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RATES OF CONVERGENCE FOR ITERATIVE METHODS FOR NONLINEAR SYSTEMS OF EQUATIONS

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ABSTRACT

Title of Thesis: Rates of Convergence for Iterative Methods for Nonlinear Systems of Equations

Robert Gary Voigt, Doctor of Philosophy, 1969

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The rate of convergence for iterative methods of the form

$$(1) \qquad x^{k+1} = G(x^k, \ldots, x^{k-m}), \; k = m, m+1, \ldots,$$

is considered. Linear convergence theorems are given and applied to some well-known iterative procedures, and necessary and sufficient conditions are given for superlinear convergence.

Several higher order rate of convergence theorems are given for procedure (1) including sufficient conditions for obtaining the precise order of convergence. These theorems, as well as other techniques, are used to give the precise order of convergence for the secant method and Steffensen's method, in particular.

RATES OF CONVERGENCE FOR ITERATIVE METHODS FOR NONLINEAR SYSTEMS OF EQUATIONS

by
Robert Gary Voigt

Dissertation submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
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INTRODUCTION

In this paper we shall be concerned with iterative methods of the form

$$(1) \qquad I: x^{k+1} = G(x^k, \ldots, x^{k-m}), k = m, m+1, \ldots,$$

where $x^o, \ldots, x^m$ are given, for finding a point x^* such that $x^* = G(x^*, \ldots, x^*)$, or perhaps, $x^* = G(x^*, y^2, \ldots, y^m)$ where the y^i are arbitrary. We shall primarily be interested in the case when the arguments of G are elements of R^n, but many of the results will hold if the arguments of G are elements of an arbitrary Banach space.

We shall investigate the rate at which the sequence $\{x^k\}$ generated by (1) converges to the point x^*. To facilitate this investigation we shall use three measures of the order of convergence denoted by $O_R(I,x^*)$, $O_Q(I,x^*)$ and $O_{WQ}(I,x^*)$. The R and Q measures have been thoroughly studied by Ortega and Rheinboldt [1970], and since many of their results will be useful in our analysis, the chapter of their forthcoming book containing these results is reprinted essentially in total in the Appendix.

The WQ measure is developed in the first chapter and stems from a measure used by Ostrowski [1966].

In Chapter I we shall be interested in distinguishing between linear and superlinear convergence of the process (1). In particular, we shall give linear convergence theorems for (1) and apply them to several well-known iterations, and we shall also give necessary and sufficient conditions for (1) to have superlinear convergence. A certain amount of work in this direction has already been done; for example, Ostrowski [1966] has given a convergence theorem for the process (1) when $m = 0$, and Ortega and Rockoff [1966] have given the rate of convergence and a necessary and sufficient condition for superlinear convergence in the R measure for the same procedure.

In Chapter II we shall be concerned with the precise order of convergence of process (1). Almost all convergence theorems found in the literature provide only a lower bound for the order of convergence. As well as giving results of this type for (1), we shall also give sufficient conditions for the order of convergence to be precisely the lower bound. To date the only result on precise order may be found in Ortega and Rheinboldt [1970] where sufficient conditions are given for Newton's method to have precisely quadratic convergence. We shall give a similar result for an n-dimensional secant method. Korganoff [1961], Schmidt [1966], and Ortega and Rheinboldt [1970] have all given convergence theorems for the n-dimensional secant method showing that the order of convergence is at least $\frac{1+\sqrt{5}}{2}$, but even for the one-dimensional secant method the precise order apparently has not been given. In addition, we shall analyze other methods including a procedure given by Traub [1964] and Steffensen's method which has been studied in n dimensions by Chen [1964], Wegge [1966], and Ortega and Rheinboldt [1970] among others.

CHAPTER I

LINEAR AND SUPERLINEAR CONVERGENCE THEOREMS

Section 1.1

Introduction

In this chapter we shall be interested in local convergence theorems for single-step and multi-step iterative procedures. By a multi-step iterative procedure we shall mean an iteration of the form

$$(1) \qquad I: x^{k+1} = G(x^k, \ldots, x^{k-m+1}), k = m - 1, m, \ldots,$$

where $G: D^m \subset (R^n)^m \to R^n$ and $x^i \in D$ for $i = 0, \ldots, m - 1$ are m given starting points. Throughout this paper D^m denotes the cartesian product of D m times, and if $x \in R^n$ then $x = (x_1, \ldots, x_n)^T$. A single-step iterative procedure is given by (1) when $m = 1$.

Of particular interest will be the asymptotic rates of convergence of these procedures with emphasis on the difference between linear and superlinear convergence as measured by two concepts, the root and quotient convergence factors denoted by R and Q respectively. A complete discussion of these factors is given in the Appendix which is basically a reproduction of parts of Chapter 9 of a forthcoming book of Ortega and Rheinboldt [1970].

It should be pointed out here that in many of the results in this paper involving the R factors a specific norm is chosen to facilitate the proof. According to Theorem A.2.2 the R factors are independent of the norm so that this in no way limits the generality of the result.

As was mentioned above, we are only interested in convergence theorems of a local nature. By local convergence of the process (1) to a point x^* we mean that the iterates of (1) converge to x^* whenever $x^0,\dots,x^{m-1}$ are sufficiently close to x^*. This is made precise by the following concept (see Ostrowski [1966]).

<u>Definition 1.1.1</u>. Let $G: D^m \subset (R^n)^m \to R^n$. Then x^* is a point of attraction of the iteration (1) if there is an open neighborhood D_o of x^* such that $D_o \subset D$ and for any $x^0,\dots,x^{m-1} \in D_o$, the iterates $\{x^k\}$ defined by (1) all lie in D and converge to x^*.

Before continuing we give some notation which will be used throughout this paper. If X and Y are Banach spaces, then $\mathfrak{L}(X,Y)$ denotes the set of linear operators from X to Y and is abbreviated $\mathfrak{L}(X)$ when $X = Y$. Thus $\mathfrak{L}(R^n)$ is the set of $n \times n$ matrices. If $A \in \mathfrak{L}(R^n)$ then we denote the elements of A by $a_{i,j}$, $i,j = 1,\ldots,n$, and the columns of A, a_i, $i = 1,\ldots,n$. If A is a diagonal matrix, then we write $A = \mathrm{diag}(d_1,\ldots,d_n)$, or $\mathrm{diag}(d_i)$ if the value of n is clear. If $A \in \mathfrak{L}(R^n)$ has eigenvalues $\lambda_1,\ldots,\lambda_n$ then the spectrum of A is denoted by $\sigma(A) = \{\lambda_1,\ldots,\lambda_n\}$ and the spectral radius of A by $\rho\{A\} = \max\limits_{1\le i\le n} |\lambda_i|$. If $a_{i,j} \ge 0 (\le 0)$ for $i,j = 1,\ldots,n$, we write $A \ge 0 (\le 0)$.

For $x \in X$ and $r > 0$ we define the set $S(x,r) = \{y \in X ; \|x-y\| < r\}$ where $\|\cdot\|$ denotes a norm on X, and $\overline{S}(x,r)$ will denote the closure of $S(x,r)$. The most frequently used norms on R^n will be the "infinity" norm

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| \text{ and the "one" norm } \|x\|_1 = \sum_{i=1}^{n} |x_i| .$$

If the norm is not specified it may be chosen arbitrarily. It will occasionally be necessary to refer to a point x in the interior of an arbitrary set D; this will be done by the notation $x \in \mathrm{int}(D)$.

We shall conclude this section with some results from the theory of Frechet differentiation. A complete discussion of this subject may be found in Vainberg [1964] or Dieudonné [1960].

Definition 1.1.2. Let $F: X \to Y$ where X and Y are real Banach spaces. If for some $x \in X$ and $A \in \mathcal{L}(X,Y)$

$$\lim_{\|h\| \to 0} \frac{\|F(x+h) - F(x) - Ah\|}{\|h\|} = 0$$

then F is said to be Frechet differentiable, or for this paper differentiable, at x with the derivative A denoted by $F'(x)$ and the differential Ah by $F'(x)h$.

It should be pointed out that if $F: R^n \to R^n$ then $F(x)$ is represented by $(f_1(x), \ldots, f_n(x))^T$ and $F'(x)$ is the Jacobian $(\partial_j f_i(x)) \equiv \left(\frac{\partial f_i(x)}{\partial x_j}\right)$.

We shall frequently be interested in partial derivatives; if $G: X^m \to Y$ then the partial derivative of G with respect to the ith vector variable will be denoted by $\underline{\partial}_i G(x^1, \ldots, x^m)$. We shall also be interested in higher derivatives. We may obtain the pth derivative of F by applying Definition 1.1.2 to $F^{[p-1]}(x)$. In this way we obtain the second differential of F which may be represented by

$$[F''(x)hk]^T = (h^T H_1(x)k, h^T H_2(x)k, \ldots, h^T H_n(x)k)^T$$

where $H_1(x), \ldots, H_n(x)$ are the Hessian matrices of $f_1, \ldots, f_n$ at x ; that is

$$H_\ell(x) = (\partial_i \partial_j f_\ell(x)) .$$

Note that in keeping with the above notation the pth partial of G with respect to the ith variable will be denoted by $\partial_i^p G(x^1, \ldots, x^m)$.

We conclude this section with the Taylor Expansion theorem which will be useful throughout this paper.

<u>Theorem 1.1.3</u>. Suppose that $F : D \subset X \to Y$ is p times continuously differentiable on a convex set $D_0 \subset D$. Then for any $x, y \in D_0$ we have

$$F(x) = F(y) + \sum_{i=1}^{p-1} \frac{1}{i!} F^{[i]}(y)(y-x)^i$$

$$+ \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} F^{[p]}(x+t(y-x))(y-x)^p dt .$$

Here $F^{[i]}(y)(y-x)^i$ denotes $F^{[i]}(y)$ operating on $(y-x)$ i times.

Since X and $\mathbb{Y}$ are arbitrary Banach spaces, Theorem 1.1.3 may be applied to the function $G: D^m \subset X^m \to \mathbb{Y}$ obtaining, when $p = 2$ for example, the following expansion

$$G(x^1,\ldots,x^m) = G(y^1,\ldots,y^m) + \sum_{j=1}^{m} \underline{\partial}_j G(y^1,\ldots,y^m)(y^j-x^j)$$

$$+ \int_0^1 (1-t)^2 \sum_{i=1}^{m} \sum_{j=1}^{m} \underline{\partial}_i \underline{\partial}_j G(x^1+t(y^1-x^1),\ldots,x^m+t(y^m-x^m))$$

$$(y^i-x^i)(y^j-x^j)\,dt \; .$$

Section 1.2

Linear Convergence Theorems

In this section we shall give linear convergence theorems for multi-step iterative procedures. The first theorem is based on the following well-known single-step result; the convergence statement is due to Ostrowski [1966] and the rate of convergence statement is due to Ortega and Rockoff [1966].

<u>Theorem 1.2.1</u>. Let $G:D \subset R^n \to R^n$. Suppose that there is a point $x^* \in \text{int}(D)$ such that $x^* = G(x^*)$ and G is differentiable at x^*. Then if $\rho = \rho\{G'(x^*)\} < 1$, x^* is a point of attraction of the iteration $I:x^{k+1} = G(x^k)$ and $R_1(I,x^*) = \rho$. Moreover, if $\rho > 0$, then $O_R(I,x^*) = O_Q(I,x^*) = 1$.

We now give the multi-step result analogous to Theorem 1.2.1.

<u>Theorem 1.2.2</u>. Let $G:D^m \subset (R^n)^m \to R^n$. Suppose that there is a point $x^* \in \text{int}(D)$ such that $x^* = G(x^*,\ldots,x^*)$ and G is differentiable at $(x^*,\ldots,x^*)$. Let $H_i = \underline{\partial}_i G(x^*,\ldots,x^*)$ for $i = 1,\ldots,m$ and

$$(1) \qquad H = \begin{pmatrix} H_1 & H_2 & \cdots & H_m \\ I & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix}.$$

Then if $\rho = \rho\{H\} < 1$, x^* is a point of attraction of the iteration 1.1-1 and $R_1(I,x^*) = \rho$. Moreover, if $\rho > 0$, then $O_R(I,x^*) = O_Q(I,x^*) = 1$.

Proof. Set $z^k = (x^k,\ldots,x^{k-m+1})$, for $k = m-1,m,\ldots,$ and $z^* = (x^*,\ldots,x^*)$ where x^k is any sequence generated by G . Define $\hat{G}:D^m \subset (R^n)^m$ by

$$\hat{G}(y^1,\ldots,y^m) = (G(y^1,\ldots,y^m),y^1,\ldots,y^{m-1}) .$$

Then $z^{k+1} = \hat{G}(z^k)$. Since G is differentiable at $(x^*,\ldots,x^*)$, we have that $\hat{G}$ is differentiable at z^* and $\hat{G}'(z^*) = H$. Thus $\rho\{G'(z^*)\} = \rho < 1$, so that by Theorem 1.2.1 z^* is a point of attraction of the iteration $I_2:z^{k+1} = \hat{G}(z^k)$ and $R_1(I_2,z^*) = \rho$. Hence, x^* is a point of attraction of the iteration I and we now show that $R_1(I,x^*) = \rho$.

Let x^k be any sequence generated by G and set $\|z^k\| = \max_{0\le i\le m-1} \|x^{k-i}\|$. Then $\|x^k-x^*\| \le \|z^k-z^*\|$; hence,

$$R_1\{x^k\} = \limsup_{k\to\infty} \|x^k-x^*\|^{1/k} \le \limsup_{k\to\infty} \|z^k-z^*\|^{1/k} = \rho .$$

Now assume that $0 < R_1\{x^k\} = \alpha < \rho$. For any i satisfying $0 \le i \le m-1$, notice that

$$\log \alpha - 1/k \log \|x^{k-i}-x^*\| = \log \alpha - \frac{1}{k-i} \log \|x^{k-i}-x^*\|$$

$$+ \frac{i}{k}\left[\frac{1}{k-i} \|x^{k-i}-x^*\|\right] .$$

Hence, $\log \alpha - \limsup_{k\to\infty} \frac{1}{k} \log \|x^{k-i}-x^*\| = 0$; or for any i satisfying $0 \le i \le m-1$, $\limsup_{k\to\infty} \|x^{k-i}-x^*\|^{1/k} = \alpha$. But

$$\begin{aligned}\rho &= \limsup_{k\to\infty} \|z^k-z^*\|^{1/k} \\ &= \limsup_{k\to\infty} \left\{\max_{0\le i\le m-1} \|x^{k-i}-x^*\|\right\}^{1/k} \\ &= \alpha\end{aligned}$$

which is a contradiction. If $\alpha = 0$ one can show in a completely analogous fashion that $\limsup_{k\to\infty} \|x^{k-i}-x^*\|^{1/k} = 0$ for any i satisfying $0 \le i \le m-1$, and hence again reach a contradiction. Thus $R_1\{x^k\} = \rho$ and since $\{x^k\}$ was arbitrary we have that $R_1(I,x^*) = \rho$.

Moreover, if $\rho > 0$ we have immediately from Definition A.2.5 that $O_R(I,x^*) = 1$ and by Theorem A.3.1 that $Q_1(I,x^*) > 0$. Then if $Q_1(I,x^*) < \infty$, by Theorem A.1.3 $Q_p(I,x^*) = \infty$ for all $p > 1$, and by Definition A.1.5, $O_Q(I,x^*) = 1$; likewise, if

$Q_1(I,x^*) = \infty$, $O_Q(I,x^*) = 1$. Hence we have shown that if $\rho > 0$, then $O_R(I,x^*) = O_Q(I,x^*) = 1$. Q.E.D.

We shall now consider various special conditions on the iteration function G which guarantee that $\rho\{H\} < 1$ where H is given by relation (1). We first give a lemma, an elegant proof of which may be found in Weinitschke [1964], about the roots of a particular polynomial.

Lemma 1.2.3. Let $\alpha_i, i = 1,\ldots,m$, be complex numbers which satisfy the condition $\sum_{i=1}^{m} |\alpha_i| < 1$. Then all of the roots of $\lambda^m = \alpha_1\lambda^{m-1} +\ldots+ \alpha_m$ are less than unity in modulus.

We now have the following result as a corollary of Theorem 1.2.2.

Corollary 1.2.4. Let $G:D^m \subset (R^n)^m \to R^n$. Suppose that there is a point $x^* \in \text{int}\,(D)$ such that $x^* = G(x^*,\ldots,x^*)$ and G is differentiable at $(x^*,\ldots,x^*)$. Set $\alpha_i = \|\underline{\partial}_i G(x^*,\ldots,x^*)\|$, for $i = 1,\ldots,m$, and assume that $\sum_{i=1}^{m} \alpha_i < 1$ or, more generally, that all of the roots of $\lambda^m = \alpha_1\lambda^{m-1} +\ldots+ \alpha_m$ are less than unity in modulus. Then x^* is a point of attraction of the iteration 1.1-1 and $R_1(I,x^*) = \rho\{H\}$ where H is given by relation (1).

Proof. If $\sum_{i=1}^{m} \alpha_i < 1$, then by Lemma 1.2.3 all of the roots of $\lambda^m = \alpha_1 \lambda^{m-1} + \ldots + \alpha_m$ are less than unity in modulus, so we may proceed with the proof under this assumption.

Let $H' = \begin{pmatrix} \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_m \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{pmatrix}$. Then by hypothesis $\rho\{H'\} < 1$, so there exists a norm $\|\cdot\|'$ such $\|H'\|' < 1$. Now we define a norm on $\mathfrak{L}\{(R^n)^m\}$ in the following way.

Let $A \in \mathfrak{L}\{(R^n)^m\}$ be given by $\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix}$ where $A_{i,j} \in \mathfrak{L}(R^n)$ for $i,j = 1,\ldots,m$, and set

$$\|A\|'' = \left\| \begin{matrix} \|A_{1,1}\| & \cdots & \|A_{1,m}\| \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \|A_{m,1}\| & \cdots & \|A_{m,m}\| \end{matrix} \right\|' .$$

It is easily verified that $\|\cdot\|''$ is indeed a norm. Thus we have

$$\|G'(x^*)\|'' = \left\| \begin{matrix} \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_m \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{matrix} \right\| < 1$$

and consequently $\rho\{H\} < 1$. The result now follows from Theorem 1.2.2. Q.E.D.

We now have a condition on $\|\underline{\partial}_i G(x^*,\ldots,x^*)\|$, $i = 1,\ldots,m$, to guarantee that $\rho\{H\} < 1$, and it is natural to ask if a condition can be given in terms of $\rho\{\underline{\partial}_i G(x^*,\ldots,x^*)\}$, $i = 1,\ldots,m$; that is, does

$$\sum_{i=1}^{m} \rho\{\underline{\partial}_i G(x^*,\ldots,x^*)\} < 1 \text{ imply that } \rho\{H\} < 1 .$$

The following example provides a negative answer.

<u>Example 1.2.5</u>. Let $H = \begin{pmatrix} H_1 & H_2 \\ I & 0 \end{pmatrix}$ where $H_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $H_2 = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$. Then $\rho\{H_1\} + \rho\{H_2\} = 0$; however, it is a simple exercise to verify that $\rho\{H\} = \alpha^{1/3}$. Hence, we can have $\sum_{i=1}^{m} \rho\{\underline{\partial}_i G(x^*,\ldots,x^*)\} = 0$ but $\rho\{H\}$ arbitrarily large.

Thus in order to insure that $\rho\{H\} < 1$ we must add an additional assumption. This we now do with the aid of the following lemma, the proof of which may be found in Polyak [1964].

Lemma 1.2.6. Let $A \in \mathfrak{L}\{(R^n)^m\}$ be given by

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix} \quad \text{where} \quad A_{i,j} \in \mathfrak{L}(R^n) \quad \text{for} \quad i,j = 1,\ldots,m .$$

Assume that the $A_{i,j}$, $i,j = 1,\ldots,m$, are mutually commutative. Then for every $\lambda \in \sigma(A)$ there exist $\lambda_{i,j} \in \sigma(A_{i,j})$, $i,j = 1,\ldots,m$, such that $\lambda \in \sigma(B)$

$$\text{where} \quad B = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,m} \\ \vdots & & \vdots \\ \lambda_{m,1} & \cdots & \lambda_{m,m} \end{pmatrix}.$$

Corollary 1.2.7. Let $G:D^m \subset (R^n)^m \to R^n$. Suppose that there is a point $x^* \in \text{int}\ (D)$ such that $x^* = G(x^*,\ldots,x^*)$ and G is differentiable at $(x^*,\ldots,x^*)$. Assume that the $\underline{\partial}_i G(x^*,\ldots,x^*)$, $i = 1,\ldots,m$, are mutually commutative and that $\sum_{i=1}^{m} \underline{\partial}_i G(x^*,\ldots,x^*) < 1$. Then x^* is a point of attraction of the iteration 1.1-1 and $R_1(I,x^*) = \rho\{H\}$ where H is given by relation (1).

<u>Proof</u>. Since the $\underline{\partial}_i G(x^*,\ldots,x^*)$, $i = 1,\ldots,m$, are mutually commutative we have by Lemma 1.2.6 that for any $\bar{\lambda} \in \sigma(H)$, $\bar{\lambda} \in \sigma\begin{pmatrix} \lambda_1, & \ldots, & & & \lambda_m \\ 1 & 0 & . & . & 0 \\ 0 & . & . & & . \\ . & . & . & & . \\ . & & . & . & . \\ 0 & . & 0 & 1 & 0 \end{pmatrix}$ for some $\lambda_i \in \sigma(\underline{\partial}_i G(x^*,\ldots,x^*))$ for $i = 1,\ldots,m$. But $\sum_{i=1}^{m} |\lambda_i| < 1$ so by Lemma 1.2.3 all of the roots of $\lambda^m = \lambda_1\lambda^{m-1} +\ldots+ \lambda_m$ are less than unity in modulus. Hence, $\rho\{H\} < 1$ and the result now follows from Theorem 1.2.2. Q.E.D.

It should be pointed out that the condition that $\sum_{i=1}^{m} \rho\{\underline{\partial}_i G(x^*,\ldots,x^*)\} < 1$ is by no means necessary for $\rho\{H\} < 1$, for consider the following example.

<u>Example 1.2.8</u>. Let $H = \begin{pmatrix} H_1 & H_2 \\ I & 0 \end{pmatrix}$ where $H_1 = \begin{pmatrix} 1/8 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $H_2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$. Then $\rho\{H_1\} + \rho\{H_2\} = 5/4 > 1$; however, it is easily verified that

$$\sigma(H) = \left\{\frac{1}{16}\ (1\pm\sqrt{193}, 1/4(1\pm\sqrt{5})\right\} .$$

Hence, $\rho\{H\} < 1$.

As will become evident later, many practical multi-step iterative procedures satisfy the following kind of fixed point condition.

$$(2) \qquad x^* = G(x^*, y^2, \ldots, y^m)$$

for all $y^j \in D, j = 2,\ldots,m$, where $G: D^m \subset (R^n)^m \to R^n$ and $x^* \in \text{int}(D)$. This condition clearly insures that $\underline{\partial}_i G(x^*,\ldots,x^*) = 0$, $i = 2,\ldots,m$, and hence the commutativity condition of Corollary 1.2.7 is trivially satisfied. We summarize this in the following result. (We also note that a more direct proof, without using Corollary 1.2.7, and hence Lemma 1.2.6, is easily given.)

<u>Corollary 1.2.9</u>. Let $G: D^m \subset (R^n)^m \to R^n$. Suppose that there is a point $x^* \in \text{int}(D)$ such that $x^* = G(x^*, y^2,\ldots,y^m)$ for all $y^j \in D$, $j = 2,\ldots,m$, and G is differentiable at x^*. Then if $\rho = \rho\{\underline{\partial}_1 G(x^*,\ldots,x^*)\} < 1$, x^* is a point of attraction of the iteration 1.1-1 with $R_1(I, x^*) = \rho$.

It is frequently of interest to consider iteration 1.1-1 in the following implicit form

$$(3) \qquad G(x^{k+1}, x^k, \ldots, x^{k-m+1}) = 0 \;, \quad k = m-1,\, m, \ldots$$

where $G: D^{m+1} \subset (R^n)^{m+1} \to R^n$. For the corresponding single-step procedure, $G(x^{k+1}, x^k) = 0$, Ortega and Rheinboldt [1970] have given the following result which requires less differentiability than a similar result of Ortega and Rockoff [1966].

Theorem 1.2.10. Suppose that $G:DxD \subset R^n xR^n \to R^n$ is continuously differentiable on an open neighborhood $D_o \subset D$. Assume that there is an $x^* \in D_o$ such that $G(x^*,x^*) = 0, \underline{\partial}_1 G(x^*,x^*)$ is non-singular, and $\rho = \rho\{-[\underline{\partial}_1 G(x^*,x^*)]^{-1}\underline{\partial}_2 G(x^*,x^*)\} < 1$. Then there is an open ball $S = S(x^*,\delta) \subset D_o$ such that the sequence $\{x^{k+1}\}$ is well-defined for any $x^o \in S$ where $G(x^{k+1},x^k) = 0$, and x^* is a point of attraction of the iteration $I:G(x^{k+1},x^k) = 0$ with $R_1(I,x^*) = \rho$.

We now give the corresponding result for iteration (3).

Theorem 1.2.11. Suppose that $G:D^{m+1} \subset (R^n)^{m+1} \to R^n$ is continuously differentiable on an open neighborhood $D_o^{m+1} \subset D^{m+1}$. Assume that there is an $x^* \in D_o$ such that $G(x^*,\ldots,x^*) = 0$, $\underline{\partial}_1 G(x^*,\ldots,x^*)$ is non-singular, and $\rho = \rho\{\overline{H}\} < 1$ where

$$(4) \qquad \overline{H} = \begin{pmatrix} (H_1)^{-1}H_2 & \cdots & \cdots & (H_1)^{-1}H_{m+1} \\ I & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 \;\; I & 0 \end{pmatrix}$$

with $H_i \equiv \underline{\partial}_i G(x^*,\ldots,x^*)$ $i = 1,\ldots,m+1$.

Then there is an open ball $S = S(x^*,\delta) \subset D_o$ such that the sequence $\{x^{k+1}\}$ is well-defined for any

$(x^o,\ldots,x^{m-1}) \in S^m$ where $G(x^{k+1},x^k,\ldots,x^{k-m+1}) = 0$, $k = m-1,m,\ldots,$ and x^* is a point of attraction of the iteration $I:G(x^{k+1},x^k,\ldots,x^{k-m+1}) = 0$ with $R_1(I,x^*) = \rho$.

Proof. Define $\hat{G}:D^m x D^m \subset (R^n)^m x(R^n)^m \to (R^n)^m$ by $\hat{G}(y^1,y^2) = (G(y_1^1,y^2),y_2^1 - y_1^2,\ldots,y_m^1 - y_{m-1}^2)$ where $y^i = (y_1^i,\ldots,y_m^i) \in D^m$ for $i = 1,2$. If $z^* = (x^*,\ldots,x^*) \in (R^n)^m$, then $\hat{G}(z^*,z^*) = 0$. It is easily verified that

$$\underline{\partial}_1\hat{G}(z^*,z^*) = \begin{pmatrix} H_1 & 0 & \cdots & & 0 \\ 0 & I & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & 0 & & I \end{pmatrix}$$

where $\underline{\partial}_1\hat{G}$ denotes the vector partial derivative of $\hat{G}$ with respect to the entire vector variable y^1. Hence $\underline{\partial}_1\hat{G}(z^*,z^*)$ is non-singular since $\underline{\partial}_1 G(x^*,\ldots,x^*)$ is non-singular. Furthermore,

$$-[\underline{\partial}_1\hat{G}(z^*,z^*)]^{-1}\underline{\partial}_2\hat{G}(z^*,z^*)$$

$$= - \begin{pmatrix} H_1^{-1} & 0 & \cdots & 0 \\ 0 & I & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & I \end{pmatrix} \begin{pmatrix} H_2 & H_3 & \cdots & H_{m+1} \\ -I & 0 & & \vdots \\ \vdots & -I & \ddots & \vdots \\ 0 & 0 & -I & 0 \end{pmatrix}$$

$$= \overline{H}.$$

Hence, $\rho\{-[\underline{\partial}_1\hat{G}(z^*,z^*)]^{-1}\underline{\partial}_2\hat{G}(z^*,z^*)\} = \rho\{\overline{H}\} = \rho < 1$. Thus by Theorem 1.2.10, there is an open ball $S = S(x^*,\delta) \subset D_o$

such that the sequence $\{z^{k+1}\}$ is well-defined for any $z^o = (x^{m-1},\ldots,x^o) \in S^m$ where $\hat{G}(z^{k+1},z^k) = 0$; that is, if we denote z^k by $(x^k,\ldots,x^{k-m+1})$, z^{k+1} is given by $(x^{k+1},x^k,\ldots,x^{k-m+2})$ where $\{x^{k+1}\}$ satisfies $G(x^{k+1},x^k,\ldots,x^{k-m+1}) = 0$. Moreover, by Theorem 1.2.10, z^* is a point of attraction of the iteration $I_1:\hat{G}(z^{k+1},z^k) = 0$, and $R_1(I_1,z^*) = \rho$; therefore, x^* is a point of attraction of the iteration $G(x^{k+1},\ldots,x^{k-m+1}) = 0$ and, by an argument completely analogous with the one used in Theorem 1.2.2, $R_1(I,x^*) = \rho$. Q.E.D.

It is clear that we could give corollaries to Theorem 1.2.11 which would be completely analogous to Corollaries 1.2.4, 1.2.7, and 1.2.9, providing sufficient conditions for $\rho\{\bar{H}\} < 1$ where $\bar{H}$ is given by relation (4). However, in order to avoid unnecessary repetition, we shall formally state without proof only the following result. It is included here because it will be used in the next section.

Corollary 1.2.12. Suppose that $G:D^{m+1} \subset (R^n)^{m+1} \to R^n$ is continuously differentiable on an open neighborhood $D_o \subset D$. Assume that there is an $x^* \in D_o$ such that $G(x^*,x^*,y^2,\ldots,y^m) = 0$ for all $y^j \in D_o, j = 2,\ldots,m$, $\underline{\partial}_1 G(x^*,\ldots,x^*)$ is non-singular, and $\rho = \rho\{-[\underline{\partial}_1 G(x^*,\ldots,x^*)]^{-1} \underline{\partial}_2 G(x^*,\ldots,x^*)\} < 1$. Then there is an open ball

$S = S(x^*, \delta) \subset D_o$ such that the sequence $\{x^{k+1}\}$ is well-defined for any $(x^o, \ldots, x^{m-1}) \in S^m$ where $G(x^{k+1}, x^k, \ldots, x^{k-m+1}) = 0$ for $k = m - 1, m, \ldots,$ and x^* is a point of attraction of the iteration $I: G(x^{k+1}, \ldots, x^{k-m+1}) = 0$ with $R_1(I, x^*) = \rho$.

Section 1.3

Some Iterative Processes and Their Asymptotic Rates of Convergence

In this section we shall use the results of the previous section to determine the asymptotic rates of convergence of several iterative processes for approximating a solution x^* of the equation $F(x) = 0$ where $F:D \subset R^n \to R^n$. This has already been done by Ortega and Rockoff [1966] for a collection of procedures, two of which will be used here for comparison purposes. The first of these procedures is the Jacobi-Newton-Process given by

$$I_1 : x^{k+1} = G_1(x^k) \ , \ k = 0,1,\ldots, \tag{1}$$

where the ith component of $G_1:D \subset R^n \to R^n$ is given by

$$g_{1i}(x) \equiv x_i - \frac{f_i(x)}{\partial_i f_i(x)} \ ;$$

the second is the Extrapolated-Gauss-Seidel-Newton-Process given by

$$I_2 : G_2(x^{k+1}, x^k) = 0 \ , \ k = 0,1,\ldots, \tag{2}$$

where the ith component of $G_2 : D^2 \subset (R^n)^2 \to R^n$ is given by

$$g_{2i}(x,y) \equiv \partial_i f_i(x_1,\ldots,x_{i-1},y_i,\ldots,y_n)(x_i - y_i)$$
$$+ \omega f_i(x_1,\ldots,x_{i-1},y_i,\ldots,y_n)$$

with ω a real constant.

Before presenting the procedures of interest in this section, we impose the following conditions on the function $F(x)$. Assume that there is a $\delta > 0$ such that F is twice continuously differentiable on $S(x^*,\delta) \subset D$. In addition, let $F'(x) \equiv B(x) - L(x) - U(x)$ for $x \in S(x^*,\delta)$ where B, L, and U are diagonal, strictly lower triangular, and strictly upper triangular matrices respectively with $B(x)$ non-singular for $x \in S(x^*,\delta)$. We shall assume that F satisfies these conditions throughout the remainder of this section.

