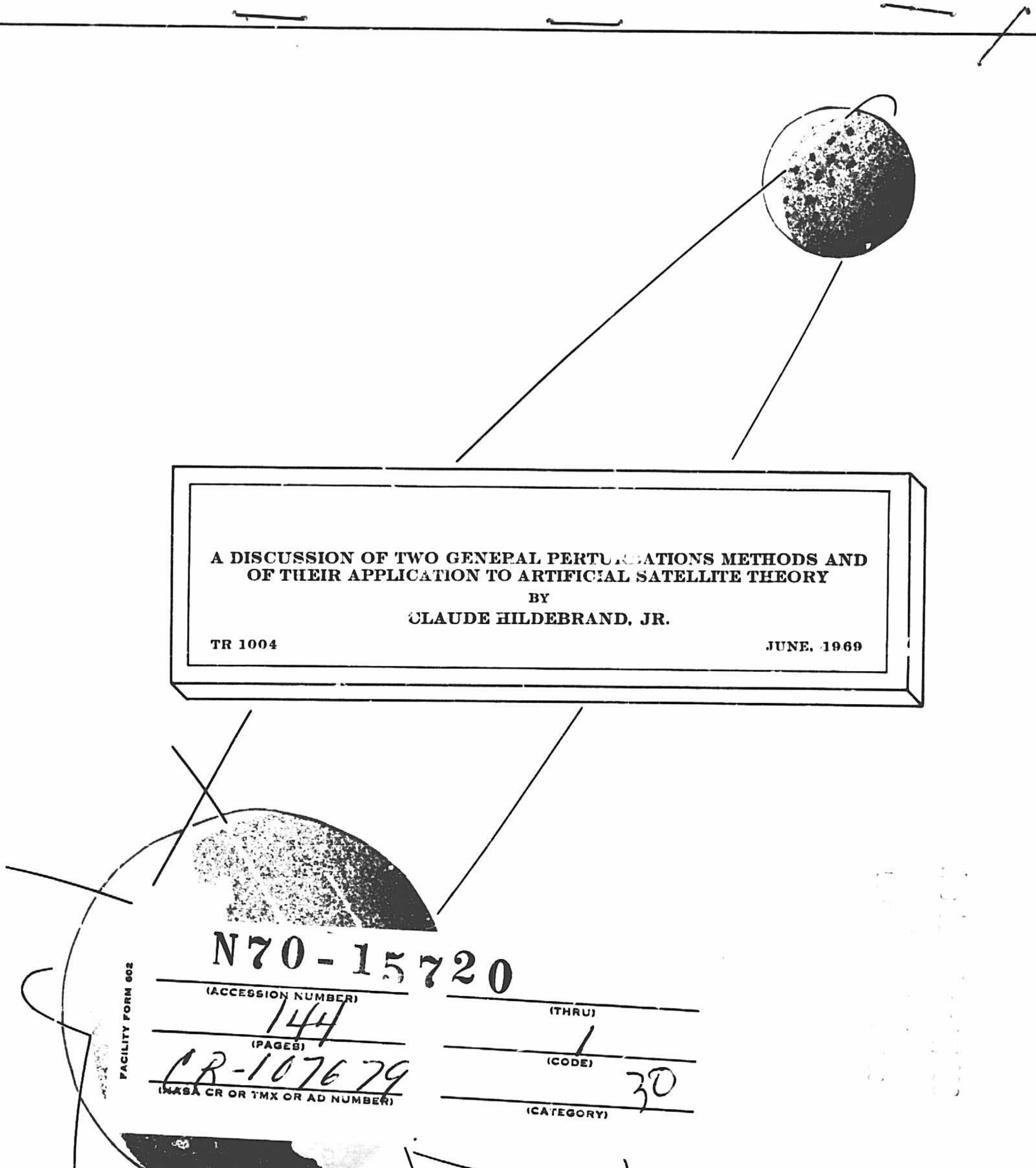

A DISCUSSION OF TWO GENERAL PERTURBATIONS METHODS AND OF THEIR APPLICATION TO ARTIFICIAL SATELLITE THEORY

BY

CLAUDE HILDEBRAND, JR.

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Professor of Aerospace Engineering

PREFACE

The launching of the first artificial satellite of the Earth in 1957 helped to promote widespread interest in the general perturbations methods of celestial mechanics and in their application to the problem of predicting the motion of artificial Earth satellites. The artificial satellite theories developed in the last decade have not only yielded important insights into satellite motion but also have proved to be extremely useful in satellite geodesy. This thesis provides an introduction to two well-known general perturbations techniques and a discussion of the development of approximate analytical solutions for the motion of artificial satellites of the Earth with the aid of these methods.

The author wishes to express his gratitude to his supervising professor Dr. Byron D. Tapley not only for guidance in the preparation of this thesis but also for being a constant source of encouragement and inspiration. He is also indebted to the other members of his committee, Dr. Wallace T. Fowler and Dr. Paul E. Russell, for their critical reading of the manuscript and their valuable suggestions. The many discussions with Dr. George H. Born and Mr. D. S. Ingram have contributed much to the progress of this research and are sincerely appreciated.

The author is pleased to acknowledge the National Science Foundation for their support of his graduate study. In addition, he would like to thank Dr. J. M. Lewallen and Dr. H. P. Decell for making it possible for him to pursue his research at the Manned Spacecraft Center during the summer of 1968, as well as Mrs. Mollie Schluter for her accurate and efficient typing of the manuscript.

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Claude Hildebrand, Jr.

The University of Texas
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ABSTRACT

This thesis is concerned with a study of the application of two general perturbations techniques to the problem of obtaining an analytical description of the motion of an artificial satellite of the Earth. The techniques (von Zeipel's method and a method of successive approximations employed by Kaula in the development of an artificial Earth satellite theory) are discussed and their characteristics are illustrated by using them to generate approximate analytical solutions for a simple dynamical system.

The artificial Earth satellite problem is formulated and the applications of the techniques mentioned above by Brouwer (Ref. 5) and Ingram (Ref. 7) to the determination of the effects of the Earth's oblateness on the motion of an artificial satellite are discussed in some detail. A comparison of the solutions indicates that Brouwer's theory appears to be superior to Ingram's for the description of long period motions since Brouwer's secular terms are more accurate than those given by Ingram and, in addition, Ingram's long period solutions are not complete. The phenomena of resonance is considered and a simple example is analyzed to demonstrate that the resonance problem is essentially a difficulty in the mathematical description of the motion.

The study closes with some comments on attempts to solve the Lunar satellite problem with the von Zeipel method. In particular, the well-known result regarding the failure of this method in the determination of analytical solutions for long term motions is noted.

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CHAPTER I

INTRODUCTION

For over 200 years general perturbations techniques have been employed by astronomers in the study of the motions of bodies within the solar system. In planetary astronomy, these methods have been used in calculating ephemerides and have played important roles in the discoveries of new planets. For example, the existence of Neptune was deduced by Le Verrier and Adams on the basis of otherwise inexplicable perturbations of the orbit of Uranus. Also, the verification of an effect predicted by general relativity, the advance of Mercury's perihelion by approximately 43" of arc each century, relies upon a discrepancy between the observed perihelion advance and the value calculated from a general perturbations solution of equations of motion of Mercury that were derived under the assumption of the validity of Newtonian mechanics. Other applications of general perturbations techniques in astronomy include studies of Saturn's rings, the motions of particles about the Sun-Jupiter and Earth-Moon libration points, and the Kirkwood gaps in the asteroid belt, as well as the development of a Lunar theory.

With the advent of the exploration of space by instrumented probes, the analytical methods which had been developed to solve problems in celestial mechanics were applied with great success to a new problem, that of predicting the motions of artificial satellites of the Earth. One of the earliest solutions describing the motions of artificial Earth satellites was given by Brouwer (Ref. 5). This solution included the effects of the second, third, fourth, and fifth zonal harmonics in the Earth's gravitational potential and was obtained by a technique known as von Zeipel's method. In the decade since the publication of the original artificial satellite theories, numerous additional studies and extensions of these theories have appeared in the literature.

The main purpose of this thesis is to organize and present the basic information required for an understanding of the von Zeipel method and of Brouwer's solution of the artificial Earth satellite problem. Secondary goals include a comparison of Brouwer's solution with the solution obtained by the method employed by Ingram in Ref. 7 and a review of the progress that has been made in recent years in completing a theory of Earth satellite motion.

Outline of Study

Chapter II is devoted to a review of relevant topics in celestial mechanics, a discussion of a technique for representing the potential of an arbitrary body, and a brief explanation of the application of canonical transformation theory to the solution of problems in dynamics.

Two general perturbations techniques, the von Zeipel method and a method of successive approximations employed by Kaula (Ref. 6) and Ingram (Ref. 7), are introduced in Chapter III. Some important characteristics of these techniques are illustrated in Chapter IV where their application to a simple example problem (the nonlinear spring) is considered.

Chapter V presents a discussion of the material in Refs. 5 and 7 concerning the application of the two general perturbations techniques to the artificial Earth satellite problem and a comparison of the results obtained using each of the methods. Also, the resonance phenomena is investigated with the aid of the simple pendulum example. Finally, some comments regarding the use of the von Zeipel method in the study of the motion of a Lunar satellite are given in Chapter VI.

CHAPTER II

PRELIMINARY TOPICS

II.1 The Two-Body Problem and Perturbed Motion

In attempting to determine the characteristics of the motion of a dynamical system, it is instructive to employ an idealized mathematical model for which a solution can be obtained analytically. Often, a study of the idealized problem yields useful information about the characteristics of the solution of the original problem.

In the case of satellite motion about an oblate planet, the idealization is the well known two-body problem of celestial mechanics. This is a consequence of the fact that the accelerations experienced by such a satellite differ only slightly from those that would occur if the planet were spherical and of constant density. Not only can the motion of the satellite be approximated by two-body motion over short time intervals, but also many relations arising in the solution of this simple problem hold for perturbed motion. For these reasons, a brief review of the two-body problem is given as an introduction to the more complicated problem.

Two-Body Problem

Consider two particles of masses M and m that attract each other according to Newton's law of universal gravitation. If $\vec{r}$ is the position vector of the mass m relative to M (see Fig. 1), the equations of motion in a non-rotating coordinate system with origin at M are

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

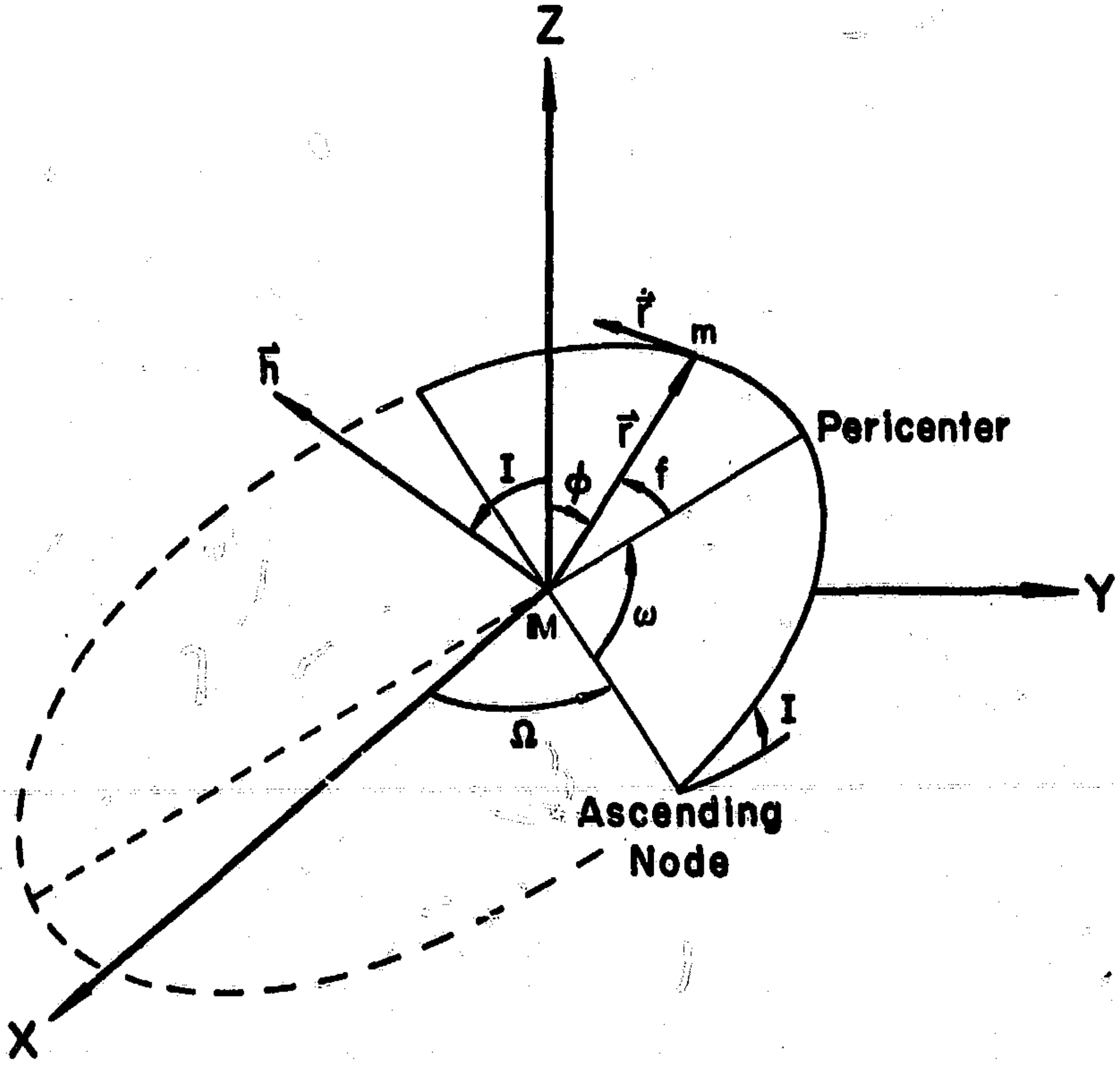


FIGURE I. Definition of Orbital Variables where μ is the gravitational constant. The relative motion of the particles has the following characteristics:

1. The particle m remains in a plane that passes through M and is perpendicular to the constant angular momentum vector $\vec{h} = \vec{r} \times \dot{\vec{r}}$.
2. The path of motion in the plane is a conic.

Since the equations of motion are a set of three second order ordinary differential equations, a complete solution can be specified by six independent constants of the motion. One commonly used set are the Keplerian elements given below.

a - semi-major axis of the conic

e - eccentricity of the conic

I - inclination of the orbital plane

Ω - longitude of the ascending node

ω - argument of pericenter

t_p - time of pericenter passage

The variables a and e determine the shape of the conic, I, Ω, and ω (shown in Fig.1) give the orientation of the orbital plane, and t_p in conjunction with the time t serves to locate the position of the particle m along its path.

Another useful set of quantities are the Delaunay variables defined below (Ref. 1).

$$
\begin{aligned}
L &= \sqrt{\mu a} & \ell &= \sqrt{\frac{\mu}{a^3}}\,(t-t_p) \\
G &= L\sqrt{1-e^2} & g &= \omega \\
H &= G \cos I & h &= \Omega
\end{aligned}
$$

These variables, which arise in the solution of the two-body problem by Hamilton-Jacobi theory, are a canonical set and are used in solving the oblate planet problem by the von Zeipel method.

The following important relations are obtained for the case of elliptical motion (Ref. 1).

$$r = |\vec{r}| = \frac{p}{1+e\cos f} = a(1-e\cos E) \tag{2.1}$$

$$p = a(1-e^2) = h^2/\mu \tag{2.2}$$

$$\tan f/2 = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \tag{2.3}$$

$$\ell = E - e\sin E = n(t-t_p) \qquad n = \sqrt{\frac{\mu}{a^3}} \tag{2.4}$$

$$\text{Energy} = -\frac{\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r} \tag{2.5}$$

$$\frac{\partial f}{\partial \ell} = \left(\frac{a}{r}\right)^2\sqrt{1-e^2} \tag{2.6}$$

$$\frac{\partial f}{\partial e} = \left(\frac{a}{r} + \frac{1}{\sqrt{1-e^2}}\right)\sin f \tag{2.7}$$

The derivations of Eqs. (2.6) and (2.7) are given in Appendix A since many celestial mechanics texts do not include them.

Perturbed Motion

Assume now that the mass $\mathbb{M}$, rather than being a particle, is an arbitrary body with a non-homogeneous mass distribution. The potential due to such a body is shown in Sec. II.2 to have the form

$$V = -\left(\frac{\mu}{r} + R\right)$$

where R is the disturbing function. The equations of motion of the particle m become

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \nabla R .$$

As before, the solution can be represented by the Keplerian elements. For perturbed motion, however, the elements are not constants, but are functions of time. The idea behind this technique is that, at each point in time, the position and velocity of the mass m uniquely determine an instantaneous two-body orbit and hence a set of Keplerian elements. Since the particle is not undergoing two-body motion, the instantaneous elements determined at different times need not be the same. The differential equations which describe the time rate of change of the Keplerian elements, known as Lagrange's planetary equations, can be derived by the variation of arbitrary constants procedure (see Ref. 1). The results are (from Ref. 2, p. 143):

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial \sigma} \qquad (2.8)$$

$$\dot{e} = \frac{1-e^2}{na^2e}\frac{\partial R}{\partial \sigma} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} \qquad (2.9)$$

$$\dot{\sigma} = -\frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} \qquad (2.10)$$

$$\dot{\Omega} = \frac{\csc I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} \qquad (2.11)$$

$$\dot{\omega} = -\frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} + \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} \qquad (2.12)$$

$$\dot{I} = \frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} - \frac{\csc I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \Omega} \qquad (2.13)$$

Here, the variable σ is defined as

$$\sigma = -nt_p .$$

A similar set of equations for the Delaunay variables can be derived with the aid of Hamilton-Jacobi perturbation theory (see Ref. 12, Chaps. 8, 9, and 11). They will be given in Sec. V.3.

The previous method for representing perturbed motion has the advantage that many of the equations derived for two-body motion, in particular Eqs. (2.1)-(2.7), are also valid along the perturbed trajectory.

One final comment must be made in regard to the equation for $\dot{\sigma}$ (Eqs. (2.10)). The disturbing function R from Sec. II.2 depends explicitly on the variables a , e , I , Ω , ω , and ℓ where

$$\ell = nt + \sigma . \tag{2.14}$$

Thus, in forming $\frac{\partial R}{\partial a}$, the dependence of ℓ upon the semi-major axis (through the mean motion, n) must be taken into account. Then,

$$\begin{aligned} \frac{\partial R}{\partial a} &= \left(\frac{\partial R}{\partial a}\right)_\ell + \frac{\partial R}{\partial \ell}\frac{\partial \ell}{\partial a} \\ &= \left(\frac{\partial R}{\partial a}\right)_\ell - \frac{3}{2}\frac{n}{a}\,t\,\frac{\partial R}{\partial \ell} . \end{aligned}$$

The notation $(\ )_\ell$ is used to indicate that ℓ is treated as being independent of the semi-major axis in performing the differentiation.

Since the disturbing function is periodic in the variables Ω , ω and ℓ , the differential equation for σ contains periodic terms with amplitudes that increase linearly with time. Terms of this form can not be conveniently treated by many perturbation methods. Therefore, the variable ℓ is usually used in place of σ. The equation for $\dot{\ell}$ can easily be derived from Eqs. (2.8), (2.10), and (2.14). It is given by the following expression:

$$\dot{\ell} = n - \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a} . \tag{2.15}$$

The only change in the remaining equations is that $\frac{\partial R}{\partial \sigma}$ must be replaced by $\frac{\partial R}{\partial \ell}$. Also, the variables ℓ and a are now treated as being independent.

II.2 The Development of the Potential

Chapter V is concerned with the problem of the motion of a particle in the gravitational field of an aspherical body. The term "aspherical" means that the potential field generated by the body differs from that produced by a homogeneous spherical body with the same mass. Since the motion of the particle is described by Lagrange's planetary equations presented in the previous section, it is necessary to determine the disturbing function R or, equivalently, the potential V due to an arbitrary distributed mass.

The Potential of an Aspherical Body

One very elegant method for obtaining the required potential is to solve Laplace's equation

$$\nabla^2 V = 0$$

in spherical coordinates (see Ref. 3). Another more direct technique is employed in Ref. 4 and will be outlined here.

Consider the distributed mass with density $D(\rho, \beta, \lambda)$ as shown in Fig. 2. The potential at a general point $P(r, \psi, \phi)$ due to the elemental mass dm is

$$dV = \frac{Gdm}{(r^2 + \rho^2 - 2\rho r \cos\gamma)^{\frac{1}{2}}}$$

where G is the universal gravitational constant. Integrating over the entire mass,

$$V = \int_\tau \frac{GD(\rho,\beta,\lambda)\ \rho^2 \sin\beta\ d\rho d\beta d\lambda}{(r^2 + \rho^2 - 2\rho r\cos\gamma)^{\frac{1}{2}}} \ . \tag{2.16}$$

If it is assumed that $\frac{\rho}{r} < 1$,

$$\begin{aligned}(r^2 + \rho^2 - 2\rho r\cos\gamma)^{-\frac{1}{2}} &= \frac{1}{r}\left(1 - 2\frac{\rho}{r}\cos\gamma + \frac{\rho^2}{r^2}\right)^{-\frac{1}{2}} \\ &= \frac{1}{r}\Big[1 + \frac{\rho}{r}\cos\gamma + \frac{\rho^2}{r^2}\left(-\frac{1}{2} + \frac{3}{2}\cos^2\gamma\right) \\ &\quad + \frac{\rho^3}{r^3}\left(-\frac{3}{2}\cos\gamma + \frac{5}{2}\cos^3\gamma\right) + \ldots\Big] \\ &= \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{\rho}{r}\right)^{\ell} P_\ell(\cos\gamma)\end{aligned} \tag{2.17}$$

where the $P_\ell(\nu)$ are the Legendre polynomials. The addition theorem for spherical harmonics can be used to write these quantities in terms of the associated Legendre functions $P_\ell^m(\nu)$. This yields

$$\begin{aligned}P_\ell(\cos\gamma) &= P_\ell(\cos\phi)\, P_\ell(\cos\beta) \\ &+ 2\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}\cos m(\psi-\lambda)\, P_\ell^m(\cos\phi)\, P_\ell^m(\cos\beta) \ .\end{aligned} \tag{2.18}$$

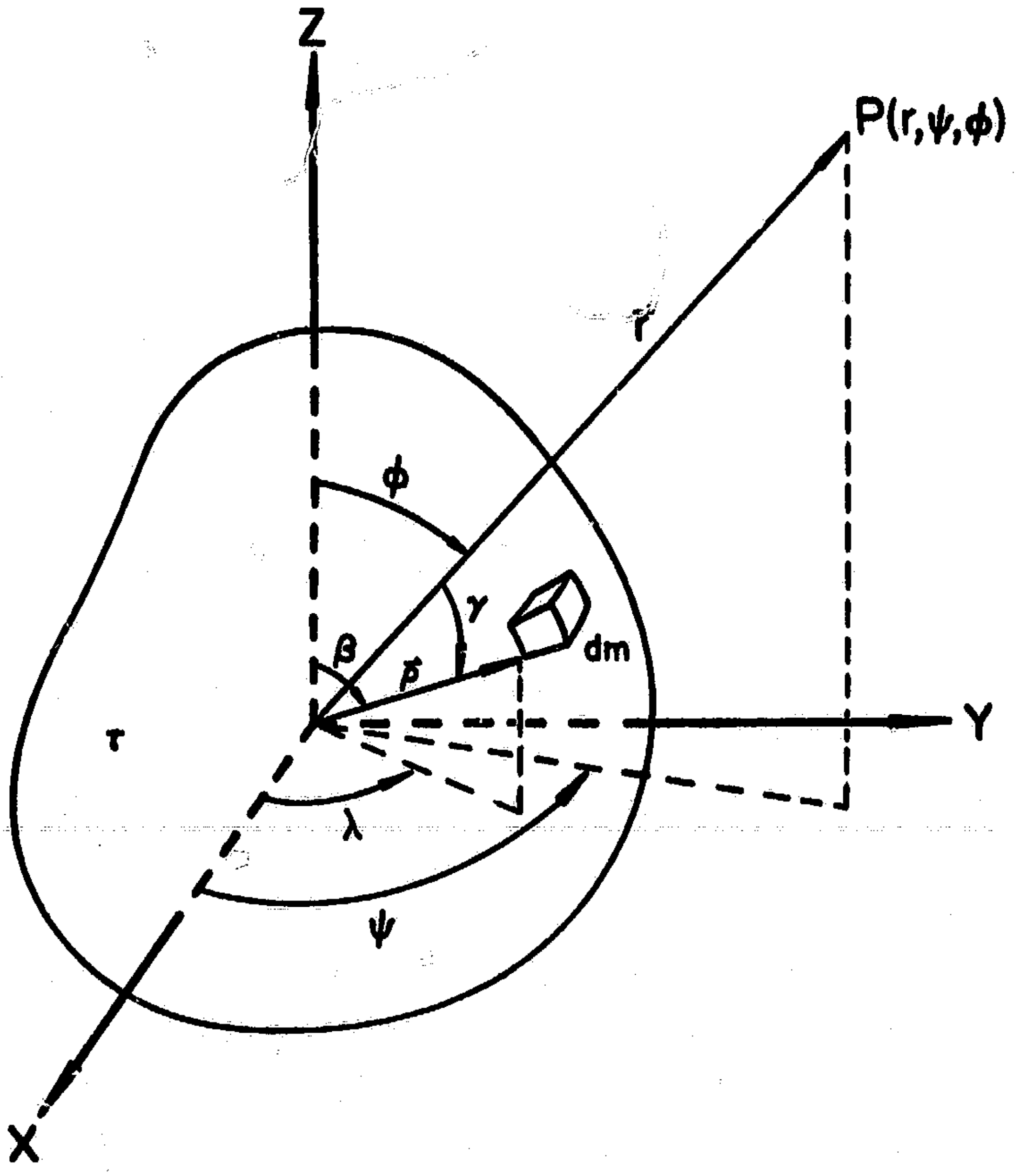


FIGURE 2. Variables Employed in the Development of the Potential

Relations for generating the Legendre polynomials and associated functions are given in Appendix B. With the aid of Eqs. (2.17) and (2.18), Eq. (2.16) becomes

$$V = \int_\tau \{\frac{G}{r} D(\rho,\beta,\lambda) \sum_{\ell=0}^{\infty} (\frac{\rho}{r})^{\ell} [P_\ell(\cos\phi)\, P_\ell(\cos\beta) + 2 \sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} \cos m(\psi-\lambda)\, P_\ell^m(\cos\phi) P_\ell^m(\cos\beta)]\rho^2 \sin\beta\} d\rho\, d\beta\, d\lambda \tag{2.19}$$

or

$$\begin{aligned} V &= -\frac{\mu}{r} - \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \frac{\mu a_e^{\ell}}{r^{\ell+1}} J_{\ell m}\, P_\ell^m(\cos\phi) \cos m(\psi-\psi_{\ell m}) \\ &= -\frac{\mu}{r} + \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} V_{\ell m} . \end{aligned} \tag{2.20}$$

The $V_{\ell m}$ defined by Eqs. (2.19) and (2.20) are called surface spherical harmonics. The quantities $J_{\ell m}$ and $\psi_{\ell m}$ are related to the volume integrals appearing in Eq. (2.19) and thus are parameters describing the distributed mass for which they are calculated. In practice, these constants cannot be determined directly for a given planet, but must be inferred from experiments such as observations of the perturbing effect of the planet on the motion of other bodies. The constant a_e is the mean equatorial radius of the central body.

