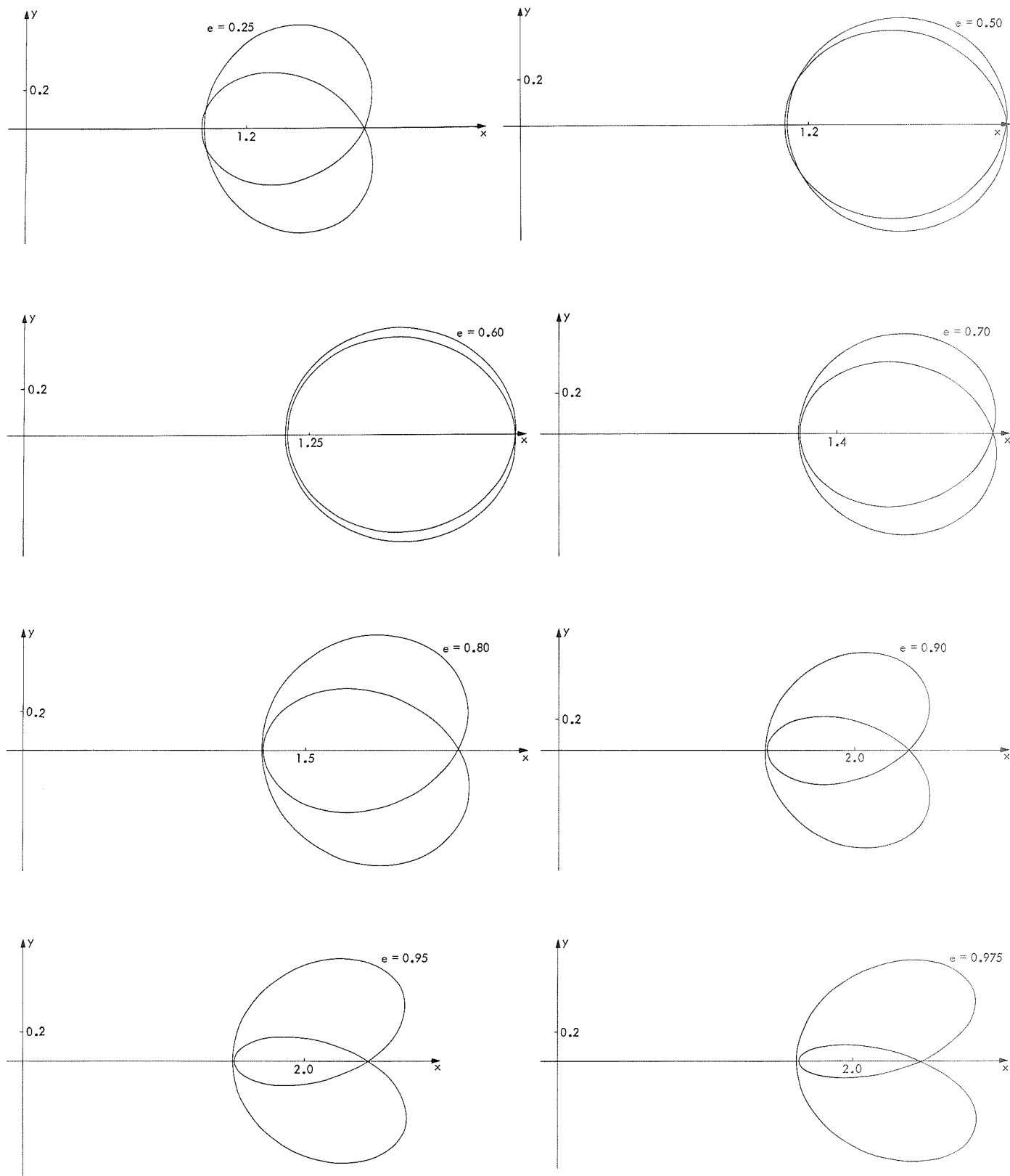


Fig. 14 (contd)

Table 13. Initial and final conditions for family 7A

ELLIPTIC PROBLEM MU=0.012155 E VARIABLE APOAPSIS FAMILY 7							
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC	
1	0.1521203	3.1607656	1.8429920	-1.5496874	0.0121550	0.0000000	
2	0.1527133	3.1574396	1.8465530	-1.5529639	0.0121550	0.0025000	
3	0.1530096	3.1557914	1.8483396	-1.5546093	0.0121550	0.0037500	
4	0.1533057	3.1541530	1.8501301	-1.5562597	0.0121550	0.0050000	
5	0.1536016	3.1525243	1.8519247	-1.5579149	0.0121550	0.0062500	
6	0.1538973	3.1509053	1.8537234	-1.5595751	0.0121550	0.0075000	
7	0.1541929	3.1492960	1.8555261	-1.5612401	0.0121550	0.0087500	
8	0.1544883	3.1476964	1.8573329	-1.5629101	0.0121550	0.0100000	
9	0.1547835	3.1461064	1.8591438	-1.5645850	0.0121550	0.0112500	
10	0.1550785	3.1445259	1.8609589	-1.5662649	0.0121550	0.0125000	
11	0.1553734	3.1429551	1.8627780	-1.5679497	0.0121550	0.0137500	
12	0.1556681	3.1413938	1.8646014	-1.5696395	0.0121550	0.0150000	
13	0.1568450	3.1352432	1.8719365	-1.5764490	0.0121550	0.0200000	
14	0.1580192	3.1292428	1.8793394	-1.5833395	0.0121550	0.0250000	
15	0.1591906	3.1233909	1.8868111	-1.5903119	0.0121550	0.0300000	
16	0.1603591	3.1176858	1.8943528	-1.5973674	0.0121550	0.0350000	
17	0.1615248	3.1121257	1.9019655	-1.6045069	0.0121550	0.0400000	
18	0.1626877	3.1067091	1.9096503	-1.6117315	0.0121550	0.0450000	
19	0.1638478	3.1014346	1.9174086	-1.6190425	0.0121550	0.0500000	
20	0.1650050	3.0963007	1.9252413	-1.6264407	0.0121550	0.0550000	
21	0.1661593	3.0913060	1.9331499	-1.6339276	0.0121550	0.0600000	
22	0.1673108	3.0864492	1.9411355	-1.6415041	0.0121550	0.0650000	
23	0.1684595	3.0817291	1.9491993	-1.6491717	0.0121550	0.0700000	
24	0.1696052	3.0771444	1.9573428	-1.6569314	0.0121550	0.0750000	
25	0.1707481	3.0726941	1.9655672	-1.6647846	0.0121550	0.0800000	
26	0.1718881	3.0683771	1.9738739	-1.6727326	0.0121550	0.0850000	
27	0.1730252	3.0641923	1.9822643	-1.6807768	0.0121550	0.0900000	
28	0.1741594	3.0601389	1.9907398	-1.6889186	0.0121550	0.0950000	
29	0.1752907	3.0562158	1.9993019	-1.6971593	0.0121550	0.1000000	
30	0.1764191	3.0524223	2.0079521	-1.7055004	0.0121550	0.1050000	
31	0.1775445	3.0487575	2.0166919	-1.7139435	0.0121550	0.1100000	
32	0.1786670	3.0452207	2.0255228	-1.7224900	0.0121550	0.1150000	
33	0.1797866	3.0418113	2.0344465	-1.7311415	0.0121550	0.1200000	
34	0.1809032	3.0385284	2.0434646	-1.7398996	0.0121550	0.1250000	
35	0.1820169	3.0353717	2.0525787	-1.7487660	0.0121550	0.1300000	
36	0.1831275	3.0323404	2.0617907	-1.7577424	0.0121550	0.1350000	
37	0.1842352	3.0294342	2.0711022	-1.7668305	0.0121550	0.1400000	
38	0.1853399	3.0266526	2.0805150	-1.7760321	0.0121550	0.1450000	
39	0.1864415	3.0239952	2.0900311	-1.7853489	0.0121550	0.1500000	
40	0.1875401	3.0214616	2.0996522	-1.7947830	0.0121550	0.1550000	
41	0.1886357	3.0190516	2.1093803	-1.8043361	0.0121550	0.1600000	
42	0.1908177	3.0146011	2.1291655	-1.8238077	0.0121550	0.1700000	
43	0.1929873	3.0106425	2.1494033	-1.8437800	0.0121550	0.1800000	
44	0.1951445	3.0071750	2.1701108	-1.8642704	0.0121550	0.1900000	
45	0.1972890	3.0041986	2.1913062	-1.8852970	0.0121550	0.2000000	
46	0.1994207	3.0017143	2.2130087	-1.9068790	0.0121550	0.2100000	
47	0.2015395	2.9997237	2.2352385	-1.9290368	0.0121550	0.2200000	
48	0.2036452	2.9982293	2.2580168	-1.9517915	0.0121550	0.2300000	
49	0.2057375	2.9972341	2.2813659	-1.9751659	0.0121550	0.2400000	
50	0.2078163	2.9967422	2.3053095	-1.9991837	0.0121550	0.2500000	
51	0.2098814	2.9967586	2.3298726	-2.0238701	0.0121550	0.2600000	
52	0.2119325	2.9972889	2.3550814	-2.0492517	0.0121550	0.2700000	
53	0.2139692	2.9983396	2.3809640	-2.0753567	0.0121550	0.2800000	
54	0.2159915	2.9999184	2.4075496	-2.1022151	0.0121550	0.2900000	
55	0.2179989	3.0020336	2.4348697	-2.1298583	0.0121550	0.3000000	
56	0.2199911	3.0046948	2.4629574	-2.1583200	0.0121550	0.3100000	
57	0.2219678	3.0079123	2.4918476	-2.1876358	0.0121550	0.3200000	
58	0.2239287	3.0116978	2.5215779	-2.2178435	0.0121550	0.3300000	
59	0.2258732	3.0160639	2.5521878	-2.2489833	0.0121550	0.3400000	
60	0.2278010	3.0210246	2.5837195	-2.2810981	0.0121550	0.3500000	

Table 13 (contd)

ELLIPTIC PROBLEM MU=0.012155 E VARIABLE APOAPSIS FAMILY 7						
NR	X0	YDOTO	X1	YDOT1	MASS RATIO	ECC
61	0.2297116	3.0265951	2.6162180	-2.3142334	0.0121550	0.3600000
62	0.2316046	3.0327919	2.6497310	-2.3484378	0.0121550	0.3700000
63	0.2334794	3.0396330	2.6843098	-2.3837631	0.0121550	0.3800000
64	0.2353354	3.0471379	2.7200087	-2.4202647	0.0121550	0.3900000
65	0.2371720	3.0553277	2.7568863	-2.4580018	0.0121550	0.4000000
66	0.2389887	3.0642254	2.7950048	-2.4970377	0.0121550	0.4100000
67	0.2407846	3.0738557	2.8344313	-2.5374404	0.0121550	0.4200000
68	0.2425591	3.0842454	2.8752375	-2.5792827	0.0121550	0.4300000
69	0.2443113	3.0954236	2.9175004	-2.6226427	0.0121550	0.4400000
70	0.2460404	3.1074215	2.9613031	-2.6676046	0.0121550	0.4500000
71	0.2477456	3.1202732	3.0067347	-2.7142589	0.0121550	0.4600000
72	0.2494257	3.1340152	3.0538917	-2.7627032	0.0121550	0.4700000
73	0.2510797	3.1486873	3.1028781	-2.8130430	0.0121550	0.4800000
74	0.2527066	3.1643325	3.1538062	-2.8653922	0.0121550	0.4900000
75	0.2543051	3.1809973	3.2067979	-2.9198743	0.0121550	0.5000000
76	0.2558739	3.1987323	3.2619855	-2.9766229	0.0121550	0.5100000
77	0.2574116	3.2175921	3.3195124	-3.0357835	0.0121550	0.5200000
78	0.2589166	3.2376365	3.3795351	-3.0975142	0.0121550	0.5300000
79	0.2603875	3.2589301	3.4422239	-3.1619875	0.0121550	0.5400000
80	0.2618223	3.2815433	3.5077649	-3.2293915	0.0121550	0.5500000
81	0.2632192	3.3055532	3.5763620	-3.2999322	0.0121550	0.5600000
82	0.2645763	3.3310436	3.6482386	-3.3738353	0.0121550	0.5700000
83	0.2658912	3.3581063	3.7236403	-3.4513489	0.0121550	0.5800000
84	0.2671616	3.3868419	3.8028373	-3.5327460	0.0121550	0.5900000
85	0.2683849	3.4173607	3.8861284	-3.6183279	0.0121550	0.6000000
86	0.2695584	3.4497839	3.9738437	-3.7084276	0.0121550	0.6100000
87	0.2706789	3.4842453	4.0663497	-3.8034149	0.0121550	0.6200000
88	0.2717433	3.5208923	4.1640539	-3.9037004	0.0121550	0.6300000
89	0.2727479	3.5598884	4.2674108	-4.0097420	0.0121550	0.6400000
90	0.2736889	3.6014145	4.3769288	-4.1220520	0.0121550	0.6500000
91	0.2745619	3.6456723	4.4931783	-4.2412046	0.0121550	0.6600000
92	0.2753624	3.6928864	4.6168014	-4.3678461	0.0121550	0.6700000
93	0.2760853	3.7433082	4.7485234	-4.5027062	0.0121550	0.6800000
94	0.2767250	3.7972199	4.8891665	-4.6466119	0.0121550	0.6900000
95	0.2772754	3.8549392	5.0396665	-4.8005040	0.0121550	0.7000000
96	0.2777297	3.9168252	5.2010930	-4.9654574	0.0121550	0.7100000
97	0.2780805	3.9832852	5.3746733	-5.1427053	0.0121550	0.7200000
98	0.2783196	4.0547833	5.5618231	-5.3336694	0.0121550	0.7300000
99	0.2784378	4.1318508	5.7641832	-5.5399970	0.0121550	0.7400000
100	0.2784251	4.2150985	5.9836655	-5.7636074	0.0121550	0.7500000
101	0.2782701	4.3052322	6.2225111	-6.0067493	0.0121550	0.7600000
102	0.2779601	4.4030728	6.4833634	-6.2720740	0.0121550	0.7700000
103	0.2774809	4.5095803	6.7693616	-6.5627298	0.0121550	0.7800000
104	0.2768164	4.6258848	7.0842611	-6.8824819	0.0121550	0.7900000
105	0.2759484	4.7533272	7.4325915	-7.2358700	0.0121550	0.8000000
106	0.2748561	4.8935101	7.8198638	-7.6284169	0.0121550	0.8100000
107	0.2735159	5.0483671	8.2528496	-8.0669066	0.0121550	0.8200000
108	0.2719001	5.2202534	8.7399588	-8.5597626	0.0121550	0.8300000
109	0.2699771	5.4120692	9.2917616	-9.1175708	0.0121550	0.8400000
110	0.2677096	5.6274301	9.9217234	-9.7538131	0.0121550	0.8500000
111	0.2650536	5.8709058	10.6472574	-10.4859223	0.0121550	0.8600000
112	0.2619565	6.1483636	11.4912723	-11.3368286	0.0121550	0.8700000
113	0.2583550	6.4674731	12.4845035	-12.3372918	0.0121550	0.8800000
114	0.2541711	6.8384717	13.6691338	-13.5295236	0.0121550	0.8900000
115	0.2493079	7.2753611	15.1046178	-14.9730126	0.0121550	0.9000000
116	0.2436419	7.7978572	16.8774426	-16.7542861	0.0121550	0.9100000
117	0.2404594	8.0997033	17.9287531	-17.8100023	0.0121550	0.9150000
118	0.2370121	8.4347116	19.1183074	-19.0040927	0.0121550	0.9200000
119	0.2332711	8.8088615	20.4745557	-20.3650155	0.0121550	0.9250000
120	0.2292023	9.2297084	22.0342278	-21.9295106	0.0121550	0.9300000

Table 13 (contd)

ELLIPTIC PROBLEM MU=0.012155 E VARIABLE APOAPSIS FAMILY 7						
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC
121	0.2247649	9.7069522	23.8455637-23.7458279	0.0121550	0.9350000	
122	0.2199099	10.2532730	25.9731719-25.8785896	0.0121550	0.9400000	
123	0.2145768	10.8855977	28.5055699-28.4163270	0.0121550	0.9450000	
124	0.2086907	11.6270829	31.5672896-31.4835904	0.0121550	0.9500000	
125	0.2021559	12.5103375	35.3391337-35.2612047	0.0121550	0.9550000	
126	0.1948485	13.5828971	40.0937490-40.0218430	0.0121550	0.9600000	
127	0.1866120	14.9166899	46.2614918-46.1958947	0.0121550	0.9650000	
128	0.1772819	16.6242127	54.5601788-54.5012097	0.0121550	0.9700000	
129	0.1668050	18.8853600	66.2729273-66.2209234	0.0121550	0.9750000	
130	0.1555807	21.9903128	83.9203405-83.8756008	0.0121550	0.9800000	

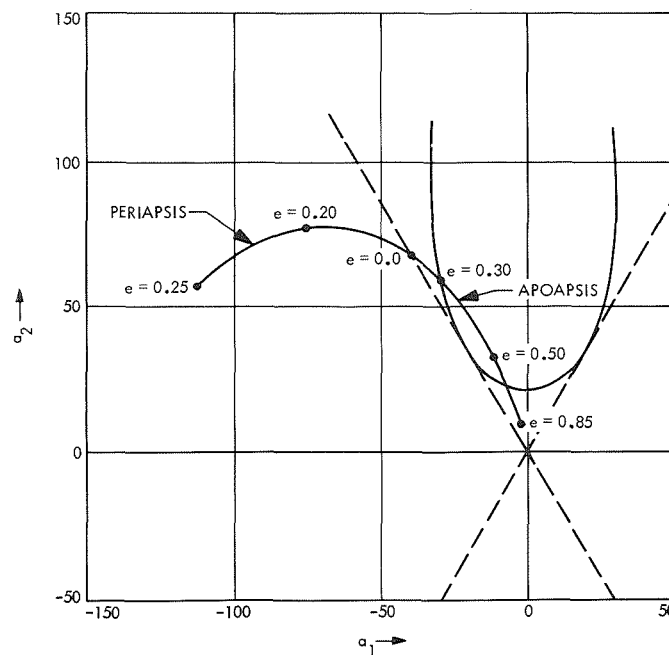


Fig. 15. Family 8 of periodic orbits

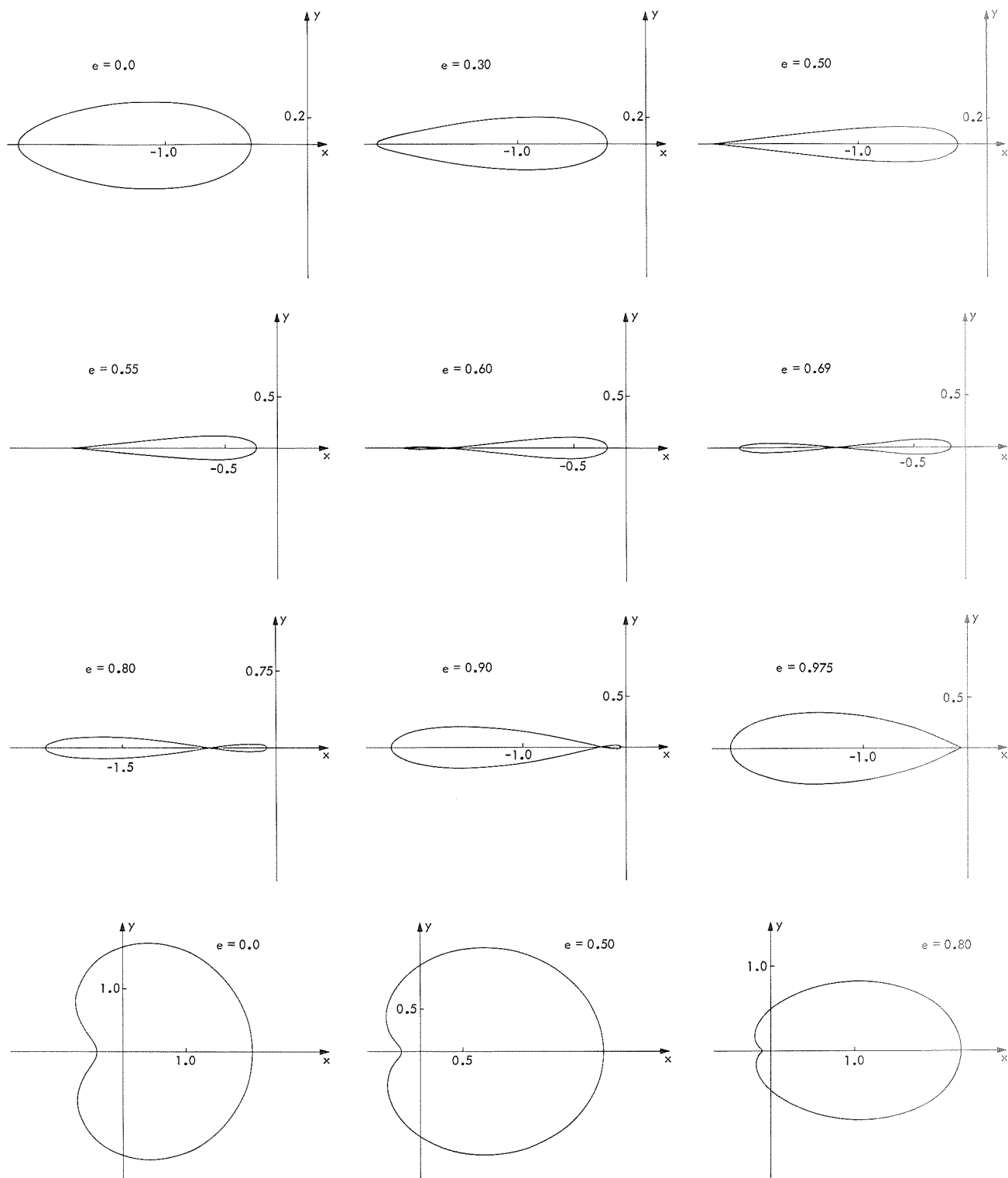


Fig. 16. Family 8P of periodic orbits

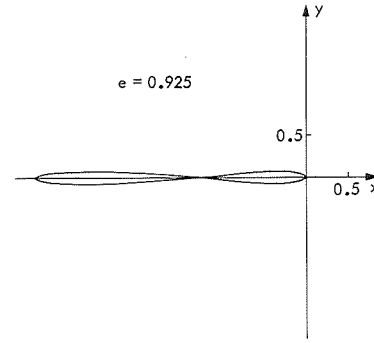
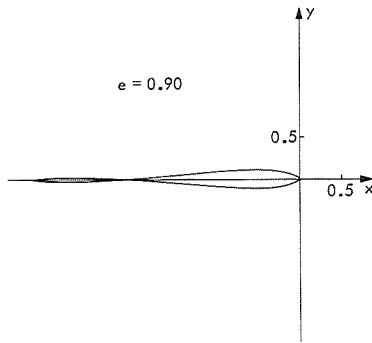
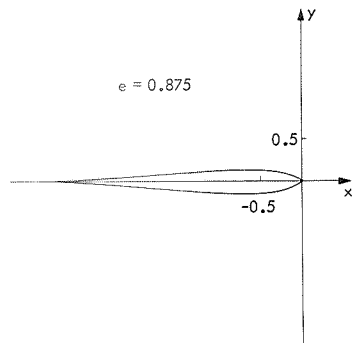
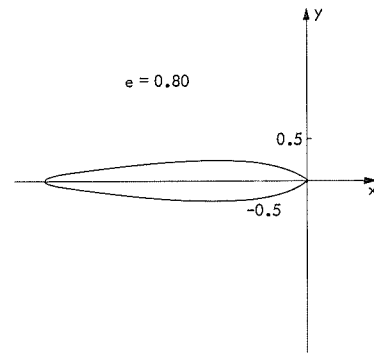
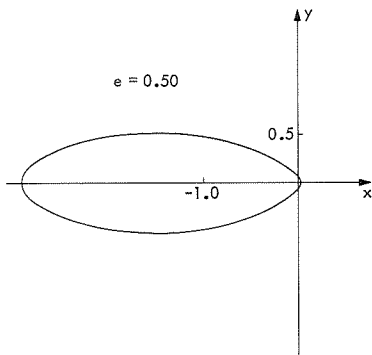
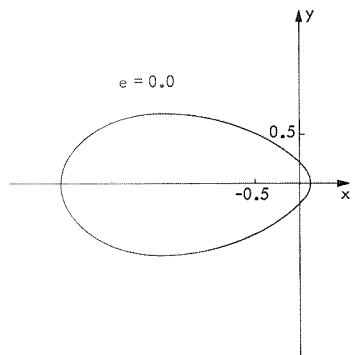
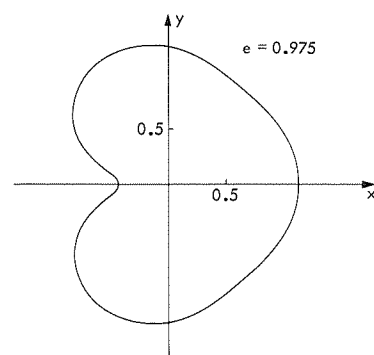
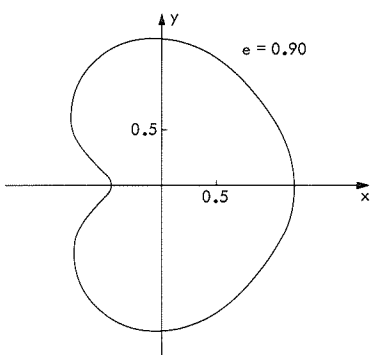
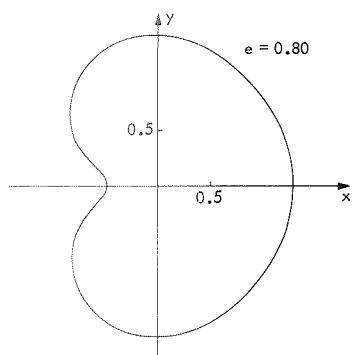
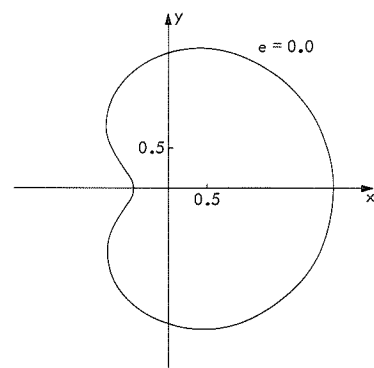
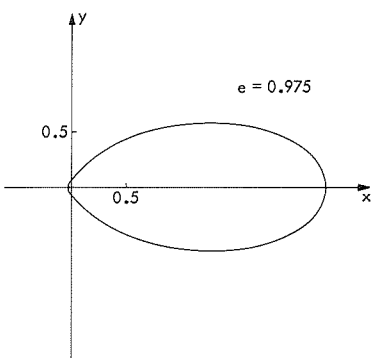
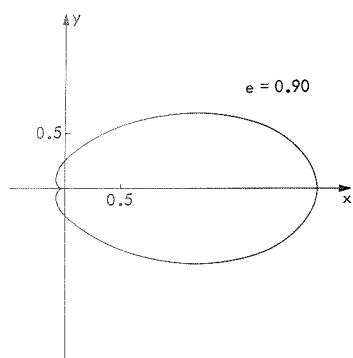


Fig. 16 (contd)

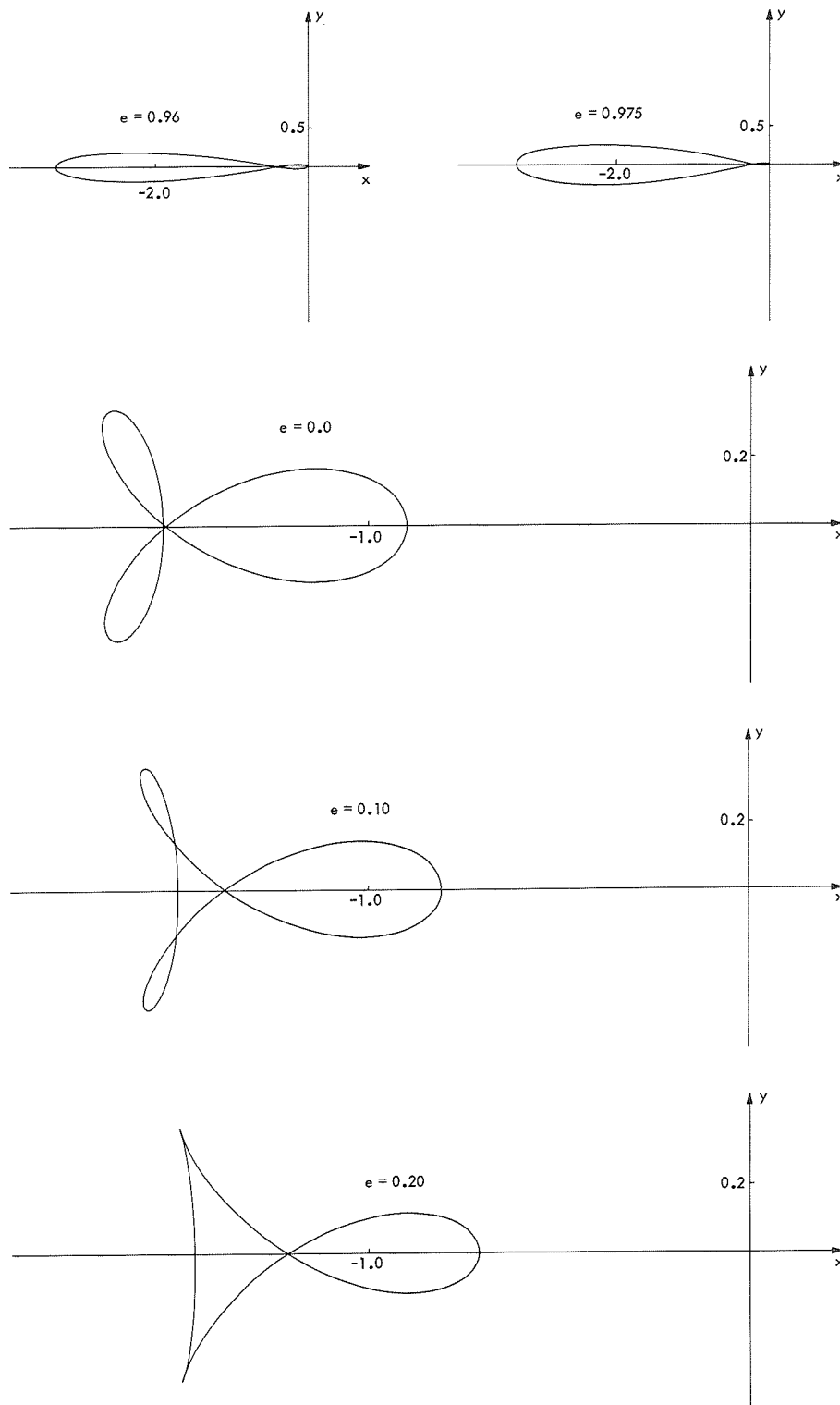


Fig. 16 (contd)

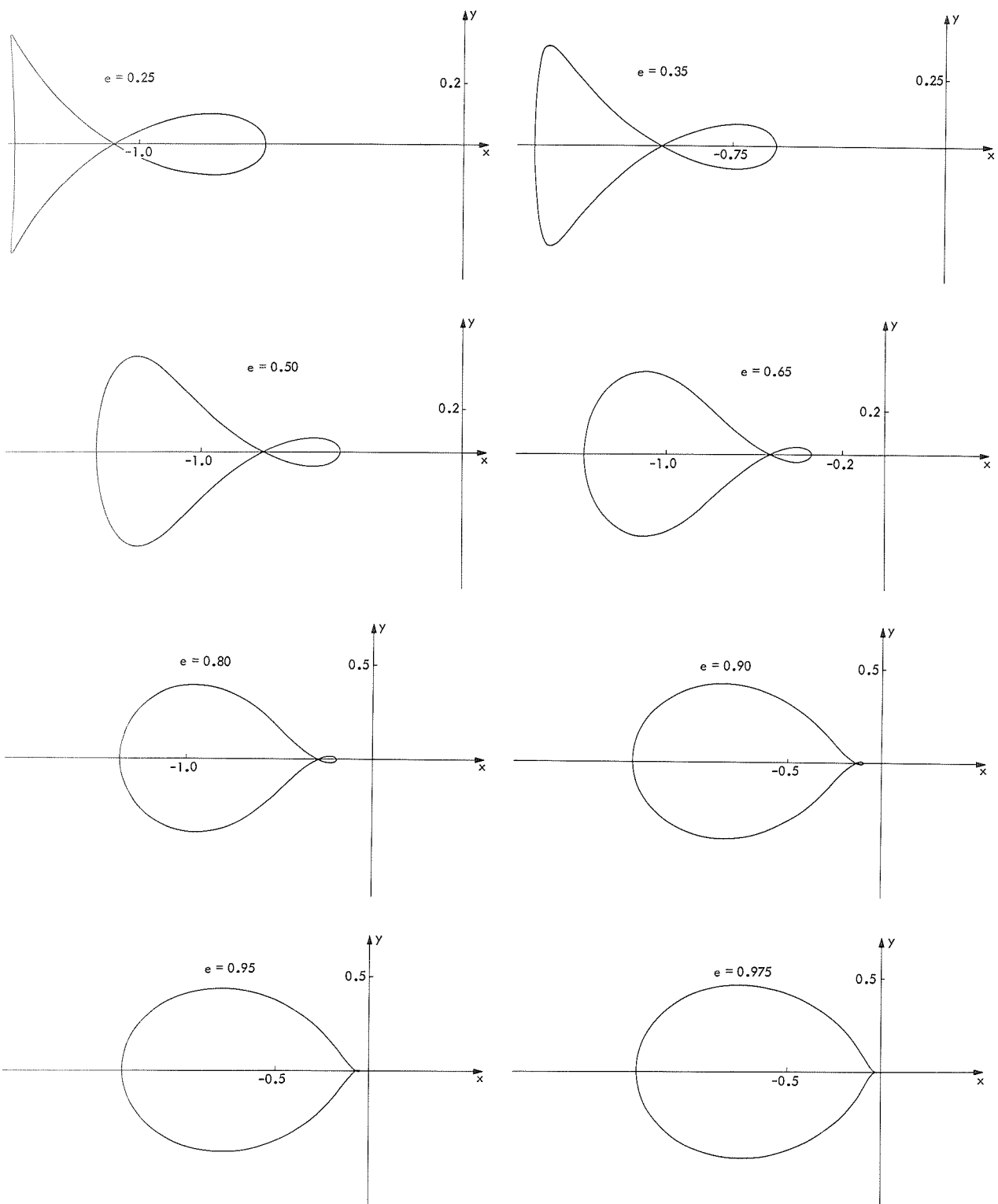


Fig. 16 (contd)

