Quarterly Progress Report

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INVESTIGATION OF SYNCHRONOUS SATELLITES FOR GEODESY

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SUMMARY

The primary efforts on this research have been to develop satellite experiments to measure continental drift and Chandler wobble. However, efforts have also been made to develop orbit determination programs employing synchronous satellites in the tracking network and to determine the least number of subsynchronous satellites that will provide continuous coverage of a given region on the ground. This report presents brief discussions of the results obtained thus far and outlines the work to be done next.

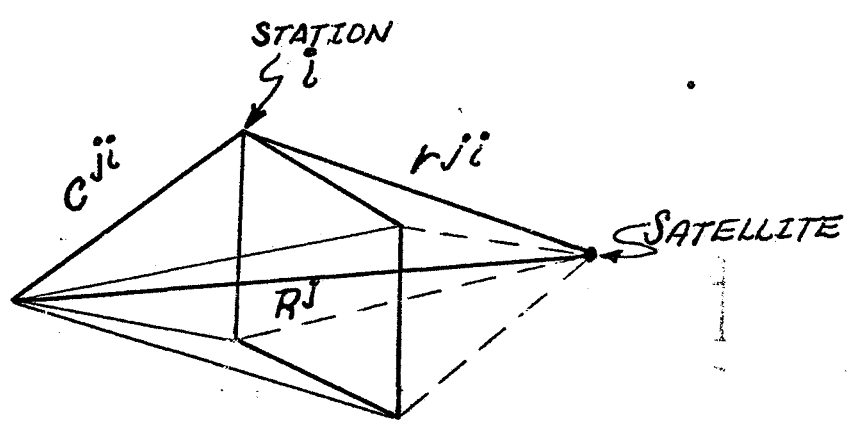
CONTINENTAL DRIFT

Introduction

It has been estimated that a Q- switched laser with an output pulse of 10ns can be used to determine the range to a satellite to a precision of 15cm. or less. A method has been developed that utilizes 12 ground stations equipped with the lasers and appropriate support equipment and one satellite at the critical inclination (63.4°). By making simultaneous observations of this satellite from 4 stations at a time for 30 satellite positions the interstation distances may be determined with again a precision of 15cm. Therefore it should be possible to obtain a direct measurement of some drift rates after only two or three years of observations.

Description of Method

The approach is briefly described below. Suppose that R^j represents the vector from the origin of some coordinate system to the satellite at the time of the jth measurement. Let c^{ji} represent the vector to the ith station and r^{ji} represent the range measurement (see sketch).



We have

$$c^{ji} + r^{ji} = R^j \qquad i = 1, \ldots, 4$$

$$j = 1, \ldots, 30$$

For a given j this provides four equations for five unknowns. In terms of components we have 120 equations for 126 unknowns (36 station coordinates and 90 satellite coordinates). The system can be made determinate by specifying six of the unknowns. This cannot be done arbitrarily but it can be done as follows. Specify the coordinates of one station (arbitrarily), then two components of the direction of motion of a second station relative to the first, and finally one component of the relative motion of a third station. For this analysis we selected.

$$\delta c_1^{12} = \delta c_2^{12} = \delta c_3^{12} = \delta c_1^{11} = \delta c_2^{11} = \delta c_3^{10} = 0 \; .$$

The range equation may be written in terms of components as

$$\sum_{k=1}^{3} (R_k^j)^2 + \sum_{k=1}^{3} (c_k^{ji})^2 - 2\sum_{k=1}^{3} R_k^j c_k^{ji} - \left|r^{ji}\right|^2 = f^j = 0 \; .$$

Perturbation equations may be written as

$$\sum_{k=1}^{3} (R_k^j - c_k^{ji})\delta c_k^{ji} - \sum_{k=1}^{3} (R_k^j - c_k^{ji})\delta R_k^j + \frac{1}{2} f^j(X^N) = 0$$

where X^N represents the vector (c,R) about which perturbations are taken. The above perturbation equation may be expressed in matrix form as

$$[A]\left\{\frac{\delta c}{\delta R}\right\} = f$$

which may be inverted to give

$$\left\{\frac{\delta c}{\delta R}\right\} = [A]^{-1} f \; .$$

This provides an iterative procecure for locating stations and determining interstation distances. The iterative procedure is described as follows:

1. Guess nominal values for R, c.
2. Obtain a complete set of range measurements r.
3. Iterate to obtain the correct values of R, c.

The interstation distances are obtained from

$$D^2(I,J) = \sum_{k=1}^{3} [C(I,K) - C(J,K)]^2$$

where D(I,J) represents the distance between stations I and J.

After once obtaining the interstation distances from one complete set of measurements additional measurements may be taken and the interstation distances may be updated. This will provide a direct measurement of drift rates.

Current Work

The sensitivity of the interstation distance to R increases as R increases; however, the uncertainties in the measurements of r decrease as R increases. Hence it is of interest to determine the best satellite altitude to minimize the sensitivity of the distance calculations to R and r. This is currently being examined. We are also examining the possibilities of equipping the satellite with the lasers and using the ground stations as targets. The cost trade-offs are being examined.

OBSERVATION OF POLAR WANDERING USING SYNCHRONOUS SATELLITES

Introduction

The scheme described in this report consists of a free synchronous satellite and two neighboring controllable satellites. The free satellite is placed in a synchronous equatorial orbit and then a maneuverable vehicle is separated from the free satellite and descends to a distance Δ inside the synchronous orbit. The controllable satellite is forced to maintain a circular synchronous orbit by continuous thrusting directed away from the center of the Earth. In addition, the controllable satellite rides on a laser beam transmitted radially outward from a point on the Earth's surface. Distance measurements are made between the free satellite and the controllable satellite over an extended period of time, and these measurements are analysed to obtain information about wandering of the Earth's pole relative to its spin axis.

The procedure just described must be refined to the extent that two controllable satellites be used instead of one. Two controllable satellites are necessary because determination of polar wandering involves the determination of two independent coordinates. In practice, it may also be necessary that each controllable satellite remain at the intersection of three laser beams rather than riding a single beam.

In order to separate out the effect of polar wandering on the satellite separation distances, it is necessary that the perturbations of the free satellite and the motion of the Earth's spin axis be known a priori. A perturbation analysis is included in the Appendix and the results are used to formulate a numerical example.

Polar Wandering Coordinates

Define the pole to be the point on the Earth's surface at which the spin axis intersects the surface on a particular date. The wandering of the pole involves a motion of the original intersection point relative to the spin axis as shown in Fig. 1.