Some simplification of Eq. (2.20) results if the origin of the coordinate system is taken at the center of mass of the body. Then, from Eqs. (2.16), (2.17), and (2.20),

$$V_1 \equiv \sum_{m=0}^{1} V_{1m} = \frac{G}{r^2} \int_\tau \rho \cos\gamma\, dm = 0$$

so that

$$\begin{aligned} V &= -\frac{\mu}{r} - \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \frac{\mu a_e^{\ell}}{r^{\ell+1}} J_{\ell m}\, P_\ell^m(\cos\phi) \cos m(\psi-\psi_{\ell m}) \\ &= -\frac{\mu}{r} - R . \end{aligned}$$

If in addition the body is axially symmetric about the Z axis, i.e., $D = D(\rho,\beta)$, it is easily shown that

$$J_{\ell m} = 0 \text{ if } m \neq 0$$

and the potential becomes

$$V = -\frac{\mu}{r} - \sum_{\ell=2}^{\infty} \frac{\mu a_e^{\ell}}{r^{\ell+1}} J_{\ell o} P_{\ell}(\cos\phi) .$$

Interpretation

Insight into the effect of the individual terms in the expansion of the potential can be obtained by replacing the actual distributed mass with an equivalent body of uniform density. The approximate shape of this equivalent mass can be determined from a study of the variation of the potential with ϕ and ψ at a constant radial distance r. Values of V that are less than $-\frac{\mu}{r}$ will indicate that the equivalent body has more mass in the region than it would have if it were a homogeneous sphere. Similarly, values of V greater than $-\frac{\mu}{r}$ indicate a mass deficit.

For the purposes of this discussion, assume $J_{\ell m} > 0$ and define

$$\overline{V}_{\ell m} = -\frac{V_{\ell m}}{\frac{\mu a_e^{\ell}}{r^{\ell+1}} J_{\ell m}} = P_{\ell}^{m}(\cos\phi) \cos m(\psi - \psi_{\ell m}) .$$

Then, $\overline{V}_{\ell m}$ can be used to demonstrate the dependence of V on ϕ or ψ. For example,

$$\overline{V}_{20} = P_2(\cos\phi) = -\frac{1}{2} + \frac{3}{2}\cos^2\phi .$$

This quantity has its maximum value at $\phi = 0^{\circ}$ and 180°, its minimum at $\phi = 90^{\circ}$, and zeros at $\cos\phi = \pm\sqrt{\frac{1}{3}}$ or $\phi \approx 55^{\circ}$ and 125°. Plotting

$\overline{V}_{20}$ radially on the surface of a sphere yields Fig. 3a. The shaded areas represent an excess of mass; the unshaded, a deficiency. For convenience, Fig. 3a will be drawn as shown in Fig. 3b. By a similar process, Fig. 3c can be obtained for $\overline{V}_{30}$. It is common practice to designate V_{20} as the prolateness ($J_{20} > 0$) or oblateness ($J_{20} < 0$) and V_{30} , for obvious reasons, as the "pear shape" effect.

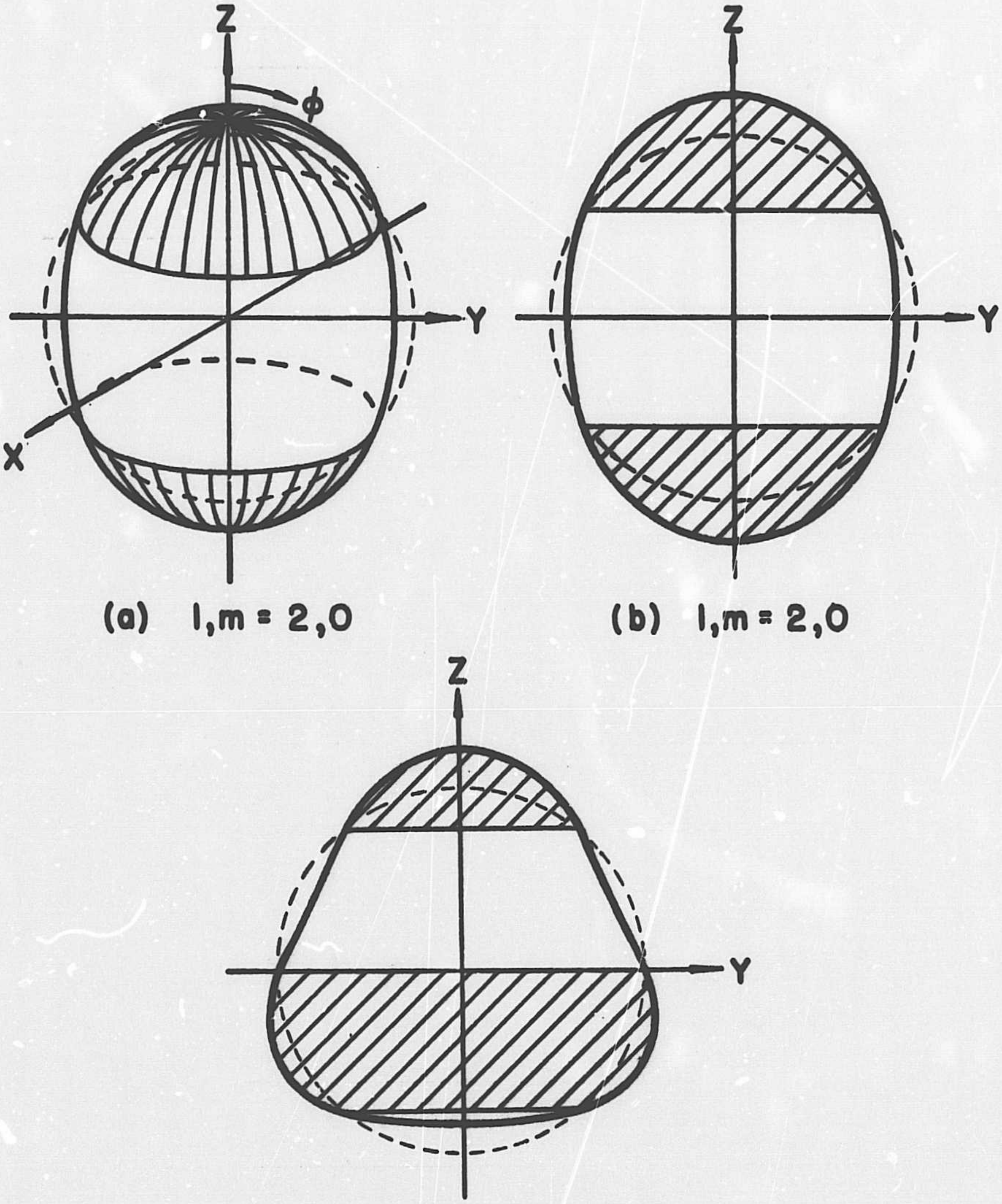
Note that all of the terms in V for which $m = 0$ are axisymmetric and thus divide the sphere into latitudinal zones. It is logical, therefore, to refer to them as zonal harmonics. The remaining terms are called tesseral harmonics.

An example of a tesseral harmonic is V_{32} sketched in Fig. 4a. It is apparent that these harmonics have zeros of both latitude and longitude. That is, whereas the zonal harmonics are latitude dependent only, the tesseral harmonics depend in addition on longitude. A special case occurs when $\ell = m$. The tesseral harmonics for which this condition is satisfied are known as sectorial harmonics because they divide the sphere into sectors. The sectorial harmonic V_{22} is sketched in Fig. 4b. Additional information on the spherical harmonics and procedures for determining $J_{\ell m}$ and $\psi_{\ell m}$ can be found in Ref. 3.

The expression for the potential given in Eq. (2.20) is not in the form employed in the analysis that will be presented in the next chapter. The following paragraphs are devoted to a discussion of the necessary modifications.

The Potential in Terms of the Orbital Elements

The form of the potential needed for a general perturbations solution is partially dictated by the particular method used. The von Zeipel technique applied by Brouwer (Ref. 5) requires only that the latitude ϕ

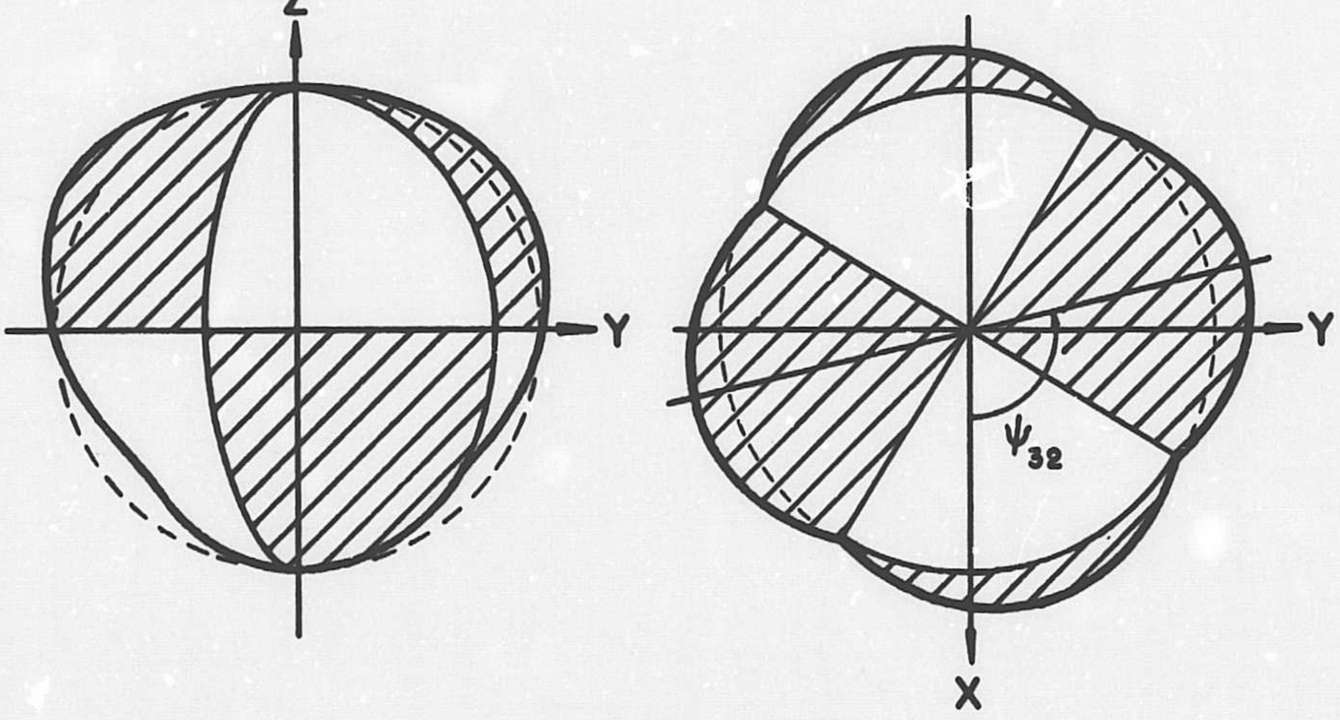


(a) l,m = 2,0

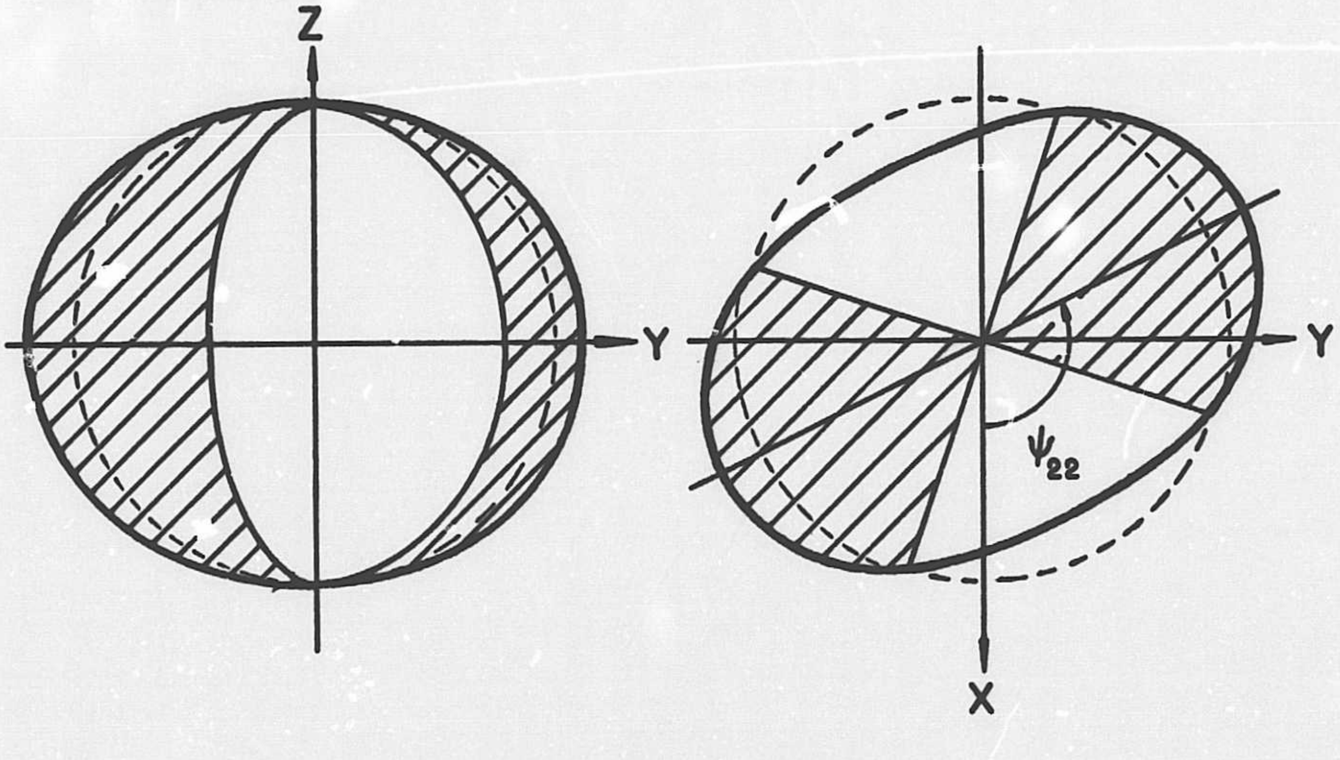
(b) l,m = 2,0

(c) l,m = 3,0

FIGURE 5. Examples of Zonal Harmonics



(a) $l,m = 3,2$



(b) $l,m = 2,2$

FIGURE 4. Examples of Tesseral Harmonics

be written in terms of the Delaunay variables and the true anomaly f , while the procedure used by Kaula (Ref. 6) and Ingram (Ref. 7) necessitates the development of the potential explicitly in terms of the modified Keplerian variables a, e, I, Ω, ω, and ℓ .

In his well-known paper, Brouwer considers only the effects of the zonal harmonics V_{20} , V_{30} , V_{40} , and V_{50} , which are the dominant terms for the Earth. Thus, the potential of the Earth is approximated by the following expression:

$$
\begin{aligned}
V = & -\frac{\mu}{r} - \frac{\mu a_e^2}{r^3} J_{20}\left(-\frac{1}{2} + \frac{3}{2}\cos^2\phi\right) \\
& - \frac{\mu a_e^3}{r^4} J_{30}\left(-\frac{3}{2}\cos\phi + \frac{5}{2}\cos^3\phi\right) \\
& - \frac{\mu a_e^4}{r^5} J_{40}\left(\frac{3}{8} - \frac{15}{4}\cos^2\phi + \frac{35}{8}\cos^4\phi\right) \\
& - \frac{\mu a_e^5}{r^6} J_{50}\left(\frac{15}{8}\cos\phi - \frac{35}{4}\cos^3\phi + \frac{63}{8}\cos^5\phi\right) .
\end{aligned}
\tag{2.21}
$$

Referring to Fig. 1 and applying a relation from spherical trigonometry yields

$$
\cos\phi = \sin I \sin(\omega+f) = \sin I \sin(g+f) . \tag{2.22}
$$

Substituting Eq. (2.22) into Eq. (2.21) and using the relationships between the Keplerian elements and the Delaunay variables leads to

$$
\begin{aligned}
V = & -\frac{\mu}{r} - \frac{\mu a_e^2}{r^3} J_{20}\left(-\frac{1}{2}\right)\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right) + \left(\frac{3}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)\cos(2g+2f)\right] \\
& - \frac{\mu a_e^3}{r^4} J_{30}\left[\left(-\frac{3}{2}\sin I + \frac{15}{8}\sin^3 I\right)\sin(g+f) \right. \\
& \left. - \frac{5}{8}\sin^3 I \sin(3g+3f)\right]
\end{aligned}
\tag{2.23}
$$

$$\begin{aligned}
&-\frac{\mu a_e^4}{r^5}\ J_{40}\left(\frac{3}{8}\right)\left[\left(\frac{3}{8}-\frac{15}{4}\frac{H^2}{G^2}+\frac{35}{8}\frac{H^4}{G^4}\right)\right.\\
&\qquad -\left(\frac{5}{6}-\frac{20}{3}\frac{H^2}{G^2}+\frac{35}{6}\frac{H^4}{G^4}\right)\cos(2g+2f)\\
&\qquad \left.+\left(\frac{35}{24}-\frac{35}{12}\frac{H^2}{G^2}+\frac{H^4}{G^4}\right)\cos(4g+4f)\right]\\
&-\frac{\mu a_e^5}{r^6}\ J_{50}\left[\left(\frac{15}{8}\sin I-\frac{105}{16}\sin^3 I+\frac{315}{64}\sin^5 I\right)\sin(g+f)\right.\\
&\qquad +\left(\frac{35}{16}\sin^3 I-\frac{315}{128}\sin^5 I\right)\sin(3g+3f)\\
&\qquad \left.+\frac{63}{128}\sin^5 I\sin(5g+5f)\right].
\end{aligned}\tag{2.23}$$

Replacing J_{20}, J_{30}, J_{40}, and J_{50} with the constants defined below gives Brouwer's potential exactly.

$$\begin{aligned}
k_2 &= -\frac{J_{20}a_e^2}{2} & k_4 &= \frac{3J_{40}a_e^4}{8}\\
A_{3,0} &= J_{30}\ a_e^3 & A_{5,0} &= J_{50}\ a_e^5
\end{aligned}\tag{2.24}$$

The derivation of the potential used by Kaula and Ingram is extremely involved and will not be given here. For the interested reader, it is outlined in Ref. 3 and discussed in detail in Refs. 8 and 9. The final result is

$$V = -\frac{\mu}{r} - \sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell}\frac{\mu a_e^{\ell}}{a^{\ell+1}}\sum_{p=0}^{\ell}F_{\ell mp}(I)\sum_{q=-\infty}^{\infty}G_{\ell pq}(e)S_{\ell mpq}(\omega,M,\Omega,\theta) \tag{2.25}$$

where

$$\begin{aligned}
S_{\ell mpq} &= J_{\ell m}\begin{bmatrix}\cos\\ \sin\end{bmatrix}_{\ell-m\ \text{odd}}^{\ell-m\ \text{even}}\Phi_{\ell mpq}\\
\Phi_{\ell mpq} &= (\ell-2p)\omega+(\ell-2p+q)M+m(\Omega-\theta-\lambda_{\ell m}).
\end{aligned}\tag{2.26}$$

The variable θ is the angle between the body fixed X axis and the inertial axis used as the reference for Ω.

Note that M is used to designate the mean anomaly to avoid confusion with the summation index ℓ . The expressions for the functions $F_{\ell mp}(I)$ and $G_{\ell pq}(e)$ are given in Appendix C. Also, tables of these functions can be found in Ref. 3.

Theoretically, the infinite series in Eq.(2.25) converges whenever $e < 1$. In practice, however, the summation on q and the summation in $G_{\ell pq}$ (which is a power series in e) are truncated to discard all terms with e raised to higher than a specified power. The infinite series then becomes a truncated power series in e that converges only for $e < 0.6627...$ (see Ref. 10).

II.3 Canonical Transformation Theory

The last section of the present chapter is concerned with a brief discussion of the theory of canonical transformations which is the basis of the von Zeipel method. More extensive treatments of this topic appear in Refs. 11, 12, and 13.