Table 14. Initial and final conditions for family 8P

ELL.RESTR.PROBL. MU=0.5 (E VARIABLE))		PERIAPSIS		FAMILY 8		
NR	X0	YDDT0	X1	YDOT1	MASS RATIO	ECC
1	-0.4017933	3.1437189	2.0396019	-1.9061162	0.5000000	0.0000000
2	-0.4019453	3.1434523	2.0358729	-1.9026045	0.5000000	0.0020000
3	-0.4021730	3.1430654	2.0303090	-1.8973689	0.5000000	0.0050000
4	-0.4025519	3.1424548	2.0211138	-1.8887278	0.5000000	0.0100000
5	-0.4029300	3.1418866	2.0120150	-1.8801915	0.5000000	0.0150000
6	-0.4033074	3.1413603	2.0030111	-1.8717586	0.5000000	0.0200000
7	-0.4036839	3.1408755	1.9941007	-1.8634279	0.5000000	0.0250000
8	-0.4040596	3.1404316	1.9852825	-1.8551980	0.5000000	0.0300000
9	-0.4044345	3.1400283	1.9765550	-1.8470677	0.5000000	0.0350000
10	-0.4048085	3.1396650	1.9679171	-1.8390358	0.5000000	0.0400000
11	-0.4051817	3.1393413	1.9593675	-1.8311009	0.5000000	0.0450000
12	-0.4055541	3.1390568	1.9509048	-1.8232621	0.5000000	0.0500000
13	-0.4059257	3.1388111	1.9425278	-1.8155180	0.5000000	0.0550000
14	-0.4062964	3.1386037	1.9342354	-1.8078676	0.5000000	0.0600000
15	-0.4066662	3.1384342	1.9260263	-1.8003097	0.5000000	0.0650000
16	-0.4070352	3.1383023	1.9178994	-1.7928434	0.5000000	0.0700000
17	-0.4074033	3.1382075	1.9098536	-1.7854674	0.5000000	0.0750000
18	-0.4077706	3.1381495	1.9018877	-1.7781809	0.5000000	0.0800000
19	-0.4085024	3.1381423	1.8861913	-1.7638720	0.5000000	0.0900000
20	-0.4092308	3.1382778	1.8708017	-1.7499089	0.5000000	0.1000000
21	-0.4099555	3.1385532	1.8557107	-1.7362842	0.5000000	0.1100000
22	-0.4106767	3.1389658	1.8409103	-1.7229907	0.5000000	0.1200000
23	-0.4113942	3.1395132	1.8263929	-1.7100217	0.5000000	0.1300000
24	-0.4121081	3.1401929	1.8121511	-1.6973708	0.5000000	0.1400000
25	-0.4128183	3.1410025	1.7981778	-1.6850317	0.5000000	0.1500000
26	-0.4135248	3.1419400	1.7844661	-1.6729988	0.5000000	0.1600000
27	-0.4142276	3.1430031	1.7710095	-1.6612663	0.5000000	0.1700000
28	-0.4149267	3.1441900	1.7578014	-1.6498292	0.5000000	0.1800000
29	-0.4156220	3.1454986	1.7448358	-1.6386823	0.5000000	0.1900000
30	-0.4163137	3.1469273	1.7321067	-1.6278209	0.5000000	0.2000000
31	-0.4170015	3.1484742	1.7196082	-1.6172406	0.5000000	0.2100000
32	-0.4176856	3.1501379	1.7073349	-1.6069373	0.5000000	0.2200000
33	-0.4183660	3.1519168	1.6952813	-1.5969069	0.5000000	0.2300000
34	-0.4190427	3.1538096	1.6834421	-1.5871457	0.5000000	0.2400000
35	-0.4197155	3.1558149	1.6718124	-1.5776505	0.5000000	0.2500000
36	-0.4203847	3.1579314	1.6603873	-1.5684178	0.5000000	0.2600000
37	-0.4210501	3.1601582	1.6491619	-1.5594449	0.5000000	0.2700000
38	-0.4217119	3.1624942	1.6381317	-1.5507290	0.5000000	0.2800000
39	-0.4223699	3.1649385	1.6272923	-1.5422676	0.5000000	0.2900000
40	-0.4230243	3.1674903	1.6166394	-1.5340586	0.5000000	0.3000000
41	-0.4236750	3.1701489	1.6061687	-1.5261000	0.5000000	0.3100000
42	-0.4243221	3.1729137	1.5958762	-1.5183901	0.5000000	0.3200000
43	-0.4249655	3.1757842	1.5857581	-1.5109274	0.5000000	0.3300000
44	-0.4256054	3.1787600	1.5758104	-1.5037107	0.5000000	0.3400000
45	-0.4262418	3.1818408	1.5660295	-1.4967392	0.5000000	0.3500000
46	-0.4268747	3.1850266	1.5564117	-1.4900121	0.5000000	0.3600000
47	-0.4275041	3.1883173	1.5469537	-1.4835291	0.5000000	0.3700000
48	-0.4281300	3.1917130	1.5376519	-1.4772900	0.5000000	0.3800000
49	-0.4287526	3.1952139	1.5285031	-1.4712951	0.5000000	0.3900000
50	-0.4293719	3.1988204	1.5195041	-1.4655450	0.5000000	0.4000000
51	-0.4299879	3.2025330	1.5106519	-1.4600403	0.5000000	0.4100000
52	-0.4306007	3.2063524	1.5019432	-1.4547823	0.5000000	0.4200000
53	-0.4312103	3.2102795	1.4933753	-1.4497725	0.5000000	0.4300000
54	-0.4318168	3.2143153	1.4849452	-1.4450129	0.5000000	0.4400000
55	-0.4324203	3.2184608	1.4766501	-1.4405056	0.5000000	0.4500000
56	-0.4330208	3.2227176	1.4684874	-1.4362534	0.5000000	0.4600000
57	-0.4336184	3.2270872	1.4604543	-1.4322595	0.5000000	0.4700000
58	-0.4342133	3.2315715	1.4525483	-1.4285274	0.5000000	0.4800000
59	-0.4348055	3.2361724	1.4447669	-1.4250614	0.5000000	0.4900000
60	-0.4353950	3.2408923	1.4371076	-1.4218660	0.5000000	0.5000000

Table 14 (contd)

ELL.RESTR.PROBL. MU=0.5 (E VARIABLE)) PERIAPSIS FAMILY 8						
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC
61	-0.4359820	3.2457338	1.4295681	-1.4189466	0.5000000	0.5100000
62	-0.4365667	3.2506997	1.4221459	-1.4163091	0.5000000	0.5200000
63	-0.4371490	3.2557932	1.4148388	-1.4139601	0.5000000	0.5300000
64	-0.4377292	3.2610180	1.4076446	-1.4119068	0.5000000	0.5400000
65	-0.4383073	3.2663779	1.4005610	-1.4101576	0.5000000	0.5500000
66	-0.4388836	3.2718772	1.3935860	-1.4087215	0.5000000	0.5600000
67	-0.4394581	3.2775208	1.3867175	-1.4076086	0.5000000	0.5700000
68	-0.4400310	3.2833140	1.3799534	-1.4068299	0.5000000	0.5800000
69	-0.4406025	3.2892627	1.3732916	-1.4063980	0.5000000	0.5900000
70	-0.4411728	3.2953732	1.3667303	-1.4063264	0.5000000	0.6000000
71	-0.4417421	3.3016528	1.3602674	-1.4066303	0.5000000	0.6100000
72	-0.4423105	3.3081094	1.3539011	-1.4073267	0.5000000	0.6200000
73	-0.4428783	3.3147516	1.3476295	-1.4084342	0.5000000	0.6300000
74	-0.4434458	3.3215892	1.3414507	-1.4099736	0.5000000	0.6400000
75	-0.4440133	3.3286328	1.3353629	-1.4119680	0.5000000	0.6500000
76	-0.4445809	3.3358944	1.3293643	-1.4144432	0.5000000	0.6600000
77	-0.4451491	3.3433872	1.3234530	-1.4174282	0.5000000	0.6700000
78	-0.4457183	3.3511259	1.3176273	-1.4209551	0.5000000	0.6800000
79	-0.4462887	3.3591269	1.3118853	-1.4250603	0.5000000	0.6900000
80	-0.4468608	3.3674087	1.3062252	-1.4297845	0.5000000	0.7000000
81	-0.4474350	3.3759920	1.3006453	-1.4351739	0.5000000	0.7100000
82	-0.4480120	3.3849000	1.2951437	-1.4412807	0.5000000	0.7200000
83	-0.4485922	3.3941591	1.2897185	-1.4481644	0.5000000	0.7300000
84	-0.4491763	3.4037991	1.2843678	-1.4558927	0.5000000	0.7400000
85	-0.4497650	3.4138539	1.2790897	-1.4645435	0.5000000	0.7500000
86	-0.4503591	3.4243624	1.2738820	-1.4742065	0.5000000	0.7600000
87	-0.4509595	3.4353692	1.2687428	-1.4849856	0.5000000	0.7700000
88	-0.4515672	3.4469258	1.2636697	-1.4970020	0.5000000	0.7800000
89	-0.4521834	3.4590923	1.2586603	-1.5103976	0.5000000	0.7900000
90	-0.4528095	3.4719386	1.2537121	-1.5253403	0.5000000	0.8000000
91	-0.4531267	3.4786420	1.2512600	-1.5334527	0.5000000	0.8050000
92	-0.4534470	3.4855474	1.2488222	-1.5420297	0.5000000	0.8100000
93	-0.4537706	3.4926676	1.2463981	-1.5511024	0.5000000	0.8150000
94	-0.4540977	3.5000165	1.2439875	-1.5607054	0.5000000	0.8200000
95	-0.4547638	3.5154634	1.2392044	-1.5816580	0.5000000	0.8300000
96	-0.4551033	3.5235968	1.2368310	-1.5930962	0.5000000	0.8350000
97	-0.4554477	3.5320302	1.2344690	-1.6052430	0.5000000	0.8400000
98	-0.4557973	3.5407863	1.2321177	-1.6181564	0.5000000	0.8450000
99	-0.4565140	3.5593711	1.2274448	-1.6465504	0.5000000	0.8550000
100	-0.4568821	3.5692596	1.2251216	-1.6621861	0.5000000	0.8600000
101	-0.4572575	3.5795918	1.2228062	-1.6789017	0.5000000	0.8650000
102	-0.4576408	3.5904079	1.2204975	-1.6968034	0.5000000	0.8700000
103	-0.4584342	3.6136807	1.2158960	-1.7366688	0.5000000	0.8800000
104	-0.4588461	3.6262491	1.2136007	-1.7589335	0.5000000	0.8850000
105	-0.4592695	3.6395276	1.2113070	-1.7829945	0.5000000	0.8900000
106	-0.4597056	3.6535959	1.2090131	-1.8090714	0.5000000	0.8950000
107	-0.4606220	3.6844910	1.2044166	-1.8683588	0.5000000	0.9050000
108	-0.4611057	3.7015574	1.2021089	-1.9022462	0.5000000	0.9100000
109	-0.4616093	3.7199019	1.1997907	-1.9395326	0.5000000	0.9150000
110	-0.4621353	3.7397127	1.1974583	-1.9807642	0.5000000	0.9200000
111	-0.4632682	3.7847075	1.1927313	-2.0779387	0.5000000	0.9300000
112	-0.4638835	3.8105339	1.1903243	-2.1358086	0.5000000	0.9350000
113	-0.4645385	3.8391554	1.1878775	-2.2016244	0.5000000	0.9400000
114	-0.4652407	3.8711661	1.1853800	-2.2772331	0.5000000	0.9450000
115	-0.4668266	3.9488063	1.1801723	-2.4687947	0.5000000	0.9550000
116	-0.4687597	4.0542798	1.1745218	-2.7458479	0.5000000	0.9650000
117	-0.4699217	4.1239666	1.1714314	-2.9384809	0.5000000	0.9700000
118	-0.4712760	4.2117288	1.1680704	-3.1909472	0.5000000	0.9750000

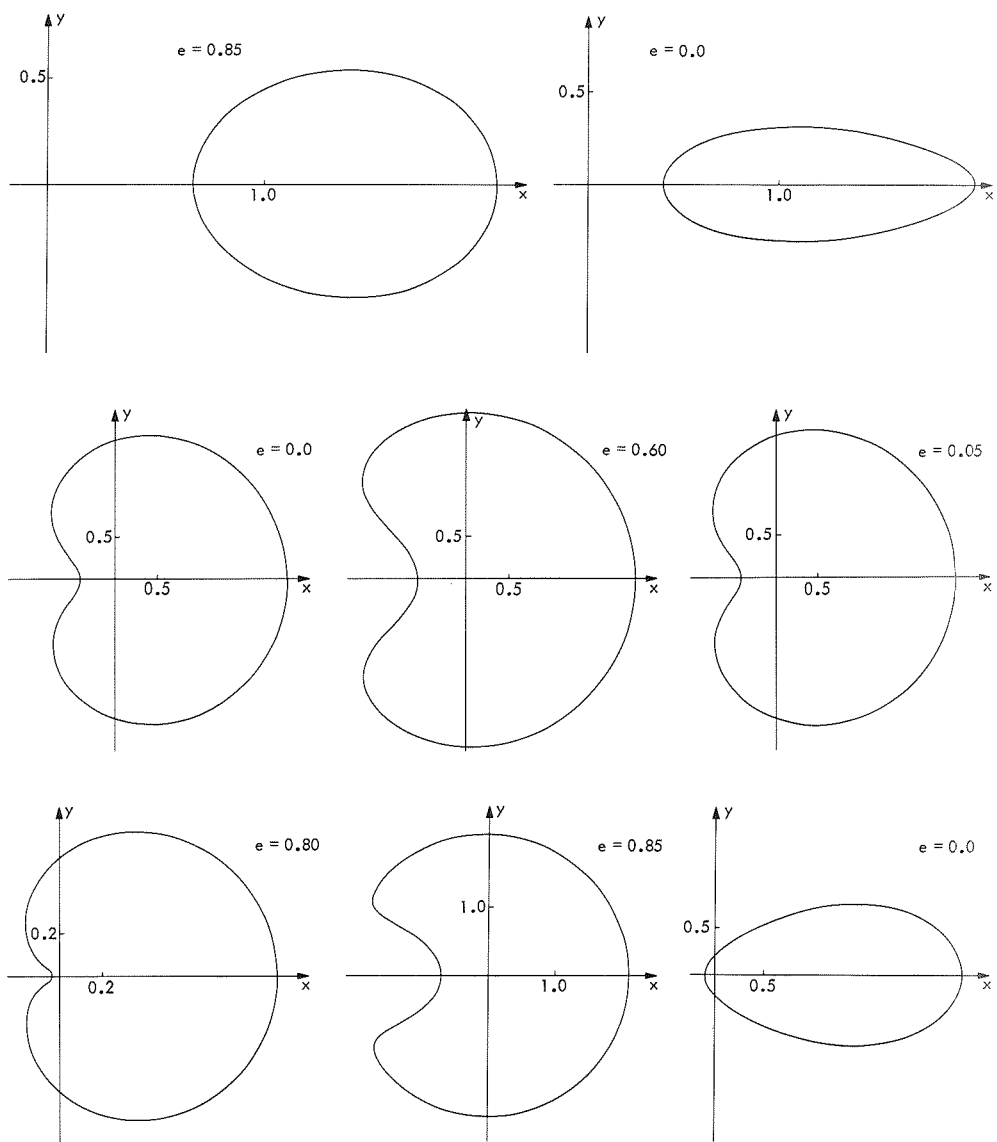


Fig. 17. Family 8A of periodic orbits

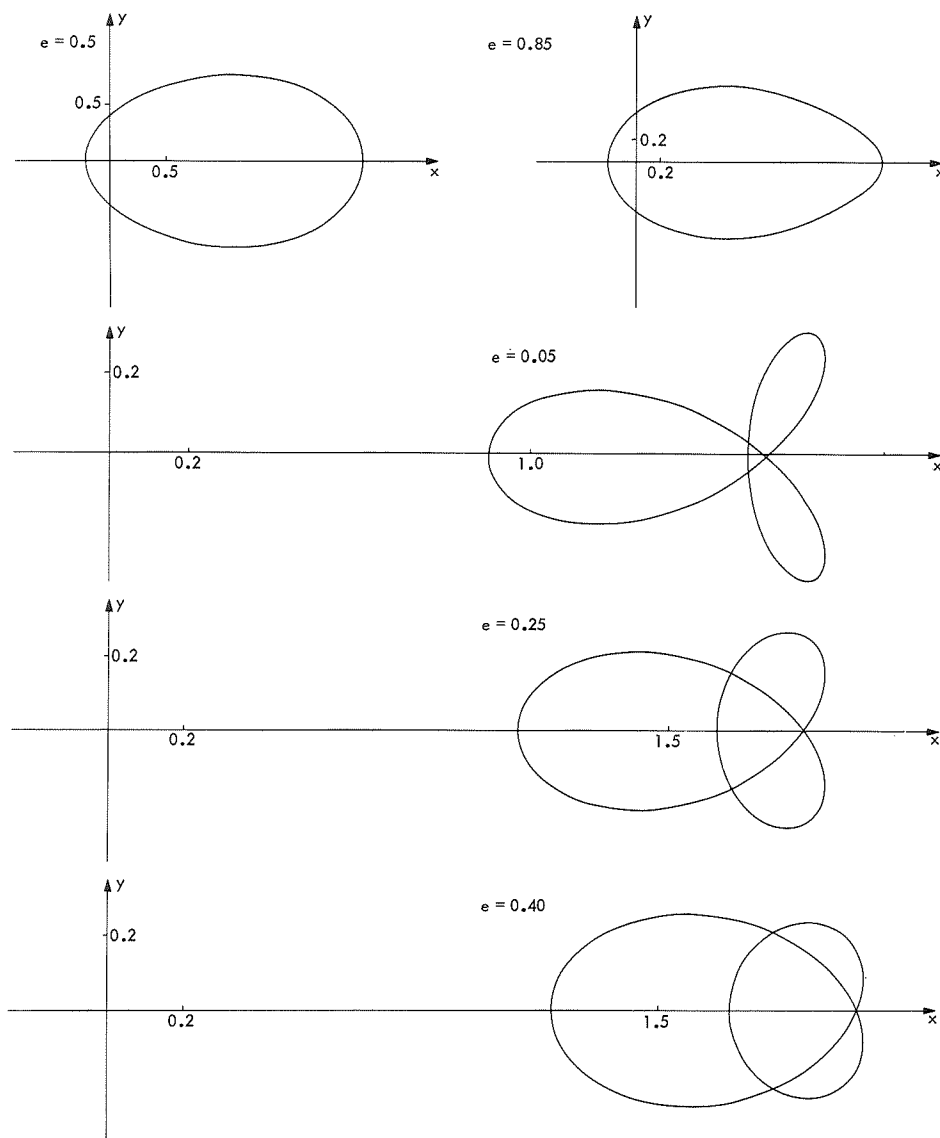


Fig. 17 (contd)

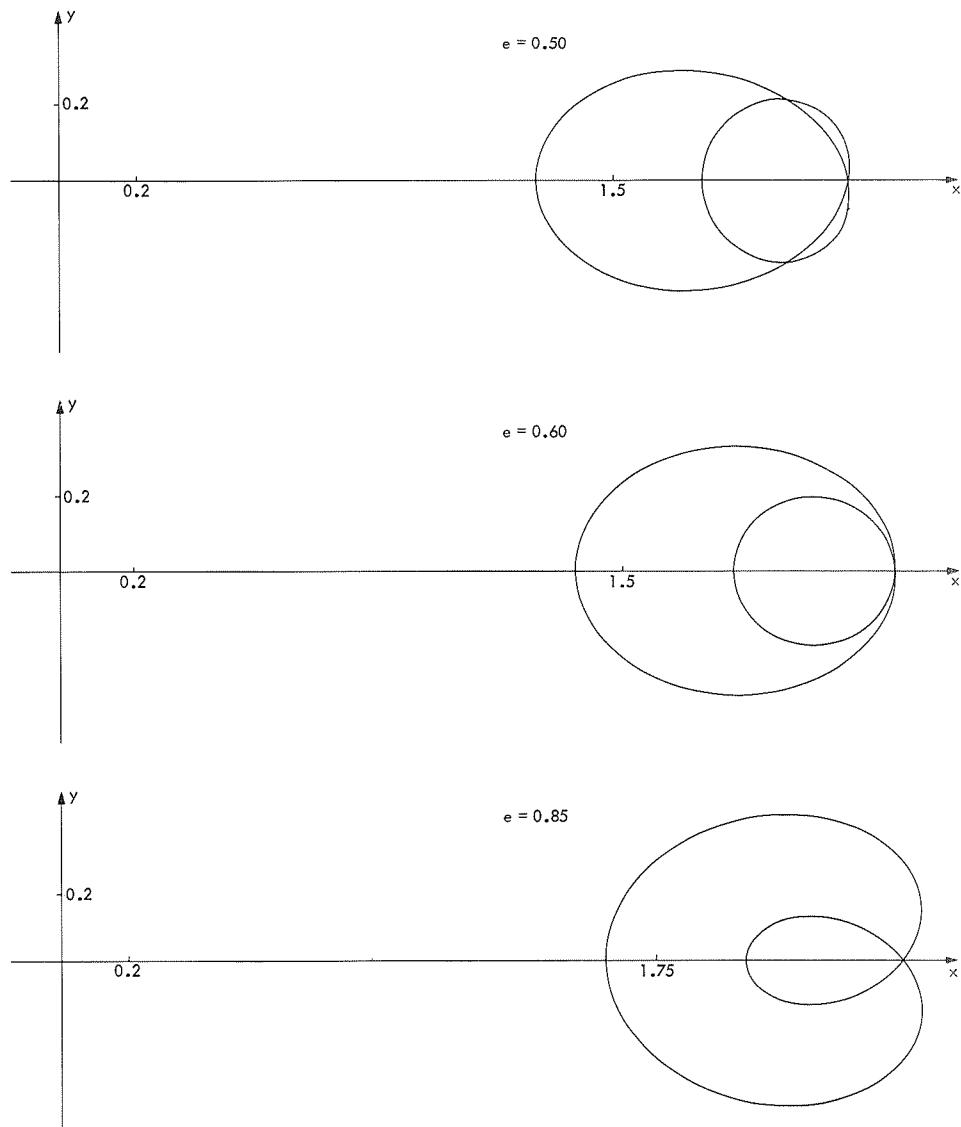


Fig. 17 (contd)

A remarkable evolution of the stability properties in this family has been found. It was possible to determine the roots of the fundamental matrix for all of the orbits up to $e = 0.85$ (Table 15). Family 8A starts in region 4, and passes (in region 2) all the way up to $e = 0.85$, except that the last computed orbit ($e = 0.85$) itself has just crossed the parabola $a_2 = a_1^2/4 + 2$ (see Fig. 2). Therefore, this orbit belongs to region 1 and is stable, unlike all of the other orbits of this family, which are unstable.

G. Family 11P

Family 11 of periodic orbits is shown in Fig. 18. Family 11P (Fig. 19) also begins at $e = 0.0$ with an orbit with period 2π taken from Strömgren's problem (in fact, also from Bartlett's class g). The initial conditions in rotating axes are

$$x_0 = -0.07084826 \quad (180a)$$

$$\dot{y}_0 = +0.82832745 \quad (180b)$$

$$\mu = 0.5 \quad (180c)$$

$$e = 0.0 \quad (180d)$$

In this family, 51 orbits have been computed with eccentricities e from 0 to 0.453 (Table 16). It is not known at present whether this family goes much farther than $e = 0.453$. The last orbits computed show increasing difficulties in the convergence of the differential correction process. The last orbit ($e = 0.453$) was obtained with very slow convergence on the differential corrections (10 iterations), whereas, for $e = 0.454$, there was definite divergence. The loss of convergence is not apparently due to a close approach with one of the primaries.

Family 11P begins with the even instability of the circular problem at $e = 0$. The orbits then move in region 4 (even-even instability), with four real, positive roots (see Fig. 2). The stability coefficient a_2 reaches a maximum of about 13400 in the neighborhood of $e = 0.18$, and then decreases all the way to 1000 at $e = 0.45$. The stability coefficient a_1 is permanently increasing from -5000 to -200 when the eccentricity goes from 0.0 to 0.45. All of the orbits of this family are thus unstable.

H. Family 11A

This family (Fig. 20) begins with the same orbit as family 11P (periapsis). About 100 periodic orbits have been found with $e = 0.0$ to $e = 0.895$ (Table 17). At this latter value of e , there is a close approach with one of the primaries.

The stability of the orbits has been determined only up to $e = 0.80$. The stability coefficients a_1 and a_2 both go through a minimum: $a_1 = -5500$ at $e = 0.18$, and $a_2 = -5200$ at $e = 0.67$. From $e = 0.0$ to 0.30, all of the orbits are in region 6; from $e = 0.30$ to 0.80, they are all in region 3 (see Fig. 2). All of the orbits of this family are thus unstable.

I. Family 10P

This is a family of periodic orbits that are symmetric with respect to both the x -axis and the y -axis in the rotating-pulsating coordinate system (Fig. 21). These orbits all belong to a mass-ratio of $\mu = 0.5$. In the circular restricted three-body problem, it is well known that only in the case of equal masses ($\mu = 0.5$) are there such periodic orbits that are symmetric with respect to both rotating coordinate axes. Strömgren had already computed several such symmetric periodic orbits in the circular problem. The family described herein shows that, in the elliptic three-body problem, periodic orbits also exist with the same type of symmetry.

Family 10P begins with $e = 0$ and goes up to $e = 0.70$, the mass-ratio μ being constant. The first orbit ($e = 0$) has been taken from Bartlett's class ν (see Ref. 18), and has the following initial conditions:

$$x_0 = 0 \quad (181a)$$

$$y_0 = -0.29524274 \quad (181b)$$

$$\dot{x}_0 = -0.31965294 \quad (181c)$$

$$\dot{y}_0 = 0 \quad (181d)$$

This orbit and all the other orbits of the family have a period 4π , but only one-quarter of the orbit ($v = 0$ to π) must be integrated because of the symmetry. The arc of the orbit that has been integrated thus starts on the y -axis with a right angle, and ends on the x -axis (also with a right angle). At the starting point on the y -axis, the two primaries are at their minimum elongation (periapsis), and the orbits are thus all " p -orbits." The orbits have all been computed in rotating-pulsating coordinates.

Family 10P contains 76 orbits, with e from 0 to 0.70 (Table 18), and these 76 orbits have all been obtained in a single computer run with an automatic program. The computation was stopped at $e = 0.70$ because of slow convergence of the differential correction process.

Table 15. Initial and final conditions for family 8A

ELL.RESTR.PROBL.	MU=0.5 (E VARIABLE))	APDAPSIS	FAMILY 8			
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC
1	-0.4017933	3.1437189	2.0396019	-1.9061162	0.5000000	0.0000000
2	-0.4016032	3.1440620	2.0442855	-1.9105301	0.5000000	0.0025000
3	-0.4014129	3.1444159	2.0489940	-1.9149712	0.5000000	0.0050000
4	-0.4012223	3.1447808	2.0537278	-1.9194394	0.5000000	0.0075000
5	-0.4010316	3.1451568	2.0584869	-1.9239351	0.5000000	0.0100000
6	-0.4008407	3.1455439	2.0632716	-1.9284585	0.5000000	0.0125000
7	-0.4006497	3.1459422	2.0680821	-1.9330097	0.5000000	0.0150000
8	-0.4004584	3.1463518	2.0729186	-1.9375889	0.5000000	0.0175000
9	-0.4002669	3.1467727	2.0777811	-1.9421962	0.5000000	0.0200000
10	-0.3994993	3.1485709	2.0974972	-1.9609118	0.5000000	0.0300000
11	-0.3987287	3.1505560	2.1176483	-1.9800946	0.5000000	0.0400000
12	-0.3979554	3.1527329	2.1382483	-1.9997582	0.5000000	0.0500000
13	-0.3971793	3.1551067	2.1593115	-2.0199164	0.5000000	0.0600000
14	-0.3964007	3.1576824	2.1808530	-2.0405841	0.5000000	0.0700000
15	-0.3956195	3.1604655	2.2028885	-2.0617767	0.5000000	0.0800000
16	-0.3948359	3.1634618	2.2254345	-2.0835102	0.5000000	0.0900000
17	-0.3940499	3.1666771	2.2485079	-2.1058017	0.5000000	0.1000000
18	-0.3932617	3.1701178	2.2721269	-2.1286688	0.5000000	0.1100000
19	-0.3924715	3.1737903	2.2963102	-2.1521301	0.5000000	0.1200000
20	-0.3916793	3.1777013	2.3210775	-2.1762051	0.5000000	0.1300000
21	-0.3908852	3.1818580	2.3464494	-2.2009143	0.5000000	0.1400000
22	-0.3900895	3.1862678	2.3724475	-2.2262791	0.5000000	0.1500000
23	-0.3892921	3.1909385	2.3990945	-2.2523222	0.5000000	0.1600000
24	-0.3884934	3.1958782	2.4264142	-2.2790672	0.5000000	0.1700000
25	-0.3876934	3.2010955	2.4544316	-2.3065390	0.5000000	0.1800000
26	-0.3868924	3.2065993	2.4831729	-2.3347640	0.5000000	0.1900000
27	-0.3860904	3.2123990	2.5126658	-2.3637696	0.5000000	0.2000000
28	-0.3852877	3.2185044	2.5429394	-2.3935849	0.5000000	0.2100000
29	-0.3844845	3.2249259	2.5740242	-2.4242406	0.5000000	0.2200000
30	-0.3836809	3.2316744	2.6059525	-2.4557689	0.5000000	0.2300000
31	-0.3828773	3.2387613	2.6387583	-2.4882039	0.5000000	0.2400000
32	-0.3820738	3.2461987	2.6724773	-2.5215815	0.5000000	0.2500000
33	-0.3812706	3.2539991	2.7071475	-2.5559397	0.5000000	0.2600000
34	-0.3804681	3.2621759	2.7428089	-2.5913185	0.5000000	0.2700000
35	-0.3796664	3.2707432	2.7795037	-2.6277605	0.5000000	0.2800000
36	-0.3788659	3.2797157	2.8172766	-2.6653105	0.5000000	0.2900000
37	-0.3780669	3.2891093	2.8561751	-2.7040162	0.5000000	0.3000000
38	-0.3772696	3.2989402	2.8962493	-2.7439279	0.5000000	0.3100000
39	-0.3764744	3.3092262	2.9375525	-2.7850991	0.5000000	0.3200000
40	-0.3756817	3.3199856	2.9801412	-2.8275868	0.5000000	0.3300000
41	-0.3748917	3.3312381	3.0240755	-2.8714513	0.5000000	0.3400000
42	-0.3741050	3.3430043	3.0694193	-2.9167567	0.5000000	0.3500000
43	-0.3733219	3.3553064	3.1162405	-2.9635716	0.5000000	0.3600000
44	-0.3725428	3.3681677	3.1646117	-3.0119687	0.5000000	0.3700000
45	-0.3717683	3.3816130	3.2146101	-3.0620258	0.5000000	0.3800000
46	-0.3709987	3.3956687	3.2663183	-3.1138259	0.5000000	0.3900000
47	-0.3702346	3.4103631	3.3198243	-3.1674576	0.5000000	0.4000000
48	-0.3694766	3.4257260	3.3752227	-3.2230159	0.5000000	0.4100000
49	-0.3687252	3.4417895	3.4326145	-3.2806025	0.5000000	0.4200000
50	-0.3679810	3.4585878	3.4921080	-3.3403262	0.5000000	0.4300000
51	-0.3672448	3.4761573	3.5538197	-3.4023044	0.5000000	0.4400000
52	-0.3665171	3.4945373	3.6178747	-3.4666626	0.5000000	0.4500000
53	-0.3657988	3.5137695	3.6844077	-3.5335365	0.5000000	0.4600000
54	-0.3650906	3.5338991	3.7535639	-3.6030719	0.5000000	0.4700000
55	-0.3643934	3.5549744	3.8255000	-3.6754264	0.5000000	0.4800000
56	-0.3637080	3.5770473	3.9003853	-3.7507703	0.5000000	0.4900000
57	-0.3630355	3.6001740	3.9784032	-3.8292879	0.5000000	0.5000000
58	-0.3623767	3.6244148	4.0597525	-3.9111790	0.5000000	0.5100000
59	-0.3617329	3.6498352	4.1446490	-3.9966605	0.5000000	0.5200000
60	-0.3611052	3.6765059	4.2333276	-4.0859684	0.5000000	0.5300000

Table 15 (contd)