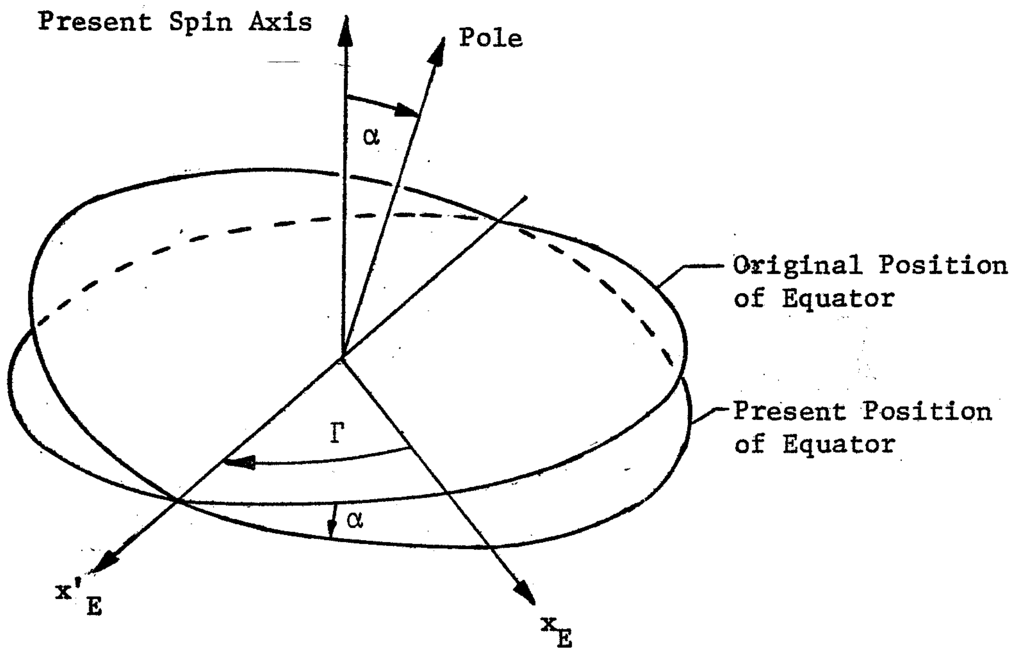


Fig. 1. Polar Wandering Coordinates.

The position of the pole at any time can be specified by the coordinates α and Γ, where α is the angular deviation of the present equator reckoned counterclockwise about the positive x_E' axis, and Γ is the longitude of the polar wandering node reckoned westward from the positive x_E axis.

Satellite Separation Distance

The motion of the Earth's spin axis in space is specified by an inclination i and a longitude Ω as shown in Fig. 2. The angle i is reckoned counterclockwise about the positive x_I axis and Ω is reckoned eastward from the reference vernal equinox, ♈. All quantities having the dimension of length are normalized with respect to the Earth's equatorial radius R, (R = 6378.39 km.).

The law of cosines is used to find D in the terms of Φ as follows,

$$D^2 = (\rho_s - \Delta)^2 + (\rho_s + \delta\rho)^2 - 2(\rho_s - \Delta)(\rho_s + \delta\rho)\cos\Phi. \qquad (1)$$

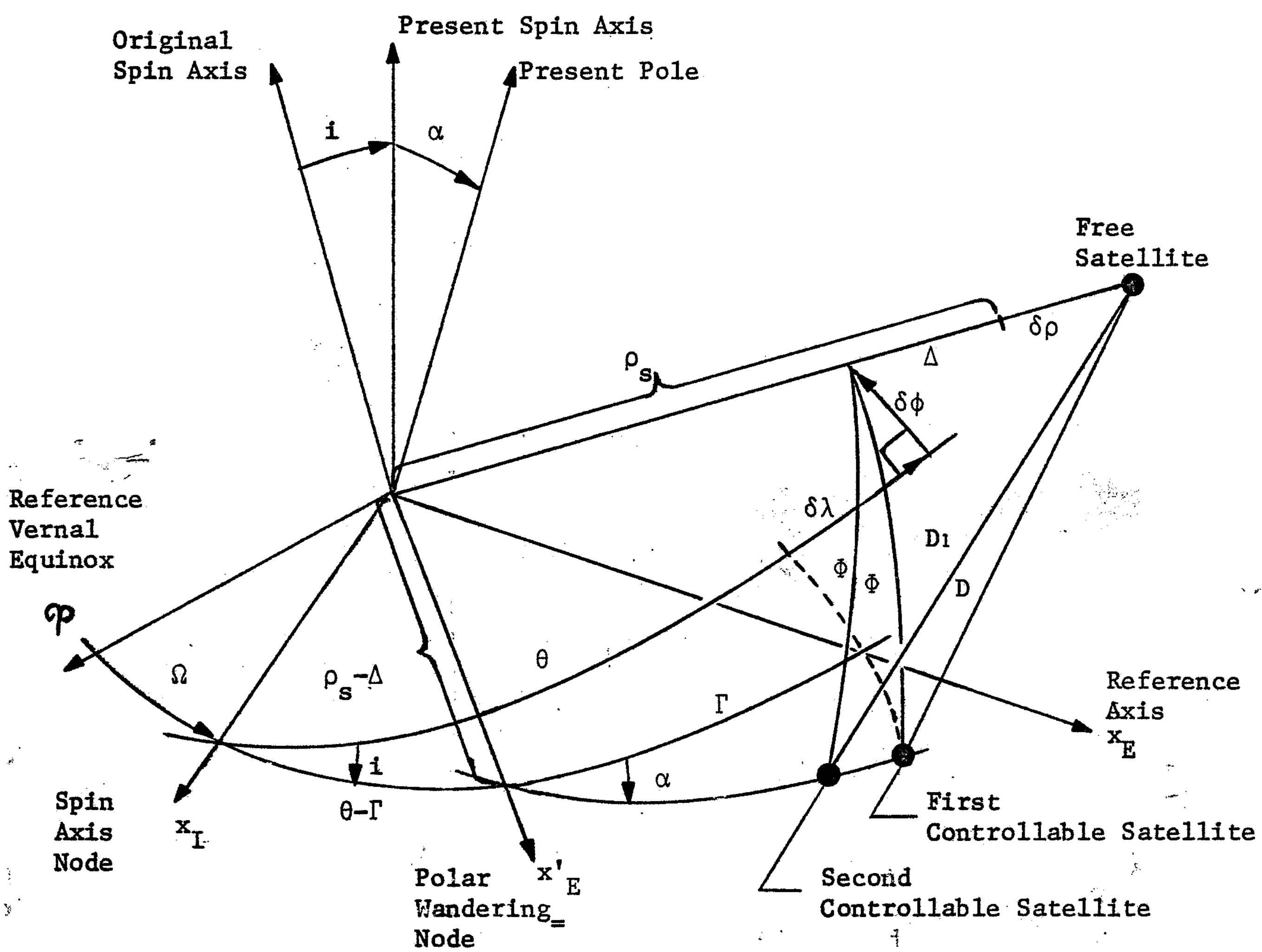


Fig. 2. Satellite Separation Distance Geometry

The solution for $\cos\Phi$ is obtained by employing the following relationships from spherical trigonometry associated with Fig. 3:

$$\cos\Phi = \cos\Theta \cos\tilde{\Phi} + \sin\Theta \sin\tilde{\Phi} \cos(\mu + i + \xi) \tag{2}$$