In classical mechanics it is shown that the equations of motion of a dynamical system under the influence of forces derivable from a (possibly time dependent) potential can be written as

$$\dot{q}^T = H_p = \begin{bmatrix} \frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} & \cdots & \frac{\partial H}{\partial p_n} \end{bmatrix}$$

$$\dot{p}^T = -H_q = -\begin{bmatrix} \frac{\partial H}{\partial q_1} & \frac{\partial H}{\partial q_2} & \cdots & \frac{\partial H}{\partial q_n} \end{bmatrix}$$

where q is an n-vector of generalized coordinates, p is an n-vector of generalized momenta, and H is a scalar function of q , p , and t called the Hamiltonian. This set of 2n ordinary differential equations of the first order are referred to as Hamilton's canonical equations. The Hamiltonian representing the dynamical system is obtained from the Lagrangian $L(q, \dot{q}, t)$ through the relations

$$p^T = L_{\dot{q}}$$

$$H(q,p,t) = p^T\dot{q} - L[q,\dot{q}(q,p,t), t] .$$

Consider now the transformation of variables

$$\begin{aligned} Q &= Q(q,p,t) \\ P &= P(q,p,t) . \end{aligned} \qquad P,Q \sim n\text{x}1 \tag{2.27}$$

This transformation is called canonical if the equations of motion in the new variables (Q,P) have the canonical form, i.e.,

$$\dot{Q}^T = K_P$$

$$\dot{P}^T = -K_Q$$

where $K(Q,P,t)$ is the new Hamiltonian. Introduction of the function $S(q,P,t)$, known as the generating function, allows the canonical transformation to be completely defined by the 2n + 1 equations

$$p^T = S_q \tag{2.28}$$

$$Q^T = S_P \tag{2.29}$$

$$H(q,p,t) + \frac{\partial S}{\partial t} = K(Q,P,t) . \tag{2.30}$$

The procedure then is to specify $K(Q,P,t)$ and to replace Q and p in Eq. (2.30) with S_P^T and S_q^T, respectively. This yields in general a non-linear partial differential equation for S. After the generating function has been found, Eqs. (2.28) and (2.29) can be used to obtain Eqs. (2.27).

Canonical transformation theory provides the following interesting approach to the problem of determining the motion of dynamical systems. Assume that the new Hamiltonian K is a function only of the new momenta P. The canonical equations become

$$\dot{P}^T = -K_Q = 0 \Longrightarrow P = \text{Cons.}$$

$$\dot{Q}^T = K_P = f(P) = \text{Cons.}\,.$$

Thus, if a transformation that eliminates the coordinates from the Hamiltonian can be found, the problem is solved. The von Zeipel technique accomplishes this elimination by a method of successive approximations.

CHAPTER III

TWO GENERAL PERTURBATIONS METHODS

The perturbation techniques that have been developed for solving problems which arise in classical celestial mechanics can be divided into two broad categories according to the form of the solution obtained. The first category, that of special perturbations, is concerned with the use of computational algorithms to generate solutions in numerical form (i.e., numerical integration). Such procedures have the disadvantages of requiring large amounts of computer time to perform the calculations and of being unsuitable for determining general characteristics of the motion. Techniques in the second category, known as general perturbations, yield solutions in analytical form and thus are not subject to the limitations enumerated above. However, these methods are usually difficult to apply and give accurate results only over limited periods of time.

A subdivision of the last category can be made on the basis of the variables employed. Some techniques are formulated in terms of variables which undergo large changes in short periods of time, such as Cartesian coordinates and velocities. Other approaches use quantities that change only slightly over long periods of time (e.g., Keplerian elements). Variables of the latter type are used in both of the procedures discussed below.

III.1 Kaula's Method

The following approach, which appears to be a modification of a well-known technique used in the development of planetary theories in classical astronomy (Ref. 12), was applied to the problem of Earth satellite motion by Kaula (Ref. 6).

The method is used to generate approximate solutions to ordinary differential equations of the form

$$\dot{\xi} = \varepsilon \sum_{\alpha} b_{\alpha}(\xi) \sin \overline{\theta}_{\alpha} + \varepsilon \sum_{\alpha,\beta} B_{\alpha\beta}(\xi) \sin \theta_{\alpha\beta} \tag{3.1}$$

$$\dot{\eta} = \varepsilon \sum_{\alpha} c_{\alpha}(\xi) \cos \overline{\theta}_{\alpha} + \varepsilon \sum_{\alpha,\beta} C_{\alpha\beta}(\xi) \cos \theta_{\alpha\beta} \tag{3.2}$$

$$\dot{\zeta} = \omega(\xi) + \varepsilon \sum_{\alpha} d_{\alpha}(\xi) \cos \overline{\theta}_{\alpha} + \varepsilon \sum_{\alpha,\beta} D_{\alpha\beta}(\xi) \cos \theta_{\alpha\beta} \tag{3.3}$$

where

$$\overline{\theta}_{\alpha} = \alpha^{T}\eta \tag{3.4}$$

$$\theta_{\alpha\beta} = \alpha^{T}\eta + \beta^{T}\zeta . \tag{3.5}$$

In the previous equations, ξ, b_{α}, and $B_{\alpha\beta}$ are m-vectors, η , c_{α} , and $C_{\alpha\beta}$ are n-vectors, and ζ , $\omega(\xi)$, d_{α} , and $D_{\alpha\beta}$ are p-vectors. The α and β are n- and p-vectors of summation indices, respectively. The scalar parameter ε is assumed to be small so that the variables ξ and η are slowly changing. In particular, ε is assumed to have a magnitude such that the $\omega(\xi)$ terms in Eq. (3.3) are large relative to the periodic terms. Finally, it is assumed that not all of the elements of the β vector vanish simultaneously.

The solution to the problem can easily be obtained when $\varepsilon = 0$ and is

$$\xi = \text{Cons.} = \overline{\xi} \tag{3.6}$$

$$\eta = \text{Cons.} = \overline{\eta} \tag{3.7}$$

$$\zeta = \text{Cons.} + \omega(\overline{\xi})t = \overline{\zeta} + \omega(\overline{\xi})t . \tag{3.8}$$

For the unperturbed motion then, all the variables are constants except the ζ_i which are linear functions of time. It is expected that the solution should not change greatly if ε is allowed to take on a small nonzero value. Thus, the expressions above could be used as a first approximation to the exact solution. A slightly better approximation is employed by Kaula and can be developed as follows.

First, choose the unperturbed solution given by Eqs. (3.6), (3.7), and (3.8) as the first approximation. Substituting these expressions for ξ, η, and ζ into the right hand sides of Eqs. (3.1) through (3.3) and integrating the result yields

$$\xi = \overline{\xi} + \varepsilon[\sum_{\alpha} b_{\alpha}(\overline{\xi}) \sin \overline{\theta}_{\alpha}]t - \varepsilon \sum_{\alpha,\beta} \frac{B_{\alpha\beta}(\overline{\xi})}{\beta^T \omega(\overline{\xi})} \cos \{\alpha^T \overline{\eta} + \beta^T [\overline{\zeta} + \omega(\overline{\xi})t]\}$$

$$\eta = \overline{\eta} + \varepsilon[\sum_{\alpha} c_{\alpha}(\overline{\xi}) \cos \overline{\theta}_{\alpha}]t + \varepsilon \sum_{\alpha,\beta} \frac{C_{\alpha\beta}(\overline{\xi})}{\beta^T \omega(\overline{\xi})} \sin \{\alpha^T \overline{\eta} + \beta^T [\overline{\zeta} + \omega(\overline{\xi})t]\}$$

$$\zeta = \overline{\zeta} + \omega(\overline{\xi})t + \varepsilon[\sum_{\alpha} d_{\alpha}(\overline{\xi}) \cos \overline{\theta}_{\alpha}]t + \frac{d\omega(\overline{\xi})}{d\overline{\xi}} \int (\xi - \overline{\xi}) dt + \varepsilon \sum_{\alpha,\beta} \frac{D_{\alpha\beta}(\overline{\xi})}{\beta^T \omega(\overline{\xi})} \sin \{\alpha^T \overline{\eta} + \beta^T [\overline{\zeta} + \omega(\overline{\xi})t]\} .$$

The terms in the solution that are multiplied by ε to the first power are called first order terms. The term containing $\frac{d\omega(\overline{\xi})}{d\overline{\xi}}$ comes from expanding $\omega(\xi)$ about $\xi = \overline{\xi}$ and retaining only the first order term in performing the integration. An examination of these results reveals that both η and ζ have first order terms that are linear in time and thus become large very rapidly. Neglecting these terms in the trigonometric arguments will produce first order errors in the frequencies of the trigonometric terms. In essence, the frequencies of the true solution will differ slightly from those of the approximate solution. This will result in very long period first order errors in all the variables as illustrated in Fig. 5. To eliminate these errors, Kaula includes first order terms that are linear in time in his first approximation, given below (Ref. 3).

$$\xi = \bar{\xi} \tag{3.9}$$

$$\eta = \bar{\eta} + \varepsilon\, c_o(\bar{\xi})t \equiv \bar{\eta} + \dot{\eta}_s t \tag{3.10}$$

$$\begin{aligned} \zeta &= \bar{\zeta} + [\omega(\bar{\xi}) + \varepsilon\, d_o(\bar{\xi})]t \\ &\equiv \bar{\zeta} + \dot{\zeta}_s t \end{aligned} \tag{3.11}$$

With this first approximation, the trigonometric arguments $\bar{\theta}_\alpha$ and $\theta_{\alpha\beta}$ become

$$\begin{aligned} \theta_\alpha &= \alpha^T(\bar{\eta} + \dot{\eta}_s t) \\ \theta_{\alpha\beta} &= \alpha^T(\bar{\eta} + \dot{\eta}_s t) + \beta^T(\bar{\zeta} + \dot{\zeta}_s t) \\ &\equiv \bar{\Theta}_{\alpha\beta} + (\alpha^T\dot{\eta}_s + \beta^T\dot{\zeta}_s)t\ . \end{aligned}$$

Substituting into the differential equations and integrating as before gives

$$\begin{aligned} \xi = \bar{\xi} &- \varepsilon \sum_{\alpha\neq 0} \frac{b_\alpha(\bar{\xi})}{\alpha^T\dot{\eta}_s} \cos[\alpha^T(\bar{\eta} + \dot{\eta}_s t)] \\ &- \varepsilon \sum_{\alpha,\beta} \frac{B_{\alpha\beta}(\bar{\xi})}{\alpha^T\dot{\eta}_s + \beta^T\dot{\zeta}_s} \cos[\bar{\Theta}_{\alpha\beta} + (\alpha^T\dot{\eta}_s + \beta^T\dot{\zeta}_s)t] \end{aligned} \tag{3.12}$$

$$\begin{aligned} \eta = \bar{\eta} + \dot{\eta}_s t &+ \varepsilon \sum_{\alpha\neq 0} \frac{c_\alpha(\bar{\xi})}{\alpha^T\dot{\eta}_s} \sin[\alpha^T(\bar{\eta} + \dot{\eta}_s t)] \\ &+ \varepsilon \sum_{\alpha,\beta} \frac{C_{\alpha\beta}(\bar{\xi})}{\alpha^T\dot{\eta}_s + \beta^T\dot{\zeta}_s} \sin[\bar{\Theta}_{\alpha\beta} + (\alpha^T\dot{\eta}_s + \beta^T\dot{\zeta}_s)t] \end{aligned} \tag{3.13}$$

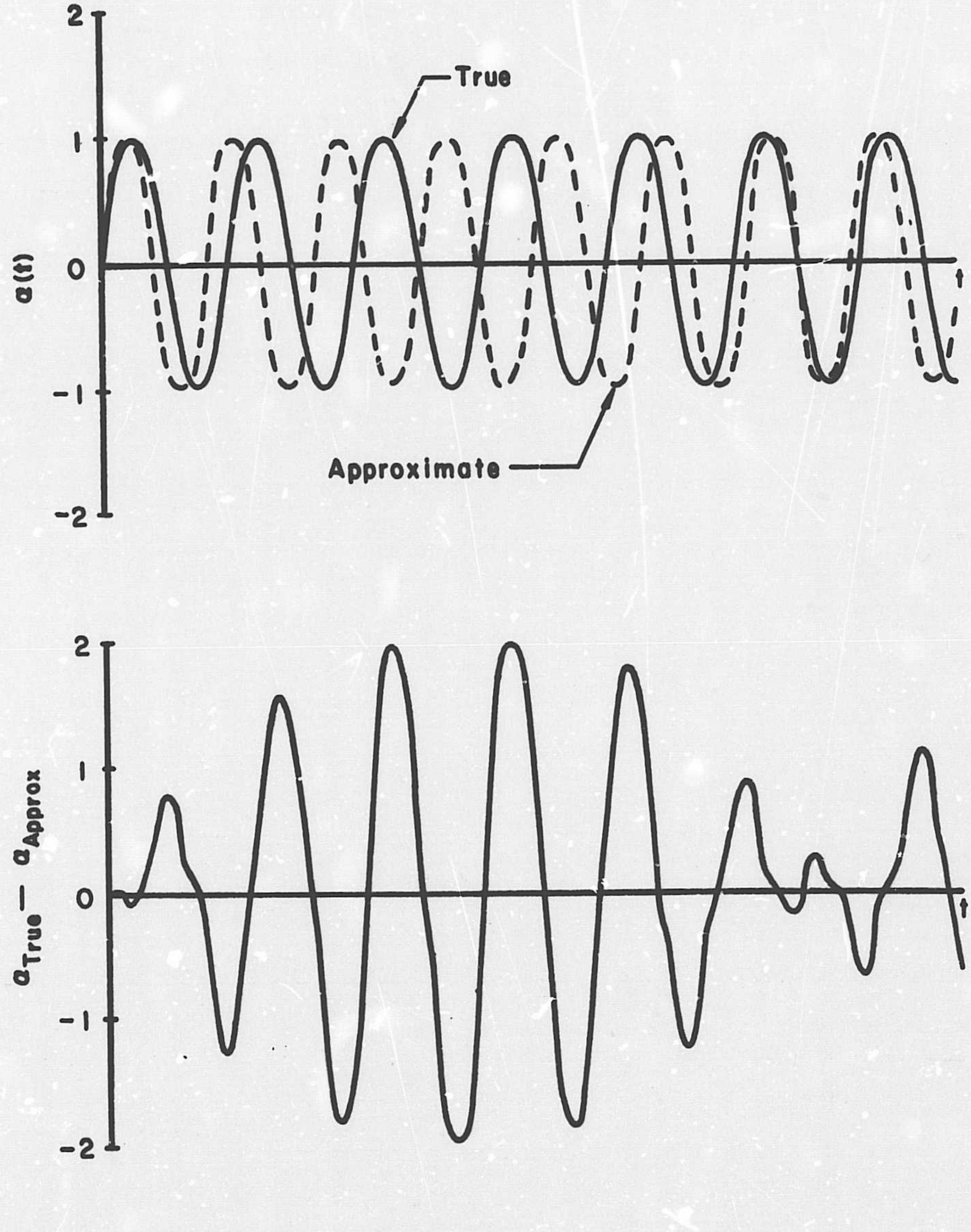


FIGURE 5. Effect of a Phase Error

$$\begin{aligned}\zeta = \overline{\zeta} + \dot{\zeta}_s t - \varepsilon \frac{d\omega}{d\overline{\xi}} \Big\{ & \sum_{\alpha\neq 0} \frac{b_\alpha(\overline{\xi})}{(\alpha^T \dot{\eta}_s)^2} \sin[\alpha^T(\overline{\eta} + \dot{\eta}_s t)] \\ & + \sum_{\alpha,\beta} \frac{B_{\alpha\beta}(\overline{\xi})}{(\alpha^T \dot{\eta}_s + \beta^T \dot{\zeta}_s)^2} \sin[\overline{\Theta}_{\alpha\beta} + (\alpha^T \dot{\eta}_s + \beta^T \dot{\zeta}_s)t] \Big\} \\ & + \varepsilon \sum_{\alpha\neq 0} \frac{d_\alpha(\overline{\xi})}{\alpha^T \dot{\eta}_s} \sin[\alpha^T(\overline{\eta} + \dot{\eta}_s t)] \\ & + \varepsilon \sum_{\alpha,\beta} \frac{D_{\alpha\beta}(\overline{\xi})}{\alpha^T \dot{\eta}_s + \beta^T \dot{\zeta}_s} \sin[\overline{\Theta}_{\alpha\beta} + (\alpha^T \dot{\eta}_s + \beta^T \dot{\zeta}_s)t] \ . \end{aligned} \tag{3.14}$$

The following types of terms appear in the solutions.

1. Secular -- The equations for η and ζ contain linear functions of the time t, known as secular terms, that are the dominant portions of the solutions for these variables. The secular terms in ζ_i are of zero order in ε while those in the η_i are of first order. Thus, the magnitudes of the ζ variables will increase much faster than those of the η variables.
2. Short period -- All of the equations contain periodic terms having $\theta_{\alpha\beta}$ as the argument of the trigonometric functions. Since a zero order secular term $(\beta^T \dot{\zeta}_s t)$ appears in $\theta_{\alpha\beta}$, the periodic terms will produce high frequency oscillations in the solution. They are therefore referred to as short period terms.
3. Long period -- The equations for ξ, η, and ζ include trigonometric functions with $\overline{\theta}_\alpha$ as the argument. The first order secular variation of $\overline{\theta}_\alpha$ (due to $\alpha^T \dot{\eta}_s t$) will produce oscillations that are much slower than the short period changes described above. For this reason, the parts of the solution periodic in $\overline{\theta}_\alpha$ are called long period terms.

The solutions, then, are given as a superposition of three different types of terms as illustrated in Fig. 6.

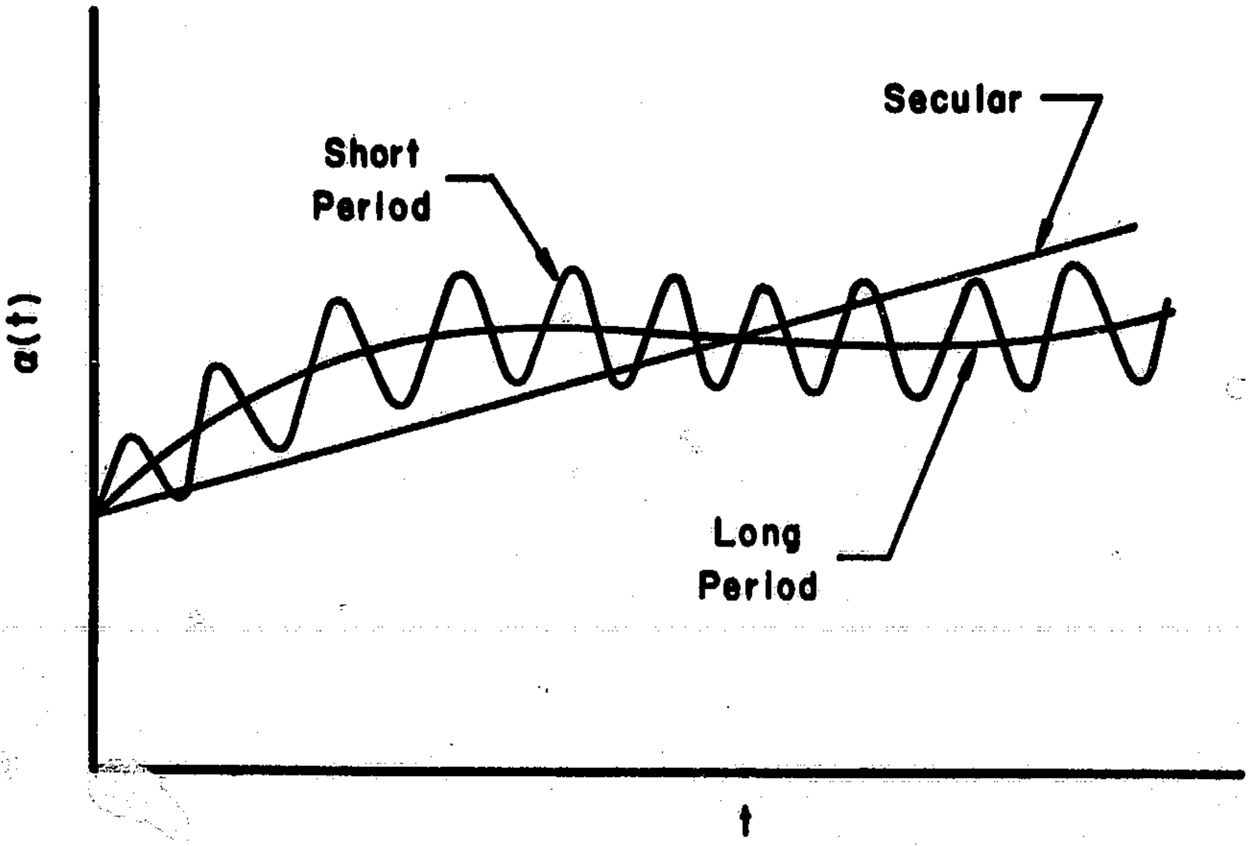


FIGURE 6. Representation of the Solutions

An interesting phenomena is observed in connection with the long period terms in the solution. Referring to Eqs. (3.12), (3.13), and (3.14), it is noted that terms of this type are multiplied by the small parameter ε and therefore appear to be of first order. However, the denominators of these expressions are $\alpha^T \dot{\eta}_s$ which is itself first order in ε . Cancelling the small parameter in the numerator and the denominator leads to long period terms of zero order (i.e. terms multiplied by ε^o). Similarly, first order long period terms will be obtained in the calculation of the second order effects. Thus, it is much more difficult to obtain first order long period terms than it is to calculate first order secular and short period terms. The importance of this characteristic will become evident in the discussion of the Earth satellite problem.

The procedure for generating second (ε^2) and higher order approximations by Kaula's method is not presented here because it is not required in the discussion of Ingram's solutions. However, it is noted that the process is similar to the one used by Kozai (Ref. 14). That is, the first order solution given by Eqs. (3.12), (3.13), and (3.14) is substituted into the right hand sides of the differential equations, the resulting terms are expanded in Taylor's series about the first approximation (Eqs. (3.9), (3.10), and (3.11)) and, for a second order solution, only terms of second order in ε are retained. Further information on this topic can be found in the discussion of Kozai's work in Ref. 15. A study of Kaula's solution and of Kozai's method has led to the tentative conclusion that both techniques are equivalent to the generalized method of averages (see Ref. 18). The higher order approximations obtained by the method of averages have the following characteristics:

1. They are combinations of secular, short period, and long period terms.
2. Long period terms of the nth order result from the calculation of effects of order $n+1$.
3. Mixed terms (trigonometric functions multiplied by time) do not occur.

The last question to be considered concerns the computation of the barred quantities (mean values) appearing in the solutions. The values of these parameters depend upon the initial conditions for the original set of differential equations. Thus, if ξ_o, η_o, and ζ_o are specified values of ξ, η, and ζ, respectively, at the initial time ($t = 0$), Eqs. (3.12), (3.13), and (3.14) (with first order long period terms included) yield

$$\bar{\xi} = \xi_o + \varepsilon \sum_{\alpha\neq 0} \frac{b_\alpha(\bar{\xi})}{\alpha^T \dot{\eta}_s} \cos \alpha^T \bar{\eta} + \varepsilon \sum_{\alpha,\beta} \frac{B_{\alpha\beta}(\bar{\xi})}{\alpha^T \dot{\eta}_s + \beta^T \dot{\zeta}_s} \cos \bar{\Theta}_{\alpha\beta}$$

$$+ \varepsilon \text{ (long period terms)}$$

$$\bar{\eta} = \eta_o - \varepsilon \sum_{\alpha\neq 0} \frac{c_\alpha(\bar{\xi})}{\alpha^T \dot{\eta}_s} \sin \alpha^T \bar{\eta}$$

$$- \varepsilon \sum_{\alpha,\beta} \frac{C_{\alpha\beta}(\bar{\xi})}{\alpha^T \dot{\eta}_s + \beta^T \dot{\zeta}_s} \sin \bar{\Theta}_{\alpha\beta} + \varepsilon \text{ (long period terms)}$$

$$\bar{\zeta} = \zeta_o + \varepsilon \frac{d\omega}{d\bar{\xi}} \left\{ \sum_{\alpha\neq 0} \frac{b_\alpha(\bar{\xi})}{(\alpha^T \dot{\eta}_s)^2} \sin \alpha^T \bar{\eta} + \sum_{\alpha,\beta} \frac{B_{\alpha\beta}(\bar{\xi})}{(\alpha^T \dot{\eta}_s + \beta^T \dot{\zeta}_s)^2} \sin \bar{\Theta}_{\alpha\beta} \right\}$$

$$+ \varepsilon \sum_{\alpha\neq 0} \frac{d_\alpha(\bar{\xi})}{\alpha^T \dot{\eta}_s} \sin \alpha^T \bar{\eta} + \varepsilon \sum_{\alpha,\beta} \frac{D_{\alpha\beta}(\bar{\xi})}{\alpha^T \dot{\eta}_s + \beta^T \dot{\zeta}_s} \sin \bar{\Theta}_{\alpha\beta}$$

$$+ \varepsilon \text{ (long period terms)} .$$

The previous equations can be solved for $\bar{\xi}$, $\bar{\eta}$, and $\bar{\zeta}$ by an iterative technique. However, it generally suffices to replace the barred quantities on the right hand sides of these equations with initial values. This is permissible because the error incurred is of order higher than the first.