ELL.RESTR.PROBL. MU=0.5 (E VARIABLE))	APOAPSIS		FAMILY 8		ECC	
NR	X0	YDOTO	X1	YDOT1	MASS RATIO	
61	-0.3604948	3.7045035	4.3260440	-4.1793597	0.5000000	0.5400000
62	-0.3599030	3.7339114	4.4230776	-4.2771150	0.5000000	0.5500000
63	-0.3593314	3.7648200	4.5247339	-4.3795410	0.5000000	0.5600000
64	-0.3587815	3.7973282	4.6313474	-4.4869740	0.5000000	0.5700000
65	-0.3582548	3.8315437	4.7432856	-4.5997829	0.5000000	0.5800000
66	-0.3577533	3.8675844	4.8609530	-4.7183738	0.5000000	0.5900000
67	-0.3572788	3.9055796	4.9847956	-4.8431943	0.5000000	0.6000000
68	-0.3568335	3.9456715	5.1153064	-4.9747394	0.5000000	0.6100000
69	-0.3564195	3.9880165	5.2530319	-5.1135575	0.5000000	0.6200000
70	-0.3560393	4.0327875	5.3985792	-5.2602577	0.5000000	0.6300000
71	-0.3556956	4.0801757	5.5526246	-5.4155186	0.5000000	0.6400000
72	-0.3553911	4.1303935	5.7159241	-5.5800986	0.5000000	0.6500000
73	-0.3551292	4.1836772	5.8893248	-5.7548470	0.5000000	0.6600000
74	-0.3549130	4.2402907	6.0737791	-5.9407192	0.5000000	0.6700000
75	-0.3545783	4.332550	6.7049441	-6.5765873	0.5000000	0.7000000
76	-0.3547464	4.3005296	6.2703616	-6.1387925	0.5000000	0.6800000
77	-0.3546334	4.3647261	6.4802890	-6.3502864	0.5000000	0.6900000
78	-0.3545862	4.5065407	6.9459054	-6.8192769	0.5000000	0.7100000
79	-0.3546623	4.5850657	7.2049817	-7.0801677	0.5000000	0.7200000
80	-0.3548125	4.6693809	7.4842562	-7.3613468	0.5000000	0.7300000
81	-0.3550434	4.7601188	7.7861398	-7.6652294	0.5000000	0.7400000
82	-0.3553621	4.8580087	8.1134373	-7.9946249	0.5000000	0.7500000
83	-0.3557768	4.9638960	8.4694312	-8.3528205	0.5000000	0.7600000
84	-0.3562964	5.0787674	8.8579863	-8.7436866	0.5000000	0.7700000
85	-0.3569309	5.2037809	9.2836844	-9.1718103	0.5000000	0.7800000
86	-0.3576916	5.3403059	9.7519969	-9.6426692	0.5000000	0.7900000
87	-0.3585912	5.4899734	10.2695097	-10.1628561	0.5000000	0.8000000
88	-0.3596442	5.6547433	10.8442206	-10.7403759	0.5000000	0.8100000
89	-0.3608670	5.8369919	11.4859359	-11.3850429	0.5000000	0.8200000
90	-0.3622784	6.0396302	12.2068081	-12.1090181	0.5000000	0.8300000
91	-0.3639002	6.2662653	13.0220771	-12.9275510	0.5000000	0.8400000
92	-0.3657577	6.5214243	13.9511105	-13.8600202	0.5000000	0.8500000

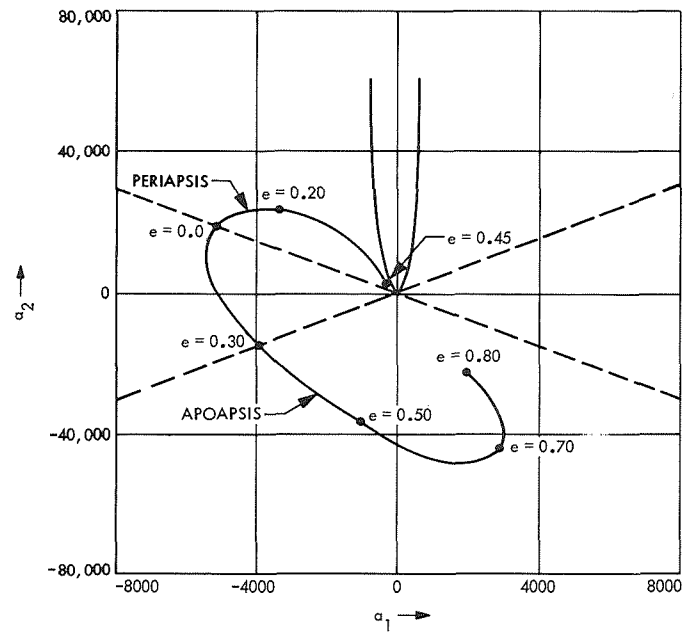


Fig. 18. Family 11 of periodic orbits

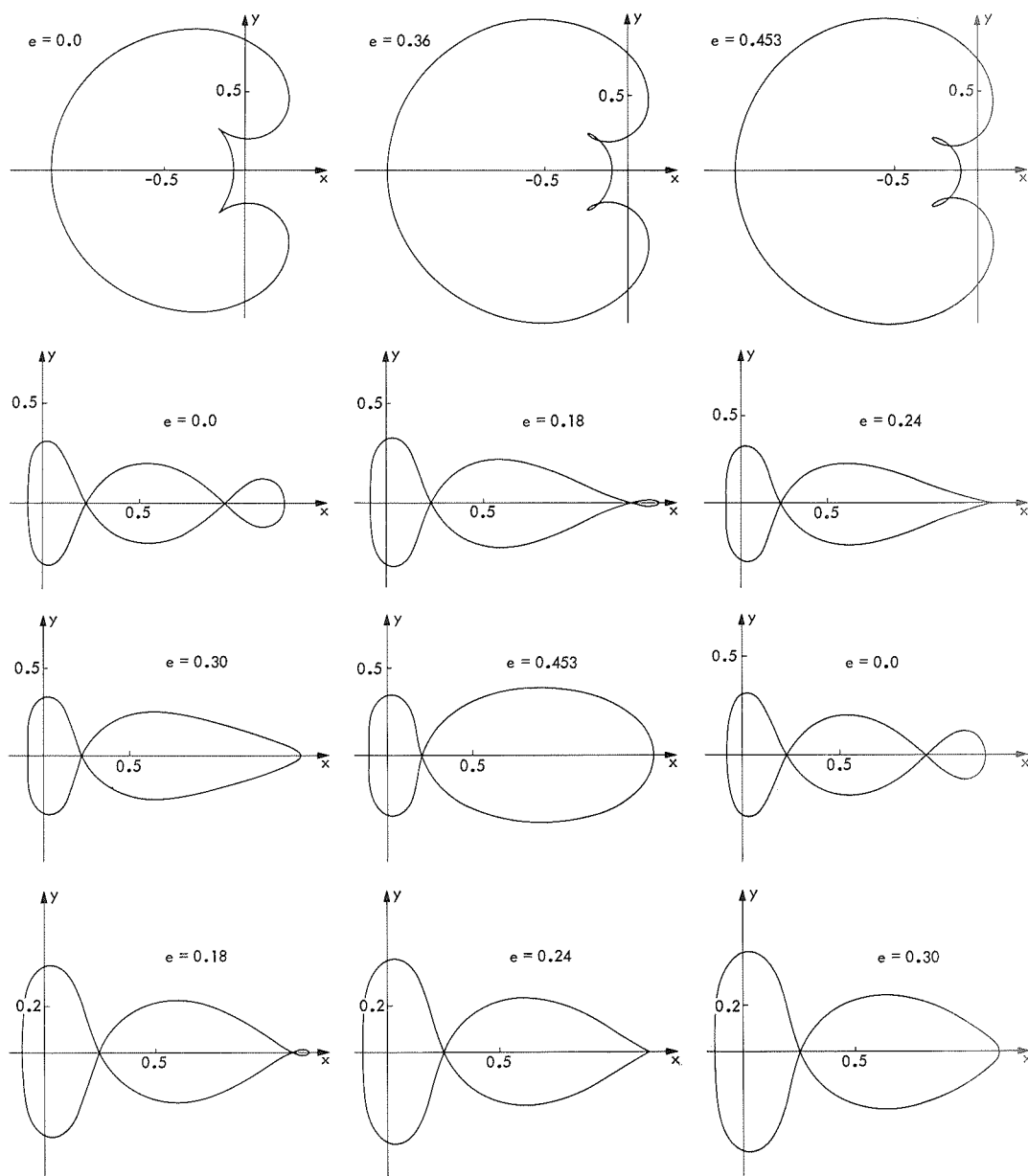


Fig. 19. Family 11P of periodic orbits

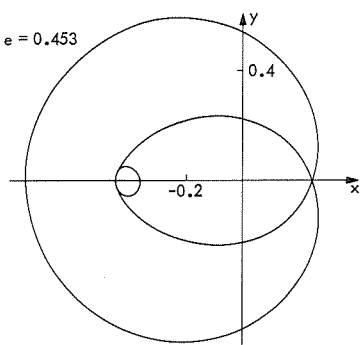
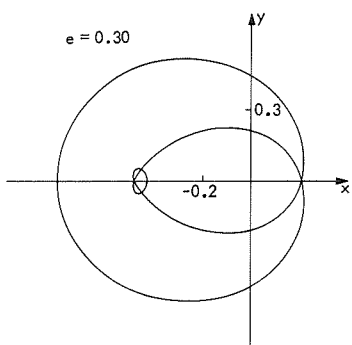
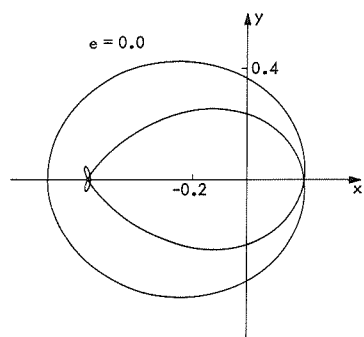
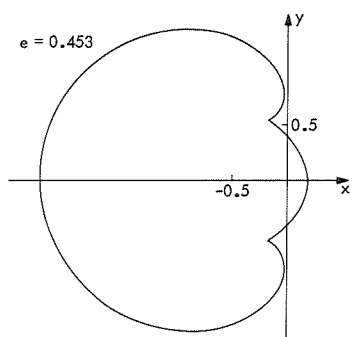
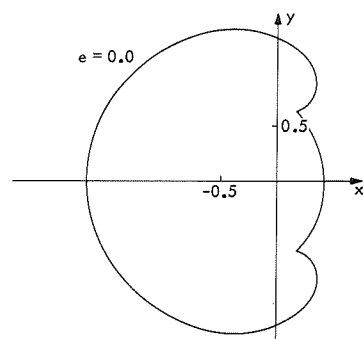
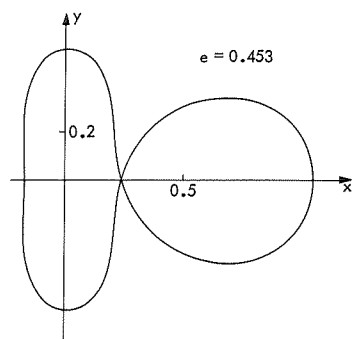


Fig. 19 (contd)

Table 16. Initial and final conditions for family 11P

ELL.RESTR.PROBL. MU=0.5 (E VARIABLE)) PERIAPSIS					FAMILY 11	
NR	X0	YDOTO	X1	YDOT1	MASS RATIO	ECC
1	-0.0708483	0.8283275	1.2236640	-0.2419082	0.5000000	0.0000000
2	-0.0714931	0.8304851	1.2218368	-0.2367393	0.5000000	0.0050000
3	-0.0721437	0.8326576	1.2200206	-0.2315685	0.5000000	0.0100000
4	-0.0728000	0.8348450	1.2182156	-0.2263955	0.5000000	0.0150000
5	-0.0734622	0.8370476	1.2164212	-0.2212194	0.5000000	0.0200000
6	-0.0741303	0.8392657	1.2146374	-0.2160397	0.5000000	0.0250000
7	-0.0748045	0.8414995	1.2128639	-0.2108557	0.5000000	0.0300000
8	-0.0761713	0.8460152	1.2093470	-0.2004725	0.5000000	0.0400000
9	-0.0775634	0.8505968	1.2058689	-0.1900647	0.5000000	0.0500000
10	-0.0789816	0.8552467	1.2024279	-0.1796269	0.5000000	0.0600000
11	-0.0804268	0.8599671	1.1990221	-0.1691535	0.5000000	0.0700000
12	-0.0818999	0.8647605	1.1956502	-0.1586389	0.5000000	0.0800000
13	-0.0834018	0.8696297	1.1923103	-0.1480771	0.5000000	0.0900000
14	-0.0849336	0.8745775	1.1890008	-0.1374619	0.5000000	0.1000000
15	-0.0864962	0.8796069	1.1857197	-0.1267868	0.5000000	0.1100000
16	-0.0880909	0.8847210	1.1824654	-0.1160448	0.5000000	0.1200000
17	-0.0897186	0.8899233	1.1792362	-0.1052289	0.5000000	0.1300000
18	-0.0913808	0.8952173	1.1760299	-0.0943313	0.5000000	0.1400000
19	-0.0930787	0.9006068	1.1728448	-0.0833440	0.5000000	0.1500000
20	-0.0948137	0.9060959	1.1696786	-0.0722583	0.5000000	0.1600000
21	-0.0965873	0.9116888	1.1665294	-0.0610651	0.5000000	0.1700000
22	-0.0984009	0.9173903	1.1633948	-0.0497544	0.5000000	0.1800000
23	-0.1002563	0.9232052	1.1602726	-0.0383155	0.5000000	0.1900000
24	-0.1021552	0.9291386	1.1571596	-0.0267367	0.5000000	0.2000000
25	-0.1040995	0.9351961	1.1540538	-0.0150057	0.5000000	0.2100000
26	-0.1060911	0.9413838	1.1509519	-0.0031087	0.5000000	0.2200000
27	-0.1081321	0.9477078	1.1478507	0.0089695	0.5000000	0.2300000
28	-0.1102249	0.9541750	1.1447467	0.0212454	0.5000000	0.2400000
29	-0.1123716	0.9607927	1.1416357	0.0337377	0.5000000	0.2500000
30	-0.1145750	0.9675685	1.1385139	0.0464666	0.5000000	0.2600000
31	-0.1168377	0.9745108	1.1353758	0.0594558	0.5000000	0.2700000
32	-0.1191626	0.9816285	1.1322161	0.0727314	0.5000000	0.2800000
33	-0.1215529	0.9889309	1.1290283	0.0863236	0.5000000	0.2900000
34	-0.1240117	0.9964282	1.1258052	0.1002670	0.5000000	0.3000000
35	-0.1265428	1.0041313	1.1225383	0.1146019	0.5000000	0.3100000
36	-0.1291498	1.0120514	1.1192170	0.1293764	0.5000000	0.3200000
37	-0.1318368	1.0202009	1.1158295	0.1446467	0.5000000	0.3300000
38	-0.1346081	1.0285925	1.1123608	0.1604819	0.5000000	0.3400000
39	-0.1374682	1.0372395	1.1087928	0.1769665	0.5000000	0.3500000
40	-0.1404219	1.0461554	1.1051020	0.1942071	0.5000000	0.3600000
41	-0.1434742	1.0553540	1.1012589	0.2123395	0.5000000	0.3700000
42	-0.1466303	1.0648479	1.0972227	0.2315480	0.5000000	0.3800000
43	-0.1498953	1.0746483	1.0929392	0.2520779	0.5000000	0.3900000
44	-0.1532743	1.0847623	1.0883287	0.2742862	0.5000000	0.4000000
45	-0.1567716	1.0951894	1.0832702	0.2987115	0.5000000	0.4100000
46	-0.1603903	1.1059120	1.0775609	0.3262535	0.5000000	0.4200000
47	-0.1641306	1.1168784	1.0708240	0.3585667	0.5000000	0.4300000
48	-0.1679856	1.1279422	1.0621817	0.3994813	0.5000000	0.4400000
49	-0.1719294	1.1385403	1.0481904	0.4638789	0.5000000	0.4500000
50	-0.1723260	1.1394828	1.0458821	0.4742526	0.5000000	0.4510000
51	-0.1727222	1.1403568	1.0430663	0.4868058	0.5000000	0.4520000
52	-0.1731180	1.1410928	1.0392112	0.5038149	0.5000000	0.4530000

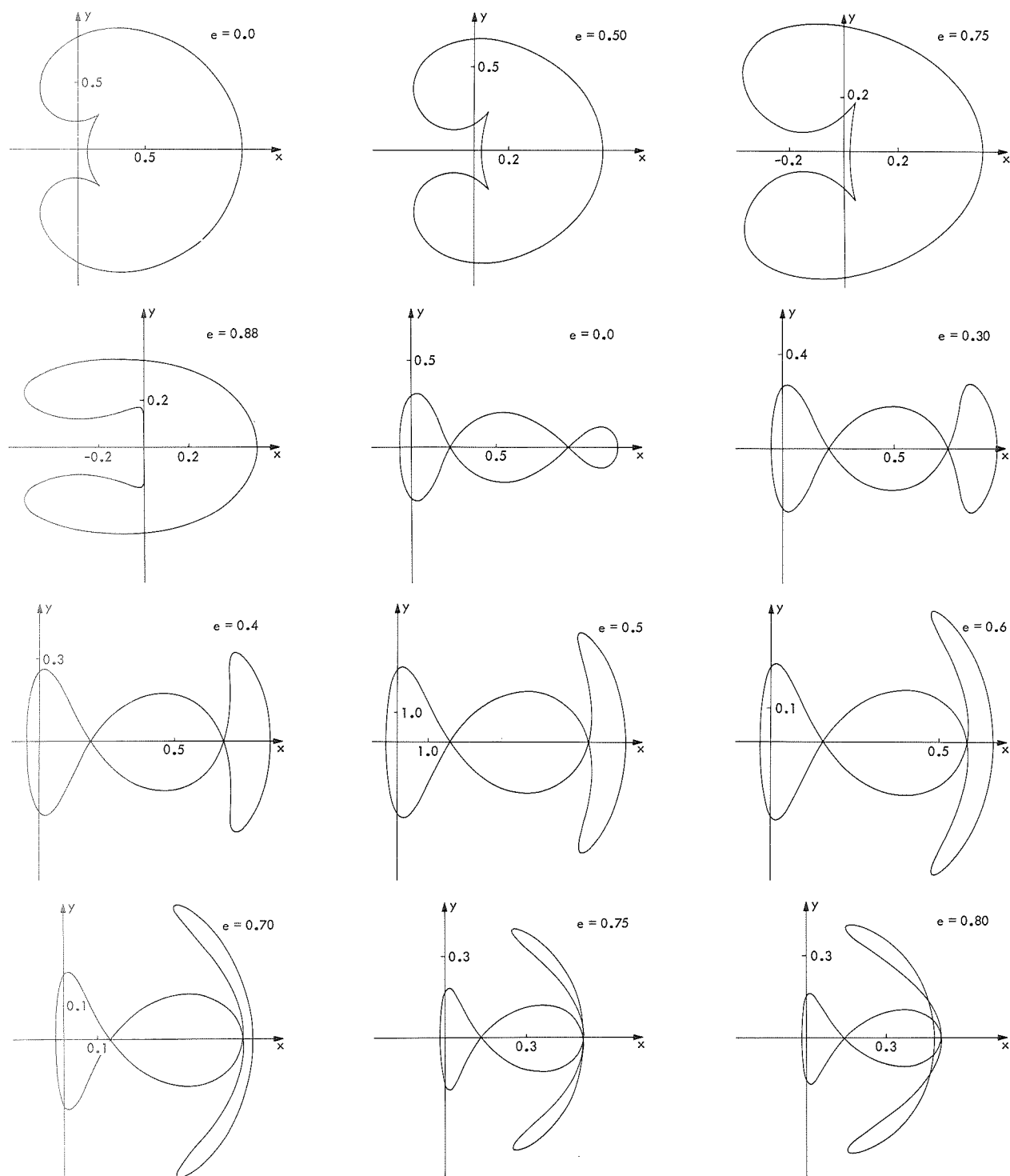


Fig. 20. Family 11A of periodic orbits

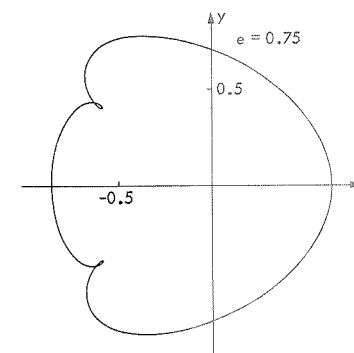
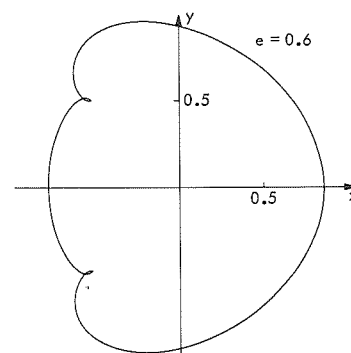
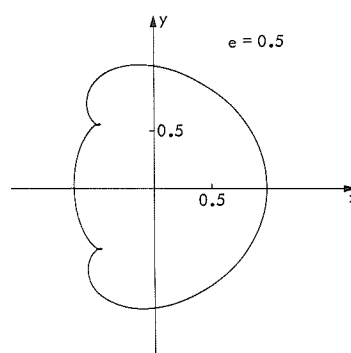
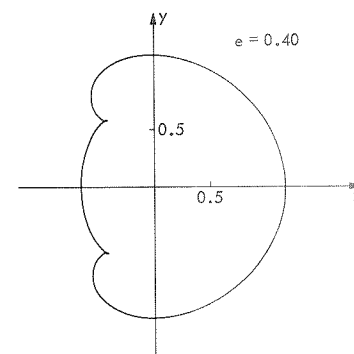
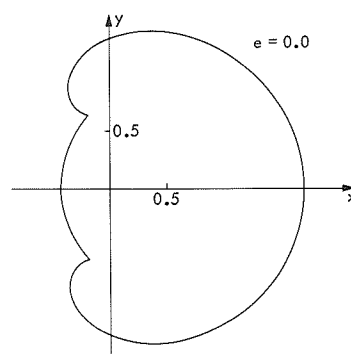
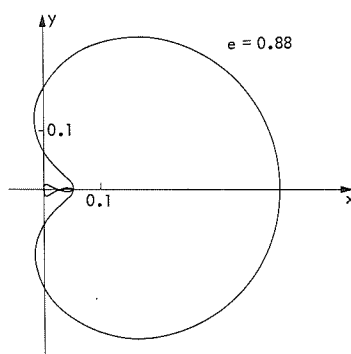
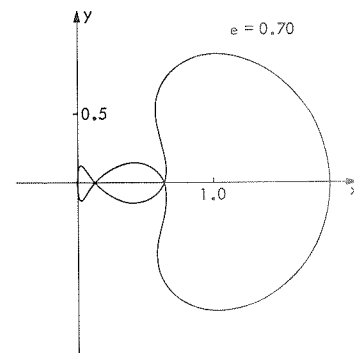
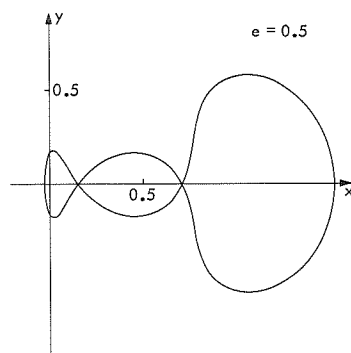
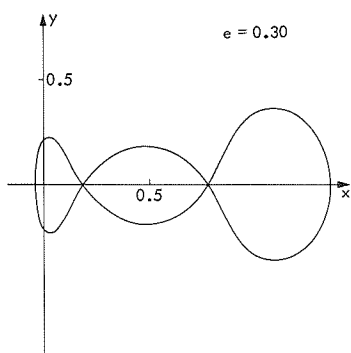
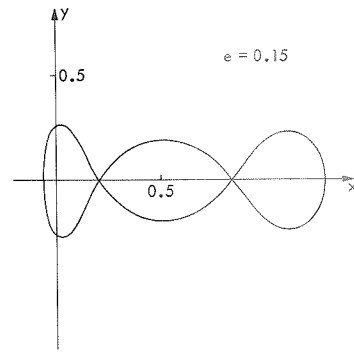
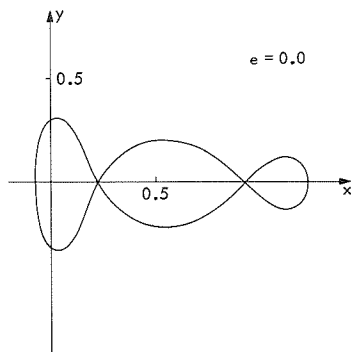
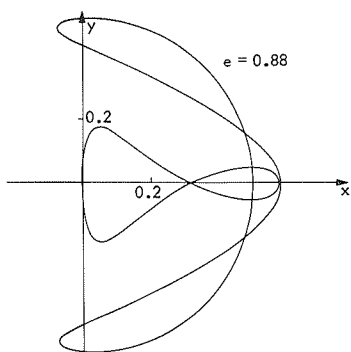


Fig. 20 (contd)

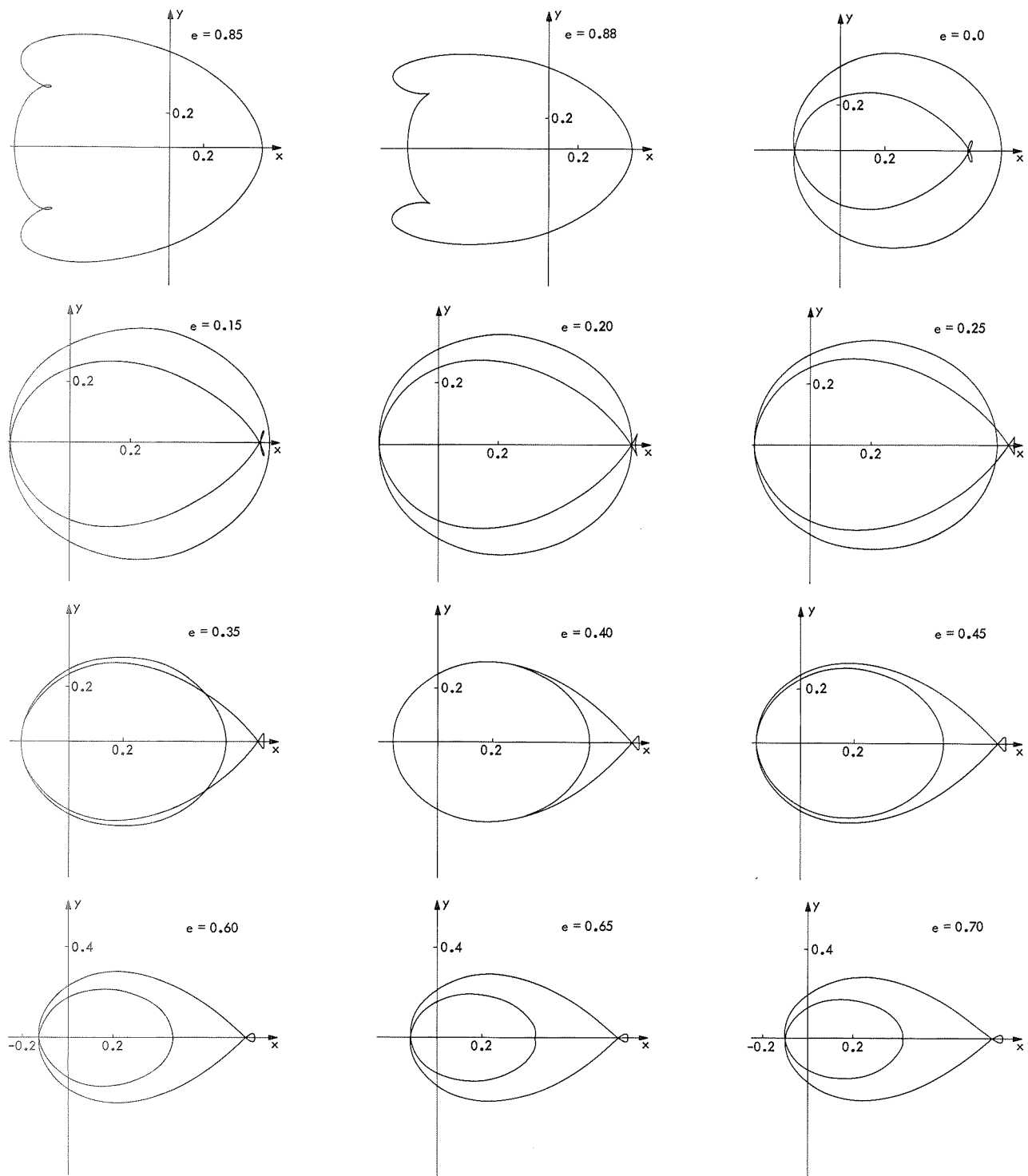


Fig. 20 (contd)

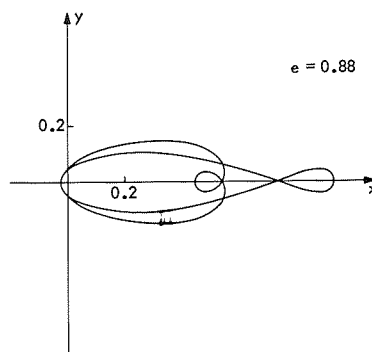
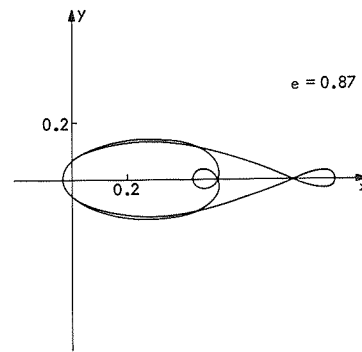
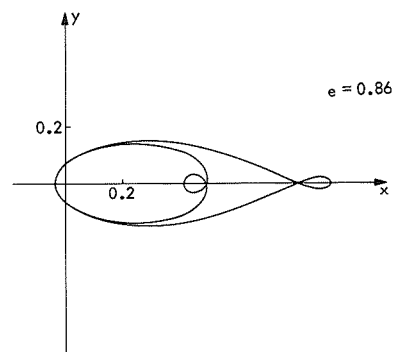
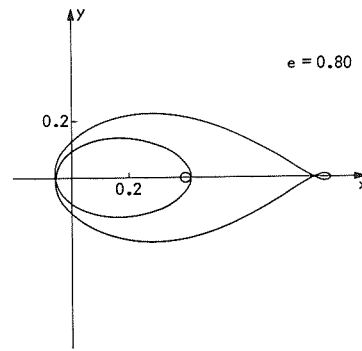
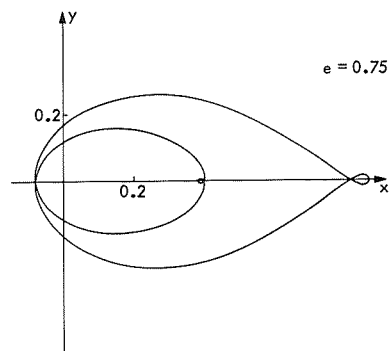


Fig. 20 (contd)

Table 17. Initial and final conditions for family 11A

ELL.RESTR.PROBL.	MU=0.5 (E	VARIABLE))	APDAPSIS	FAMILY 11		
NR	X0	YDOT0	X1	YDOT1	MASS RATIO	ECC
1	-0.0708483	0.8283275	1.2236640	-0.2419082	0.5000000	0.0000000
2	-0.0702090	0.8261843	1.2255026	-0.2470759	0.5000000	0.0050000
3	-0.0695752	0.8240555	1.2273528	-0.2522431	0.5000000	0.0100000
4	-0.0689469	0.8219408	1.2292148	-0.2574102	0.5000000	0.0150000
5	-0.0683240	0.8198400	1.2310889	-0.2625779	0.5000000	0.0200000
6	-0.0670939	0.8156795	1.2348740	-0.2729172	0.5000000	0.0300000
7	-0.0658843	0.8115723	1.2387100	-0.2832657	0.5000000	0.0400000
8	-0.0646947	0.8075171	1.2425987	-0.2936278	0.5000000	0.0500000
9	-0.0635246	0.8035126	1.2465420	-0.3040079	0.5000000	0.0600000
10	-0.0623734	0.7995575	1.2505418	-0.3144107	0.5000000	0.0700000
11	-0.0612406	0.7956506	1.2546001	-0.3248404	0.5000000	0.0800000
12	-0.0601258	0.7917910	1.2587190	-0.3353014	0.5000000	0.0900000
13	-0.0590284	0.7879774	1.2629005	-0.3457983	0.5000000	0.1000000
14	-0.0579482	0.7842090	1.2671468	-0.3563354	0.5000000	0.1100000
15	-0.0568846	0.7804849	1.2714603	-0.3669173	0.5000000	0.1200000
16	-0.0558372	0.7768043	1.2758431	-0.3775484	0.5000000	0.1300000
17	-0.0548056	0.7731663	1.2802977	-0.3882333	0.5000000	0.1400000
18	-0.0537896	0.7695704	1.2848267	-0.3989767	0.5000000	0.1500000
19	-0.0527886	0.7660157	1.2894326	-0.4097832	0.5000000	0.1600000
20	-0.0518024	0.7625018	1.2941182	-0.4206576	0.5000000	0.1700000
21	-0.0508307	0.7590281	1.2988863	-0.4316047	0.5000000	0.1800000
22	-0.0498730	0.7555941	1.3037397	-0.4426297	0.5000000	0.1900000
23	-0.0489291	0.7521994	1.3086817	-0.4537375	0.5000000	0.2000000
24	-0.0479987	0.7488437	1.3137155	-0.4649335	0.5000000	0.2100000
25	-0.0470815	0.7455266	1.3188444	-0.4762229	0.5000000	0.2200000
26	-0.0461772	0.7422478	1.3240719	-0.4876115	0.5000000	0.2300000
27	-0.0452856	0.7390071	1.3294018	-0.4991048	0.5000000	0.2400000
28	-0.0444063	0.7358044	1.3348380	-0.5107089	0.5000000	0.2500000
29	-0.0435392	0.7326396	1.3403846	-0.5224298	0.5000000	0.2600000
30	-0.0426839	0.7295126	1.3460459	-0.5342741	0.5000000	0.2700000
31	-0.0418403	0.7264233	1.3518266	-0.5462482	0.5000000	0.2800000
32	-0.0410080	0.7233719	1.3577313	-0.5583592	0.5000000	0.2900000
33	-0.0401870	0.7203584	1.3637652	-0.5706143	0.5000000	0.3000000
34	-0.0393768	0.7173830	1.3699337	-0.5830210	0.5000000	0.3100000
35	-0.0385774	0.7144458	1.3762425	-0.5955873	0.5000000	0.3200000
36	-0.0377885	0.7115472	1.3826977	-0.6083215	0.5000000	0.3300000
37	-0.0370099	0.7086874	1.3893057	-0.6212322	0.5000000	0.3400000
38	-0.0362414	0.7058668	1.3960733	-0.6343288	0.5000000	0.3500000
39	-0.0354829	0.7030857	1.4030079	-0.6476210	0.5000000	0.3600000
40	-0.0347340	0.7003447	1.4101171	-0.6611190	0.5000000	0.3700000
41	-0.0339947	0.6976443	1.4174094	-0.6748336	0.5000000	0.3800000
42	-0.0332646	0.6949849	1.4248936	-0.6887766	0.5000000	0.3900000
43	-0.0325438	0.6923672	1.4325792	-0.7029600	0.5000000	0.4000000
44	-0.0318318	0.6897919	1.4404764	-0.7173970	0.5000000	0.4100000
45	-0.0311287	0.6872597	1.4485963	-0.7321016	0.5000000	0.4200000
46	-0.0304341	0.6847713	1.4569507	-0.7470887	0.5000000	0.4300000
47	-0.0297480	0.6823274	1.4655522	-0.7623742	0.5000000	0.4400000
48	-0.0290700	0.6799291	1.4744148	-0.7779753	0.5000000	0.4500000
49	-0.0284001	0.6775770	1.4835533	-0.7939104	0.5000000	0.4600000
50	-0.0277381	0.6752721	1.4929840	-0.8101995	0.5000000	0.4700000
51	-0.0270837	0.6730152	1.5027245	-0.8268641	0.5000000	0.4800000
52	-0.0264367	0.6708074	1.5127939	-0.8439273	0.5000000	0.4900000
53	-0.0257971	0.6686495	1.5232132	-0.8614146	0.5000000	0.5000000
54	-0.0251645	0.6665424	1.5340054	-0.8793535	0.5000000	0.5100000
55	-0.0245388	0.6644870	1.5451957	-0.8977739	0.5000000	0.5200000
56	-0.0239198	0.6624842	1.5568119	-0.9167088	0.5000000	0.5300000
57	-0.0233071	0.6605345	1.5688845	-0.9361943	0.5000000	0.5400000
58	-0.0227007	0.6586387	1.5814474	-0.9562701	0.5000000	0.5500000
59	-0.0221002	0.6567971	1.5945383	-0.9769800	0.5000000	0.5600000
60	-0.0215054	0.6550101	1.6081991	-0.9983726	0.5000000	0.5700000