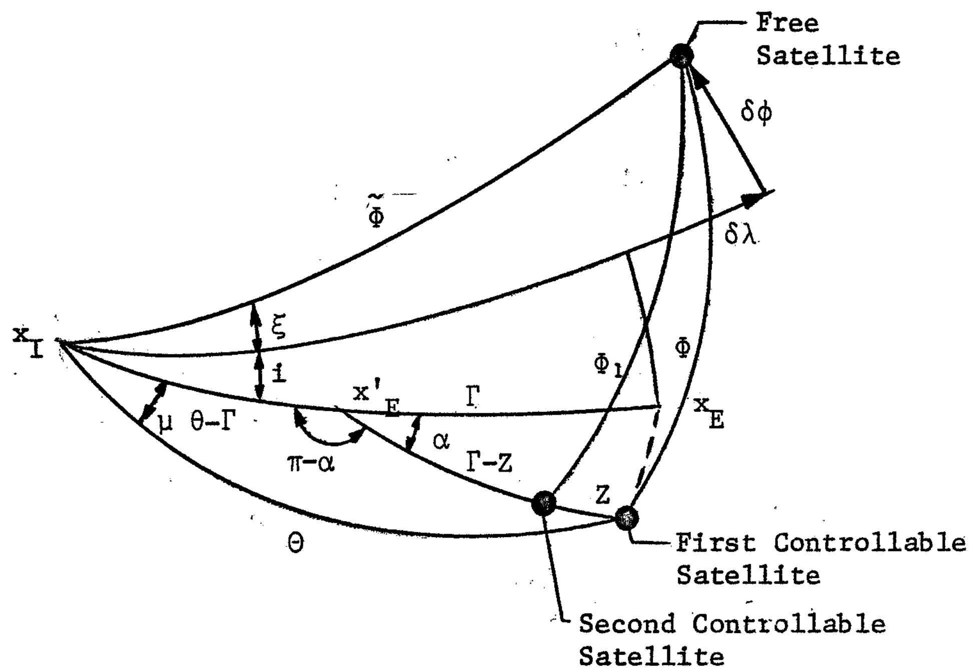


Fig. 3. Spherical Triangles for Separation Distance Analysis

$$\cos\Theta = \cos(\theta-\Gamma) \cos\Gamma - \sin(\theta-\Gamma) \sin\Gamma \cos\alpha \quad , \tag{3}$$

$$\sin\mu = \frac{1}{\sin\Theta} \sin\alpha \sin\Gamma \quad , \tag{4}$$

$$\cos\mu = \frac{\cos\Gamma - \cos\Theta \cos(\theta-\Gamma)}{\sin\Theta \sin(\theta-\Gamma)} \quad , \tag{5}$$

$$\cos\tilde{\Phi} = \cos(\theta+\delta\lambda) \cos(\delta\phi) \quad , \tag{6}$$

$$\cos\xi = \tan(\theta+\delta\lambda) \cot\tilde{\Phi} \quad . \tag{7}$$

After a considerable amount of algebraic manipulation, it is found that

$$
\begin{aligned}
\cos\Phi = {} & \cos(\theta+\delta\lambda)\cos(\delta\phi)\left[\cos\theta\ (\cos^2\Gamma + \cos\alpha\ \sin^2\Gamma)\right.\\
& \left. + \sin\theta\ \sin\Gamma\ \cos\Gamma(1 - \cos\alpha)\right]\\
& + \left[\sin(\theta+\delta\lambda)\ \cos i\ \cos(\delta\phi) = \sin i\ \sin(\delta\phi)\right] .\\
& \cdot \left[\sin\theta\ (\cos^2\Gamma + \cos\alpha\ \sin^2\Gamma) - \cos\theta\ \sin\Gamma\ \cos\Gamma\right.\\
& \quad \left.(1 - \cos\alpha)\right]\\
& - \sin\alpha\ \sin\Gamma\left[\sin i\ \sin(\theta+\delta\lambda)\ \cos(\delta\phi)\right.\\
& \left. + \cos i\ \sin(\delta\phi)\right] . \qquad (8)
\end{aligned}
$$

The next step is to substitute series expansions for all of the small angles, $\delta\lambda$, $\delta\phi$, α, and i appearing in Equation 8. If terms through second order in the small angles are retained, the resulting expression for $\cos\Phi$ is

$$
\begin{aligned}
\cos\Phi = {} & 1 - \frac{\alpha^2}{2}\sin^2\Gamma - \frac{1}{2}(\delta\lambda^2 + \delta\phi^2)\\
& - \sin^2\theta\ \frac{i^2}{2} - i\ \sin\theta\,(\delta\phi)\\
& - \alpha\sin\Gamma(i\ \sin\theta + \delta\phi). \qquad (9)
\end{aligned}
$$

Equation 9 for $\cos\Phi$ is now substituted into Equation (1) for D^2 and terms through second order in the small angles and $\delta\rho$ are retained to give

$$
\begin{aligned}
D^2 = {} & \Delta^2 + 2\Delta\delta\rho + \delta\rho^2 + 2(\rho_s - \Delta)\rho_s\left[\sin^2\Gamma\ \frac{\alpha^2}{2}\right.\\
& + \frac{1}{2}(\delta\lambda^2 + \delta\phi^2) + \frac{i^2}{2}\sin^2\theta + i(\delta\phi)\ \sin\theta\\
& \left. + \alpha\sin\Gamma\ (i\ \sin\theta + \delta\phi)\right]. \qquad (10)
\end{aligned}
$$

Equation 10 is a quadratic in $\alpha\sin\Gamma$. The solution is

$$
\alpha\sin\Gamma = -\ (i\ \sin\theta + \delta\phi) \pm \sqrt{\frac{D^2-(\Delta+\delta\rho)^2}{{\rho_s}^2} - \delta\lambda^2}\ . \qquad (11)
$$

The second controllable satellite is spaced an angular distance Z away from the first controllable satellite. The expressions for arc and separation distance D_1 corresponding to the second controllable satellite are

$$\cos\Phi_1 = \cos Z\ (1 - \frac{\alpha^2}{2}\sin^2\Gamma) + \frac{\alpha^2}{2}\sin Z\ \sin\Gamma\ \cos\Gamma$$
$$- \frac{1}{2}(\delta\lambda^2 + \delta\phi^2)\cos Z - \sin Z(\delta\lambda)$$
$$- \frac{i^2}{2}\sin\theta\ \sin(\theta - Z) - i(\delta\phi)\sin(\theta - Z)$$
$$- \alpha\sin(\Gamma - Z)(i\ \sin\theta + \delta\phi), \tag{12}$$

and

$$D_1{}^2 = 2\rho_s\ (\rho_s - \Delta)(1 - \cos Z) + \Delta^2 + 2\rho_s\delta\rho + \delta\rho^2$$
$$- 2\delta\rho(\rho_s - \Delta)(\cos Z - \sin Z\ \delta\lambda)$$
$$+ 2\rho_s\ (\rho_s - \Delta)\ [\cos Z\ \frac{1}{2}\ \alpha^2\sin^2\Gamma$$
$$- \frac{1}{2}\ \alpha^2\sin Z\ \sin\Gamma\ \cos\Gamma + \frac{1}{2}(\delta\lambda^2 + \delta\phi^2)\cos Z$$
$$+ \sin Z(\delta\lambda) + \frac{1}{2}\ i^2\ \sin\theta\ \sin(\theta - Z)$$
$$+ i(\delta\phi)\sin(\theta - Z) + \alpha i\ \sin\theta\ \sin(\Gamma - Z)$$
$$+ \alpha(\delta\phi)\sin(\Gamma - Z)]\ . \tag{13}$$

Equation 13 is linear in $\alpha\cos\Gamma$. The solution for $\alpha\cos\Gamma$ is

$$\alpha\cos\Gamma = \frac{1}{\sin Z[\frac{1}{2}(\alpha\sin\Gamma) + i\ \sin\theta + \delta\phi]}\left\{\frac{1}{2\rho_s(\rho_s - \Delta)}[\Delta^2\right.$$
$$+ 2\rho_s\delta\rho + \delta\rho^2 - 2\delta\rho(\rho_s - \Delta)(\cos Z - \sin Z\ \delta\lambda) - D_1{}^2]$$
$$+ 1 - \cos Z + \frac{1}{2}\cos Z\ (\alpha\sin\Gamma)^2$$
$$+ \frac{1}{2}(\delta\lambda^2 + \delta\phi^2)\cos Z + \sin Z\ \delta\lambda$$
$$+ \frac{1}{2}\ i^2\ \sin\theta\ \sin(\theta - Z) + i\ \delta\phi\ \sin(\theta - Z)$$
$$\left. + i\ \sin\theta\ \cos Z\ (\alpha\sin\Gamma) + \delta\phi\ \cos Z\ (\alpha\sin\Gamma)\right\}. \tag{14}$$

This completes the separation distance analysis.