This completes the discussion of Kaula's technique. The von Zeipel method is presented in the following section.

III.2 The von Zeipel Method

The problem of determining solutions of the system of canonical equations

$$\dot{q}^T = H_p$$

$$\dot{p}^T = -H_q$$

was briefly discussed in Section II.3. It was noted that the problem is trivial if the Hamiltonian is independent of the coordinates q . Thus, the solution can be obtained by finding a canonical transformation to a new set of variables such that the new Hamiltonian is a function of the momenta only. In general, however, this procedure is difficult to apply because the required transformation is described by a generating function which is the solution of a nonlinear partial differential equation. The von Zeipel method is a technique for determining this generating function by successive approximations.

At this point the notation will be changed to make it correspond more closely to that used in celestial mechanics. In the following paragraphs, the generalized coordinates will be designated by ℓ rather than q , the generalized momenta will be designated by L , and the function $F(L,\ell)$ defined by

$$F(L,\ell) = -H(\ell,L)$$

will be used in place of H. It is common practice in celestial mechanics to refer to F as the Hamiltonian. Note that, without loss of generality, the Hamiltonian is assumed to be explicitly independent of the time t (see p. 531 of Ref. 1). With these modifications, the canonical equations become

$$\begin{aligned} \dot{L}^T &= F_\ell \\ \dot{\ell}^T &= -F_L . \end{aligned} \tag{3.15}$$

In the following discussion of the von Zeipel method, consideration will be given only to Hamiltonians of the type encountered in the usual problems in celestial mechanics. That is, F will be assumed to have the form

$$F(L,\ell) = F_o(L) + \varepsilon F_1(L,\ell) + \varepsilon^2 F_2(L,\ell) + \dots \tag{3.16}$$

where ε is a small parameter and the $F_i (i = 1,2,\dots)$ are periodic of period 2π in the components of ℓ.

The procedure, then, is to make a canonical transformation from (L,ℓ) to (L',ℓ') such that the new Hamiltonian F^* is independent of ℓ'. The transformation is to be specified by the generating function $S(L',\ell)$ and is therefore given by the equations

$$L = S_\ell^T \qquad \ell' = S_{L'}^T \tag{3.17}$$

$$F(S_\ell^T,\ell) = F^*(L') . \tag{3.18}$$

It is assumed that S and F* can be expanded in powers of the small parameter ε , i.e.,

$$\begin{aligned} S &= S_o + \varepsilon S_1 + \varepsilon^2 S_2 + \ldots \\ F^* &= F^*_o + \varepsilon F^*_1 + \varepsilon^2 F^*_2 + \ldots . \end{aligned} \tag{3.19}$$

Since the solution to the problem is known when $\varepsilon = 0$, the canonical transformation is chosen to be the identity transformation for this case. Thus,

$$S_o = (L')^T \ell .$$

With the previous assumptions, Eqs. (3.17) become

$$\begin{aligned} L &= L' + \varepsilon\left(\frac{\partial S_1}{\partial \ell}\right)^T + \varepsilon^2\left(\frac{\partial S_2}{\partial \ell}\right)^T + \ldots \\ \ell' &= \ell + \varepsilon\left(\frac{\partial S_1}{\partial L'}\right)^T + \varepsilon^2\left(\frac{\partial S_2}{\partial L'}\right)^T + \ldots . \end{aligned} \tag{3.20}$$

Equations defining S and F* may now be developed as follows. Substituting the expressions for L , ℓ' , F , and F* from Eqs. (3.16), (3.19), and (3.20) into Eq. (3.18) yields

$$\begin{aligned} & F_o[L' + \varepsilon\left(\frac{\partial S_1}{\partial \ell}\right)^T + \varepsilon^2\left(\frac{\partial S_2}{\partial \ell}\right)^T + \ldots] + \varepsilon F_1[L' + \varepsilon\left(\frac{\partial S_1}{\partial \ell}\right)^T + \varepsilon^2\left(\frac{\partial S_2}{\partial \ell}\right)^T + \ldots, \ell] \\ & \quad + \varepsilon^2 F_2[L' + \varepsilon\left(\frac{\partial S_1}{\partial \ell}\right)^T + \varepsilon^2\left(\frac{\partial S_2}{\partial \ell}\right)^T + \ldots, \ell] + \ldots \\ & \quad = F^*_o(L') + \varepsilon F^*_1(L') + \varepsilon^2 F^*_2(L') + \ldots . \end{aligned}$$

Expanding each term in a Taylor's series about $L = L'$ and $\ell' = \ell$ gives

$$F_o(L') + \varepsilon \frac{\partial F_o}{\partial L'}\left(\frac{\partial S_1}{\partial \ell}\right)^T + \varepsilon^2 \frac{\partial F_o}{\partial L'}\left(\frac{\partial S_2}{\partial \ell}\right)^T + \frac{1}{2}\varepsilon^2 \frac{\partial S_1}{\partial \ell}\frac{\partial^2 F_o}{\partial L'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^T$$
$$+ \varepsilon F_1(L', \ell) + \varepsilon^2 \frac{\partial F_1}{\partial L'}\left(\frac{\partial S_1}{\partial \ell}\right)^T + \varepsilon^2 F_2(L', \ell) + \ldots \qquad (3.21)$$
$$= F_o^*(L') + \varepsilon F_1^*(L') + \varepsilon^2 F_2^*(L') + \ldots$$

where, for example,

$$\frac{\partial F_o}{\partial L'} \equiv \left.\frac{\partial F_o}{\partial L}\right|_{L=L'} .$$

Equations (3.22), (3.23), and (3.24) below then follow from Eq. (3.21) by the process of equating terms containing the same power of the small parameter ε .

$$F_o(L') = F_o^*(L') \qquad (3.22)$$

$$\frac{\partial F_o}{\partial L'}\left(\frac{\partial S_1}{\partial \ell}\right)^T + F_1(L', \ell) = F_1^*(L') \qquad (3.23)$$

$$\frac{\partial F_o}{\partial L'}\left(\frac{\partial S_2}{\partial \ell}\right)^T + \frac{1}{2}\frac{\partial S_1}{\partial \ell}\frac{\partial^2 F_o}{\partial L'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^T + \frac{\partial F_1}{\partial L'}\left(\frac{\partial S_1}{\partial \ell}\right)^T + F_2(L',\ell) = F_2^*(L') \qquad (3.24)$$
$$\vdots$$

The zero order portion of the new Hamiltonian F^* is obtained immediately from Eq. (3.22). Both S_1 and F_1^* must be determined from Eq. (3.23). However, this single equation is not adequate for determining these functions uniquely.

Equation (3.23) can be solved once either F_1^* or S_1 is specified. It may be possible now to exploit this arbitrariness by choosing F_1^* or S_1 in such a manner that the amount of work involved in developing a solution is reduced or that the accuracy of the final solution is enhanced. As will become evident in the discussion of the example problem in Chapter IV, the second consideration suggests assuming that S_1 is periodic in the components of ℓ. Inasmuch as F_1^* is a function of L' only, it may be chosen to be the portion of the left hand side of Eq. (3.23) that is independent of ℓ, i.e.,

$$F_1^* = \frac{1}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \left[\frac{\partial F_o}{\partial L'} \left(\frac{\partial S_1}{\partial \ell}\right)^T + F_1\right] d\ell_1\, d\ell_2 \ldots d\ell_n$$

$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} F_1\, d\ell_1\, d\ell_2 \ldots d\ell_n \equiv F_{1s}$$

where F_{1s} is called the "secular" part of F_1. Then, Eq. (3.23) is replaced by the equations

$$\begin{aligned} F_{1s} &= F_1^* \\ \frac{\partial F_o}{\partial L'}\left(\frac{\partial S_1}{\partial \ell}\right)^T &= -(F_1 - F_{1s}) \equiv -F_{1p} . \end{aligned} \tag{3.25}$$

The quantity F_{1p} is purely periodic in ℓ and therefore is referred to as the periodic portion of F_1. Saying that F_{1p} is purely periodic in ℓ means

1. F_{1p} is periodic in ℓ
2. The secular part of F_{1p} is zero.

Mathematically, these conditions are

1. $F_{1p}(L, \ell_1, \ell_2, \ldots \ell_m + 2\pi, \ldots \ell_n) = F_{1p}(L, \ell)$ for $m = 1,2,\ldots n$
2. $\dfrac{1}{(2\pi)^n} \displaystyle\int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} F_{1p}\, d\ell_1\, d\ell_2 \ldots d\ell_n = 0$.

Note that the partial differential equation for S_1 yields a generating function that is also periodic in ℓ as was required. The solution of this partial differential equation can be obtained, for example, by the method of characteristics (see Ref. 16).

With S_1 known, Eq. (3.24) can be solved for S_2 and F_2^* in the same manner. The process is continued until the generating function and the Hamiltonian have been determined to the desired order. Hence, if F^* and S are given to order m in ε, the solution is

$$\dot{L}' = (F^*_{\ell'})^T = 0 \Longrightarrow L' = \text{Cons.}$$

$$\dot{\ell}' = -(F^*_{L'})^T = \text{Cons.} = \omega_o(L') + \varepsilon\omega_1(L') + \dots + \varepsilon^m\omega_m(L')$$

$$\Longrightarrow \ell' = \ell'_o + (\omega_o + \varepsilon\omega_1 + \dots + \varepsilon^m\omega_m)t$$

$$L = L' + \varepsilon\left(\frac{\partial S_1}{\partial \ell}\right)^T + \dots + \varepsilon^m\left(\frac{\partial S_m}{\partial \ell}\right)^T$$

$$\ell = \ell'_o + (\omega_o + \varepsilon\omega_1 + \dots + \varepsilon^m\omega_m)t - \varepsilon\left(\frac{\partial S_1}{\partial L'}\right)^T - \dots - \varepsilon^m\left(\frac{\partial S_m}{\partial L'}\right)^T .$$

In common with the results given in Section III.1, the solution is a combination of periodic and secular terms.

It is important to note that under certain circumstances the procedure presented above fails to produce the desired transformation to a new Hamiltonian containing momenta only (Ref. 17). Consider, for example, a Hamiltonian of the form

$$F(L_1,L_2,\ell_1,\ell_2) = F_o(L_1) + \varepsilon\, F_1(L_1,L_2,\ell_1,\ell_2)$$

with F_1 periodic of period 2π in ℓ_1 and ℓ_2. The new Hamiltonian $F^*(L_1', L_2')$ and the generating function S are to be obtained (through first order) from Eqs. (3.22) and (3.23). From the first equation the zero order portion of F^* is found to be

$$F_o^*(L_1') = F_o(L_1') .$$

Both S_1 and F_1^* must be determined from Eq. (3.23) which becomes

$$\frac{\partial F_o}{\partial L_1'} \frac{\partial S_1}{\partial \ell_1} + F_1(L_1', L_2', \ell_1, \ell_2) = F_1^*(L_1', L_2')$$

or

$$\frac{\partial S_1}{\partial \ell_1} = -\left(\frac{\partial F_o}{\partial L_1'}\right)^{-1} [F_1(L_1', L_2', \ell_1, \ell_2) - F_1^*(L_1', L_2')] . \tag{3.26}$$

Since S_1 is assumed to be periodic in ℓ_1, the derivative $\frac{\partial S_1}{\partial \ell_1}$ is purely periodic in this variable. It follows from Eq. (3.26) that

$$F_1^*(L_1', L_2') = \frac{1}{2\pi} \int_0^{2\pi} F_1(L_1', L_2', \ell_1, \ell_2) d\ell_1 . \tag{3.27}$$

But the average of F_1 with respect to ℓ_1 may be a function of the coordinate ℓ_2 while F_1^* must contain momenta only. Thus, if the function obtained by averaging F_1 with respect to ℓ_1 contains ℓ_2, Eq. (3.27) can not be satisfied and the procedure fails. It is this difficulty which prevents the determination of a complete solution to the Lunar satellite problem by von Zeipel's method (see Chapter VI).

Several good discussions of the von Zeipel method may be found in the literature. Giacaglia (Ref. 17) has derived expressions that can be used to determine the generating function and the new Hamiltonian to any order.

The relation between the von Zeipel method and the generalized method of averaging is developed by Morrison (Ref. 18). In Ref. 19, Hutchinson gives an introductory discussion of the method and applies it to a simple problem.

III.3 Small Divisors and Resonance

Inherent in the preceding perturbation techniques is a difficulty that has not yet been discussed but which has been important in the development of many theories in celestial mechanics. This is the problem of small divisors or, as it is also known, the problem of resonance.

Consider the solutions obtained in Sec. III.1. An examination of Eqs. (3.12), (3.13), and (3.14) reveals that the expressions for the periodic perturbations contain denominators of the form

$$\psi = \alpha^T \dot{\eta}_{\underline{s}} + \beta^T \dot{\zeta}_{\underline{s}} .$$

In most cases, there are a large number of values of the indices α and β in the range of the summations for which the quantity ψ is very nearly zero. For these values of the indices, the corresponding terms in the solutions become very large and the convergence of the formal series expansion is endangered. This leads to serious questions about the validity of the series solution.

A similar circumstance can arise in the determination of the generating functions required in the von Zeipel method (Ref. 20). For example, assume that F_{1p} in Eq. (3.25) is represented by a Fourier cosine series expansion in the variables $\ell_1, \ell_2, \ldots \ell_n$. The partial differential equation for S_1 becomes

$$\nu_1 \frac{\partial S_1}{\partial \ell_1} + \nu_2 \frac{\partial S_1}{\partial \ell_2} + \ldots + \nu_n \frac{\partial S_1}{\partial \ell_n} = \sum_{\alpha} C_\alpha(L) \cos(\alpha_1 \ell_1 + \alpha_2 \ell_2 + \ldots + \alpha_n \ell_n)$$

where

$$\nu_i \equiv \frac{\partial F_o}{\partial L_i'} .$$

The solution of this equation is

$$S_1 = \sum_{\alpha} \frac{C_\alpha(L)}{\alpha_1\nu_1+\alpha_2\nu_2+\ldots+\alpha_n\nu_n} \sin(\alpha_1\ell_1 + \alpha_2\ell_2 + \ldots + \alpha_n\ell_n) .$$

Again a denominator that can become small appears.

Brown (Ref. 21) has analyzed the phenomena of resonance as it arises in the problem of pendulum motion in a uniform gravitational field. He found that two distinct types of motion are possible, resonant and non-resonant. The pendulum is in a state of resonance when it is oscillating about the stable equilibrium position. Non-resonant motion corresponds to the case in which the pendulum is making complete revolutions about its support. Brown shows that the formal series solution obtained for non-resonant motion is not valid for resonant motion. Indeed, a small divisor appears in the series as a resonant state is approached. Thus, the appearance of a small divisor seems to indicate that the formal series solution is not valid in the region under consideration. This problem will be treated further in the discussion of the artificial Earth satellite theories in Chapter V.

CHAPTER IV

THE NONLINEAR SPRING PROBLEM

Familiarity with the general perturbations techniques described in Chapter III can be gained by applying them to a physically meaningful problem for which an analytical solution can be obtained with a minimum of algebraic manipulation. The following paragraphs are devoted to the development of the analytical solutions for the motion of a simple dynamical system and to comparisons of numerical results.

IV.1 Problem Formulation

Consider the problem of determining the motion of a mass moving on a smooth horizontal surface and attached to a rigid support by a nonlinear spring. If the force exerted on the mass by the spring is proportional to a linear combination of the displacement (x) of the spring from its equilibrium position and the cube of this displacement, the equation of motion of the mass has the form

$$\ddot{x} = -k^2x - \varepsilon x^3 \qquad (4.1)$$

where $k \ (\neq 0)$ and $\varepsilon \ (\geq 0)$ are parameters characterizing the stiffness of the spring. It is easily shown that the system is conservative, the energy being given by

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}k^2x^2 + \tfrac{1}{4}\varepsilon x^4 = \text{constant} . \qquad (4.2)$$

Since ε is assumed to be non-negative and k is assumed to be non-zero, Eq. (4.2) indicates that the magnitude of x is bounded.

As is evident from Eq. (4.1), the acceleration acting on the mass can be separated into two components, a linear term $(-k^2x)$ and a nonlinear term $(-\varepsilon x^3)$. The ratio of these accelerations is

$$\delta = \frac{-\varepsilon x^3}{-k^2x} = \varepsilon \frac{x^2}{k^2}$$

$$\Longrightarrow \delta_{max} = \frac{\varepsilon}{k^2} x^2_{max} .$$

Now, for given values of k^2 and x^2_{max} , δ_{max} can be made as small as desired by adjusting ε . That is, ε can be chosen to make the linear portion of the acceleration dominate the nonlinear portion. It is important to note that the value of ε required to achieve this condition will depend on the initial conditions for the motion.

The equation of motion of the spring mass system (Eq. (4.1)) can be rewritten as

$$\ddot{x} + k^2x = -\varepsilon x^3 . \qquad (4.3)$$

Then, for small values of ε , the problem can be treated as a simple harmonic oscillator subject to a small perturbing acceleration. The solution of the perturbed problem is expected to differ only slightly from that of the unperturbed problem for this case.

Before the general perturbation techniques of the previous chapter can be applied to this problem, the equation of motion must be put in the form required by these methods. Using an approach similar to the variation of parameters, define the new variables L and ℓ by

$$x = \sqrt{\frac{2L}{k}} \cos \ell \qquad (4.4)$$

$$\dot{x} = -\sqrt{2kL} \sin \ell \qquad (4.4)$$

The quantities L and ℓ are closely related to action-angle variables (Ref. 22). Substituting Eqs. (4.4) into Eq. (4.3) yields the differential equations

$$\begin{aligned} \dot{L} &= \varepsilon \frac{L^2}{k^2} (\sin 2\ell + \frac{1}{2} \sin 4\ell) \\ \dot{\ell} &= k + \varepsilon \frac{L}{k^2} (\frac{3}{4} + \cos 2\ell + \frac{1}{4} \cos 4\ell) . \end{aligned} \qquad (4.5)$$

Observe that, for $\varepsilon = 0$, the solution of the previous equations is

$$\begin{aligned} L &= \text{constant} \\ \ell &= kt + \text{constant} . \end{aligned}$$

For $\varepsilon \neq 0$, Eqs. (4.5) are of the form required by Kaula's method (see Eqs. (3.1), (3.2), and (3.3)).

In order to obtain the canonical form required by the von Zeipel method, the energy integral (Eq. (3.2)) can be used to express the Hamiltonian as a function of the variables L and ℓ. Following astronomical convention, the Hamiltonian F is defined as the negative of the total energy. Thus,

$$\begin{aligned} F = -E &= - \tfrac{1}{2} \dot{x}^2 - \tfrac{1}{2} k^2 x^2 - \tfrac{1}{4} \varepsilon x^4 \\ &= - kL - \varepsilon \frac{L^2}{k^2} \cos^4 \ell \end{aligned}$$

or

$$F = - kL - \varepsilon \frac{L^2}{k^2} (\frac{3}{8} + \frac{1}{2} \cos 2\ell + \frac{1}{8} \cos 4\ell) . \qquad (4.6)$$

Comparison of Eq. (4.6) with Eq. (3.16) shows that the Hamiltonian given above satisfies the required conditions. It is easily verified that Eqs. (4.5) can be written in the canonical form

$$\dot{L} = F_\ell \qquad \dot{\ell} = -F_L \; .$$

In summary, for the application of Kaula's method, the dynamical system is represented by the equations

$$\begin{aligned} \dot{L} &= \varepsilon \frac{L^2}{k^2} (\sin 2\ell + \frac{1}{2} \sin 4\ell) \\ \dot{\ell} &= k + \varepsilon \frac{L}{k^2} (\frac{3}{4} + \cos 2\ell + \frac{1}{4} \cos 4\ell) \; . \end{aligned} \tag{4.7}$$

For the von Zeipel method, the form of the equations is

$$\dot{L} = F_\ell \qquad \dot{\ell} = -F_L \tag{4.8}$$

where

$$F = -kL - \varepsilon \frac{L^2}{k^2} (\frac{3}{8} + \frac{1}{2} \cos 2\ell + \frac{1}{8} \cos 4\ell) \; .$$

In both cases, ε is assumed small.

IV.2 Solution by Kaula's Method

Following the procedure of Section III.1, assume

$$\begin{aligned} L &= \bar{L} \\ \ell &= \bar{\ell} + \dot{\ell}_s t = \bar{\ell} + k(1 + \varepsilon \frac{3}{4} \frac{\bar{L}}{k^3}) t \end{aligned} \tag{4.9}$$

where $\bar{L}$ and $\bar{\ell}$ are constants. Substituting Eqs. (4.9) into the right hand sides of Eqs. (4.7) and integrating gives

$$L = \bar{L} - \varepsilon \frac{\bar{L}^2}{2k^2 \dot{\ell}_s} [\cos 2(\bar{\ell} + \dot{\ell}_s t) + \frac{1}{4} \cos 4(\bar{\ell} + \dot{\ell}_s t)] \tag{4.10}$$

$$\ell = \bar{\ell} + \dot{\ell}_s t + \varepsilon \frac{\bar{L}}{2k^2\dot{\ell}_s}[\sin 2(\bar{\ell}+\dot{\ell}_s t) + \frac{1}{8}\sin 4(\bar{\ell}+\dot{\ell}_s t)] \,. \tag{4.11}$$

The quantities $\bar{L}$ and $\bar{\ell}$ are computed from the known initial conditions L_E and ℓ_E by using the equations below.

$$\bar{L} = L_E + \varepsilon \frac{L_E^2}{2k^2\dot{\ell}_{sE}}(\cos 2\ell_E + \frac{1}{4}\cos 4\ell_E) \tag{4.12}$$

$$\bar{\ell} = \ell_E - \varepsilon \frac{L_E}{2k^2\dot{\ell}_{sE}}(\sin 2\ell_E + \frac{1}{8}\sin 4\ell_E) \tag{4.13}$$

Here, $\dot{\ell}_{sE}$ is $\dot{\ell}_s$ evaluated at epoch.