Table 17 (contd)

ELL.RESTR.PROBL.	MU=0.5 (E	VARIABLE))	APDAPSIS	FAMILY 11		
NR	XO	YDDTO	X1	YDOT1	MASS RATIO	ECC
61	-0.0209160	0.6532777	1.6224768	-1.0205021	0.5000000	0.5800000
62	-0.0203317	0.6515994	1.6374238	-1.0434286	0.5000000	0.5900000
63	-0.0197522	0.6499745	1.6530995	-1.0672199	0.5000000	0.6000000
64	-0.0191772	0.6484018	1.6695710	-1.0919518	0.5000000	0.6100000
65	-0.0186062	0.6468793	1.6869144	-1.1177103	0.5000000	0.6200000
66	-0.0180388	0.6454043	1.7052169	-1.1445929	0.5000000	0.6300000
67	-0.0174747	0.6439730	1.7245783	-1.1727108	0.5000000	0.6400000
68	-0.0169132	0.6425806	1.7451138	-1.2021918	0.5000000	0.6500000
69	-0.0163539	0.6412209	1.7669573	-1.2331831	0.5000000	0.6600000
70	-0.0157960	0.6398858	1.7902649	-1.2658555	0.5000000	0.6700000
71	-0.0152389	0.6385656	1.8152203	-1.3004090	0.5000000	0.6800000
72	-0.0146818	0.6372480	1.8420412	-1.3370785	0.5000000	0.6900000
73	-0.0141238	0.6359186	1.8709870	-1.3761428	0.5000000	0.7000000
74	-0.0135638	0.6345603	1.9023702	-1.4179351	0.5000000	0.7100000
75	-0.0130006	0.6331536	1.9365701	-1.4628575	0.5000000	0.7200000
76	-0.0124328	0.6316773	1.9740515	-1.5113991	0.5000000	0.7300000
77	-0.0118590	0.6301102	2.0153901	-1.5641616	0.5000000	0.7400000
78	-0.0112771	0.6284346	2.0613061	-1.6218924	0.5000000	0.7500000
79	-0.0106852	0.6266430	2.1127105	-1.6855299	0.5000000	0.7600000
80	-0.0100809	0.6247502	2.1707682	-1.7562650	0.5000000	0.7700000
81	-0.0094614	0.6228148	2.2369836	-1.8356240	0.5000000	0.7800000
82	-0.0088235	0.6209771	2.3133158	-1.9255781	0.5000000	0.7900000
83	-0.0081638	0.6195207	2.4023274	-2.0286853	0.5000000	0.8000000
84	-0.0074782	0.6189705	2.5073655	-2.1482591	0.5000000	0.8100000
85	-0.0067618	0.6202322	2.6327527	-2.2885449	0.5000000	0.8200000
86	-0.0060082	0.6247621	2.7839428	-2.4548627	0.5000000	0.8300000
87	-0.0052072	0.6347131	2.9675846	-2.6536719	0.5000000	0.8400000
88	-0.0043416	0.6529777	3.1915298	-2.8926113	0.5000000	0.8500000
89	-0.0033809	0.6831124	3.4650655	-3.1808004	0.5000000	0.8600000
90	-0.0022736	0.7293410	3.7999312	-3.5299395	0.5000000	0.8700000
91	-0.0009317	0.7970914	4.2128522	-3.9569061	0.5000000	0.8800000
92	0.0008073	0.8949025	4.7310377	-4.4893070	0.5000000	0.8900000
93	0.0019244	0.9598935	5.0439827	-4.8096513	0.5000000	0.8950000

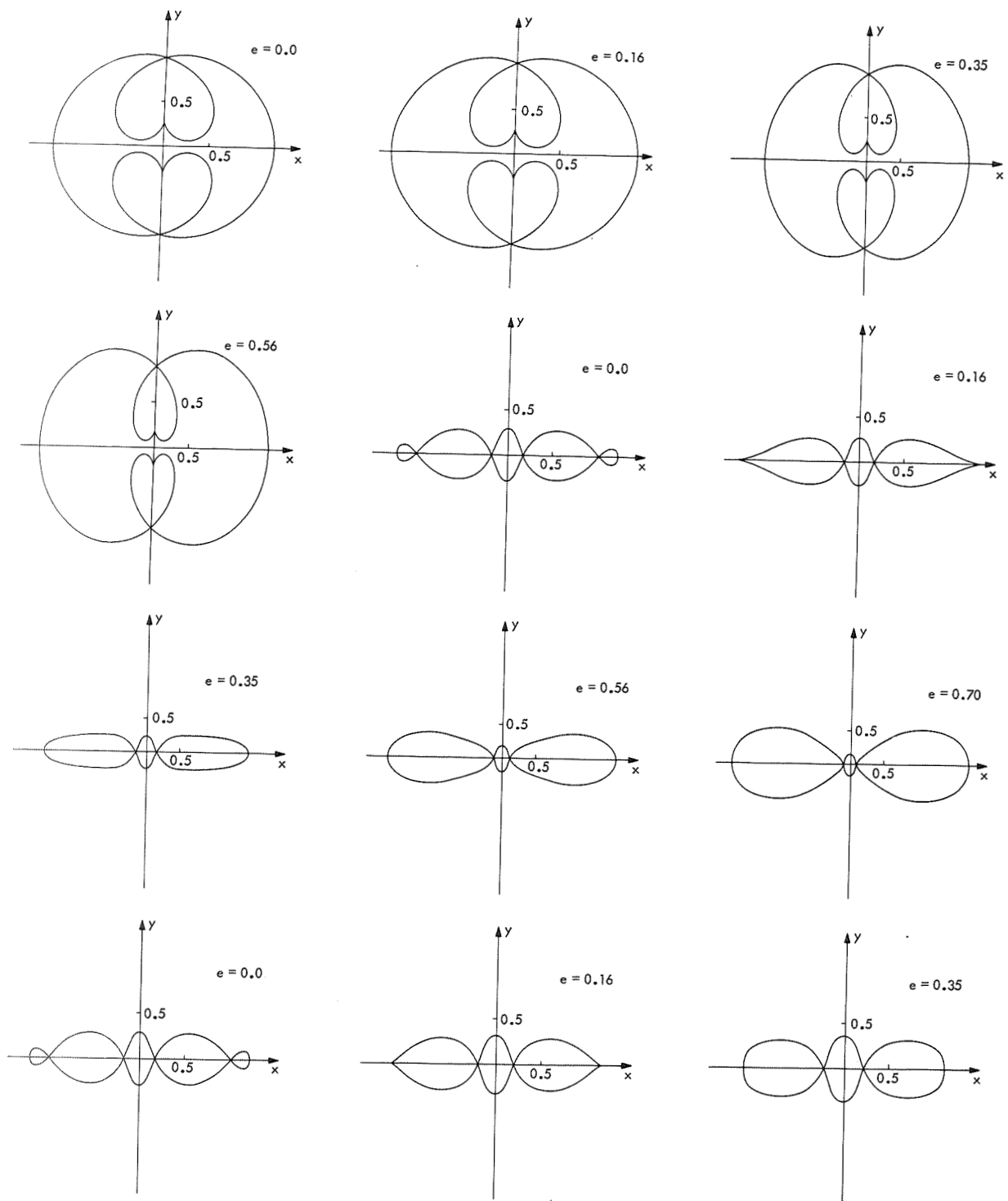


Fig. 21. Family 10P of periodic orbits

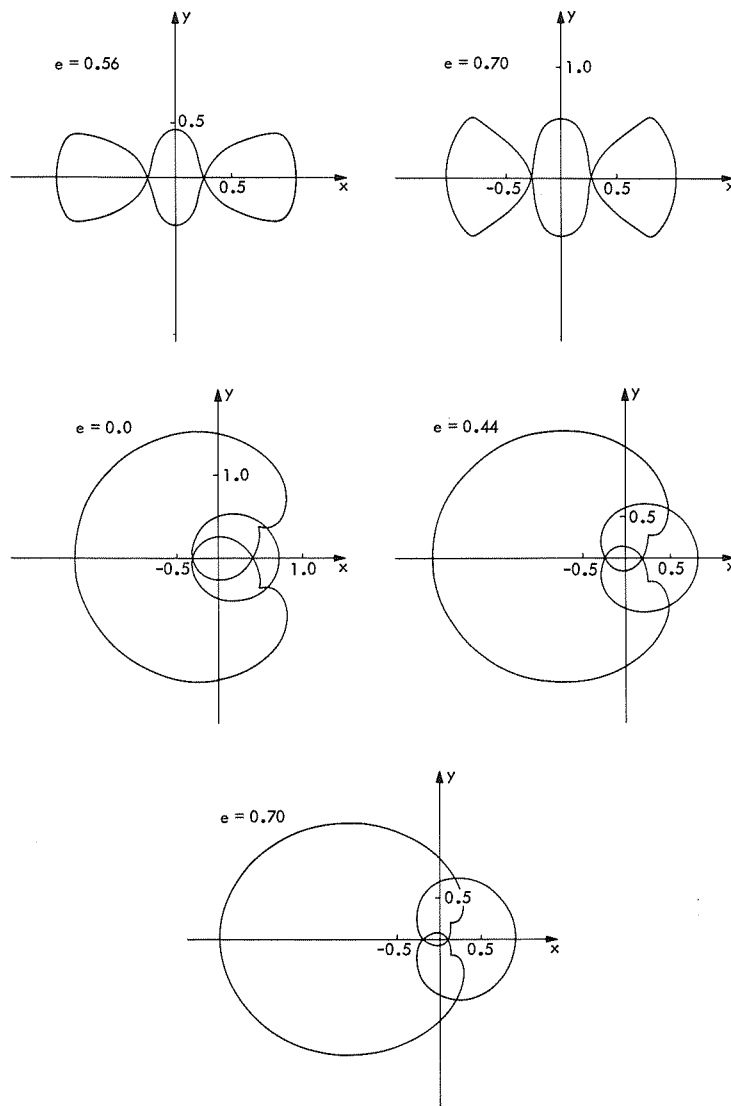


Fig. 21 (contd)

Table 18. Initial and final conditions for family 10P

ELL.RESTR.PROBL. MU=0.5 (E VARIABLE)) X AND Y SYM. FAMILY 10						
NR	YO	XDDTO	X1	YDOT1	MASS RATIO	ECC
1	-0.2952427	-0.3196529	-1.2283965	0.1762734	0.5000000	0.0000000
2	-0.2961463	-0.3207342	-1.2264245	0.1705527	0.5000000	0.0050000
3	-0.2970537	-0.3218168	-1.2244687	0.1648409	0.5000000	0.0100000
4	-0.2979651	-0.3229008	-1.2225285	0.1591371	0.5000000	0.0150000
5	-0.2988807	-0.3239862	-1.2206052	0.1534424	0.5000000	0.0200000
6	-0.2998002	-0.3250729	-1.2186971	0.1477549	0.5000000	0.0250000
7	-0.3007239	-0.3261610	-1.2168045	0.1420746	0.5000000	0.0300000
8	-0.3016517	-0.3272505	-1.2149273	0.1364012	0.5000000	0.0350000
9	-0.3025837	-0.3283413	-1.2130653	0.1307343	0.5000000	0.0400000
10	-0.3035199	-0.3294335	-1.2112188	0.1250740	0.5000000	0.0450000
11	-0.3044604	-0.3305270	-1.2093863	0.1194182	0.5000000	0.0500000
12	-0.3063543	-0.3327181	-1.2057661	0.1081232	0.5000000	0.0600000
13	-0.3082659	-0.3349145	-1.2022036	0.0968461	0.5000000	0.0700000
14	-0.3101956	-0.3371164	-1.1986975	0.0855836	0.5000000	0.0800000
15	-0.3121436	-0.3393238	-1.1952467	0.0743326	0.5000000	0.0900000
16	-0.3141105	-0.3415366	-1.1918501	0.0630897	0.5000000	0.1000000
17	-0.3160966	-0.3437549	-1.1885066	0.0518516	0.5000000	0.1100000
18	-0.3181025	-0.3459789	-1.1852150	0.0406150	0.5000000	0.1200000
19	-0.3201286	-0.3482084	-1.1819749	0.0293768	0.5000000	0.1300000
20	-0.3221755	-0.3504438	-1.1787847	0.0181332	0.5000000	0.1400000
21	-0.3242436	-0.3526850	-1.1756437	0.0068809	0.5000000	0.1500000
22	-0.3263335	-0.3549323	-1.1725510	-0.0043837	0.5000000	0.1600000
23	-0.3284459	-0.3571857	-1.1695058	-0.0156641	0.5000000	0.1700000
24	-0.3305813	-0.3594453	-1.1665071	-0.0269639	0.5000000	0.1800000
25	-0.3327404	-0.3617115	-1.1635541	-0.0382873	0.5000000	0.1900000
26	-0.3349238	-0.3639844	-1.1606461	-0.0496377	0.5000000	0.2000000
27	-0.3371323	-0.3662641	-1.1577822	-0.0610193	0.5000000	0.2100000
28	-0.3393667	-0.3685510	-1.1549617	-0.0724362	0.5000000	0.2200000
29	-0.3416276	-0.3708452	-1.1521839	-0.0838926	0.5000000	0.2300000
30	-0.3439161	-0.3731471	-1.1494480	-0.0953928	0.5000000	0.2400000
31	-0.3462329	-0.3754570	-1.1467535	-0.1069414	0.5000000	0.2500000
32	-0.3485789	-0.3777751	-1.1440994	-0.1185431	0.5000000	0.2600000
33	-0.3509552	-0.3801019	-1.1414853	-0.1302027	0.5000000	0.2700000
34	-0.3533628	-0.3824378	-1.1389104	-0.1419253	0.5000000	0.2800000
35	-0.3558027	-0.3847831	-1.1363740	-0.1537163	0.5000000	0.2900000
36	-0.3582762	-0.3871382	-1.1338756	-0.1655811	0.5000000	0.3000000
37	-0.3607846	-0.3895036	-1.1314144	-0.1775258	0.5000000	0.3100000
38	-0.3633288	-0.3918803	-1.1289899	-0.1895553	0.5000000	0.3200000
39	-0.3659106	-0.3942683	-1.1266015	-0.2016773	0.5000000	0.3300000
40	-0.3685313	-0.3966683	-1.1242484	-0.2138981	0.5000000	0.3400000
41	-0.3711924	-0.3990810	-1.1219302	-0.2262246	0.5000000	0.3500000
42	-0.3738957	-0.4015071	-1.1196460	-0.2386645	0.5000000	0.3600000
43	-0.3766430	-0.4039474	-1.1173953	-0.2512255	0.5000000	0.3700000
44	-0.3794360	-0.4064026	-1.1151775	-0.2639162	0.5000000	0.3800000
45	-0.3822769	-0.4088737	-1.1129917	-0.2767456	0.5000000	0.3900000
46	-0.3851679	-0.4113615	-1.1108372	-0.2897233	0.5000000	0.4000000
47	-0.3881113	-0.4138671	-1.1087140	-0.3028586	0.5000000	0.4100000
48	-0.3911096	-0.4163916	-1.1066203	-0.3161637	0.5000000	0.4200000
49	-0.3941656	-0.4189361	-1.1045558	-0.3296498	0.5000000	0.4300000
50	-0.3972822	-0.4215019	-1.1025196	-0.3433298	0.5000000	0.4400000
51	-0.4004626	-0.4240905	-1.1005106	-0.3572173	0.5000000	0.4500000
52	-0.4037105	-0.4267033	-1.0985281	-0.3713271	0.5000000	0.4600000
53	-0.4070294	-0.4293420	-1.0965704	-0.3856761	0.5000000	0.4700000
54	-0.4104238	-0.4320083	-1.0946368	-0.4002814	0.5000000	0.4800000
55	-0.4138983	-0.4347044	-1.0927257	-1.0908355	0.5000000	0.4900000
56	-0.4174578	-0.4374323	-0.4151629	-0.4303421	0.0000000	0.5000000
57	-0.4211082	-0.4401945	-1.0889644	-0.4458430	0.5000000	0.5100000
58	-0.4248558	-0.4429935	-1.0871102	-0.4616926	0.5000000	0.5200000
59	-0.4287078	-0.4458325	-1.0852709	-0.4779202	0.5000000	0.5300000
60	-0.4326722	-0.4487147	-1.0834430	-0.4945605	0.5000000	0.5400000

Table 18 (contd)

ELL.RESTR.PROBL. MU=0.5 (E VARIABLE)) X AND Y SYM. FAMILY 10							
NR	Y0	XDOT0	X1	YDOT1	MASS RATIO	ECC	
61	-0.4367584	-0.4516439	-1.0816235	-0.5116522	0.5000000	0.5500000	
62	-0.4409769	-0.4546243	-1.0798080	-0.5292397	0.5000000	0.5600000	
63	-0.4453402	-0.4576609	-1.0779915	-0.5473751	0.5000000	0.5700000	
64	-0.4498628	-0.4607593	-1.0761678	-0.5661195	0.5000000	0.5800000	
65	-0.4545622	-0.4639262	-1.0743290	-0.5855459	0.5000000	0.5900000	
66	-0.4594595	-0.4671695	-1.0724648	-0.6057430	0.5000000	0.6000000	
67	-0.4645808	-0.4704987	-1.0705622	-0.6268205	0.5000000	0.6100000	
68	-0.4699592	-0.4739257	-1.0686035	-0.6489178	0.5000000	0.6200000	
69	-0.4756377	-0.4774657	-1.0665644	-0.6722167	0.5000000	0.6300000	
70	-0.4816747	-0.4811385	-1.0644104	-0.6969639	0.5000000	0.6400000	
71	-0.4881523	-0.4849718	-1.0620903	-0.7235097	0.5000000	0.6500000	
72	-0.4951948	-0.4890059	-1.0595226	-0.7523847	0.5000000	0.6600000	
73	-0.5030064	-0.4933061	-1.0565675	-0.7844678	0.5000000	0.6700000	
74	-0.5119728	-0.4979931	-1.0529487	-0.8214318	0.5000000	0.6800000	
75	-0.5230283	-0.5033547	-1.0479696	-0.8673676	0.5000000	0.6900000	
76	-0.5409435	-0.5108416	-1.0379172	-0.9434420	0.5000000	0.7000000	

No attempt has yet been made to determine the reason for this slow convergence or to prolongate the family for eccentricities higher than 0.70. The stability and the characteristic exponents of these orbits have not been determined.

J. A Family of Periodic Collision Orbits

Using the regularized computer program, it was possible to find a family of 69 periodic collision orbits. The integrations were made with a predictor-corrector method (eighth-order) and Birkhoff regularized coordinates in the neighborhood of the primaries (within a radius of 0.07 canonical unit of distance from each primary). The starting procedure used for the Adams-Moulton integration was Runge-Kutta-Merson, of fourth order, with automatic error control and step adjusting. Outside of the two circular areas of radius 0.07 around the two primaries, the integrations were performed with the recurrent power-series method. Generally, the terms up to twelfth order have been used in the power series, with an integration step of 0.03, as compared with a step of 0.005 in the Adams-Moulton method.

The principles of differential corrections for periodic ejection orbits have been explained previously. In all of the numerical work described herein, the collision angle θ has been kept at zero, and only collisions from the primary $m_2 (= \mu)$ have been studied. At first, some attempts were made to find a periodic collision orbit corresponding to equal masses, but these attempts were unsuccessful. It was then decided to run a series of collision orbits in the equal-mass problem.

The eccentricities e were varied from 0.1 to 0.5, and the energy constant was taken between 0 and -1.5 . This allowed the determination of the interesting regions of energies shown in Fig. 22. Orbits with energies larger than about -0.7 all escape from the system, whereas orbits with energies below -1.2 are close satellite orbits, and have been disregarded. The region that has then been studied in more detail is the white region in the center of Fig. 22. The following procedure was then performed to find approximate initial conditions for periodic orbits: the two lines $y_F = 0$ and $\dot{x}_F = 0$ were constructed by examining the final values y_F and $\dot{x}_F$ of the variables y and x (at the end of a half revolution, $v = \pi$). The intersection of these two lines then corresponds to a symmetric periodic orbit. An intersection was found close

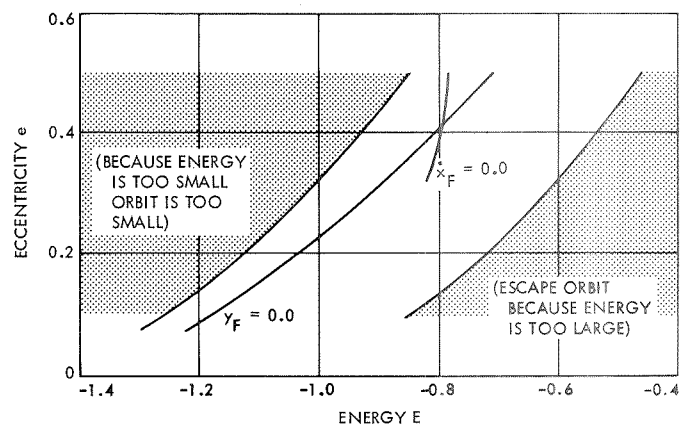


Fig. 22. Ejection orbits (from $m_2 = \mu$)
with $\theta = 0.0$, $\mu = 0.5$

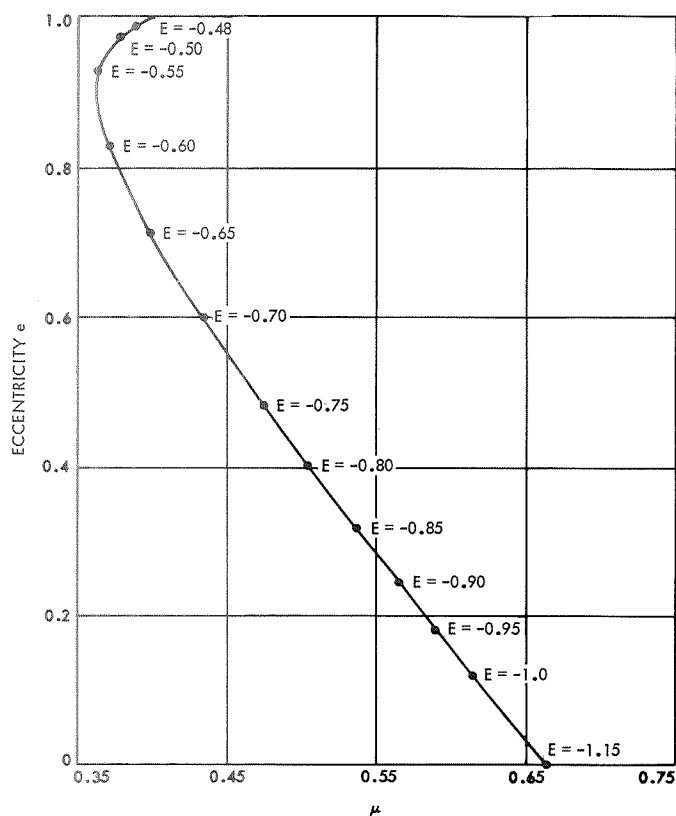


Fig. 23. Family of ejection orbits (e vs μ)

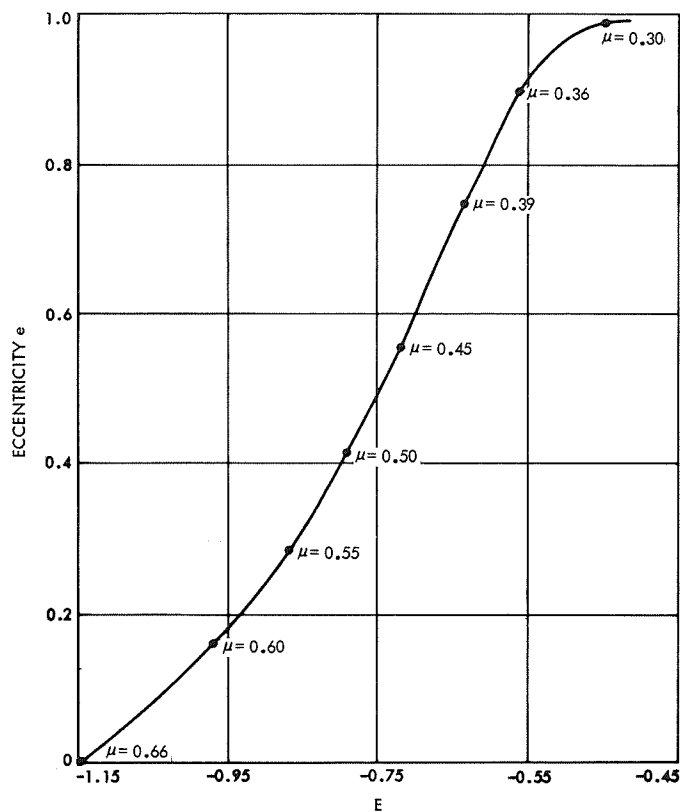


Fig. 24. Family of ejection orbits (e vs E)

to $E = -0.79$, $e = +0.42$. Starting from these values, the exact periodic orbit was then obtained with the differential correction program.

Once the first periodic orbit had been found, corresponding to $\mu = 0.5$, it was not difficult to follow the corresponding family of periodic orbits along increasing and decreasing mass-ratios. The variations of the eccentricity e along this line are shown in Figs. 23 and 24. At the eccentricity $e = 0.0$, the family joins a classical collision orbit of the circular three-body problem, with period 2π . At the other end of this family ($e = 1$), there

seems to be a triple collision of both primaries and the satellite. The last orbit computed corresponds to $e = 0.995$, $\mu = 0.40$. It is obvious that, to be able to study the end of this family numerically, a program is necessary that can handle (and regularize eventually) the triple-collision phenomenon.

Figures 25 and 26 show (in Birkhoff coordinates) two extreme orbits of the family, with $e = 0$ and $e = 0.98$. Figure 27 shows some of the collision orbits represented in different coordinate systems. The final conditions for this family are listed in Table 19.

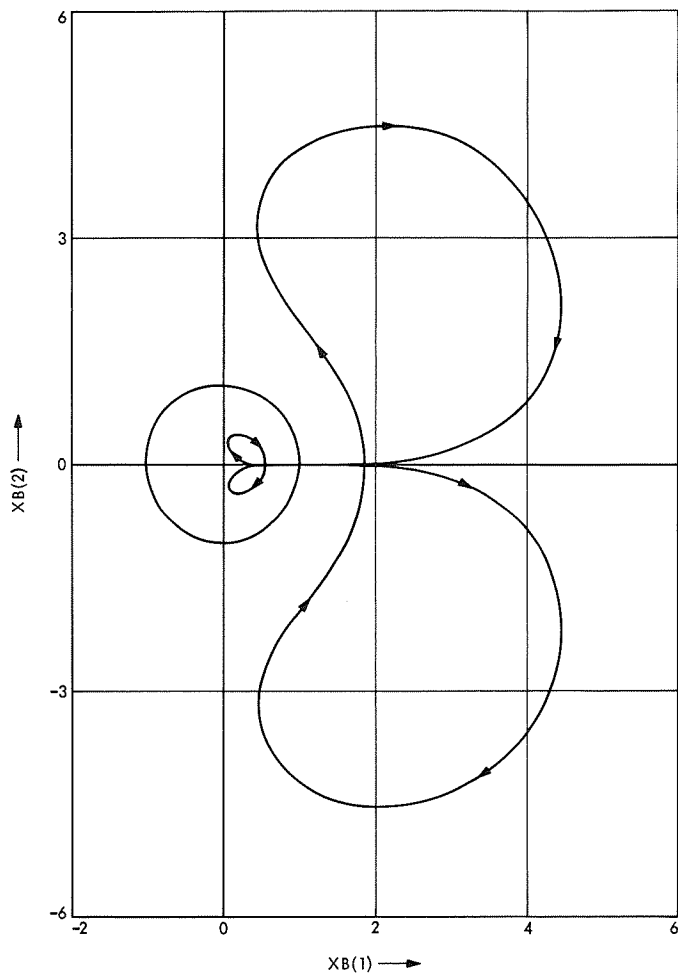


Fig. 25. An extreme orbit of the family of periodic collision orbits ($e = 0$)

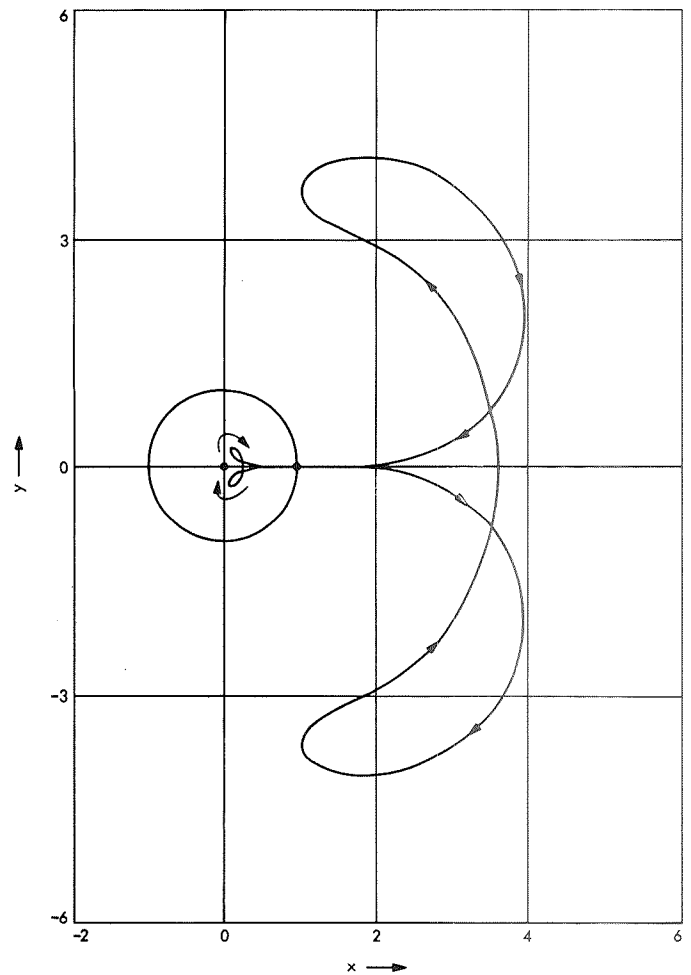


Fig. 26. An extreme orbit of the family of periodic collision orbits ($e = 0.98$)