Now it is easily verified that

$$G_1'(x^*) = B^{-1}(x^*)[L(x^*)+U(x^*)]$$

and that

$$-[\underline{\partial}_1 G_2(x^*,x^*)]^{-1}\underline{\partial}_2 G_2(x^*,x^*) = [B(x^*)-\omega L(x^*)]^{-1}[(1-\omega)B(x^*)+\omega U(x^*)] .$$

Thus if

(3) $\qquad \rho_1 = \rho\{G_1'(x^*)\} < 1$, and

(4) $\qquad \rho_2 = \rho\{-[\underline{\partial}_1 G_2(x^*,x^*)]^{-1}\underline{\partial}_2 G_2(x^*,x^*)\} < 1$

then by Theorems 1.2.1 and 1.2.10, x^* is a point of attraction of the procedures I_1 and I_2 with $R_1(I_1,x^*) = \rho_1$ and $R_1(I_2,x^*) = \rho_2$, respectively. This result may be found in Ortega and Rockoff [1966].

Procedures I_1 and I_2 were found by composing the Jacobi and the Extrapolated-Gauss-Seidel processes with the one-dimensional Newton's method. We shall now obtain additional procedures by using other well-known one dimensional processes in place of the Newton process. Thus we have the Jacobi-Secant-Process given by

(5) $$I_3 : x^{k+1} = G_3(x^k, x^{k-1}) \ , \ k = 1,2,\ldots,$$

where the ith component of $G_3 : D^2 \subset (R^n)^2 \to R^n$ is given by

$$g_{3i}(x,y) \equiv \begin{cases} x_i - \dfrac{(y_i - x_i) f_i(x)}{f_i(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n) - f_i(x)} & \text{for } y_i \neq x_i \ , \\ x_i - \dfrac{f_i(x)}{\partial_i f_i(x)} & \text{for } y_i = x_i \ . \end{cases}$$

Likewise, the Extrapolated-Gauss-Seidel-Secant-Process is given by

(6) $$I_4 : G_4(x^{k+1}, x^k, x^{k-1}) \ , \ k = 1,2,\ldots,$$

where the ith component of $G_4 : D^3 \subset (R^n)^3 \to R^n$ is given by

$$g_{4i}(x,y,z) \equiv \begin{cases} \dfrac{f_i(\overline{z}) - f_i(\overline{y})}{z_i - y_i} (x_i - y_i) + \omega f_i(\overline{y}) & \text{for } z_i \neq y_i \\ \partial_i f_i(\overline{y})(x_i - y_i) + \omega f_i(\overline{y}) & \text{for } z_i = y_i \end{cases}$$

with $\overline{z} \equiv (x_1,\ldots,x_{i-1},z_i,y_{i+1},\ldots,y_n)$,

$\overline{y} \equiv (x_1,\ldots,x_{i-1},y_i,\ldots,y_n)$, and ω a real constant.

Clearly, in procedures I_3 and I_4, the secant method has replaced Newton's method in procedures I_1 and I_2. We now give a convergence theorem for procedures I_3 and I_4.

Theorem 1.3.1. Let I_3 and I_4 be the iterative procedures defined by equations (5) and (6), respectively. Assume that ρ_1 and ρ_2 satisfy relations (3) and (4) respectively. Then x^* is a point of attraction of the procedures I_3 and I_4, and $R_1(I_3,x^*) = \rho_1$ and $R_1(I_4,x^*) = \rho_2$.

Proof. From equation (5) we see that G_3 has the form

$$G_3(x,y) \equiv x - A_3(x,y)^{-1}F(x)$$

with $A_3(x,y) \equiv \mathrm{diag}(d_i(x,y))$, where

$$d_i(x,y) = \begin{cases} \dfrac{f_i(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)-f_i(x)}{y_i-x_i} & \text{for } y_i \neq x_i \\ \partial_i f_i(x) \quad \text{for } y_i = x_i , \end{cases}$$

and from equation (6) G_4 has the form

$$G_4(x,y,z) \equiv A_4(x,y,z)(x-y) + \omega\overline{F}(x,y)$$

with $A_4(x,y,z) \equiv \mathrm{diag}(d_i(x,y,z))$, where

$$d_i(x,y,z) = \begin{cases} \dfrac{f_i(\overline{z})-f_i(\overline{y})}{z_i-y_i} & \text{for } z_i \neq y_i , \\ \partial_i f_i(\overline{y}) & \text{for } z_i = y_i , \end{cases}$$

and with the ith component of $\overline{F}$ given by

$$\overline{f}_i(x,y) = f_i(\overline{y})$$

where $\overline{z}$ and $\overline{y}$ are defined in relation (6).

Then clearly $x^* = G_3(x^*,y)$ for all $y \in D$; likewise, $G_4(x^*,x^*,z) = 0$ for all $z \in D$. Furthermore, it is easily verified that

$$\begin{aligned} \underline{\partial}_1 G_3(x^*,x^*) &= I - B(x^*)^{-1}F'(x^*) \\ &= B(x^*)^{-1}[L(x^*)+U(x^*)] \end{aligned}$$

and that

$$\begin{aligned} &-[\underline{\partial}_1 G_4(x^*,x^*,x^*)]^{-1}\underline{\partial}_2 G_4(x^*,x^*,x^*) \\ &\qquad = [B(x^*)+\omega\underline{\partial}_1\overline{F}(x^*,x^*)]^{-1}[-B(x^*)+\omega\underline{\partial}_2\overline{F}(x^*,x^*)] \\ &\qquad = [B(x^*)-\omega L(x^*)]^{-1}[(1-\omega)B(x^*)+\omega U(x^*)] \ . \end{aligned}$$

Hence, by hypothesis, we have $\rho\{\underline{\partial}_1 G_3(x^*,x^*)\} = \rho_1 < 1$ and $\rho\{-[\underline{\partial}_1 G_4(x^*,x^*,x^*)]^{-1}\underline{\partial}_2 G_4(x^*,x^*,x^*)\} = \rho_2 < 1$. Then using Corollaries 1.2.10 and 1.2.13 we have that x^* is a point of attraction of the procedures I_3 and I_4 with $R_1(I_3,x^*) = \rho_1$ and $R_1(I_4,x^*) = \rho_2$. Q.E.D.

It is interesting to note that although the order of convergence of Newton's method is at least quadratic while that of the secant method is at least $\frac{1+\sqrt{5}}{2}$, whenever the two are composed with the Jacobi or Extrapolated-Gauss-Seidel processes, the resulting procedures have precisely the same rate of convergence. Thus the use of Newton's method would appear to offer no advantages over the use of the secant method; in fact, if the derivative evaluation were troublesome, the secant method would be superior.

At this point one might be of the opinion that no matter what process was used in connection with the Jacobi and Extrapolated-Gauss-Seidel processes, the resulting procedures would have the same rate of convergence as when Newton's method is used. In order to show that this is not the case we shall now give the procedures obtained when Newton's method is replaced by the linearly convergent method of false position. Thus we have the Jacobi-False-Position-Process given by

(7) $$I_5 : x^{k+1} = G_5(x^k) \quad k = 0,1,\ldots$$

where the ith component of $G_5 : D \subset R^n \to R^n$ is given by

$$g_{5i}(x) \equiv \begin{cases} x_i - \dfrac{(z_i - x_i) f_i(x)}{f_i(x_1,\ldots,x_{i-1},z_i,x_{i+1},\ldots,x_n) - f_i(x)} \\ \text{for} \quad z_i \neq x_i \;, \\ x_i - \dfrac{f_i(x)}{\partial_i f_i(x)} \quad \text{for} \quad z_i = x_i \;, \end{cases}$$

with $z \in D$ a fixed vector. Likewise we have the Extrapolated-Gauss-Seidel-False-Position-Process given by

(8) $$I_6 : G_6(x^{k+1}, x^k) = 0 \;,\; k = 0,1,\ldots,$$

where the ith component of $G_6 : D^2 \subset (R^n)^2 \to R^n$ is given by

$$g_{6i}(x,y) \equiv \begin{cases} \dfrac{f_i(\overline{z}) - f_i(\overline{y})}{z_i - y_i}(x_i - y_i) + \omega f_i(\overline{y}) \quad \text{for} \quad z_i = y_i \;, \\ \partial_i f_i(\overline{y})(x_i - y_i) + \omega f_i(\overline{y}) \quad \text{for} \quad z_i = y_i \;, \end{cases}$$

with $\overline{z}$ and $\overline{y}$ defined in relation (6) where $z \in D$ is a fixed vector, and with ω a real constant.

We now give a convergence theorem for procedures I_5 and I_6.

Theorem 1.3.2. Let I_5 and I_6 be the iterative procedures defined by equations (7) and (8), respectively. Assume that ρ_1 and ρ_2 satisfy relations (3) and (4), respectively. Then there are points z^1 and $z^2 \in D$ such that x^* is a point of attraction of the procedures I_5 and I_6. Moreover, in general, $R_1(I_5,x^*) \neq R_1(I_3,x^*)$ and $R_1(I_6,x^*) \neq R_1(I_4,x^*)$.

Proof. From equation (7) we see that G_5 has the form

$$G_5(x) \equiv x - A_5(x)^{-1}F(x)$$

with $A_5(x) \equiv \mathrm{diag}(d_i(x))$, where

$$d_i(x) = \begin{cases} \dfrac{f_i(x_1,\ldots,x_{i-1},z_i^1,x_{i+1},\ldots,x_n)-f_i(x)}{z_i^1-x_i} \\ \text{for} \quad z_i^1 \neq x_i, \\ \partial_i f_i(x) \quad \text{for} \quad z_i^1 = x_i, \end{cases}$$

and from equation (8) G_6 has the form

$$G_6(x,y) \equiv A_6(x,y)(x-y) + \omega\overline{F}(x,y)$$

with $A_6(x,y) \equiv \mathrm{diag}(d_i(x,y))$, where

$$d_i(x,y) = \begin{cases} \dfrac{f_i(\overline{z})-f_i(\overline{y})}{z_i^2-y_i} & \text{for } z_i^2 \neq y_i , \\ \partial_i f_i(\overline{y}) & \text{for } z_i^2 = y_i , \end{cases}$$

and with the ith component of $\overline{F}$ given by

$$\overline{f}_i(x,y) = f_i(\overline{y})$$

where $\overline{z}$ and $\overline{y}$ are defined in relation (6). Then it easily verified that

$$G_5'(x^*) = I - A_5(x^*)^{-1}[B(x^*)-L(x^*)-U(x^*)]$$

and that

$$-[\underline{\partial}_1 G_6(x^*,x^*)]^{-1}\underline{\partial}_2 G_6(x^*,x^*)$$
$$= -[A_6(x^*,x^*)-\omega L(x^*)]^{-1}[-A_6(x^*,x^*)+\omega(B(x^*)-U(x^*))] .$$

Since $A_5(x^*)$ and $A_6(x^*,x^*)$ are approximations of $B(x^*)$, since (3) and (4) hold, and since the eigenvalues of a matrix are continuous functions of the elements of the matrix, there exist points z^1 and $z^2 \in D$ such that $\rho\{G_5'(x^*)\} < 1$ and $\rho\{-[\underline{\partial}_1 G_6(x^*,x^*)]^{-1}\underline{\partial}_2 G_6(x^*,x^*)\} < 1$. It now follows from Theorems 1.2.1 and 1.2.10 that x^* is a point of attraction of the iterations I_5 and I_6. Furthermore, since, in general, $\rho\{G_5'(x^*)\} \neq \rho_1$ and $\rho\{-[\underline{\partial}_1 G_6(x^*,x^*)]^{-1}\underline{\partial}_2 G_6(x^*,x^*)\} \neq \rho_2$ we have that $R_1(I_5,x^*) \neq R_1(I_3,x^*)$ and $R_1(I_6,x^*) \neq R_1(I_4,x^*)$. Q.E.D.

From the previous theorem it is clear that the rate of convergence of procedures I_5 and I_6 depends on the points z^1 and z^2, respectively. Depending on the problem in question, it is possible that the choice of the points z^1 and z^2 could either improve or worsen the rate of convergence of procedures I_5 and I_6 as compared with I_1 and I_2.

To this point nothing has been said about conditions which will guarantee that relations (3) and (4) hold; however, it is possible to give classes of problems for which these relations are satisfied. This is done, for example, by Ortega and Rockoff [1966] for problems of the form

$$F(x) \equiv Ax + \varphi(x) = 0 \ ,$$

where $A \in \mathfrak{L}(R^n)$ and $\varphi : D \subset R^n \to R^n$.

Section 1.4

Necessary and Sufficient Conditions

for R-Suerplinear Convergence

In Section 1.2 we were concerned with theorems which gave a linear order of convergence for an iterative procedure; that is, if I denotes the iterative procedure $x^{k+1} = G(x^k,\ldots,x^{k-m+1})$ where $x^* = G(x^*,\ldots,x^*)$ we gave theorems in which $0 < R_1(I,x^*) < 1$ and $0 < Q_1(I,x^*) < \infty$. In this section we shall be concerned with theorems which pertain to an R-superlinear order of convergence for an iterative process; that is, theorems in which $R_1(I,x^*) = 0$. In later sections we shall discuss Q-superlinear convergence.

In Theorem 1.2.2 we saw that if $G:D^m \subset (R^n)^m \to R^n$ is differentiable at $x^* \in \text{int}\,(D)$ where $x^* = G(x^*,\ldots,x^*)$ and $\rho\{H\} < 1$ where

$$(1) \quad H = \begin{pmatrix} H_1 & , \ldots & & & & H_m \\ I & & 0 & \cdots & & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & 0 & I & 0 \end{pmatrix}$$

with $H_i \equiv \underline{\partial}_i G(x^*,\ldots,x^*)$, $i = 1,\ldots,m$, then $R_1(I,x^*) = \rho\{H\}$. Thus Theorem 1.2.2 gives us an immediate answer to the question of when a procedure is R-superlinear; that is, $R_1(I,x^*) = 0$ if and only if $\rho\{H\} = 0$. Notice that this gives no information on $Q_1(I,x^*)$, for by Theorem A.3.1 $R_1(I,x^*) \le Q_1(I,x^*)$. In fact, the following example shows that a procedure may be R-superlinear but only linear in the Q measure.

Example 1.4.1. Let $G:R^2 \to R^2$ be defined by $G(x) = (x_2,x_1^2)^T$. Then $x^* = (0,0)^T$ is a point of attraction of the iteration $I_1:x^{k+1} = G(x^k)$ and

$R_1(I_1,x^*) = \rho\left\{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\} = 0$. Hence the procedure is R-superlinear. However, consider $x^0 = (\alpha,0)^T$ with $|\alpha|<|$. Then

$$x^k = \begin{cases} \left(\alpha^{2^{k/2}},0\right)^T & \text{for } k \text{ even,}\\ \left(0,\alpha^{2^{\frac{k+1}{2}}}\right)^T & \text{for } k \text{ odd}. \end{cases}$$

Thus for k odd and $\|x\| = \|x\|_1$ we have

$$\frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} = \frac{|\alpha^{2^{\frac{k+1}{2}}}|}{|\alpha^{2^{\frac{k+1}{2}}}|} = 1 .$$

Hence $\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} \geq 1$ and since we have exhibited a sequence for which $Q_1(\{x^k\}) > 0$, we have that $Q_1(I_1,x^*) > 0$; that is, the procedure is linearly convergent in the Q measure.

It should be pointed out that in the previous example, and in many of the results to follow in this chapter, a particular norm is used; however, as was pointed out in Section 1.1 for the R factors, by Theorem A.1.6, the relation $Q_1(I,x^*) = 0$ is independent of norm so the results remain valid for any norm.

Section 1.5

Sufficient Conditions for Q-Superlinear Convergence

It is possible to give a very simple sufficient condition for a procedure to have Q-superlinear convergence; that is, for an iterative procedure I to satisfy $Q(I,x^*) = 0$. Let $G: D^m \subset (R^n)^m \to R^n$ with $x^* \in \text{int}(D)$ satisfying $x^* = G(x^*,\dots,x^*)$, and assume that every sequence $\{x^k\}$ generated by the iterative process $I: x^{k+1} = G(x^k,\dots,x^{k-m+1})$ satisfies

$$\|x^{k+1}-x^*\| \le \alpha_k \|x^k - x^*\|$$

for all $k \ge k_o$. Then it is clear from the definition of Q_1 that if $\limsup_{k\to\infty} \alpha_k = 0$, $Q_1(I,x^*) = 0$.

This kind of result is cumbersome in that it requires knowledge about the behavior of every sequence generated by the iterative procedure. It is much more desirable to have conditions on the iteration function G itself rather than to have a condition which must be satisfied by every sequence. As was pointed out in the previous section, Theorem 1.2.2 provides the more desirable condition on G for the R measure, and the remainder of this chapter is devoted to investigating such conditions for the Q measure.

In Example 1.4.1, we gave an iteration function G for which $\rho\{G'(x^*)\} = 0$ but G was not Q-superlinearly convergent. As was noted $G'(x^*) \not\equiv 0$; the following simple result shows that if $G'(x^*) = 0$, then indeed G is Q-superlinearly convergent. This result may be found in Ortega and Rheinboldt [1970].

<u>Theorem 1.5.1</u>. Let $G:D \subset R^n \to R^n$ with $x^* \in \text{int}(D)$ satisfying $G(x^*) = x^*$. Assume that G is differentiable at x^* and that $G'(x^*) = 0$. Then x^* is a point of attraction of the iteration $I:x^{k+1} = G(x^k)$ and $Q_1(I,x^*) = 0$.

<u>Proof</u>. The point of attraction result follows from Theorem 1.2.1.

Now let $\{x^k\}$ be any sequence generated by G which converges to x^* . Then

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} = \limsup_{k\to\infty} \frac{\|G(x^k)-G(x^*)-G'(x^*)(x^k-x^*)\|}{\|x^k-x^*\|}$$
$$= 0$$

since G is differentiable at x^* . Since $\{x^*\}$ was arbitrary $Q_1(I,x^*) = 0$. Q.E.D.

It is natural to expect that Theorem 1.5.1 can be extended to multi-step operators where the condition $G'(x^*) \equiv 0$ is replaced by $\underline{\partial}_i G(x^*,\ldots,x^*) \equiv 0 \quad i = 1,\ldots,m$; however, the following simple example shows that the condition $\underline{\partial}_i G(x^*,\ldots,x^*) \equiv 0$, $i = 1,\ldots,m$, is not sufficient for $Q_1(I,x^*) = 0$ where $I:x^{k+1} = G(x^k,\ldots,x^{k-m+1})$.

<u>Example 1.5.2.</u> Let $G:R^2 \times R^2 \to R^2$ be defined by $G(x,y) = (x_1^2, y_2^4)^T$. Then for $x^* = (0,0)^T$, $x^* = G(x^*,x^*)$ and $\underline{\partial}_i G(x^*,x^*) \equiv 0$ for $i = 1,2$. Let $x^o = (0,\alpha)^T$, $x' = (\alpha^4,0)^T$ and it is easy to verify that if $I:x^{k+1} = G(x^k,x^{k-1})$ then

$$x^k = \begin{cases} (\alpha^{2^{k+1}}, \alpha^{2^k})^T & \text{for } k \text{ even} \\ (\alpha^{2^{k+1}}, 0)^T & \text{for } k \text{ odd ;} \end{cases}$$

thus if $|\alpha| < 1$, the sequence $\{x^k\}$ converges to x^* . However, for k odd and $\|x\| = \|x\|_\infty$,

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} \geq \limsup_{k\to\infty} \frac{|\alpha^{2^{k+1}}|}{|\alpha^{2^{k+1}}|} = 1 ,$$

and $Q_1(I,x^*) \geq 1$.

Thus we see that Theorem 1.5.1 provides a sufficient condition for Q-superlinear convergence for single-step operators which is too weak for multi-step operators; however, the following theorem shows that for multi-step operators of a particular form, the condition is sufficient.

<u>Theorem 1.5.3.</u> Let $G:D^m \subset (R^n)^m \to R^n$ with $x^* \in$ int (D) satisfying $x^* = G(x^*,y^2,\ldots,y^m)$ for all $y^j \in D$,

$j = 2,\ldots,m$. Assume that $\underline{\partial}_1 G$ exists on $D_0 \subset D$ and is continuous at $(x^*,\ldots,x^*) \in D_0^m$, and that $\underline{\partial}_1 G(x^*,\ldots,x^*) \equiv 0$. Then x^* is a point of attraction of the procedure $I: x^{k+1} = G(x^k,\ldots,x^{k-m+1})$ and $Q_1(I,x^*) = 0$.

Proof. It is easily verified that $\underline{\partial}_i G(x^*,\ldots,x^*) \equiv 0$ for $i = 1,\ldots,m$, so x^* is a point of attraction of procedure I by Corollary 1.2.9. Furthermore, if $\{x^k\}$ is any sequence generated by I converging to x^* we have by the Taylor Expansion Theorem 1.1.3 that

$$\frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} = \frac{\|G(x^k,\ldots,x^{k-m+1})-G(x^*,x^{k-1},\ldots,x^{k-m+1})\|}{\|x^k-x^*\|}$$

$$\leq \int_0^1 \|\underline{\partial}_1 G(x^*+t(x^k-x^*),\ x^{k-1},\ \ldots,\ x^{k-m+1})$$

$$- \underline{\partial}_1 G(x^*,\ldots,x^*)\|dt .$$

Thus by the continuity of $\underline{\partial}_1 G$ at $(x^*,\ldots,x^*)$ we have

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} = 0 ;$$

hence $Q_1(I,x^*) = 0$. Q.E.D.

Although, as has been pointed out, the condition that $\underline{\partial}_i G(x^*,\ldots,x^*) \equiv 0$ for $i = 1,\ldots,m$ is not sufficient, in general, to imply that $Q_1(I,x^*) = 0$ where I is a multi-step iteration, the following example shows that the condition $G'(x) = 0$ is actually not necessary for $Q_1(I,x^*) = 0$ where I is a single-step iteration.

Example 1.5.4. Let $G:R^2 \to R^2$ be defined by $G(x) = (x_1^2+x_2, x_1^4-x_2^2)^T$. Then for $x^* = (0,0)^T$, $x^* = G(x^*)$ and $G'(x^*) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \not\equiv 0$, so x^* is a point of attraction for the iteration $I:x^{k+1} = G(x^k)$. Let $\{x^k\}$ be any sequence generated by I converging to x^* and let $\|x\| = \|x\|_1$. Then

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} = \limsup_{k\to\infty} \frac{\|G(x^k)-G(x^*)\|}{\|x^k-x^*\|}$$

$$\le \limsup_{k\to\infty} \frac{\|G'(x^*)x^k\|}{\|x^k\|} + \limsup_{k\to\infty} o\|x^k\|$$

$$\le \limsup_{k\to\infty} \frac{|(x_1^k)^4-(x_2^k)^2|}{|(x_1^k)^2+x_2^k|}$$

$$\le \limsup_{k\to\infty} |(x_1^k)^2-x_2^k| = 0.$$

Hence $Q_1(I,x^*) = 0$.

Thus we would like to weaken the condition $G'(x^*) \equiv 0$ but still obtain a result that would extend to multistep operators. The natural thing to do is to return to the condition $\rho\{G'(x^*)\} = 0$, and since, as was seen in Example 1.4.1, $\rho\{G'(x^*)\} = 0$ is not sufficient for

Q-superlinear convergence, impose additional assumptions on G. The first condition we impose is that the second derivative be singular at x^*, that is, $G''(x^*)hh = 0$ for some non-zero $h \in R^n$. However, the following example shows that this is not sufficient for Q-superlinear convergence.

<u>Example 1.5.5</u>. Let $G:R^2 \to R^2$ be defined as in Example 1.4.1. Then for $x^* = (0,0)^T$, $G(x^*) = x^*$ and all derivatives greater than the second of G at x^* are 0. Now

$$G^{[2]}(x^*) = \left[\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix} \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right]$$

which is singular, for $G^{[2]}(x^*)hh = 0$ if h is of the form $(0,h_2)^T$ where $h_2 \neq 0$. As was seen in Example 1.4.1 for $I:x^{k+1} = G(x^k)$, $Q_1(I,x^*) \geq 1$. In fact the previous example shows that even a stronger condition is not sufficient, namely, requiring that the second derivative be singular at x^* and <u>all</u> succeeding derivatives be zero at x^*.

Thus we are led to demanding that the second through the mth derivatives of G be zero at x^*; however, the following example again shows that this is not sufficient for Q-superlinear convergence.

Example 1.5.6. Let $G:R^2 \to R^2$ be defined by $G(x) = (x_1^m - x_2, x_2^m)^T$ where m is even. Then for $x^* = (0,0)^T$, $G(x^*) = x^*$ and all derivatives greater than the first of G at x^* are zero except for $G^{[m]}(x^*)$. Let $x^0 = (0,\alpha)^T$ and it is easy to see that for $I: x^{k+1} = G(x^k)$

$$x^k = \begin{cases} (0, \alpha^{m^k})^T & \text{for } k \text{ even,} \\ (\alpha^{m^{k-1}}, \alpha^{m^k})^T & \text{for } k \text{ odd;} \end{cases}$$

thus if $|\alpha| < 1$ the sequence $\{x^k\}$ converges to x^*. However for k even and $\|x\| = \|x\|_\infty$

$$\limsup_{k\to\infty} \frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|} \geq \limsup_{k\to\infty} \frac{|\alpha^{m^k}|}{|\alpha^{m^k}|} = 1 \,,$$

and hence $Q_1(I,x^*) \geq 1$. It should be pointed out that for m odd, an analogous argument shows that $Q_1(I,x^*) \geq 1$ for the iteration function $G(x) = (-x_1^m - x_2, x_2^m)^T$ with $x^0 = (0,-|\alpha|)^T$.

Again we have shown that an even stronger condition is not sufficient, for there is only one derivative of G other than the first which is non-zero. Thus we are led to assuming that G is infinitely differentiable at x^* with all derivatives, other than the first, zero at x^*. Amazingly enough, this is still not sufficient for Q-superlinear convergence as the following example shows.

Example 1.5.7. Let $G:R^2 \to R^2$ be defined by $G(x) = (x_2, f(x_1))^T$ where $f(0) = 0$ and f is infinitely differentiable with all derivatives equal to zero at $x_1 = 0$. Then for $x^* = (0,0)^T$, $G(x^*) = x^*$ and since $\rho\{G'(x^*)\} = 0$, x^* is a point of attraction of the process $I:x^{k+1} = G(x^k)$ so we may choose α so that the sequence generated by I with $x^o = (0,\alpha)^T$ converges to x^*. For this choice of x^o we have $x^1 = (f(\alpha),0)^T$, $x^2 = (0,f(f(\alpha))^T$, $x^3 = (f(f(\alpha)),0)^T$, and it is easily seen by induction that for k even, $x^k = (0,f(x_1^{k-1}))^T$ and $x^{k+1} = (f(x_1^{k-1}),0)^T$. Thus if we assume that k is even and let $\|x\| = \|x\|_\infty$ we have

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} \geq \limsup_{k\to\infty} \frac{|f(x_1^{k-1})|}{|f(x_1^{k-1})|} = 1 ,$$

and hence $Q_1(I,x^*) \geq 1$. It is not difficult to satisfy the conditions on f for it is well-known that

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

is infinitely differentiable at $x = 0$ and all derivatives are zero at $x = 0$.

Considering the previous example it is clear that, at least in the area of superlinear convergence, the Q measure may not give a realistic measure of the asymptotic rate of convergence. The definition of the Q measure requires that one consider a ratio of successive terms, and in view of the previous example and other examples in this Chapter, this appears to be too restrictive. Thus we are led to the following measure of convergence which relaxes the requirement that the ratio of successive terms be used.

Definition 1.5.8. Let $\{x^k\} \subset R^n$ be any convergent sequence with limit x^*. Then the quantities

$$WQ_{r,p}\{x^k\} = \begin{cases} 0 & \text{if } x^k = x^* \text{ for all } k \geq k_o , \\ \limsup\limits_{k\to\infty} \dfrac{\|x^{k+1}-x^*\|}{\|x^{k-r}-x^*\|^p} & \text{if } x^k \neq x^* \text{ for all but finitely many } k , \\ \infty & \text{otherwise} , \end{cases}$$

defined for all $p \in [1,\infty)$ and all integers $r \geq 0$ are the weak quotient convergence factors of $\{x^k\}$. If I is an iterative process with limit point x^* and $C(I,x^*)$ is the set of all sequences generated by I converging to x^*, then $WQ_{r,p}(I,x^*) = \sup\{WQ_{r,p}\{x^k\} \mid \{x^k\} \in C(I,x^*)\}$ are the weak quotient convergence factors, or WQC factors, of I at x^*.

Ostrowski [1966] has considered the WQC factors with $p = 1$ and uses the term weakly linearly convergent for any process whose WQC factors satisfy $WQ_{r,1}(I,x^*) < 1$ for some $r \geq 0$.

We shall call a process weakly superlinearly convergent if $WQ_{r,1}(I,x^*) = 0$ for some $r \geq 0$. In keeping with this, we have the concept of weak quotient order.

<u>Definition 1.5.9.</u> Let $WQ_{r,p}(I,x^*)$ be the WQC factors of the iterative process I at x^*. Then

$$O_{WQ}(I,x^*) = \begin{cases} \infty \text{ if } WQ_{r,p}(I,x^*) = 0 \text{ for some } r \geq 0 \text{ and} \\ \quad \text{all } p \in [1,\infty) \\ \sup\{p\in[1,\infty) \mid WQ_{r,p}(I,x^*) = 0 \text{ for some } r \geq 0\} \\ 1 \text{ if } WQ_{r,p}(I,x^*) \neq 0 \text{ for all } r \geq 0 \text{ and all} \\ \quad p \in [1,\infty) . \end{cases}$$

is the weak quotient order of I at x^*.

In keeping with the usual conventions, if $O_{WQ}(I,x^*) = 2$ we shall call I a weakly quadratically convergent process.

In terms of the weak quotient order we are able to give sufficient conditions for weak superlinear and higher weak orders of convergence of an iterative process in terms of the derivatives of the iteration function. Before doing this we give the following lemmas about the derivatives of composite operators and products of operators.

<u>Lemma 1.5.10</u>. Assume that $G:D \subset R^n \to D$ is differentiable on D. Then $[G^r]'(x) = G'(G^{r-1}(x))G'(G^{r-2}(x))\ldots G'(x)$ for every $x \in D$ and any integer $r \geq 1$.

<u>Proof</u>. The proof is by induction. The result clearly holds for $r = 1$. Assume that the result holds for $p = 1,\ldots,r-1$. Then

$$[G^r]'(x) = [G(G^{r-1})]'(x)$$

and by the Chain Rule which may be found, for example, in Vainberg [1964]

$$\begin{aligned}[G^r]'(x) &= G'(G^{r-1}(x))[G^{r-1}]'(x) \\ &= G'(G^{r-1}(x))G'(G^{r-2}(x))\ldots G'(G(x))G'(x)\end{aligned}$$

for any $x \in D$. This completes the induction argument and the proof. Q.E.D.

<u>Lemma 1.5.11</u>. Assume that $A_i:D \subset R^n \to \mathfrak{L}(R^n)$ are differentiable at $x \in D$ for $i = 1,2$. Then the differential of the product $A_1(x)A_2(x)$ is $A_1'(x)hA_2(x) + A_1(x)A_2'(x)h$.

<u>Proof</u>.

$$\lim_{\|h\|\to 0} \frac{\|A_1(x+h)A_2(x+h) - A_1(x)A_2(x) - A_1'(x)hA_2(x) - A_1(x)A_2'(x)h\|}{\|h\|}$$

$$\leq \lim_{\|h\|\to 0} \frac{\|A_1(x+h)A_2(x+h) - A_1(x)A_2(x+h) - A_1'(x)hA_2(x+h)\|}{\|h\|}$$

$$+ \frac{\|A_1'(x)hA_2(x+h) - A_1'(x)hA_2(x)\|}{\|h\|}$$

$$+ \frac{\|A_1(x)A_2(x+h) - A_1(x)A_2(x) - A_1(x)A_2'(x)h\|}{\|h\|}$$

$$\le \lim_{\|h\|\to 0} \frac{\|A_1'(x)\| \ \|h\| \ \|A_2(x+h) - A_2(x)\|}{\|h\|} = 0 .$$

Hence

$$A_1'(x)hA_2(x) + A_1(x)A_2'(x)h$$

is the differential of $A_1(x)A_2(x)$. Q.E.D.