IV.3 Solution by the von Zeipel Method

The Hamiltonian is

$$\begin{aligned} F &= -kL - \varepsilon \frac{L^2}{k^2}(\frac{3}{8} + \frac{1}{2}\cos 2\ell + \frac{1}{8}\cos 4\ell) \\ &= F_o + \varepsilon F_1 \end{aligned} \tag{4.14}$$

where L is the generalized momentum and ℓ is the generalized coordinate. Following Section III.2, a canonical transformation to the new variables L' and ℓ' is made such that

$$F(L,\ell) = F^*(L') = F_o^*(L') + \varepsilon F_1^*(L') + \varepsilon^2 F_2^*(L') + \dots \,. \tag{4.15}$$

The generating function $S(L',\ell)$ is expanded in powers of the small parameter and Eqs. (3.17) are used to write L and ℓ' in terms of L' and ℓ. Substituting the results into Eq. (4.15) and expanding in Taylor's series yields the following relations (through second order in ε) which are analogous to Eqs. (3.22), (3.23), and (3.24).

$$F_o(L') = F_o^*(L') \tag{4.16}$$

$$\frac{\partial F_o}{\partial L'}\frac{\partial S_1}{\partial \ell} + F_1(L',\ell) = F_1^*(L') \tag{4.17}$$

$$\frac{\partial F_o}{\partial L'}\frac{\partial S_2}{\partial \ell} + \frac{1}{2}\frac{\partial^2 F_o}{\partial L'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^2 + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial \ell} + F_2(L',\ell) = F_2^*(L') \tag{4.18}$$

From Eqs. (4.14) and (4.16),

$$F_o^*(L') = - kL' . \tag{4.19}$$

Equation (4.17) becomes

$$-k\frac{\partial S_1}{\partial \ell} - \frac{L'^2}{k^2}\left(\frac{3}{8} + \frac{1}{2}\cos 2\ell + \frac{1}{8}\cos 4\ell\right) = F_1^*(L') . \tag{4.20}$$

The assumption that S_1 is periodic in ℓ is satisfied by choosing F_1^* to be the portion of the left hand side of Eq. (4.20) that is independent of ℓ, i.e.,

$$F_1^*(L') = - \frac{3}{8}\frac{L'^2}{k^2} . \tag{4.21}$$

Then, the equation for S_1 is

$$\frac{\partial S_1}{\partial \ell} = - \frac{L'^2}{2k^3}\left(\cos 2\ell + \frac{1}{4}\cos 4\ell\right) \tag{4.22}$$

which can easily be integrated to give

$$S_1 = - \frac{L'^2}{4k^3}\left(\sin 2\ell + \frac{1}{8}\sin 4\ell\right) + \hat{S}_1(L') . \tag{4.23}$$

The arbitrary function of L', $\hat{S}_1(L')$, can be taken as zero since its value does not affect the final results.

At this point, Eqs. (3.20) and (4.23) can be used to obtain the relations between the primed and unprimed variables through first order. They are

$$L = \frac{\partial S}{\partial \ell} = L' - \varepsilon \frac{L'^2}{2k^3}(\cos 2\ell + \frac{1}{4}\cos 4\ell) \tag{4.24}$$

$$\ell' = \frac{\partial S}{\partial L'} = \ell - \varepsilon \frac{L'}{2k^3}(\sin 2\ell + \frac{1}{8}\sin 4\ell) . \tag{4.25}$$

The new Hamiltonian through first order is

$$F^*(L') = -kL' - \varepsilon \frac{3}{8}\frac{L'^2}{k^2} . \tag{4.26}$$

The accuracy of the solution can be increased by including the second order secular term in ℓ since neglecting this term leads to an error of the form $\varepsilon^2 t$ which may become significant in a short period of time. For this reason the new Hamiltonian is calculated through second order. Choosing F_2^* to be the portion of the left hand side of Eq. (4.18) that is independent of ℓ yields

$$F_2^* = \frac{17}{64}\frac{L'^3}{k^5} .$$

Thus, through terms of the second order,

$$F^* = -kL' - \varepsilon \frac{3}{8}\frac{L'^2}{k^2} + \varepsilon^2 \frac{17}{64}\frac{L'^3}{k^5} . \tag{4.27}$$

The differential equations for L' and ℓ' are

$$\dot{L}' = F^*_{\ell'} = 0 \Longrightarrow L' = \text{constant} \tag{4.28}$$

$$\begin{aligned} \dot{\ell}' &= -F^*_{L'} = k + \varepsilon \frac{3}{4}\frac{L'}{k^2} - \varepsilon^2 \frac{51}{64}\frac{L'^2}{k^5} \\ &\Longrightarrow \ell' = \ell'_0 + k(1 + \varepsilon \frac{3}{4}\frac{L'}{k^3} - \varepsilon^2 \frac{51}{64}\frac{L'^2}{k^6})t . \end{aligned} \tag{4.29}$$

Equations (4.24), (4.25), (4.28), and (4.29) represent a complete first order solution of the nonlinear spring problem ("complete" means that second order secular effects are included). However, Eqs. (4.24) and (4.25) are not in the proper form for numerical calculations. The necessary change is the replacement of ℓ in the trigonometric terms with ℓ'. This is permissible because the error introduced is of second order. The final results obtained by the application of the von Zeipel method are

$$\begin{aligned} L &= L' - \varepsilon \frac{L'^2}{2k^3}\left(\cos 2\ell' + \frac{1}{4}\cos 4\ell'\right) \\ \ell &= \ell' + \varepsilon \frac{L'}{2k^3}\left(\sin 2\ell' + \frac{1}{8}\sin 4\ell'\right) \\ \ell' &= \ell'_o + k\left(1 + \varepsilon \frac{3}{4}\frac{L'}{k^3} - \varepsilon^2 \frac{51}{64}\frac{L'^2}{k^6}\right)t \ . \end{aligned} \tag{4.30}$$

The constants L' and ℓ'_o are evaluated by inverting the previous equations at the initial time.

The justification for requiring that S be periodic in the components of ℓ now becomes clear. If the right hand side of the partial differential equation for S_1 (Eq. (4.22)) contains a term that is a function of L' only, S_1 will have a term that is linear in ℓ. Since ℓ increases rapidly with time, this term will cause S_1 to become large and the assumption that εS_1 is of first order will be violated. Requiring S to be periodic in the components of ℓ prevents the appearance of such secular terms.

The analysis given above was extended to obtain a complete second order solution so that the effects of including higher order periodic terms could be demonstrated. The second order solution is

$$\begin{aligned}
L &= L' - \varepsilon \frac{L'^2}{2k^3}(\cos 2\ell' + \frac{1}{4}\cos 4\ell') \\
&\quad + \varepsilon^2 \frac{L'^3}{2k^6}(\frac{17}{32} + \frac{21}{16}\cos 2\ell' + \frac{3}{16}\cos 4\ell' - \frac{1}{16}\cos 6\ell') \\
\ell &= \ell' + \varepsilon \frac{L'}{2k^3}(\sin 2\ell' + \frac{1}{8}\sin 4\ell') \\
&\quad - \varepsilon^2 \frac{L'^2}{4k^6}(\frac{25}{8}\sin 2\ell' + \frac{1}{32}\sin 4\ell' - \frac{1}{8}\sin 6\ell' - \frac{1}{128}\sin 8\ell') \\
\ell' &= \ell'_o + k(1 + \varepsilon \frac{3}{4}\frac{L'}{k^3} - \varepsilon^2 \frac{51}{64}\frac{L'^2}{k^6} + \varepsilon^3 \frac{375}{256}\frac{L'^3}{k^9})t \ .
\end{aligned} \tag{4.31}$$

For convenience, the equations necessary for evaluating L' and ℓ'_o through second order are given below.

$$\begin{aligned}
L' &= L_E + \varepsilon \frac{L_E^2}{2k^3}(\cos 2\ell_E + \frac{1}{4}\cos 4\ell_E) \\
&\quad + \varepsilon^2 \frac{L_E^3}{2k^6}(\frac{17}{32} - \frac{3}{4}\cos 2\ell_E - \frac{3}{16}\cos 4\ell_E) \\
\ell'_o &= \ell_E - \varepsilon \frac{L_E}{2k^3}(\sin 2\ell_E + \frac{1}{8}\sin 4\ell_E) \\
&\quad - \varepsilon^2 \frac{L_E^2}{64k^6}(49 \sin 2\ell_E + \frac{17}{2}\sin 4\ell_E + \sin 6\ell_E \\
&\quad + \frac{1}{8}\sin 8\ell_E)
\end{aligned}$$

IV.4 Comparison and Interpretation of Results

The first and second order solutions of the nonlinear spring problem developed in Section IV.3 were programmed for the UNIVAC 1108 computer at the NASA Manned Spacecraft Center and the results of the calculations were compared with those obtained by numerical integration of the equations of motion (Eqs. (4.5)). For this study, the following values were used:

$$L_E = 1.0$$

$$\ell_E = 0.0$$

$$k = 2\pi/10 \simeq 0.6283184/\text{sec.}$$

The values of ε ranged from 0.0001 to 0.1.

The trajectories generated by numerical integration of the equations of motion with $\varepsilon = 0.001$ over a time interval of 50 seconds (5 cycles for the unperturbed oscillator) are given in Figs. 7 and 8. These figures show that the momentum L undergoes periodic oscillations of constant amplitude about a mean value of approximately 1.002. Also, the change in ℓ is primarily secular with $\dot{\ell} \simeq 0.63/\text{sec.}$ as compared to $\dot{\ell} = k = 0.6283184/\text{sec.}$ for the unperturbed case. Superimposed on this secular change are periodic oscillations with amplitudes that are too small to be seen in the figure.

The trajectories for other values of ε appear very similar to the one presented here, the differences being increasing amplitudes and decreasing periods of the oscillations as ε becomes larger, and an increase in the slope of the ℓ vs. t curve.

The first case for which a comparison of the exact and first order general perturbation solutions is made is $\varepsilon = 0.0001$. The quantity ΔL, the true solution minus the first order solution, is plotted over a time interval of 50 seconds in Fig. 9. For both the solution by Kaula's method and the solution by von Zeipel's method, the error is periodic with nearly constant amplitude over the time interval considered. The amplitude of the

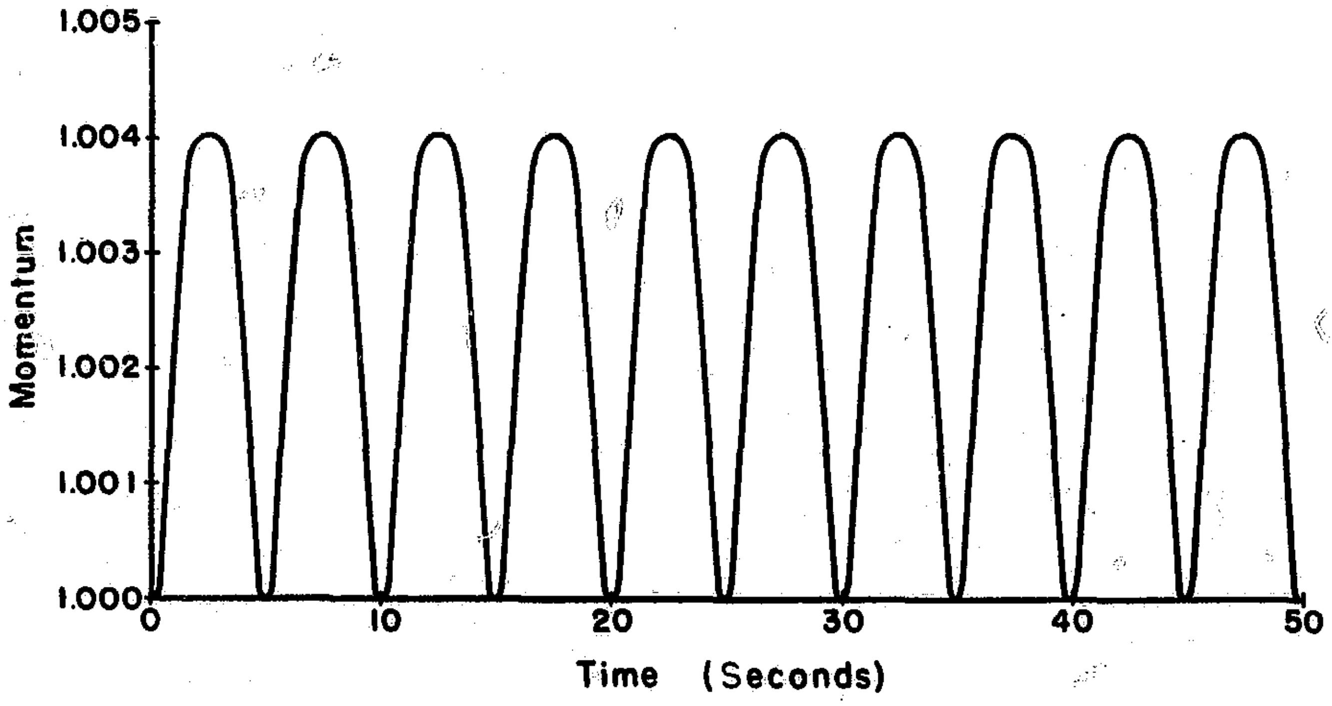


FIGURE 7. Momentum vs. Time for $\epsilon = 0.001$

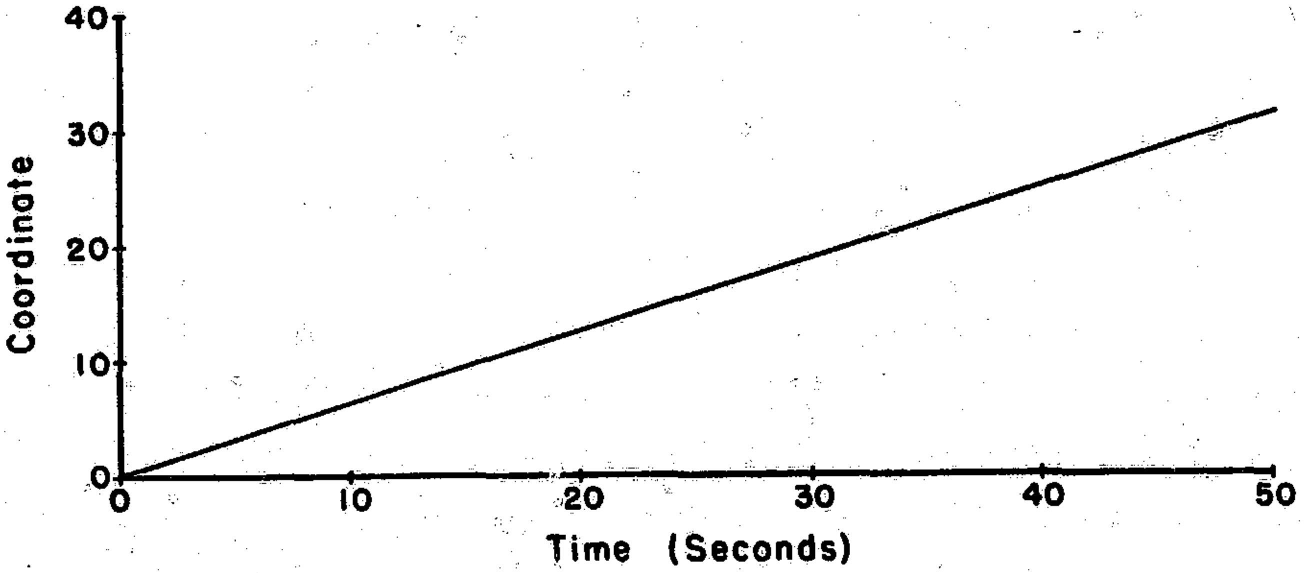


FIGURE 8. Coordinate vs. Time for $\epsilon = 0.001$

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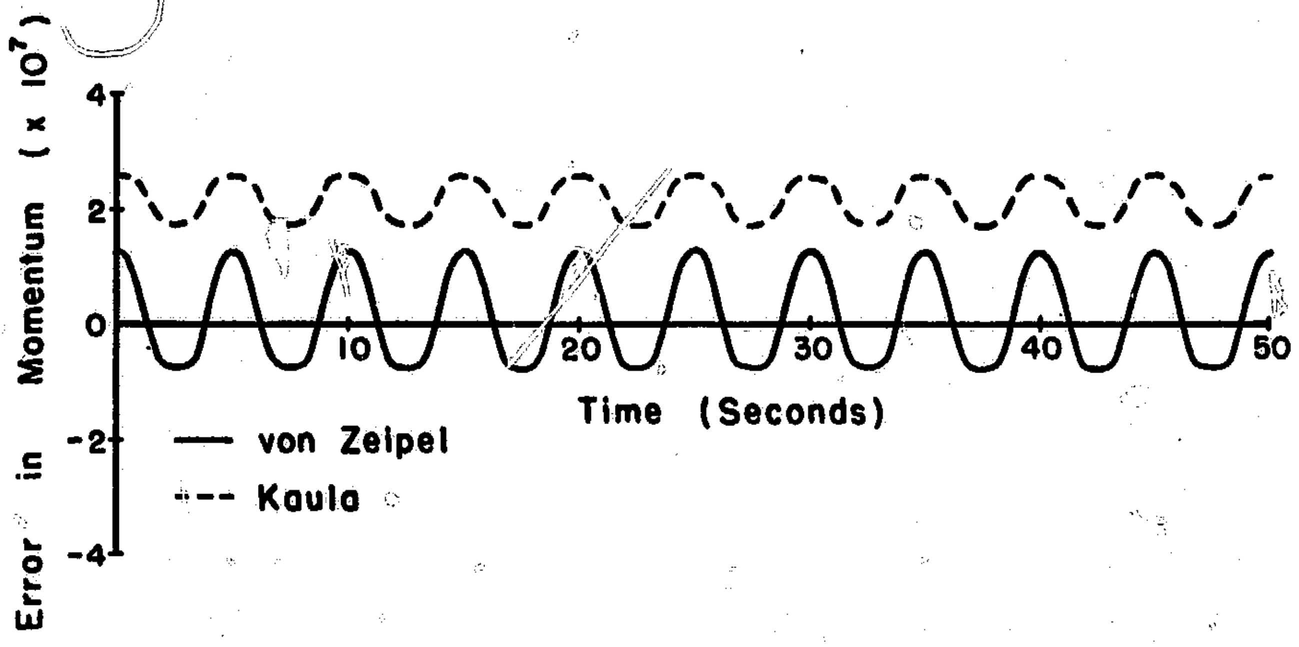

FIGURE 9. Error in Momentum vs. Time for ϵ = 0.0001

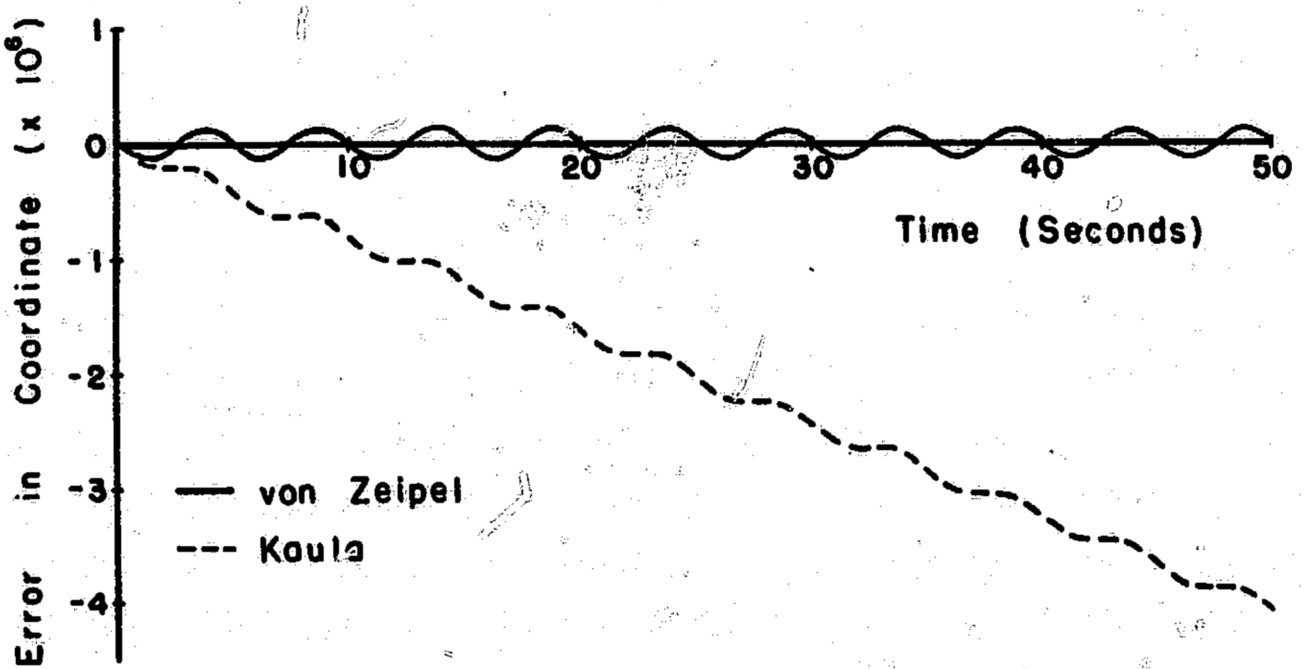

FIGURE 10. Error in Coordinate vs. Time for ϵ = 0.0001

error in the Kaula solution is approximately 5×10^{-8} while that of the von Zeipel solution is 10^{-7}. Consider now the second order von Zeipel solution (Eqs. (4.31)). The second order term in L, which is not included in the first order solution, is periodic in ℓ' and has a magnitude approximated by

$$\varepsilon^2 \frac{L'^3}{k^6} \simeq \frac{\varepsilon^2}{k^6} \simeq 16\ \varepsilon^2 = 1.6 \times 10^{-7} .$$

Thus, for this value of ε, both the nature of the errors in the first order solution for L and their magnitudes can be determined from the second order term.

A discussion of the relatively large bias in the error in the Kaula solution for L will be deferred until later. The only comment to be made at this point is that the bias is of second order (2×10^{-7}) and therefore negligible in the first order solution.

Proceeding to the analysis of the errors in the coordinate ℓ, one observes from Fig. 10 that both solutions yield an error component which is periodic and, in addition, the Kaula solution produces a secular component. The periodic error can be attributed to neglecting the second order periodic term. This term in the von Zeipel solution for ℓ has a magnitude approximated by

$$\varepsilon^2 \frac{L'^3}{k^6} \simeq \frac{\varepsilon^2}{k^6} \simeq 16\ \varepsilon^2 = 1.6 \times 10^{-7}$$

compared to an error of 10^{-7} shown in the figure. Since the secular error in the von Zeipel solution is very small, it might be expected that the secular error in the Kaula solution results from neglecting the second order secular term. From the von Zeipel solution, this error after 50 seconds is approximately

$$\varepsilon^2 \frac{L'^2}{k^5}(50) \approx 8\ \varepsilon^2(50) \approx 4 \times 10^{-6} .$$

Again, the predicted error is in excellent agreement with the observed value of 4.1×10^{-6}. The third order secular error present in the first order von Zeipel solution is predicted to be

$$\varepsilon^3 \frac{L'^3}{k^8}(50) \approx 32\ \varepsilon^3(50) \approx 1.5 \times 10^{-9} ,$$

a value that is too small to be noticeable in the figure.

Based on the discussion of the errors given above, the following observations can be made:

1. The first order error in L is periodic of nearly constant amplitude.
2. Neglecting the second order secular term in ℓ produces a significant secular error which becomes ten times as large as the second order periodic error in this relatively brief time interval.
3. The magnitude of the error can be predicted from the magnitude of the first neglected term in the solution. That is, for a first order solution, the error incurred is approximated by the magnitude of the second order terms.

So far, the previous conclusions are based entirely on the analysis of the errors for $\varepsilon = 0.0001$. The effect of changing ε will now be examined.