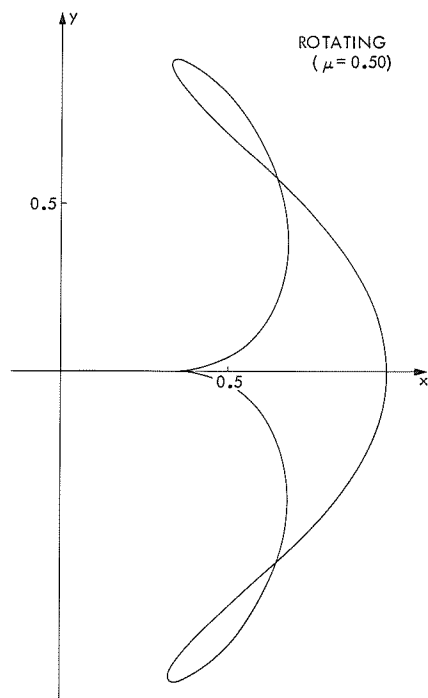
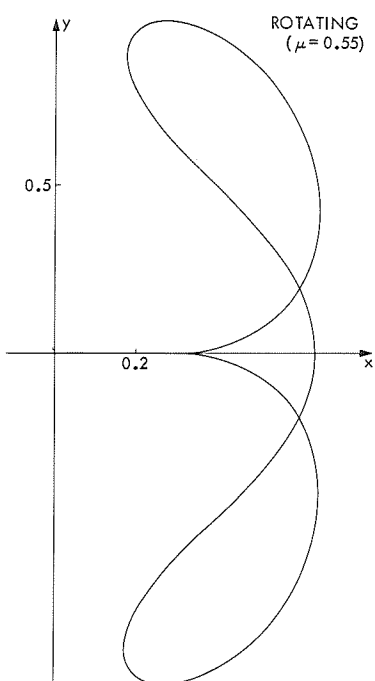
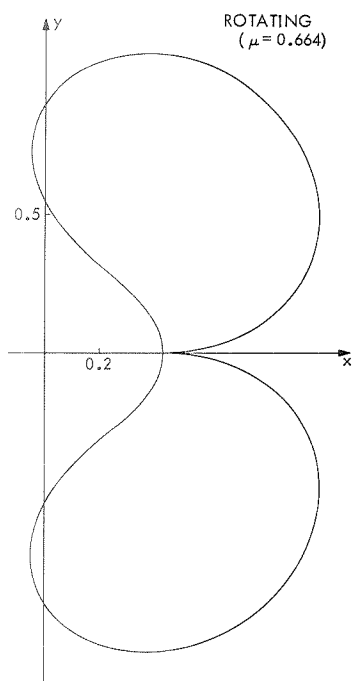
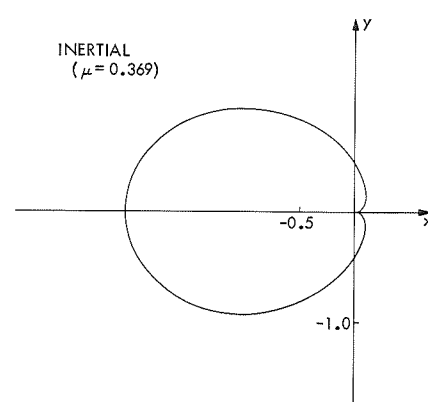
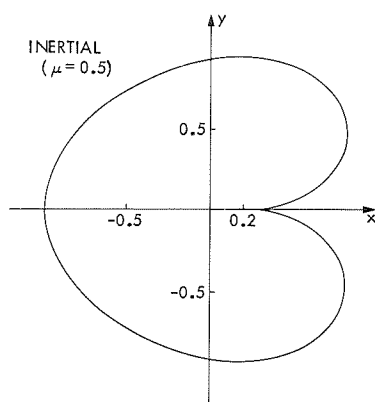
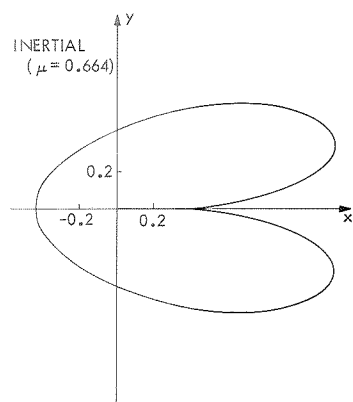


Fig. 27. Periodic collision orbits in different coordinate systems

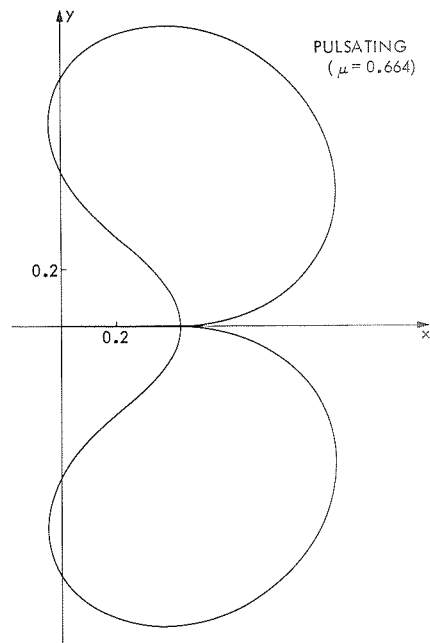
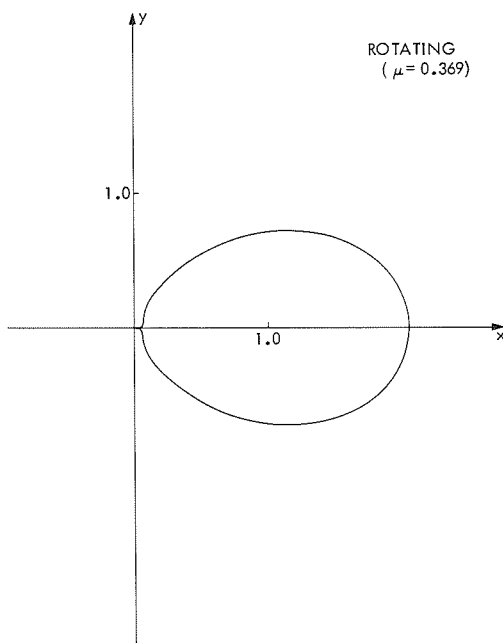
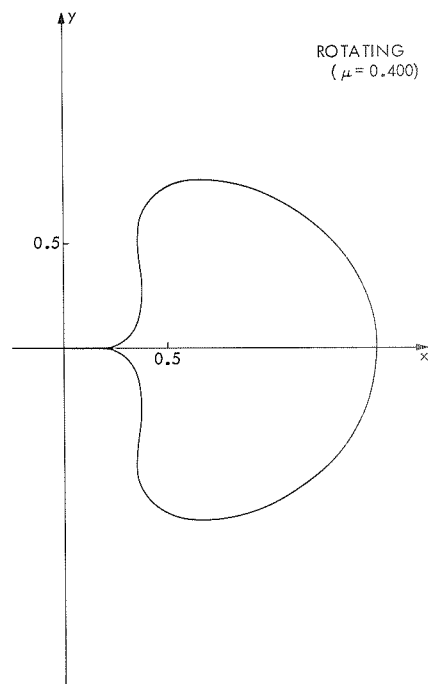
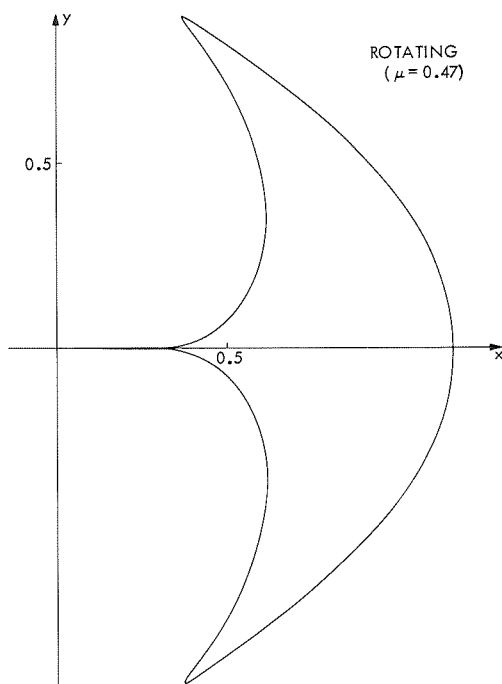


Fig. 27 (contd)

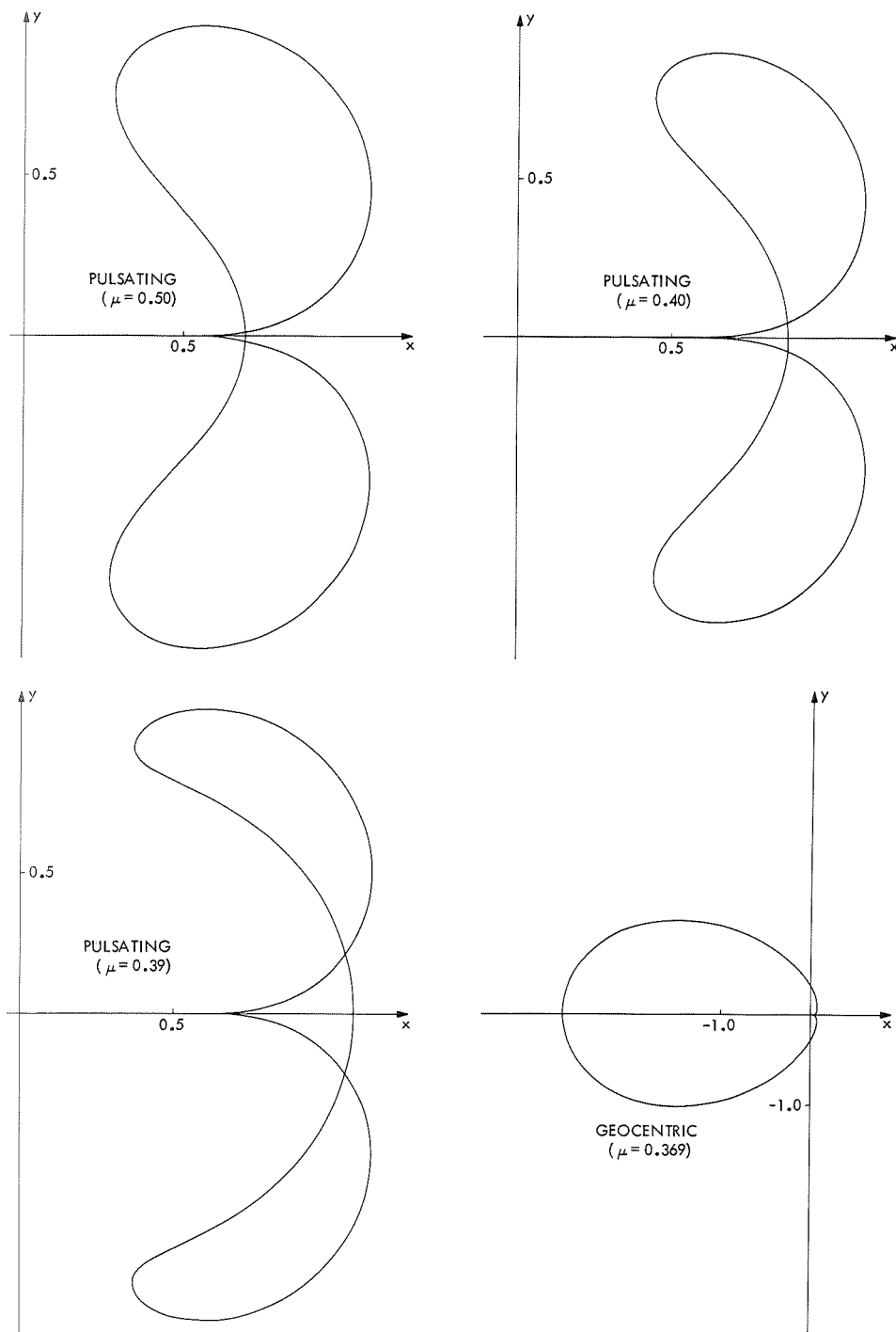


Fig. 27 (contd)

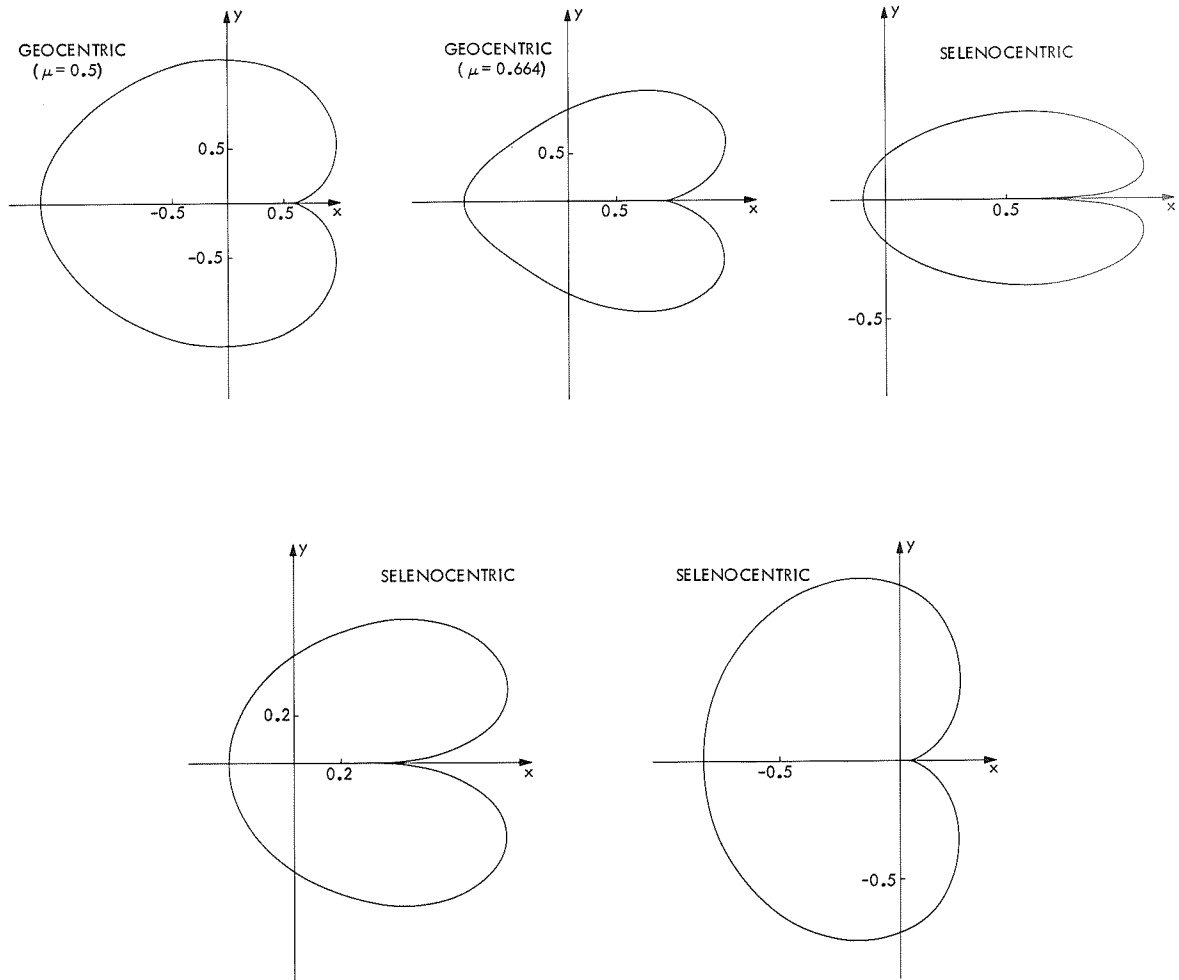


Fig. 27 (contd)

Table 19. Final conditions for periodic collision orbits

PERIODIC COLLISION ORBITS					
NR	X1	YDOT1	ENERGY	MASS RATIO	ECC
1	0.4317709	3.5154248	-1.1470386	0.6641096	-0.0000000
2	0.4422967	3.4560406	-1.1248080	0.6570000	0.0174910
3	0.4452673	3.4397299	-1.1187117	0.6550000	0.0224155
4	0.4482423	3.4235877	-1.1126825	0.6530000	0.0273420
5	0.4512215	3.4076114	-1.1067194	0.6510000	0.0322704
6	0.4542051	3.3917987	-1.1008213	0.6490000	0.0372008
7	0.4571929	3.3761471	-1.0949873	0.6470000	0.0421334
8	0.4601851	3.3606543	-1.0892163	0.6450000	0.0470682
9	0.4752110	3.2854956	-1.0612741	0.6350000	0.0717771
10	0.4903469	3.2139907	-1.0347706	0.6250000	0.0965512
11	0.5055945	3.1459079	-1.0096007	0.6150000	0.1214000
12	0.5209562	3.0810431	-0.9856676	0.6050000	0.1463340
13	0.5364350	3.0192178	-0.9628819	0.5950000	0.1713645
14	0.5520341	2.9602770	-0.9411609	0.5850000	0.1965040
15	0.5598800	2.9318458	-0.9306753	0.5800000	0.2091188
16	0.5677575	2.9040875	-0.9204274	0.5750000	0.2217661
17	0.5756673	2.8769886	-0.9104082	0.5700000	0.2344478
18	0.5836100	2.8505372	-0.9006093	0.5650000	0.2471659
19	0.5915865	2.8247225	-0.8910223	0.5600000	0.2599226
20	0.5995974	2.7995348	-0.8816390	0.5550000	0.2727201
21	0.6076438	2.7749661	-0.8724517	0.5500000	0.2855607
22	0.6157265	2.7510093	-0.8634525	0.5450000	0.2984472
23	0.6238467	2.7276591	-0.8546340	0.5400000	0.3113820
24	0.6320055	2.7049113	-0.8459889	0.5350000	0.3243681
25	0.6402042	2.6827633	-0.8375099	0.5300000	0.3374084
26	0.6484443	2.6612140	-0.8291898	0.5250000	0.3505063
27	0.6567272	2.6402639	-0.8210217	0.5200000	0.3636651
28	0.6650549	2.6199155	-0.8129987	0.5150000	0.3768885
29	0.6734291	2.6001731	-0.8051139	0.5100000	0.3901806
30	0.6818520	2.5810431	-0.7973603	0.5050000	0.4035455
31	0.6852354	2.5735644	-0.7942941	0.5030000	0.4089129
32	0.6860825	2.5717103	-0.7935307	0.5025000	0.4102567
33	0.6869301	2.5698625	-0.7927684	0.5020000	0.4116013
34	0.6877783	2.5680210	-0.7920073	0.5015000	0.4129468
35	0.6886270	2.5661857	-0.7912474	0.5010000	0.4142930
36	0.6894762	2.5643568	-0.7904887	0.5005000	0.4156400
37	0.6903260	2.5625342	-0.7897312	0.5000000	0.4169879
38	0.6911763	2.5607178	-0.7889749	0.4995000	0.4183366
39	0.6920272	2.5589078	-0.7882197	0.4990000	0.4196861
40	0.6928786	2.5571042	-0.7874657	0.4985000	0.4210365
41	0.6937305	2.5553069	-0.7867129	0.4980000	0.4223877
42	0.6971437	2.5481817	-0.7837129	0.4960000	0.4278010
43	0.6988537	2.5446576	-0.7822196	0.4950000	0.4305128
44	0.7074380	2.5274276	-0.7748186	0.4900000	0.4441257
45	0.7160823	2.5108617	-0.7675211	0.4850000	0.4578323
46	0.7247902	2.4949811	-0.7603199	0.4800000	0.4716393
47	0.7424141	2.4653825	-0.7461761	0.4700000	0.4995836
48	0.7603504	2.4389009	-0.7323241	0.4600000	0.5280252
49	0.7786518	2.4159184	-0.7186934	0.4500000	0.5570476
50	0.7973892	2.3969803	-0.7052024	0.4400000	0.5867579
51	0.8166596	2.3828802	-0.6917513	0.4300000	0.6172978
52	0.8366019	2.3748101	-0.6782115	0.4200000	0.6488630
53	0.8574241	2.3746475	-0.6644036	0.4100000	0.6817391
54	0.8794613	2.3855670	-0.6500552	0.4000000	0.7163739
55	0.9033125	2.4135514	-0.6346959	0.3900000	0.7535523
56	0.9302562	2.4721542	-0.6173412	0.3800000	0.7949235
57	0.9458395	2.5235941	-0.6071866	0.3750000	0.8183975
58	0.9642965	2.6070881	-0.5949220	0.3700000	0.8455941
59	0.9910076	2.7898253	-0.5763913	0.3650000	0.8833366
60	1.0006883	2.8816913	-0.5693343	0.3640000	0.8963937

Table 19 (contd)

PERIODIC COLLISION ORBITS					
NR	X1	YDOT1	ENERGY	MASS RATIO	ECC
61	1.0083154	2.9671155	-0.5636017	0.3635774	0.9063937
62	1.0161692	3.0697326	-0.5575095	0.3635184	0.9163937
63	1.0243012	3.1951282	-0.5509627	0.3639060	0.9263937
64	1.0327810	3.3518316	-0.5438277	0.3648572	0.9363937
65	1.0417075	3.5536714	-0.5359065	0.3665460	0.9463937
66	1.0512286	3.8249412	-0.5268868	0.3692469	0.9563937
67	1.0615840	4.2134702	-0.5162289	0.3734351	0.9663937
68	1.0732118	4.8315714	-0.5028633	0.3800552	0.9763937
69	1.0870994	6.0419078	-0.4840867	0.3915046	0.9863937

V. Computer Programs

A. Introduction

The periodic orbits described previously were obtained with the use of several more or less automatic computer programs. These programs were all written in FORTRAN 4 language, but use a limited number of machine-language subroutines taken from the JPL library. Because of the presence of some nonstandard subroutines in these programs, and because of a few characteristics that are related to JPL's particular hardware configurations, it has not been judged useful to include herein a complete description of all of the computer programs.

Some of the nonstandard subroutines that are not described herein are for timing of the computations and for plotting of the orbits; others are standard subroutines, e.g., those for matrix inversion or for step-wise numerical integration. Every reader who is interested in the integration of orbits will certainly have his own programs for numerical integration or for matrix inversion. Thus, it was decided to describe herein a few subroutines that are directly related to the elliptic restricted three-body problem.

These subroutines do not all belong to the same program, but they have been used in different main programs. To give an example of the logic used in the computing efforts, two main programs are also described.

Some of the descriptions given below are not quite complete, but they are complemented by the presence of many comments in the programs. It is hoped that, with the information provided herein and with the FORTRAN 4 listings, any programmer will be able to construct fairly rapidly a running program for the elliptic restricted three-body problem.

B. Derivative Subroutines

To integrate the equations of motion numerically, it is necessary to program the derivatives of the dependent variables (or the right-hand members of the differential equations), and to insert this coding in the main program or in the subroutine that is used for numerical integration. The subroutines described below give the derivatives for a few different coordinate systems or integration methods. In all cases, the differential equations are reduced to first-order equations.

The first four subroutines (DERIV) are to be used with predictor-corrector or Runge-Kutta types of integration methods. For completeness, a short fourth-order classical Runge-Kutta integration subroutine is also given; however, the user should preferably replace it with a more efficient integrator. Each call to the Runge-Kutta subroutine will perform a single integration step, and this subroutine will call subroutine DERIV four times. Subroutine RUNGE-KUTTA may be used for any number N of simultaneous first-order differential equations.

1. Subroutine DERIV with inertial coordinates. This subroutine is for the numerical integration of the equations of motion, Eq. (13), in barycentric inertial coordinates. This subroutine may be used for all eccentricities, including $e = +1$. It is prepared for five dependent variables, the last of which is also the independent variable t (time). The subroutine calls subroutine KEPLER to transform the mean anomaly AM in the eccentric anomaly ECAN. Subroutine KEPLER uses the Newtonian iteration method. In case the Newtonian iteration method does not converge, a bisection method is used. This can only happen for an eccentricity of $+1$ and for some exceptional values of the mean anomaly AM.

Subroutine DERIV uses the following constants, which are in COMMON:

$$RU = \text{mass-ratio } \mu = m_2 \quad (182a)$$

$$UMM = 1 - \mu = m_1 \quad (182b)$$

$$E = \text{eccentricity } e \quad (182c)$$

$$Q = (1 - e^2)^{1/2} \quad (182d)$$

2. Subroutine DERIV with rotating-pulsating coordinates. This subroutine uses the pulsating coordinates (x, y) described in Eq. (33). The independent variable is the true anomaly v . The distance r between the primaries is obtained here as a function of the true anomaly rather than by integrating a differential equation. The first four equations give $(x, y, \dot{x}, \dot{y})$, the fifth gives the time t , and the sixth gives the true anomaly v . No Kepler equation has to be solved with this set of variables, the primaries having fixed locations on the x -axis. This subroutine uses the following constants, which must be in COMMON:

$$RU = \text{mass-ratio } \mu = m_2 \quad (183a)$$

$$UMM = 1 - \mu = m_1 \quad (183b)$$

$$XII = x_1 = -\mu = -m_2 \quad (183c)$$

$$XI2 = x_2 = 1 - \mu = m_1 \quad (183d)$$

$$E = \text{eccentricity } e \quad (183e)$$

$$P = 1 - e^2 \quad (183f)$$

$$Q = (1 - e^2)^{1/2} \quad (183g)$$

3. Subroutine DERIV with regularized Birkhoff coordinates. This is the subroutine that has been used in the regularized program; it integrates differential Eqs. (110), (116), and (117). Here the differential equations are again in the first-order form. The first four equations give $(\xi, \eta, \dot{\xi}, \dot{\eta})$. The last three equations give the energy E , true anomaly v , and time t . The regularized independent variable s is not explicitly present in the differential equations. The following constants are used by the subroutine, and must be in COMMON:

$$AM2 = \mu = m_2 \text{ (mass-ratio)} \quad (184a)$$

$$AM1 = 1 - \mu = m_1 \quad (184b)$$

$$XII = x_1 = -\mu = -m_2 \quad (184c)$$

$$XI2 = x_2 = 1 - \mu = m_1 \quad (184d)$$

$$E = \text{eccentricity } e \quad (184e)$$

$$P = 1 - e^2 \quad (184f)$$

$$SP = (1 - e^2)^{1/2} \quad (184g)$$

$$XO = x_0 = \frac{1}{2} - \mu = \frac{(x_1 + x_2)}{2} \quad (184h)$$

4. Subroutine DERIV with rotating-pulsating variables and with variational equations. This subroutine uses the same variables and the same formulation as the subroutine described in Section V-B-3. The first four variables are the same as before. The fifth to twentieth variables are four times the four variations. The 16 variables (5 to 20) will thus give the fundamental matrix $\mathbf{R}$ with partial derivatives. The initial conditions for the 16 variables should form a unit-matrix. It should be noted that good results will be obtained for the solutions of the variational equations only if a good integration subroutine is available. In particular, if the simple Runge-Kutta subroutine is used, no high precision should be expected. On the other hand, there will also be an appreciable loss of precision in the solution of the variational equations in cases with close approaches between the satellite and the primaries, or between the primaries themselves (for high values of e).

The following constants must be present in COMMON for this subroutine:

$$RU = \text{mass-ratio } \mu = m_2 \quad (185a)$$

$$UMM = 1 - \mu = m_1 \quad (185b)$$

$$XII = x_1 = -\mu = -m_2 \quad (185c)$$

$$XI2 = x_2 = 1 - \mu = m_1 \quad (185d)$$

$$E = \text{eccentricity } e \quad (185e)$$

$$P = 1 - e^2 \quad (185f)$$

$$Q = (1 - e^2)^{1/2} \quad (185g)$$

5. Subroutine ERPSV for recurrent power series. This subroutine implements the solution of both the equations of motion and four sets of variational equations with

the recurrent power-series method and with rotating-pulsating variables. The equations that are programmed are Eqs. (139) and (141) and those listed in Tables 1 and 2. The notations of these equations are closely followed in the FORTRAN 4 program; for this reason, very little explanation is necessary here.

In the listing of subroutine ERPSV, it can be seen that only a minimum number of DO-loops has been used. It was considered preferable to write these long lists of similar statements because, on many compilers or computers, this will actually result in a more efficient object program from the point of view of the speed of calculations. Subroutine ERPSV could have been programmed in a much shorter way, but this would be possible only by using more nested DO-loops and a slightly longer computer time. In other words, the present subroutine has been designed for efficiency in speed more than in memory occupation.

As aforementioned in Section III, variables P_1 through P_{18} are relative to the equations of motion, whereas variables P_{19} through P_{50} are relative to the variational equations. A user interested only in the equations of motion could remove from the subroutine all of the statements that are relative to the variational equations; i.e., variables P_{19} through P_{50} .

The input to the subroutine consists of the integration step DT for the independent variable v (true anomaly) and initial values for 11 components of the vector $\mathbf{X}$ ($\mathbf{X}_1 \cdots \mathbf{X}_7$ have the meaning $x, y, \dot{x}, \dot{y}, r, \dot{r}, t$, whereas $\mathbf{X}_{10} \cdots \mathbf{X}_{13}$ are $r_1, r_2, r_1^{-3}, r_2^{-3}$). The components $\mathbf{X}_8, \mathbf{X}_9$ are $z, \dot{z}$, and are not used in the two-dimensional program.

The output after every integration step consists of the new values of the 11 parameters $\mathbf{X}_1 \cdots \mathbf{X}_7$ and $\mathbf{X}_{10} \cdots \mathbf{X}_{13}$, together with the matrix $\mathbf{A}$ containing the fundamental matrix (partial derivatives).

Subroutine ERPSV is designed to operate in the following three ways, as far as the input is concerned, according to the input parameter INIT:

- (1) INIT = 1 for the first integration step of the orbit. Initial conditions must be given for $\mathbf{X}_1$ through $\mathbf{X}_7$ only.
- (2) INIT = 2 for all succeeding integration steps. Initial conditions must be given for variables $\mathbf{X}_1 \cdots \mathbf{X}_7$ and $\mathbf{X}_{10} \cdots \mathbf{X}_{13}$. In fact, the values of these variables were computed by the preceding integration step.

- (3) INIT = 3 if the values of the series must be computed without recomputing the coefficients of the series. This option is useful if points must be computed for odd values of the independent variable v between the regularly spaced integration points.

In this subroutine, it can be seen that the equations of motion do not explicitly contain the independent variable v (true anomaly), and that the value of r (distance between the primaries) is obtained by solving the differential equation verified by r .

The dimension statements of subroutine ERPSV are such that only power series with a maximum of 20 terms can be used.

The following constants are needed by this subroutine, and should be in the proper locations in COMMON:

$$\text{RU} = \text{mass-ratio } \mu = m_2 \quad (186a)$$

$$\text{UMM} = 1 - \mu = m_1 \quad (186b)$$

$$\text{XII} = x_1 = -\mu = -m_2 \quad (186c)$$

$$\text{XII2} = x_2 = 1 - \mu = m_1 \quad (186d)$$

$$\text{E} = \text{eccentricity } e \quad (186e)$$

$$\text{P} = 1 - e^2 \quad (186f)$$

$$\text{Q} = (1 - e^2)^{1/2} \quad (186g)$$

$$\text{NT} = n = \text{number of terms used in power series} \quad (186h)$$

$$\text{NTM1} = n - 1 \quad (186i)$$

C. Conversion Subroutines

In the program listings, a few subroutines are included that have been used to convert different coordinate systems, mainly for the purpose of input of initial conditions or output of the orbits in different forms.

The first four subroutines correspond to some of the transformations described in Section II. They essentially transform four quantities (two coordinates and two velocity components) into the four corresponding quantities in another coordinate system. These subroutines contain no COMMON statements, and all input and

output are transmitted through the parameter list of the CALL statements. The exact function of each subroutine is as follows:

- (1) PULROT: transforms pulsating coordinates XP into rotating coordinates XR according to Eq. (65).
- (2) ROTPUL: transforms rotating coordinates XR into pulsating coordinates XP according to Eq. (64).
- (3) ROTIN: transforms rotating coordinates XR into inertial (barycentric) coordinates X according to Eq. (63).
- (4) INROT: transforms inertial coordinates X into rotating coordinates XR according to Eq. (62).

The description of the input and output parameters is given in detail in the comments. It can be seen that, for the last two subroutines, the true anomaly TRUE is among the input subroutines. Because of this parameter, another small subroutine has been written to compute the true anomaly (TRUEAN). It is joined to the listings with subroutines. This subroutine first calls subroutine KEPLER to obtain the eccentric anomaly from the mean anomaly; next it converts the eccentric anomaly into the true anomaly v . The value of v at exit is between 0 and 2π .

Some other conversion subroutines included in the listings are related to the Birkhoff regularization. The two most important subroutines convert between Cartesian rotating-pulsating median coordinates and velocity components X, and regularized Birkhoff coordinates and velocity components XB. Subroutine BIRCR transforms from Birkhoff to Cartesian coordinates, whereas subroutine CARBR converts from Cartesian to Birkhoff coordinates.

Subroutine CARBR must compute the square root of a complex number, and this is done in the small auxiliary subroutine COMSQR. In subroutine COMSQR, the complex arithmetic features that are available in many modern FORTRAN 4 compilers are not used. To have maximum flexibility, no COMMON statements have been used in subroutines BIRCR, CARBR, and COMSQR.

The two possible determinations of the square root of the complex number treated in subroutine COMSQR correspond to the two points in the Birkhoff plane that are associated with a single point in the Cartesian plane (the Birkhoff transformation being 2 to 1). In all of the

work that has been done, only one determination has been used, but the other determination could easily be obtained by changing the signs of A and B in subroutine COMSQR. It should be noted that subroutine BIRCR cannot be used when $r_1 = 0$ or $r_2 = 0$ because the velocity components computed by this subroutine contain r_1 and r_2 in the denominators.

D. Differential Correction Subroutines

Two main programs are described below, each of which uses a different technique for differential corrections. In one program, the variational equations are solved to obtain the necessary partial derivatives. In the other program (which is regularized), finite differences between a nominal orbit and perturbed orbits are used to obtain the partial derivatives. The operations related to the generation of the nominal orbit, the perturbed orbits, the partial derivatives, and the corrections to the initial conditions have been grouped in a single general-purpose differential correction subroutine called SEARCH. This subroutine, along with its listing, will be briefly described.

Subroutine SEARCH calls a matrix-inversion subroutine, which is not given here. Instead of the general matrix-inversion subroutine, a short subroutine is given for the solution of a system with two equations because, in the present problem of finding symmetric periodic orbits in the three-body problem, differential corrections in only two variables were necessary. If a more general matrix-inversion subroutine is added, however, subroutine SEARCH may be considered as a general-purpose subroutine for linear differential corrections in n variables.

The general problem consists of adjusting n initial parameters in such a way that n final parameters have a preassigned value. This problem can be expressed by a system of n generally very complicated equations, which are first linearized and solved by classical elimination or matrix inversion. Let x_i ($i = 1, \dots, n$) be the initial unknown parameters for which the values must be found. Also, let y_i be the final parameters for which the desired values Y_i are given. Then y_i must be considered as functions of x_i :

$$y_i = F_i(x_1, x_2, \dots, x_n) \quad (186)$$

and the problem consists of solving the following system in x_i :

$$F_i(x_1, x_2, \dots, x_n) - Y_i = 0 \quad (187)$$

At first a standard or nominal solution is computed, using as initial conditions the given approximation $x_i^{(0)}$. This nominal solution has the final values $y_i^{(0)}$. The deviations between the nominal solution and the desired ideal solution are thus given by

$$\Delta y_i = Y_i - y_i^{(0)} \quad (188)$$

Next, the partial derivatives are computed by perturbing successively each $x_j^{(0)}$ by the given small quantity δ_j . Each perturbed orbit will then give one column of the complete square matrix of partial derivatives:

$$\frac{\partial y_i}{\partial x_j} = \frac{y_{ij} - y_i^{(0)}}{\delta_j} \quad (189)$$

where y_{ij} is the final value of the orbit with the j th initial parameter $x_j^{(0)}$ perturbed by δ_j .