Lemma 1.5.12. Assume that $G:D \subset R^n \to R^n$ is p-times differentiable on D where $p \ge 2$, and that there is a point $x^* \in D$ such that $x^* = G(x^*)$ and $G^{[k]}(x^*) = 0$ for all $k = 2,\ldots,p$. Then $(G^r)^{[k]}(x^*) = 0$ for any integer $r \ge 1$ and all $k = 2,\ldots,p$.

Proof. Let k be any integer satisfying $2 \le k \le p$. Then by Lemma 1.5.10

$$(G^r)^{[k]}(x) = [G'(G^{r-1})G'(G^{r-2})\ldots G']^{[k-1]}(x) .$$

Using Lemma 1.5.11 and the Chain Rule repeatedly we have

$$(G^r)^{[k]}(x) = G^{[k]}(G^{r-1}(x))(G^{r-1})^{[k-1]}(x)\{G'(G^{r-2}(x)\ldots G'(x)\}$$
$$+ G'(G^{r-1}(x))G^{[k]}(G^{r-2}(x))(G^{r-2})^{[k-1]}(x)$$

$$\cdot \{G'(G^{r-3}(x))\ldots G'(x)\} + \ldots$$

$$+ \{G'(G^{r-1}(x))\ldots G'(G(x))\}G^{[k]}(x) \ .$$

Thus using the fact that $G(x^*) = x^*$ and $G^{[k]}(x^*) = 0$ gives $(G^r)^{[k]}(x^*) = 0$. Q.E.D.

We now characterize weak convergence in terms of the derivatives of the iteration function.

Theorem 1.5.13. Let $G: D \subset R^n \to R^n$ with $x^* \in \text{int}(D)$ satisfy $x^* = G(x^*)$. Assume that for some integer $p \geq 1$, G is differentiable at x^* up to and including order $p + 1$ with $\rho\{G'(x^*)\} = 0$, $G^{[i]}(x^*) = 0$ for i satisfying $2 \leq i \leq p$, and $G^{[p+1]}$ continuous on D . Then x^* is a point of attraction of the iterative process $I: x^{k+1} = G(x^k)$ and the weak quotient order is at least p . Moreover, if in addition G is infinitely differentiable at x^* with all derivatives greater than the first equal to zero at x^*, then the weak quotient order is infinite.

Proof. The point of attraction result follows from Theorem 1.2.1. Let $\{x^k\}$ be any sequence generated by I converging to x^* and let p be the integer for which the hypotheses are satisfied. Since $\rho\{G'(x^*)\} = 0$ there is an $r \geq 1$ such that $[G'(x^*)]^r = 0$. Then by the Taylor Expansion Theorem 1.1.3 we have

$$\|x^{k+1}-x^*\| = \|G^r(x^{k-r+1})-G^r(x^*)\|$$

$$= \|[G^r]'(x^*)(x^{k-r+1}-x^*) + \sum_{i=2}^{p} [G^r]^{[i]}(x^*)(x^{k-r+1}-x^*)^i$$

$$+ \int_0^1 \frac{(1-t)^p}{p!} [G^r]^{[p+1]}(x^*+t(x^{k-r+1}-x^*))(x^{k-r+1}-x^*)^{p+1}dt\|.$$

Using Lemmas 1.5.10 and 1.5.11 and the fact that $x^* = G(x^*)$ gives

$$(1) \quad \|x^{k+1}-x^*\| \le \|[G'(x^*)]^r(x^{k-r+1}-x^*) + \sum_{i=2}^{p} B_i(x^*)(x^{k-r+1}-x^*)^i\|$$

$$+ \frac{1}{(p+1)!} \sup_{0\le t\le 1} \|[G^r]^{[p+1]}(x^*+t(x^{k-r+1}-x^*))\| \|x^{k-r+1}-x^*\|^{p+1}$$

where $B_i(x^*)(x^{k-r+1})^i$ represents the ith differential of G^r evaluated at x^*. From Lemma 1.5.12 we have that $B_i(x^*) = 0$ for $i = 2,\ldots,p$ since $G^{[i]}(x^*) = 0$ for $i = 2,\ldots,p$.

Consequently,

$$(2) \quad \frac{\|x^{k+1}-x^*\|}{\|x^{k-r+1}-x^*\|^{p_o}} \le \frac{1}{(p+1)!} \sup_{0\le t\le 1} \|[G^r]^{[p+1]}(x^*+t(x^{k-r+1}-x^*))\|$$

$$\cdot \|x^{k-r+1}-x^*\|^{p-p_o+1}.$$

Hence, $WQ_{r-1,p_o}(I,x^*) = 0$ for all $p_o \leq p$, so by Definition 1.5.9 $O_{WQ}(I,x^*) \geq p$.

Now we assume that G is infinitely differentiable at x^* with all derivatives, greater than the first zero at x^*, and that for some p_o there is no $r \geq 0$ such that $WQ_{r,p_o}(I,x^*) = 0$. Then relation (1) with p replaced by p_o leads to a contradiction in relation (2). Hence, for all $p \geq 1$ there is an $r \geq 0$ such that $WQ_{r,p}(I,x^*) = 0$, and thus by Definition 1.5.9, the weak quotient order is infinite. Q.E.D.

Notice that if G is analytic with all of its derivatives, except the first, zero at x^*, then inequality (1) can be written as

$$\|x^{k+1}-x^*\| = \|[G^r]'(x^*)(x^{k-r+1}-x^*) + \sum_{i=2}^{\infty} B_i(x^*)(x^{k-r+1}-x^*)^i\| .$$

Then as in the theorem, $B_i(x^*)(x^{k-r+1}-x^*)^i \equiv 0$ for all i so that $\|x^{k+1}-x^*\| = 0$, and hence, $Q_p(I,x^*) = 0$ for all $p \geq 1$. However, the infinite order of convergence for such a process is uninteresting since the assumptions on G imply that G is linear.

Section 1.6

Necessary Conditions for Q-Superlinear Convergence

We have already seen that one condition necessary for Q-superlinear convergence is that $\rho\{G'(x^*)\} = 0$, for otherwise we have $Q_1(I,x^*) \geq R_1(I,x^*) = \rho\{G'(x^*)\} > 0$. However, as was pointed out by Example 1.4.1, there must be a stronger condition which is also necessary. We have also seen in Example 1.5.4 that $G'(x^*) \equiv 0$ is not a necessary condition for Q-superlinear convergence. Therefore, as in Section 5, one might be tempted to impose additional conditions on the higher derivatives of G , but in view of the following examples this would seem to be a difficult task.

<u>Example 1.6.1</u>. Let $G:R^2 \to R^2$ be defined by $G(x) = (x_1^{2m}-x_2, (x_1^{2m}-x_2)^{2m})^T$. Then for $x^* = (0,0)^T$ $G(x^*) = x^*$ and since $\rho\{G'(x^*)\} = 0$, x^* is a point of attraction of the iteration $I:x^{k+1} = G(x^k)$. Now let $\{x^k\}$ be any sequence generated by G converging to x^* .

Then we have

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} = \limsup_{k\to\infty} \frac{\|G(x^k)-G(x^*)\|}{\|x^k-x^*\|}$$

$$\le \limsup_{k\to\infty} \frac{\|G'(x^*)x^k\|}{\|x^k\|} + \limsup_{k\to\infty} o(\|x^k\|)$$

$$\le \limsup_{k\to\infty} \frac{[(x_1^k)^{2m}-x_2^k]^{2m}}{|(x_1^k)^{2m}-x_2^k|}$$

where $\|x\| = \|x\|_\infty$. Thus $\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} = 0$ and hence $Q_1(I,x^*) = 0$.

It is particularly interesting to compare the previous example with Example 1.5.6. In Example 1.5.6 the iteration was shown to have Q-linear convergence, yet an examination of the derivatives of the two iteration functions at x^* shows that they are identical up to the 2mth derivative, being 0 ; from the 2mth to the $(2m)^{2mth}$ derivative they are again both 0 ; and in both cases all derivatives above the $(2m)^{2mth}$ derivative are 0 . Thus by increasing m it is possible to give two iteration functions whose derivatives agree to an arbitrarily high order, yet one has Q-linear convergence while the other has Q-superlinear convergence.

Another interesting aspect of Example 1.6.1 is that $g_1(x)$ is a factor of $g_2(x)$. One might think that this is a significant fact in obtaining the superlinear convergence, but the following example provides an iteration function whose components are not factors and yet it has Q-superlinear convergence.

Example 1.6.2. Let $G:R^2 \to R^2$ be defined by $G(x) = (x_1^2+x_2, x_1^4+x_2^2)^T$. Then for $x^* = (0,0)^T$ $G(x^*) = x^*$ and since $\rho\{G'(x^*)\} = 0$, x^* is a point of attraction of the iteration $I: x^{k+1} = G(x^k)$. Now let $\{x^k\}$ be any sequence generated by G which converges to x^* . Then as in Example 1.6.1 we have

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} \le \limsup_{k\to\infty} \frac{\|G'(x^*)x^k\|}{\|x^k\|} .$$

Now since $g_2(x) \ge 0$ we may assume that $x_2^k \ge 0$ so that if $\|x\| = \|x\|_\infty$ we have

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} \le \limsup_{k\to\infty} \frac{|(x_1^k)^4+(x_2^k)^2|}{|(x_1^k)^2+(x_2^k)|}$$

$$\le \lim_{x\to 0} \frac{|x_1^4+x_2^2|}{|x_1^2+x_2|} .$$

We shall now show that $\lim_{x\to 0} \frac{|x_1^4+x_2^2|}{|x_1^2+x_2|} = 0$.

Let ϵ satisfy $0 < \epsilon \le 1$ and assume that $\|x\| \le \epsilon$. Then

$$|x_1^4+x_2^2| \le |\epsilon^2 x_1^2+\epsilon x_2| \le \epsilon|x_1^2+x_2|$$

so that

$$\frac{|x_1^4+x_2^2|}{|x_1^2+x_2|} \le \epsilon$$

whenever $\|x\| \le \epsilon$. Since $\{x^k\}$ was an arbitrary sequence generated by G converging to x^* we have $Q_1(I,x^*) = 0$.

In view of the previous examples no attempt will be made to give a necessary condition for Q-superlinear convergence in terms of higher derivatives of the iteration function G . However a necessary condition can be given utilizing the Jordan Form of the matrix $G'(x^*)$.

<u>Theorem 1.6.3</u>. Assume that $G:D \subset R^n \to R^n$ is differentiable on the convex set D and for some constant L

$$\|G'(x)-G'(x^*)\| \le L\|x-x^*\|$$

for all $x \in D$. Assume that $x^* \in D$ is a point of attraction of the iteration $I:x^{k+1} = G(x^k)$. Furthermore assume that

$$PG'(x^*)P^{-1} = (a_{i,j}) \ne 0$$

is in Jordan Form with $a_{j_1,j_1+1} = a_{j_2,j_2+1} = \ldots = a_{j_m,j_m+1} = 1$ and all other $(a_{i,j}) = 0$. Let

$$S_i = \{x \in D; |x_i - x_i^*| \geq |x_{j_1+1} - x_{j_1+1}^*|, \ldots, |x_{j_m+1} - x_{j_m+1}^*|\}$$

$i = 1,\ldots,n$, $i \neq j_1 + 1,\ldots,j_m + 1$, and

$$D_o = \{x \in D; \{x^k\} = \{G^k(x)\} \text{ converges to } x^*\}.$$

Assume that $Q_1(I,x^*) = 0$. Then for every $x \in D_o$ there is a k_o such that $G^k(x) \in P^{-1}S$ for every $k \geq k_o$ where

$$S = \bigcup_{i=1}^{n} S_i .$$

<u>Proof</u>. Let $x^o \in D_o$. Then since $Q_1(I,x^*) = 0$,

$$\limsup_{k\to\infty} \frac{\|G(x^k) - G(x^*)\|}{\|x^k - x^*\|} = 0 .$$

Define a norm $\|\cdot\|^*$ by $\|z\|^* = \|P^{-1}z\|$. Since by Theorem A.1.6 $Q_1(I,x^*) = 0$ is independent of norm

(1) $$\limsup_{k\to\infty} \frac{\|P[G(x^k) - G(x^*)]\|^*}{\|P(x^k - x^*)\|^*} = 0 .$$

Define $\hat{G}: PD \subset R^n \to R^n$ by $\hat{G}(y) = PG(P^{-1}y)$. Then from (1), with $y^k = Px^k$ and $y^* = Px^*$,

$$0 = \limsup_{k\to\infty} \frac{\|PG(P^{-1}y^k)-PG(P^{-1}x^*)\|^*}{\|y^k-y^*\|^*}$$

$$= \limsup_{k\to\infty} \frac{\|\hat{G}(y^k)-\hat{G}(y^*)\|^*}{\|y^k-y^*\|^*}$$

$$= \limsup_{k\to\infty} \frac{\|y^{k+1}-y^*\|^*}{\|y^k-y^*\|^*} .$$

Thus $Q_1(I_2,y^*) = 0$ where $I_2 : y^{k+1} = \hat{G}(y^k)$.

Now we assume that the statement of the theorem is false; that is, there exists an $x^o \in D_o$ and a sequence of indices k_ℓ for which $G^{k_\ell}(x^o) \notin P^{-1}S$, or there exists a $y^o = Px^o$ and a sequence of indices k_ℓ for which $PG^{k_\ell}(P^{-1}y^o) \notin S$, or there exists a $y^o \in PD_o$ and a sequence of indices k_ℓ for which $\hat{G}^{k_\ell}(y^o) \notin S$. Thus the subsequence $\{y^{k_\ell}\}$ of $\{y^k\}$ satisfies $y^{k_\ell} \notin S$ for every $\ell = 1,2,\ldots$. Therefore, the sequence $\{y^{k_\ell}\}$ contains a subsequence, which for simplicity we also denote by $\{y^{k_\ell}\}$, satisfying

$$\|y^{k_\ell}-y^*\| = |y^{k_\ell}_{j'+1}-y^*_{j'+1}|$$

for some $j' \in \{j_1,\ldots,j_m\}$ where $\|y\| = \|y\|_\infty$. Then by the Taylor Expansion Theorem 1.1.3 we have

$$0 = Q_1(I_2,y^*) \geq \limsup_{k\to\infty} \frac{\|y^{k+1}-y^*\|}{\|y^k-y^*\|}$$

$$= \limsup_{k\to\infty} \frac{\|\hat{G}(y^k)-\hat{G}(y^*)\|}{\|y^k-y^*\|}$$

$$\geq \limsup_{k\to\infty} \frac{\|\hat{G}'(y^*)(y^k-y^*)\|}{\|y^k-y^*\|}$$

$$- \limsup_{k\to\infty} \frac{\left\|\int_0^1 [\hat{G}'(y^*+t(y^k-y^*))-\hat{G}'(y^*)](y^k-y^*)\,dt\right\|}{\|y^k-y^*\|}$$

$$\geq \limsup_{\ell\to\infty} \frac{\|\hat{G}'(y^*)(y^{k_\ell}-y^*)\|}{\|y^{k_\ell}-y^*\|}$$

$$\geq \limsup_{\ell\to\infty} \frac{|y^{k_\ell}_{j'+1}-y^*_{j'+1}|}{|y^{k_\ell}_{j'+1}-y^*_{j'+1}|} = 1 .$$

Since this is a contradiction the theorem is proved. Q.E.D.

Now the following example, which is quite similar to Example 1.4.1, shows that the conditions of Theorem 1.6.3 are not sufficient to guarantee Q-superlinear convergence.

Example 1.6.4. Let $G:D \subset R^2 \to R^2$ be defined as in Example 1.4.1 where $D = \{x \in R^2; x_1 \geq x_2 \geq x_1^2\}$. Thus D is a convex subset of R^2 . Let $D_o = \{x \in D; x_1 < 1\}$. Let $x^o = (\alpha,\beta)^T \in D_o$. Then $G(x^o) = (\beta,\alpha^2)^T \in D_o$.

Hence $G:D_o \to D_o$ so that the iteration $I:x^{k+1} = G(x^k)$ is well defined for $x^o \in D_o$ and, furthermore, $\{x^k\}$ converges to $x^* = (0,0)^T$ since

$$x^k = \begin{cases} (\alpha^{2^{k/2}}, \beta^{2^{k/2}})^T & \text{for } k \text{ even}, \\ (\alpha^{2^{\frac{k-1}{2}}}, \beta^{2^{\frac{k+1}{2}}}) & \text{for } k \text{ odd}. \end{cases}$$

Since $G'(x^*) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $S = \{x \in D; x_1 \geq x_2\}$. Thus for any $x^o \in D_o$, $x^{k+1} = G(x^k) \in S$ for all k . Hence the conditions of Theorem 1.6.3 are satisfied, but as was shown in Example 1.4.1, $Q_1(I,x^*) \geq 1$.

The following theorem extends Theorem 1.6.3 to multistep operators.

Theorem 1.6.5. Assume that $G:D^m \subset (R^n)^m \to R^n$ is differentiable on the convex set D^m and satisfies, for some constant L,

$$\|G'(x^1,\ldots,x^m) - G'(x^*,\ldots,x^*)\| \leq L\|(x^1-x^*,\ldots,x^m-x^*)\|$$

for all $(x^1,\ldots,x^m) \in D^m$. Assume that $x^* \in D$ is a point of attraction of the iteration $I_1:x^{k+1} = G(x^k,\ldots,x^{k-m+1})$. Furthermore, assume that

$$PHP^{-1} = (a_{i,j}) \neq 0$$

is in Jordan Form, where H is given by relation 1.2-1, with $a_{j_1,j_1+1} = \dots = a_{j_\ell,j_\ell+1} = 1$ and all other $a_{i,j} = 0$.

Let

$$S_i = \left\{z \in D^m ; |z_i - z_i^*| \ge |z_{j_1+1} - z_{j_1+1}^*|, \dots, |z_{j_\ell+1} - z_{j_\ell+1}^*|\right\},$$

$i = 1,\dots,mn$, $i \ne j_1+1,\dots,j_\ell+1$ where $z^* = (x^*,\dots,x^*)$,

and

$$D_o = \{(x^o,\dots,x^{m-1}) \in D^m ; \{x^k\} = \{G^k(x^o,\dots,x^{m-1})\} \text{ converges to } x^*\} .$$

Assume that $Q_1(I_1,x^*) = 0$. Then for every $z \in D_o$ there is a k_o such that

$$(G^k(z),G^{k-1}(z),\dots,G^{k-m}(z))^T \in P^{-1}S$$

for every $k \ge k_o$ where $S = \bigcup_{i=1}^{mn} S_i$.

<u>Proof</u>. Define the sequence $\{z^k\}$ by $z^k = (x^k,\dots,x^{k-m+1})$. Then $z^{k+1} = \hat{G}(z^k)$ where $\hat{G}(z^k) = (G(z^k),x^k,\dots,x^{k-m+2})$. As in Theorem 1.2.2 $\hat{G}'(z^*) = H$ so by hypothesis $\hat{G}$ satisfies all of the conditions of Theorem 1.6.3.

Now let $\{z^k\}$ be any sequence generated by I_2 converging to z^*. Let $\|z\| = \max_{1\le i\le m} \|x^i\|$ where $z = (x^1,\dots,x^m)$.

Since by hypothesis $Q_1(I_1,x^*) = 0$ there is a k_o such that $\|x^{k+1}-x^*\| \le \|x^k-x^*\|$ for every $k \ge k_o$. Hence,

$$\limsup_{k\to\infty} \frac{\|z^{k+1}-z^*\|}{\|z^k-z^*\|} = \limsup_{k\to\infty} \frac{\|x^{k-m+2}-x^*\|}{\|x^{k-m+1}-x^*\|} = 0 .$$

Thus $Q_1(I_2,z^*) = 0$, and by Theorem 1.6.3, for every $z \in D_o$ there is a k_o such that $\hat{G}^k(z) \in P^{-1}S$ for every $k \ge k_o$; or using the definition of $\hat{G}$, we have for every $z \in D_o$ there is a k_o such that

$$(G^k(z), G^{k-1}(z), \ldots, G^{k-m}(z))^T \in P^{-1}S$$

for every $k \ge k_o$. Q.E.D.

At the beginning of this section it was noted in Example 1.6.1 that it was not necessary that $G'(x^*) \equiv 0$ for Q-superlinear convergence; however, one might wonder if this is true when the iteration function has a particular form. For example, we are generally interested in finding a zero, x^*, of $F(x)$ and the iteration function frequently has the form $G(x) = x - A(x)F(x)$ where $A: R^n \to \mathfrak{L}(R^n)$. Unfortunately even for so special a G it is not necessary that $G'(x^*) \equiv 0$ for Q-superlinear convergence as the following example shows.

<u>Example 1.6.6.</u> Let $F:R^2 \to R^2$ be twice differentiable and satisfy $F(0) = 0$, $f_1(x) = x_1$, $\partial_1 f_2(0) = -1$, and $\partial_2 f_2(0) = 1$. Define $G:R^2 \to R^2$ by $G(x) = x - IF(x)$. Then $G'(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0$ and $(0,0)^T$ is a point of attraction of the iteration $I:x^{k+1} = G(x^k)$. Let $\{x^k\}$ be any sequence generated by I converging to $x^* = (0,0)^T$. Then $\{x^k\}$ is of the form $(0,\alpha_k)^T$ and as in Example 1.6.1 we have

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|} \le \limsup_{k\to\infty} \frac{\|G'(x^*)x^k\|}{\|x^k\|} + \limsup_{k\to\infty} o(\|x^k\|)$$

$$\le \limsup_{k\to\infty} \frac{\|(0,0)^T\|}{\|(0,\alpha_k)^T\|}$$

$$= 0 .$$

Hence $Q_1(I,x^*) = 0$.

CHAPTER II

PRECISE HIGHER ORDER OF CONVERGENCE THEOREMS

Section 2.1

Introduction

In the previous chapter we were concerned with the concept of superlinear order of convergence for iterative procedures. In this chapter we shall again be interested in superlinear convergence, but the emphasis will be on giving precise values to the order of convergence as opposed to the general results of the previous chapter which simply allow one to distinguish between linear and superlinear convergence.

We shall be particularly interested in the two-point secant method for finding a zero, x^*, of $F(x)$ where $F:D \subset R^n \to R^n$. This method is given by

$$x^{k+1} = G(x^k, x^{k-1}) \qquad k = 1,2,\ldots, \tag{1}$$

where $G:D^2 \subset (R^n)^2 \to R^n$ is defined by $G(x,y) = x - J^{-1}(x,y-x)F(x)$ with the jth column of

$$J:D^2 \subset (R^n)^2 \to \mathcal{L}(R^n)$$

given by

$$J_j(x,y-x) = \begin{cases} \dfrac{1}{y_j-x_j}\left[F\left(x+\beta\sum_{i=1}^{j-1}(y_i-x_i)e_i+(y_j-x_j)e_j\right)\right. \\ \left. -F\left(x+\beta\sum_{i=1}^{j-1}(y_i-x_i)e_i\right)\right] & \text{for } y_j \neq x_j , \\ F'\left(x+\beta\sum_{i=1}^{j-1}(y_i-x_i)e_i\right)e_j & \text{for } y_j = x_j , \end{cases}$$

for $\beta \in [0,1]$. Korganoff [1961] has given a convergence theorem showing the order of convergence to be at least $\frac{1+\sqrt{5}}{2}$ for the case $\beta = 0$, and Schmidt [1966] obtained similar results for the case $\beta = 1$. In Section 3 we shall attempt to give the precise order of convergence of procedure (1) for any $\beta \in [0,1]$.

The remainder of this chapter will be devoted to convergence theorems where the order of convergence is superlinear and to applications of these theorems to specific iterative processes.

Section 2.2

An Order of Convergence Theorem for Multi-Step Iterative Procedures

We begin this section with a theorem which is useful in determining the order of convergence of multi-step iterative processes. The result is due to Ortega and Rheinboldt [1970] and is based on results of Ostrowski [1966] and the following lemma, the proof of which may be found in Ostrowski [1966] or Traub [1964].

<u>Lemma 2.2.1</u>. For any integer $m \geq 1$, the polynomial $p_m(t) = t^{m+1} - t^m - \alpha$ has a unique positive root τ_m for $\alpha = 1$ or 2. Moreover $\tau_m \in (1,2]$, $\tau_m > \tau_{m+1}$, and $\lim_{m\to\infty} \tau_m = 1$.

The proof of the following theorem is given in A.2.9.

<u>Theorem 2.2.2</u>. Let I be an iterative process with limit x^* and let $\gamma_j, j = 0,1,\ldots,m$, be non-negative constants. If for every sequence $\{x^k\}$ generated by I with $\lim_{k\to\infty} x^k = x^*$ there is a k_o depending on $\{x^k\}$ such that

$$\|x^{k+1}-x^*\| \leq \|x^k-x^*\| \sum_{j=0}^{m} \gamma_j \|x^{k-j}-x^*\| \tag{1}$$

for all $k \geq k_o$, then $O_R(I,x^*) \geq \tau$ where τ is the unique positive root of $t^{m+1} - t^m - 1 = 0$. In addition, if there is a $\beta > 0$, a k_o, and some sequence $\{x^k\}$ generated by I with $\lim_{k\to\infty} x^k = x^*$ such that $\|x^k - x^*\| > 0$ for all $k \geq k_o - m$ and

(2) $$\|x^{k+1} - x^*\| \geq \beta \|x^k - x^*\| \, \|x^{k-m} - x^*\| \quad \text{for all } k \geq k_o ,$$

then $O_R(I,x^*) = \tau$.

Perhaps the first thing that one notices about Theorem 2.2.2 is that it says nothing about $O_Q(I,x^*)$. One might expect that if 2.2-1 is satisfied then $O_Q(I,x^*) \geq \tau$; however, the following example shows that this is not true.

Example 2.2.3. Let I be the iterative process defined by

(3) $$x^{k+1} = \begin{cases} x^k x^{k-1} & \text{for } k \text{ even}, \\ (x^k)^{\frac{1}{\alpha-1}} x^{k-1} & \text{for } k \text{ odd}, \end{cases}$$

where $x^o, x^1 \in R^1$ and α satisfies $1 < \alpha < \tau = \frac{1+\sqrt{5}}{2}$. Then for $|x^o|, |x^1| < 1$ the sequence $\{x^k\}$ generated by I converges to $x^* = 0$ and satisfies

$$|x^{k+1}| \leq |x^k|\ |x^{k-1}|$$

for all $k \geq 0$ since $\frac{1}{\alpha-1} > 1$. However, we shall now show that $O_Q(I,x^*) \leq \alpha$. Consider the sequence $\{x^k\}$ generated by I satisfying (3) with k even. Then

$$Q_\alpha(I,x^*) \geq \limsup_{k\to\infty} \frac{|x^{k+1}-x^*|}{|x^k-x^*|^\alpha} = \limsup_{k\to\infty} \frac{|x^k|\,|x^{k-1}|}{|x^k|^\alpha}$$

$$\geq \limsup_{k\to\infty} \frac{|x^{k-1}|}{|x^k|^{\alpha-1}}$$

$$\geq \limsup_{k\to\infty} \frac{|x^{k-1}|}{|x^{k-1}|\,|x^{k-2}|^{\alpha-1}} = \infty \ .$$

Hence $O_Q(I,x^*) \leq \alpha < \tau$.

Theorem 2.2.2 can be used to give a lower bound for the order of convergence of the two-point secant method given by Equation 2.1-1. This is done in Ortega and Rheinboldt [1970].

As has been pointed out in previous sections, it is frequently advantageous to impose conditions on the iteration function to obtain convergence results rather than on the sequences generated by that iteration function. It is this approach which is taken in the following theorem.

Theorem 2.2.4. Let I denote the iteration $x^{k+1} = G(x^k, \ldots, x^{k-m+1})$ where $G: D^m \to R^n$ with D an open subset of R^n. Assume that there is a point $x^* \in \mathrm{int}(D)$ such that $x^* = G(x^*, y^2, \ldots, y^m)$ for all $y^j \in D$, $j = 2, \ldots, m$, that $\partial_1 Gh$ exists as a differential on D and satisfies

$$\|\partial_1 G(y^1, \ldots, y^m)h - \partial_1 G(x^*, \ldots, x^*)h\| \leq \sum_{i=1}^{m} \alpha_i \|y^i - x^*\| \|h\|$$

for all $y^i \in D$, $i = 1, \ldots, m$, and that $\partial_1 G(x^*, \ldots, x^*) = 0$. Then x^* is a point of attraction of the procedure I and $O_R(I, x^*) \geq \tau$ where τ is the unique positive root of $t^m - t^{m-1} - 1 = 0$. Furthermore, assume that $\partial_m \partial_1 G$ exists as a vector partial derivative at $(x^*, \ldots, x^*)$ and that $\partial_m \partial_1 G(x^*, \ldots, x^*) h^1 h^m \neq 0$ for all $h^1, h^m \in R^n$, $h^1, h^m \neq 0$. Then $O_R(I, x^*) = \tau$.

Proof. It follows immediately from $x^* = G(x^*, y^2, \ldots, y^m)$ that $\partial_j G(x^*, \ldots, x^*) = 0$ for $j = 2, \ldots, m$, so by Theorem 1.2.2 x^* is a point of attraction of the procedure I. Let $\{x^k\}$ be any sequence generated by I which converges to x^*. Choose k_o sufficiently large so that $x^{k-m+1} \in D$ for all $k \geq k_o$. Then by the Taylor Expansion Theorem 1.1.3

$$(4)\quad \|x^{k+1}-x^*\| = \|G(x^k,\ldots,x^{k-m+1})-G(x^*,x^{k-1},\ldots,x^{k-m+1})\|$$

$$\leq \int_0^1 \|[\underline{\partial}_1 G(x^*+t(x^k-x^*),x^{k-1},\ldots,x^{k-m+1})$$

$$-\underline{\partial}_1 G(x^*,\ldots,x^*)](x^k-x^*)\|dt$$

$$\leq \left[\tfrac{1}{2}\,\alpha_1\|x^k-x^*\|+\sum_{j=2}^{m}\alpha_j\|x^{k-j+1}-x^*\|\right]\|x^k-x^*\|$$

where $\alpha_1,\ldots,\alpha_m$ are the Lipschitz constants for $\underline{\partial}_1 Gh$. Thus relation 2.2-1 is satisfied for all $k \geq k_o$, so by Theorem 2.2.2 $O_R(I,x^*) \geq \tau$ where τ is the unique positive root of $t^m - t^{m-1} - 1 = 0$.