Location of the Polar Wandering Node

The location of the polar wandering node (i.e., the x_E^1 axis) is known once $\tan\Gamma$ is specified. In order to obtain $\tan\Gamma$, Equation 11 for $\alpha\sin\Gamma$ is divided by Equation 14 for $\alpha\cos\Gamma$. It is noteworthy that the ambiguous $\pm$ sign in the $\alpha\sin\Gamma$ equation cancels out in the algebraic manipulations and the result can be written unambiguously as

$$
\begin{aligned}
\tan\Gamma = \frac{1}{2}\sin Z \Big[\frac{D^2-(\Delta+\delta\rho)^2}{\rho_s^2} - \delta\lambda^2 - (i\sin\theta+\delta\phi)^2\Big] \\
\cdot\Big\{\frac{1}{2\rho_s(\rho_s-\Delta)}\big[\Delta^2 + 2\rho_s\,\delta\rho + \delta\rho^2 \\
- 2\delta\rho(\rho_s-\Delta)(\cos Z - \sin Z\,\delta\lambda) - D_1{}^2\big] \\
+ 1 - \cos Z + \frac{1}{2}\cos Z\ \frac{D^2-(\Delta+\delta\rho)^2}{s} \\
+ \sin Z\,\delta\lambda - \frac{1}{2} i^2 \sin\theta\cos\theta\sin Z \\
- i\,\delta\phi\cos\theta\sin Z\Big\}^{-1} .
\end{aligned}
\qquad (15)
$$

Sensitivity Relations

It is apparent that inaccuracies in the values used for known quantities will introduce corresponding erros in the values calculated for α and Γ. The sensitivities, $\partial\alpha/\partial q_j$ and $\partial\Gamma/\partial q_j$, (where q_j denotes one of the ten known quantities $\delta\rho$, $\delta\lambda$, $\delta\phi$, D, D_1, i, θ, ρ_s, Δ, and Z), are obtained from the expressions for $\alpha\sin\Gamma$ and $\alpha\cos\Gamma$ as follows:

$$
\frac{\partial(\alpha\sin\Gamma)}{\partial q_j} = \sin\Gamma\,\frac{\partial\alpha}{\partial q_j} + \alpha\cos\Gamma\,\frac{\partial\Gamma}{\partial q_j}
$$

$$
\frac{\partial(\alpha\cos\Gamma)}{\partial q_j} = \cos\Gamma\,\frac{\partial\alpha}{\partial q_j} - \alpha\sin\Gamma\,\frac{\partial\Gamma}{\partial q_j} .
$$

Therefore,

$$\frac{\partial\alpha}{\partial q_j} = \sin\Gamma\,\frac{\partial(\alpha\sin\Gamma)}{\partial q_j} + \cos\Gamma\,\frac{\partial(\alpha\cos\Gamma)}{\partial q_j} \tag{16}$$

$$\frac{\partial\Gamma}{\partial q_j} = \frac{1}{\alpha}\cos\Gamma\,\frac{\partial(\alpha\sin\Gamma)}{\partial q_j} - \frac{1}{\alpha}\sin\Gamma\,\frac{\partial(\alpha\cos\Gamma)}{\partial q_j} \tag{17}$$

These sensitivity relations are applied to Equations 11 and 14. For convenience, u and S are defined as

$u = \pm\ 1$; the ambiguity in Equation .

$$S = u\sqrt{\frac{1}{\rho_s^{\,2}}\left[D^2 - (\delta\rho + \Delta)^2\right] - \delta\lambda^2} \quad .$$

The sensitivity formulas are as follows:

$$\frac{\partial\alpha}{\partial(\delta\rho)} = -\frac{\sin\Gamma\ (\Delta+\delta\rho)}{\rho_s^{\,2}\ S} + \frac{2\ \cos\Gamma}{\sin Z\ (i\ \sin\theta + \delta\phi + S)\rho_s^{\,2}}$$
$$\left\{\rho_s + \delta\rho - (\cos Z - \delta\lambda\sin Z)(\rho_s-\Delta) - \cos Z\ (\delta\rho+\Delta) + \frac{1}{2S}\sin Z\ (\delta\rho+\Delta)(\alpha\cos\Gamma)\right\} \tag{18}$$

$$\frac{\partial\Gamma}{\partial(\delta\rho)} = -\frac{\cos\Gamma\ (\Delta+\delta\rho)}{\alpha\rho_s^{\,2}\ S} - \frac{2\ \sin\Gamma}{\alpha\sin Z\ (i\ \sin\theta + \delta\phi + S)\rho_s^{\,2}}$$
$$\left\{\rho_s + \delta\rho - (\cos Z - \delta\lambda\ \sin Z)(\rho_s-\Delta) - \cos Z\ (\delta\rho+\Delta) + \frac{\sin Z}{2S}\ (\delta\rho+\Delta)(\alpha\cos\Gamma)\right\} \tag{19}$$

$$\frac{\partial\alpha}{\partial(\delta\lambda)} = -\frac{\sin\Gamma\ \delta\lambda}{S} + \frac{2\ \cos\Gamma}{i\ \sin\theta + \delta\phi + S}\left\{1 + \frac{\delta\rho}{\rho_s} + \frac{\delta\lambda(\alpha\cos\Gamma)}{2\ S}\right\} \tag{20}$$

$$\frac{\partial\Gamma}{\partial(\delta\lambda)} = -\frac{\cos\Gamma\ \delta\lambda}{\alpha\ S} - \frac{2\ \sin\Gamma}{\alpha(i\ \sin\theta + \delta\phi + S}\left\{1 + \frac{\delta\rho}{\rho_s} + \frac{\delta\lambda(\alpha\cos\Gamma)}{2\ S}\right\} \tag{21}$$

$$\frac{\partial\alpha}{\partial(\delta\phi)} = -\sin\Gamma - \left\{\frac{2i\ \cos\theta + (\alpha\cos\Gamma)}{i\ \sin\theta + \delta\phi + S}\right\} \cos\Gamma \tag{22}$$

$$\frac{\partial\Gamma}{\partial(\delta\phi)} = -\frac{1}{\alpha}\cos\Gamma + \left\{\frac{2i\ \cos\theta + (\alpha\cos\Gamma)}{i\ \sin\theta + \delta\phi + S}\right\} \frac{1}{\alpha}\sin\Gamma \tag{23}$$

$$\frac{\partial\Gamma}{\partial(D)} = \frac{\sin\Gamma\ D}{\rho_s{}^2 S} + \frac{2\cos\Gamma\ D}{\rho_s{}^2\ \sin Z(i\ \sin\theta + \delta\phi + S)}\left\{1 - \frac{1}{2S}\sin Z(\alpha\cos\Gamma)\right\} \tag{24}$$