The solution of linear system

$$\Delta y_i = \sum_j \frac{\partial y_i}{\partial x_j} \Delta x_j \quad (190)$$

then gives the corrections Δx_j , which must be applied to the initial conditions $x_j^{(0)}$. Because of the nonlinearity of the problem, the whole process has to be repeated several times. At each iteration, the values of $|\Delta y_i|$ or $|\Delta x_i|$ are compared with some given $\epsilon_i > 0$ to decide whether the problem has converged with the required precision.

The input and output of the subroutine are described in detail in the comments. The meaning of some of the most important FORTRAN variables is as follows:

X = initial value for nominal solution $x_i^{(0)}$ (input)
 = new computed initial values $x_i^{(0)} + \Delta x_i$ (output)

G = array with perturbed initial conditions $x_i^{(0)} + \delta_i$ (output)

YO = final values $y_i^{(0)}$ of nominal solution (input)

YF = desired final values Y_i (input)

DEL = increments δ_i for perturbed solutions (input)

EPS = vector with precision constants ϵ_i (input)

EP = ϵ = limit for determinant of matrix; if determinant is smaller than ϵ in absolute value, matrix is declared singular (input)

E. Main Program With Variational Equations and Recurrent Power Series

In the FORTRAN 4 listings given below, two different main programs are included. The main program described in this section may be characterized by the following four main points:

- (1) Solves simultaneously the equations of motion and the variational equations in rotating-pulsating coordinates.
- (2) Uses the method of recurrent power series for the numerical integration.
- (3) Uses a least-squares approach for the differential corrections of periodic orbits.
- (4) Has no regularization.

This program is also characterized by two types of input cards that are accepted and by two completely different functions that it can accomplish. The input is normally done with a single NAMELIST statement containing about 30 quantities. These quantities all have a nominal value built into the program in such a way that only the parameters that are different from the nominal value have to be read in. If several successive cases are treated in one computer run, the computations never alter the input quantities; therefore, on succeeding cases, only parameters that are different from the preceding case must be punched on cards. This input philosophy allows a great flexibility, and allows the running of many cases with a minimum of preparation of input cards.

Another type of input whose possibilities are very interesting is made with "recapitulation cards." Each time a periodic orbit is found by the program, a set of four cards is punched in full double precision. These four cards contain the initial conditions $(x_0, y_0, \dot{x}_0, \dot{y}_0)$, the final conditions $(x_1, y_1, \dot{x}_1, \dot{y}_1)$, and the two parameters (μ, e) —also the energy E in the case of periodic collision orbits). These are punched and read according to the FORMAT: 3D24.17, I8, and contain the information in the following order:

- (1) Card one: $x_0, x_1, \mu, 1$.
- (2) Card two: $y_0, y_1, e, 2$.
- (3) Card three: $x_0, x_1, E, 3$.
- (4) Card four: $y_0, y_1, 0.0, 4$.

Recapitulation cards of this type are compatible with work previously done by the author on the circular restricted three-body problem. These four recapitulation

cards may be used, together with the NAMELIST input, to define all of the initial conditions for each computer run.

The program is capable of running in two different modes, which are called "Job One" ($JN = 1$) and "Job Two" ($JN = 2$). The two different jobs are described in the paragraphs that follow. Subsequent paragraphs describe some other operations that are done by the main program.

1. $JN = 1$. The program computes a list with a given number of orbits (nonperiodic) equally spaced in any of the initial parameters. If an orbit in the list is almost periodic, differential corrections are started to find the periodic orbit and punch it on four recapitulation cards. Then the normal list continues. Previous recapitulation cards may be used to read in the initial conditions that are necessary to begin the list.

2. $JN = 2$. The program computes a list with periodic orbits equally spaced in any given parameter. Each orbit is "made periodic" by differential corrections. Approximate initial conditions must be given only for the first orbit in the list. This first approximation may also be on recapitulation cards. Each periodic orbit that is found will be punched on four recapitulation cards, and will also be added to an extrapolation table. This table is thus used for extrapolations to find a good initial guess for the next periodic orbit. In this way, the program gradually builds up a table with "past experience"; as the number of periodic orbits increases in the table, a better and better initial guess will be obtained for the new periodic orbits. An initial extrapolation table with periodic orbits on recapitulation cards (from a previous computer run) may also be entered in the table at the beginning, and the list will then start from there. At the first orbit that is not converging, the whole list will be interrupted.

The interpolation-extrapolation subroutine INPOL, which has been used, uses the Lagrange-Aitken method; the FORTRAN listing of this subroutine is given in Table 22.

3. *Computing the variable-integration step.* The recurrent power-series method is used with a variable-integration step. This step is made smaller when the satellite approaches one of the primaries. The effective step that is used is $\bar{r}_1 \bar{r}_2 h$. The distances $\bar{r}_1$ and $\bar{r}_2$ are only approximations, retained from the preceding integration step. The nominal integration step h is an input quantity.

If the factor $\bar{r}_1 \bar{r}_2$ is larger than 2, it is not applied, and the integration step is taken as $2h$. If the factor $\bar{r}_1 \bar{r}_2$ is too small (e.g., smaller than 0.02), the integration is stopped. Each call to subroutine ERPSV thus increases the independent variable v (true anomaly) by the quantity $\bar{r}_1 \bar{r}_2 h$, and computes the corresponding values of all of the dependent variables.

4. *Logic of the differential correction process.* This logic is contained in the main program; however, this operation also involves calling several subroutines, especially for solving systems of linear equations. The differential correction may operate in two different ways according to two different periodicity criteria. The program may search for periodic orbits by attempting to "close the orbit"; i.e., to match the initial and final conditions. This method may also be used for symmetric periodic orbits; for these, however, the program may also proceed in a different way—by attempting to end the orbit by a perpendicular crossing with a *syzygy*-axis. The user may choose either of these periodicity criteria.

If the program is asked to "close the orbit," there will generally be a differential correction in four variables, but this number may be restricted to three if the user keeps one of the four initial parameters fixed by "zeroing out" one entire matrix column of the equations of conditions. The system of linearized differential-correction equations will then be solved by a method that is essentially equivalent to the least-squares method. In general, when the matrix of the linear system is too nearly singular, a least-squares solution will be made. In the circular problem, this matrix becomes singular for each periodic orbit. This may also happen (for different reasons) in the elliptic problem, but it has been observed that the least-squares approach gives a satisfactory solution to this problem. Subroutines CORL2 and CORR, which are relative to these operations, are considered too lengthy to be included in this report.

5. *Additional operations for periodic orbits.* When a periodic orbit is found, the program must perform a few operations that are related to this periodic orbit. The orbit will be added to the extrapolation table (if $JN = 2$); the coefficients of the characteristic equation will be computed (with subroutine CARP04); and the four recapitulation cards with initial and final conditions will be punched.

6. *Computer time.* The computer time (in seconds) is printed for each periodic orbit, and for each list of orbits. For this purpose, use is made of the standard built-in

7094 clock. Subroutines CLOCKI and CLOCT, which are relative to the timing, are too machine-dependent to be listed in this report.

F. A Regularized Main Program

This main program has the following principal characteristics:

- (1) Solves the equations of motion in two different ways, according to the region in which the particle is found. In two circular regions with given radius around each primary, the regularized Birkhoff coordinates are used; in the remaining part of the plane, the ordinary pulsating variables are used.
- (2) The integration in Birkhoff variables is done with a Runge-Kutta or a predictor-corrector method. In pulsating variables, it is done with the recurrent power-series method, with a variable step.
- (3) No variational equations are solved. The partial derivatives and the differential corrections are obtained by subroutine SEARCH, and are adapted to symmetric periodic orbits only.

The input philosophy with NAMELIST and with recapitulation cards is similar to that described for the preceding program. The regularized program is also designed to run in the same two modes—"Job One" for a list of nonperiodic orbits (with eventual interpolation for a periodic orbit) and "Job Two" for a list of periodic orbits.

The differential-correction logic is in the main program. The logic of the switching between regularized and pulsating variables has been collected in subroutine ORBIT. The function of this subroutine is to compute a single complete orbit at each call. Subroutine ORBIT contains the different input conversions, the conversions related to the switching of regions, and the necessary interpolations to end the orbit at the desired value $k\pi$ of the true anomaly v (even in regularized coordinates where v is not the independent variable).

The families of periodic orbits, along with their principal characteristics, are listed in Table 20. Table 21 is a list of the subroutines. Table 22 is a printout of the computer program.

Table 20. Families of periodic orbits

No.	Family group	Number of orbits	Mass-ratio μ	Eccentricity e
1	1P	13	0.5	1.0
2	3P	23	Variable	1.0
3	4P	26	0.5	Variable
4	5P	22	Variable	1.0
5	9P	12	0.5	Variable
6	6A	118	Variable (0.5-0)	1.0
7	12A	152	0.5	Variable (0-1.0)
8	7P	131	0.012155	Variable
9	7A	130	0.012155	Variable
10	8P	118	0.5	Variable
11	8A	92	0.5	Variable
12	10P	76	0.5	Variable
13	11P	52	0.5	Variable
14	11A	93	0.5	Variable
15	Periodic collision orbits	69	Variable	Variable

Table 21. List of subroutines

Subroutine	Function
MAIN	Main program with variational equations
ERPSV	Elliptic recurrent power series with variational equations
DERIV	Derivatives for equations of motion and variational equations
RUK	Runge-Kutta integration
KEPLER	Solution of Kepler equation
TRUEAN	Computes true anomaly v
PULROT	Converts pulsating to rotating coordinates
ROTPUL	Converts rotating to pulsating coordinates
ROTIN	Converts rotating to inertial coordinates
INROT	Converts inertial to rotating coordinates
DERIV	Computes derivatives for equations in pulsating coordinates
DERIV	Computes derivatives for equations in inertial coordinates
MAIN	Main program with Birkhoff regularization
ORBIT	Computes one orbit in Cartesian or Birkhoff coordinates
DERIV	Computes derivatives for equations in Birkhoff coordinates
BIRCR	Converts Birkhoff to Cartesian coordinates
CARBR	Converts Cartesian to Birkhoff coordinates
COMSQR	Square root of complex number
SEARCH	Linear differential corrections
SOLVE2	Solution of linear system with two variables
INPOL	Interpolation with Lagrange-Aitken method

Table 22. Computer program

```

$IBFTC R3BVPS
C   ELLIPTIC RESTR.3-BODY/VARIATIONAL EQUATIONS/RECURRENT POWER SERIES
C   JN   =JOB NUMBER
C       =1-COMPUTE LIST OF ORBITS,WITH SEARCH FOR PERIODIC ORBITS
C         OR COMPUTE A LIST OF(PERIODIC)ORBITS FROM(APPROXIMATE)
C         INITIAL CONDITIONS ON RECAP CARDS
C       =2-GENERATE LIST OF PERIODIC ORBITS BY EXTRAPOLATIONS
C   NT   =NUMBER OF TERMS IN POWER SERIES
C   NPRIN=PRINT STEP
C       ONE LINE WILL BE PRINTED EVERY NPRIN INTEGRATION STEPS.
C   NDIFC=MAXIMUM NUMBER OF DIFFERENTIAL CORRECTIONS ALLOWED
C         TO MAKE AN ORBIT PERIODIC
C   MNO  =MAXIMUM NUMBER OF ORBITS TO COMPUTE IN LIST ,IF JN=1
C       =MAXIMUM NUMBER OF PERIODIC ORBITS TO BE GENERATED ,IF JN=2
C   MNPS=MAXIMUM NUMBER OF POINTS USED IN EXTRAPOLATIONS,WHEN JN=2
C       WHEN MNPS=0,NO EXTRAPOLATION TABLE IS USED
C       AT ANY TIME,ONLY THE LAST MNPS GENERATED ORBITS ARE IN THE
C       EXTRAPOLATION TABLE.
C   NCOL=0 DO NOT READ INITIAL CONDITIONS FROM RECAP CARDS
C       =1 OR 2 RECAP CARDS ARE USED FOR INITIAL CONDITIONS
C       1=INITIAL CONDITIONS ARE IN COLS 1 TO 24
C       2=INITIAL CONDITIONS ARE IN COLS 25 TO 48
C   MCOL=0 WHEN NO INITIAL TABLE FOR EXTRAPOLATION IS READ IN
C       =1 OR 2 WHEN TABLE FOR EXTRAPOLATION IS READ IN
C       1=COLS 1 TO 24 , 2=COLS 25 TO 48
C   IMUE=0 WHEN E AND RU ON RECAP CARDS ARE NOT USED
C       =NOT 0 WHEN E AND RU ON RECAP CARDS ARE USED
C       MU IS IN COLS 49 TO 72 ,ON CARD ONE
C       E  IS IN COLS 49 TO 72 ,ON CARD TWO
C       THE FORMAT OF RECAPITULATION CARDS IS (3D24.17,I8)
C       THERE ARE FOUR RECAPITULATION CARDS PER PERIODIC ORBIT
C   I1   =VARIABLE TO BE INCREMENTED WHEN GENERATING LIST OF ORBITS
C       I1 IS FROM 1 TO 8.(1,2,3,4=X,Y,XDOT,YDOT, 7=MU,8=E)
C   IS1  =FIRST VARIABLE TO SEARCH ON (GENERALLY 1)
C   IS2  =SECOND VARIABLE TO SEARCH ON (GENERALLY 4)
C   IF1  =FIRST VARIABLE TO DRIVE TO 0.0 (GENERALLY 2)
C   IF2  =SECOND VARIABLE TO DRIVE TO 0.0 (GENERALLY 3)
C   N1   =TYPE OF PERIODICITY CRITERION USED
C       =1=TRY TO FIND A PERPENDICULAR CROSSING OF THE X OR Y AXIS
C         THIS IS A TWO BY TWO SEARCH
C       =2=TRY TO CLOSE THE ORBIT (THIS IS AN N*N SEARCH)
C         THE LINEAR SYSTEM IS SOLVED WITH THE LEAST SQUARES METHOD
C         WHEN N1=1,IS1 AND IS2 ARE USED,AND ZRO IS NOT USED
C         WHEN N1=2,ZRO IS USED AND IS1,IS2 ARE NOT USED
C   X(1 TO 4)=INITIAL CONDITIONS =X,Y,XDOT,YDOT.
C   RMU   =MU=MASS-RATIO
C   E     =ECCENTRICITY(OF THE PRIMARIES)
C   H     =NOMINAL INTEGRATION STEP
C   TRUIN=0.000 WHEN START AT PERIAPSIS
C       =1.000 WHEN START AT APOAPSIS
C   HORB  =NUMBER OF HALF-ORBITS TO BE COMPUTED(NUMBER OF TIMES PI)
C   DELTA=INCREMENT FOR LIST (TO BE ADDED TO X(I1))
C   EPSL1=PRECISION FACTOR TO DECIDE IF A SEARCH FOR A PERIODIC ORBIT
C         HAS TO BE STARTED (USED IF JN=1 ONLY)
C   PRECIS=PRECISION TO DECIDE WHETHER AN ORBIT IS PERIODIC OR NOT
C   ZRO   =4-VECTOR WITH ONES OR ZEROS USED TO ZERO OUT
C         SOME COLUMNS OF THE MATRIX WITH PARTIALS
C   FOR INSTANCE
C       USE ZRO=1.,1.,1.,1. FOR NON SYMMETRIC ORBITS
C       USE ZRO=1.,0.,0.,1. FOR OX-SYMMETRIC ORBITS
C       USE ZRO=0.,1.,1.,0. FOR OY-SYMMETRIC ORBITS
C       ZRO=0.,1.,1.,1. FOR NON-SYMMETRIC ORBITS WHEN NO CORRECTION DX

```

Table 22 (contd)

```

C      IN THE VARIABLE X IS ALLOWED .
C      R1MIN=COLLISION LIMIT FOR R1
C      R2MIN=COLLISION LIMIT FOR R2
C      DXMIN=MINIMUM CORRECTION DX ALLOWED
C      DXMAX=MAXIMUM CORRECTION DX ALLOWED
C      THE DIFFERENTIAL CORRECTION ITERATIONS WILL BE INTERRUPTED IF ONE
C      OF THE COMPONENTS OF THE PROPOSED CORRECTION VECTOR DX
C      IS OUTSIDE THE LIMITS DXMIN AND DXMAX, IN ABSOLUTE VALUE
      DOUBLE PRECISION X(13),AMAT(4,4),RMU,UMU,DV,ECOF(8),E,P,Q,S1,S2,PI
1    1,TRUIN,TRUFI,XI1,XI2,R,H,R1R2, T, TRUE, ZERO,DX(6),DEV,PRECIS,
2    ZRO(8),HORB, ERROR,DUMY(3),PNP(10),TABLE1(20),TABLE2(20),TABLE3(20
3    3),XPI(8),DELTA,DXMIN,DETER,EPSL1,R1MIN,R2MIN,TABLE4(20),TABLEX(20)
4    4,DXMAX,A23(2,3)
      DIMENSION NBR(4)
      COMMON RMU,UMU,XI1,XI2,E,P,Q,S1,S2,PI,NT,NTM1,NPRIN
36   FORMAT (1H0,19H THE TOTAL TIME IS ,F11.3, 9H SECONDS )
191  FORMAT(1H ,1P6D18.11,I5,1PD9.1)
194  FORMAT(1H ,25H A4,A3,A2,A1,A0
195  FORMAT(1H ,5D24.17)
197  FORMAT(1H1,20I5)
206  FORMAT(1H ,4D25.16)
308  FORMAT(1H ,5D24.17)
703  FORMAT(1H0,25H NO PERIODIC ORBIT FOUND
704  FORMAT(1H ,13,22H PERIODIC ORBITS FOUND
706  FORMAT(1H )
707  FORMAT(1H ,19H SUGGESTED CHANGES=,4D16.9)
708  FORMAT(1H0,14HNEAR COLLISION ,2D25.15)
709  FORMAT(1H1,3HJN=,I1,2X3HNT=,I2,2X6HNPRIN=,I4,2X6HNDIFC=,I2,2X4HMNO
1    1=,I3,2X5HMNPS=,I2,2X5HNCOL=,I1,2X5HMCOL=,I1,2X5HIMUE=,I1,2X3HI1=,I
2    21,2X4HIS1=,I1,2X4HIS2=,I1,2X3HN1=,I1,2X4HIF1=,I1,2X4HIF2=,I1)
710  FORMAT(1H ,2HH=,1PD10.3,8H, TRUIN=,D8.1,7H, HORB=,D9.2,8H, DELTA=,
1    1D9.2,8H, EPSL1=,D8.1,9H, PRECIS=,D8.1)
711  FORMAT(1H ,4HZRO=,4(1PD8.1),5X6HR1MIN=,D8.1,3X6HR2MIN=,D8.1,
1    15X6HDXMIN=,D8.1,3X6HDXMAX=,D8.1)
712  FORMAT(1H0,4HRMU=,1PD24.16,3X2HE=,D24.16)
713  FORMAT(1H ,12H THE TIME IS ,F11.3, 9H SECONDS )
977  FORMAT(3D24.17,I8)
      CALL FPTRP(0,0,0)
      NBR(1)=1
      NBR(2)=2
      NBR(3)=3
      NBR(4)=4
      PI=3.141592653589793
      ZERO=0.000
      NPOIN=2000
      NAMELIST/NAMES/X,RMU,E,TRUIN,NCOL,N1, NT ,H,HORB,NPRIN,NDIFC,JN,
1    1MCOL,MNO,MNPS,IMUE,I1,IS1,IS2,DELTA,ZRO,EPSL1,R1MIN,R2MIN
2    2,PRECIS,DXMIN,DXMAX,IF1,IF2
C      THE BUILT-IN VALUES FOR THE INPUT PARAMETERS ARE
      JN=1
      NT=12
      NPRIN=200
      NDIFC=8
      MNO=3
      MNPS=0
      NCOL=0
      MCOL=0
      IMUE=0
      I1=7
      IS1=1
      IS2=4
      IF1=2

```

Table 22 (contd)

```

      IF2=3
      N1=1
      DO 900 I=1,9
900  X(I)=0.0D0
      X(1)=0.10248881D0
      X(4)=3.5891975D0
      RMU=0.012155D0
      E=0.2D0
      H=0.02D0
      TRUIN=0.0D0
      HORB=1.0D0
      DELTA=.01D0
      EPSL1=0.01D0
      PRECIS=1.0D-12
      DXMIN=0.1D-14
      DXMAX=0.1D0
      DO 615 I=1,4
      ZRO(I) =1.0D0
615  ZRO(I+4)=0.0D0
      R1MIN=0.02D0
      MPOIN=MPOIN+1
      IF(MPOIN-NPOIN)610,611,611
      R2MIN=0.01D0
C   HERE TO START A NEW CASE
      5 CALL CLOCKI
      READ(5,NAMES)
      WRITE(6,709)JN,NT,NPRIN,NDIFC,MNO,MNPS,NCOL,MCOL,IMUE,I1,IS1,IS2,N
11,IF1,IF2
      WRITE(6,710)H,TRUIN,HORB,DELTA,EPSL1,PRECIS
      T=PI*HORB
      NTM1=NT-1
      X(7)=RMU
      X(8)=E
      DO 501 I=1,8
501  XPI(I)=X(I)
      WRITE(6,711)(ZRO(I),I=1,4),R1MIN,R2MIN,DXMIN,DXMAX
      IF(NCOL.EQ.0) GO TO 800
C   READ-IN RECAP CARDS WITH INITIAL CONDITIONS
12  READ(5,977)DUMY(1),DUMY(2),DUMY(3),NNN
      IF(NNN.EQ.99) GO TO 41
      XPI(1)=DUMY(NCOL)
      IF(IMUE.NE.0) XPI(7)=DUMY(3)
      RMU=XPI(7)
      READ(5,977)DUMY(1),DUMY(2),DUMY(3)
      XPI(2)=DUMY(NCOL)
      IF(IMUE.NE.0) XPI(8)=DUMY(3)
      E=XPI(8)
      READ(5,977)DUMY(1),DUMY(2)
      XPI(3)=DUMY(NCOL)
      READ(5,977)DUMY(1),DUMY(2)
      XPI(4)=DUMY(NCOL)
800  NNO=0
      NPS=0
      IF(MCOL.EQ.0) GO TO 600
C   READ-IN INITIAL EXTRAPOLATION TABLE ON RECAP CARDS
13  READ(5,977)DUMY(1),DUMY(2),PNP(7),NNN
      IF(NNN.EQ.99)GO TO 600
      PNP(1)=DUMY(MCOL)
      READ(5,977)DUMY(1),DUMY(2),PNP(8)
      PNP(2)=DUMY(MCOL)
      READ(5,977)DUMY(1),DUMY(2)
      PNP(3)=DUMY(MCOL)

```

Table 22 (contd)

```

      READ(5,977)DUMY(1),DUMY(2)
      PNP(4)=DUMY(MCOL)
      NPS=NPS+1
      IF(N1.EQ.1) GO TO 630
      TABLEX(NPS)=PNP(I1)
      TABLE1(NPS)=PNP(1)
      TABLE2(NPS)=PNP(2)
      TABLE3(NPS)=PNP(3)
      TABLE4(NPS)=PNP(4)
      WRITE(6,308)TABLEX(NPS),TABLE1(NPS),TABLE2(NPS)
      *
      ,TABLE3(NPS),TABLE4(NPS)
      GO TO 13
630  TABLEX(NPS)=PNP(I1)
      TABLE1(NPS)=PNP(IS1)
      TABLE2(NPS)=PNP(IS2)
      WRITE(6,308)TABLEX(NPS),TABLE1(NPS),TABLE2(NPS)
      GO TO 13
600  IF(MNPS.EQ.0) GO TO 14
C   EXTRAPOLATE FIRST GUESS FOR NEW PERIODIC ORBIT.
      IF(N1.EQ.1) GO TO 631
      IF(NPS-1)14,632,633
632  XPI(1)=TABLE1(1)
      XPI(2)=TABLE2(1)
      XPI(3)=TABLE3(1)
      XPI(4)=TABLE4(1)
      GO TO 14
633  IF(ZRO(1).EQ.1.0D0)
      *CALL INPOL(TABLEX,TABLE1,XPI(I1),XPI(1),NPS)
      IF(ZRO(2).EQ.1.0D0)
      *CALL INPOL(TABLEX,TABLE2,XPI(I1),XPI(2),NPS)
      IF(ZRO(3).EQ.1.0D0)
      *CALL INPOL(TABLEX,TABLE3,XPI(I1),XPI(3),NPS)
      IF(ZRO(4).EQ.1.0D0)
      *CALL INPOL(TABLEX,TABLE4,XPI(I1),XPI(4),NPS)
      GO TO 14
631  IF(NPS-1)14,15,16
      15 XPI(IS1)=TABLE1(1)
      XPI(IS2)=TABLE2(1)
      GO TO 14
      16 CALL INPOL(TABLEX,TABLE1,XPI(I1),XPI(IS1),NPS)
      CALL INPOL(TABLEX,TABLE2,XPI(I1),XPI(IS2),NPS)
C   PRELIMINARY COMPUTATIONS FOR EACH ORBIT.
      14 RMU=XPI(7)
      E=XPI(8)
      P=1.0D0-E*E
      Q=DSQRT(P)
      UMU=1.0D0-RMU
      XI1=-RMU
      XI2=UMU
      TRUFI=PI*TRUIN+PI*HORB
      WRITE(6,712)RMU,E
      MDIFC=0
      100 TRUE=TRUIN*PI
      R=P/(1.0D0+E*DCOS(TRUE))
      DO 21 I=1,4
      21 X(I)=XPI(I)
      X(5)=R
      X(6)=0.0D0
      X(7)=0.0D0
      S1=DSQRT((X(1)-XI1)**2+X(2)**2)
      S2=DSQRT((X(1)-XI2)**2+X(2)**2)

```

Table 22 (contd)

```

      ERROR=0.0D0
      DO 700 I=1,4
      DO 700 J=1,4
      AMAT(I,J)=0.0D0
700  IF(I.EQ.J) AMAT(I,J)=1.0D0
      MPOIN=0
      MPRIN=0
      WRITE(6,191) (X(I),I=1,4),X( 7),TRUE,MPOIN,ERROR
      MDIFC=MDIFC+1
      INIT=1
C   COMPUTE INTEGRATION STEP
607  R1R2=S1*S2
      DV=H
      IF(R1R2.LT.2.0D0) DV=H*R1R2
C   CALL ERPSV DO PERFORM ONE INTEGRATION STEP
612  CALL ERPSV(X,AMAT,DV,INIT)
      TRUE=TRUE+DV
      S1=X(10)
      S2=X(11)
C   CHECK IF COLLISION WITH ONE OF PRIMARIES
      IF(S1.GT.R1MIN.AND.S2.GT.R2MIN) GO TO 625
      WRITE(6,708)S1,S2
      GO TO (410,118),JN
625  MPRIN=MPRIN+1
610  IF(MPRIN-NPRIN)608,609,609
C   ESTIMATE LOCAL INTEGRATION ERROR
609  ERROR=0.25D0*(DABS( X(10)**(-3)- X(12))
      1+DABS( X(11)**(-3)- X(13))
      2+DABS(DSQRT(( X(1)-XI1)**2+ X(2)**2)- X(10))
      3+DABS(DSQRT(( X(1)-XI2)**2+ X(2)**2)- X(11)) )
      WRITE(6,191)(X(I),I=1,4),X(7),TRUE,MPOIN,ERROR
      MPRIN=0
608  IF(DABS(TRUFI-TRUE)-DV)606,607,607
606  DV=TRUFI-TRUE
      MPR=MPOIN
      MPOIN =9998
      GO TO 612
C   ESTIMATE LOCAL INTEGRATION ERROR FOR FINAL POINT
611  ERROR=0.25D0*(DABS( X(10)**(-3)- X(12))
      1+DABS( X(11)**(-3)- X(13))
      2+DABS(DSQRT(( X(1)-XI1)**2+ X(2)**2)- X(10))
      3+DABS(DSQRT(( X(1)-XI2)**2+ X(2)**2)- X(11)) )
      WRITE(6,191)(X(I),I=1,4),X(7),TRUE,MPR ,ERROR
C   START DIFFERENTIAL CORRECTIONS
      IF(N1.NE.1)GO TO 619
      DUMY(1)=DABS(X(IF1))+DABS(X(IF2))
      GO TO 624
619  DUMY(1)=0.0D0
      DO 620 I=1,4
620  DUMY(1)=DUMY(1)+DABS(X(I)-XPI(I))*ZRO(I)
624  IF(JN.NE.1)GO TO 618
      IF(DUMY(1).GT.EPSL1) GO TO 410
618  IF(N1.EQ.2)GO TO 622
      A23(1,1)=AMAT(IF1,IS1)
      A23(1,2)=AMAT(IF1,IS2)
      A23(2,1)=AMAT(IF2,IS1)
      A23(2,2)=AMAT(IF2,IS2)
      A23(1,3)=-X(IF1)
      A23(2,3)=-X(IF2)
      CALL CORL2(A23,DX)
      DEV=DABS(DX(1))+DABS(DX(2))
      WRITE(6,707)(DX(I),I=1,2)

```

Table 22 (contd)

```

      XPI(IS1)=XPI(IS1)+DX(1)
      XPI(IS2)=XPI(IS2)+DX(2)
      GO TO 623
622  CALL CORR(4,AMAT,XPI,X,DX,ZRO)
      WRITE(6,707)(DX(I),I=1,4)
      DO 22 I=1,4
22   XPI(I)=XPI(I)+DX(I)
      DEV=0.0D0
      DO 701 I=1,4
701  DEV=DEV+ DABS(DX(I))
623  WRITE(6,706)
C    CHECK IF PERIODIC ORBIT FOUND OR IF DIVERGENCE
      IF(DUMY(1).LT.PRECIS) GO TO 702
      IF(DEV.LT.DXMIN) GO TO 613
      IF(DEV.GT.DXMAX) GO TO 613
      IF(MDIFC.LT.NDIFC) GO TO 100
613  WRITE(6,703)
      GO TO (410,118),JN
702  DO 205 I=1,4
205  WRITE(6,206) (AMAT(I,J),J=1,4)
C    COMPUTE COEFFICIENTS OF CHARACTERISTIC EQUATION
      CALL CARPO4(AMAT,ECOF)
      WRITE(6,194)
      WRITE(6,195)(ECOF(I),I=1,5)
      NN00=NN0+1
      WRITE(6,704) NN00
C    RECORD MACHINE TIME FOR THIS PERIODIC ORBIT
      CALL CLOCT(PARTIM,TOTTIM)
      WRITE(6,713)PARTIM
      IF(MNPS.EQ.0) GO TO 17
C    STORE PERIODIC ORBIT IN EXTRAPOLATION TABLE
      IF(N1.EQ.1) GO TO 626
      IF(NPS.LT.MNPS) GO TO 627
      DO 628 I=2,NPS
      TABLE(I-1)=TABLE(I)
      TABLE1(I-1)=TABLE1(I)
      TABLE2(I-1)=TABLE2(I)
      TABLE3(I-1)=TABLE3(I)
628  TABLE4(I-1)=TABLE4(I)
      GO TO 629
627  NPS=NPS+1
629  TABLE(NPS)=XPI(I1)
      TABLE1(NPS)=XPI(1)
      TABLE2(NPS)=XPI(2)
      TABLE3(NPS)=XPI(3)
      TABLE4(NPS)=XPI(4)
      GO TO 17
626  IF(NPS.LT.MNPS) GO TO 18
      DO 19 I=2,NPS
      TABLE1(I-1)=TABLE1(I)
      TABLE2(I-1)=TABLE2(I)
19   TABLE(I-1)=TABLE(I)
      GO TO 20
18   NPS=NPS+1
20   TABLE(NPS)=XPI(I1)
      TABLE1(NPS)=XPI(IS1)
      TABLE2(NPS)=XPI(IS2)
C    PUNCH 4 RECAP CARDS FOR PERIODIC ORBIT
17  PUNCH 977,XPI(1),X (1),RMU,NBR(1),
      1      XPI(2),X (2),E ,NBR(2),
      2      XPI(3),X (3),ECOF(2),NBR(3),
      3      XPI(4),X (4),ECOF(3),NBR(4)