Now assume that $\underline{\partial}_m\underline{\partial}_1 G(x^*,\ldots,x^*)h^1h^m \neq 0$ for all $h^1,h^m \in R^n$, $h^1,h^m \neq 0$. Then since $\|\underline{\partial}_m\underline{\partial}_1 G(x^*,\ldots,x^*)h^1h^m\|$ is a non-zero, continuous function of h^1 and h^m on the compact set $\{h^1\in R^n;\|h^1\| = 1\} \times \{h^m\in R^n;\|h^m\| = 1\}$, there is a positive constant C such that $\|\underline{\partial}_m\underline{\partial}_1 G(x^*,\ldots,x^*)h^1h^m\| \geq C\|h^1\|\;\|h^m\|$ for all non-zero $h^1,h^m \in R^n$. Returning to equation (4) we have

$$\|x^{k+1}-x^*\| = \|\int_0^1 [\underline{\partial}_1 G(x^*+t(x^k-x^*),x^{k-1},\ldots,x^{k-m+1})$$

$$-\ \underline{\partial}_1 G(x^*,x^{k-1},\ldots,x^{k-m+1})$$

$$+ \underline{\partial}_1 G(x^*, x^{k-1}, \ldots, x^{k-m+1}) - \ldots$$

$$+ \underline{\partial}_1 G(x^*, \ldots, x^*, x^{k-m+1})$$

$$- \underline{\partial}_1 G(x^*, \ldots, x^*)](x^k - x^*)\,dt\| .$$

Adding and subtracting $\underline{\partial}_m \underline{\partial}_1 G(x^*, \ldots, x^*)(x^{k-m+1} - x^*)(x^k - x^*)$, and using the Lipschitz condition for $\underline{\partial}_1 G$ on the first $m - 1$ differences of $\underline{\partial}_1 G$ gives

$$(5) \quad \|x^{k+1} - x^*\| \geq \|\underline{\partial}_m \underline{\partial}_1 G(x^*, \ldots, x^*)(x^{k-m+1} - x^*)(x^k - x^*)\|$$
$$- \|\underline{\partial}_1 G(x^*, \ldots, x^*, x^{k-m+1}) - \underline{\partial}_1 G(x^*, \ldots, x^*)$$
$$- \underline{\partial}_m \underline{\partial}_1 G(x^*, \ldots, x^*)(x^{k-m+1} - x^*)\| \; \|x^k - x^*\|$$
$$- \frac{1}{2}\alpha_1 \|x^k - x^*\|^2 - \|x^k - x^*\| \sum_{j=2}^{m-1} \alpha_j \|x^{k-j+1} - x^*\| .$$

Now since $\underline{\partial}_m \underline{\partial}_1 G$ exists at $(x^*, \ldots, x^*)$ there is a $k_1 \geq k_o$ such that for every $k \geq k_1$

$$\|\underline{\partial}_1 G(x^*, \ldots, x^*, x^{k-m+1}) - \underline{\partial}_1 G(x^*, \ldots, x^*)$$
$$- \underline{\partial}_m \underline{\partial}_1 G(x^*, \ldots, x^*)(x^{k-m+1} - x^*)\| \leq \frac{c}{4} \|x^{k-m+1} - x^*\| .$$

Furthermore, since by Theorem 1.5.3 $Q_1(I, x^*) = 0$, there is a $k_2 \geq k_1$ such that $\|x^{k-j} - x^*\| \leq \delta \|x^{k-m+1} - x^*\|$ for

$j = 0,1,\ldots,m-2$ and all $k \geq k_2$ where δ is chosen so that $\delta\left[\frac{1}{2}\alpha_1 + \sum_{j=2}^{m-1} \alpha_j\right] < C/4$. Returning to relation (5) we may now write

$$\|x^{k+1}-x^*\| \geq C\|x^{k-m+1}-x^*\| \; \|x^k-x^*\| - \frac{C}{4}\|x^{k-m+1}-x^*\| \; \|x^k-x^*\|$$

$$- \|x^{k-m+1}-x^*\| \; \|x^k-x^*\| \delta\left[\frac{1}{2}\alpha_1 + \sum_{j=2}^{m-1}\alpha_j\right]$$

$$\geq \frac{C}{2}\|x^{k-m+1}-x^*\| \; \|x^k-x^*\| \quad \text{for all } k \geq k_2 .$$

Hence relation 2.2-2 is satisfied and we have by Theorem 2.2.2 that $O_R(I,x^*) = \tau$. Q.E.D.

The condition that $x^* = G(x^*,y^2,\ldots,y^m)$ for all $y^j \in D$, $j = 2,\ldots,m$, is, of course, very restrictive. However, many iterative procedures for finding a point x^* to satisfy $F(x^*) = 0$ are of the form $x^{k+1} = x^k - A(x^k,\ldots,x^{k-m+1})F(x^k) \equiv G(x^k,\ldots,x^{k-m+1})$ where $A:D^m \subset (R^n)^m \to \mathfrak{L}(R^n)$. Then it is clear that $x^* = G(x^*,y^2,\ldots,y^m)$ for all $y^j \in D$, $j = 2,\ldots,m$. In particular the two-point secant method given by Equation 2.1-1 falls into this category with $A(x^k,x^{k-1}) \equiv J^{-1}(x^k,x^{k-1}-x^k)$.

In the next section we shall attempt to apply Theorem 2.2.4 to this method.

There is another broad class of operators which satisfy the condition $x^* = G(x^*, y^2, \ldots, y^m)$ for all $y^j \in D$, $j = 2, \ldots, m$, and that is the single-step operators, the case when $m = 1$. We shall now apply Theorem 2.2.4 to the single-step operators obtaining a result which may be found in Ortega and Rheinboldt [1970] under slightly stronger assumptions on G.

Corollary 2.2.5. Let I denote the iteration $x^{k+1} = G(x^k)$ where $G: D \to R^n$ with D an open subset of R^n. Assume that there is a point $x^* \in \mathrm{int}(D)$ for which $G(x^*) = x^*$, that G is differentiable on D and G' satisfies

$$\|G'(x)h - G'(x^*)h\| \le \alpha \|x - x^*\| \; \|h\| ,$$

and that $G'(x^*) = 0$. Then x^* is a point of attraction of the procedure I with $O_R(I, x^*) \ge 2$. Furthermore, assume that G'' exists at x^* and that $G''(x^*)hh \ne 0$ for all non-zero $h \in R^n$. Then $O_R(I, x^*) = 2$.

Proof. The conditions of the first part of Theorem 2.2.4 are easily seen to be satisfied for $m = 1$; hence, x^* is a point of attraction of the procedure I with $O_R(I,x^*) \geq \tau$ where τ is the root of $\tau - 1 - 1 = 0$. Thus $O_R(I,x^*) \geq 2$. Likewise the conditions of the second part of Theorem 2.2.4 are satisfied so that $O_R(I,x^*) = 2$. Q.E.D.

It should be pointed out that the version of Corollary 2.2.5 due to Ortega and Rheinboldt [1970] also gives information about $O_Q(I,x^*)$, namely, from the first part, that $O_Q(I,x^*) \geq 2$ and, from the second, that $O_Q(I,x^*) = 2$. This result cannot be obtained from Theorem 2.2.4 because, as was shown in Example 2.2.3, Theorem 2.2.2 gives no information about $O_Q(I,x^*)$.

One might wonder if the condition giving the precise order statement in Theorem 2.2.4 and its corollary might be weakened to requiring only that $\underline{\partial}_m \underline{\partial}_1 G(x^*,\ldots,x^*)h \neq 0$ for all non-zero $h \in R^n$. The following example shows that this is not possible.

<u>Example 2.2.6</u>. Let $G:R^2 \to R^2$ be defined by $G(x) = (0, x_1 x_2 + x_2^m)^T$ for $m \geq 3$. Then for $x^* = (0,0)^T$ $G(x^*) = x^*$ and $G'(x^*) = 0$. Thus x^* is a point of attraction of the iteration $I:x^{k+1} = G(x^k)$. Now let $\{x^k\}$ be any sequence generated by G which converges to x^*. Then for any $k \geq 1$, $x^k = (0, \gamma^{m^{k-1}})^T$ for some constant γ with $|\gamma| < 1$, so we have immediately that

$$\limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|^m} = \limsup_{k\to\infty} \frac{|\gamma^{m^k}|}{|\gamma^{m^{k-1}}|^m} = 1 .$$

Since $\{x^k\}$ is any sequence generated by G converging to x^*, we have $Q_m(I,x^*) = 1$ so that $O_Q(I,x^*) = m$. In addition we have

$$0 < R_m\{x^k\} = \limsup_{k\to\infty} \|x^k-x^*\|^{1/m^k} = \limsup_{k\to\infty} |\gamma^{m^{k-1}}|^{1/m^k} < 1 ,$$

so that since $O_R(I,x^*) \geq O_Q(I,x^*)$, $O_R(I,x^*) = m$. However, note that

$$G''(x^*) = \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

so that $G''(x^*)\,h = \begin{pmatrix} 0 & h_2 \\ 0 & h_1 \end{pmatrix} \neq 0$ for $h \neq 0$. Hence $G''(x^*)h \neq 0$ for $h \neq 0$ is not sufficient to guarantee that $O_R(I,x^*) \leq 2$.

The same kind of example may be given if G is a multi-step operator for let $G: R^2 \times R^2 \to R^2$ be defined by $G(x,y) = (0, x_1y_2+x_2y_1+x_2^m)^T$. Then as in the above example it is easy to show that $\partial_2\partial_1 G(x^*,x^*)\,h \neq 0$ for all non-zero $h \in R^2$, but that $O_R(I,x^*) = O_Q(I,x^*) = m$

where I is the iteration $x^{k+1} = G(x^k, x^{k-1})$.

As was pointed out in Chapter I it is not necessary that $G'(x^*) = 0$ or $\underline{\partial}_1 G(x^*, x^*) = 0$ to have Q-superlinear convergence. A slight modification of the above iteration function yields an even stronger result. Let $G: R^2 \to R^2$ be defined by $G(x) = (0, x_1 + x_1 x_2 + x_2^m)^T$ and $\hat{G}: R^2 \times R^2 \to R^2$ be defined by $\hat{G}(x,y) = (0, x_1 + x_1 y_2 + x_2 y_1 + x_2^m)^T$ for $m \geq 3$.

Then $G'(x^*) = \underline{\partial}_1 \hat{G}(x^*, x^*) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$, but as is easily verified for the iterations $I_1 : x^{k+1} = G(x^k)$ and $I_2 : x^{k+1} = \hat{G}(x^k, x^{k-1})$, $O_R(I_1, x^*) = O_Q(I_1, x^*) = O_R(I_2, x^*) = O_Q(I_2, x^*) = m$. Thus even though $G'(x^*) \neq 0$ or $\underline{\partial}_1 \hat{G}(x^*, x^*) \neq 0$ we may have an iteration with an arbitrarily high order of convergence.

Section 2.3

A Sufficient Condition for the Precise Order of Convergence of the Two-Point Secant Method

In this section we shall attempt to determine the precise order of convergence for the two-point secant method introduced in Section 2.1. Therefore we shall be concerned with the procedure I given by

$$(1) \qquad x^{k+1} = G(x^k, x^{k-1}) \equiv x^k - J^{-1}(x^k, x^{k-1}-x^k)F(x^k)$$

where $J(x^k, x^{k-1}-x^k)$ is defined by relation 2.1-1. As was mentioned in the previous section, Ortega and Rheinboldt [1970] have shown that the order of convergence of procedure (1) is at least $\frac{1+\sqrt{5}}{2}$. Before attempting to show that this is the exact order, we shall give a lemma pertaining to the partial derivatives of the iteration function.

Lemma 2.3.1. Assume that $F: D \subset R^n \to R^n$ is twice continuously differentiable on D with $x^* \in \mathrm{int}(D)$ such that $F(x^*) = 0$ and $F'(x^*)$ is non-singular, and that for some constant α_1

$$(2) \qquad \|F''(x) - F''(x^*)\| \le \alpha_1 \|x - x^*\|$$

for all $x \in D$. Let $G: D x D \subset R^n x R^n \to R^n$ be defined by $G(x,y) = x - J^{-1}(x,y-x)F(x)$ where $J(x,y-x)$ is given by Equation 2.1-1. Then there exists an $r > 0$ with $\overline{S}(x^*,r) \subset D$, and constants L_1 and L_2 such that

$$(3) \qquad \|\underline{\partial}_1 G(x,y)h - \underline{\partial}_1 G(x^*,y)h\| \le L_1 \|x-x^*\| \; \|h\|$$

and

$$(4) \quad \|\underline{\partial}_2\underline{\partial}_1 G(x^*,y)hk - \underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)hk\| \le L_2 \|y-x^*\| \; \|h\| \; \|k\|$$

for all $x,y \in \overline{S}(x^*,r)$.

Proof. It is clear that $J(x,y-x)$ is continuous at $x = y = x^*$ and $J(x^*,0) = F'(x^*)$. Thus there exists an $r > 0$ such that $F'(x)$ and $J(x,y-x)$ are non-singular for any $x,y \in \overline{S}(x^*,r) \subset D$. Clearly,

$$(5) \qquad \underline{\partial}_1 G(x,y)h = h - \underline{\partial}_1[J^{-1}(x,y-x)]hF(x) - J^{-1}(x,y-x)F'(x)h$$

provided that $\underline{\partial}_1[J^{-1}(x,y-x)]$ exists. We shall now verify that $\underline{\partial}_1[J^{-1}(x,y-x)]$ does indeed exist and is continuous on $\overline{S}(x^*,r)$. Applying Lemma 1.5.6 to the identity $J^{-1}(x,y-x)J(x,y-x) \equiv I$ gives

(6) $\quad \underline{\partial}_1[J^{-1}(x,y-x)]h = -J^{-1}(x,y-x)\underline{\partial}_1 J(x,y-x)hJ^{-1}(x,y-x)$

for all $x,y \in \overline{S}(x^*,r)$, so we need only calculate $\underline{\partial}_1 J(x,y-x)$. Since $J(x,y-x)$ is a matrix, $\underline{\partial}_1 J(x,y-x)$ is a bilinear operator composed of n $n \times n$ blocks, and one obtains the jth column of the ith block by taking the partial derivative of the i,jth component of $J(x,y-x)$, denoted by $a_{i,j}(x,y-x)$, with respect to the first vector variable. From relation 2.1-1 we see that

$$a_{i,j}(x,y-x) = \frac{f_i(x(j)+(y_j-x_j)e_j)-f_i(x(j))}{y_j-x_j}$$

for $y_j \neq x_j$ where $x(j) = x + \beta \sum_{\ell=1}^{j-1} (y_\ell - x_\ell)e_\ell$, so by direct computation the k,jth component of the ith block of $\underline{\partial}_1 J(x,y-x)$ is given by

$$(7) \quad \partial_{x_k} a_{i,j}(x,y-x) = \begin{cases} \dfrac{[\partial_k f_i(x(j)+(y_j-x_j)e_j)-\partial_k f_i(x(j))]}{y_j-x_j}(1-\beta) , \\ \quad \text{for } k < j , \\ \dfrac{f_i(x(j)+(y_j-x_j)e_j)-f_i(x(j))}{(y_j-x_j)^2} - \dfrac{\partial_j f_i(x(j))}{(y_j-x_j)} , \\ \quad \text{for } k = j \\ \dfrac{\partial_k f_i(x(j)+(y_j-x_j)e_j)-\partial_k f_i(x(j))}{(y_j-x_j)} \\ \quad \text{for } k > j \end{cases}$$

where ∂_{x_k} denotes the partial derivative with respect to the x_kth variable.

Now if $y_j = x_j$ the differentiation cannot be done directly; however,

$$\partial_{x_k} a_{i,j}(x,y-x) = \lim_{t\to 0} \frac{1}{t}[a_{i,j}(x+te_k, y-x-te_k) - a_{i,j}(x,y-x)]$$

$$= \lim_{t\to 0} \frac{1}{t}[\partial_j f_i(x(j)+te_k-\beta te_k) - \partial_j f_i(x(j))]$$

(8) $$= \partial_k \partial_j f_i(x(j))(1-\beta) \quad \text{for} \quad k < j$$

$$\partial_{x_j} a_{i,j}(x,y-x) = \lim_{t\to 0} \frac{1}{t}\left[\frac{f_i(x(j)+te_j) - f_i(x(j))}{t} - \partial_j f_i(x(j))\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[\frac{1}{2}\partial_j^2 f_i(x(j)+\theta te_j)t\right] \quad \text{for} \quad \theta \in [0,1]$$

(9) $$= \frac{1}{2}\partial_j^2 f_i(x(j)) \quad \text{for} \quad k = j .$$

In the same manner that we obtained equation (8) we have

(10) $$\partial_{x_k} a_{i,j}(x,y-x) = \partial_k \partial_j f_i(x(j)) \quad \text{for} \quad k > j .$$

From equations (7), (8), (9), and (10) it is easy to see that $\partial_{x_k} a_{i,j}(x,y-x)$ are continuous on $\bar{S}(x^*,r)$ for all $i,j,k = 1,\ldots,n$. Thus $\underline{\partial}_1 a_{i,j}(x,y-x)$ exists and is continuous on $\bar{S}(x^*,r)$ with the components given by equations (7), (8), (9), and (10).

Since F is twice continuously differentiable on D, there is some constant α_2 such that

$$(11) \qquad \|F'(x)-F'(y)\| \le \alpha_2\|x-y\|$$

for all $x,y \in \bar{S}(x^*,r)$. Now consider the i,jth component of

$$J(x,y-x) - J(x^*,y-x^*)$$

where $i < j$ and $x_j \neq y_j \neq x_j^*$. Then using the integral form of the Mean Value Theorem and Lipschitz condition (11) we have

$$\left| \frac{f_i(x(j)+(y_j-x_j)e_j)-f_i(x(j))}{y_j-x_j} - \frac{f_i(x^*+\beta\sum_{\ell=1}^{j-1}(y_\ell-x_\ell^*)e_\ell+(y_j-x_j^*)e_j)-f_i(x^*+\beta\sum_{\ell=1}^{j-1}(y_\ell-x_\ell^*)e_\ell)}{y_j-x_j^*} \right|$$

$$= \left| \int_0^1 \Big[\partial_j f_i(x(j)+t(y_j-x_j)e_j) - \partial_j f_i(x^*+\beta\sum_{\ell=1}^{j-1}(y_\ell-x_\ell^*)e_\ell+t(y_j-x_j^*)e_j)\Big]dt \right|$$

$$\le \int_0^1 \|x-\beta\sum_{\ell=1}^{j-1}x_\ell e_\ell-tx_je_j-x^*+\beta\sum_{\ell=1}^{j-1}x_\ell^*e_\ell+tx_j^*e_j\|dt \le \left(\frac{3}{2}+\beta\right)\|x-x^*\|$$

In an analogous fashion we may obtain a similar inequality when $i = j$ or $i > j$; it is even simpler to obtain the same type of inequality for the case $x_j = y_j$ and all $i,j = 1,\dots,n$. Thus we have for some constant α_3

$$(12) \qquad \|J(x,y-x) - J(x^*,y-x^*)\| \le \alpha_3\|x-x^*\|$$

for all $x,y \in \bar{S}(x^*,r)$. Now since $J^{-1}(x,y-x)$ exists and is continuous for all $x,y \in \bar{S}(x^*,r)$, we have from the Lipschitz condition (12) that for some constant α_4

$$\begin{aligned} &\|J^{-1}(x,y-x)-J^{-1}(x^*,y-x^*)\| \\ &\qquad \le \|J^{-1}(x,y-x)\| \ \|J(x^*,y-x^*)-J(x,y-x)\| \ \|J^{-1}(x^*,y-x^*)\| \\ (13) \qquad &\qquad \le \alpha_4\|x-x^*\| \end{aligned}$$

for all $x,y \in \bar{S}(x^*,r)$.

Then from equation (5) and the fact that $F(x^*) = 0$ we have

$$\begin{aligned} \|\underline{\partial}_1 G(x,y)h-\underline{\partial}_1 G(x^*,y)h\| &= \|-\underline{\partial}_1[J^{-1}(x,y-x)]hF(x)-J^{-1}(x,y-x)F'(x)h \\ &\quad + J^{-1}(x^*,y-x^*)F'(x^*)h\| \\ (14) \qquad &\le \|-\underline{\partial}_1[J^{-1}(x,y-x)]h\|\|F(x)-F(x^*)\| \\ &\quad + \|J^{-1}(x,y-x)-J^{-1}(x^*,y-x^*)\| \ \|F'(x^*)\| \ \|h\| \\ &\quad + \|J^{-1}(x^*,y-x^*)\| \ \|F'(x^*)-F'(x)\| \ \|h\| \ . \end{aligned}$$

Now using the fact that $\underline{\partial}_1[J^{-1}(x,y-x)]$ is continuous, we may bound $\underline{\partial}_1[J^{-1}(x,y-x)]$ on $\bar{S}(x^*,r)$; likewise, we may bound $F'(x)$ and $J^{-1}(x^*,y-x^*)$ on $\bar{S}(x^*,r)$. Then using Lipschitz conditions (11) and (13) in inequality (14) we have for some constant L_1

$$\|\underline{\partial}_1 G(x,y)h-\underline{\partial}_1 G(x^*,y)h\| \le L_1\|x-x^*\| \ \|h\|$$

for all $x,y \in \bar{S}(x^*,r)$.

Now we turn our attention to the Lipschitz condition (4). In what follows we shall need an expression for $\underline{\partial}_2 J(x,y-x)$. By means of the technique used to obtain equations (7), (8), (9), and (10), we have that the k,jth component of the ith block of $\underline{\partial}_2 J(x,y-x)$ is given by

$$(15)\ \partial_{y_k} a_{i,j}(x,y-x) = \begin{cases} \dfrac{[\partial_k f_i(x(j)+(y_j-x_j)e_j)-\partial_k f_i(x(j))]}{y_j-x_j}\,\beta\ , & \text{for } k<j \\[2ex] \dfrac{\partial_j f_i(x(j)+(y_j-x_j)e_j)}{y_j-x_j} - \dfrac{f_i(x(j)+(y_j-x_j)e_j)-f_i(x(j))}{(y_j-x_j)^2}\ , & \text{for } k=j \\[2ex] 0\ , & \text{for } k>j \end{cases}$$

where $x_j \neq y_j$, and

$$(16) \quad \partial_{y_k} a_{i,j}(x,y-x) = \begin{cases} \partial_k \partial_j f_i(x(j))\beta \ , & \text{for } k < j \\ \frac{1}{2} \partial_j^2 f_i(x(j)) \ , & \text{for } k = j \\ 0 \ , \text{ for } k > j & \end{cases}$$

where $x_j = y_j$. It is clear that $\partial_{y_k} a_{i,j}(x,y-x)$ are continuous on $\bar{S}(x^*,r)$ for all $i,j,k = 1,\ldots,n$. Hence, $\underline{\partial}_2 J(x,y-x)$ exists and is continuous on $\bar{S}(x^*,r)$ with the components given by equations (15) and (16).

Now as in equation (6) we have

$$(17) \quad \underline{\partial}_2[J^{-1}(x,y-x)]h = J^{-1}(x,y-x)\underline{\partial}_2 J(x,y-x)hJ^{-1}(x,y-x)$$

so that $\underline{\partial}_2[J^{-1}(x,y-x)]$ is continuous on $\bar{S}(x^*,r)$. Let $\varepsilon > 0$ be arbitrary and choose $\|k\| < r$. Then from equation (5) and the fact that $\underline{\partial}_2[J^{-1}(x^*,y-x^*)]$ exists, we have

$$\|\underline{\partial}_1 G(x^*,y+k)h - \underline{\partial}_1 G(x^*,y)h + \underline{\partial}_2[J^{-1}(x^*,y-x^*)]kF'(x^*)h\|$$

$$= \|-J^{-1}(x^*,y+k-x^*)F'(x^*)h + J^{-1}(x^*,y-x^*)F'(x^*)h$$

$$+ \underline{\partial}_2[J^{-1}(x^*,y-x^*)]kF'(x^*)h\|$$

$$\leq \|J^{-1}(x^*,y+k-x^*) - J^{-1}(x^*,y-x^*) - \underline{\partial}_2[J^{-1}(x^*,y-x^*)]k\|$$

$$\cdot \ \|F'(x^*)\| \ \|h\| \leq \varepsilon\|k\| \ \|F'(x^*)\| \ \|h\|$$

Thus since ε was arbitrary we have that

$$(18) \qquad \underline{\partial}_2\underline{\partial}_1 G(x^*,y)hk = -\underline{\partial}_2[J^{-1}(x^*,y-x^*)]k\, F'(x^*)h$$

for all $y \in \bar{S}(x^*,r)$.

Now by means of a technique analogous to the one used to obtain the Lipschitz condition (12), using relation (2) and expressions (15) and (16) we obtain for some constant α_5

$$(19) \qquad \|\underline{\partial}_2 J(x^*,y-x^*) - \underline{\partial}_2 J(x^*,0)\| \leq \alpha_5 \|y-x^*\|$$

for all $y \in \bar{S}(x^*,r)$. Then from equation (17) we have

$$\|\underline{\partial}_2[J^{-1}(x^*,y-x^*)]h - \underline{\partial}_2[J^{-1}(x^*,0)]h\|$$

$$= \| -J^{-1}(x^*,y-x^*) \quad \underline{\partial}_2 J(x^*,y-x^*)h\, J^{-1}(x^*,y-x^*)$$

$$+ J^{-1}(x^*,0) \;\underline{\partial}_2 J(x^*,0)h\, J^{-1}(x^*,0)\|$$

$$(20) \qquad = \|[J^{-1}(x^*,y-x^*) - J^{-1}(x^*,0)]\;\underline{\partial}_2 J(x^*,y-x^*)h\, J^{-1}(x^*,y-x^*)$$

$$- J^{-1}(x^*,0)\;\{[\underline{\partial}_2 J(x^*,0)-\underline{\partial}_2 J(x^*,y-x^*)]h\, J^{-1}(x^*,0)$$

$$- \underline{\partial}_2 J(x^*,y-x^*)[J^{-1}(x^*,y-x^*)-J^{-1}(x^*,0)]\}\|.$$

Then by means of the technique used to obtain the Lipschitz condition (13) we have for some constant α_6

$$(21) \qquad \|J^{-1}(x^*,y-x^*) - J^{-1}(x^*,0)\| \leq \alpha_6 \|y-x^*\|$$

for all $y \in \bar{S}(x^*,r)$. Furthermore, using Lipschitz conditions (19) and (21) in equation (2) and bounding the continuous functions $\underline{\partial}_2 J(x^*,y-x^*)$, $J^{-1}(x^*,y-x^*)$, and $J^{-1}(x^*,0)$ on $\bar{S}(x^*,r)$ gives for some constant α_7

$$(22) \qquad \|\underline{\partial}_2[J^{-1}(x^*,y-x^*)]h-\underline{\partial}_2[J^{-1}(x^*,0)]h\| \le \alpha_7\|y-x^*\|$$

for all $y \in \bar{S}(x^*,r)$. Then using expression (18) we have

$$(23) \qquad \|\underline{\partial}_2\underline{\partial}_1 G(x^*,y)hk-\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)hk\|$$
$$\le \|-\underline{\partial}_2[J^{-1}(x^*,y-x^*)]k+\underline{\partial}_2[J^{-1}(x^*,0)]k\| \; \|F'(x^*)\| \; \|h\| .$$

Or from Lipschitz condition (22) we have for some constant L_2

$$\|\underline{\partial}_2\underline{\partial}_1 G(x^*,y)hk-\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)hk\| \le L_2\|y-x^*\| \; \|h\| \; \|k\| ,$$

for all $y \in \bar{S}(x^*,r)$. Q.E.D.

Now we return to the two-point secant method given by relation (1). Using the Lipschitz condition (3), the fact that $\underline{\partial}_2\underline{\partial}_1 G(x^*,y)$ is continuous and, therefore, bounded on $\bar{S}(x^*,r)$, and the Taylor Expansion Theorem 1.1.3 we have for some constant L_3 that

$$\|\underline{\partial}_1 G(y^1,y^2)h-\underline{\partial}_1 G(x^*,x^*)h\| \le \|\underline{\partial}_1 G(y^1,y^2)h-\underline{\partial}_1 G(x^*,y^2)h\|$$

$$+ \|\underline{\partial}_1 G(x^*,y^2)h-\underline{\partial}_1 G(x^*,x^*)h\|$$

$$\le L_1\|y^1-x^*\| \; \|h\| + \int_0^1 \|\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*+t(y^2-x^*))\|$$

$$\cdot \; \|h\| \; \|y^2-x^*\|dt$$

$$\le L_3[\|y^1-x^*\| + \|y^2-x^*\|]\|h\| \; . \tag{24}$$

Then we could use Theorem 2.2.4 to conclude that the two-point secant method has local convergence and $O_R(I,x^*) \ge \frac{1+\sqrt{5}}{2}$. However, to obtain the Lipschitz condition (24) we would have to assume that F satisfies

$$\|F''(x)-F''(x^*)\| \le a\|x-x^*\|$$

and Ortega and Rheinboldt [1970] have obtained the same result assuming only that F' is Lipschitzian on some neighborhood of x^* . Thus as an application to the two-point secant method, our only interest in Theorem 2.2.4 is the statement about the precise order of convergence.

Unfortunately, we shall now see that the two-point secant method does not satisfy the condition $\underline{\partial}_m\underline{\partial}_1 G(x^*,\ldots,x^*)hk \ne 0$ for all non-zero $h,k \in R^n$ of Theorem 2.2.4. From equation (18) we have

$$\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)hk = -\underline{\partial}_2[J^{-1}(x^*,0)]kF'(x^*)h$$

where from equation (17)

$$\underline{\partial}_2[J^{-1}(x^*,0)]k = -F'(x^*)^{-1}\underline{\partial}_2 J(x^*,0)kF'(x^*)^{-1} .$$

Thus we have

$$\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)hk = F'(x^*)^{-1}\underline{\partial}_2 J(x^*,0)kh$$

where $\underline{\partial}_2 J(x^*,0)$ is given by equation (16). If we now choose $k = e_n$ and $h = e_1 + \ldots + e_{n-1}$, it is easily verified that $\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)hk = 0$. Consequently, Theorem 2.2.4 cannot be applied to the two-point secant method to obtain the precise order of convergence.

It is interesting to note that Theorem 2.2.4 does apply to the one-dimensional secant method, for here

$$\partial_2\partial_1 G(x^*,x^*) = \frac{f''(x^*)}{2f'(x^*)} \quad \text{so if} \quad f''(x^*) \neq 0 , \ \partial_2\partial_1 G(x^*,x^*)hk \neq 0$$

for all non-zero $h,k \in R^1$. Hence the precise order of convergence for the one-dimensional secant method is $\frac{1+\sqrt{5}}{2}$. This result appears to have been overlooked although the one-dimensional method has been studied intensively.

At this point a comment about the precise order of convergence statement of Theorem 2.2.4 is in order.

The proof actually shows that if the conditions of the theorem are satisfied, then every sequence generated by the iterative procedure has the precise order τ. This is a stronger result than $O_R(I,x^*) = \tau$, for $O_R(I,x^*) = \tau$ when there is only one sequence $\{x^k\}$ for which $0 < R_\tau \{x^k\} < 1$. Thus the failure of the application of Theorem 2.․.4 simply means that every sequence generated by the two-point secant method may not converge with rate $\frac{1+\sqrt{5}}{2}$. In what follows we shall give a sufficient condition on F for G to generate one sequence which has rate precisely $\frac{1+\sqrt{5}}{2}$, but first we give a lemma pertaining to a recursive relation which will play an important role in the generation of that sequence.

Lemma 2.3.2. Let σ_k be a sequence of integers defined by $\sigma_{k+1} = \sigma_k + \sigma_{k-1} + 1$ where $\sigma_0 = 0$ and $\sigma_1 = 1$. Let $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Then

$$\sigma_k/\lambda_1^k < \frac{\lambda_2-2}{\lambda_2-\lambda_1}$$

for $k = 0,1,\ldots,$ and

$$\lim_{k\to\infty} \sigma_k/\lambda_1^k = \frac{\lambda_2-2}{\lambda_2-\lambda_1} .$$

Proof. For the homogeneous equation $\sigma_k - \sigma_{k-1} - \sigma_{k-2} = 0$, σ_k is given by $\alpha\lambda_1^k + \beta\lambda_2^k$ where α,β are constants.

Thus the non-homogeneous equation is satisfied by $\sigma_k = \alpha\lambda_1^k + \beta\lambda_2^k - 1$. Now using the initial conditions $\sigma_0 = 0$, $\sigma_1 = 1$ we find that $\alpha = \frac{\lambda_2 - 2}{\lambda_2 - \lambda_1}$ and $\beta = \frac{2-\lambda_1}{\lambda_2-\lambda_1}$.

Hence,

$$\frac{\sigma_k}{\lambda_1^k} = \frac{\lambda_2-2}{\lambda_2-\lambda_1} + \frac{2-\lambda_1}{\lambda_2-\lambda_1}\frac{\lambda_2^k}{\lambda_1^k} - \frac{1}{\lambda_1^k} .$$

Thus it will follow that

$$\frac{\sigma_k}{\lambda_1^k} < \frac{\lambda_2-2}{\lambda_2-\lambda_1}$$

for $k = 0,1,\ldots,$ if we show that

$$\frac{2-\lambda_1}{\lambda_2-\lambda_1}\lambda_2^k - 1 < 0$$

for $k = 0,1,\ldots$. But since $-1 < \lambda_2 < 0$,

$$\frac{2-\lambda_1}{\lambda_2-\lambda_1}\lambda_2^k < \frac{2-\lambda_1}{\lambda_2-\lambda_1}\lambda_2 = \frac{\sqrt{5}-2}{\sqrt{5}} < 1$$

for $k = 0,1,\ldots$. Furthermore, since $\left|\frac{\lambda_2}{\lambda_1}\right| < 1$, from equation (25) we have

$$\lim_{k\to\infty} \frac{\sigma_k}{\lambda_1^k} = \frac{\lambda_2-2}{\lambda_2-\lambda_1} . \quad \text{Q.E.D.}$$

We now give a sufficient condition for the two-point secant method to have precisely $\frac{1+\sqrt{5}}{2}$ order of convergence.