The error in L for the case $\varepsilon = 0.001$ is presented in Fig. 11. The curves appear to have forms similar to those for $\varepsilon = 0.0001$, the major difference being the slightly erratic behavior and increasing

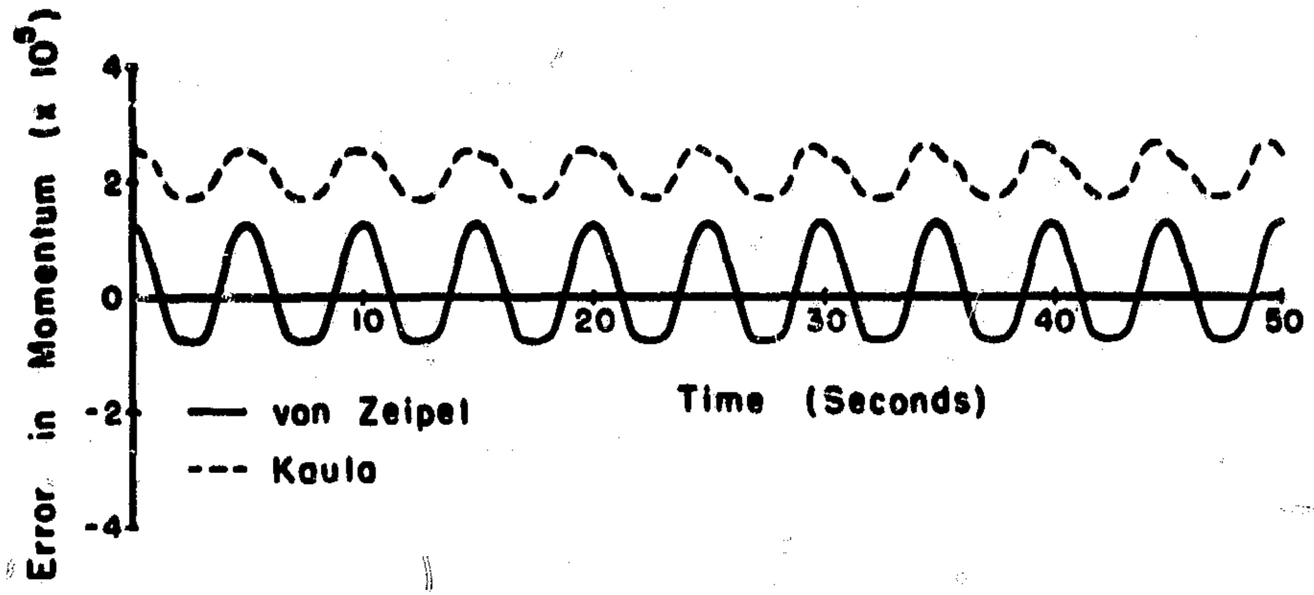

FIGURE 11. Error in Momentum vs. Time for ϵ = 0.001

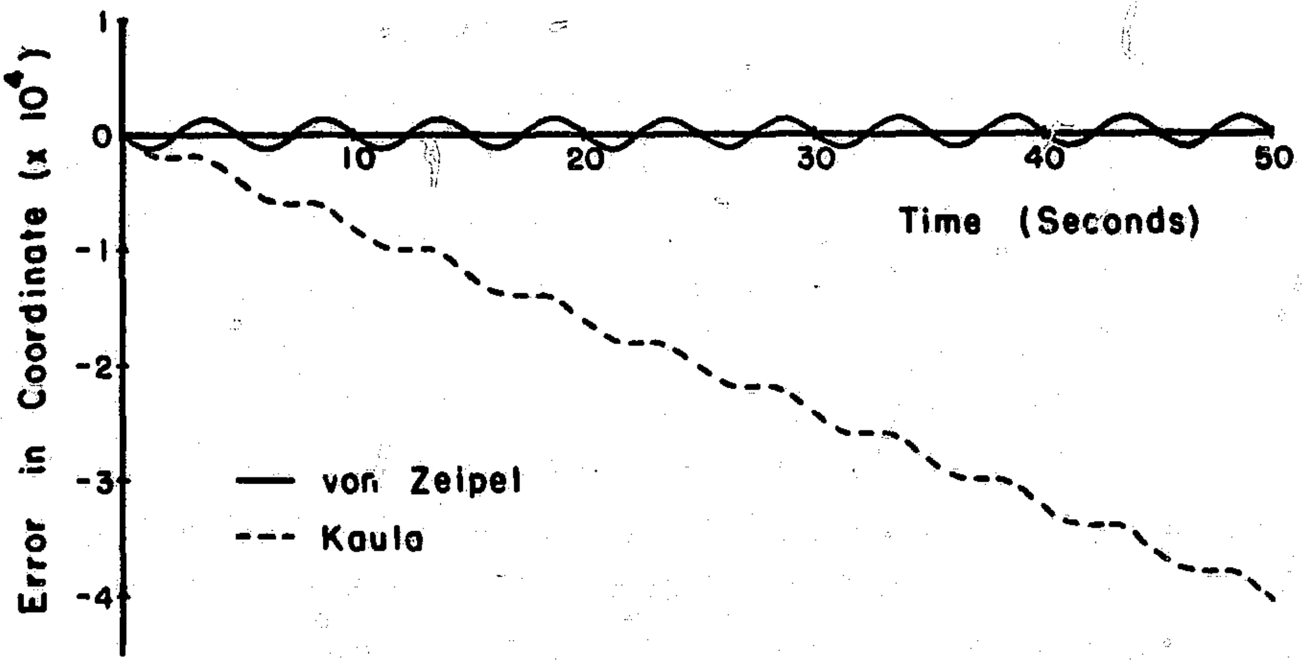

FIGURE 12. Error in Coordinate vs. Time for ϵ = 0.001

amplitude in the error in the Kaula solution. The predicted error, given by

$$\varepsilon^2 \frac{L'^3}{k^6} \approx 1.6 \times 10^{-5} ,$$

is in good agreement with the observed error. Note that the periodic error in the first order solution is 100 times as large as it was for the previous value of ε . As before, the error in ℓ is mainly periodic for the von Zeipel solution, while the error in the Kaula solution has a large secular component. The predicted value of the periodic error is

$$\varepsilon^2 \frac{L'^3}{k^6} \approx 1.6 \times 10^{-5}$$

and the predicted secular error in the Kaula solution after 50 seconds is

$$\varepsilon^2 \frac{L'^2}{k^5} (50) \approx 4 \times 10^{-4} ,$$

both in close agreement with the observed values.

The preceding analysis leads to the conclusion that the three characteristics of the errors enumerated for $\varepsilon = 0.0001$ also hold for $\varepsilon = 0.001$. A fourth observation resulting from a comparison of the magnitudes of the errors in the two cases is:

4. The size of the second order error increases as ε increases (since it is proportional to ε^2).

Continuing to the case of $\varepsilon = 0.01$, two features are worthy of note. The first is the pronounced increase in the amplitude of the error in the Kaula solution for L (Fig. 13); the second is the appearance of the third order secular error in the von Zeipel solution for ℓ (Fig. 14). The magnitudes of all the errors except that in the Kaula solution for L

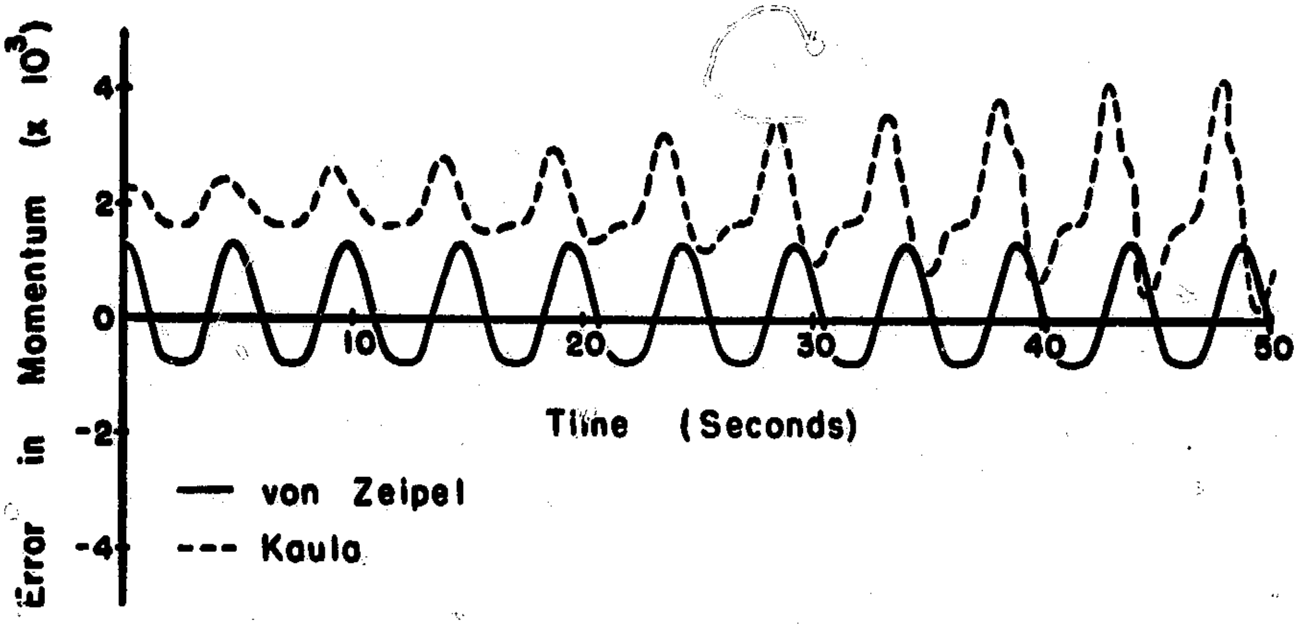

FIGURE 13. Error in Momentum vs. Time for ϵ = 0.01

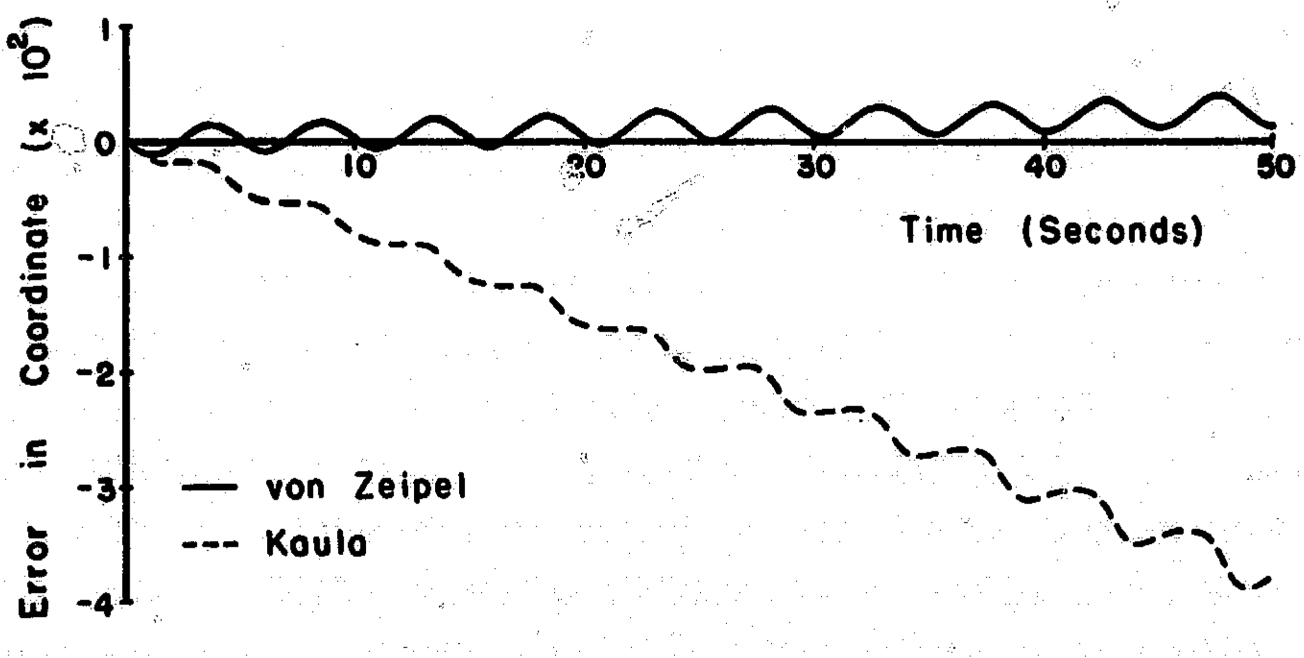

FIGURE 14. Error in Coordinate vs. Time for ϵ = 0.01

can be predicted from the first neglected term in the solutions. The growth of the amplitude of the error in L can be explained by consideration of the case $\varepsilon = 0.1$.

The value $\varepsilon = 0.1$ represents a strongly perturbed system, the ratio of the nonlinear perturbing acceleration to the linear term having a maximum of approximately 0.8. Thus, it is to be expected that a first order solution will be very inaccurate. This is verified by the results given in Figures 15 and 16. For this value of ε , both solutions yield periodic errors whose amplitudes vary with time and both solutions produce large secular errors in ℓ . The relation between these effects is indicated by Eqs. (4.24) and (4.30), and by Fig. 17.

In Equation (4.24), the angular variable in the von Zeipel solution for L is ℓ . In the solution programmed (Eqs. (4.30)), the angular variable ℓ is replaced by ℓ' , the secular variation of ℓ . Any error in ℓ' will produce a secular change in the quantity $\ell - \ell'$. This, in turn, causes a phase difference between the solutions for L obtained from Eq. (4.24) and the first of Eqs. (4.30). Figure 17 shows that the predicted phase difference does actually occur. For the present case, the true and approximate solutions are 180° out of phase after a period of 26 seconds which should be equal to the time required for $2\ell - 2\ell'$ to reach 180° or approximately the time required for ΔL to reach 90° (1.57 rad.). This conclusion is verified by Fig. 16. Such a phase difference can have a significant effect on the errors in L because, regardless of the accuracy of the amplitude of the solution for L , the error can have a value equal to twice the amplitude of the oscillations in the true solution. The previous characteristic, which will also appear for smaller values of ε over longer periods of time, in conjunction with the large

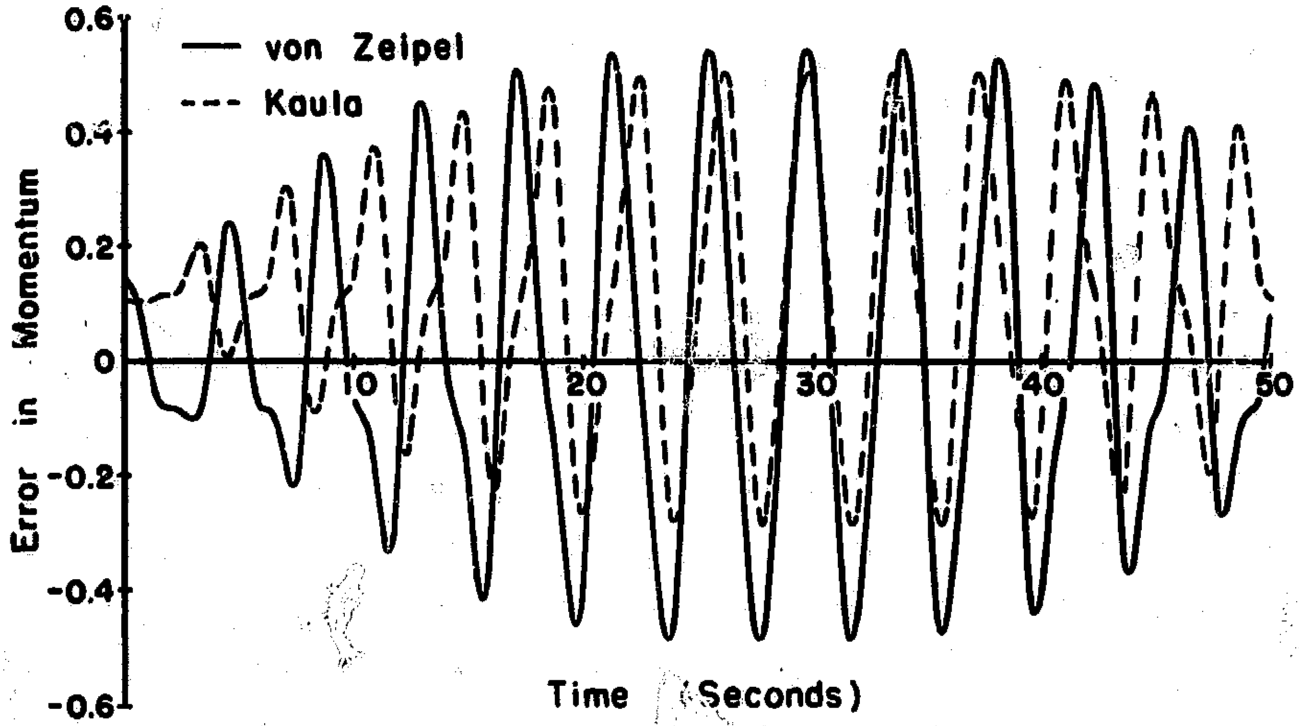

FIGURE 15. Error in Momentum vs. Time for $\epsilon = 0.1$

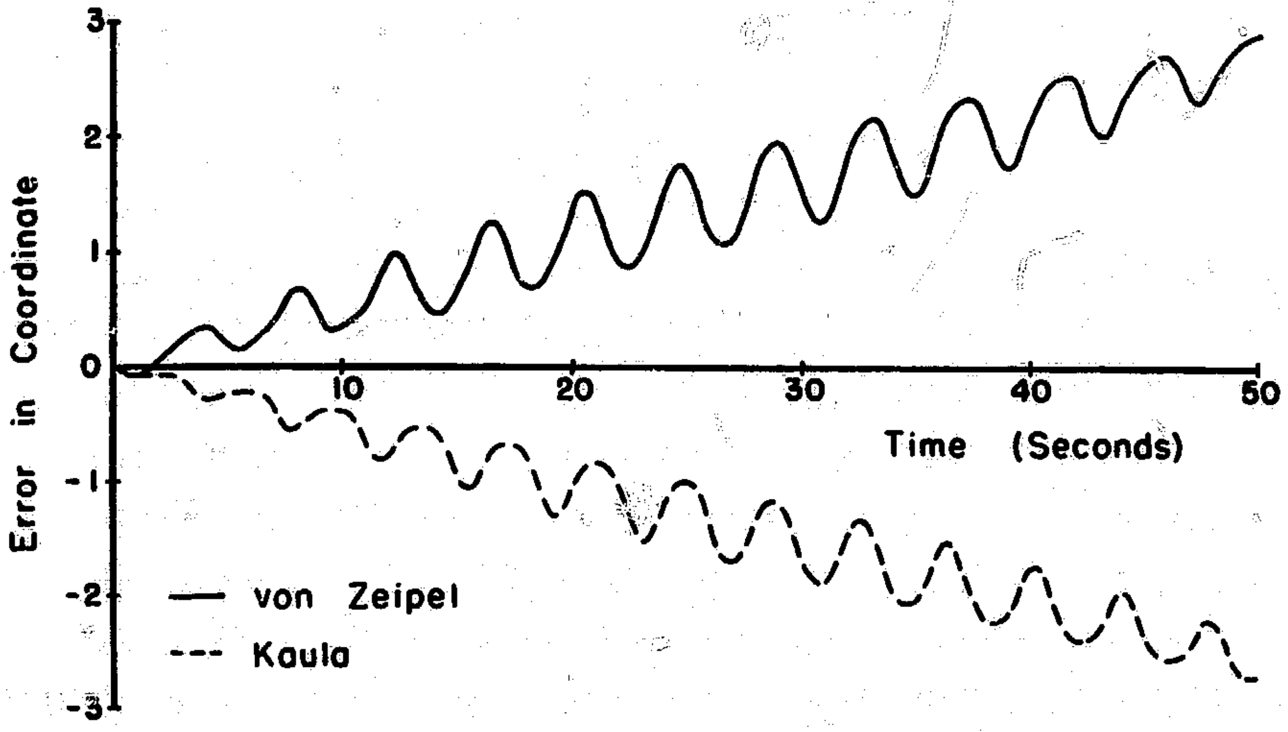

FIGURE 16. Error in Coordinate vs. Time for $\epsilon = 0.1$

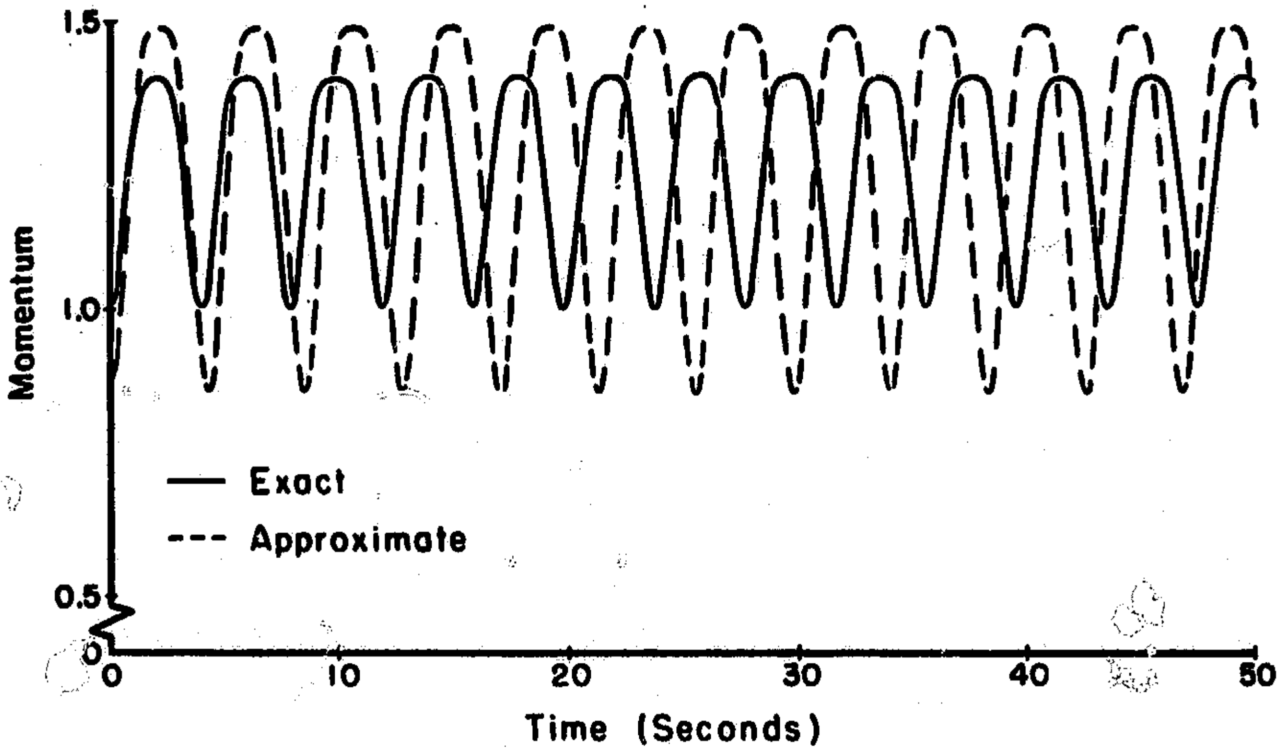

FIGURE 17. Momentum vs. Time for $\epsilon = 0.1$

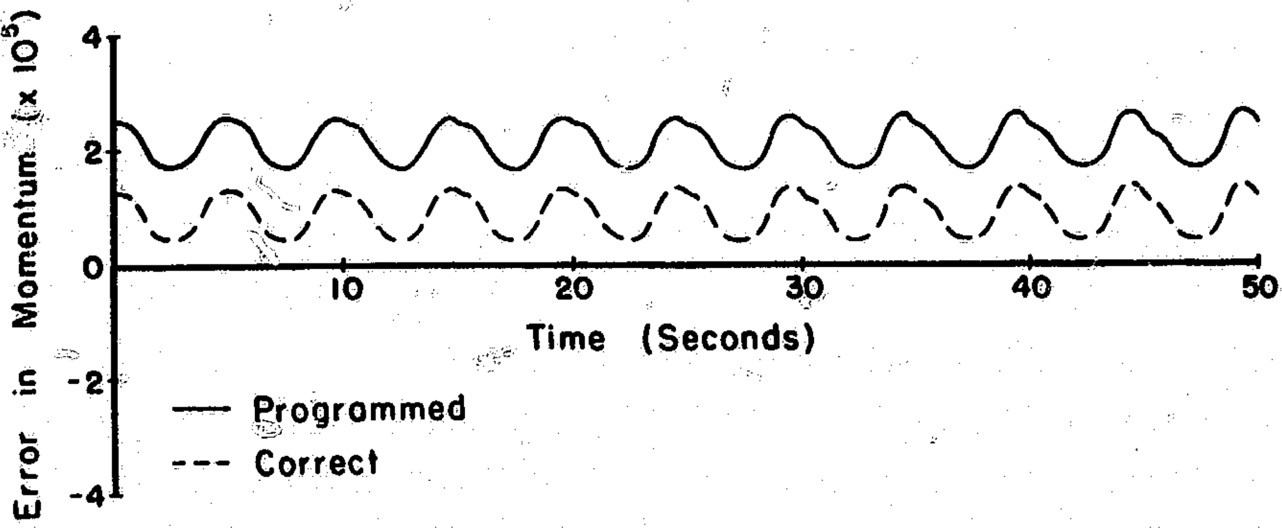

FIGURE 18. Comparison of Biases of the Errors in the Momentum (ϵ = 0.001)

secular errors in ℓ leads to the conclusion that it is very important to determine the secular portion of the solution for ℓ as accurately as possible.