```

Table 22 (contd)

```

410 NNO=NNO+1
    IF(NNO-MNO)107,118,118
C   PERTURB THE LIST-PARAMETER BY DELTA
107 XPI(I1)=XPI(I1)+DELTA
    WRITE(6,706)
    GO TO 600
118 IF(NCOL)12,41,12
    41 CALL CLOCT(PARTIM,TOTTIM)
    WRITE(6,36)TOTTIM
    GO TO 5
END

$IBFTC ERPS.
    SUBROUTINE ERPSV(X,A,DT,INIT)
C   INPUT=INIT,DT,X(INITIAL VALUES),A(FINAL VALUES).
C   OUTPUT=X(FINAL VALUES) AND A(FINAL VALUES).
C   A=MATRIX WITH 16 PARTIAL DERIVATIVES.
C   DT=INTEGRATION STEP
C   THE INDEPENDENT VARIABLE(TRUE ANOMALY)IS NOT EXPLICITLY IN THE
C   EQUATIONS OF MOTION HERE
C   INIT=1 FOR THE FIRST STEP OF THE ORBIT
C   INIT=2 FOR THE FOLLOWINGS STEPS
C   INIT=3 TO COMPUTE THE VALUE OF THE POWER SERIES ONLY
C   WHEN INIT=1 OR 2 THE COEFFICIENTS AND THE VALUES OF THE
C   POWER SERIES ARE EVALUATED
C   THE 13 COMPONENTS OF X ARE
C   X(1)=X,X(2)=Y,X(3)=XDOT,X(4)=YDOT,
C   X(5)=R,X(6)=RDOT,X(7)=TIME,X(8) AND X(9) ARE NOT USED
C   X(10)=R1 , X(11)=R2 , X(12)=R1**(-3) , X(13)=R2**(-3).
C   TO START AN ORBIT,ONLY X(1)...X(7) MUST BE GIVEN,NOT X(10)...X(13)
C   AT EXIT FROM EACH INTEGRATION STEP,X(1)...X(13) ARE COMPUTED.
C   DOUBLE PRECISION X,DT,P1,P2,P3,P4,P5,P6,P7,P8,P9,P10,P11,P12,P13,
1P14,P15,P16,P17,P18,RU,UMM,XI1,XI2,E,P,Q,S1,S2,PI,FN,FQ,SOM
2,A,P19,P20,P21,P22,P23,P24,P25,P26,P27,P28,P29,P30,P31,P32,P33,P34
3,P35,P36,P37,P38,P39,P40,P41,P42,P43,P44,P45,P46,P47,P48,P49,P50
    COMMON RU,UMM,XI1,XI2,E,P,Q,S1,S2,PI,NT,NTM1
    DIMENSION X(13),P1(20),P2(20),P3(20),P4(20),P5(20),P6(20),P7(20),
1P8(20),P9(20),P10(20),P11(20),P12(20),P13(20),P14(20),P15(20),
2P16(20),P17(20),P18(20)
3,P19(20),P20(20),P21(20),P22(20),P23(20),P24(20),P25(20),P26(20),
4 P27(20),P28(20),P29(20),P30(20),P31(20),P32(20),P33(20),P34(20),
5 P35(20),P36(20),P37(20),P38(20),P39(20),P40(20),P41(20),P42(20),
6 P43(20),P44(20),P45(20),P46(20),P47(20),P48(20),P49(20),P50(20),
7A(4,4)
    GO TO(1,2,3),INIT
1  INIT=2
    P10(1)=DSQRT((X(1)-XI1)**2+X(2)**2)
    P11(1)=DSQRT((X(1)-XI2)**2+X(2)**2)
    P12(1)=1.0D0/P10(1)**3
    P13(1)=1.0D0/P11(1)**3
    GO TO 4
2  P10(1)=X(10)
    P11(1)=X(11)
    P12(1)=X(12)
    P13(1)=X(13)
4  P1(1)=X(1)
    P2(1)=X(2)
    P4(1)=X(3)
    P5(1)=X(4)
    P7(1)=X(5)
    P8(1)=X(6)

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Table 22 (contd)

```

P9(1)=X(7)
P14(1)=1.0D0-UMM*P12(1)-RU*P13(1)
P15(1)=P1(1)*P14(1)+RU*UMM*(P13(1)-P12(1))
P16(1)=P2(1)*P14(1)
P18(1)=P7(1)**2
P19(1)=A(1,1)
P20(1)=A(2,1)
P21(1)=A(3,1)
P22(1)=A(4,1)
P23(1)=A(1,2)
P24(1)=A(2,2)
P25(1)=A(3,2)
P26(1)=A(4,2)
P27(1)=A(1,3)
P28(1)=A(2,3)
P29(1)=A(3,3)
P30(1)=A(4,3)
P31(1)=A(1,4)
P32(1)=A(2,4)
P33(1)=A(3,4)
P34(1)=A(4,4)
P35(1)=(P1(1)-XI1)/P10(1)
P36(1)=(P1(1)-XI2)/P11(1)
P37(1)=(P2(1))/P10(1)
P38(1)=(P2(1))/P11(1)
P39(1)=-1.0D0+3.0D0*P35(1)**2
P40(1)=-1.0D0+3.0D0*P36(1)**2
P41(1)=-1.0D0+3.0D0*P37(1)**2
P42(1)=-1.0D0+3.0D0*P38(1)**2
P43(1)=      +3.0D0*P37(1)*P35(1)
P44(1)=      +3.0D0*P38(1)*P36(1)
P45(1)=+1.0D0+UMM*P12(1)*P39(1)+RU*P13(1)*P40(1)
P46(1)=      +UMM*P12(1)*P43(1)+RU*P13(1)*P44(1)
P47(1)=+1.0D0+UMM*P12(1)*P41(1)+RU*P13(1)*P42(1)
P48(1)=P7(1)*P45(1)/P
P49(1)=P7(1)*P46(1)/P
P50(1)=P7(1)*P47(1)/P
DO 111 N=1,NTM1
NP1=N+1
FN=N
P1(NP1)=P4(N)/FN
P2(NP1)=P5(N)/FN
P4(NP1)=+2.0D0*P5(N)
P5(NP1)=-2.0D0*P4(N)
P7(NP1)=P8(N)/FN
P8(NP1)=P18(N)
P9(NP1)=P18(N)/(FN*Q)
P10(NP1)=RU*P4(N)
P11(NP1)=-UMM*P4(N)
P12(NP1)=0.0D0
P13(NP1)=0.0D0
SOM=0.0D0
DO 112 NP=1,N
NQ=N-NP+1
P4(NP1)=P4(NP1)+P7(NP)*P15(NQ)/P
P5(NP1)=P5(NP1)+P7(NP)*P16(NQ)/P
P8(NP1)=P8(NP1)+2.0D0*P8(NP)*P8(NQ)-P7(NP)*P18(NQ)/P
112 SOM=SOM+P1(NP)*P4(NQ)+P2(NP)*P5(NQ)
P4(NP1)=P4(NP1)/FN
P5(NP1)=P5(NP1)/FN
P10(NP1)=P10(NP1)+SOM
P11(NP1)=P11(NP1)+SOM

```

Table 22 (contd)

```

      IF(N.EQ.1) GO TO 113
      DO 114 NP=2,N
      NQ=N-NP+1
      NQP1=NQ+1
      FQ=NQ
      P8(NP1)=P8(NP1)-FQ*P7(NP)*P8(NQP1)
      P10(NP1)=P10(NP1)-FQ*P10(NP)*P10(NQP1)
      P11(NP1)=P11(NP1)-FQ*P11(NP)*P11(NQP1)
      P12(NP1)=P12(NP1)-FQ*P10(NP)*P12(NQP1)
114  P13(NP1)=P13(NP1)-FQ*P11(NP)*P13(NQP1)
113  P8(NP1)=P8(NP1)/(FN*P7(1))
      P10(NP1)=P10(NP1)/(FN*P10(1))
      P11(NP1)=P11(NP1)/(FN*P11(1))
      DO 115 NP=1,N
      NQ=N-NP+1
      NQP1=NQ+1
      FQ=NQ
      P12(NP1)=P12(NP1)-3.0D0*FQ*P12(NP)*P10(NQP1)
115  P13(NP1)=P13(NP1)-3.0D0*FQ*P13(NP)*P11(NQP1)
      P12(NP1)=P12(NP1)/(FN*P10(1))
      P13(NP1)=P13(NP1)/(FN*P11(1))
      P14(NP1)=-UMM*P12(NP1)-RU*P13(NP1)
      P15(NP1)=RU*UMM*(P13(NP1)-P12(NP1))
      P16(NP1)=0.0D0
      P18(NP1)=0.0D0
      DO 119 NP=1,NP1
      NQ=N-NP+2
      P15(NP1)=P15(NP1)+P1(NP)*P14(NQ)
      P16(NP1)=P16(NP1)+P2(NP)*P14(NQ)
119  P18(NP1)=P18(NP1)+P7(NP)*P7(NQ)
      P19(NP1)=P21(N)/FN
      P20(NP1)=P22(N)/FN
      P21(NP1)=+2.0D0*P22(N)
      P22(NP1)=-2.0D0*P21(N)
      P23(NP1)=P25(N)/FN
      P24(NP1)=P26(N)/FN
      P25(NP1)=+2.0D0*P26(N)
      P26(NP1)=-2.0D0*P25(N)
      P27(NP1)=P29(N)/FN
      P28(NP1)=P30(N)/FN
      P29(NP1)=+2.0D0*P30(N)
      P30(NP1)=-2.0D0*P29(N)
      P31(NP1)=P33(N)/FN
      P32(NP1)=P34(N)/FN
      P33(NP1)=+2.0D0*P34(N)
      P34(NP1)=-2.0D0*P33(N)
      DO 200 NP=1,N
      NQ=N-NP+1
      P21(NP1)=P21(NP1)+P48(NP)*P19(NQ)+P49(NP)*P20(NQ)
      P22(NP1)=P22(NP1)+P49(NP)*P19(NQ)+P50(NP)*P20(NQ)
      P25(NP1)=P25(NP1)+P48(NP)*P23(NQ)+P49(NP)*P24(NQ)
      P26(NP1)=P26(NP1)+P49(NP)*P23(NQ)+P50(NP)*P24(NQ)
      P29(NP1)=P29(NP1)+P48(NP)*P27(NQ)+P49(NP)*P28(NQ)
      P30(NP1)=P30(NP1)+P49(NP)*P27(NQ)+P50(NP)*P28(NQ)
      P33(NP1)=P33(NP1)+P48(NP)*P31(NQ)+P49(NP)*P32(NQ)
200  P34(NP1)=P34(NP1)+P49(NP)*P31(NQ)+P50(NP)*P32(NQ)
      P21(NP1)=P21(NP1)/FN
      P22(NP1)=P22(NP1)/FN
      P25(NP1)=P25(NP1)/FN
      P26(NP1)=P26(NP1)/FN
      P29(NP1)=P29(NP1)/FN
      P30(NP1)=P30(NP1)/FN

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Table 22 (contd)

```

P33(NP1)=P33(NP1)/FN
P34(NP1)=P34(NP1)/FN
P35(NP1)=P1(NP1)
P36(NP1)=P1(NP1)
P37(NP1)=P2(NP1)
P38(NP1)=P2(NP1)
DO 201 NP=2,NP1
NQ=N-NP+2
P35(NP1)=P35(NP1)-P10(NP)*P35(NQ)
P36(NP1)=P36(NP1)-P11(NP)*P36(NQ)
P37(NP1)=P37(NP1)-P10(NP)*P37(NQ)
201 P38(NP1)=P38(NP1)-P11(NP)*P38(NQ)
P35(NP1)=P35(NP1)/P10(1)
P36(NP1)=P36(NP1)/P11(1)
P37(NP1)=P37(NP1)/P10(1)
P38(NP1)=P38(NP1)/P11(1)
P39(NP1)=0.0D0
P40(NP1)=0.0D0
P41(NP1)=0.0D0
P42(NP1)=0.0D0
P43(NP1)=0.0D0
P44(NP1)=0.0D0
P45(NP1)=0.0D0
P46(NP1)=0.0D0
P47(NP1)=0.0D0
P48(NP1)=0.0D0
P49(NP1)=0.0D0
P50(NP1)=0.0D0
DO 202 NP=1,NP1
NQ=N-NP+2
P39(NP1)=P39(NP1)+3.0D0*P35(NP)*P35(NQ)
P40(NP1)=P40(NP1)+3.0D0*P36(NP)*P36(NQ)
P41(NP1)=P41(NP1)+3.0D0*P37(NP)*P37(NQ)
P42(NP1)=P42(NP1)+3.0D0*P38(NP)*P38(NQ)
P43(NP1)=P43(NP1)+3.0D0*P37(NP)*P35(NQ)
202 P44(NP1)=P44(NP1)+3.0D0*P38(NP)*P36(NQ)
DO 203 NP=1,NP1
NQ=N-NP+2
P45(NP1)=P45(NP1)+UMM*P12(NP)*P39(NQ)+RU*P13(NP)*P40(NQ)
P46(NP1)=P46(NP1)+UMM*P12(NP)*P43(NQ)+RU*P13(NP)*P44(NQ)
203 P47(NP1)=P47(NP1)+UMM*P12(NP)*P41(NQ)+RU*P13(NP)*P42(NQ)
DO 204 NP=1,NP1
NQ=N-NP+2
P48(NP1)=P48(NP1)+P7(NP)*P45(NQ)/P
P49(NP1)=P49(NP1)+P7(NP)*P46(NQ)/P
204 P50(NP1)=P50(NP1)+P7(NP)*P47(NQ)/P
111 CONTINUE
3 X(1)=P1(NT)
X(2)=P2(NT)
X(3)=P4(NT)
X(4)=P5(NT)
X(5)=P7(NT)
X(6)=P8(NT)
X(7)=P9(NT)
X(10)=P10(NT)
X(11)=P11(NT)
X(12)=P12(NT)
X(13)=P13(NT)
A(1,1)=P19(NT)
A(2,1)=P20(NT)
A(3,1)=P21(NT)
A(4,1)=P22(NT)

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Table 22 (contd)

```

A(1,2)=P23(NT)
A(2,2)=P24(NT)
A(3,2)=P25(NT)
A(4,2)=P26(NT)
A(1,3)=P27(NT)
A(2,3)=P28(NT)
A(3,3)=P29(NT)
A(4,3)=P30(NT)
A(1,4)=P31(NT)
A(2,4)=P32(NT)
A(3,4)=P33(NT)
A(4,4)=P34(NT)
DO 120 K=2,NT
L=NT-K+1
X(1)=X(1)*DT+P1(L)
X(2)=X(2)*DT+P2(L)
X(3)=X(3)*DT+P4(L)
X(4)=X(4)*DT+P5(L)
X(5)=X(5)*DT+P7(L)
X(6)=X(6)*DT+P8(L)
X(7)=X(7)*DT+P9(L)
X(10)=X(10)*DT+P10(L)
X(11)=X(11)*DT+P11(L)
X(12)=X(12)*DT+P12(L)
X(13)=X(13)*DT+P13(L)
A(1,1)=A(1,1)*DT+P19(L)
A(2,1)=A(2,1)*DT+P20(L)
A(3,1)=A(3,1)*DT+P21(L)
A(4,1)=A(4,1)*DT+P22(L)
A(1,2)=A(1,2)*DT+P23(L)
A(2,2)=A(2,2)*DT+P24(L)
A(3,2)=A(3,2)*DT+P25(L)
A(4,2)=A(4,2)*DT+P26(L)
A(1,3)=A(1,3)*DT+P27(L)
A(2,3)=A(2,3)*DT+P28(L)
A(3,3)=A(3,3)*DT+P29(L)
A(4,3)=A(4,3)*DT+P30(L)
A(1,4)=A(1,4)*DT+P31(L)
A(2,4)=A(2,4)*DT+P32(L)
A(3,4)=A(3,4)*DT+P33(L)
A(4,4)=A(4,4)*DT+P34(L)
120 CONTINUE
RETURN
END

$IBFTC DERIV.
SUBROUTINE DERIV(X,F)
C SUBROUTINE DERIV WITH PULSATING COORDINATES
C (EQUATIONS OF MOTION + VARIATIONAL EQUATIONS)
C INPUT =X =
C X(1 THROUGH 4)=X,Y,XDOT,YDOT
C X(5 THROUGH 20)=VARIATIONAL EQUATIONS
C X(21)=TIME
C X(22)=TRUE ANOMALY
C F=DERIVATIVES OF X
DOUBLE PRECISION F(22),C9,C10,S12,S22,YS,V3,X(22),C1,C2,C3,C4,C5,C
16,C7,S1,S2,R,P,UMM,XI1,RU,XI2,Q,DFXX,DFXY,DFYY,E
COMMON RU,UMM,XI1,XI2,E,P,Q,S1,S2
C1=X(1)-XI1
C2=X(1)-XI2
C3=X(2)*X(2)

```

Table 22 (contd)

```

C4=1.0D0+E*DCOS(X(22))
R=P/C4
S12=C1*C1+C3
S22=C2*C2+C3
S1=DSQRT(S12)
S2=DSQRT(S22)
C5=UMM/S1
C6=RU/S2
C7=1.0D0/C4
V3=C5/S12+C6/S22
C9=C5*C1
C10=C6*C2
F(1)=X(3)
F(2)=X(4)
F(3)=2.0D0*X(4)+C7*(X(1)-C9/S12-C10/S22)
F(4)=-2.0D0*X(3)+C7*(X(2)-C5*X(2)/S12-C6*X(2)/S22)
DFXX=C7*(1.0D0-V3+3.0D0*(C9*C1/S12/ S12+C10*C2/S22/S22))
DFXY=3.0D0*C7*(X(2)*C9/S12/S12+X(2)*C10/S22/S22)
DFYY=C7*(1.0D0-V3+3.0D0*(C3*C5/S12/S12+C3*C6/S22/S22))
DO 100 I=5,17,4
F(I)=X(I+2)
F(I+1)=X(I+3)
F(I+2)=DFXX*X(I)+DFXY*X(I+1)+2.0D0*X(I+3)
100 F(I+3)=DFXY*X(I)+DFYY*X(I+1)-2.0D0*X(I+2)
F(21)=R*R/Q
F(22)=1.0D0
RETURN
END

$IBFTC DERIV.
      SUBROUTINE DERIV(X,F)
C      SUBROUTINE DERIV FOR PULSATING CARTESIAN COORDINATES.
C INPUT =X(6)=X,Y,XDOT,YDOT,TIME .
C OUTPUT=CORRESPONDING DERIVATIVES.
C SOME OTHER INPUT CONSTANTS ARE IN COMMON.
      DOUBLE PRECISION R,C7,RU,XI2,Q,UMM,E,S1,XI1,P,S2,S13,C2, V,C3,S23
      1,C1,C4,C5,C8, C6,X(10),F(10),PI
      2,CHI,ECAN,AM,COSE,SINE,COSV,SINV,RHO1,RHO2,CX1,CX2,CY1,CY2
      COMMON RU,UMM,XI1,XI2,E,P,Q,S1,S2,PI,NT,NTM1,CHI,ECAN,COSV,SINV,NK
      C1=X(1)-XI1
      C2=X(1)-XI2
      C3=X(2)*X(2)
      C4=1.0D0+E*DCOS(X(6))
      R=P/C4
      S1=DSQRT(C1*C1+C3)
      S13=S1*S1*S1
      S2=DSQRT(C2*C2+C3)
      S23=S2*S2*S2
      C5=UMM*C1/S13+RU*C2/S23
      C6=UMM/S13+RU/S23
      C7=1.0D0/C4
      C8=1.0D0-C7
      F(1)=X(3)
      F(2)=X(4)
      F(3)=2.0D0*X(4)+C7*(X(1)-C5)
      F(4)=-2.0D0*X(3)+C7*X(2)*(1.0D0-C6)
      F(5)=R*R/Q
      F(6)=1.0D0
      RETURN
      END

```

Table 22 (contd)

```

$IBFTC DERIV.
  SUBROUTINE DERIV(X,F)
C    SUBROUTINE DERIV FOR INERTIAL CARTESIAN COORDINATES.
C    INPUT =X(6)=X,Y,XDOT,YDOT,TIME,TRUEANOMALY.
C    OUTPUT=CORRESPONDING DERIVATIVES.
C    SOME OTHER INPUT CONSTANTS ARE IN COMMON.
      DOUBLE PRECISION R,C7,RU,XI2,Q,UMM,E,S1,XI1,P,S2,S13,C2, V,C3,S23
1,C1,C4,C5,C8, C6,X(10),F(10),PI
2,CHI,ECAN,AM,COSE,SINE,COSV,SINV,RHO1,RHO2,CX1,CX2,CY1,CY2
      COMMON RU,UMM,XI1,XI2,E,P,Q,S1,S2,PI,NT,NTM1,CHI,ECAN,COSV,SINV,NK
      AM=X(5)+CHI
      CALL KEPLER(AM,E,ECAN,0.1D-12,NK,12)
      SINE=DSIN(ECAN)
      COSE=DCOS(ECAN)
      COSV=(-E+COSE)
      SINV=(Q*SINE)
      CX1=-RU *COSV
      CX2=+UMM*COSV
      CY1=-RU *SINV
      CY2=+UMM*SINV
      S1=DSQRT((X(1)-CX1)**2+(X(2)-CY1)**2)
      S13=S1**3
      S2=DSQRT((X(1)-CX2)**2+(X(2)-CY2)**2)
      S23=S2**3
      F(1)=X(3)
      F(2)=X(4)
      F(3)=-UMM*(X(1)-CX1)/S13-RU*(X(1)-CX2)/S23
      F(4)=-UMM*(X(2)-CY1)/S13-RU*(X(2)-CY2)/S23
      F(5)=+1.0D0
      RETURN
      END

$IBFTC KEPLR.
  SUBROUTINE KEPLER(AM,E,EA,EPS,NIT,MAX)
C    SOLVES KEPLER EQUATTION
      DOUBLE PRECISION AM,E,EA,EPS,EMIN,EMAX,SLOPE,DE,E0,FE
      E0=EA
      EMIN=AM-E
      EMAX=AM+E
C    HERE WITH NEWTON ITERATION METHOD
      NIT=0
3  SLOPE=1.0D0-E*DCOS(E0)
   IF(SLOPE.LT.1.D-3) SLOPE=1.D-3
   DE=(E0-E*DSIN(E0)-AM)/SLOPE
   EA=E0-DE
   IF(EA.LT.EMIN) EA=EMIN
   IF(EA.GT.EMAX) EA=EMAX
   IF(DABS(DE)-EPS) 1,1,2
2  E0=EA
   NIT=NIT+1
   IF(NIT-MAX)3,4,4
C    HERE WITH BISECTION METHOD (WHEN E=1.0)
4  NIT=0
5  EA=.5D0*(EMIN+EMAX)
   FE=EA-E*DSIN(EA)-AM
   IF(FE)101,1,103
101 EMIN=EA
   GO TO 104
103 EMAX=EA

```

Table 22 (contd)

```

104 IF((EMAX-EMIN)-0.1D-12) 1,1,5
  1 RETURN
  END
  NIT=0
  1 RETURN

$IBFTC RUK.
  SUBROUTINE RUK(X,DT,N,U,F,D)
C  RUNGE-KUTTA 4TH ORDER, CLASSICAL                                RUK00030
C  INTEGRATES A SYSTEM WITH N DIFFERENTIAL EQUATIONS (FIRST-ORDER)
C  WITH N VARIABLES X ,WITH INTEGRATION STEP DT
C  U,F,D=TEMPORARY STORAGE(IN CALLING LIST BECAUSE OF VARIABLE DIMENSION)
C  12 OPERATIONS PER STEP AND PER VARIABLE                        RUK00040
  DIMENSION X(N),U(N),F(N),D(N)
  CALL DERIV(X,D)
  DO 101 I=1,N                                                    RUK00100
    D(I)=D(I)*DT
  101 U(I)=X(I)+0.5D0*D(I)
  CALL DERIV(U,F)
  DO 102 I=1,N                                                    RUK00140
    F(I)=F(I)*DT
    D(I)=D(I)+2.0D0*F(I)
  102 U(I)=X(I)+0.5D0*F(I)
  CALL DERIV(U,F)
  DO 103 I=1,N                                                    RUK00180
    F(I)=F(I)*DT
    D(I)=D(I)+2.0D0*F(I)
  103 U(I)=X(I)+F(I)                                              RUK00200
  CALL DERIV(U,F)
  DO 104 I=1,N                                                    RUK00220
    F(I)=F(I)*DT
  104 X(I)=X(I)+(D(I)+F(I))/6.0D0
  RETURN                                                         RUK00240
  END                                                            RUK00250

$IBFTC TRUANO
  SUBROUTINE TRUEAN(TIME,V,R)
C  COMPUTES THE TRUE ANOMALY V AND DISTANCE R
C  INPUT =TIME
C  DOUBLE PRECISION AM,E,ECAN,COSE,SINE,R,COSV, SINV,V,Q,PI
  1,TIME,RU,UMM,XI1,XI2,P,S1,S2,CHI
  COMMON RU,UMM,XI1,XI2,E,P,Q,S1,S2,PI,NT,NTM1,CHI,ECAN,COSV,SINV,NK
  AM=TIME+CHI
  CALL KEPLER(AM,E,ECAN,0.1D-12,NK,12)
  SINE=DSIN(ECAN)
  COSE=DCOS(ECAN)
  R=1.0D0-E*COSE
  IF(R-1.0D-10) 401,401,402
  402 COSV=(-E+COSE)/R
  SINV=(Q*SINE)/R
  GO TO 403
  401 COSV=+1.0D0
  SINV=+0.0D0
  403 CALL ARC(COSV,SINV,V)
C  ARC=ARC TANGENT SUBROUTINE(0 TO 360)
  RETURN
  END

$IBFTC INROT.

```

Table 22 (contd)

```

SUBROUTINE INROT(X,XR,TRUE,R,Q)
C   CONVERTS INERTIAL X TO ROTATING COORDINATES XR
C   INPUT=X,TRUE,R,Q   (X=X,Y,XPRIME,YPRIME)
C   OUTPUT=XR =X,Y,XPRIME,YPRIME
C   TRUE=TRUE ANOMALY
C   R=DISTANCE BETWEEN PRIMARIES
C   Q=SQRT(1.0-E**2)
C   DOUBLE PRECISION X,XR,COV,Q,SIV,C1,TRUE,R
C   DIMENSION X(10),XR(10)
C   COV=DCOS(TRUE)
C   SIV=DSIN(TRUE)
C   C1=Q/R**2
C   XR(1)=+X(1)*COV+X(2)*SIV
C   XR(2)=-X(1)*SIV+X(2)*COV
C   XR(3)=+XR(2)*C1+X(3)*COV+X(4)*SIV
C   XR(4)=-XR(1)*C1-X(3)*SIV+X(4)*COV
C   XR(5)=X(5)
C   XR(6)=X(6)
C   XR(7)=X(7)
C   XR(8)=X(8)
C   XR(9)=X(9)
C   RETURN
C   END

$IBFTC ROTIN.
SUBROUTINE ROTIN(XR,X,TRUE,R,Q)
C   CONVERTS ROTATING XR TO INERTIAL COORDINATES X
C   INPUT=XR,TRUE,R,Q   (XR=X,Y,XPRIME,YPRIME)
C   OUTPUT=X =X,Y,XPRIME,YPRIME
C   TRUE =TRUE ANOMALY
C   R=DISTANCE BETWEEN PRIMARIES
C   Q=SQRT(1.0-E**2)
C   DOUBLE PRECISION COV,Q,SIV,C2,C1,XR,X,TRUE,R
C   DIMENSION XR(10),X(10)
C   COV=DCOS(TRUE)
C   SIV=DSIN(TRUE)
C   X(1)=XR(1)*COV-XR(2)*SIV
C   X(2)=XR(1)*SIV+XR(2)*COV
C   C1=XR(3)-XR(2)*Q/R**2
C   C2=XR(4)+XR(1)*Q/R**2
C   X(3)=C1*COV-C2*SIV
C   X(4)=C1*SIV+C2*COV
C   X(5)=XR(5)
C   X(6)=XR(6)
C   X(7)=XR(7)
C   X(8)=XR(8)
C   X(9)=XR(9)
C   RETURN
C   END

$IBFTC ROTPL.
SUBROUTINE ROTPUL(XR,XP,R,RP,Q)
C   CONVERTS ROTATING XR TO PULSATING XP COORDINATES
C   INPUT= XR,R,RP,Q.
C   OUTPUT=XP
C   XR=X,Y,XPRIME,YPRIME   XP=X,Y,XDOT,YDOT
C   R=DISTANCE BETWEEN TWO PRIMARIES
C   RP=RPRIME
C   Q=SQRT(1.0-E**2)
C   DOUBLE PRECISION Q,XR,XP,R,RP

```

Table 22 (contd)

```

      DIMENSION XR(10),XP(10)
      XP(1)=XR(1)/R
      XP(2)=XR(2)/R
      XP(3)=(R*XR(3)-RP*XR(1))/Q
      XP(4)=(R*XR(4)-RP*XR(2))/Q
      XP(5)=XR(5)
      XP(6)=XR(6)
      XP(7)=XR(7)
      XP(8)=XR(8)
      XP(9)=XR(9)
      RETURN
      END

$IBFTC PULRT.
      SUBROUTINE PULROT(XP,XR,R,RD,Q)
C      CONVERTS PULSATING XP TO ROTATING XR COORDINATES
C      INPUT=XP,R,RD,Q.
C      OUTPUT=XR
C      XP=X,Y,XDOT,YDOT      XR=X,Y,XPRIME,YPRIME
C      R=DISTANCE BETWEEN PRIMARIES
C      RD=RDOT
C      Q=SQRT(1.0-E**2)
      DOUBLE PRECISION XP,XR,C1,Q,R,RD
      DIMENSION XP(10),XR(10)
      XR(1)=R*XP(1)
      XR(2)=R*XP(2)
      C1=Q/R**2
      XR(3)=(RD*XP(1)+R*XP(3))*C1
      XR(4)=(RD*XP(2)+R*XP(4))*C1
      XR(5)=XP(5)
      XR(6)=XP(6)
      XR(7)=XP(7)
      XR(8)=XP(8)
      XR(9)=XP(9)
      RETURN
      END

$IBFTC MAINE3
C      REGULARIZED RESTRICTED ELLIPTIC THREE-BODY PROBLEM R.BROUCKE 5/67
C      INP=INTEGER INPUT VECTOR(DIMENSION=22)
C      1=JN=JOB NUMBER
C      1=COMPUTE A LIST OF ORBITS,WITH SEARCH FOR PERIODIC ORBITS
C      2=GENERATE A LIST OF PERIODIC ORBITS (BY EXTRAPOLATIONS)
C      2=INPUT CODE (1 TO 4)
C      1=INPUT WITH CARTESIAN BARYCENTRIC ROTATING PULSATING
C      2=INPUT WITH BIRKHOFF COORDINATES
C      3=INPUT FOR EJECTION FROM M1 (GIVE ONLY RNP(5) AND RNP(15))
C      4=INPUT FOR EJECTION FROM M2 (GIVE ONLY RNP(5) AND RNP(15))
C      4=PRINT STEP
C      ONE LINE WILL BE PRINTED EVERY INP(4) INTEGRATION STEPS
C      7 =NCOL=0 WHEN NO RECAP CARDS ARE USED FOR INITIAL CONDITIONS
C      =1 OR 2 WHEN RECAP CARDS USED FOR INITIAL CONDITIONS
C      1=USE COLS 1 TO 24 FOR INITIAL CONDITIONS
C      2=USE COLS 25 TO 48 FOR INITIAL CONDITIONS
C      8=MAX=MAXIMUM NUMBER OF ITERATIONS IN SEARCH
C      9=MNO=MAXIMUM NUMBER OF ORBITS IN LIST ,WHEN JN=1
C      =MAXIMUM NUMBER OF PERIODIC ORBITS,WHEN JN=2
C      10=I1=VARIABLE TO BE INCREMENTED IN LIST,WHEN JN=1 (1 TO 8)
C      =VARIABLE TO BE PERTURBED WHEN JN=2 (1 TO 8)
C      13=IS1= FIRST VARIABLE TO SEARCH ON

```

Table 22 (contd)

```

C      14=IS2=SECOND VARIABLE TO SEARCH ON
C      15      =0 WHEN E AND MU ON RECAP CARDS ARE NOT USED
C              =NOT 0 WHEN E AND MU ON RECAP CARDS ARE USED
C      16      =MCOI=0 WHEN NO INITIAL TABLE FOR EXTRAPOLATION IS READ IN
C              =1 OR 2 WHEN TABLE FOR EXTRAPOLATION IS READ IN
C      17=MNPS =MAXIMUM NUMBER OF POINTS USED IN EXTRAPOLATIONS
C      18=NT=NUMBER OF TERMS USED IN POWER SERIES
C      19=MAXIMUM NUMBER OF INTERSECTIONS WITH X-AXIS (BUILT-IN=2)
C      20=MAXIMUM NUMBER OF INTEGRATION STEPS ALLOWED (BUILT-IN=1000)
C      RNP=REAL INPUT VECTOR (DIMENSION=22)
C          1,2,3,4=X,Y,VX,VY(NOT USED FOR COLLISION ORBITS)
C          5=ENERGY (BARYCENTRIC)(USED ONLY FOR COLLISION ORBITS)
C          7=MASS-RATIO
C          8=ECCENTRICITY
C          9=EPSILON FOR MATRIX INVERSION
C      10=PRECISION FOR FIRST VARIABLE TO SEARCH ON
C      11=PRECISION FOR SECOND VARIABLE TO SEARCH ON
C      12=FIRST INCREMENT FOR PARTIAL DERIVATIVES IN SEARCH
C      13=SECOND INCREMENT FOR PARTIAL DERIVATIVES IN SEARCH
C      14=NUMBER OF HALF REVOLUTIONS
C      15=COLLISION ANGLE IN BIRKHOFF PLANE(IN DEGREES)
C      16=0.0 FOR START AT PERIAPSIS, AND 1.0 FOR APDAPSIS
C      17=INCREMENT FOR LIST
C      18=INTEGRATION STEP FOR BIRKHOF COORDINATES
C      19=NECESSARY PRECISION TO START SEARCH WHEN JN=1
C      20=NOMINAL INTEGRATION STEP WITH POWER SERIES
C      21=R1MIN =R1-LIMIT FOR SWITCHING OF VARIABLES
C      22=R2MIN =R2-LIMIT FOR SWITCHING OF VARIABLES
C      INTEGER INP(22),NBR(4)
C      DOUBLE PRECISION RNP(22),XP(22),XB(22),XPI(22),XPIR(22),
C      1TABLE1(20),TABLE2(20),TABLE3(20),DUMY(3),PNP(20) ,
C      2XS(2),Y(2),YO(2),G(2),YF(2)
C      3,U,UM1,XI1,XI2,E,P,SP,R1,R2,PI,XO
C      COMMON U,UM1,XI1,XI2,E,P,SP,R1,R2,PI,NT,NTM1,XO
C      36 FORMAT (1H0,19H THE TOTAL TIME IS ,F11.3, 9H SECONDS )
C      111 FORMAT(1H0,7H ALARM=,I2)
C      308 FORMAT(1H ,4D24.17,I4)
C      313 FORMAT(25X14HNO CONVERGENCE)
C      314 FORMAT(26X13HDETERMINANT=0)
C      400 FORMAT(1H0,22H PERIODIC ORBIT FOUND      )
C      977 FORMAT(3D24.17,I8)
C      978 FORMAT(1H0)
C      980 FORMAT(1H ,6(1PD20.13))
C      981 FORMAT(1H ,24I5)
C      988 FORMAT(1H1,67HREGULARIZED ELLIPTIC RESTRICTED THREE BODY PROBLEM -
C      1 TWO DIMENSIONS)
C      DATA(NBR(I),I=1,4)/1,2,3,4/
C      DO 601 I=1,22
C      INP(I)=0
C      601 RNP(I)=0.000
C      CALL FPTRP(0,0,0)
C      PI=3.141592653589793
C      YF(1)=0.000
C      YF(2)=0.000
C      NAMELIST/NAMES/INP,RNP
C      BUILT-IN VALUES FOR INPUT PARAMETERS
C      INP(1)=2
C      INP(2)=3
C      INP(4)=20
C      INP(7)=0
C      INP(8)=5
C      INP(9)=3

```

Table 22 (contd)