<u>Theorem 2.3.3</u>. Let $G: D\times D \subset R^n \times R^n \to R^n$ be the two-point secant iteration function given by relation (1), where $F: D \subset R^n \to R^n$ is twice continuously differentiable on the convex, compact set D satisfying for some constant α_1

$$\|F''(x)-F''(x^*)\| \le \alpha_1 \|x-x^*\|$$

for all $x \in D$. Assume that for some $x^* \in D$, $F(x^*) = 0$, $F'(x^*)$ is non-singular, and the following relations hold:

(26) $$F'(x^*)^{-1}[\partial_i^2 f_1(x^*),\ldots,\partial_i^2 f_n(x^*)]^T > 0$$

for all $i = 1,\ldots,n$, and

(27) $$\beta F'(x^*)^{-1}[\partial_i\partial_j f_1(x^*),\ldots,\partial_i\partial_j f_n(x^*)]^T \ge 0$$

for all $i,j = 1,\ldots,n$ with $i < j$. Then $O_R(I,x^*) = \frac{1+\sqrt{5}}{2}$.

<u>Proof</u>. It is clear that $x^* = G(x^*,y)$ for all $y \in D$. Choose $r' > 0$ satisfying Lemma 2.3.1. Then from relation (24) we have for some constant L_3 that

$$\|\underline{\partial}_1 G(y^1,y^2)h-\underline{\partial}_1 G(x^*,x^*)h\| \le L_3[\|y^1-x^*\| + \|y^2-x^*\|]\|h\|$$

for all $y^1, y^2 \in \bar{S}(x^*, r)$. We showed in the proof of Lemma 2.3.1 that

$$\underline{\partial}_1 G(x,y)h = h - J^{-1}(x,y-x)\underline{\partial}_1 J(x,y-x)hJ^{-1}(x,y-x)F(x)$$
$$- J^{-1}(x,y-x)F'(x)h ;$$

therefore, since $J(x^*,0) \equiv F'(x^*)$ we have $\underline{\partial}_1 G(x^*,x^*)h = 0$. Thus the conditions of the first part of Theorem 2.2.4 are satisfied so that x^* is a point of attraction of the iteration (1) and $O_R(I,x^*) \geq \lambda_1$ where $\lambda_1 = \frac{1+\sqrt{5}}{2}$ is the unique positive root of $t^2 - t - 1 = 0$.

We shall now generate a sequence which converges with rate precisely $\frac{1+\sqrt{5}}{2}$. Since x^* is a point of attraction there is an r satisfying $0 < r \leq r'$ such that for any $x^o, x^1 \in S(x^*, r)$, the sequence $\{x^k\}$ generated by G converges to x^*; furthermore, r may be chosen so that if

$x, y \in \bar{S}(x^*, r)$, then $x + \beta \sum_{k=1}^{j-1} (y_k - x_k)e_k + (y_j - x_j)e_j \in D$ for

any $j = 1,\ldots,n$. Then from Lemma 2.3.1 we have that

(28) $\|\underline{\partial}_1 G(x^*+t(y-x^*),y)h - \underline{\partial}_1 G(x^*,y)h\| \leq L_1 t\|y-x^*\| \ \|h\|$

(29) $\|\underline{\partial}_2\underline{\partial}_1 (x^*,x^*+t(y-x^*))hk - \underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)hk\| \leq L_2 t\|y-x^*\|\|h\|\|k\|$

for all $y \in \bar{S}(x^*, r)$.

Now the ℓth component of $\partial_2 \partial_1 G(x^*, x^*)hk$ is given by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{\ell,k} b_{k,j,i} k_i h_j$$ where we have set $F'(x^*)^{-1} = (a_{\ell,k})$

and $\partial_2 J(x^*, 0) = (b_{k,j,i})$ where k denotes the block and j and i the row and column, respectively, within the kth block. Then using the expression for $\partial_2 J(x^*, 0)$ given by equation (16) and condition (26) of the theorem, it is easy to see that all of the coefficients of $h_i k_i$, $i = 1, \ldots, n$ in each component of $\partial_2 \partial_1 G(x^*, x^*)hk$ are strictly positive; therefore, if

$$2\alpha = \min\left\{ \min_{1 \le \ell, i \le n} \sum_{k=1}^{n} a_{\ell,k} b_{k,i,i}, 1 \right\}$$ then we have $0 < \alpha < 1$.

Now set

$$\xi = \max(L_1, L_2, 1) ,$$

$$\eta = \min\left\{ \frac{\alpha}{2\xi} \frac{(1/2\alpha)^{\frac{\lambda_2 - 2}{\lambda_2 - \lambda_1}}}{(2\xi)^{2\left(\frac{\lambda_2 - 2}{\lambda_2 - \lambda_1}\right)}} , r^{\frac{1}{\lambda_1 - 1}} \right\}, \quad \lambda_1 = \frac{1 + \sqrt{5}}{2} , \quad \lambda_2 = \frac{1 - \sqrt{5}}{2} ,$$

and notice that $\eta < 1$. Choose $\gamma > 0$ so that $2\xi\gamma < 1$ and $\gamma^{\lambda_1 - 1} < \eta$; and notice that $\gamma < r$ and $\gamma^{\lambda_1} < \eta\gamma$.

At this point we are ready to generate the desired sequence. Choose x^o so that $\gamma = x_i^o - x_i^*$, $i = 1,\ldots,n$; choose x^1 so that $\frac{1}{2}\alpha\gamma^{\lambda_1} < x_i^1 - x_i^* < \gamma^{\lambda_1}$, $i = 1,\ldots,n$. We shall now show by induction that

$$x_\ell^k - x_\ell^* > (\tfrac{1}{2}\alpha)^{\sigma_k} \gamma^{\lambda_1^k} , \text{ for all } \ell = 1,\ldots,n , \tag{30}$$

$$\|x^k - x^*\| < (2\xi)^{\sigma_k} \gamma^{\lambda_1^k} \tag{31}$$

and

$$\|x^k - x^*\| \le \|x^{k-1} - x^*\| \tag{32}$$

for all k where σ_k satisfies the recurrence relation $\sigma_{k+1} = \sigma_k + \sigma_{k-1} + 1$ with initial conditions $\sigma_0 = 0$, $\sigma_1 = 1$. Now (30), (31), and (32) are satisfied for $k = 0,1$ where $\|x\| = \|x\|_\infty$. Assume (30), (31), and (32) hold for $p = 0,1,\ldots,k$. Then by Taylors Expansion Theorem 1.1.3 we have, since $\underline{\partial}_1 G(x^*,x^*) = 0$,

$$\begin{aligned} x^{k+1} - x^* &= G(x^k,x^{k-1}) - G(x^*,x^{k-1}) \\ &= [\underline{\partial}_1 G(x^*,x^{k-1}) - \underline{\partial}_1 G(x^*,x^*)](x^k - x^*) \\ &\quad + \int_0^1 [\underline{\partial}_1 G(x^* + t(x^k - x^*),x^{k-1}) - \underline{\partial}_1 G(x^*,x^{k-1})]\,(x^k - x^*)dt \end{aligned} \tag{33}$$

$$= \underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)(x^{k-1}-x^*)(x^k-x^*)$$

$$+ \int_0^1 [\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*+t(x^{k-1}-x^*)) - \underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)](x^{k-1}-x^*)(x^k-x^*)\,dt$$

$$+ \int_0^1 [\underline{\partial}_1 G(x^*+t(x^k-x^*),x^{k-1}) - \underline{\partial}_1 G(x^*,x^{k-1})](x^k-x^*)\,dt\,.$$

We have already noted that the ℓth component of $\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)(x^{k-1}-x^*)(x^k-x^*)$ is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{n} a_{\ell,p} b_{p,j,i}(x_i^k-x_i^*)(x_j^{k-1}-x_j^*)\ ;$$

so since every component of x^k-x^* and $x^{k-1}-x^*$ is strictly positive, and since by condition (27) and the representation (16) for $\underline{\partial}_2 J(x^*,0)$, we have $\sum_{p=1}^{n} a_{\ell,p} b_{p,j,i} \geq 0$ for all $\ell,j,i = 1,\ldots,n$, $j \neq i$, then we have that every component of $\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)(x^{k-1}-x^*)(x^k-x^*)$ is strictly greater than $\alpha(x_i^{k-1}-x_i^*)(x_i^k-x_i^*)$ for all i ; that is,

$$[\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)(x^{k-1}-x^*)(x^k-x^*)]_\ell > \alpha(x_i^{k-1}-x_i^*)(x_i^k-x_i^*)$$

for $i, \ell = 1,\ldots,n$. Therefore, if i is chosen so that

$\|x^k-x^*\| = x_i^k - x_i^*$, using inequalities (28) and (29) in relation (33) we may write

$$x_\ell^{k+1}-x_\ell^* > \alpha(x_i^{k-1}-x_i^*)(x_i^k-x_i^*) - L_2/2\|x^{k-1}-x^*\|^2\|x^k-x^*\|$$
$$- L_1/2\|x^k-x^*\|^2$$

for $\ell = 1,\ldots,n$. Then using the induction hypotheses and the fact that $\lambda_1^{k+1} = \lambda_1^k + \lambda_1^{k-1}$ we have

$$x_\ell^{k+1}-x_\ell^* > \alpha\left(\frac{1}{2}\alpha\right)^{\sigma_{k-1}} \gamma^{\lambda_1^{k-1}} \left(\frac{1}{2}\alpha\right)^{\sigma_k} \gamma^{\lambda_1^k}$$
$$\cdot\left[1 - \frac{\xi}{2\alpha}\frac{\|x^{k-1}-x^*\|^2}{x_i^{k-1}-x_i^*} - \frac{\xi}{2\alpha}\frac{\|x^k-x^*\|}{x_i^{k-1}-x_i^*}\right]$$
$$> \alpha\left(\frac{1}{2}\alpha\right)^{\sigma_k+\sigma_{k-1}} \gamma^{\lambda_1^{k+1}}$$
$$\cdot\left[1 - \frac{\xi}{2\alpha}\left(\frac{(2\xi)^{2\sigma_{k-1}}\gamma^{2\lambda_1^{k-1}}}{\left(\frac{1}{2}\alpha\right)^{\sigma_{k-1}}\gamma^{\lambda_1^{k-1}}} + \frac{(2\xi)^{\sigma_k}\gamma^{\lambda_1^k}}{\left(\frac{1}{2}\alpha\right)^{\sigma_{k-1}}\gamma^{\lambda_1^{k-1}}}\right)\right].$$

Then, since $\gamma^2 < \gamma^{\lambda_1} < \eta\gamma$ and hence, in particular, $\gamma^{\lambda_1^k} < (\eta\gamma)^{\gamma_1^{k-1}}$, we have

$$x_\ell^{k+1}-x_\ell^* > \alpha\left(\frac{1}{2}\,\alpha\right)^{\sigma_k+\sigma_{k-1}\,\gamma^{\lambda_1^{k+1}}}$$

$$\cdot\left\{1-\frac{\xi}{2\alpha}\left[\left(\frac{(2\xi)^{\frac{2\sigma_{k-1}}{\lambda_1^{k-1}}\eta\gamma}}{\left(\frac{1}{2}\,\alpha\right)^{\frac{\sigma_{k-1}}{\lambda_1^{k-1}}\gamma}}\right)^{\lambda_1^{k-1}}+\left(\frac{(2\xi)^{\frac{\sigma_k}{\lambda_1^{k-1}}\eta\gamma}}{\left(\frac{1}{2}\,\alpha\right)^{\frac{\sigma_{k-1}}{\lambda_1^{k-1}}\gamma}}\right)^{\lambda_1^{k-1}}\right]\right\}$$

Now we have shown in Lemma 2.3.2 that

$$\frac{\sigma_j}{\lambda_1^j} < \frac{\lambda_2-2}{\lambda_2-\lambda_1}$$

for all $j = 0,1,\ldots$. Thus using the definition of η we have

$$x_\ell^{k+1}-x_\ell^* > \alpha\left(\frac{1}{2}\,\alpha\right)^{\sigma_k+\sigma_{k-1}\,\gamma^{\lambda_1^{k+1}}}$$

$$\cdot\left\{1-\frac{\xi}{2\alpha}\left[\left(\frac{(2\xi)^{2\frac{\lambda_2-2}{\lambda_2-\lambda_1}\eta}}{\left(\frac{1}{2}\,\alpha\right)^{\frac{\lambda_2-2}{\lambda_2-\lambda_1}}}\right)^{\lambda_1^{k-1}}+\left(\frac{(2\xi)^{\lambda_1\frac{\lambda_2-2}{\lambda_2-\lambda_1}\eta}}{\left(\frac{1}{2}\,\alpha\right)^{\frac{\lambda_2-2}{\lambda_2-\lambda_1}}}\right)^{\lambda_1^{k-1}}\right]\right\}$$

$$> \alpha\left(\frac{1}{2}\alpha\right)^{\sigma_k+\sigma_{k-1}} \gamma^{\lambda_1^{k+1}} \left\{1-\left[\frac{\xi}{2\alpha}\left(\frac{\alpha}{2\xi}\right)^{\lambda_1^{k-1}} + \left(\frac{\alpha}{2\xi}\right)^{\lambda_1^{k-1}}\right]\right\}$$

$$> \left(\frac{1}{2}\alpha\right)^{\sigma_k+\sigma_{k-1}+1} \gamma^{\lambda_1^{k+1}} = \left(\frac{1}{2}\alpha\right)^{\sigma_{k+1}} \gamma^{\lambda_1^{k+1}} ,$$

for $\ell = 1,\ldots,n$, which is (30) with k replaced by $k + 1$.

Now from equation (33) we see that

$$\|x^{k+1}-x^*\| \le \xi\|x^{k-1}-x^*\| \; \|x^k-x^*\| + \xi/2 \; \|x^k-x^*\|^2$$

or, using induction hypothesis (32), that

$$(34) \quad \|x^{k+1}-x^*\| \le 2\xi\|x^{k-1}-x^*\| \; \|x^k-x^*\| \left(\frac{1}{2}+\frac{1}{4}\right)$$

Then using the induction hypothesis (31) we have

$$\|x^{k+1}-x^*\| < 2\xi(2\xi)^{\sigma_{k-1}} \gamma^{\lambda_1^{k-1}} (2\xi)^{\sigma_k} \gamma^{\lambda_1^k} = (2\xi)^{\sigma_{k+1}} \gamma^{\lambda_1^{k+1}} .$$

Furthermore, since

$$2\xi\|x^{k-1}-x^*\| \le 2\xi\gamma < 1$$

we have from relation (34) that $\|x^{k+1}-x^*\| < \|x^k-x^*\|$.
This completes the induction argument so that relations (30), (31), and (32) hold for all k .

We shall now show that the sequence $\{x^k\}$ has the precise rate of $\frac{1+\sqrt{5}}{2}$. By relation (30) and Lemma 2.3.2 it follows that

$$R_{\lambda_1}\{x^k\} = \limsup_{k\to\infty} \|x^k - x^*\|^{1/\lambda_1^k}$$

$$> \limsup_{k\to\infty} \left(\frac{1}{2}\alpha\right)^{\sigma_k/\lambda_1^k} \gamma$$

$$> \left(\frac{1}{2}\alpha\right)^{\frac{\lambda_2 - 2}{\lambda_2 - \lambda_1}} \gamma > 0$$

Therefore, since we have already seen that for any sequence $\{x^k\}$, $R_{\lambda_1}\{x^k\} < \infty$, the precise rate of $\{x^k\}$ is $\frac{1+\sqrt{5}}{2}$. Hence $O_R(I,x^*) = \frac{1+\sqrt{5}}{2}$. Q.E.D.

In the case that conditions (26) and (27) are satisfied by the relations < 0 and ≤ 0, respectively, a completely analogous proof may be given. The only change is that the starting values x^0, x^1 must satisfy $x_i^0 - x_i^* = -\gamma$ and $-\gamma^{\lambda_1} < x_i^1 - x_i^* < -|\frac{1}{2}\alpha\gamma^{\lambda_1}|$ for $i = 1,\ldots,n$, where α satisfies $2\alpha = -\min\left\{\min_{1\leq \ell, i\leq n} |\sum_{k=1}^{n} a_{\ell,k} b_{k,j,i}|, 1\right\}$. The induction argument then proceeds as before with the appropriate changes in relations (30) and (31).

Note that condition (27) is automatically satisfied if $\beta = 0$. Thus the sufficient conditions for the precise order of convergence of the two-point secant method is much less restrictive on F if the particular method in question is given by $\beta = 0$.

In Section 5 of this chapter we shall give a condition on F in which equality is assumed in condition (26). This will be shown to be a sufficient condition for quadratic convergence of the two-point secant method. Thus it would seem to be difficult to significantly weaken condition (26) and still obtain the precise rate $\frac{1+\sqrt{5}}{2}$.

It should be pointed out that it is possible to give meaningful problems for which condition (26) is satisfied. For example, standard discretizations of

$$\Delta u = e^{u}$$

where Δ is the Laplacian operator lead to systems for which $F'(x^*)$ has a non-negative inverse and $F''(x^*) \geq 0$, with the diagonal terms of each block of $F''(x^*)$ positive. For further details the reader is referred to Greenspan and Parter [1965] or Ortega and Rockoff [1966].

Section 2.4

Additional Order of Convergence Theorems for Multi-Step Methods

In this section we shall give theorems analogous to those given in Section 2.2; however, by imposing stronger conditions we shall obtain higher orders of convergence than those obtained in Section 2.2. The first theorem is similar to Theorem 2.2.2.

Theorem 2.4.1. Let I be an iterative process with limit x^* and let $\gamma_j, j = 0,1,\ldots,m$ be non-negative constants. If for every sequence $\{x^k\}$ generated by I with $\lim_{k\to\infty} x^k = x^*$ there is a k_0 depending on $\{x^k\}$ such that

$$(1) \quad \|x^{k+1}-x^*\| \le \gamma_0\|x^k-x^*\|^2 + \|x^k-x^*\| \sum_{j=1}^{m} \gamma_j\|x^{k-j}-x^*\|^2$$

for all $k \ge k_0$, then $O_R(I,x^*) \ge \tau$ where τ is the unique positive root of $t^{m+1} - t^m - 2 = 0$. In addition, if there is a $\gamma > 0$, a k_0 and some sequence $\{x^k\}$ generated by I with $\lim_{k\to\infty} x^k = x^*$ such that $\|x^k-x^*\| > 0$ for all

$k \ge k_0 - m$ and

(2) $$\|x^{k+1}-x^*\| \geq \gamma\|x^{k-m}-x^*\|^2\|x^k-x^*\|$$

for all $k \geq k_0$, then $O_R(I,x^*) = \tau$.

Proof. Assume that $\{x^k\}$ is any sequence generated by I converging to x^*, that (1) holds for $k_0 = m$ and that $\|x^k-x^*\| \leq \beta^2$ for $k = 0,\ldots,m$, where β satisfies

$$\gamma_0\beta + \sum_{j=1}^{m} \gamma_j\beta^3 < 1 \text{ and } 0 < \beta < 1 . \text{ Set } \epsilon_k = \|x^k-x^*\| .$$

We shall show by induction that

(3) $$\epsilon_{k+1} \leq \beta^{\sigma_{k+1}}$$

where σ_{k+1} satisfies $\sigma_{k+1} = \sigma_k + 2\sigma_{k-m} - 3$ with $\sigma_i = 2$ for $i = 0,\ldots,m$. In the argument we shall need the fact that $1 \leq 2\sigma_k - \sigma_{k+1}$ for all k which we now verify by induction. Clearly $\sigma_{m+1} = 3$ so that $1 \leq 2\sigma_m - \sigma_{m+1}$. Assume that $1 \leq 2\sigma_p - \sigma_{p+1}$ for $p = 0,1,\ldots k-1$. Since $\sigma_{k+1} = \sigma_k + 2\sigma_{k-m} - 3$, $\sigma_{k-m} \leq \sigma_{k-1}$, and $\sigma_{k-m} \leq 2\sigma_{k-m-1} - 1$ we have

$$\sigma_{k+1} \leq \sigma_k + \sigma_{k-1} + 2\sigma_{k-m-1} - 3 - 1 = 2\sigma_k - 1$$

which verifies that $1 \leq 2\sigma_k - \sigma_{k+1}$ for all k.

Now we return to the verification of relation (3).

By hypothesis

$$\epsilon_{m+1} \le \gamma_0 \epsilon_m^2 + \epsilon_m \sum_{j=1}^{m} \gamma_j \epsilon_{m-j}^2$$

$$\le \gamma_0 \beta^4 + \beta^6 \sum_{j=1}^{m} \gamma_j$$

$$\le \beta^3 \quad \text{since} \quad \gamma_0 \beta + \sum_{j=1}^{m} \gamma_j \beta^3 < 1 \ .$$

Then since $\sigma_{m+1} = 3$ we have $\epsilon_{m+1} \le \beta^{\sigma_{m+1}}$. Now assume that $\epsilon_p \le \beta^{\sigma_p}$ for all $p = 0,\ldots,k$. Then

$$\epsilon_{k+1} \le \gamma_0 \epsilon_k^2 + \epsilon_k \sum_{j=1}^{m} \gamma_j \epsilon_{k-j}^2$$

$$\le \gamma_0 \beta^{2\sigma_k} + \beta^{\sigma_k} \sum_{j=1}^{m} \gamma_j \beta^{2\sigma_{k-j}}$$

$$\le \gamma_0 \beta^{2\sigma_k} + \beta^{\sigma_k} \beta^{2\sigma_{k-m}} \sum_{j=1}^{m} \gamma_j$$

$$= \beta^{\sigma_{k+1}} \left[\gamma_0 \beta^{2\sigma_k - \sigma_{k+1}} + \beta^3 \sum_{j=1}^{m} \gamma_j \right]$$

$$\le \beta^{\sigma_{k+1}} \quad \text{since} \quad 2\sigma_k - \sigma_{k+1} \ge 1 \ .$$

We next show that the σ_k satisfy

(4) $$\sigma_k \geq \alpha\tau^k + 3/2$$

where $\alpha = \frac{1}{2}\tau^{-m}$ and τ , the unique positive root of $t^{m+1} - t^m - 2 = 0$, is well defined by Lemma 2.2.1 and satisfies $1 < \tau \leq 2$. Note that for $k = 0,\ldots,m$, relation (4) holds since $\tau > 1$. Assume that (4) holds for $p = 0,\ldots,k-1$. Then

$$\begin{aligned}
\sigma_k &= \sigma_{k-1} + 2\sigma_{k-m-1} - 3 \\
&\geq \alpha\tau^{k-1} + 3/2 + 2\alpha\tau^{k-m-1} + 3 - 3 \\
&\geq \alpha\tau^{k-m-1}[\tau^m - 2] + 3/2 \\
&\geq \alpha\tau^k + 3/2 .
\end{aligned}$$

Now for any sequence $\{x^k\}$ generated by I with $\lim_{k\to\infty} x^k = x^*$ and satisfying relation (1), there is a k_0' such that relation (3) holds for all $k \geq k_0'$. Thus from relation (4) we have

$$\limsup_{k\geq k_0'} \epsilon_k^{1/\tau^k} \leq \limsup_{k\geq k_0'} \beta^{\alpha + \frac{3}{2\tau^k}} .$$

Therefore, $\limsup_{k\to\infty} \|x^k - x^*\|^{1/\tau^k} \leq \beta^\alpha < 1$ and we have that $O_R(I,x^*) \geq \tau$.

Now suppose that (2) holds for some sequence $\{x^k\}$ generated by I with $\epsilon_k > 0$ for all $k \geq k_0 - m$ and $\lim_{k\to\infty} x^k = x^*$. As is clear from the first part we may assume without loss of generality that $k_0 = m$ and $\epsilon_k \leq 1/\gamma^{\frac{1}{2}}$ for all k. Set $\eta_k = \gamma^{\frac{1}{2}}\epsilon_k$ and $\eta = \min(\eta_0,\ldots,\eta_m)$; then $0 < \eta < 1$. We shall show by induction that

$$(5) \qquad \eta_k \geq \eta^{\mu_k}$$

where μ_k satisfies $\mu_{k+1} = \mu_k + 2\mu_{k-m}$ with $\mu_k = 1$ for $k = 0,\ldots,m$. Clearly (5) holds for $k = 0,\ldots,m$. Assume that (5) holds for $p = 0,\ldots,k-1$. Then

$$\begin{aligned}\eta_k = \gamma^{\frac{1}{2}}\epsilon_k &\geq \gamma^{\frac{1}{2}}\gamma\epsilon_{k-1}\epsilon_{k-m-1}^2 \\ &= \eta_{k-1}\eta_{k-m-1}^2 \\ &\geq \eta^{\mu_{k-1}+2\mu_{k-m-1}} = \eta^{\mu_k}\end{aligned}$$

and we have that (5) holds for all k. Since $\tau > 1$ $\mu_i \leq \tau^i$ for $i = 0,\ldots,m$. Assume that $\mu_p \leq \tau^p$ for $p = 0,\ldots,k-1$. Then

$$\mu_k = \mu_{k-1} + 2\mu_{k-m-1} \geq \tau^{k-1} + 2\tau^{k-m-1} = \tau^k .$$

Thus $\mu_k \le \tau^k$ for all k. Hence $\eta_k \ge \eta^{\tau^k}$; therefore,

$$\limsup_{k\to\infty} \|x^k - x^*\|^{1/\tau^k} = \limsup_{k\to\infty} \left(\frac{1}{\gamma^{\frac{1}{2}}}\eta_k\right)^{1/\tau^k} \ge \eta > 0$$

Thus $O_R(I,x^*) = \tau$. Q.E.D.

We are now in a position to give the counterpart to Theorem 2.2.4.

<u>Theorem 2.4.2</u>. Let I denote the iteration $x^{k+1} = G(x^k,\ldots,x^{k-m+1})$ where $G:D^m \to (R^n)^m$ with D an open subset of R^n . Assume that there is a point $x^* \in D$ such that $x^* = G(x^*,y^2,\ldots,y^m)$ for all $y^j \in D$, $j = 2,\ldots,m$, and that $\underline{\partial}_j\underline{\partial}_1 Gh^1h^j$ exist as vector partial differentials on D for $j = 2,\ldots,m$. Furthermore, assume that for all $y^j \in D$ $j = 1,\ldots,m$, $\|\underline{\partial}_1 G(y^1,\ldots,y^m)h - \underline{\partial}_1 G(x^*,y^2,\ldots,y^m)h\| \le \alpha\|y^1-x^*\|\ \|h\|$, that for all $y^i \in D$ $i = j,\ldots,m$,

$$\|\underline{\partial}_j\underline{\partial}_1 G(x^*,\ldots,x^*,y^j,\ldots,y^m)h^1h^j - \underline{\partial}_j\underline{\partial}_1 G(x^*,\ldots,x^*)h^1h^j\| \le \sum_{i=j}^{m} \alpha_i\|y^i-x^*\|\ \|h^1\|\ \|h^j\|$$

for $j = 2,\ldots,m$, and that $\underline{\partial}_1 G(x^*,\ldots,x^*) = \underline{\partial}_j\underline{\partial}_1 G(x^*,\ldots,x^*) = 0$ for $j = 2,\ldots,m$. Then x^* is a point of attraction of the procedure I and $O_R(I,x^*) \geq \tau$ where τ is the unique positive root of $t^m - t^{m-1} - 2 = 0$. Moreover, assume that $\underline{\partial}_1^2 G$ exists on D and is continuous at $(x^*,\ldots,x^*)$, that $\underline{\partial}_m^2\underline{\partial}_1 G$ exists at $(x^*,\ldots,x^*)$, that $\underline{\partial}_1^2 G(x^*,\ldots,x^*) \neq 0$, and that for some constant $c > 0$

$$\|\underline{\partial}_1 G(x^*,\ldots,x^*)h^1h^1+\underline{\partial}_m^2\underline{\partial}_1 G(x^*,\ldots,x^*)h^1h^mh^m\|$$

$$\geq c[\|h^1\|^2+\|h^1\|\ \|h^m\|^2]$$

for all $h^1, h^m \in R^n$. Then $O_R(I,x^*) = \tau$.

<u>Proof</u>. It is easy to see that $\underline{\partial}_j G(x^*,\ldots,x^*) = 0$ for $j = 1,\ldots,m$ so by Theorem 1.2.2, x^* is a point of attraction of the procedure I. Let $\{x^k\}$ be any sequence generated by the procedure which converges to x^*. According to Theorem 1.5.3, we may take k_0 large enough so that $\|x^{k-m+2}-x^*\| \leq \|x^{k-m+1}-x^*\| \leq r$ for all $k \geq k_0$ where $\overline{S}(x^*,r) \subset D$. Then by the Taylor Expansion Theorem 1.1.3

$$\|x^{k+1}-x^*\| = \|G(x^k,\ldots,x^{k-m+1})-G(x^*,x^{k-1},\ldots,x^{k-m+1})\|$$

$$\leq \|[\underline{\partial}_1 G(x^*,x^{k-1},\ldots,x^{k-m+1})-\underline{\partial}_1 G(x^*,\ldots,x^*)](x^k-x^*)\|$$

$$+ \|\int_0^1 [\underline{\partial}_1 G(x^*+t(x^k-x^*),x^{k-1},\ldots,x^{k-m+1})$$

$$- \underline{\partial}_1 G(x^*,x^{k-1},\ldots,x^{k-m+1})](x^k-x^*)dt\|$$

$$\leq \left\| \int_0^1 [\underline{\partial}_2\underline{\partial}_1 G(x^*, x^*+t(x^{k-1}-x^*), x^{k-2}, \ldots, x^{k-m+1})(x^{k-1}-x^*)\right.$$

$$\left. + \ldots + \underline{\partial}_m\underline{\partial}_1 G(x^*, \ldots, x^*, x^*+t(x^{k-m+1}-x^*))(x^{k-m+1}-x^*)](x^k-x^*)dt \right\|$$

$$+ \frac{\alpha}{2} \|x^k-x^*\|^2$$

$$= \left\| \int_0^1 [(\underline{\partial}_2\underline{\partial}_1 G(x^*, x^*+t(x^{k-1}-x^*), \ldots, x^{k-m+1})\right.$$

$$- \underline{\partial}_2\underline{\partial}_1 G(x^*, \ldots, x^*))(x^{k-1}-x^*)$$

$$+ \ldots + (\underline{\partial}_m\underline{\partial}_1 G(x^*, \ldots, x^*, x^*+t(x^{k-m+1}-x^*))$$

$$\left. - \underline{\partial}_m\underline{\partial}_1 G(x^*, \ldots, x^*))(x^{k-m+1}-x^*)](x^k-x^*)dt \right\| + \alpha/2\|x^k-x^*\|^2$$

$$\leq \Big\{ \Big[\frac{\alpha_2}{2} \|x^{k-1}-x^*\| + \alpha_3\|x^{k-2}-x^*\| + \ldots + \alpha_m\|x^{k-m+1}-x^*\|\Big]\|x^{k-1}-x^*\|$$

$$+ \Big[\frac{\alpha_3}{2} \|x^{k-2}-x^*\| + \alpha_4\|x^{k-3}-x^*\| + \ldots + \alpha_m\|x^{k-m+1}-x^*\|\Big]\|x^{k-2}-x^*\|$$

$$+ \ldots + \frac{\alpha_m}{2} \|x^{k-m+1}-x^*\| \, \|x^{k-m+1}-x^*\|\Big\}\|x^k-x^*\| + \alpha/2\|x^k-x^*\|^2 \, .$$

Using the fact that $\|x^{k-m+2}-x^*\| \leq \|x^{k-m+1}-x^*\|$ for all $k > k_0$ we have

$$\|x^{k+1}-x^*\| \leq \gamma_o\|x^k-x^*\|^2 + \|x^k-x^*\| \sum_{j=1}^{m-1} \gamma_j\|x^{k-j}-x^*\|^2$$

Thus by Theorem 2.4.1

$O_R(I,x^*) \geq \tau$ where τ is the unique positive root of $t^m - t^{m-1} - 2 = 0$.