$$\frac{\partial\Gamma}{\partial(D)} = \frac{\cos\Gamma\ D}{\alpha\rho_s{}^2 S} - \frac{2\cos\Gamma\ D}{\alpha\rho_s{}^2\ \sin Z(i\ \sin\theta + \delta\phi + S)}$$

$$\cdot\left\{1 - \frac{1}{2S}\sin Z(\alpha\cos\Gamma)\right\} \tag{25}$$

$$\frac{\partial\alpha}{\partial(D_1)} = -\frac{2D_1\ \cos\Gamma}{\rho_s{}^2\ \sin Z(i\ \sin\theta + \delta\phi + S)} \tag{26}$$

$$\frac{\partial\Gamma}{\partial(D_1)} = \frac{2D\ \ \sin}{\alpha\rho_s{}^2\ \sin Z(i\ \sin\theta + \delta\phi + S)} \tag{27}$$

$$\frac{\partial\alpha}{\partial(i)} = -\sin\Gamma\ \sin\theta - \cos\Gamma\left\{\frac{\cos\theta(i\ \sin\theta + \delta\phi) + \sin\theta(\alpha\cos\Gamma)}{i\ \sin\theta + \delta\phi + S}\right\} \tag{28}$$

$$\frac{\partial\Gamma}{\partial(i)} = -\frac{1}{\alpha}\cos\Gamma\ \sin\theta + \frac{1}{\alpha}\sin\Gamma\left\{\frac{\cos\theta(i\ \sin\theta + \delta\phi) + \sin\theta(\alpha\cos\Gamma)}{i\ \sin\theta + \delta\phi + S}\right\} \tag{29}$$

$$\frac{\partial\alpha}{\partial(\theta)} = -\sin\Gamma\ i\ \cos\theta - \frac{\cos\Gamma}{i\ \sin\theta + \delta\phi + S}\left\{i^2\cos 2\theta\right.$$

$$\left. - 2i\ \delta\phi\ \sin\theta + i\ \cos\theta(\alpha\cos\Gamma)\right\} \tag{30}$$

$$\frac{\alpha\Gamma}{\partial(\theta)} = -\frac{1}{\alpha}\cos\Gamma\ i\ \cos\theta + \frac{\sin\Gamma}{\alpha(i\ \sin\theta + \delta\phi + S)}\left\{i^2\cos 2\theta\right.$$

$$\left. - 2i\ \delta\phi\ \sin\theta + i\ \cos\ (\alpha\cos\Gamma)\right\} \tag{31}$$

$$\frac{\partial\alpha}{\partial(\rho_s)} = -\frac{\sin\Gamma}{\rho_s{}^3 S}\left[D^2 - (\delta\rho + \Delta)^2\right] + \frac{2\cos\Gamma}{\rho_s{}^3(i\sin\theta + \delta\phi + S}$$

$$\Big\{\rho_s\delta\rho(1 - \cos Z + \delta\lambda\sin Z) - \Delta^2 - 2\rho_s\delta\rho - \delta\rho^2$$

$$+ 2\delta\rho(\rho_s - \Delta)(\cos Z - \delta\lambda\sin Z) + D_1{}^2$$

$$- \left[D^2 - (\delta\rho+\Delta)^2\right]\left[\cos Z - \frac{1}{2S}\sin Z\ (\alpha\cos\Gamma)\right]\Big\} \tag{32}$$

$$\frac{\partial\Gamma}{\partial(\rho_s)} = -\frac{c\Gamma}{\alpha\rho_s{}^3 S}\left[D^2 - (\delta\rho+\Delta)^2\right] - \frac{2\sin\Gamma}{\alpha\rho_s{}^3(i\sin\theta + \delta\phi + S}$$

$$\Big\{\rho_s\delta\rho(1 - \cos Z + \delta\lambda\sin Z) - \Delta^2 - 2\rho_s\delta\rho - \delta\rho^2$$

$$+ 2\delta\rho(\rho_s - \Delta)(\cos Z - \delta\lambda\sin Z) + D_1^2$$

$$- \left[D^2 - (\delta\rho+\Delta)^2\right]\left[\cos Z - \frac{1}{2S}\sin Z\ (\alpha\cos\Gamma)\right]\Big\} \tag{33}$$

$$\frac{\partial\alpha}{\partial(\Delta)} = -\frac{\sin\Gamma(\delta\rho+\Delta)}{\rho_s{}^2 S} + \frac{2\cos\Gamma}{\rho_s{}^2\sin Z(i\sin\theta + \delta\phi + S}$$

$$\Big\{\Delta + \delta\rho(\cos Z - \delta\lambda\sin Z) - (\delta\rho+\Delta)\Big[\cos Z - \frac{1}{2S}\sin Z$$

$$(\alpha\cos\Gamma)\Big]\Big\} \tag{34}$$

$$\frac{\partial\Gamma}{\partial(\Delta)} = -\frac{\cos\Gamma(\delta\rho+\Delta)}{\alpha\rho_s{}^2 S} - \frac{2\sin\Gamma}{\alpha\rho_s{}^2\sin Z(i\sin\theta + \delta\phi + S)}$$

$$\Big\{\Delta + \delta\rho\ (\cos Z - \delta\lambda\sin Z) - (\delta\rho+\Delta)\ \Big[\cos Z$$

$$- \frac{1}{2S}\sin Z\ (\alpha\cos\Gamma)\Big]\Big\} \tag{35}$$

$$\frac{\partial\alpha}{\partial(Z)} = \frac{2\cos\Gamma}{\sin Z(i\sin\theta + \delta\phi + S} \left\{ \frac{1}{\rho_s^2}\delta\rho(\rho_s - \Delta)(\sin Z + \delta\lambda\cos Z) \right.$$

$$+ \sin Z - \frac{1}{2}\sin Z(i\sin\theta + \delta\phi + S) - \frac{1}{2}\sin Z(\delta\lambda^2 - \delta\phi^2)$$

$$+ \delta\lambda\cos Z - \frac{1}{2}i^2\sin\theta\cos\theta\cos Z$$

$$+ \frac{1}{2}i^2\sin^2\theta\sin Z - i\,\delta\phi\cos\theta\cos Z$$

$$+ i\,\delta\phi\sin\theta\sin Z - \sin Z(i\sin\theta + \delta\phi)S$$

$$\left. - \frac{1}{2}\cos Z(i\sin\theta + \delta\phi + S)(\alpha\cos\Gamma) \right\} \tag{36}$$

$$\frac{\partial\Gamma}{\partial(Z)} = -\frac{2\sin\Gamma}{\alpha\sin Z(i\sin\theta + \delta\phi + S} \left\{ \frac{1}{\rho_s^2}\delta\rho(\rho_s - \Delta)(\sin Z + \delta\lambda\cos Z) \right.$$

$$+ \sin Z - \frac{1}{2}\sin Z(i\sin\theta + \delta\phi + S) - \frac{1}{2}\sin Z(\delta\lambda^2 - \delta\phi^2)$$

$$+ \delta\lambda\cos Z - \frac{1}{2}i^2\sin\theta\cos\theta\cos Z$$

$$+ \frac{1}{2}i^2\sin^2\theta\sin Z - i\,\delta\phi\cos\theta\cos Z$$

$$+ i\,\delta\phi\sin\theta\sin Z - \sin Z(i\sin\theta + \delta\phi)S$$

$$\left. - \frac{1}{2}\cos Z(i\sin\theta + \delta\phi + S)(\alpha\cos\Gamma) \right\} \tag{37}$$

The sensitivity formulas listed above make it possible to determine the acceptability of a proposed set of uncertainties in the ten known parameters, $\delta\rho$, $\delta\lambda$, $\delta\phi$, D, D_1, i, θ, ρ_s, Δ, and Z. It is desirable that the sensitivities be as small as possible and the sensitivity formulas can be used in this respect to find the best value of θ (i.e., the Earth's angular position) at which to perform the distance measurements.