```

INP(10)=7
INP(13)=5
INP(14)=8
INP(15)=0
INP(16)=0
INP(17)=6
INP(18)=12
INP(19)=2
INP(20)=1000
RNP(1)=0.02D0
RNP(2)=0.0D0
RNP(3)=0.0D0
RNP(4)=6.3D0
RNP(5)=-0.216D0
RNP(7)=0.012155D0
RNP(8)=+0.670D0
RNP(9)=0.1D-8
RNP(10)=0.1D-8
RNP(11)=0.1D-8
RNP(12)=0.1D-6
RNP(13)=0.1D-6
RNP(14)=1.0D0
RNP(15)=0.0D0
RNP(16)=0.0D0
RNP(17)=0.01D0
RNP(18)=0.02D0
RNP(19)=0.10D0
RNP(20)=0.02D0
RNP(21)=0.08D0
RNP(22)=0.05D0
C  START ONE CASE
102 READ(5,NAMES)
    CALL CLOCKI
    WRITE(6,988)
    WRITE(6,981)(INP(I),I=1,20)
    WRITE(6,980)(RNP(I),I=1,22)
C  PRELIMINARY CALCULATIONS
    JN =INP(1)
    IN=INP(2)
    NPRIN=INP(4)
    NCOL=INP(7)
    MAX=INP(8)
    MNO=INP(9)
    I1 =INP(10)
    IS1=INP(13)
    IS2=INP(14)
    MCOL=INP(16)
    MNPS=INP(17)
    NT=INP(18)
    NTM1=NT-1
    DO 10 I=1,22
10  XPI(I)=RNP(I)
    IF(NCOL.EQ.0) GO TO 11
C  READ-IN INITIAL CONDITIONS ON RECAP CARDS
12 READ(5,977)DUMY(1),DUMY(2),DUMY(3),NNN
    IF(NNN.EQ.99) GO TO 108
    WRITE(6,978)
    XPI(1)=DUMY(NCOL)
    IF(INP(15).NE.0) XPI(7)=DUMY(3)
    READ(5,977)DUMY(1),DUMY(2),DUMY(3)
    XPI(2)=DUMY(NCOL)
    IF(INP(15).NE.0) XPI(8)=DUMY(3)

```

Table 22 (contd)

```

      READ(5,977) DUMY(1), DUMY(2), XPI(5)
      XPI(3)=DUMY(MCOL)
      READ(5,977) DUMY(1), DUMY(2)
      XPI(4)=DUMY(MCOL)
11  NNO=0
      NPS=0
      IF(MCOL.EQ.0) GO TO 600
C   READ INITIAL EXTRAPOLATION TABLE ON RECAP CARDS
13  READ(5,977) DUMY(1), DUMY(2), PNP(7), NNN
      IF(NNN.EQ.99) GO TO 24
      PNP(1)=DUMY(MCOL)
      READ(5,977) DUMY(1), DUMY(2), PNP(8)
      PNP(2)=DUMY(MCOL)
      READ(5,977) DUMY(1), DUMY(2), PNP(5)
      PNP(3)=DUMY(MCOL)
      READ(5,977) DUMY(1), DUMY(2)
      PNP(4)=DUMY(MCOL)
      IF(NPS.LT.MNPS) GO TO 21
      DO 22 I=2, MNPS
        TABLE1(I-1)=TABLE1(I)
        TABLE2(I-1)=TABLE2(I)
22  TABLE3(I-1)=TABLE3(I)
      GO TO 23
21  NPS=NPS+1
23  TABLE1(NPS)=PNP(I1)
      TABLE2(NPS)=PNP(IS1)
      TABLE3(NPS)=PNP(IS2)
      GO TO 13
24  IF(NPS.EQ.0) GO TO 600
      XPI(I1)=TABLE1(NPS)+RNP(17)
      DO 25 I=1, NPS
25  WRITE(6,308) TABLE1(I), TABLE2(I), TABLE3(I)
600 IF(INP(17).EQ.0) GO TO 14
C   EXTRAPOLATION FOR FIRST GUESS OF NEW PERIODIC ORBIT
      IF(NPS=1) 14,15,16
15  XPI(IS1)=TABLE2(1)
      XPI(IS2)=TABLE3(1)
      GO TO 14
16  CALL INPOL(TABLE1, TABLE2, XPI(I1), XPI(IS1), NPS)
      CALL INPOL(TABLE1, TABLE3, XPI(I1), XPI(IS2), NPS)
C   COMPUTE ONE COMPLETE ORBIT (BY CALL ORBIT)
14  CALL ORBIT(XPI, XP, XB, INP, NPRIN, IALARM)
      IF(IALARM.GT.0) GO TO 110
      MC=0
      GO TO (300,301), JN
300 IF(DABS(XP(2))-RNP(19)) 302,302,106
302 IF(DABS(XP(3))-RNP(19)) 303,303,106
303 DO 305 I=1,9
305 XPIR(I)=XPI(I)
301 IF(DABS(XP(2))-RNP(10)) 306,306,307
306 IF(DABS(XP(3))-RNP(11)) 309,309,307
C   BEGIN SEARCH FOR PERIODIC ORBIT
307 XS(1)=XPI(IS1)
      XS(2)=XPI(IS2)
      MC=1
      NAL=2
      YO(1)=XP(2)
      YO(2)=XP(3)
100 WRITE(6,308) XPI(IS1), XPI(IS2), YO(1), YO(2), MC
101 CALL SEARCH(XS, G, YO, Y, YF, RNP(12), RNP(10), 2, MAX, 2, NAL, MC, RNP(9), LL)
      GO TO (106,2,2,4,5,6), NAL
2   IF((+DABS(XPI(IS1)-G(1))) .GT. 1.0D0) GO TO 5

```

Table 22 (contd)

```

      IF((+DABS(XPI(IS2)-G(2))).GT.1.0D0) GO TO 5
      XPI(IS1)=G(1)
      XPI(IS2)=G(2)
      CALL ORBIT(XPI,XP,XB,INP,0,IALARM)
      IF(IALARM.GT.0) GO TO 110
      GO TO (106,3,7),NAL
3    IF(DABS(XP(2)).GT.DABS(YO(1))) GO TO 109
      IF(DABS(XP(3)).GT.DABS(YO(2))) GO TO 109
      YO(1)=XP(2)
      YO(2)=XP(3)
      GO TO 100
7    Y(1)=XP(2)
      Y(2)=XP(3)
      GO TO 101
309  WRITE(6,308)XPI(IS1),XPI(IS2),XP(2),XP(3),MC
      GO TO 401
4    XPI(IS1)=XS(1)
      XPI(IS2)=XS(2)
401  WRITE (6,400)
      IF(INP(17).EQ.0) GO TO 8
C    STORE PERIODIC ORBIT IN EXTRAPOLATION TABLE
      IF(NPS.LT.MNPS) GO TO 18
      DO 19 I=2,NPS
        TABLE1(I-1)=TABLE1(I)
        TABLE2(I-1)=TABLE2(I)
19    TABLE3(I-1)=TABLE3(I)
      GO TO 20
18    NPS=NPS+1
20    TABLE1(NPS)=XPI(I1)
      TABLE2(NPS)=XPI(IS1)
      TABLE3(NPS)=XPI(IS2)
C    PUNCH 4 RECAP CARDS FOR PERIODIC ORBIT
8    PUNCH 977,XPI(1),XP(1),XPI(7),NBR(1),
1      XPI(2),XP(2),XPI(8),NBR(2),
2      XPI(3),XP(3),XPI(5),NBR(3),
3      XPI(4),XP(4),XPI(15),NBR(4)
310  GO TO (311,106),JN
311  DO 312 I=1,9
312  XPI(I)=XPIR(I)
      GO TO 106
109  WRITE(6,308)XPI(IS1),XPI(IS2),XP(2),XP(3),MC
5    WRITE(6,313)
      GO TO(311,118),JN
6    WRITE(6,314)
      GO TO(311,118),JN
110  WRITE(6,111) IALARM
      GO TO (106,118),JN
106  NNO=NNO+1
      IF(NNO-MNO)107,118,118
C    PERTURB LIST PARAMETER
107  XPI(I1)=XPI(I1)+RNP(17)
      WRITE(6,978)
      GO TO 600
118  IF(NCOL)12,108,12
108  CALL CLOCT(PARTIM,TOTTIM)
      WRITE (6,36)TOTTIM
      GO TO 102
      END

$IBFTC ORBIT.
      SUBROUTINE ORBIT(XI,X,XB,INP,NPRIN,IALARM)

```

Table 22 (contd)

```

C INPUT=XI=INITIAL CONDITIONS
C XI(22)=22 INPUT CONSTANTS+INITIAL CONDITIONS.(=RNP)
C INP=INTEGER INPUT VECTOR (ONLY INP(2) IS USED).
C NPRIN =PRINT-STEP (NO PRINTING IF ZERO) (=INPUT)
C INP(20)=20 INTEGER INPUT CONSTANTS
C OUTPUT=X,XB AND IALARM
C X(6)=OUTPUT=FINAL POINT OF THE ORBIT,IN CARTESIAN COORDINATES
C XB(6)=OUTPUT=FINAL POINT OF THE ORBIT,IN BIRKHOFF COORDINATES
C X AND XB(1 TO 7) =X,Y,XDOT,YDOT/ENERGY/TRUE ANOMALY/TIME
C IALARM IS NON-ZERO IN CASE OF AN ABNORMAL ENDING OF THE ORBIT
C DOUBLE PRECISION ABC,X,XB,E,P,UM1,U,XO,SP,PI,RAD,EPS4,XS2,DS,SF,YS
1,R1,R2,EP1,S,XB6PR,SPREC,DSS,XI,ENER,A(7),B(7),C(7),YACT,YPREC
2,XI1,XI2,DVC,XIS,ETS,SUM,DIF,EX,XSYS,CCC,R1R2,TRUE,ERROR
C DIMENSION X(22),XB(22),XI(22),INP(20)
C COMMON U,UM1,XI1,XI2,E,P,SP,R1,R2,PI,NT,NTM1,XO
26 FORMAT(1H ,5(1PD21.13),1PD9.1)
603 FORMAT(10X3HMU=,D24.17,5X4HECC=,D24.17)
989 FORMAT(1H ,6(1PD14.7),2(1PD18.11))
RAD=180.D0/PI
IALARM=0
MTEMP=0
INTER=0
IN=INP(2)
MTOTAL=0
ERROR=0.0D0
DO 10 I=1,5
10 X(I)=XI(I)
XB(7)=0.0D0
X(7)=0.0D0
S=0.0D0
MPRIN=0
EPS4=1.0D-13
U=XI(7)
UM1=1.0D0-U
XI1=-U
XI2=UM1
XO=.5D0-U
XS2=0.5D0*XO*XO
E=XI(8)
IF(NPRIN.GT.0)WRITE(6,603)U,E
P=1.0D0-E*E
IF(P.LE.0.0D0) GO TO 12
SP=DSQRT(P)
DS=XI(18)
SF=PI*(XI(14)+XI(16))
C CONVERT INITIAL CONDITIONS
GO TO (1,1,3,4),IN
1 YS=X(2)*X(2)
R1=DSQRT((X(1)+U)**2+YS)
R2=DSQRT((X(1)-1.0D0+U)**2+YS)
X(6)=XI(16)*PI
EP1=1.0D0/(1.0D0+E*DCOS(X(6)))
ENER=.5D0*(X(3)**2+X(4)**2)-EP1*(.5D0*(X(1)*X(1)+YS)+UM1/R1+U/R2)
C CHECK IN WHICH REGION INTEGRATION HAS TO START
IF(R1.GT.XI(21).AND.R2.GT.XI(22)) GO TO 28
IF(NPRIN.GT.0)
*WRITE(6,989)X(1),X(2),X(3),X(4),X(6),XB(7),ENER,ENER
X(5)=ENER+XS2*EP1
C CONVERT TO BIRKHOFF COORDINATES
X(1)=X(1)-XO
CALL CARBR(X,XB)
YACT=X(2)

```

Table 22 (contd)

```

      GO TO 17
28  TRUE=X(6)
      X(5)=P*EP1
      X(6)=E*DSIN(X (6))*X(5)**2/P
      INIT=1
      IF (NPRIN.GT.0)
      *WRITE(6,26)(X(I),I=1,4), ENER, ERROR
      GO TO 18
3   XB(1)=-1.0D0
      XB(2)=0.0D0
      XB(6)=XI(16)*PI
      EP1=1.0D0/(1.0D0+E*DCOS(XB(6)))
      XB(3)=DSQRT(2.0D0*UM1*EP1)*DCOS( XI(15)/RAD)
      XB(4)=DSQRT(2.0D0*UM1*EP1)*DSIN( XI(15)/RAD)
      XB(5)=X (5)+XS2*EP1
      R1=0.0D0
      R2=1.0D0
      GO TO 5
4   XB(1)=1.0D0
      XB(2)=0.0D0
      XB(6)=XI(16)*PI
      EP1=1.0D0/(1.0+E*DCOS(XB(6)))
      XB(3)=DSQRT(2.0D0*U*EP1)*DCOS( XI(15)/RAD)
      XB(4)=DSQRT(2.0D0*U*EP1)*DSIN( XI(15)/RAD)
      XB(5)=X (5)+XS2*EP1
      R1=1.0D0
      R2=0.0D0
5   S=0.0D0
      YACT=0.0D0
      IF (NPRIN.GT.0) GO TO 9
C  CHECK IF A CHANGE OF REGION IS NECESSARY
8   IF (R1.LT.XI(21).OR.R2.LT.XI(22)) GO TO 17
      CALL BIRCR(XB,X)
      X(1)=X(1)+X0
      TRUE=XB(6)
      X(5)=P/(1.0D0+E*DCOS(XB(6)))
      X(6)=E*DSIN(XB(6))*X(5)**2/P
      X(7)=XB(7)
      INIT=1
      GO TO 18
C  INTEGRATION WITH BIRKHOFF COORDINATES
17  XB6PR=XB(6)
      YPREC=YACT
      SPREC=S
      CALL RUK(XB,DS,7,A,B,C)
      S=S+DS
      MTOTAL=MTOTAL+1
      YACT=XB(2)*(XB(1)**2+XB(2)**2-1.0D0)
      IF (YPREC*YACT.LT.0.0D0) INTER=INTER+1
      IF (INTER.GE.INP(19)) GO TO 30
      IF (MTOTAL.GE.INP(20)) GO TO 30
      ABC=XB(1)**2+XB(2)**2
      IF (ABC.LT.0.001D0) GO TO 13
      IF (ABC.GT.1000.D0) GO TO 14
      IF (XB(5).GT.50.D0) GO TO 15
      IF (XB(6)-SF)6,134,134
C  INTEGRATION OF END OF ORBIT IN BIRKHOFF COORDINATES
134 DSS=(SPREC-S)*(XB(6)-SF)/(XB(6)-XB6PR)
      XB6PR=XB(6)
      SPREC=S
      CALL RUK (XB,DSS,7,A,B,C)
      S=S+DSS

```

Table 22 (contd)

```

MTEMP=MTEMP+1
IF (MTEMP.GT.15) GO TO 16
ABC=XB(1)**2+XB(2)**2
IF (ABC.LT.0.001D0) GO TO 13
IF (ABC.GT.1000.D0) GO TO 14
IF (XB(5).GT.50.D0) GO TO 15
IF (DABS(DSS)-EPS4 ) 7,7,142
142 IF (DABS(XB(6)-SF)-EPS4) 7,7,134
C CONVERT TO CARTESIAN COORDINATES
7 CALL BIRCR(XB,X)
X(1)=X(1)+X0
EP1=1.0D0/(1.0D0+E*DCOS(X(6)))
YS=X(2)*X(2)
R1=DSQRT((X(1)+U)**2+YS)
R2=DSQRT((X(1)-1.0D0+U)**2+YS)
ENER=.5D0*(X(3)**2+X(4)**2)-EP1*(.5D0*(X(1)*X(1)+YS)+UM1/R1+U/R2)
X(5)=X(5)-XS2*EP1
IF (NPRIN.EQ.0) GO TO 11
WRITE(6,989)X(1),X(2),X(3),X(4),X(6),XB(7),X(5),ENER
GO TO 11
6 IF (NPRIN.EQ.0) GO TO 8
MPRIN=MPRIN+1
IF (MPRIN-NPRIN) 8,9,9
9 CALL BIRCR(XB,X)
X(1)=X(1)+X0
EP1=1.0D0/(1.0D0+E*DCOS(X(6)))
YS=X(2)*X(2)
R1=DSQRT((X(1)+U)**2+YS)
R2=DSQRT((X(1)-1.0D0+U)**2+YS)
ENER=.5D0*(X(3)**2+X(4)**2)-EP1*(.5D0*(X(1)*X(1)+YS)+UM1/R1+U/R2)
X(5)=X(5)-XS2*EP1
WRITE(6,989)X(1),X(2),X(3),X(4),X(6),XB(7),X(5),ENER
MPRIN=0
GO TO 8
C CHECK IF A CHANGE OF REGION IS NECESSARY
19 IF (R1.GT.XI(21).AND.R2.GT.XI(22)) GO TO 18
YS=X(2)*X(2)
R1=DSQRT((X(1)+U)**2+YS)
R2=DSQRT((X(1)-UM1)**2+YS)
XB(7)=X(7)
X(6)=TRUE
EP1=1.0D0/(1.0D0+E*DCOS(X(6)))
X(5)=.5D0*(X(3)**2+X(4)**2)-EP1*(.5D0*(X(1)*X(1)+YS)+UM1/R1+U/R2)
X(5)=X(5)+XS2*EP1
X(1)=X(1)-X0
CALL CARBR(X,XB)
S=0.0D0
YACT=X(2)
GO TO 17
C INTEGRATION WITH POWER SERIES
18 DVC=XI(20)
YPREC=X(2)
R1R2=R1*R2
IF (R1R2-2.0D0) 20,20,21
20 DVC=R1R2*XI(20)
21 IF (DABS(SF-TRUE)-DVC) 22,23,23
22 DVC=SF-TRUE
MTOTAL=9999
23 CALL ERPS(X,DVC,INIT)
YACT=X(2)
MTOTAL=MTOTAL+1
TRUE=TRUE+DVC

```

Table 22 (contd)

```

R1=X(10)
R2=X(11)
IF(NPRIN.EQ.0) GO TO 24
MPRIN=MPRIN+1
IF(MPRIN-NPRIN) 24,25,25
25 MPRIN=0
ERROR=0.25D0*(DABS(X(10)**(-3)-X(12))
1+DABS(X(11)**(-3)-X(13))
2+DABS(DSQRT((X(1)-XI1)**2+X(2)**2)-X(10))
3+DABS(DSQRT((X(1)-XI2)**2+X(2)**2)-X(11)))
WRITE(6,26)(X(I),I=1,4),X(7),ERROR
24 IF(X(2)*YPREC.LT.0.0D0) INTER=INTER+1
IF(INTER.GE.INP(19)) GO TO 30
IF(MTOTAL.GT.INP(20).AND.MTOTAL.LT.9999) GO TO 30
IF(MTOTAL-9999)19,27,27
27 IF(NPRIN.EQ.0) GO TO 29
ERROR=0.25D0*(DABS(X(10)**(-3)-X(12))
1+DABS(X(11)**(-3)-X(13))
2+DABS(DSQRT((X(1)-XI1)**2+X(2)**2)-X(10))
3+DABS(DSQRT((X(1)-XI2)**2+X(2)**2)-X(11)))
WRITE(6,26)(X(I),I=1,4),X(7),ERROR
29 X(1)=X(1)-X0
CALL CARBR(X,XB)
X(1)=X(1)+X0
GO TO 11
12 IALARM=1
GO TO 11
13 IALARM=2
GO TO 11
14 IALARM=3
GO TO 11
15 IALARM=4
GO TO 11
16 IALARM=5
GO TO 11
30 IALARM=6
11 RETURN
END

```

\$IBFTC DERIV.

C SUBROUTINE DERIV FOR BIRCKOFF COORDINATES

SUBROUTINE DERIV(X,F)

C INPUT=X(7)=XI,ETA,XIDOT,ETADOT,ENERGY,TRUEANOMALY,TIME

C OUTPUT=F(7)=CORRESPONDING DERIVATIVES

```

DOUBLE PRECISION E,P,AM1,AM2,X0,SP,X,F, XIS,ETS,S,D,EX,XSYS,C,R1
1,R2,RR,AJ,SS,DJDXI,DJDET,DXDXI,DXDET,DX2DX,DX2DE,DR12X,DR12E,S1,DR
21DE,DR2DX,DR2DE,ECV,EPI,RRV,R,C2,A,B,DJVDX,DJVDE,ENER,DR1DX
Z,XI1,XI2,PI
DIMENSION X(7),F(7)
COMMON AM2,AM1,XI1,XI2,E,P,SP,R1,R2,PI,NT,NTM1,X0
XIS=X(1)*X(1)
ETS=X(2)*X(2)
S=XIS+ETS
D=XIS-ETS
EX=X(1)/4.0D0*(1.0D0+1.0D0/S)
XSYS=(S+1.0D0/S+2.0D0*D/S)/16.0D0
C=4.0D0*DSQRT(S)
R1=(S+1.0D0+2.0D0*X(1))/C
R2=(S+1.0D0-2.0D0*X(1))/C
RR=R1*R2
AJ=RR/S

```

Table 22 (contd)

```

SS=S*S
C=4.0D0*S*SS
DJD XI=X(1)*(2.0D0*D-S-1.0D0)/C
DJDET=X(2)*(2.0D0*D+S-1.0D0)/C
DXD XI=(1.0D0-D/SS)/4.0D0
DXDET=-X(1)*X(2)/(2.0D0*SS)
C=8.0D0*SS
DX2DX=X(1)/C*(SS+4.0D0*ETS-1.0D0)
DX2DE=X(2)/C*(SS-4.0D0*XIS-1.0D0)
DR12X=X(1)/C*(SS-4.0D0*ETS-1.0D0)
DR12E=X(2)/C*(SS+4.0D0*XIS-1.0D0)
C=4.0D0*DSQRT(SS*S)
S1=S-1.0D0
DR1DX=(S1*X(1)+2.0D0*ETS)/C
DR1DE=(S1-2.0D0*X(1))*X(2)/C
DR2DX=(S1*X(1)-2.0D0*ETS)/C
DR2DE=(S1+2.0D0*X(1))*X(2)/C
ECV=E*DCOS(X(6))
C=1.0D0+ECV
EP1=1.0D0/C
R=P/C
C2=.5D0*XSYS+X0*EX
RRV=AM1*R2+AM2*R1+RR*C2
A=-2.0D0*EP1/SS
B=EP1/S
DJVDX=A*X(1)*RRV+B*(AM1*DR2DX+AM2*DR1DX+C2*DR12X+RR*(.5D0*DX2DX+X0
1*DXD XI))
DJVDE=A*X(2)*RRV+B*(AM1*DR2DE+AM2*DR1DE+C2*DR12E+RR*(.5D0*DX2DE+X0
1*DXDET))
F(1)=X(3)
F(2)=X(4)
F(3)=2.0D0*AJ*X(4)+DJVDX+X(5)*DJD XI
F(4)=-2.0D0*AJ*X(3)+DJVDE+X(5)*DJDET
F(5)=-RRV*E*DSIN(X(6))/(S*C*C)
F(6)=AJ
F(7)=RR*R/R/(S*SP)
RETURN
END

$IBFTC CARBR.
SUBROUTINE CARBR(XC,XB)
C CONVERTS CARTESIAN COORDINATES TO BIRKHOFF
C INPUT =CARTESIAN MEDIAN=X =X,Y,XDOT,YDOT.
C OUTPUT= BIRKHOFF =XB=XI,ETA,XIDOT,ETADOT.
DOUBLE PRECISION XC,XB,A,B,U,V,XIS,ETS,S,D,SS,DXD XI,DYDET,DXDET,DY
1DXI
DIMENSION XC(6),XB(6)
A=XC(1)*XC(1)-XC(2)*XC(2)-.25D0
B=2.0D0*XC(1)*XC(2)
CALL COMSQR(A,B,U,V)
XB(1)=(XC(1)+U)*2.0D0
XB(2)=(XC(2)+V)*2.0D0
XIS=XB(1)*XB(1)
ETS=XB(2)*XB(2)
S=XIS+ETS
D=XIS-ETS
SS=S*S
DXD XI=(1.0D0-D/SS)*.25D0
DYDET=DXD XI
DXDET=-(XB(1)*XB(2))/(2.0D0*SS)
DYD XI=-DXDET

```

Table 22 (contd)

```

XB(3)=DYDET*XC(3)-DXDET*XC(4)
XB(4)=-DYDXI*XC(3)+DXDXI*XC(4)
XB(5)=XC(5)
XB(6)=XC(6)
RETURN
END

$IBFTC BIRCR.
SUBROUTINE BIRCR(XB,XC)
C   CONVERTS BIRKHOFF COORDINATES TO CARTESIAN FOR OUTPUT
C   INPUT=BIRKHOFF=XB=XI,ETA,XIDOT,ETADOT.
C   OUTPUT=CARTESIAN=X = X,Y,XDOT,YDOT(=CARTESIAN MEDIAN)
DOUBLE PRECISION XB,XC,XIS,ETS,S,SS,D,C,R1,R2,AJ,DXDXI,DYDET,DXDET
1,DYDXI
DIMENSION XB(6),XC(6)
XIS=XB(1)*XB(1)
ETS=XB(2)*XB(2)
S=XIS+ETS
SS=S*S
D=XIS-ETS
C=.25D0/S**.5D0
R1=C*(S+1.0D0+2.0D0*XB(1))
R2=C*(S+1.0D0-2.0D0*XB(1))
AJ=R1*R2/S
DXDXI=.25D0*(1.0D0-D/SS)
DYDET=DXDXI
DXDET=-XB(1)*XB(2)/(2.0D0*SS)
DYDXI=-DXDET
XC(1)=XB(1)*(1.0D0+1.0D0/S)*.25D0
XC(2)=XB(2)*(1.0D0-1.0D0/S)*.25D0
XC(3)=(DXDXI*XB(3)+DXDET*XB(4))/AJ
XC(4)=(DYDXI*XB(3)+DYDET*XB(4))/AJ
XC(5)=XB(5)
XC(6)=XB(6)
RETURN
END

$IBFTC COMSQ.
SUBROUTINE COMSQR (AA,BB,A,B)
COMPUTES SQUARE ROOT(A,B) OF COMPLEX NUMBER(AA,BB)
DOUBLE PRECISION TEMPO,AA,BB,A,B
TEMPO=DSQRT(AA**2+BB**2)
IF(TEMPO.LT.DABS(AA)) TEMPO=DABS(AA)
A=DSQRT(.5D0*(AA+TEMPO))
IF (BB) 10,20,20
10 B=-DSQRT(0.5D0*(-AA+TEMPO))
GO TO 30
20 B=DSQRT(.5D0*(-AA+TEMPO))
30 RETURN
END

$IBFTC SEARCH.
SUBROUTINE SEARCH(X,G,YO,Y,YF,DEL,EPS,N,M,NP,NAL,MC,EP,L)
C   X CONTAINS THE INITIAL VALUES OF THE N UNKNOWNNS (INPUT)
C   AND THEIR COMPUTED VALUE (OUTPUT)
C   G IS AN ARRAY WHERE SEARCH STORES THE VALUES OF X FOR
C   COMPUTATION OF NOMINAL AND VARIED SOLUTION, WITH NAL=2 (G=OUTPUT)
C   YO(INPUT)=NOMINAL SOLUTION (FINAL VALUES)
C   Y=ARRAY WHERE THE MAIN PROGRAM HAS TO STORE THE VARIED

```

Table 22 (contd)

```

C      FINAL VALUES, WHEN NAL=3 (INPUT)
C      YF=(INPUT) DESIRED FINAL VALUES OF Y
C      DEL=ARRAY WITH PERTURBATIONS FOR X(INPUT)
C      EPS=ARRAY WITH ACCURACY PARAMETERS FOR X(NP=1)
C      OR FOR Y (NP=2) (INPUT)
C      N=NUMBER OF VARIABLES (INPUT)
C      M=MAXIMUM NUMBER OF ITERATIONS (INPUT)
C      NAL=1.FOR INITIAL START
C          2.(SET=2 BY SUBROUTINE SEARCH)=NOMINAL VALUES YO HAVE TO BE
C             COMPUTED WITH X. THEN SEARCH HAS TO BE CALLED AGAIN.
C          3.(SET=3 BY SUBROUTINE SEARCH)=VARIED VALUES Y HAVE TO BE
C             COMPUTED WITH THE VARIED VALUES G OF X.
C             THEN CALL SEARCH AGAIN.
C          4.END OF SEARCH.CONVERGENCE IN LESS THAN M ITERATIONS
C          5.END OF SEARCH. NO CONVERGENCE
C          6.END OF SEARCH. DETERMINANT WAS ZERO IN MATRIX INVERSION
C      MC=ACTUAL NUMBER OF ITERATIONS (OUTPUT)
C      EP=EPSILON FOR MATRIX INVERSION (INPUT)
C      A=MATRIX (N,N1) OF PARTIALS (OUTPUT) (N1=N+1)
C      L=INDEX OF COLUMN OF PARTIAL DERIVATIVES(OUTPUT)
C      DOUBLE PRECISION X,G,YO,Y,YF,DEL,EPS,EP,A,DETER
C      DIMENSION          X(2),G(2),YO(2),Y(2),YF(2),DEL(2),EPS(2),A(2,3)
C      GO TO (1,2,3),NAL
1      MC=1
100     NAL=2
        DO 126 I=1,N
126     G(I)=X(I)
        GO TO 999
        2 DO 111 I=1,N
111     A(I,N+1)=YF(I)-YO(I)
        IF(NP-2)116,113,116
113     DO 114 I=1,N
        IF(DABS(A(I,N+1))-EPS(I))114,116,116
114     CONTINUE
        NAL=4
        GO TO 999
116     L=1
300     DO 118 J=1,N
118     G(J)=X(J)
        G(L)=X(L)+DEL(L)
        NAL=3
        GO TO 999
        3 DO 117 K=1,N
117     A(K,L)=(Y(K)-YO(K))/DEL(L)
        L=L+1
        IF(L-N) 300,300,301
301     CALL SOLVE2(A,EP,DETER)
        IF(DETER)125,124,125
125     DO 119 I=1,N
119     X(I)=X(I)+A(I,N+1)
        IF(NP-1)120,121,120
121     DO 122 I=1,N
        IF(DABS(A(I,N+1))-EPS(I))122,120,120
122     CONTINUE
        NAL=4
        GO TO 999
120     MC=MC+1
        IF(MC-M)100,100,123
123     NAL=5
        GO TO 999
124     NAL=6
999     RETURN

```

Table 22 (contd)

```

END

$IBFTC SOLV2.
      SUBROUTINE SOLVE2(A,EP,DETER)
C   INPUT=A,EP
C   A=GIVEN 2*3 MATRIX
C   EP =PRECISION FOR ZERO-DETERMINANT
C   OUTPUT=DETER AND A
C   DETER=DETERMINANT
C   (THE LAST COLUMN IS THE SOLUTION OF THE SYSTEM)
      DOUBLE PRECISION A,EP,DETER,B
      DIMENSION A(2,3),B(2,3)
900  FORMAT(10X,3D16.8)
      WRITE(6,900)(A(1,K),K=1,3)
      WRITE(6,900)(A(2,K),K=1,3)
      DETER=A(1,1)*A(2,2)-A(1,2)*A(2,1)
      IF(DABS(DETER)-EP)100,101,101
101  B(1,1)=+A(2,2)/DETER
      B(1,2)=-A(1,2)/DETER
      B(2,1)=-A(2,1)/DETER
      B(2,2)=+A(1,1)/DETER
      B(1,3)=+B(1,1)*A(1,3)+B(1,2)*A(2,3)
      B(2,3)=+B(2,1)*A(1,3)+B(2,2)*A(2,3)
      A(1,1)=B(1,1)
      A(1,2)=B(1,2)
      A(2,1)=B(2,1)
      A(2,2)=B(2,2)
      A(1,3)=B(1,3)
      A(2,3)=B(2,3)
      GO TO 999
100  DETER=0.000
999  RETURN
      END

$IBFTC INPOL.          POLYNOMIAL INTERPOLATION,AITKEN
C   INPOL.             POLYNOMIAL INTERPOLATION,AITKEN
      SUBROUTINE INPOL(G,H,X,Y,N)
C   R.A.BROUCKE,JPL,1966
C   G()                TABLE OF ABSCISSAE
C   H()                TABLE OF ORDINATES
C   X                  INPUT ARGUMENT
C   Y                  OUTPUT INTERPOLATED VALUE
C   N                  NO. OF POINTS TO BE USED
      DOUBLE PRECISION G,H,X,Y,F
      DIMENSION F(30),G(30),H(30)
      DO 100 M=1,N
100  F(M)=H(M)
      K=N-1
      DO 111 J=1,K
      L=J+1
      DO 111 I=L,N
111  F(I)=((X-G(J))*F(I)-(X-G(I))*F(J))/(G(I)-G(J))
      Y=F(N)
      RETURN
      END

```

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