Now let $\{x^k\}$ be any sequence generated by I converging to x^*, and assume that the hypotheses of the second part of the theorem are satisfied. Since $\partial_1^2 G$ is continuous at $(x^*,\ldots,x^*)$ there is a k_1 such that

$$(6) \qquad \sup_{0\le t\le 1} \|\partial_1^2 G(x^*+t(x^k-x^*),x^{k-1},\ldots,x^{k-m+1}) - \partial_1^2 G(x^*,\ldots,x^*)\| \leq C$$

for all $k \geq k_1$; since $\partial_m^2\partial_1 G$ exists at $(x^*,\ldots,x^*)$, there is a $k_2 \geq k_1$ such that

$$(7) \qquad \|\partial_m\partial_1 G(x^*,\ldots,x^*,x^*+t(x^{k-m+1}-x^*)) - \partial_m\partial_1 G(x^*,\ldots,x^*) - \partial_m^2\partial_1 G(x^*,\ldots,x^*)t(x^{k-m+1}-x^*)\| \leq \frac{C}{4}\, t\|x^{k-m+1}-x^*\|$$

for all $k \geq k_2$; and finally by Theorem 1.5.3, there is a $k_3 \geq k_2$ such that

$$(8) \qquad \|x^{k-j}-x^*\| \leq \delta\|x^{k-m+1}-x^*\|$$

for $j = 0,\ldots,m-2$ and all $k \geq k_3$ where δ is chosen so that

$$\delta \sum_{j=1}^{m-2} \gamma_j < \frac{c}{8} \tag{9}$$

Then, again using the Taylor Expansion Theorem 1.1.3, we have

$$\|x^{k+1}-x^*\| = \|[\underline{\partial}_1 G(x^*,x^{k-1},\ldots,x^{k-m+1})-\underline{\partial}_1 G(x^*,\ldots,x^*)](x^k-x^*)$$

$$+\int_0^1 (1-t)\underline{\partial}_1^2 G(x^*+t(x^k-x^*),x^{k-1},\ldots,x^{k-m+1})(x^k-x^*)^2 dt\|.$$

Adding and subtracting $t\underline{\partial}_m^2\underline{\partial}_1 G(x^*,\ldots,x^*)(x^{k-m+1}-x^*)^2(x^k-x^*)$ and $(1-t)\underline{\partial}_1^2 G(x^*,\ldots,x^*)(x^k-x^*)^2$, using the Taylor Expansion Theorem 1.1.3, and recalling that $\underline{\partial}_j\underline{\partial}_1 G(x^*,\ldots,x^*) = 0$, $j = 2,\ldots,m$, gives

$$\|x^{k+1}-x^*\| = \|\int_0^1 [(1-t)\underline{\partial}_1^2 G(x^*,\ldots,x^*)(x^k-x^*)^2$$

$$+t\underline{\partial}_m^2\underline{\partial}_1 G(x^*,\ldots,x^*)(x^{k-m+1}-x^*)^2(x^k-x^*)]dt$$

$$+\int_0^1 (1-t)[\underline{\partial}_1^2 G(x^*+t(x^k-x^*),x^{k-1},\ldots,x^{k-m+1})$$

$$-\underline{\partial}_1^2 G(x^*,\ldots,x^*)](x^k-x^*)^2 dt$$

$$+\int_0^1 \{[\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*+t(x^{k-1}-x^*),x^{k-2},\ldots,x^{k-m+1})$$

$$-\underline{\partial}_2\underline{\partial}_1 G(x^*,\ldots,x^*)](x^{k-1}-x^*)$$

$$+\ldots+[\underline{\partial}_{m-1}\partial_1 G(x^*,\ldots,x^*,x^*+t(x^{k-m+2}-x^*),x^{k-m+1})$$

$$-\underline{\partial}_{m-1}\underline{\partial}_1 G(x^*,\ldots,x^*)](x^{k-m+2}-x^*)\}(x^k-x^*)\,dt$$

$$+\int_0^1[\underline{\partial}_m\underline{\partial}_1 G(x^*,\ldots,x^*,x^*+t(x^{k-m+1}-x^*)-\underline{\partial}_m\underline{\partial}_1 G(x^*,\ldots,x^*)$$

$$-\underline{\partial}_m^2\underline{\partial}_1 G(x^*,\ldots,x^*)t(x^{k-m+1}-x^*)](x^{k-m+1}-x^*)(x^k-x^*)\,dt\| .$$

Now using the estimates (6), (7), and (8) and the Lipschitz conditions for $\underline{\partial}_j\underline{\partial}_1 G$ for $j = 2,\ldots,m-1$ we have

$$\|x^{k+1}-x^*\| \geq \frac{C}{2}[\|x^k-x^*\|^2+\|x^{k-m+1}-x^*\|^2\|x^k-x^*\|]$$

$$-\frac{C}{2}\|x^k-x^*\|^2-\|x^k-x^*\|\sum_{j=2}^{m-1}\gamma_j\|x^{k-j+1}-x^*\|^2$$

$$-\frac{C}{8}\|x^{k-m+1}-x^*\|^2\|x^k-x^*\|$$

$$\geq \frac{3C}{8}\|x^{k-m+1}-x^*\|^2\|x^k-x^*\|-\|x^{k-m+1}-x^*\|^2\|x^k-x^*\|\delta\sum_{j=2}^{m-1}\gamma_j .$$

Then using the estimate (9) we have

$$\|x^{k+1}-x^*\| \geq \gamma\|x^{k-m+1}-x^*\|^2\|x^k-x^*\|$$

for all $k \geq k_3$. Thus by Theorem 2.4.1 $O_R(I,x^*) = \tau$. Q.E.D.

It should be pointed out that if we assume that $\underline{\partial}_1^2 G(x^*,\ldots,x^*) = 0$ an argument analogous to the one used in the second part of the previous theorem shows that $O_R(I,x^*) = \tau$; thus nothing is gained by the additional assumption $\underline{\partial}_1^2 G(x^*,\ldots,x^*) = 0$.

Section 2.5

Order of Convergence for a Class of Iterative Procedures

In this section we shall apply the results of Sections 2.2 and 2.4 to an iteration function of a specific, but still quite general, form. Let I denote the iterative procedure

$$(1) \qquad x^{k+1} = G(x^k, x^{k-1}), \quad k = 1,2,\ldots$$

where $G: D \times D \subset R^n \times R^n \to R^n$ has the form $G(x,y) = x - A^{-1}(x,y)F(x)$ with $A \equiv (a_{i,j}): D \times D \to \mathfrak{L}(R^n)$ defined by

$$(2) \quad a_{i,j}(x,y) = \begin{cases} \dfrac{f_i\left(x+\beta \sum_{k=1}^{j-1} h_k(x,y)e_k + h_j(x,y)e_j\right) - f_i\left(x+\beta \sum_{k=1}^{j-1} h_k(x,y)e_k\right)}{h_j(x,y)} \\ \quad \text{for } h_j(x,y) \neq 0, \\ \partial_j f_i\left(x+\beta \sum_{k=1}^{j-1} h_k(x,y)e_k\right) \quad \text{for } h_j(x,y) = 0. \end{cases}$$

Here $\beta \in [0,1]$ and $h_j: D \times D \to R^1$, for $j = 1,\ldots,n$ are given functions. We shall also be interested in the case where h_j depends only on x making I a single-step iterative procedure. Before applying previous results to I we need a lemma pertaining to the vector partial derivatives of G . This lemma is similar to Lemma 2.3.1.

Lemma 2.5.1. Assume that $F:D \subset R^n \to R^n$ is twice continuously differentiable on D with $x^* \in \mathrm{int}(D)$ such that $F(x^*) = 0$ and $F'(x^*)$ is non-singular, and that for some constant α_1

$$(3) \qquad \|F''(x)-F''(x^*)\| \le \alpha_1\|x-x^*\|$$

for all $x \in D$. Let $G:DxD \subset R^n x R^n \to R^n$ be defined by $G(x,y) = x - A^{-1}(x,y)F(x)$ where A is given by relation (2). Assume that $h_j(x,y)$ is twice continuously differentiable on DxD for $j = 1,\ldots,n$. Then there exists an $r > 0$ with $\overline{S}(x^*,r) \subset D$ such that for some constants L_1, L_2 , and L_3 we have

$$(4) \quad \|\underline{\partial}_1 G(x,y)k-\underline{\partial}_1 G(x^*,y)k\| \le L_1\|x-x^*\|\|k\|$$

$$(5) \quad \|\underline{\partial}_2\underline{\partial}_1 G(x^*,y)k^1k^2-\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)k^1k^2\| \le L_2\|y-x^*\|\|k^1\|\|k^2\|$$

$$(6) \quad \|\underline{\partial}_1 G(x,y)k-\underline{\partial}_1 G(x^*,x^*)k\| \le L_3[\|x-x^*\|+\|y-x^*\|]\|k\|$$

for all $x,y \in \overline{S}(x^*,r)$.

Proof. Since the proof of this lemma is very similar to the proof of Lemma 2.3.1, it will only be sketched here. As in Equations 2.3-5 and 2.3-6 we have

$$(7)\quad \underline{\partial}_1 G(x,y)k = k - \underline{\partial}_1[A^{-1}(x,y)]kF(x) - A^{-1}(x,y)F'(x)k$$

where

$$(8)\quad \underline{\partial}_1[A^{-1}(x,y)]k = -A^{-1}(x,y)\underline{\partial}_1 A(x,y)kA^{-1}(x,y)$$

for all $x,y \in \overline{S}(x^*,r)$ where r is chosen so that $A^{-1}(x,y)$ and $F'(x)^{-1}$ exist on $\overline{S}(x^*,r)$. Now if we set $(a_{i,j}(x,y)) = A(x,y)$ we have by direct computation that the k,jth component of the ith block of $\underline{\partial}_1 A(x,y)$ is given by

$$\begin{aligned}(9)\quad \partial_{x_k} a_{i,j}(x,y) = & \sum_{\ell=1}^{j-1} \frac{\partial_\ell f_i(x(j)+h_j(x,y)e_j)-\partial_\ell f_i(x(j))}{h_j(x,y)} \partial_{x_k} h_\ell(x,y)\beta \\ & + \partial_j f_i(x(j)+h_j(x,y)e_j)\partial_{x_k} h_j(x,y) \\ & + \frac{\partial_k f_i(x(j)+h_j(x,y)e_j)-\partial_k f_i(x(j))}{h_j(x,y)} \\ & - \frac{f_i(x(j)+h_j(x,y)e_j)-f_i(x,y)}{h_j(x,y)^2} \partial_{x_k} h_j(x,y)\end{aligned}$$

for $h_j(x,y) \neq 0$ where $x(j) = x + \beta \sum_{\ell=1}^{j-1} h_\ell(x,y) e_\ell$. If

$h_j(x,y) = 0$, the situation is more complicated:

$$\partial_{x_k} a_{i,j}(x,y) = \lim_{t\to\infty} \frac{1}{t} [a_{i,j}(x+te_k,y) - a_{i,j}(x,y)]$$

$$= \lim_{t\to 0} \frac{1}{t} \left[\frac{f_i(x+te_k+\beta \sum_{\ell=1}^{j-1} h_\ell(x+te_k,y) e_\ell + h_j(x+te_k,y) e_k)}{h_j(x+te_k,y)} \right.$$

$$\left. - \frac{f_i(x+te_k+\beta \sum_{\ell=1}^{j-1} h_\ell(x+te_k,y) e_\ell)}{h_j(x+te_k,y)} - \partial_j f_i(x(j)) \right]$$

$$= \lim_{t\to 0} \left\{ \left[\frac{f_i(x+te_k+\beta \sum_{\ell=1}^{j-1} h_\ell(x+te_k,y) e_\ell + h_j(x+te_k,y) e_j - f_i(x+te_k+\beta \sum_{\ell=1}^{j-1} h_\ell(x+te_k,y) e_\ell)}{h_j(x+te_k,y)^2} \right. \right.$$

$$\left. - \frac{\partial_j f_i(x+te_k+\beta \sum_{\ell=1}^{j-1} h_\ell(x+te_k,y) e_\ell)}{h_j(x+te_k,y)} \right] \frac{h_j(x+te_k,y) - h_j(x,y)}{t}$$

$$+ \frac{\partial_j f_i(x+te_k+\beta \sum_{\ell=1}^{j-1} h_\ell(x+te_k,y) e_\ell) - \partial_j f_i(x+\beta \sum_{\ell=1}^{j-1} h_\ell(x+te_k,y) e_\ell)}{t}$$

$$+ \left[\frac{\partial_j f_i(x+\beta \sum_{\ell=1}^{j-1} h_\ell(x+te_k,y) e_\ell) - \partial_j f_i(x+\beta h_1(x,y) + \beta \sum_{\ell=2}^{j-1} h_\ell(x+te_k,y) e_\ell)}{\beta[h_1(x+te_k,y) - h_1(x,y)]} \right]$$

$$\cdot\ \beta\left[\frac{h_1(x+te_k,y)-h_1(x,y)}{t}\right] + \cdots$$

$$+\left[\frac{\partial_j f_i(x+\beta\sum_{\ell=1}^{j-2} h_\ell(x,y)+\beta h_{j-1}(x+te_k,y)e_{j-1})-\partial_j f_i(x(j))}{\beta[h_{j-1}(x+te_k,y)-h_{j-1}(x,y)]}\right]$$

$$\left.\cdot\ \beta\left[\frac{h_{j-1}(x+te_k,y)-h_{j-1}(x,y)}{t}\right]\right\}$$

$$\text{(10)} \quad = \sum_{\ell=1}^{j-1} \partial_\ell \partial_j f_i(x(j))\partial_{x_k} h_\ell(x,y)\beta + \partial_k \partial_j f_i(x(j))$$

$$+ \frac{1}{2}\partial_j^2 f_i(x(j))\partial_{x_k} h_j(x,y)$$

for $h_j(x,y) = 0$. Then it is easily verified that $\partial_{x_k} a_{i,j}(x,y)$ are continuous on $\bar{S}(x^*,r)$ for all $i,j,k = 1,\ldots,n$; hence, $\underline{\partial}_1 A(x,y)$ exists and is continuous on $\bar{S}(x^*,r)$, and is given by equations (9) and (10).

Now since $h_j(x,y)$ is twice continuously differentiable on D we have for some constant α_1

$$\text{(11)} \qquad |h_j(x,y)-h_j(x^*,y)| \le \alpha_1\|x-x^*\|$$

for all $x,y \in \bar{S}(x^*,r)$ and $j = 1,\ldots,n$. Using relation (11) and the technique used to obtain the Lipschitz condition 2.3-12, we have for some constant α_2

$$(12) \qquad \|A^{-1}(x,y)-A^{-1}(x^*,y)\| \le \alpha_2\|x-x^*\|$$

for all $x,y \in \overline{S}(x^*,r)$. If $\|\underline{\partial}_1 G(x,y)k-\underline{\partial}_1 G(x^*,y)k\|$ is now written as in relation 2.3-14 the Lipschitz condition (4) follows from relation (12) as in Lemma 2.3.1.

The verification of the Lipschitz condition (5) closely parallels the argument used to verify the Lipschitz condition 2.3-4 of Lemma 2.3.1 although the calculations are more complicated. We first exhibit $\underline{\partial}_2 A(x,y)$. As in equations (9) and (10) the k,jth component of the ith block is given by

$$(13) \quad \partial_{y_k} a_{i,j}(x,y) = \sum_{\ell=1}^{j-1}\left[\frac{\partial_\ell f_i(x(j)+h_j(x,y)e_j)-\partial_\ell f_i(x(j))}{h_j(x,y)}\right]\partial_{y_k} h_\ell(x,y)\beta$$

$$+ \frac{\partial_j f_i(x(j)+h_j(x,y)e_j)}{h_j(x,y)} \quad \partial_{y_k} h_j(x,y)$$

$$- \frac{f_i(x(j)+h_j(x,y)e_j)-f_i(x(j))}{h_j(x,y)^2} \quad \partial_{y_k} h_j(x,y)$$

for $h_j(x,y) \neq 0$, and

$$(14) \qquad \partial_{y_k} a_{i,j}(x,y) = \sum_{\ell=1}^{j-1} \partial_\ell \partial_j f_i(x(j))\partial_{y_k} h_\ell(x,y)\beta$$

$$+ \frac{1}{2}\partial_j^2 f_i(x(j))\partial_{y_k} h_j(x,y)$$

for $h_j(x,y) = 0$.

Now by means of a technique completely analogous with the one used to obtain Equation 2.3-18 we have that

$$(15) \qquad \underline{\partial}_2\underline{\partial}_1 G(x^*,y)k^1k^2 = -\underline{\partial}_2[A^{-1}(x^*,y)]k^2F'(x^*)k^1$$

Since $h_j(x,y)$ is twice continuously differentiable we have for some constant α_3 that

$$(16) \qquad \left|\partial_{y_k} h_j(x^*,y) - \partial_{y_k} h_j(x^*,x^*)\right| \le \alpha_3 \|y-x^*\|$$

for all $y \in \bar{S}(x^*,r)$ and all $k,j = 1,\ldots,n$. Now by means of a technique analogous to the one used to obtain the Lipschitz condition 2.3-12, using the Lipschitz conditions (3), (11), and (16) and expressions (13) and (14), we have for some constant α_4

$$(17) \qquad \|\underline{\partial}_2 A(x^*,y) - \underline{\partial}_2 A(x^*,x^*)\| \le \alpha_4 \|y-x^*\|$$

for all $y \in \bar{S}(x^*,r)$. Then proceeding as in Lemma 2.3.1 through relations 2.3-21, 2.3-22, and 2.3-23 we obtain for some constant α_5

$$(18) \qquad \|\underline{\partial}_2[A^{-1}(x^*,y)]k - \underline{\partial}_2[A^{-1}(x^*,x^*)]k\| \le \alpha_5 \|y-x^*\| \; \|k\|$$

for all $y \in \bar{S}(x^*,r)$. Finally, using equation (15) and the Lipschitz condition (17) in the manner of relation 2.3-23 yields the Lipschitz condition (5).

We can now obtain Lipschitz condition (7) in precisely the same way that Lipschitz condition 2.3-24 was obtained. Q.E.D.

We are now in a position to characterize the order of convergence of procedure (1) in terms of the functions $h_j(x,y)$, $j = 1,\ldots,n$.

Theorem 2.5.2. Let I denote the iterative procedure defined by (1) and (2). Assume that there is a point $x^* \in D$, an open subset of R^n , for which $F(x^*) = 0$ and $F'(x^*)$ is non-singular. Assume that F is twice continuously differentiable on D and satisfies, for some constant a ,

$$\|F''(x)-F''(x^*)\| \leq a\|x-x^*\|$$

for all $x \in D$. Assume that $h_j : DxD \to R^1$ is twice continuously differentiable on DxD for $j = 1,\ldots,n$, and that $h_j(x^*,x^*) = 0$ for $j = 1,\ldots,n$. Then x^* is a point of attraction of the iterative procedure I and $O_R(I,x^*) \geq \frac{1+\sqrt{5}}{2}$.

If, in addition, $\partial_{y_i} h_j(x^*,x^*) = 0$ for $i,j = 1,\ldots,n$, then $O_R(I,x^*) \geq 2$.

Proof. It is clear that $x^* = G(x^*,y)$ for all $y \in D$, and since $h_j(x,y) \to 0$ as $x,y \to x^*$, it is easily verified that $\underline{\partial}_1 G(x^*,x^*) = 0$. Furthermore, from Lemma 2.5.1 there exists an $r > 0$ such that

$$\|\underline{\partial}_1 G(x,y)k - \underline{\partial}_1 G(x^*,x^*)k\| \le L_1[\|x-x^*\| + \|y-x^*\|]\|k\|$$

for all $x,y \in \bar{S}(x^*,r) \subset D$. Thus it follows from Theorem 2.2.4 that x^* is a point of attraction of the iteration I and $O_R(I,x^*) \ge \frac{1+\sqrt{5}}{2}$.

Now assume that in addition the $h_j(x,y)$ satisfy $\partial_{y_i} h_j(x^*,x^*) = 0$ for $i,j = 1,\ldots,n$. From Lemma 2.5.1 there exists an $r > 0$ such that

$$\|\underline{\partial}_1 G(x,y)k - \underline{\partial}_1 G(x^*,y)k\| \le L_1\|x-x^*\| \; \|k\|$$

and

$$\|\underline{\partial}_2\underline{\partial}_1 G(x^*,y)k^1k^2 - \underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)k^1k^2\| \le L_2\|y-x^*\| \; \|k^1\| \; \|k^2\|$$

for all $x,y \in \bar{S}(x^*,r) \subset D$. Also from Lemma 2.5.1, equation (15), we have

$$\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)k^1k^2 = -\underline{\partial}_2[A^{-1}(x^*,x^*)]k^2F'(x^*)k^1$$

But as in equation (8)

$$\underline{\partial}_2[A^{-1}(x^*,x^*)]k^2 = -A^{-1}(x^*,x^*)\underline{\partial}_2 A(x^*,x^*)k^2 A^{-1}(x^*,x^*)$$

where, using the fact that $h_j(x^*,x^*) = 0$ for $j = 1,\ldots,n$ in equation (2), it is easy to see that $A^{-1}(x^*,x^*) \equiv F'(x^*)^{-1}$. Thus we have

$$\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)k^1k^2 = F'(x^*)^{-1}\underline{\partial}_2 A(x^*,x^*)k^2k^1$$

where the p,jth element of the ith block of $\underline{\partial}_2 A(x^*,x^*)$ is given by relations (13) and (14). Hence, from the form of $\underline{\partial}_2 A(x^*,x^*)$ since $\partial_{y_i} h_j(x^*,x^*) = 0$ for $i,j = 1,\ldots,n$, we have $\underline{\partial}_2 A(x^*,x^*) = 0$. Thus $\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*) = 0$ so the result follows from Theorem 2.4.2. Q.E.D.

It should be pointed out that we could now give a precise order theorem for the iteration I similar to Theorem 2.3.3 in which conditions (26) and (27) would be phrased in terms of the $h_j(x^*,x^*)$ $j = 1,\ldots,n$; however, in view of the complicated calculations required by Theorem 2.3.3 this result will not be given here. In the event that $h_j(x,y) \equiv h_j(x)$ $j = 1,\ldots,n$, that is I is a single-step procedure, it is not difficult to give a precise order result as may be seen in the following theorem.

<u>Theorem 2.5.3.</u> Let I denote the iterative procedure defined by (1) and (2). Assume that there is a point $x^* \in D$, an open subset of R^n, for which $F(x^*) = 0$ and $F'(x^*)$ is non-singular. Assume that F is twice continuously differentiable on D and satisfies, for some constant a,

$$\|F''(x)-F''(x^*)\| \leq a\|x-x^*\|$$

for all $x \in D$. Assume that $h_j : D \to R^1$ is twice continuously differentiable on D for $j = 1,\ldots,n$, and that $h_j(x^*) = 0$ for $j = 1,\ldots,n$. Then x^* is a point of attraction of the procedure I and $O_R(I,x^*) \geq 2$.

If, in addition,

$$(19) \qquad [2A'(x^*)-F''(x^*)]kk \neq 0$$

for all non-zero $k \in R^n$, then $O_R(I,x^*) = 2$.

<u>Proof.</u> It is easy to see that all of the hypotheses of Theorem 2.5.2 are satisfied so that x^* is a point of attraction of the procedure I and $O_R(I,x^*) \geq 2$.

Now we assume that condition (19) is satisfied. From equation (7) we have

$$G'(x)k = k - (A^{-1})'(x)kF(x) - A^{-1}(x)F'(x)k$$

for all $x \in \bar{S}(x^*, r)$ where r is chosen so that $A^{-1}(x)$ and $F'(x)^{-1}$ exist on $\bar{S}(x^*, r)$. We shall now show that

$$(20)\qquad G''(x^*)k^1k^2 = -(A^{-1})'(x^*)k^1F'(x^*)k^2$$
$$- (A^{-1})'(x^*)k^2F'(x^*)k^1 - A^{-1}(x^*)F''(x^*)k^1k^2 .$$

We have

$$\|G'(x^*+k)h-G'(x^*)h+(A^{-1})'(x^*)hF'(x^*)k+(A^{-1})'(x^*)kF'(x^*)h$$
$$+A^{-1}(x^*)F''(x^*)hk\|$$
$$= \|-(A^{-1})'(x^*+k)hF(x^*+k)-A^{-1}(x^*+k)F'(x^*+k)h+A^{-1}(x^*)F'(x^*)h$$
$$+(A^{-1})'(x^*)hF'(x^*)k+(A^{-1})'(x^*)kF'(x^*)h+A^{-1}(x^*)F''(x^*)hk\|$$
$$\le \|(A^{-1})'(x^*+k)h\| \ \|F(x^*+k)-F(x^*)-F'(x^*)k\|$$
$$+\|(A^{-1})'(x^*)h-(A^{-1})'(x^*+k)h\| \ \|F'(x^*)k\|$$
$$+\|(A^{-1})(x^*+k)-A^{-1}(x^*)-(A^{-1})'(x^*)k\| \ \|-F'(x^*)h\|$$
$$+\|-A^{-1}(x^*+k)\| \ \|F'(x^*+k)h-F'(x^*)h-F''(x^*)hk\|$$
$$+\|A^{-1}(x^*)-A^{-1}(x^*+k)\| \ \|F''(x^*)hk\| .$$

Now let $\epsilon > 0$ be arbitrary. $F'(x)$ is non-singular for $x \in \bar{S}(x^*, r)$ and, as was shown in Lemma 2.5.1, $(A^{-1})'$ is continuous at x^*. Hence, we may choose k sufficiently small so that $x^* + k \in \bar{S}(x^*, r)$, that $\|(A^{-1})'(x^*+k) - (A^{-1})'(x^*)\| < \epsilon$, and that $\|A^{-1}(x^*+k) - A^{-1}(x^*)\| < \epsilon$;

furthermore, since F is twice differentiable and A^{-1} is differentiable we have for k sufficiently small that

$\|F(x^*+k)-F(x^*)-F'(x^*)k\| \le \epsilon\|k\|$,

$\|F'(x^*+k)h-F'(x^*)h-F''(x^*)hk\| \le \epsilon\|h\| \; \|k\|$, and

$\|A^{-1}(x^*+k)-A^{-1}(x^*)-(A^{-1})'(x^*)k\| \le \epsilon\|k\|$. Thus we have

$$\begin{aligned}
&\|G'(x^*+k)h-G'(x^*)h+(A^{-1})'(x^*)hF'(x^*)k+(A^{-1})'(x^*)kF'(x^*)h \\
&\qquad +A^{-1}(x^*)F''(x^*)hk\| \\
&\qquad \le \sup_{\|k\|\le r} \|(A^{-1})'(x^*+k)\| \; \|h\|\epsilon\|k\| + \epsilon\|F'(x^*)\| \; \|k\| \; \|h\| \\
&\qquad +\epsilon\|k\| \; \|F'(x^*)\| \; \|h\| + \sup_{\|k\|\le r} \|A^{-1}(x^*+k)\|\epsilon\|h\| \; \|k\| \\
&\qquad +\epsilon\|F''(x^*)\| \; \|h\| \; \|k\| .
\end{aligned}$$

Thus since ϵ was arbitrary $G''(x^*)$ is given by equation (20). Furthermore, as in Theorem 2.5.2,

$$(A^{-1})'(x^*)k = -F'(x^*)^{-1}A'(x^*)kF'(x^*)^{-1}$$

so that $G''(x^*)$ has the form

$$G''(x^*)kk = 2F'(x^*)^{-1}A'(x^*)kk - F'(x^*)^{-1}F''(x^*)kk$$

Then since $2A'(x^*)kk - F''(x^*)kk \neq 0$ for all non-zero $k \in R^n$, $G''(x^*)kk \neq 0$ for all non-zero $k \in R^n$. Thus by Corollary 2.2.5 $O_R(I,x^*) = 2$. Q.E.D.

It is interesting to note that if $h_j(x) \equiv 0$ for $j = 1,\ldots,n$ then procedure (1) reduces to Newton's method. Then from equation (10) we see that $A'(x^*) = F''(x^*)$ so that condition (19) reduces to $F''(x^*)kk \neq 0$ for all non-zero $k \in R^n$. This is precisely the condition that Ortega and Rheinboldt [1970] have given for Newton's method to have exact quadratic convergence.

Section 2.6

Orders of Convergence of Some Specific Iterative Procedures

In this section we shall apply some of the theorems given in earlier sections of this chapter to some specific iterative procedures. In Section 2.3 we gave a sufficient condition for the two-point secant method, defined by relation 2.1-1, to have exactly $\frac{1+\sqrt{5}}{2}$ order of convergence. This condition involved restrictions on the second partial derivative of the function F. We shall now give a set of conditions on the second partial derivatives of F which will be sufficient to guarantee that the two-point secant method has at least quadratic convergence.

<u>Theorem 2.6.1</u>. Let I denote the two-point secant iteration $x^{k+1} = G(x^k, x^{k-1})$ where $G: DxD \subset R^n x R^n \to R^n$ is defined by $G(x,y) = x - J^{-1}(x,y-x)F(x)$ with $J(x,y-x)$ given by relation 2.1-1. Assume that $F: D \subset R^n \to R^n$ is twice continuously differentiable on the open subset D and that there exists an $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)$ is non-singular. Furthermore, suppose that for some constant a

$$\|F''(x)-F''(x^*)\| \le a\|x-x^*\|$$

for all $x \in D$, that

(1) $$F'(x^*)^{-1}(\partial_i^2 f_1(x^*),\ldots,\partial_i^2 f_n(x^*))^T = 0$$

for all $i = 1,\ldots,n$, and that

(2) $$\beta F'(x^*)(\partial_i\partial_j f_1(x^*),\ldots,\partial_i\partial_j f_n(x^*))^T = 0$$

for all $i,j = 1,\ldots,n$, with $i < j$. Then x^* is a point of attraction and $O_R(I,x^*) \ge 2$.

<u>Proof</u>. As in Theorem 2.3.3 the conditions of Theorem 2.2.4 are satisfied so that x^* is a point of attraction of the iteration I.

We shall now show that $\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*) = 0$. From equations 2.3-18 and 2.3-17 and the fact that $J^{-1}(x^*,0) = F'(x^*)^{-1}$ we have that

$$\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)k^1k^2 = F'(x^*)^{-1}\underline{\partial}_2 J(x^*,0)k^2k^1.$$

Then as in Theorem 2.3.3 the ℓ th component of $\underline{\partial}_2\underline{\partial}_1 G(x^*,x^*)k^1k^2$ is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{\ell,k}b_{k,j,i}k_i^2k_j^1$$

where $F'(x^*)^{-1} \equiv (a_{\ell,k})$ and $\underline{\partial}_2 J(x^*,0) \equiv (b_{k,j,i})$ where k denotes the block, and j and i the row and column, respectively, of the kth block. But by using the expression for $\underline{\partial}_2 J(x^*,0)$ given by equation 2.3-16 it is easy to verify that

$$\sum_{k=1}^{n} a_{\ell,k} b_{k,j,i} = \begin{cases} [F'(x^*)^{-1}]_\ell (\beta \partial_i \partial_j f_1(x^*), \ldots, \beta \partial_i \partial_j f_n(x^*))^T \\ \text{for } i < j , \\ [F'(x^*)^{-1}]_\ell \left(\frac{1}{2} \partial_i^2 f_1(x^*), \ldots, \frac{1}{2} \partial_i^2 f_n(x^*)\right)^T \\ \text{for } i = j , \\ 0 \quad \text{for } i > j , \end{cases}$$

where $[F'(x^*)]_\ell$ denotes the ℓth row of $F'(x^*)$. Thus using conditions (1) and (2) we have that $\underline{\partial}_2 \underline{\partial}_1 G(x^*,x^*) k^1 k^2 = 0$ for all $k^1, k^2 \in R^n$. By means of Lemma 2.5.1 with $h_j(x,y) = y_j - x_j$ for $j = 1,\ldots,n$, the remaining conditions of Theorem 2.4.2 are satisfied so that $O_R(I,x^*) \geq 2$. Q.E.D.

It is interesting to note that, as in Theorem 2.3.3, the conditions of Theorem 2.6.1 are much less restrictive if the two-point secant method in question has $\beta = 0$.

We shall now give an apparently new two-step method which has at least quadratic convergence and eliminates the need for either relation (1) or (2) in the previous theorem.

Theorem 2.6.2. Let I denote the iteration $x^{k+1} = G(x^k, x^{k-1})$ where $G: D \times D \subset R^n \times R^n \to R^n$ is defined by $G(x,y) = x - A^{-1}(x,y)F(x)$ with $A(x,y)$ given by relation 2.5-2 where $h(x,y) = (h_1(x,y), \ldots, h_n(x,y))^T$ is given by $h_j(x,y) = (y_j - x_j)^2$ for $j = 1, \ldots, n$. Assume that there is a point $x^* \in D$, an open subset of R^n, for which $F(x^*) = 0$ and $F'(x^*)$ is non-singular. Assume that F is twice continuously differentiable on D satisfying, for some constant a,

$$\|F''(x) - F''(x^*)\| \leq a\|x - x^*\|$$

for all $x \in D$. Then x^* is a point of attraction of the iterative procedure I and $O_R(I, x^*) \geq 2$.