Spacing of the Second Controllable Satellite

The arc Z, separating the two controllable satellites has been left arbitrary in all of the foregoing equations. It may be seen from Equation 18 that Z must be a very small angle in order that $\partial\alpha/\partial(\delta\rho)$ have order-of-magnitude one. This result is fortunate since close spacing of all three satellites will simplify the launching procedure and also require fewer ground stations for transmission of the laser beams.

Calculation of the Polar Wandering Angle, α

In order to evaluate the polar wandering angle, α, the ambiguity, ± 1, in Equation 11 must first be resolved. This can be accomplished by making two distance measurements from the free satellite to the first controllable satellite; one at $\theta = 0$ and one at $\theta = \pi$.

The separation distance, D, can be visualized, to second-order accuracy, in terms of the plane triangle shown in Fig. 4.

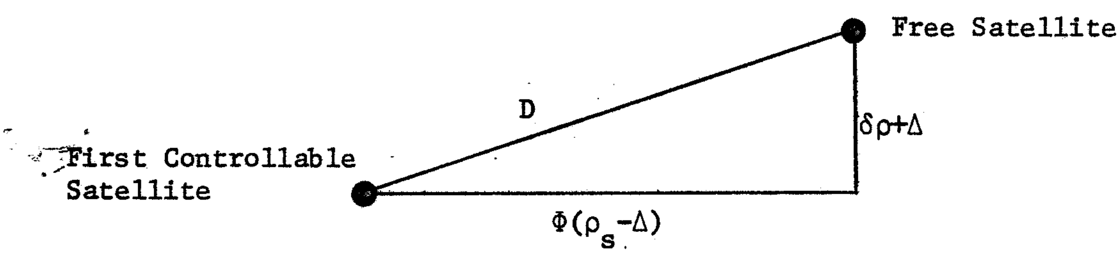


Fig. 4. Interpretation of Terms in Equation 11.

If $\theta = 0$ or $\theta = \pi$, Equation 11 can be written as

$$\alpha\sin\Gamma = -\delta\phi \pm \sqrt{\Phi^2 - \delta\lambda^2}. \tag{38}$$

The geometry involved in the distance measurement of the node corresponding to the positive x_I axis is shown in Fig. 5, which is

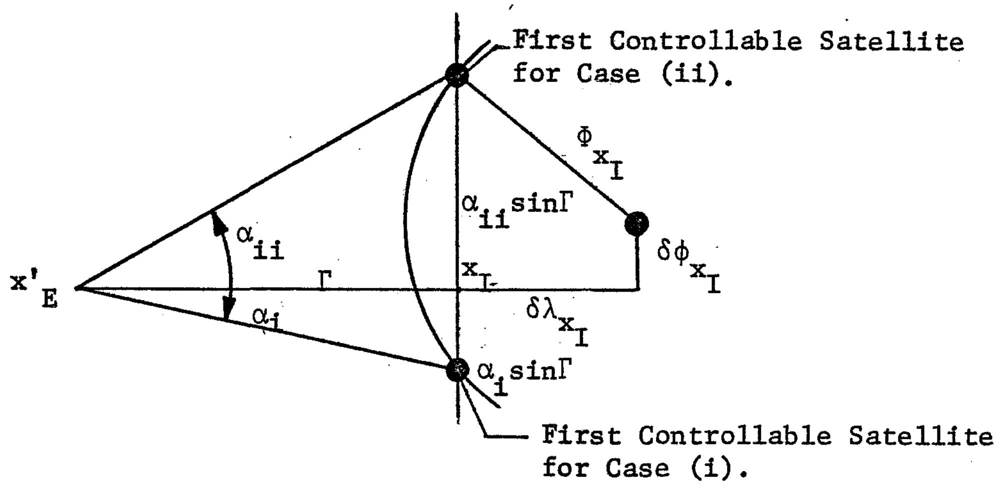


Fig. 5. Node Corresponding to the Positive x_I Axis

a projection of the indicated arcs and points upon a unit sphere. Measurement of D at the x_I node gives the arc Φ_{x_I}, and one of the following two cases must be true:

$$\text{Case (i)} \quad ; \quad \alpha\sin\Gamma = -\delta\phi_{x_I} + \sqrt{\Phi^2_{x_I} - \delta\lambda^2_{x_I}} \quad , \tag{39}$$

$$\text{Case (ii)} \quad ; \quad \alpha\sin\Gamma = -\delta\phi_{x_I} - \sqrt{\Phi^2_{x_I} - \delta\lambda^2_{x_I}} \quad . \tag{40}$$

To determine which case holds true, a second distance measurement is made at the node corresponding to the $- x_I$ axis. The situation is shown in Fig. 6.

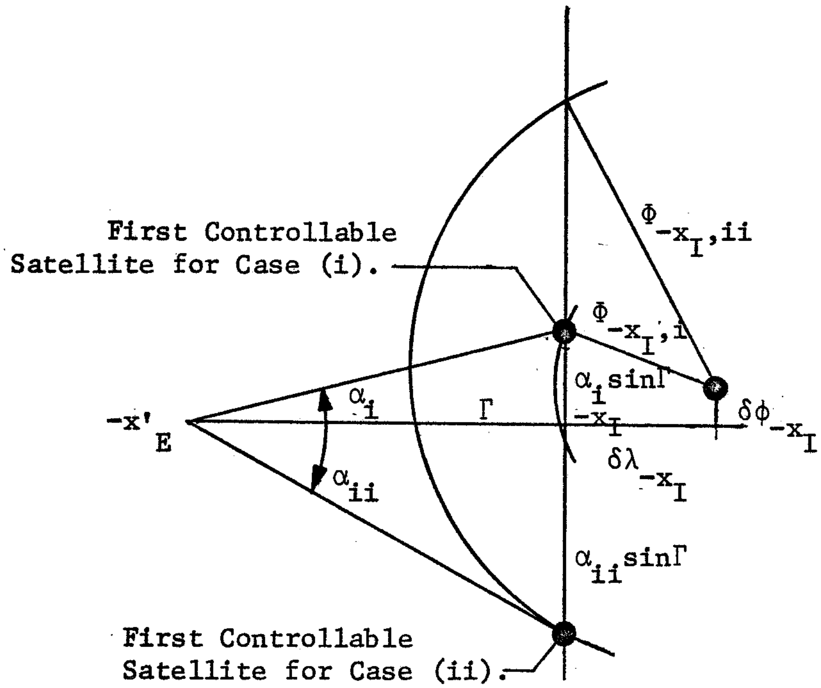


Fig. 6. Node Corresponding to the Negative x_I Axis.

If $\Phi_{-x_I,i}$ is measured at the $-x_I$ node, then case (i) must be true. If $\Phi_{-x_I,ii}$ is measured at the $-x_I$ node, then case (ii) must be true.