Unfortunately, in most cases the accurate determination of the secular variation in ℓ is difficult. For example, for the present value of ε, the second and third order secular errors in ℓ are large and of the same magnitude. Hence, an accurate solution for ℓ would require the calculation of secular terms of much higher than third order.

At this point the bias of the error in the Kaula solution for L mentioned earlier will be considered in more detail.

It is suspected that the bias of the error in L is due to an error in the mean value of L. This mean value ($\overline{L}$ in the Kaula solution) from Eq. (4.12) is

$$\overline{L} = L_E + \varepsilon \frac{L_E^2}{2k^2\omega} (\cos 2\ell_E + \frac{1}{4} \cos 4\ell_E)$$

where

$$\omega = \dot{\ell}_{sE} = k(1 + \varepsilon \frac{3}{4} \frac{L_E}{k^3}) .$$

In the expression that was programmed, ω was incorrectly taken as

$$\omega' = \dot{\ell}_E \equiv k + \varepsilon \frac{L_E}{k^2} (\frac{3}{4} + \cos 2\ell_E + \frac{1}{4} \cos 4\ell_E) .$$

Since ω and ω' differ by a term of the first order, the values of $\overline{L}$ calculated with these values of ω will differ by second order. The magnitude of this error can be predicted as follows.

Let $\overline{L}$ be the mean value of L based on ω and $\overline{L}'$ the mean value based on ω'. Then, with $L_E = 1.0$ and $\ell_E = 0$,

$$\overline{L} - \overline{L}' = \frac{\varepsilon}{2k^3}\left(\frac{5}{4}\right)\left[\left(1 + \frac{3}{4}\frac{\varepsilon}{k^3}\right)^{-1} - \left(1 + 2\frac{\varepsilon}{k^3}\right)^{-1}\right]$$

$$= \frac{25}{32}\frac{\varepsilon^2}{k^6} + O(\varepsilon^3) \approx \frac{25}{2}\varepsilon^2 .$$

For $\varepsilon = 0.001$,

$$\overline{L} - \overline{L}' \approx 10^{-5} .$$

The bias in ΔL for both the correct and the programmed solutions is shown in Fig. 18. The biases differ by 10^{-5} as predicted. Further reduction of the error in the mean value can be achieved by setting $\omega = k$. The equation for the mean value then is the same as that given by the von Zeipel solution, and comparison of Figs. 11 and 18 shows that the bias in this error is the smallest. However, the bias is of second order for either of the possible values of ω and is therefore negligible in a first order solution. The remainder of this section is devoted to a study of the nature of the errors over extended periods of time.

Figure 19 presents the envelope of the oscillations in ΔL over an interval of 5000 seconds (500 cycles of the unperturbed oscillator) for $\varepsilon = 0.001$. The error in the von Zeipel solution has a nearly constant amplitude while the error in the Kaula solution experiences an increase in amplitude which, for the 50 second time interval discussed previously, was observed only for higher values of ε . As before, this amplitude change is due to a phase difference caused by the second order secular error in ℓ . The amplitude of ΔL will reach a maximum of 0.004 (100 times the predicted second order error) in the time it takes $\Delta\ell$ to become 90° (approx. 200,000 seconds).

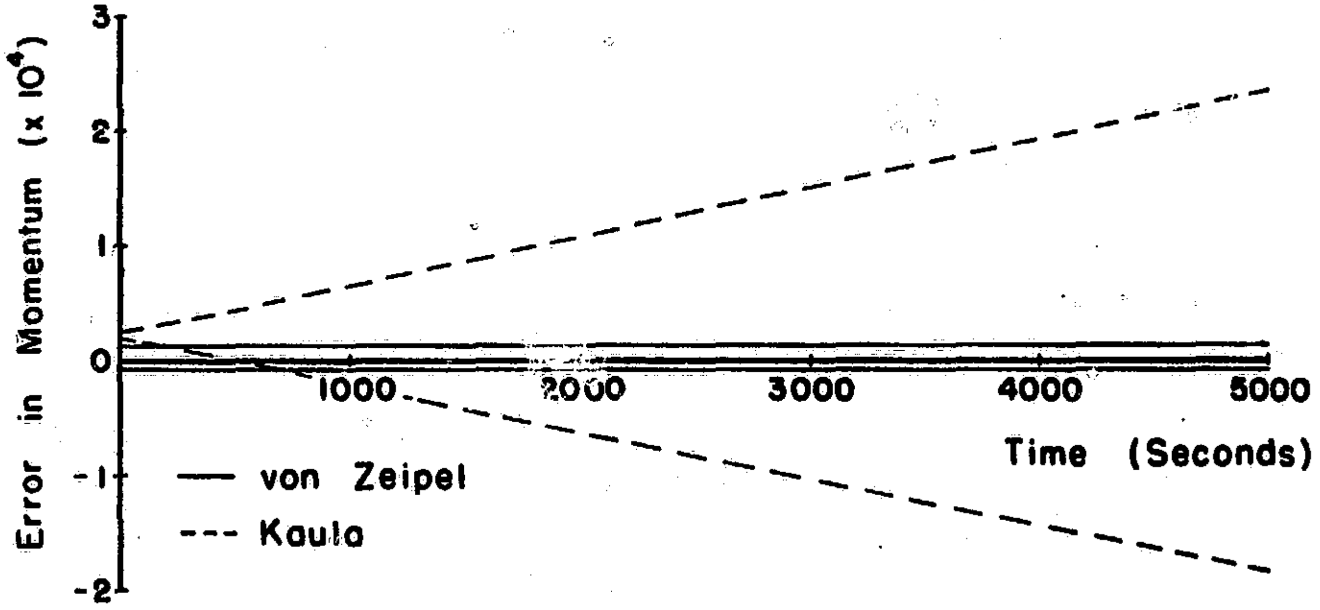

FIGURE 19. Envelope of Errors in Momentum for ϵ = 0.001

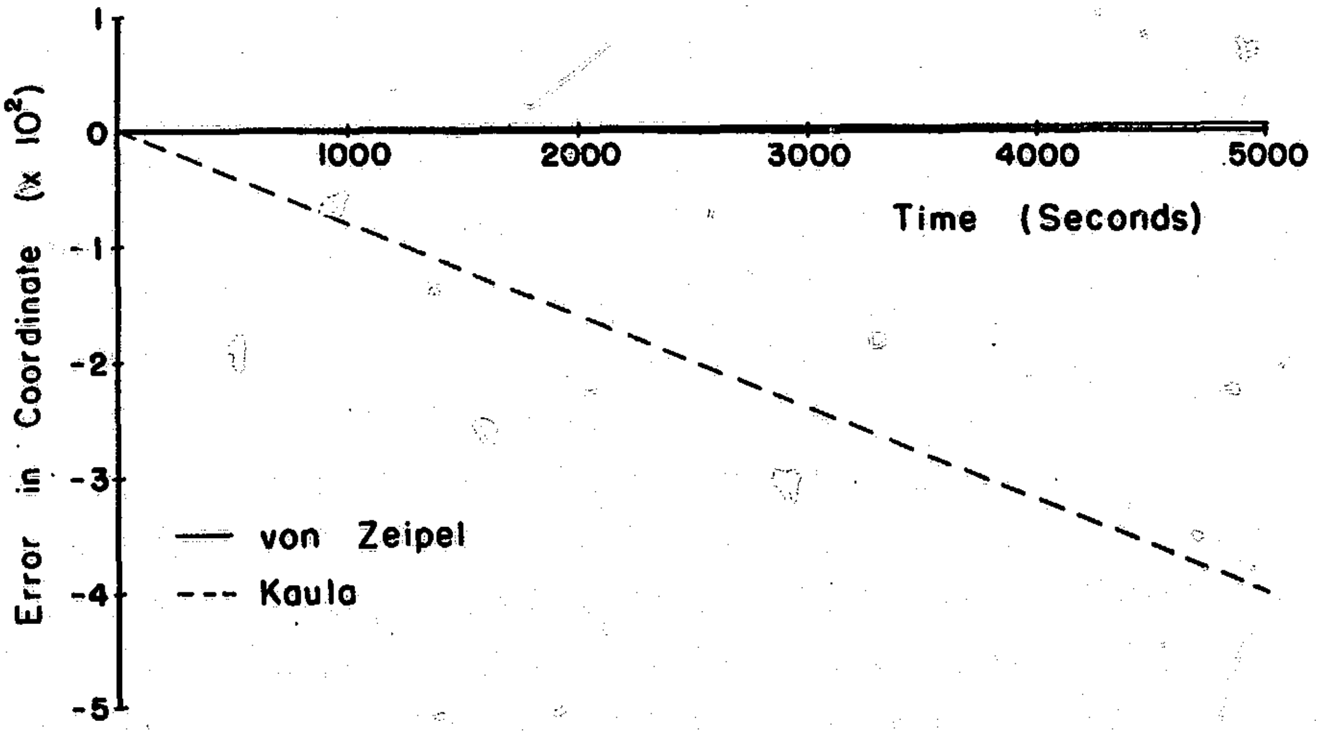

FIGURE 20. Long Term Error in Coordinate for ϵ = 0.001

Similar results are presented in Figs. 21 and 22 for $\varepsilon = 0.01$. In this case, however, the third order error in the von Zeipel solution is large enough to produce an observable increase in the amplitude of the error in the solution for L . Theoretically, the error in the Kaula solution for L should reach a maximum of 0.04 after 2100 seconds. This prediction agrees well with the computational results.

Finally, Figures 23 and 24 present comparisons of the results of first and second order solutions obtained by the von Zeipel method for $\varepsilon = 0.001$. The second order solution yields errors in L and ℓ which are significantly smaller than those for the first order solution. As before, the error in the second order solution is approximated by the magnitude of the first neglected term (i.e., third order periodic and fourth order secular terms).

The analysis presented in this section leads to the following conclusions:

1. The magnitude of the error in the first order solution increases as ε increases.
2. For small values of ε , the error can be predicted from the magnitude of the first neglected term if the effect of the phase error in the solution for the momentum L is taken into account.
3. Secular errors in the coordinate ℓ have the greatest effect on the overall accuracy of the solution.
4. The increased accuracy of the von Zeipel solution as compared to the Kaula solution is due almost entirely to the inclusion of the second order secular term in the von Zeipel solution for ℓ .

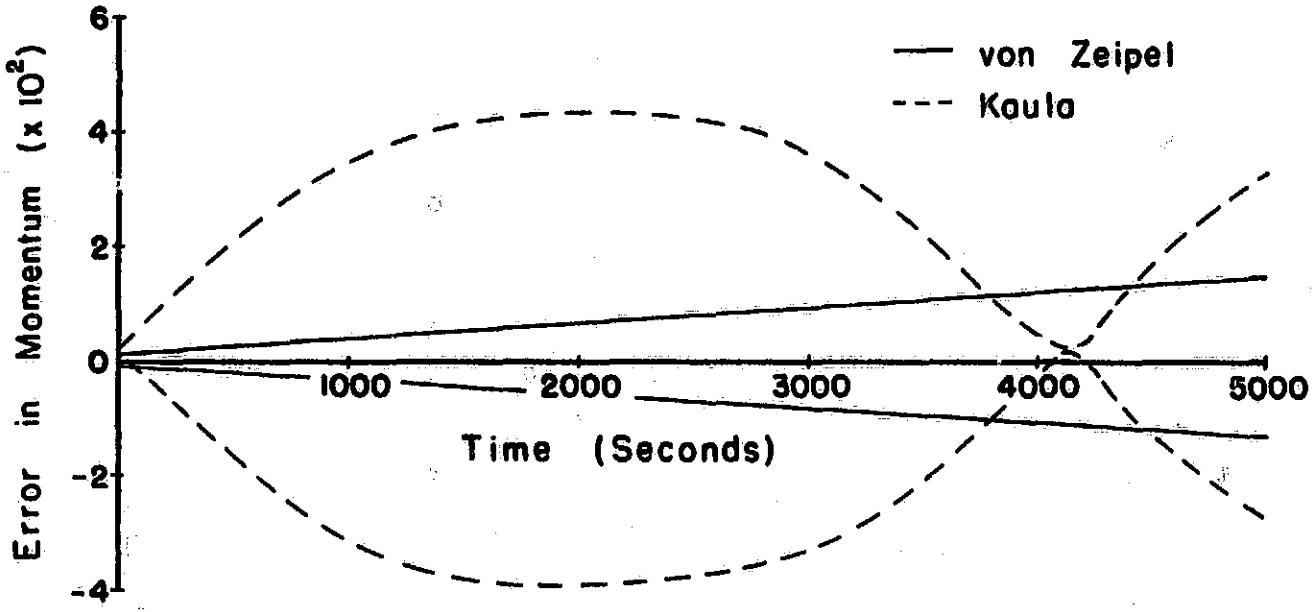

FIGURE 21. Envelope of Errors in Momentum for ϵ = 0.01

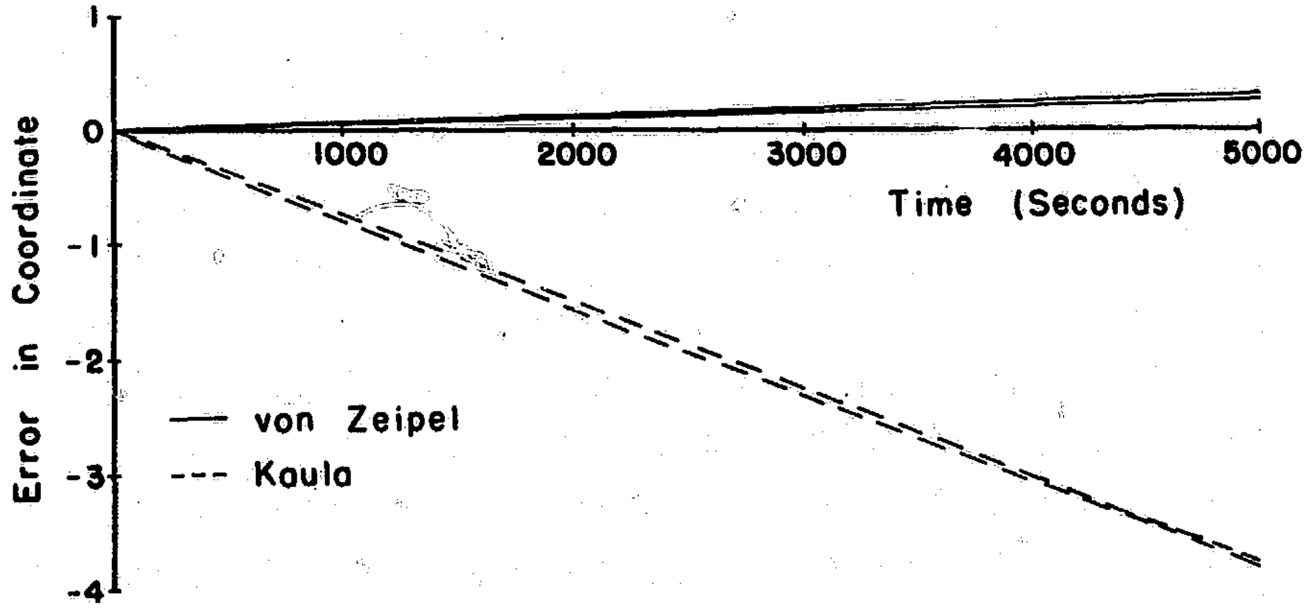

FIGURE 22. Envelope of Errors in Coordinate for ϵ = 0.01

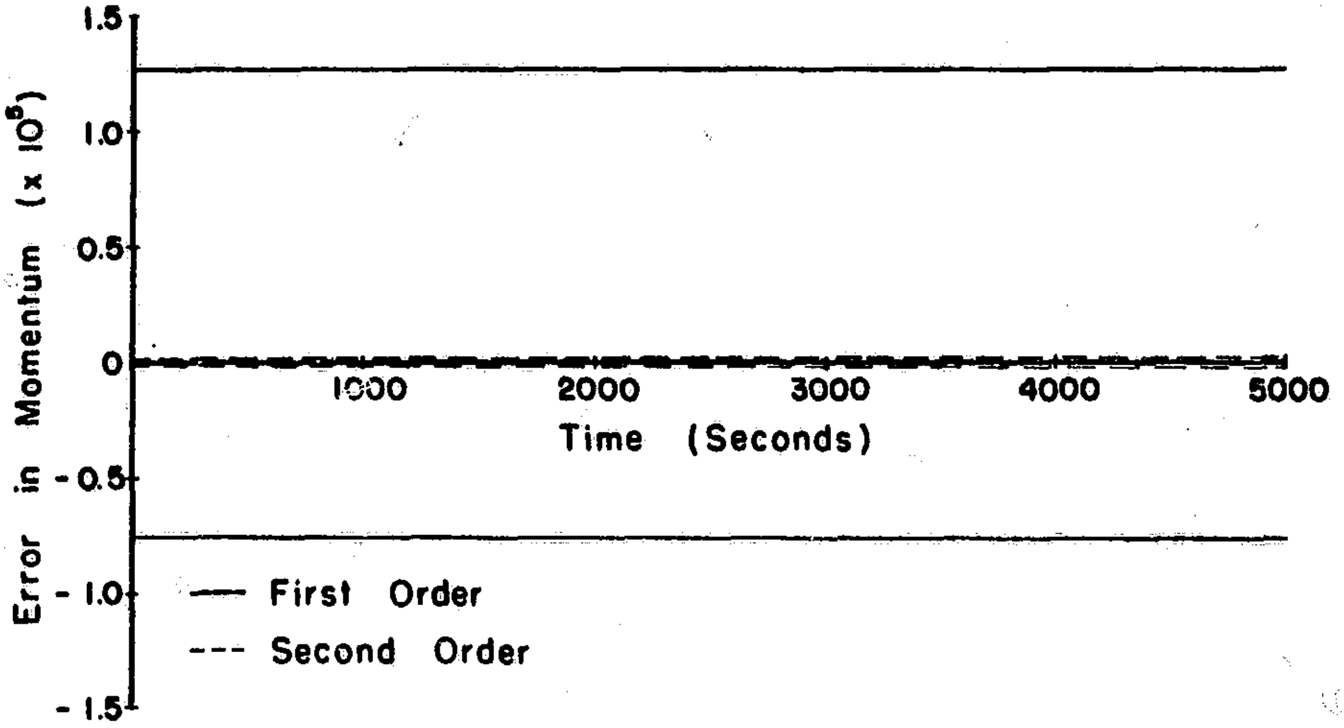

FIGURE 23. Comparison of First and Second Order Solutions for Momentum (ϵ = 0.001)

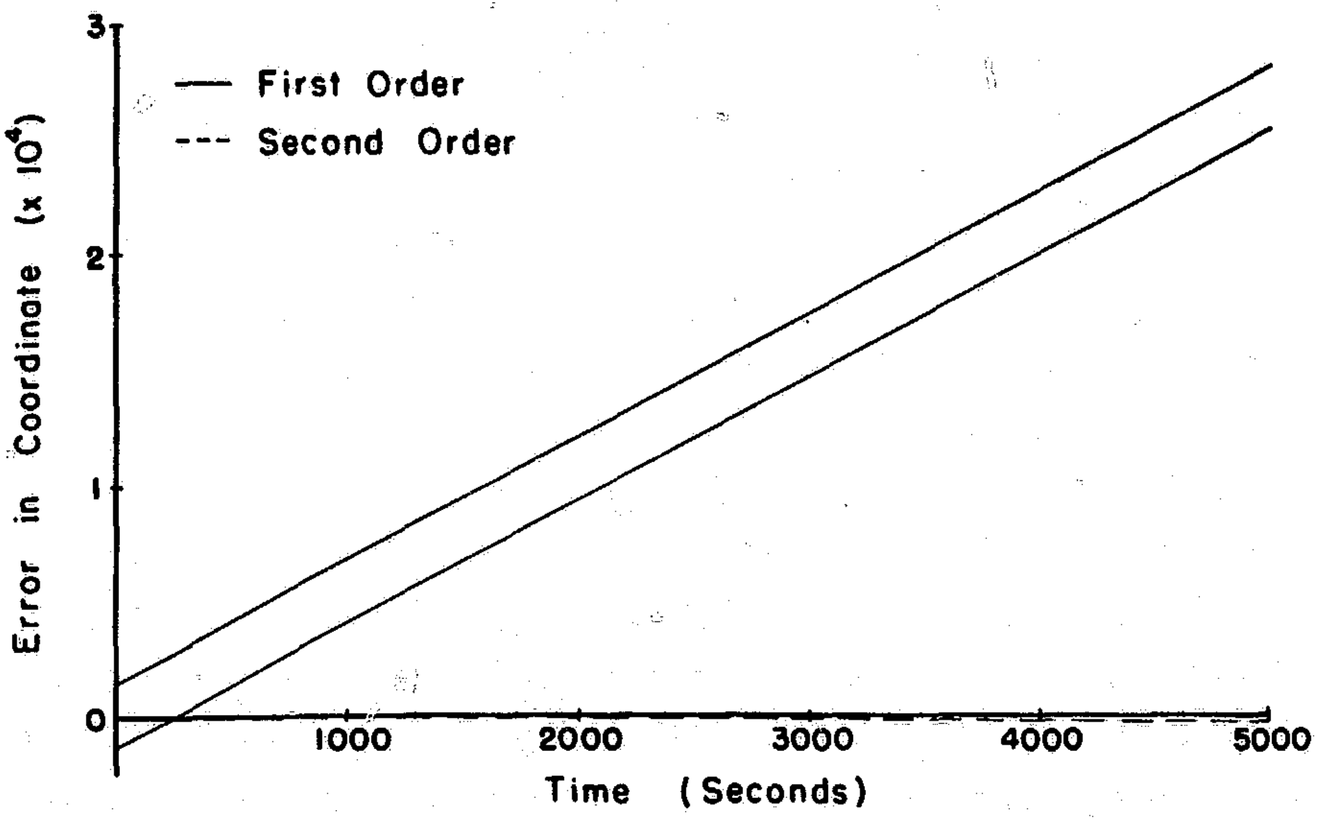

FIGURE 24. Comparison of First and Second Order Solutions for Coordinate (ϵ = 0.001)

CHAPTER V

THE ARTIFICIAL EARTH SATELLITE PROBLEM

One of the purposes of Chapters II, III, and IV has been to present the material required in the following discussion of two existing general perturbations solutions describing the motion of an artificial satellite of the Earth. The problem of selecting the perturbations to be included in the analysis is considered in Section V.1. Sections V.2 and V.3 are devoted to the developments of the general perturbations solutions with comparisons of the results being given in Section V.4. Finally, studies of the phenomena of resonance as it appears in this problem are briefly reviewed in Section V.5.

V.1 The Principal Perturbations

The forces influencing the motion of an artificial satellite of the Earth are due to the following causes (Ref. 23):

1. the Earth's gravitational field,
2. the gravitational fields of the Sun, Moon, and planets,
3. the Earth's atmosphere,
4. the Earth's magnetic field,
5. solar radiation and
6. charged and uncharged particles (i.e., ions from the Sun or the atmosphere and micrometeorites).

The third influence listed, atmospheric drag, is extremely difficult to treat by analytical methods for several reasons. Since the characteristics of the atmosphere are not fixed, such things as density variations that are dependent upon the position relative to the Sun (the diurnal bulge) and those that are due to fluctuations in the intensity of solar activity must be considered (Ref. 24). In addition, the drag forces depend in a very complex way on the shape, orientation, position, and velocity of the satellite. Therefore, in the solutions presented here, the motion of the satellite is assumed to occur in a region in which the accelerations due to atmospheric drag are negligible.

The effects of the Earth's gravitational field will be considered next. In Chapter II, the expression for the potential of an arbitrary body was written in infinite series form (Eq.(2.20)). For the Earth, the dominant term in this expansion is the first which represents the potential of an inverse square force field and will be called the central force term. Of the remaining terms, the most important is V_{20} (representing the oblateness) which is a factor of 10^{-3} smaller than the central force term. All other terms in the potential are a factor of 10^{-6} smaller than the dominant term (Ref. 25).