Proof. It is clear that $h_j(x,y) \to 0$ as $x, y \to x^*$ and that $h_j(x,y)$ is twice continuously differentiable on D, for $j = 1, \ldots, n$. Furthermore,

$$\partial_{y_j} h_j(x,y) = \begin{cases} 0 & \text{if } i \neq j , \\ 2(y_j - x_j) & \text{if } i = j ; \end{cases}$$

hence, $\partial_{y_j} h_j(x^*, x^*) = 0$ for all $i, j = 1, \ldots, n$. Thus the hypotheses Theorem 2.5.2 are satisfied and the result follows. Q.E.D.

We shall now turn our attention to a single-step iterative procedure of the form $x^{k+1} = x^k - A^{-1}(x^k)F(x^k)$ where again $A(x)$ is given by relation 2.5-2. If we set $h_j(x) = f_j(x)$ for $j = 1,\ldots,n$ we obtain a well-known version of the Steffensen iteration. This iteration has been studied by several authors. Wegge [1966] has given a convergence theorem for the case $\beta = 0$ in which he obtains at least quadratic convergence; Schmidt [1966] has given a similar result for the case $\beta = 1$. The differentiability requirements of these results are weakened by Ortega and Rheinboldt [1970] in a theorem which handles the iteration for any $\beta \in [0,1]$. We shall now give a convergence theorem for the Steffensen iteration in which we obtain precisely quadratic convergence.

<u>Theorem 2.6.3</u>. Let I denote the iteration $x^{k+1} = G(x^k)$ where $G:D \subset R^n \to R^n$ is defined by $G(x) = x - A^{-1}(x^k)(x)$ with $A(x)$ given by relation 2.5-2 where $h_j(x) \equiv f_j(x)$ for $j = 1,\ldots,n$. Assume that there is a point $x^* \in D$, an open subset of R^n, for which $F(x^*) = 0$ and $F'(x^*)$ is non-singular. Assume that F is twice continuously differentiable on D and satisfies, for some constant a,

$$\|F''(x)-F''(x^*)\| \le a\|x-x^*\|$$

for all $x \in D$. Furthermore, assume that

$$(3) \qquad [F''(x^*)+B(x^*)]kk \neq 0$$

for all non-zero $k \in R^n$ where the k,jth element of the ith block of the bilinear operator $B(x^*)$ is given by

$$2\sum_{\ell=1}^{j-1} \partial_\ell \partial_j f_i(x^*)\partial_k f_i(x^*)\beta + \partial_j^2 f_i(x^*)\partial_k f_j(x^*) .$$

Then x^* is a point of attraction of I and $O_R(I,x^*) = 2$.

<u>Proof</u>. Clearly the hypotheses of the first part of Theorem 2.5.3 are satisfied so that x^* is a point of attraction of the iteration I and $O_R(I,x^*) \ge 2$.

From relation 2.5-10 of Lemma 2.5.1 we have that the k,jth element of the ith block of $A'(x^*)$ has the form

$$\sum_{\ell=1}^{j-1} \partial_\ell \partial_j f_i(x^*)\partial_k f_\ell(x^*)\beta + \partial_k \partial_j f_i(x^*) + \frac{1}{2}\partial_j^2 f_i(x^*)\partial_k f_i(x^*) .$$

Hence

$$2A'(x^*) - F''(x^*) \equiv F''(x^*) + B(x^*) ;$$

therefore, by condition (3)

$$[2A'(x^*) - F''(x^*)]kk \neq 0$$

for all non-zero $k \in R^n$ and it follows from Theorem 2.5.3 that $O_R(I,x^*) = 2$. Q.E.D.

It should be pointed out that condition (3) is greatly simplified in the case $\beta = 0$, for then the k,jth component of the ith block of $B(x^*)$ has the form $\partial_j^2 f_i(x^*)\partial_k f_j(x^*)$. In this case, condition (3) is very similar to the condition given for the quadratic convergence of Newton's method at the end of the preceding section.

We shall complete this section with an investigation of an iterative procedure formulated by Traub [1964] called the Newton-Secant iteration. This iteration, for finding a zero, x^* , of a function $F:D \subset R^n \to R^n$ is given by

$$(4) \qquad I_1: x^{k+1} = G(x^k) \ , \ k = 0,1,\ldots$$

where $G:D \subset R^n \to R^n$ is defined by $G(x) = x - A^{-1}(x)F(x)$ with $A(x)$ given by relation 2.5-2 where $\beta = 0$ and $h_j(x) \equiv -e_j^T F'(x)^{-1}F(x)$ for $j = 1,\ldots,n$. For $n = 1$ the Newton-Secant iteration reduces to

$$I_2: x^{k+1} = x^k - \frac{-f(x^k)/f'(x^k)}{f(x^k - f(x^k)/f'(x^k)) - f(x^k)} f(x^k), \ k = 0,1,\ldots$$

Traub shows that I_2 is convergent to x^* and that $O_Q(I_2,x^*) \geq 3$; hence, by Theorem A.3.2 $O_R(I_2,x^*) \geq 3$. However, Traub does not include an analysis of the n-dimensional procedure given by equation (4). Thus we now give a convergence statement for procedure I_1 including sufficient conditions for the procedure to have precisely quadratic convergence.

Theorem 2.6.4. Let I_1 be the iterative procedure defined by relation (4). Assume that there is a point $x^* \in D$, an open subset of R^n for $n \geq 2$, for which $F(x^*) = 0$ and $F'(x^*)$ is non-singular. Assume that F is three times continuously differentiable on D . Then x^* is a point of attraction of the iterative procedure I_1 and $O_R(I,x^*) \geq 2$. Moreover, if

$$[F''(x^*)+B(x^*)]kk \neq 0 \tag{5}$$

for all non-zero $k \in R^n$ where the ith block of the bilinear operator $B(x^*)$ is given by

$$-\text{diag}(\partial_j^2 f_i(x^*)) , \tag{6}$$

then $O_R(I_1,x^*) = 2$.

Proof. Since F is three times continuously differentiable on D we have for some constant a

$$\|F''(x)-F''(x^*)\| \le a\|x-x^*\|$$

for all $x \in D$. Since $F'(x^*)$ is non-singular, there is an $r > 0$ such that $F'(x)$ is non-singular for all $x \in \bar{S}(x^*,r) \subset D$. Thus since F is three times continuously differentiable on D, h_j is twice continuously differentiable on $\bar{S}(x^*,r)$ for $j = 1,\ldots,n$. Since $F(x^*) = 0$, $h_j(x^*) = 0$ for $j = 1,\ldots,n$. Thus the hypotheses of the first part of Theorem 2.5.3 are satisfied so that x^* is a point of attraction of the iteration I_1 and $O_R(I_1,x^*) \ge 2$.

Now assume that condition (5) is satisfied. Since $h_j(x) = -e_j^T F'(x)^{-1}F(x)$ we have

$$\partial_k h_j(x) = \partial_k[-e_j^T F'(x)^{-1}]F(x) - e_j^T F'(x)^{-1}\partial_k F(x)$$

for $k,j = 1,\ldots,n$. Or

$$\partial_k h_j(x^*) = -e_j^T F'(x^*)^{-1}F'(x^*)^{-1}e_k$$

$$= \begin{cases} -1 & \text{for } k = j \\ 0 & \text{for } k \ne j . \end{cases}$$

Thus from relation 2.5-10 we have that

$$A'(x^*) = F''(x^*) + \frac{1}{2}B(x^*)$$

where $B(x^*)$ is given by relation (6). Therefore, by condition (5)

$$[2A'(x^*)-F''(x^*)]kk = [F''(x^*)+B(x^*)]kk \neq 0$$

for all non-zero $k \in R^n$. It now follows from Theorem 2.5.3 that $O_R(I_1,x^*) = 2$. Q.E.D.

It is interesting to note that the natural extension of procedure I_2 to higher dimensions does not preserve the order of convergence exhibited by the one-dimensional procedure.

Section 2.7

The Method of False Position

In this section we discuss two extensions of the method of false position or regula falsi. In one dimension the procedure is given by

$$x^{k+1} = x^k - \left[\frac{f(\bar{x})-f(x^k)}{\bar{x}-x^k}\right]^{-1} f(x^k) \qquad k = 0,1,\ldots \tag{1}$$

where $\bar{x}$ is held fixed throughout the iteration. Notice that (1) is just the secant method with $\bar{x}$ replacing x^{k-1} in each iteration step. Thus we shall obtain two extensions of the method of false position by holding certain points fixed in two n-dimensional secant methods. One of these methods is the two-point secant method; we shall now introduce the other method.

Let I denote the iteration

$$I: x^{k+1} = G(x^k, x^{k,1}, \ldots, x^{k,n}) \ , \ k = 0,1,\ldots \tag{2}$$

where $x^o, x^{o,1}, \ldots, x^{o,n}$ are given and $G: D^{n+1} \subset (R^n)^{n+1} \to R^n$ is defined by $G(x^k, \ldots, x^{k-n}) = x^k - J^{-1}(x^k, H_k)F(x^k)$ with

$J:D\times D_H \subset R^n \times \mathfrak{L}(R^n) \to \mathfrak{L}(R^n)$ given by

$$(3) \quad J(x^k,H_k) = (F(x^k+H_k e_1)-F(x^k),\ldots,F(x^k+H_k e_n)-F(x^k))(H_k)^{-1}$$

where $H_k \in D_H$ is given by

$$(4) \qquad H_k \equiv (x^{k,1}-x^k,\ldots,x^{k,n}-x^k) .$$

We shall call procedure (2) the interpolatory secant method when $x^{k,i} = x^{k-i}$ for $i = 1,\ldots,n$. For a more complete development of this procedure the reader is referred to Ortega and Rheinboldt [1970].

The first rigorous analysis of the interpolatory secant method was given by Bittner [1959] and then Tornheim [1964] gave a rate of convergence statement. These results are improved by Ortega and Rheinboldt [1970] using the following concept due to Ortega [1967].

Definition 2.7.1. Let $J:D\times D_H \subset R^n \times \mathfrak{L}(R^m,R^n) \to \mathfrak{L}(R^n)$. Then J is a consistent approximation to F' on D_o if $D_o \subset D$, 0 is a limit point of D_H , and

$$\lim_{\substack{H\to 0 \\ H\in D_H}} J(x,H) = F'(x)$$

uniformly for all $x \in D_o$. In addition, J is a strongly consistent approximation to F' on D_o if there are constants c and $\delta > 0$ such that

$$\|F'(x) - J(x,H)\| \le c\|H\|$$

for all $x \in D_o$, $H \in D_H$ such that $\|H\| < \delta$.
Now in general one cannot show that $J(x,H)$ as defined by equation (3) is a consistent approximation. In fact, Ortega and Rheinboldt [1970] have shown that no matter how small one takes $\|H\|$, $\|F'(x) - J(x,H)\|$ may be arbitrarily large. In spite of this, it is possible to show that $J(x,H)$ is a consistent approximation to $F'(x)$ if H is suitably restricted. This has been done by Ortega and Rheinboldt [1970] by means of uniformly non-singular matrices, a concept suggested by a result of Bittner [1959], and is included here for the sake of completeness.

<u>Definition 2.7.2</u>. A family of matrices $\mathfrak{F} \in \mathfrak{L}(R^n)$ is uniformly non-singular if there exists a constant $\gamma > 0$ such that for any $H = (h_1, \ldots, h_n) \in \mathfrak{F}$, $\|h_i\| \ne 0$ for $i = 1, \ldots, n$, and

$$\left|\det\left(\frac{h_1}{\|h_1\|}, \ldots, \frac{h_n}{\|h_n\|}\right)\right| \ge \gamma .$$

Notice that any set of non-singular diagonal matrices is uniformly non-singular.

<u>Theorem 2.7.3.</u> Let $F : D \subset R^n \to R^n$ be continuously differentiable on the open set D and let $\mathfrak{J} \in \mathfrak{L}(R^n)$ be a uniformly non-singular family of matrices such that the null matrix is a limit point of $\mathfrak{J}$. Then for any compact subset D_o of D, there is a constant $r > 0$ such that the mapping J given by equations (3) and (4) is well defined on $D_o x D_H$ where $D_H = \{H \in \mathfrak{J} ; \|H\| \le r\}$, and is a consistent approximation of F' on D_o.

A convergence theorem for the interpolatory secant method is now possible under the rather restrictive assumptions that the x^k are all well defined, $\|H_k\|$ remain suitably small for all k and $H_k \in K(\gamma)$ for all k where

$$(5) \qquad K(\gamma) = \left\{H \in \mathfrak{L}(R^n) ; \left|\det\left(\frac{h_1}{\|h_1\|}, \ldots, \frac{h_n}{\|h_n\|}\right)\right| \ge \gamma > 0\right\}.$$

This result may be found in Ortega and Rheinboldt [1970].

We shall now return to the method of false position and consider the two-point secant method given by equation 2.1-1. To obtain equation (1), x^{k-1} was replaced by a fixed point $\bar{x}$, and we now do the exact same thing in equation 2.1-1; that is, with the jth column of the matrix $J(x,h)$ defined by

$$(6)\quad J_j(x,h) = \begin{cases} 1/h_j\left[F\left(x+\beta\sum_{i=1}^{j-1} h_i e_i + h_j e_j\right) - F\left(x+\beta\sum_{i=1}^{j-1} h_i e_i\right)\right], & h_j \neq 0 \\ F'\left(x+\beta\sum_{i=1}^{j-1} h_i e_i\right)e_j\ , & h_j = 0 \end{cases}$$

where, the first method of false position is given by

$$(7)\qquad x^{k+1} = x^k - J^{-1}(x^k,\overline{x}-x^k)F(x^k) \equiv G(x^k)$$

with $\overline{x}$ held fixed.

Before giving a convergence theorem for the first method of false position, we need the following lemma about consistent approximations which were defined in Definition 2.7.1. This lemma may be found in Ortega and Rheinboldt [1970].

<u>Lemma 2.7.4</u>. Assume that $J(x,H)$ is a consistent approximation to $F'(x)$ on some set $D \subset R^n$ and that F' is continuous at $x^* \in D$. Then there exists an $r > 0$ such that if $\|H\| \leq r$, $J^{-1}(x,H)$ exists for all $x \in S(x^*,r)$.

Proof. Set $\beta = \|F'(x^*)^{-1}\|$ and choose $\alpha > 0$ so that $\alpha\beta < 1$. Since $J(x,H)$ is a consistent approximation, there is an $r_1 > 0$ such that whenever $\|H\| \le r_1$

$$\|F'(x)-J(x,H)\| \le \alpha/2$$

for every $x \in S(x^*,r_1) \subset D$. Since F' is continuous at x^*, there is an r satisfying $0 < r \le r_1$ such that

$$\|F'(x)-F'(x^*)\| \le \alpha/2$$

for every $x \in S(x^*,r)$. Hence, whenever $\|H\| \le r$

$$\|F'(x^*)-J(x,h)\| \le \alpha$$

for every $x \in S(x^*,r)$. Thus by the well-known Banach Lemma, $J^{-1}(x,H)$ exists for all $x \in S(x^*,r)$ whenever $\|H\| \le r$. Q.E.D.

Theorem 2.7.5. Let the iteration $I:x^{k+1} = G(x^k)$ be given by equation (7). Assume that $F:D \subset R^n \to R^n$ is continuously differentiable on D where $F(x^*) = 0$ and $F'(x^*)$ is non-singular. Then there is an $r > 0$ such that if $\overline{x} \in S(x^*,r)$, x^* is a point of attraction of the iteration I and

$$R_1(I,x^*) = \rho\{I-J^{-1}(x^*,\overline{x}-x^*)F'(x^*)\}.$$

Proof. By the previous lemma, there exists an $r_1 > 0$ such that whenever $\|\bar{x}-x\| \leq r_1$, $J^{-1}(x,\bar{x}-x)$ exists for all $x \in S(x^*,r_1) \subset D$.

We shall now show that

$$G'(x^*) = I - J^{-1}(x^*,\bar{x}-x^*)F'(x^*) .$$

Let $\epsilon > 0$ be arbitrary. Since $J(x,\bar{x}-x)$ is continuous at $x = x^*$ we may choose $\|g\|$ sufficiently small so that $x^* + g$, $\bar{x} - x^* - g \in \bar{S}(x^*,r_1)$ and

$$\|J^{-1}(x^*+g,\bar{x}-x^*-g) - J^{-1}(x^*,\bar{x}-x^*)\| \leq \epsilon$$

Then using the fact that $F(x^*) = 0$ and that F is differentiable, and the Taylor Expansion Theorem 1.1.3 we have

$$\|G(x^*+g)-G(x^*)-[I-J^{-1}(x^*,\bar{x}-x^*)F'(x^*)]g\|$$

$$= \|-J^{-1}(x^*+g,\bar{x}-x^*-g)F(x^*+g)+J^{-1}(x^*,\bar{x}-x^*)F'(x^*)g\|$$

$$\leq \|J^{-1}(x^*+g,\bar{x}-x^*-g)-J^{-1}(x^*,\bar{x}-x^*)\| \; \|F(x^*+g)-F(x^*)\|$$

$$+ \|J^{-1}(x^*,\bar{x}-x^*)\| \; \|F(x^*+g)-F(x^*)-F'(x^*)g\|$$

$$\leq \epsilon\int_0^1 \|F'(x^*+tg)\| \; \|g\|dt + \|J^{-1}(x^*,\bar{x}-x^*)\|\epsilon\|g\| .$$

Then bounding the continuous function $F'(x^*+tg)$ on $\overline{S}(x^*,r_1)$ gives, for some constant $a \geq 0$,

$$\|G(x^*+g)-G(x^*)-[I-J^{-1}(x^*,\overline{x}-x^*)F'(x^*)]g\| \leq a\varepsilon\|g\|$$

which completes the verification of the derivative of G .

Now since

$$J(x^*,\overline{x}-x^*) \to F'(x^*) \quad \text{as} \quad \overline{x} \to x^*$$

and the eigenvalues of a matrix are continuous functions of the elements of the matrix, there exists an r satisfying $0 < r \leq r_1$ such that

$$\rho\{I-J^{-1}(x^*,\overline{x}-x^*)F'(x^*)\} < 1$$

for all $\overline{x} \in S(x^*,r)$. Hence by Theorem 1.2.1 x^* is a point of attraction of the iteration I and $R_1(I,x^*) = \rho\{I-J^{-1}(x^*,\overline{x}-x^*)F'(x^*)\}$. Q.E.D.

Now, in general, we cannot obtain a superlinear rate of convergence for the first method of false position no matter how close to x^* we choose $\overline{x}$. This is made precise in the following theorem.

<u>Theorem 2.7.6</u>. Let $F:D \subset R^n \to R^n$ be continuously differentiable on D and have at least one component f_i

which is non-linear in every neighborhood of x^* contained in D where $F(x^*) = 0$. Assume that $F'(x^*)^{-1}$ exists. Then for every $r > 0$ such that $S(x^*,r) \subset D$ there is a point $\bar{x} \in S(x^*,r)$ such that $\rho\{I-J^{-1}(x^*,\bar{x}-x^*)F'(x^*)\} \neq 0$.

<u>Proof</u>. Let $r > 0$ be chosen so that $S(x^*,r) \subset D$ but otherwise arbitrary. Then there is a j, $1 \leq j \leq n$ such that $\partial_j f_i(x^*) \neq \partial_j f_i(\bar{x})$ where $\bar{x} = (x_1^*,\ldots,x_{j-1}^*,\xi_j,x_{j+1}^*,\ldots,x_n^*)^T \in S(x^*,r)$. Then from equation (6), we see that

$$J(x^*,\bar{x}-x^*) = F'(x^*) + ue_j^T$$

where $u = (\partial_j f_1(y^1)-\partial_j f_1(x^*),\ldots,\partial_j f_n(y^n)-\partial_j f_n(x^*))^T$ and the y^i are chosen by means of the Mean Value Theorem so that

$$\partial_j f_i(y^i) = \frac{f_i(\bar{x})-f_i(x^*)}{\xi_j-x_j^*}$$

for $i = 1,\ldots,n$. Then from the Sherman-Morrison formula, which may be found, for example, in Householder [1964], we have

$$J^{-1}(x^*,\bar{x}-x^*) = F'(x^*)^{-1} - \frac{F'(x^*)^{-1}ue_j^TF'(x^*)^{-1}}{1+e_j^TF'(x^*)^{-1}u}$$

or

$$J^{-1}(x^*,\bar{x}-x^*)F'(x^*) = I - \frac{F'(x^*)^{-1}ue_j^T}{1+e_j^TF'(x^*)^{-1}u} .$$

At this point by means of the continuity of F' we choose ξ_j close enough to x_j^* so that

$$(8) \qquad e_j^TF'(x^*)^{-1}(\partial_jf_1(y^1),\ldots,\partial_jf_n(y^n))^T = \alpha$$

where $\alpha \notin \{0,1\}$. Now since

$$F'(x^*)^{-1}u = F'(x^*)^{-1}(\partial_jf_1(y^1),\ldots,\partial_jf_n(y^n))^T - e_j ,$$

we have

$$1 + e_j^TF'(x^*)^{-1}u = \alpha .$$

Hence

$$I - J^{-1}(x^*,\bar{x}-x^*)F'(x^*)$$

$$= \frac{1}{\alpha} [F'(x^*)^{-1}(\partial_jf_1(y^1),\ldots,\partial_jf_n(y^n))^T-e_j]e_j^T .$$

But then the matrix $I - J^{-1}(x^*,\bar{x}-x^*)F'(x^*)$ has all zero columns except for the jth column, so the eigenvalues of $I - J^{-1}(x^*,\bar{x}-x^*)F'(x^*)$ are zero and the jth component of the jth column, or from equation (8) the eigenvalues are zero and $\frac{1}{\alpha}(\alpha-1)$. Thus since $\alpha \neq 1, \rho\{I-J^{-1}(x^*,\bar{x}-x^*)F'(x^*)\}$ $\neq 0$ and the theorem is proved. Q.E.D.

Next we turn our attention to the second method of false position which we shall obtain from the interpolatory secant method given by procedure (2). Now in the first method of false position we held one point fixed throughout the iteration in order to obtain a single-step method, but in the interpolatory method we must hold n points fixed in order to obtain a single-step method. Hence the second method of false position is given by procedure (2) with

$$H_k = (x^{-1}-x^k,\dots,x^{-n}-x^k)$$

where x^{-i} are n fixed points.

It is possible to give a convergence theorem for the second method of false position if the fixed points x^{-i}, $i = 1,\dots,n$, are suitably chosen; however, before giving the theorem we need a lemma which will guarantee that certain matrices are uniformly non-singular, that is, they are contained in the set $K(\gamma)$ given by equation (5).

<u>Lemma 2.7.7</u>. Let $A \in \mathfrak{L}(R^n)$ be non-singular. Then for any non-singular diagonal matrix D , $AD \in K(\gamma)$ where

$$0 < \gamma \leq |\det A| \prod_{j=1}^{n} 1/\|a_j\| \quad \text{with} \quad A = (a_1,\dots,a_n) .$$

<u>Proof</u>. Let $H = (h_1,\ldots,h_n) = AD = A\,\mathrm{diag}(d_1,\ldots,d_n)$. Then

$$\left|\det\left(\frac{h_1}{\|h_1\|},\ldots,\frac{h_n}{\|h_n\|}\right)\right| = \left|\det\left(\frac{d_1a_1}{|d_1|\|a_1\|},\ldots,\frac{d_na_n}{|d_n|\|a_n\|}\right)\right|$$

$$= |\det A|\ \left|\prod_{j=1}^{n} d_j\right|\ \left|\prod_{j=1}^{n}\frac{1}{|d_j|\|a_j\|}\right|$$

$$\geq \gamma$$

Hence $AD \in K(\gamma)$. Q.E.D.

We are now in a position to give a convergence theorem for the second method of false position under the assumption that the fixed points x^{-i}, $i = 1,\ldots,n$ have a special form.

<u>Theorem 2.7.8</u>. Let $I: x^{k+1} = G(x^k)$ be the iteration (2) and assume that $F: D \subset R^n \to R^n$ is continuously differentiable on D with $F'(x^*)^{-1}$ existing where $F(x^*) = 0$. Let $A = (a_1,\ldots,a_n) \in \mathfrak{L}(R^n)$ be non-singular. Let $x^{-i} = x^* + t_ia_i$, $t_i \neq 0$ for $i = 1,\ldots,n$. Then there is an $r > 0$ such that for $|t_i| \leq r$, $i = 1,\ldots,n$, x^* is a point of attraction of the iteration I with

$$H_k \equiv (x^{-1}-x^k,\ldots,x^{-n}-x^k)\ ,$$

and $R_1(I,x^*) = \rho\{I-J^{-1}(x^*,H(t))F'(x^*)\}$ where

$$H(t) \equiv (x^{-1}-x^{*},\ldots,x^{-n}-x^{*}) .$$

Proof. Set $H(t) = AD$ where $D = \mathrm{diag}(t_1,\ldots,t_n)$. Let $\mathfrak{J} = \{H(t) \; ; \; t_i \neq 0 \;\text{ for any }\; i = 1,\ldots,n\}$. Then by Lemma 2.7.7 $\mathfrak{J}$ is a uniformly non-singular family of matrices, and it is clear that the null matrix is a limit point of $\mathfrak{J}$. Hence, by Theorem 2.7.3, $J(x^*,H(t))$ is a consistent approximation to $F'(x^*)$ on $D_o = x^* x D_H$ where

$$D_H = \{H(t) \in \mathfrak{J} ; \|H(t)\| < a \quad \text{for some} \quad a > 0\} ,$$

and by Lemma 2.7.4 there is an $r_1 > 0$ such that for $|t_i| \leq r_1$, $i = 1,\ldots,n$, $J^{-1}(x^*,H(t))$ exists.

As in Theorem 2.7.5 it is easy to verify that

$$G'(x^*) = I - J^{-1}(x^*,H(t))F'(x^*)) .$$

Thus since

$$J(x^*,H(t)) \to F'(x^*) \quad \text{as} \quad t \to 0$$

and the eigenvalues of a matrix are continuous functions of the elements of the matrix, there exists an r satisfying $0 < r \leq r_1$, such that

$$\rho\{I-J^{-1}(x^*,H(t))F'(x^*)\} < 1$$

for all $|t_i| \leq r$. Hence the result follows from Theorem 1.2.1. Q.E.D.

It should be pointed out that although the condition on the x^{-i} of the previous theorem cannot be checked prior to application of the second method of false position, the condition is not restrictive, for consider n vectors x^{-i} , $i = 1,\ldots,n$, such that $B \equiv (x^{-1},\ldots,x^{-n})$ is non-singular. Then the condition of the theorem requires that there exists a non-singular matrix AD such that

$$B = (x^*+t_1a_1,\ldots,x^*+t_na_n)$$

where $A = (a_1,\ldots,a_n)$ and $D = \mathrm{diag}(t_1,\ldots,t_n)$. But if $1 + e^TB^{-1}x^* \neq 0$ where e is the n-vector $(-1,\ldots,-1)^T$, then by the Sherman-Morrison formula (see Theorem 2.7.6) $B-(x^*,\ldots,x^*)$ is non-singular so that

$$AD \equiv B - (x^*,\ldots,x^*)$$

satisfies the condition of the theorem.

APPENDIX A

In this appendix we reproduce a large portion of Chapter 9 of a forthcoming book of Ortega and Rheinboldt [1970].

Section A.1

The Quotient Convergence Factors

Let I denote an iterative process and let x^* be one of its limit points. Our aim is to attach to I precise indicators of the asymptotic rate of convergence of the process at x^*. We first define one such indicator for an arbitrary convergent sequence which need not have been generated by an iterative process.

<u>Definition A.1.1</u>. Let $\{x^k\} \subset R^n$ be any convergent sequence with limit x^*. Then the quantities

$$Q_p\{x^k\} = \begin{cases} 0 & , \text{ if } x^k = x^*, \ \forall\, k \geq k_o , \\ \limsup\limits_{k\to\infty} \dfrac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|^p} & , \text{ if } x^k \neq x^* \text{ for all but finitely many } k , \\ +\infty & , \text{ otherwise} \end{cases}$$

defined for all $p \in [1,\infty)$, are the <u>quotient convergence factors</u>, or <u>QC-factors</u> for short, of $\{x^k\}$ with respect to the norm $\|\cdot\|$ on R^n.

Note that if $Q_p = Q_p\{x^k\} < +\infty$ for some $p \in [1,\infty)$, then for any $\epsilon > 0$ there exists a k_o such that

$$(1) \qquad \|x^{k+1} - x^*\| \le \gamma \| x^k - x^* \| , \quad \forall k \ge k_o$$

holds with $\gamma = Q_p + \epsilon$.

When considering not just one sequence but an iterative process I, it is desirable that the rate-of-convergence indicator measure the worst possible asymptotic rate of convergence of any sequence of I with the same limit point.

Definition A.1.2. Let $C(I,x^*)$ denote the set of all sequences generated by an iterative process I with limit point x^*. Then

$$Q_p(I,x^*) = \sup\{Q_p\{x^k\} \mid \{x^k\} \in C(I,x^*)\} , \quad 1 \le p < +\infty ,$$

are the QC-factors of I at x^* with respect to the norm in which the $Q_p\{x^k\}$ are computed.

For a given iterative process I and limit point x^* we consider first the behavior of $Q_p(I,x^*)$ as a function of p. The following basic result shows that Q_p is a monotonically increasing function of p which takes on only the values 0 and ∞ except at possibly one point.

Theorem A.1.3. Let $Q_p(I,x^*), p \in [1,\infty)$, denote the QC-factors of an iterative process at x^*, in some fixed norm on R^n . Then exactly one of the following conditions holds:

(a) $Q_p(I,x^*) = 0$, $\forall\ p \in [1,\infty)$;

(b) $Q_p(I,x^*) = \infty$, $\forall\ p \in [1,\infty)$;

(c) There exists a $p_o \in [1,\infty)$ such that

$Q_p(I,x^*) = 0$, $\forall\ p \in [1,p_o)$, and

$Q_p(I,x^*) = \infty$, $\forall\ p \in (p_o,\infty)$.

<u>Proof</u>. Let $\{x^k\} \in C(I,x^*)$ be any sequence generated by I which converges to x^* , and set $\epsilon_k = \|x^k - x^*\|$, $k = 0,1,\ldots$. If $\epsilon_k = 0$ for all but finitely many k then $Q_p\{x^k\} = 0$, $p \in [1,\infty)$. Assume therefore, that $\epsilon_k > 0$ for $k \geq k_o$, and suppose further that $Q_p\{x^k\} < +\infty$ for some $p \in (1,\infty)$. Then we have for any $q \in [1,p)$,

$$Q_q\{x^k\} = \limsup_{k\to\infty} \frac{\epsilon_{k+1}\epsilon_k^{p-q}}{\epsilon_k^p} \leq Q_p\{x^k\} \limsup_{k\to\infty} \epsilon_k^{p-q} = 0 ,$$

which in turn implies that $Q_p\{x^k\} = +\infty$ whenever $Q_q\{x^k\} > 0$ and $p > q \geq 1$; in fact, if $Q_p < +\infty$, then we just saw that $Q_q = 0$. Hence one and only one of the properties (a), (b), (c) holds for any sequence $\{x^k\} \in C(I,x^*)$.

Now assume that neither (a) nor (b) holds, and set $p_o = \inf\{p \in [1,\infty) \mid Q_p(I,x^*) = +\infty\}$. Suppose there is a

$p > p_o$ such that $Q_p(I,x^*) < +\infty$, that is, $Q_p\{x^k\} < +\infty$ for every sequence $\{x^k\}$ generated by I and converging to x^*. Then, by the definition of p_o, there exists a $p' \in [p_o,p)$ such that $Q_{p'}(I,x^*) = +\infty$ and hence we have $Q_{p'}\{x^k\} > 0$ for some $\{x^k\} \in C(I,x^*)$. This implies that $Q_p\{x^k\} = +\infty$ which is a contradiction, and therefore it follows that $Q_p(I,x^*) = +\infty$ for all $p \in (p_o,\infty)$. On the other hand, if $p_o > 1$ and $Q_p(I,x^*) \neq 0$ for some $p \in [1,p_o)$, then it follows by a similar argument that $Q_{p'}(I,x^*) = +\infty$ for $p' \in (p,p_o)$ and this would contradict the definition of p_o. Hence, we have $Q_p(I,x^*) = 0$ for all $p \in [1,p_o)$. Q.E.D.