Once the ambiguity is resolved by the preceding argument, $|\alpha|$ can be calculated using

$$|\alpha| = + \sqrt{(\alpha\sin\Gamma)^2 + (\alpha\cos\Gamma)^2}\,, \tag{41}$$

where $(\alpha\sin\Gamma)$ and $(\alpha\cos\Gamma)$ are given by Equations 11 and 14. The sign of α is known from the results of the two nodal measurements described above.

Numerical Example

The results of the perturbation analysis indicate that perturbations of the free satellite on the order of 1 kilometer are to be expected. Due primarily to the effects of lunisolar precession, the inclination, i, of the spin axis changes on the order of 50" arc per year [1][1]. The polar wandering angle, α, has a magnitude of approximately 0.2" arc [2]. The following parameters are used to formulate a numerical example:

$$\delta\rho = 1 \text{ km}/6378 \text{ km} = 1.57 \times 10^{-4}$$

$$\delta\lambda = 2 \times 10^{-5} \text{ rad.}$$

$$\delta\phi = 2 \times 10^{-5} \text{ rad.}$$

$$\Delta = 10 \text{ km}/6378 \text{ km} = 1.57 \times 10^{-3}$$

$$\rho_s = 42{,}100 \text{ km}/6378 \text{ km} = 6.60$$

$$i = 50'' \text{ arc} = 2.4 \times 10^{-4} \text{ rad.}$$

$$\theta = \pi/2$$

$$\alpha = ''0.2063 \text{ arc} = 10^{-6} \text{ rad.}$$

$$Z = 2 \times 10^{-4} \text{ rad.}$$

$$\Gamma = \pi/2$$

The corresponding separation distance D is calculated using Equation 10,

$$D = 2.45 \times 10^{-3} \text{ .}$$

In kilometers,

$$D^* = 15.6 \text{ km.}$$

The sensitivities to uncertainty in the radial perturbation, $\delta\rho$, are calculated using Equations 18 and 19. The results are

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[1] Identifies listing in reference section.

$$\frac{\partial\Gamma}{\partial(\delta\rho)} = -8.85 \times 10^4 \quad ,$$

$$\frac{\partial\alpha}{\partial(\delta\rho)} = -1.52 \times 10^{-1} \quad .$$

These sensitivities can be stated in appropriate units as

$$\frac{\partial\Gamma}{\partial(\delta\rho)^*} = -\ 0.80 \text{ degrees/meter} \quad ,$$

$$\frac{\partial\alpha}{\partial(\delta\rho)^*} = -\ 0.005 \text{ arc seconds/meter} \quad ,$$

where the star superscripts indicate that the quantities are no longer dimensionless.

The sensitivities of α and Γ to uncertainty in the radial perturbation are not the best that can be achieved using three synchronous satellites in the manner described. All of the sensitivities are dependent upon both the controllable satellite spacing, Z, and the position of the Earth at the time of measurement, θ. By suitable adjustments in the parameters θ and Z, it is expected that the polar wandering coordinates α and Γ can be measured to a higher accuracy than that obtainable from presently existing techniques.

Appendix

The results of the perturbation analysis are partially summarized below.

Lunar Perturbation in the Radial Direction:

0.6 day term - - - - - - - - - - - - - 1.38 km.

0.5 day term - - - - - - - - - - - - - 1.24 km.

long period term - - - - - - - - - - - 3.50 km.

Solar Radiation Perturbation in the North-South Direction:

daily term - - - - - - - - - - - - - 2.34×10^{-8} rad.

long period term with daily oscillation - - - - - - - - 2.10×10^{-8} rad.

secular term after 1 year - - - - - 2.63×10^{-3} rad.

Oblateness Perturbations:

Depend on initial displacement from equilibrium.

References

1. Ehricke, K. A., "Space Flight, I. Environment and Celestial Mechanics," Van Nostrand Co., Inc., 1960. (page 383).

2. I.A.U. Symposium No. 32, Continental Drift, Ed. by W. Markowitz and B. Guinot, (page 77).

ORBIT DETERMINATION

Introduction

The increasing requirements for precision in orbit determination of geodetic satellites have led to an interest in examining the potential of utilizing synchronous satellites in a tracking network. The synchronous satellite offers the advantage of not having to propagate signals through the atmosphere. By using laser ranging techniques from a synchronous satellite (satellites) it may be possible to obtain very precise orbit determinations.

System

The system being examined consists of a synchronous satellite equipped with a laser ranging device. The satellite makes range measurements to a subsynchronous satellite at discrete points. These range measurements are used to obtain a best estimate for the current orbit parameters.

Problems being Examined

1. The type of laser system. Efforts are being made to determine the best state of the art laser system in terms of accuracy, weight, cost, etc.

2. The number of measurements. It is desired to determine the least number of measurements required in order to obtain a satisfactory estimate of the orbit parameters.

3. Sequential estimation and search. If the frequencies used are not near the optical range the antenna size increases significantly. Therefore it is best to operate with a very directional type system. This

requires that the laser ranging system be able to acquire the satellite being tracked. This may involve a sequential search technique, by which is meant a sequential procedure for locating one or more objects in one or more cells, a cell being a subregion of a search domain. Information will be acquired by means of a sensor system, and "sequential" implies that the results of past and present sensor outputs (observations) guide the search strategy in the future. We are interested in determining the optimal sequential search strategy, where by optimal we mean that the objective is achieved in minimum time or minimum cost or a combination of the two. Extensions will include the case where the sensors are noisy and where the objects have a degree of maneuverability.

4. Tracking. Once a satellite is located by the search technique it will be necessary to track the satellite so that range measurements may be made. The type of control system to best accomplish the tracking aspect is being examined.

COVERAGE
(DATA RELAY SATELLITE)

The problem considered herein deals with the continuous coverage of every point in a specified area on the surface of some celestial body by a system of artificial satellites. In particular, it is desired to determine the minimum number of satellites, and their orbits, necessary to form a system to carry out the above objective.

It is immediately seen that one satellite in synchronous orbit will fulfill all the requirements. There are, however, difficulties involved with a satellite at synchronous altitude. Two examples of these difficulties are: resolution, if there is an on-board camera, and power requirements for communication with ground stations. These difficulties may not be critical or insurmountable, but they do lend motivation to the constraints that will presently be placed on the solution to the coverage problem.

At the outset, it will be attempted to develop a system consisting of the minimum number of spin-stabilized satellites in subsynchronous (~5000 mi) orbits which will maintain constant coverage of some portion of the earth's surface. After this is accomplished, the solution parameters will be generalized in such a way as to make the solution applicable to a Mars mission, or any similar celestial body.

For purposes of preliminary study, one satellite in an arbitrary circular orbit will be considered. The earth will be taken to be a sphere. It is hoped that by determining the satellite antenna trace on the earth's surface, one may be enabled to develop a computer program that would "put together" the traces of all the satellites and determine the number of satellites, and their orbits, required to yield a composite trace that covers all of the specified area at all times. Figure 1 below depicts the "addition of traces" to give total coverage.