The primary motivation for introducing the QC-factors of an iterative process is to have a precise means of comparing the rate of convergence of different iterations. We utilize the QC-factors in this connection as follows:

<u>Definition A.1.4</u>. Let I_1 and I_2 denote two iterative processes with the same limit point x^*, and let $Q_p(I_1,x^*)$ and $Q_p(I_2,x^*)$ be the corresponding QC-factors computed in the same norm on R^n. Then I_1 is <u>Q-faster</u> than I_2 at x^* if there is a $p \in [1,\infty)$ such that $Q_p(I_1,x^*) < Q_p(I_2,x^*)$.

Note that by Theorem A.1.3 the concept of "Q-faster" is well-defined, that is, it is not possible to find $p, p' \in [1,\infty)$ such that $Q_p(I_1,x^*) < Q_p(I_2,x^*)$ but $Q_{p'}(I_1,x^*) > Q_{p'}(I_2,x^*)$. This follows from the fact that if, say, $p' > p$ and $Q_p(I_2,x^*) > Q_p(I_1,x^*)$ then $Q_p(I_2,x^*) > 0$ and hence $Q_{p'}(I_2,x^*) = +\infty$. Note furthermore that Theorem A.1.3 also shows the transitivity of the "Q-faster" relation; that is, if I_1 is Q-faster than I_2 and I_2 is Q-faster than I_3 then I_1 is Q-faster than I_3.

The concept of "Q-faster" depends upon the norm on R^n and it is possible that in one norm a process I_1 is Q-faster than another process I_2 while in another norm I_2 is Q-faster than I_1. This simply reflects the fact that when $\{x^k\}$ is an arbitrary convergent sequence such that $0 < Q_p\{x^k\} < +\infty$ then the magnitude of $Q_p\{x^k\}$ is dependent upon the norm. However, there is a very important norm-independent measure that we now introduce.

<u>Definition A.1.5</u>. Let $Q_p(I,x^*)$ be the QC-factors of the iterative process I at x^* in some norm on R^n. Then

$$O_Q(I,x^*) = \begin{cases} +\infty & , \text{ if } Q_p(I,x^*) = 0, \ \forall\, p \in [1,\infty)\ , \\ \inf\{p\in[1,\infty) \mid Q_p(I,x^*) = +\infty\}, & \text{otherwise,} \end{cases}$$

is the Q-order of I at x^*.

Theorem A.1.6. Let $Q_p(I,x^*)$ be the QC-factors of the iterative process I at x^*. Then the three relations $Q_p(I,x^*) = 0$, $0 < Q_p(I,x^*) < +\infty$, and $Q_p(I,x^*) = +\infty$, are independent of the norm on R^n. Hence the Q-order of I at x^* is also independent of the norm.

Proof. Let $\{x^k\} \subset R^n$ be any sequence that converges to x^* and denote by $Q_p\{x^k\}$ and $Q'_p\{x^k\}$ the QC-factors of $\{x^k\}$ in the norms $\|\cdot\|$ and $\|\cdot\|'$. Suppose that $x^k \neq x^*$, $\forall\, k \geq k_o$, otherwise we would have, by definition, $Q_p\{x^k\} = 0$ in every norm. Since on R^n all norms are equivalent we have that there exist constants $d \geq c > 0$ such that

(2) $$d\|x\| \geq \|x\|' \geq c\|x\|\ ,\ \forall\, x \in R^n\ ;$$

therefore

(3) $$Q_p\{x^k\} = \limsup_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|^p}$$

$$\leq \limsup_{k\to\infty} \frac{d^p}{c} \frac{\|x^{k+1}-x^*\|'}{\|x^k-x^*\|'^{\,p}} = \frac{d^p}{c}\, Q'_p\{x^k\}\ .$$

Hence $Q_p\{x^k\} = 0$ whenever $Q'_p\{x^k\} = 0$, and $Q'_p\{x^k\} = +\infty$ whenever $Q_p\{x^k\} = +\infty$. This shows that the relations $Q_p\{x^k\} = 0, +\infty$ are both norm-independent and consequently, $0 < Q_p\{x^k\} < +\infty$ must also be norm-independent.

Now suppose that $Q_p(I,x^*) = 0$ in the norm $\|\cdot\|$. Then for any sequence $\{x^k\}$ generated by I and converging to x^*, we must have $Q_p\{x^k\} = 0$. But we have just seen that this remains valid in any norm and therefore $Q_p(I,x^*) = 0$ in any norm. On the other hand, suppose that $Q_p(I,x^*) = +\infty$ in the norm $\|\cdot\|$. Then (3) implies an immediate contradiction to the proposition that $Q'_p(I,x^*) < \infty$ in another norm $\|\cdot\|'$. It follows then that the set $\{p\in[1,\infty) \mid Q_p(I,x^*) = +\infty\}$, and hence also the Q-order $O_Q(I,x^*)$, is independent of the norm. Q.E.D.

As direct consequences of Theorems A.1.3 and A.1.6 we obtain the following important observations.

<u>Theorem A.1.7</u>. Let I_1 and I_2 be iterative processes with limit x^*. If $O_Q(I_1,x^*) > O_Q(I_2,x^*)$, then I_1 is Q-faster than I_2 at x^* in every norm.

<u>Theorem A.1.8</u>. Let I be an iterative process with limit x^*. If $Q_p(I,x^*) < +\infty$ for some $p \in [1,\infty)$ then $O_Q(I,x^*) \geq p$. If $Q_q(I,x^*) > 0$ for some $q \in [1,\infty)$, then $O_Q(I,x^*) \leq q$.

Whenever we have $Q_p(I,x^*) < +\infty$ for some iterative process I with limit x^*, an estimate of the form (1) holds, with $\gamma = Q_p + \epsilon$ and any $\epsilon > 0$, for every sequence $\{x^k\}$ generated by I that converges to x^*; however, the index k_o will, in general, depend on $\{x^k\}$. If $Q_p(I,x^*) = +\infty$, then this estimate fails.

Iterative processes with Q-orders 1, 2, or 3 play an especially important role in the theory, and we introduce some additional terminology that will be useful on occasion. Whenever $Q_1(I,x^*) = 0$, we say that the process has <u>Q-superlinear</u> convergence at x^*, while if $0 < Q_1(I,x^*) < 1$ in some norm, the convergence is called <u>Q-linear</u>. Note that Theorem A.1.3 implies that any process of Q-order greater than one is Q-superlinear, and that by Theorem A.1.6 the concept of Q-superlinear convergence is norm independent. In contrast, the Q-linear convergence of a process I depends on the norm. But we can say that for a Q-linearly convergent process I there is always a $\gamma \in (0,1)$ and a norm such that (1) holds for any sequence generated by I and converging to x^*; that is, there is a norm in which the error vectors ultimately decrease at each step by a factor $\gamma < 1$. Any process I for which $Q_1(I,x^*) \geq 1$ in some norm is called <u>Q-sublinear</u> in that norm.

Similarly, any process I of Q-order two for which $0 < Q_2(I,x^*) < +\infty$ is called Q-quadratic at x^*, while for $Q_2(I,x^*) = 0$ or $Q_2(I,x^*) = +\infty$ we refer to Q-superquadratic or Q-subquadratic convergence, respectively. Finally, an analogous terminology is sometimes applied to Q-order three convergence with "quadratic" replaced by "cubic".

All of the results of this section extend to Banach spaces except for Theorem A.1.6.

Section A.2

The Root Convergence Factors

In this section, we consider another measure of the rate of convergence of iterative processes; it is obtained by taking appropriate roots of successive errors rather than quotients.

Definition A.2.1. Let $\{x^k\} \subset R^n$ be any sequence that converges to x^*. Then the numbers

$$R_p\{x^k\} = \begin{cases} \limsup\limits_{k\to\infty} \|x^k - x^*\|^{\frac{1}{k}} & , \text{ if } p = 1 \\ \\ \limsup\limits_{k\to\infty} \|x^k - x^*\|^{1/p^k} & , \text{ if } p > 1 \end{cases}$$

are the root-convergence factors, or RC-factors for short, of the sequence. If I is an iterative process with limit point x^* and $C(I, x^*)$ is the set of all sequences generated by I which converge to x^*, then

$$R_p(I,x^*) = \sup\{R_p\{x^k\} \mid \{x^k\} \in C(I,x^*)\} \ , \ 1 \le p < \infty \ ,$$

are the RC-factors of I at x^*.

Note that when $\{x^k\}$ converges to x^*, there is always a $k_o \ge 0$ such that

$$0 \le \|x^k - x^*\| < 1 \quad , \quad k \ge k_o$$

and hence we have $0 \le R_p\{x^k\} \le 1$ for all $p \ge 1$.

In contrast to the QC-factors of a sequence, the RC-factors are always independent of the norm on R^n.

<u>Theorem A.2.2</u>. Let $\{x^k\} \subset R^n$ be any sequence that converges to x^*. Then $R_p\{x^k\}$ is independent of the norm on R^n for any $p \in [1,\infty)$.

<u>Proof</u>. Let $\|\cdot\|$ and $\|\cdot\|'$ be any two norms on R^n and let $\{\gamma_k\}$ be any sequence of positive real numbers with $\lim_{k\to\infty} \gamma_k = 0$. Then the inequality A.1-2 again holds with certain constants $d \ge c > 0$ and, since $\lim_{k\to\infty} a^{\gamma_k} = 1$ for any real $a > 0$, we have

$$\begin{aligned}
\limsup_{k\to\infty} \|x^k - x^*\|^{\gamma_k} &\le \lim_{k\to\infty} \left(\frac{1}{c}\right)^{\gamma_k} \limsup_{k\to\infty} \|x^k - x^*\|'^{\gamma_k} \\
&\le \lim_{k\to\infty} \left(\frac{d}{c}\right)^{\gamma_k} \limsup_{k\to\infty} \|x^k - x^*\|^{\gamma_k} \\
&= \limsup_{k\to\infty} \|x^k - x^*\|^{\gamma_k} \ . \quad \text{Q.E.D.}
\end{aligned}$$

As an immediate corollary of Theorem A.2.2, it follows, of course, that $R_p(I,x^*)$ is also always independent of the norm.

The following result shows that as functions of p the RC-factors behave in a manner analogous to the QC-factors.

Theorem A.2.3. Let I be an iterative process with limit x^*. Then exactly one of the following conditions holds:

(a) $R_p(I,x^*) = 0$, $\forall\ p \in [1,\infty)$.

(b) $R_p(I,x^*) = 1$, $\forall\ p \in [1,\infty)$.

(c) There is a $p_o \in [1,\infty)$ such that $R_p(I,x^*) = 0$, $\forall\ p \in [1,p_o)$, and $R_p(I,x^*) = 1$, $\forall\ p \in (p_o,\infty)$.

Proof. Let $\{x^k\}$ be an arbitrary sequence that converges to x^*, and set $\gamma_{1k} = \frac{1}{k}$, $\gamma_{pk} = 1/p^k$ for $p > 1$, $k = 1,2,\ldots$. Then $\lim\limits_{k\to\infty} \gamma_{qk}/\gamma_{pk} = \infty$ whenever $1 \le q < p$. Now suppose that $R_p\{x^k\} < 1$ for some $p \in (1,\infty)$ and choose $\epsilon > 0$ so that $R_p\{x^k\} + \epsilon = \alpha < 1$. Next set $\epsilon_k = \|x^k - x^*\|$ and choose $k_o \ge 0$ so that

$$\epsilon_k^{\gamma_{pk}} \le \alpha \quad , \quad \forall\ k \ge k_o .$$

Then we find that for any $q \in [1,p)$

$$R_q\{x^k\} = \limsup_{k\to\infty} (\epsilon_k^{\gamma_{pk}})^{\gamma_{qk}/\gamma_{pk}} \le \lim_{k\to\infty} \alpha^{\gamma_{qk}/\gamma_{pk}} = 0 ,$$

that is, $R_q\{x^k\} = 0$ whenever $q < p$ and $R_p\{x^k\} < 1$. This in turn also shows that $R_q\{x^k\} = 1$ whenever $q > p$ and $R_p\{x^k\} > 0$. Hence one and only one of the properties (a), (b), or (c) holds for any convergent sequence. Now assume that neither (a) nor (b) holds; then $p_o = \inf\{p \in [1,\infty) \mid R_p(I,x^*) = 1\}$ is well-defined. Suppose there is a $p > p_o$ such that $R_p(I,x^*) < 1$. Then $R_p\{x^k\} < 1$ for all $\{x^k\} \in C(I,x^*)$ while, by definition of p_o, there exists a $p' \in [p_o,p)$ such that $R_{p'}(I,x^*) = 1$. In particular, therefore, $R_{p'}\{x^k\} > 0$ for some sequence in $C(I,x^*)$ which by the first part of the proof implies that $R_p\{x^k\} = 1$ for this sequence, contrary to assumption. Thus $R_p(I,x^*) = 1$ for $p > p_o$, and similarly it follows that $R_p(I,x^*) = 0$ for $p < p_o$. Q.E.D.

We may now proceed along the same lines as in Section A.1.

Definition A.2.4. Let I_1 and I_2 be two iterative processes with limit x^*. Then I_1 is R-faster than I_2 at x^* if there exists a $p \in [1,\infty)$ such that $R_p(I_1,x^*) < R_p(I_2,x^*)$.

Note that Theorem A.2.3 shows that the concept of R-faster is well-defined, while Theorem A.2.2 ensures that when I_1 is R-faster than I_2 in some norm, the same relation holds in every norm on R^n.

Definition A.2.5. Let I be an iterative process with limit point x^*. Then the quantity

$$O_R(I,x^*) = \begin{cases} \infty & , \text{ if } R_p(I,x^*) = 0 \, , \\ & \quad \forall\, p \in [1,\infty) \\ \inf\{p\in[1,\infty) \,|\, R_p(I,x^*) = 1\} & , \text{ otherwise} \end{cases}$$

is called the R-order of I at x^*.

Again Theorems A.2.2 and A.2.3 show that the R-order is well-defined and norm-independent. Moreover, the following results are immediate consequences of Theorem A.2.3.

Theorem A.2.6. Let I_1 and I_2 be iterative processes with the same limit point x^*. If $O_R(I_1,x^*) > O_R(I_2,x^*)$ then I_1 is R-faster than I_2 at x^*.

Theorem A.2.7. Let I be an iterative process with limit x^*. If $R_p(I,x^*) < 1$ for some $p \in [1,\infty)$ then $O_R(I,x^*) \geq p$. If $R_q(I,x^*) > 0$ for some $q \in [1,\infty)$ then $O_R(I,x^*) \leq q$.

We note that whenever $R_p = R_p(I,x^*) < 1$, then for any $\varepsilon > 0$ with $R_p + \varepsilon < 1$, there is a $k_o \geq 0$, depending on each sequence $\{x^k\} \in C(I,x^*)$ such that either

$$\|x^k - x^*\| \le (R_p+\varepsilon)^{p^k} \quad , \forall\ k \ge k_o\ , \quad \text{if} \quad p > 1\ ,$$

or

$$\|x^k - x^*\| \le (R_1+\varepsilon)^{k} \quad , \forall\ k \ge k_o\ , \quad \text{if} \quad p = 1$$

In the latter case, the convergence of any sequence generated by I and converging to x^* is ultimately as rapid as a geometric progression with ratio $R_1 + \varepsilon < 1$.

If $0 < R_1(I,x^*) < 1$ we shall say that the convergence of I at x^* is R-linear, while for $R_1 = 1$ or $R_1 = 0$ we call the convergence R-sublinear or R-superlinear, respectively. Similarly, if $0 < R_2(I,x^*) < 1$ we say that the convergence is R-quadratic at x^* . Note that in this case these concepts are norm-independent.

We conclude this section with a result about R-orders which will be of value in connection with multi-step iterative processes. For this we need first the following lemma.

Lemma A.2.8. For any integer $m \ge 1$, the polynomial $p_m(t) = t^{m+1} - t^m - 1$ has a unique positive root τ_m . Moreover, $\tau_m \in (1,2)$, $\tau_m > \tau_{m+1}$, and $\lim_{m\to\infty} \tau_m = 1$.

Proof. Since $p_m(1) = -1$ and $p_m(2) = 2^m - 1 > 0$, there is a root τ_m in $(1,2)$ and, by Descartes' rule of signs, there can be no other positive root. To show that τ_m is monotone decreasing with m, note that $p_{m+1}(\tau_m) = \tau_m - 1 > 0$ so that $\tau_{m+1} \in (1,\tau_m)$. Finally, suppose that $\lim_{m\to\infty} \tau_m = \tau > 1$. Then, for m sufficiently large, $p_m(\tau) = \tau^m(\tau-1) - 1 > 0$ and hence p_m has a root in $(1,\tau)$ which is a contradiction. Q.E.D.

Theorem A.2.9. Let I be an iterative process and $C(I,x^*)$ the set of sequences generated by I which converge to the limit x^*. Furthermore, let $\gamma_0,\gamma_1,\ldots,\gamma_m$ be non-negative constants. If for any $\{x^k\} \in C(I,x^*)$ there is a k_0 such that

$$(1) \qquad \|x^{k+1}-x^*\| \le \|x^k-x^*\| \sum_{j=0}^{m} \gamma_j \|x^{k-j}-x^*\| , \quad \forall\ k \ge k_0 (\ge m) ,$$

then $O_R(I,x^*) \ge \tau$ where τ is the unique positive root of $t^{m+1} - t^m - 1 = 0$. If, in addition, there exists a $\beta > 0$ and some sequence $\{x^k\} \in C(I,x^*)$ such that for a $k_0 \ge m$

$$(2) \qquad \|x^{k+1}-x^*\| \ge \beta\|x^k-x^*\| \ \|x^{k-m}-x^*\| > 0 , \quad \forall\ k \ge k_0 ,$$

then $O_R(I,x^*) = \tau$.

Proof. We may assume that $\gamma = \sum_{j=0}^{m} \gamma_j > 0$ or else the first result is trivial and (2) could not hold.

Let $\{x^k\} \in C(I,x^*)$ and set $\epsilon_k = \|x^k - x^*\|$, $\eta_k = \epsilon_k \gamma$,

and $\delta_j = \gamma_j/\gamma$. Then $\sum_{j=0}^{m} \delta_j = 1$ and the sequence $\{\eta_k\}$

satisfies

(3) $$\eta_{k+1} \le \eta_k \sum_{j=0}^{m} \delta_j \eta_{k-j} \quad , \quad \forall\ k \ge k_o \ge m\ .$$

Since $\{\epsilon_k\}$ converges to zero, we have $\eta_k \le \eta < 1$ for $k \ge k'(\ge k_o)$. Hence it follows from (3) that

$$\eta_{k'+m+1} \le \sum_{j=0}^{m} \delta_j \eta \le \eta^2 \ ,$$

$$\eta_{k'+m+2} \le \delta_o \eta^4 + \eta^2 \sum_{j=1}^{m} \delta_j \eta \le \eta^3 \ ,$$

and by induction

(4) $$\eta_{k'+i} \le \eta^{\mu_i} \ , \ i = 0,1,\ldots,$$

where

(5) $$\mu_{i+1} = \mu_i + \mu_{i-m} \ , \ i = m, m+1, \ldots, \ \mu_o = \mu_1 = \ldots = \mu_m = 1\ .$$

In fact, if (4) holds up to some $i \ge m$ then

$$\eta_{k'+i+1} \le \eta^{\mu_i} \sum_{j=0}^{m} \delta_j \eta^{\mu_{i-j}} \le \eta^{\mu_i + \mu_{j-m}} \ ,$$

since the μ_i are monotonically increasing.

We next show that the μ_i satisfy

(6) $$\mu_i \geq \alpha\tau^i \ , \ i = 0,1,\ldots$$

where $\alpha = \tau^{-m}$. In fact, since by Lemma A.2.8 , $\tau > 1$, (6) holds for $i = 0,1,\ldots,m$ and if it is valid up to some $i \geq m$ then

$$\mu_{i+1} = \mu_i + \mu_{i-m} \geq \alpha\tau^i + \alpha\tau^{i-m} = \alpha\tau^{i+1}(\tau^{-1}+\tau^{-m-1}) = \alpha\tau^{i+1}$$

which completes the induction. Therefore (4) and (6) together imply that

$$\epsilon_{k'+i} \leq \gamma^{-1}\eta_{k'+i} \leq \gamma^{-1}\eta^{\alpha\tau^i} \ , \ \forall\ i \geq 0$$

so that

$$R_\tau\{x^k\} = \limsup_{i\to\infty} \epsilon_{k'+i}^{\frac{1}{\tau^{k'+i}}} \leq \eta^{\frac{\alpha}{\tau^{k'}}} < 1 .$$

Hence for any $\epsilon > 0$ we have $R_{\tau-\epsilon}\{x^k\} = 0$ for all $\{x^k\} \in C(I,x^*)$, and thus $R_{\tau-\epsilon}(y,x^*) = 0$, or by Theorem A.2.7, since ϵ is arbitrary, $O_R(I,x^*) \geq \tau$.

Now suppose that (2) holds for some $\{x^k\} \in C(I,x^*)$; then we have for $\eta_k = \beta\epsilon_k = \beta\|x^k-x^*\|$,

(7) $$\eta_{k+1} \geq \eta_k \eta_{k-m} , \quad k \geq k_o .$$

Let $k' \geq k_o$ be such that $\eta_k \leq \eta < 1$, $\forall\, k \geq k'$ and set $\eta = \min(\eta_{k'},\ldots,\eta_{k'+m})$. Then, similarly as before,

(8) $$\eta_{k'+i} \geq \eta^{\mu_i} , \; i = 0,1,\ldots$$

where $\{\mu_i\}$ is defined by (5). In fact, (8) clearly holds for $i = 0,\ldots,m$ and if it is valid for $i \geq m$ then

$$\eta_{k'+i+1} \geq \eta_{k'+i}\eta_{k'+i-m} \geq \eta^{\mu_i \mu_{i-m}} .$$

Moreover, we have

(9) $$\mu_i \leq \tau^i , \; i = 0,1,\ldots$$

since this is certainly true for $i = 0,\ldots,m$ because of $\tau > 1$, and if (9) holds for some $i \geq m$, then

$$\mu_{i+1} = \mu_i + \mu_{i-m} \leq \tau^i + \tau^{i-m} = \tau^{i+1}(\tau^{-1}+\tau^{-m-1}) = \tau^{i+1} .$$

Therefore, altogether

$$\eta_{k'+i} \geq \eta^{\mu_i} \geq \eta^{\tau^i} , \; \forall\, i \geq 0 ,$$

or

$$R_\tau\{x^k\} \geq \eta^{\frac{1}{\tau^{k'}}} > 0 ,$$

so that also $R_\tau(y,x^*) > 0$, and by Theorem A.2.7, $O_R(I,x^*) \leq \tau$. Q.E.D.

All of the results of this section extend to Banach spaces except for Theorem A.2.2.

Section A.3

Relations Between the R- and Q-Convergence Factors

We turn now to the important question of the relation between $Q_p(I,x^*)$ and $R_p(I,x^*)$. When $\{x^k\} \subset R^n$ converges to x^* and $0 < Q_p\{x^k\} < +\infty$ as well as $0 < R_p\{x^k\} < 1$ for some $p > 1$, then we can always achieve either one of the relations $R_p\{x^k\} < Q_p\{x^k\}$ or $R_p\{x^k\} > Q_p\{x^k\}$ by selecting a suitable norm. It is an important fact, however, that this cannot be done if $p = 1$.

<u>Theorem A.3.1</u>. Let $\{x^k\} \subset R^n$ be a sequence convergent to x^* . Then $R_1\{x^k\} \le Q_1\{x^k\}$ in every norm. Consequently, if I is an iterative process with limit point x^* , then also $R_1(I,x^*) \le Q_1(I,x^*)$ in every norm.

<u>Proof</u>. Assume that $Q_1\{x^k\} < \infty$ and set $\epsilon_k = \|x^k - x^*\|$. Then for any $\epsilon > 0$ and $\gamma = Q\{x^k\} + \epsilon$, there is a $k_o \ge 0$ such that

$$\epsilon_k \le \gamma \epsilon_{k-1} \le \dots \le \gamma^{k-k_o} \epsilon_{k_o} \ , \ \forall \ k \ge k_o \ .$$

Therefore,

$$R_1\{x^k\} \le \gamma \limsup_{k\to\infty} \left[\frac{\epsilon_{k_o}}{\gamma^{k_o}}\right]^{\frac{1}{k}} = \gamma$$

and, since ϵ was arbitrary, it follows that $R_1\{x^k\} \le Q_1\{x^k\}$. But then, if $C(I,x^*)$ again denotes all sequences generated by I and converging to x^* , we find that

$$R_1(I,x^*) = \sup \{R_1\{x^k\} | \{x^k\} \in C(I,x^*)\}$$

$$\le \sup \{Q_1\{x^k\} | \{x^k\} \in C(I,x^*)\} = Q_1(I,x^*) \ . \text{ Q.E.D.}$$

As an immediate consequence of Theorem A.3.1, we see that $R_1\{x^k\}$ is a lower bound for any possible constant γ in the estimate

$$\|x^{k+1}-x^*\| \le \gamma \|x^k-x^*\| , \forall k \ge k_o .$$

As mentioned before, for $p > 1$ the relation between R_p and Q_p will in general depend on the norm. Nevertheless, we can always compare the R- and Q-orders.

Theorem A.3.2. Let I be an iterative process with limit x^*. Then

$$O_Q(I,x^*) \le O_R(I,x^*) .$$

Proof. Suppose that $\{x^k\} \subset R^n$ converges to x^*. We show first that $Q_p\{x^k\} < \infty$ for $p > 1$ implies that $R_p\{x^k\} < 1$. Set $\epsilon_k = \|x^k-x^*\|$, $k = 0,1,\ldots,$ and, for a given $\epsilon > 0$, let $\gamma = Q_p\{x^k\} + \epsilon$. Then there is a k_o such that

$$\epsilon_{k+1} \le \gamma\epsilon_k^p \le \ldots \le \gamma^{1+p+\ldots+p^{k-k_o}} \epsilon_{k_o}^{p^{k-k_o+1}} , \forall k \ge k_o ,$$

and hence

$$\epsilon_{k+1}^{1/p^{k+1}} \le \gamma^{\sum_{j=k_o+1}^{k+1} 1/p^j} \epsilon_{k_o}^{1/p^{k_o}} \le (\gamma'\epsilon_{k_o})^{1/p^{k_o}}$$

where $\gamma' = \max(1,\gamma^{\frac{1}{p-1}})$. But, since $\lim_{k\to\infty} \epsilon_k = 0$, we may assume that k_o has been chosen so that $\gamma'\epsilon_{k_o} < 1$; then

$$R_p\{x^k\} \le (\gamma'\epsilon_{k_o})^{1/p^{k_o}} < 1 .$$

Now suppose that $q = O_Q(I,x^*) > O_R(I,x^*) = r$. Then it follows immediately from Theorems A.1.3, A.2.3, and the definition of the orders, that $Q_p(I,x^*) = 0$ and $R_p(I,x^*) = 1$ for all $p \in (r,q)$. Hence for any sequence $\{x^k\}$ generated by I and converging to x^* , we have $Q_{p'}\{x^k\} = 0$ for $p' = (r+q)/2$, and, by the first part of the proof, $R_{p'}\{x^k\} < 1$. Then it follows from Theorem A.2.3 that $R_p\{x^k\} = 0$ for any $p \in (r,p')$ and all $\{x^k\}$ generated by I ; that is, $R_p(I,x^*) = 0$, which is a contradiction. Q.E.D.

In many instances the orders will be identical. The following result gives one simple sufficient condition for this.

<u>Theorem A.3.3</u>. Let I be an iterative process and $C(I,x^*)$ the set of sequences generated by I which converge to the limit x^* . Suppose there exists a $p \in [1,\infty)$ and a constant c_2 such that for any $\{x^k\} \in C(I,x^*)$ there is a k_o such that

(1) $$\|x^{k+1}-x^*\| \le c_2 \|x^k-x^*\|^p , \forall k \ge k_o .$$

Then $0_R(I,x^*) \geq 0_Q(I,x^*) \geq p$. On the other hand, if there is a constant $c_1 > 0$ and some sequence $\{x^k\} \in C(I,x^*)$ such that

$$\text{(2)} \qquad \|x^{k+1}-x^*\| \geq c_1 \|x^k-x^*\|^p > 0 \ , \ \forall\, k \geq k_o = k_o(\{x^k\}) \ ,$$

then $0_Q(I,x^*) \leq 0_R(I,x^*) \leq p$. Hence if (1) and (2) hold then $0_Q(I,x^*) = 0_R(I,x^*) = p$.

<u>Proof</u>. If (1) holds, then $Q_p\{x^k\} \leq c_2$ for all $\{x^k\} \in C(I,x^*)$, so that $Q_p(I,x^*) \leq c_2 < +\infty$ and by Theorems A.1.8 and A.3.2 , $0_R(I,x^*) \geq 0_Q(I,x^*) \geq p$. Now suppose that (2) holds for some sequence $\{x^k\} \in C(I,x^*)$. Then $\epsilon_k = \|x^k-x^*\| > 0$ for $k \geq k_o$ and

$$\epsilon_{k+1} \leq c_1^{1+p+\dots+p^{k-k_o}} \epsilon_{k_o}^{p^{k-k_o+1}} \ , \ \forall\, k \geq k_o \ .$$

Therefore, if $p = 1$

$$R_1\{x^k\} \geq \lim_{k\to\infty} \left(c_1^{k-k_o+1} \epsilon_{k_o}\right)^{\frac{1}{k}} = c_1 > 0$$

while, if $p > 1$

$$\epsilon_{k+1}^{1/_p k+1} \geq c_1^{\sum_{j=k_o}^{k} 1/_p j+1} \epsilon_{k_o}^{1/_p k_o} \geq \min\left(1, c_1^{\frac{1}{p-1}}\right) \epsilon_{k_o}^{1/_p k_o} > 0 \ ,$$

so that again $R_p\{x^k\} > 0$. Hence $R_p(I,x^*) > 0$ and, by Theorems A.2.7 and A.3.2, $0_Q(I,x^*) \le 0_R(I,x^*) \le p$. Q.E.D.

The results of this section extend to Banach spaces.

LITERATURE CITED

L. Bittner, A Generalization of the Secant Process, Wiss. Z. Techn. Hochsch. Dresden 9, (1959), pp. 325-329.

K. Chen, Generalization of Steffensen's Method to Operator Equations, Comment. Math. Univ. Carolinae 5, (1964), pp. 47-77.

J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.

D. Greenspan and S. Parter, Mildly Nonlinear Elliptic Partial Differential Equations and Their Numerical Solution II, Numer. Math. 7, (1965), pp. 129-146.

A. Householder, The Theory of Matrices in Numerical Analysis, Blaisdell, New York, 1964.

A. Korganoff, Algebre nonlineare, Vol. 1, Methodes de Cacul Numerique, Dunod, Paris, 1961.

J. Ortega, Notes on Newton and Secant Methods in n Dimensions, Tech. Note, IBM Federal Systems Division, 1967.

J. Ortega and W. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, in press, Academic Press, New York, 1970.

J. Ortega and M. Rockoff, Nonlinear Difference Equations and Gauss-Seidel Type Iterative Methods, SIAM J. Numer. Anal. 3, (1966), pp. 497-513.

A. Ostrowski, Solutions of Equations and Systems of Equations, Second Edition, Academic Press, New York, 1966.

B. Polyak, Some Methods of Speeding Up the Convergence of Iteration Methods, Z. Vych. Mat. 4, (1964), pp. 791-803.

J. Schmidt, Rate of Convergence of the Regula Falsi and Steffensen Processes in a Banach Space, ZAMM 46, (1966), pp. 146-148.

L. Tornheim, Convergence of Multi-Point Iterative Methods, J. Assoc. Comput. Mach. 11, (1964), pp. 210-220.

J. Traub, Iterative Methods for the Solution of Equations, Prentice Hall, Englewood Cliffs, N. J., 1964.

S. Ul'm, A Generalization of Steffensen's Method for Solving Nonlinear Operator Equations, Z. Vycisl, Mat. i Mat. Fiz. 4, (1964), pp. 1093-1097.

M. Vainberg, Variational Methods for the Study of Nonlinear Operators, Transl. A. Feinstein, Holden-Day, San Francisco, 1964.

L. Wegge, On a Discrete Version of the Newton-Raphson Method, SIAM J. Numer. Anal. 3, (1966), pp. 134-142.

H. Weinitschke, On a Class of Iteration Procedures, Numer. Math. 6, (1964), pp. 395-404.