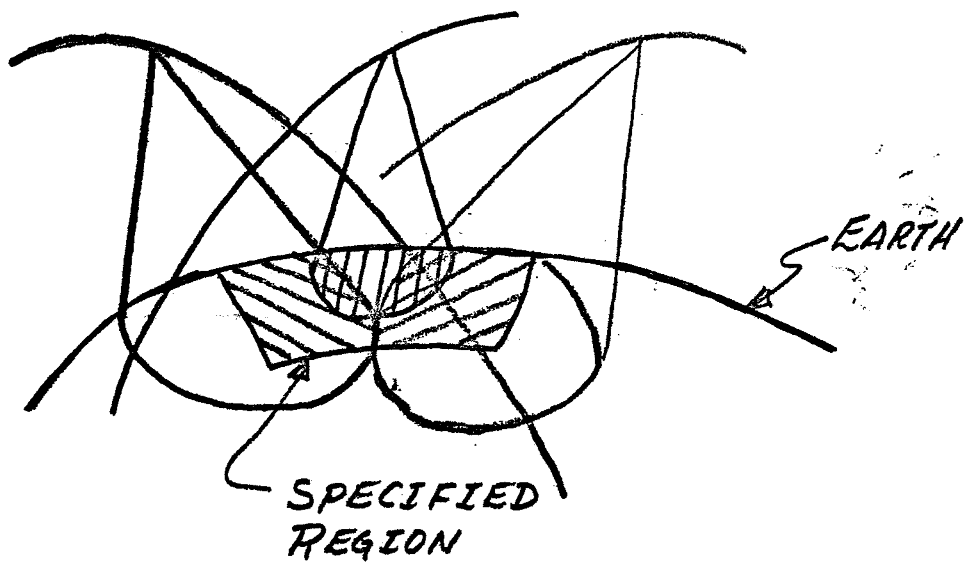

Fig. 1. Total Coverage

In order to be able to write a computer program, one must first arrive at a mathematical model. In the case of one satellite in a circular orbit this was done by attempting to derive parametric equations for the satellite trace on the earth's surface. This amounts to finding the intersection of the satellite "beam" (taken to be a cone) and the earth (taken to be a sphere) and writing it in parametric form.

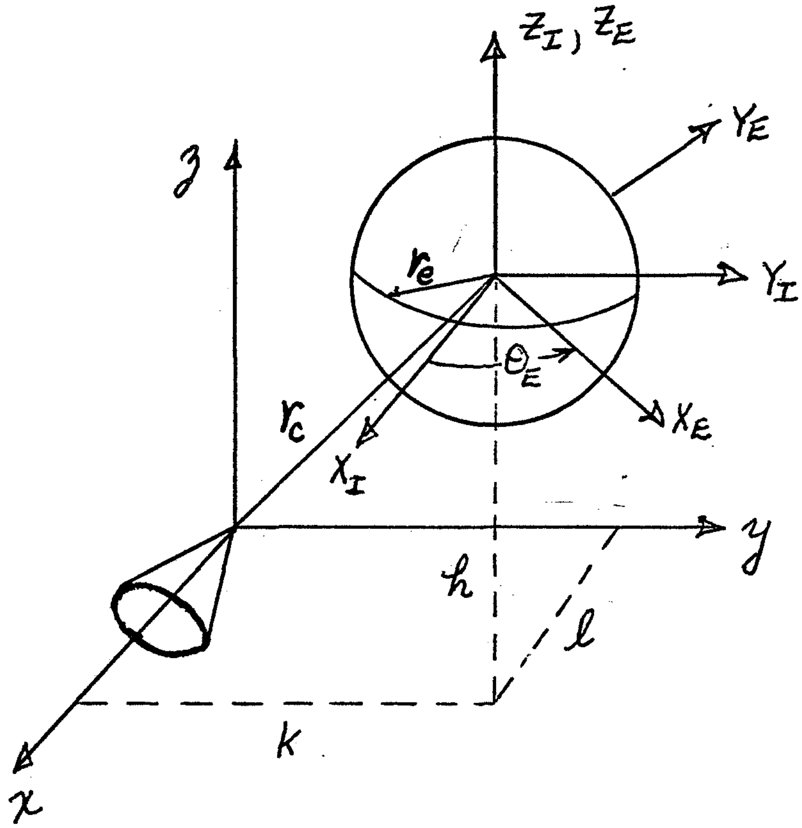

Fig. 2. The Situation Illustrated.

As shown in Figure 2, there are several possible coordinate systems to work with. X_I, Y_I, Z_I are inertial axes; X_E, Y_E, Z_E are earth-fixed axes; and x, y, z are vehicle-fixed axes. Since the ground trace is being sought, it is reasonable to suppose that the parametric equations describing the intersection of the cone and sphere will be of the form

$$X_E = X_E\ (Z_E,\ \text{parameters})$$

and

$$Y_E = Y_E\ (Z_E,\ \text{parameters}).$$

To arrive at these equations, one must solve the sphere equation and the cone equation simultaneously. Since the cone has only axial symmetry, many additional terms are introduced if its equation is written in the earth-fixed system. For this reason, the sphere equation will be written in the vehicle coordinate system since this involves only translations due to the symmetry of the sphere.

Thus,

$$\text{cone:}\quad y^2 + z^2 - Kx^2 = 0$$

$$\text{sphere:}\quad (x-\ell)^2 + (y-k)^2 + (z-h)^2 = r_e^2 \;;$$

Performing the indicated operations in the second equation, and subtracting the first from the second yields, after rearrangement, an equation linear in y. If this result is then applied to the cone equation and the quadratic formula is invoked, one arrives at z as a function x and parameters only. Further, if the resultant equation for z is substituted into the equation linear in y, one obtains y as a function of x and parameters only. These equations are given below:

$$z = \frac{\frac{h}{k^2}\,(bx^2 - 2\ell x - c) \pm \left\{\frac{h^2}{k^4}\,(\qquad)^2 - 4a\,\frac{1}{k^2}\,[\qquad\qquad]\right\}^{\frac{1}{2}}}{2a}$$

$$y = bx^2 - 2\ell x - c - 2h\left[\frac{\frac{h}{k^2}\,(bx^2 - 2\ell x - c) \pm \left\{\frac{h^2}{k^4}\,(\qquad)^2 - \frac{4a}{k^2}\,[\qquad\qquad]\right\}^{\frac{1}{2}}}{2a}\right]\Big/ 2k$$

where $\quad a = (1 + \frac{h^2}{k^2}),\quad b = (1 + K),\quad c = (r_e^2 - r_c^2)$

$$(\qquad) = (bx^2 - 2\ell x - c),$$

$$[\qquad] = (\tfrac{b}{4})^2\, x^2 - \ell b x^3 + \tfrac{1}{2}(\ell^2 bc - 2k^2 K)\, x^2 + \ell c x + \frac{c^2}{4}\;,$$

and $k \neq 0$

It remains only to transform these equations to the earth fixed coordinates. Since the satellite is spin stablized and x is chosen parallel to X_I, it remains always parallel. If the craft is chosen axially symmetric about the cone axis one may consider $y \,||\, Y_I$ and $z \,||\, Z_I$ at all times. Thus, three simple translations transform the above equations to the inertial system. A rotation through Θ_E transforms the equations to the earth-fixed system, which is the desired result. This last rotation, however, breaks down the parametric form of the equations. It is thought, therefore, that it may be sufficient to obtain the trace in the inertial system, and then "slide" the trace over on the earth's surface by using polar coordinates.

All of the above material has been tested for the special case of the cone's axis coincident with an earth radius. The trace was found to be a circle, as it should be.