Compared to the central force component of the Earth's gravitational attraction, all of the other forces acting on the satellite (except possibly atmospheric drag which will not be considered) are small. Thus, these forces are treated as perturbations causing the satellite to deviate slightly from the Keplerian orbit it would follow if the perturbations were absent.

Of the influences remaining, the only ones that can not be neglected in most instances are the third body effects of the Sun and Moon. For a satellite at an altitude of approximately 1000 miles, these perturbing forces are a factor of 10^{-7} smaller than the central force (Ref. 26) and should be considered if the higher harmonics of the Earth's gravitational field are included in the problem. Since treating these effects only complicates the analysis without revealing any new features of the techniques, they will be neglected here. A discussion of such lunisolar perturbations is given by Kozai (Ref. 27).

Then, the artificial Earth satellite problem considered in the following sections is defined to be the problem of determining the motion of an artificial satellite under the influence of the Earth's gravitational field.

V.2 Kaula's Method as Applied by Ingram

Using a variation of the technique originally employed by Kaula (Ref. 6), Ingram (Ref. 7) has developed a solution for the motion of an artificial satellite which includes the effects of the attraction of a third body and any combination of harmonics in the central body potential. Although the solution was formulated for propagating the orbital elements of Lunar satellites, it also has been used for Earth satellite studies. A modification of the solution to yield more accurate results for the effects of the third body has been performed by Born (Ref. 28).

Ingram used a set of non-singular elements in his analysis in order to avoid the zero eccentricity and zero inclination difficulties encountered with the Keplerian elements. However, the modified Keplerian set (a, e, I, Ω, ω, and M) is chosen in the following presentation to facilitate the comparison of the results with those obtained by Brouwer. In addition, the treatment of the third body perturbations is not included.

There were two reasons for deciding to study Ingram's solutions instead of better known results such as Kozai's (Ref. 14). First, the technique used to generate the analytical solution seemed very simple compared to the von Zeipel method and yet gave accurate results. Second, the solutions are used for predicting the motion of both Earth and Lunar satellites while, as will be indicated in Chapter VI, the von Zeipel method cannot be used to determine a complete solution to the Lunar satellite problem (at least not in terms of Delaunay variables).

The analysis begins with Lagrange's planetary equations given below.

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial M}$$

$$\dot{e} = \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega}$$

$$\dot{I} = \frac{\cos I}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial \Omega}$$

$$\dot{\Omega} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial I}$$

$$\dot{\omega} = -\frac{\cos I}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial I} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e}$$

$$\dot{M} = n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}$$

The disturbing function R is, from Eqs. (2.25) and (2.26),

$$\begin{aligned} R &= -(V + \frac{\mu}{r}) \\ &= \sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\frac{\mu a_e^{\ell}}{a^{\ell+1}}\sum_{p=0}^{\ell}F_{\ell mp}(I)\sum_{q=-\infty}^{\infty}G_{\ell pq}(e)S_{\ell mpq}(\omega,M,\Omega,\theta) \qquad (5.1)\end{aligned}$$

with

$$S_{\ell mpq} = J_{\ell m}\begin{bmatrix}\cos\\ \sin\end{bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}} \Phi_{\ell mpq}$$

$$\Phi_{\ell mpq} = (\ell-2p)\omega + (\ell-2p+q)M + m(\Omega-\theta-\lambda_{\ell m}) .$$

Note that the summation in R on the index q is to be truncated in accordance with the discussion at the end of Sec.II.2. The variable θ appearing in $\Phi_{\ell mpq}$ is the angle between the X axis of the body fixed coordinates (Fig. 2) and the inertial reference axis for Ω, both axes lying in the equatorial plane. Thus, if the central body is spinning about the Z axis with constant angular velocity $\dot\theta$,

$$\theta = \theta_E + \dot\theta t$$

where θ_E is the value of θ at $t = 0$.

Substituting Eq. (5.1) into Lagrange's planetary equations and letting

$$\sum_{\ell mpq} \equiv \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=-\infty}^{\infty}$$

yields the following expressions for the rates of change of the elements.

$$\dot a = \sum_{\ell mpq} \frac{2na_e^\ell}{a^{\ell-1}} J_{\ell m} F_{\ell mp} G_{\ell pq} (\ell-2p+q) \begin{bmatrix} -\sin \\ \cos \end{bmatrix} \Phi_{\ell mpq} \tag{5.2}$$

$$\dot e = \sum_{\ell mpq} \frac{na_e^\ell}{a^{\ell}} J_{\ell m} F_{\ell mp} G_{\ell pq} \left[\frac{1-e^2}{e} (\ell-2p+q) - \frac{\sqrt{1-e^2}}{e} (\ell-2p) \right] \begin{bmatrix} -\sin \\ \cos \end{bmatrix} \Phi_{\ell mpq} \tag{5.3}$$

$$\dot I = \sum_{\ell mpq} \frac{na_e^\ell}{a^{\ell}} J_{\ell m} F_{\ell mp} \frac{G_{\ell pq}}{\sqrt{1-e^2}} \left[(\ell-2p)\cot I - m \csc I \right] \begin{bmatrix} -\sin \\ \cos \end{bmatrix} \Phi_{\ell mpq} \tag{5.4}$$

$$\dot \Omega = \sum_{\ell mpq} \frac{na_e^\ell}{a^{\ell}} J_{\ell m} \frac{dF_{\ell mp}}{dI} \csc I \frac{G_{\ell pq}}{\sqrt{1-e^2}} \begin{bmatrix} \cos \\ \sin \end{bmatrix} \Phi_{\ell mpq} \tag{5.5}$$

$$\dot \omega = \sum_{\ell mpq} \frac{na_e^\ell}{a^{\ell}} J_{\ell m} \left(F_{\ell mp} \frac{\sqrt{1-e^2}}{e} \frac{dG_{\ell pq}}{de} - \frac{dF_{\ell mp}}{dI} \cot I \frac{G_{\ell pq}}{\sqrt{1-e^2}} \right) \begin{bmatrix} \cos \\ \sin \end{bmatrix} \Phi_{\ell mpq} \tag{5.6}$$

$$\dot M = n + \sum_{\ell mpq} \frac{na_e^\ell}{a^{\ell}} J_{\ell m} F_{\ell mp} \left[2(\ell+1) G_{\ell pq} - \frac{1-e^2}{e} \frac{dG_{\ell pq}}{de} \right] \begin{bmatrix} \cos \\ \sin \end{bmatrix} \Phi_{\ell mpq} \tag{5.7}$$

It must now be shown that these equations have forms corresponding to those of Eqs. (3.1), (3.2), and (3.3). Consider the equation for $\dot{M}$. For an artificial satellite of the Earth, the first term (n, the mean motion) is much larger than the remaining terms. A term of this magnitude appears only in Eq. (5.7). Thus, the mean anomaly M must be analogous to ζ in Eq. (3.3) and, following the definitions of Sec. III.1, the short period terms are those whose trigonometric arguments contain M. These terms are specified by the condition $\ell-2p+q \neq 0$.

The coefficients of the trigonometric functions in the short period terms in the rate equation for ζ are functions only of the variables ξ. Since the corresponding coefficients in Eq. (5.7) contain only a, e, and I, these variables must be analogous to the components of the ξ vector. The rate equations for a, e, and I will contain only periodic terms because the coefficients of the terms vanish when the indices are chosen to make the trigonometric arguments zero, i.e., when the following conditions are satisfied simultaneously.

$$\begin{aligned} &1.\quad \ell - 2p + q = 0 \\ &2.\quad \ell - 2p = 0 \\ &3.\quad m = 0 \end{aligned} \tag{5.8}$$

The rate equations for ξ have the same characteristic so that the correspondence drawn between a, e, and I and the components of ξ is valid.

The remaining variables, Ω and ω, appear in the trigonometric arguments of all the periodic terms and are therefore analogous to the components of the η vector. These variables are distinguished from the components of the ξ vector by the secular terms in their rate equations. As found above, such terms will not occur in Eqs. (3.1) but can occur in Eqs. (3.2) when all of the elements of the α vector are zero. The conditions which will produce these terms in the equations for $\dot{\Omega}$ and $\dot{\omega}$ are given by Eqs. (5.8).

In the previous discussion, the equivalences given below were established.

$$\xi = \begin{bmatrix} a \\ e \\ I \end{bmatrix} \qquad \eta = \begin{bmatrix} \Omega \\ \omega \end{bmatrix} \qquad \zeta = M$$

Hence, Equations (5.2) through (5.7) are in the form required for the application of Kaula's method.

The first step in the solution is the determination of the secular terms in the equations for Ω , ω , and M . These terms are obtained by selecting values of the indices satisfying Eqs. (5.8). For example, for $\ell = 2$ and $m = 0$ the secular rates are

$$\begin{aligned} \dot{\Omega}_s &= \frac{na_e^2}{a^2} J_{20} \frac{dF_{201}}{dI} \csc I \frac{G_{210}}{\sqrt{1-e^2}} \\ &= \frac{3na_e^2}{2a^2(1-e^2)^2} J_{20} \cos I \end{aligned} \tag{5.9}$$

$$\begin{aligned} \dot{\omega}_s &= \frac{na_e^2}{a^2} J_{20} \left(F_{201} \frac{\sqrt{1-e^2}}{e} \frac{dG_{210}}{de} - \frac{dF_{201}}{dI} \cot I \frac{G_{210}}{\sqrt{1-e^2}} \right) \\ &= \frac{3na_e^2}{4a^2(1-e^2)^2} J_{20}(1-5 \cos^2 I) \end{aligned} \tag{5.10}$$

$$\begin{aligned} \dot{M}_s &= n + \frac{na_e^2}{a^2} J_{20} F_{201} \left(6\, G_{210} - \frac{1-e^2}{e} \frac{dG_{210}}{de}\right) \\ &= n + \frac{3na_e^2}{4a^2} (1-e^2)^{-3/2} J_{20}(1 - 3 \cos^2 I) \ . \end{aligned} \tag{5.11}$$

The expressions for F_{201} and G_{210} were taken from Ref. 3 and are

$$F_{201} = \frac{3}{4}\sin^2 I - \frac{1}{2}$$

$$G_{210} = (1-e^2)^{-3/2} \quad .$$

In accordance with Eqs. (3.9), (3.10), and (3.11), the variables on the right hand sides of the differential equations are assumed to be approximated by

$$\begin{aligned} a &= \bar{a} \\ e &= \bar{e} \\ I &= \bar{I} \\ \Omega &= \bar{\Omega} + \dot{\Omega}_s(\bar{a}, \bar{e}, \bar{I})t \\ \omega &= \bar{\omega} + \dot{\omega}_s(\bar{a}, \bar{e}, \bar{I})t \\ M &= \bar{M} + n_o t \end{aligned} \qquad (5.12)$$

where the barred quantities are mean values which are determined from the initial conditions. Following Ingram, the secular rate in M has been chosen as n_o rather than $\dot{M}_s$. The quantity n_o is given by

$$n_o = \left(\frac{\mu}{a_o^3}\right)^{\frac{1}{2}}$$

with a_o being defined by the equation

$$\frac{\mu}{2a_o} = \frac{\mu}{2a} + R \ . \qquad (5.13)$$

As long as R is independent of the time (i.e., contains zonal harmonics only), a_o is a constant which can be calculated from the epoch values of the elements. However, Ingram uses a constant a_o based upon epoch conditions even though his disturbing function contains time. This turns out to be a good approximation because the a_o determined from Eq. (5.13) for the time dependent disturbing function experiences changes that are negligibly small over the time interval of interest. The justification for choosing the secular variation in M in this manner is not clear at this point and will be discussed further in Sec. IV.4 where n_o will be shown to be essentially equivalent to $\dot{M}_s$.

Continuing with the procedure for generating the solutions, Eqs. (5.12) are substituted into the right hand sides of Eqs. (5.2) through (5.7) and the resulting equations are integrated with respect to time. For example, the rate equation for the semi-major axis becomes

$$\dot{a} = \sum_{\ell mpq} \frac{2\bar{n}a_e^{\ell}}{\bar{a}^{\ell-1}} J_{\ell m} F_{\ell mp}(\bar{I})\, G_{\ell pq}(\bar{e})\, (\ell-2p+q) \begin{bmatrix} -\sin \\ \cos \end{bmatrix} \Phi_{\ell mpq}$$

where

$$\bar{n} = \left(\frac{\mu}{\bar{a}^3}\right)^{\frac{1}{2}}$$

and

$$\begin{aligned} \Phi_{\ell mpq} &= (\ell-2p)\bar{\omega} + (\ell-2p+q)\bar{M} + m(\bar{\Omega}-\theta_E-\lambda_{\ell m}) \\ &\qquad + [(\ell-2p)\dot{\omega}_s + (\ell-2p+q)n_o + m(\dot{\Omega}_s-\dot{\theta})]t \\ &\equiv \bar{\Phi}_{\ell mpq} + \dot{\Phi}^{s}_{\ell mpq}\, t \ . \end{aligned}$$

Integrating with respect to time yields

$$a = \bar{a} + \sum_{\ell mpq} \frac{2\bar{n}a_e^{\ell}}{\bar{a}^{\ell-1}} J_{\ell m} F_{\ell mp}(\bar{I})\, G_{\ell pq}(\bar{e})\, \frac{(\ell-2p+q)}{\dot{\Phi}^{s}_{\ell mpq}} \begin{bmatrix} \cos \\ \sin \end{bmatrix} \Phi_{\ell mpq} \ .$$

Recall that, as was noted in Sec. III.1, the variation of the mean motion with the semi-major axis must be considered in integrating the differential equation for M . This leads to a term of the form

$$- \frac{3}{2}\frac{\bar{n}}{\bar{a}} \int_t (a-\bar{a})dt \; .$$

The integral above can be evaluated after the approximate solution for the semi-major axis has been obtained. Also, the secular terms in the solution for M are to be replaced by $n_o t$.

The solutions for the case in which only the effects of the oblateness are considered are given below.

$$a = \bar{a} + \frac{2\bar{n}a_e^2}{\bar{a}} J_{20} \sum_{p=0}^{2} \sum_{q=-\infty}^{\infty} F_{20p}(\bar{I})\, G_{2pq}(\bar{e})\, \frac{(2-2p+q)}{\dot{\Phi}^s_{20pq}} \cos \Phi_{20pq} \tag{5.14}$$

$$e = \bar{e} + \frac{\bar{n}a_e^2}{\bar{a}^2} J_{20} \sum_{p=0}^{2} \sum_{q=-\infty}^{\infty} \frac{F_{20p}(\bar{I})G_{2pq}(\bar{e})}{\dot{\Phi}^s_{20pq}} \left[\frac{1-\bar{e}^2}{\bar{e}} (2-2p+q) - \frac{\sqrt{1-\bar{e}^2}}{\bar{e}} (2-2p) \right] \cos \Phi_{20pq} \tag{5.15}$$

$$I = \bar{I} + \frac{\bar{n}a_e^2}{\bar{a}^2} J_{20} \sum_{p=0}^{2} \sum_{q=-\infty}^{\infty} F_{20p}(\bar{I}) \cot \bar{I}\, \frac{G_{2pq}(\bar{e})}{\sqrt{1-\bar{e}^2}}\, \frac{(2-2p)}{\dot{\Phi}^s_{20pq}} \cos \Phi_{20pq} \tag{5.16}$$

$$\Omega = \bar{\Omega} + \frac{3\bar{n}a_e^2 \cos \bar{I}}{2\bar{a}^2(1-\bar{e}^2)^2}\, t + \frac{\bar{n}a_e^2}{\bar{a}^2} J_{20} \sum_{p=0}^{2} \sum_{q=-\infty}^{\infty} \frac{dF_{20p}(\bar{I})}{d\bar{I}} \csc \bar{I}\, \frac{G_{2pq}(\bar{e})}{\sqrt{1-\bar{e}^2}}\, \frac{\sin \Phi_{20pq}}{\dot{\Phi}^s_{20pq}} \tag{5.17}$$

$$\omega = \bar{\omega} + \frac{3\bar{n}a_e^2}{4\bar{a}^2(1-\bar{e}^2)^2} J_{20}(1-5 \cos^2 \bar{I})t + \frac{\bar{n}a_e^2}{\bar{a}^2} J_{20} \left[F_{20p}(\bar{I})\, \frac{\sqrt{1-\bar{e}^2}}{\bar{e}}\, \frac{dG_{2pq}(\bar{e})}{d\bar{e}} - \frac{dF_{20p}(\bar{I})}{d\bar{I}} \cot \bar{I}\, \frac{G_{2pq}(\bar{e})}{\sqrt{1-\bar{e}^2}} \right] \frac{\sin \Phi_{20pq}}{\dot{\Phi}^s_{20pq}} \tag{5.18}$$

$$\begin{aligned} M = {} & \overline{M} + n_o t \\ & + \frac{\overline{n} a_e^2}{\overline{a}^2} J_{20} \sum_{p=0}^{2} \sum_{q=-\infty}^{\infty} F_{20p}(\overline{I}) \left\{ 3 \left[2 + \frac{(2-2p+q)}{\dot{\Phi}^s_{20pq}} \overline{n} \right] G_{2pq}(\overline{e}) \right. \\ & \left. - \frac{1-\overline{e}^2}{\overline{e}} \frac{dG_{2pq}(\overline{e})}{d\overline{e}} \right\} \frac{\sin \Phi_{20pq}}{\dot{\Phi}^s_{20pq}} \end{aligned} \qquad (5.19)$$

In the expressions above,

$$\Phi_{20pq} = (2-2p)\,(\overline{\omega}+\dot{\omega}_s t) + (2-2p+q)\,(\overline{M}+n_o t)$$

and the summations on p and q in the equations for Ω, ω, and M are limited to those values of the indices which do not simultaneously satisfy the conditions

1. $2 - 2p + q = 0$
2. $2 - 2p = 0$.

As before, the barred quantities in the solutions are calculated by inverting the equations at the initial time.

V.3 Brouwer's Solution by the von Zeipel Method

The following paragraphs contain a discussion of one of the earliest and best known solutions to the artificial Earth satellite problem. It was developed by Brouwer (Ref. 5) and appeared in the same issue of the Astronomical Journal as the contributions of Kozai (Ref. 14) and Garfinkel (Ref. 29). In the original paper, Brouwer presented a complete first order solution (i.e., second order secular terms were included) for the effects of the second, third, fourth, and fifth zonal harmonics in the Earth's potential. Only the effects of the second zonal harmonic will be considered here since the effects of the higher zonal harmonics are relatively easy to determine. This does not mean that these harmonics can be neglected. In fact, as will be indicated later, they will yield terms of the first order in the final solution.

The Equations of Motion

The orbit of the satellite is to be described by the Delaunay variables defined in Sec. II.1. The equations of motion are

$$\begin{aligned} \dot{L} &= F_\ell & \dot{\ell} &= -F_L \\ \dot{G} &= F_g & \dot{g} &= -F_G \\ \dot{H} &= F_h & \dot{h} &= -F_H \end{aligned}$$

where the Hamiltonian $F(L,G,H,\ell,g,h)$ can be determined as follows.

From Eqs. (2.23) and (2.24) the potential for the problem is

$$\begin{aligned} V &= -\frac{\mu}{r} - \frac{\mu k_2}{r^3}\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right) + \left(\frac{3}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)\cos\,(2g+2f)\right] \\ &= -\frac{\mu}{r} - R \;. \end{aligned} \tag{5.20}$$

Since the potential is independent of the time, the system is conservative and, following astronomical convention, the Hamiltonian is equal to the negative of the total energy. Using the equation

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \nabla R$$

the total energy is found to be given by

$$E = \frac{v^2}{2} - \frac{\mu}{r} - R \;.$$

With the aid of Eq. (2.5) the Hamiltonian can be written as

$$F = -E = \frac{\mu}{2a} + R \tag{5.21}$$

or, writing the semi-major axis in terms of the Delaunay variable L,

$$F = \frac{\mu^2}{2L^2} + \frac{\mu^4 k_2}{L^6}\left[\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3} + \left(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}\cos(2g+2f)\right] . \qquad (5.22)$$

The form of the Hamiltonian given by Eq. (5.22) is the one employed in the development of the solution. However, to show that the Hamiltonian satisfies the requirements of the von Zeipel method, F must be written explicitly in terms of the mean anomaly ℓ. Applying procedures given in Refs. 8 and 9, terms of the form

$$\left(\frac{a}{r}\right)^n \cos m f ,$$

where m and n are integers, can be expanded into Fourier series in ℓ. From Ref. 5, the expansions required here are

$$\frac{a^3}{r^3} = \frac{L^3}{G^3} + \sum_{j=1}^{\infty} 2P_j(e)\cos j\ell \equiv \frac{L^3}{G^3} + \sigma_1$$

$$\frac{a^3}{r^3}\cos(2g+2f) = \sum_{j=-\infty}^{\infty} Q_j(e)\cos(2g+j\ell) \equiv \sigma_2 .$$

The periodic functions σ_1 and σ_2 are both purely periodic in ℓ since it can be shown that

$$Q_o(e) = 0 .$$

Then, the Hamiltonian is given explicitly in terms of the mean anomaly by

$$F = \frac{\mu^2}{2L^2} + \frac{\mu^4 k_2}{L^6}\left[\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right)\left(\frac{L^3}{G^3}+\sigma_1\right) + \left(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\right)\sigma_2\right] .$$

Choosing k_2 as the small parameter corresponding to ε in Eq. (3.16), the zero and first order portions of the Hamiltonian are identified as follows:

$$F_o = \frac{\mu^2}{2L^2} \tag{5.23}$$

$$\begin{aligned} F_1 &= \frac{\mu^4 k_2}{L^6}\left[\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}+\left(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}\cos(2g+2f)\right] \\ &= \frac{\mu^4 k_2}{L^6}\left[\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right)\left(\frac{L^3}{G^3}+\sigma_1\right)+\left(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\right)\sigma_2\right] . \end{aligned} \tag{5.24}$$

In contrast to the convention in Sec. III.2, the small parameter is retained in F_1 so that

$$F = F_o(L) + F_1(L,G,H,\ell,g) . \tag{5.25}$$

Inasmuch as F_o is a function of L only and F_1 is periodic in ℓ and g (h does not appear), the Hamiltonian satisfies the requirements for the application of the von Zeipel method.

Outline of Brouwer's Method

Brouwer uses the von Zeipel method presented in Sec. III.2 to perform a sequence of canonical transformations from the original variables to the new variables L'', G'', H'', ℓ'', g'', and h'' in such a manner that the new Hamiltonian F^{**} is a function only of L'', G'', and H''. The integration of the resulting system of canonical equations is trivial. The first transformation is required to yield a new Hamiltonian F^* that contains the new variables L', G', H', and g', but not the variable ℓ', through second order. A second transformation is then generated to produce a Hamiltonian F^{**} containing only L'', G'', and H'' through second order.

The Short Period Terms

It is now desired to determine a canonical transformation from the variables L, G, H, ℓ, g, and h to the variables L', G',

H', ℓ', g', and h' such that the new Hamiltonian F^* does not contain ℓ'. The transformation is specified by the equations

$$\begin{aligned} L &= \frac{\partial S}{\partial \ell} & \ell' &= \frac{\partial S}{\partial L'} \\ G &= \frac{\partial S}{\partial g} & g' &= \frac{\partial S}{\partial G'} \\ H &= \frac{\partial S}{\partial h} & h' &= \frac{\partial S}{\partial H'} \end{aligned} \tag{5.26}$$

and

$$F(L,G,H,\ell,g) = F_o(L) + F_1(L,G,H,\ell,g) = F^*(L',G',H',g') . \tag{5.27}$$

Applying the von Zeipel technique, the generating function S and the new Hamiltonian F^* are assumed to be expandable in power series in the small parameter k_2. Thus, using subscripts to indicate the order of the term, S and F^* are given by

$$S(L',G',H',\ell,g,h) = S_o + S_1 + S_2 + \ldots \tag{5.28}$$

$$F^*(L',G',H',g') = F_o^* + F_1^* + F_2^* + \ldots . \tag{5.29}$$

For $k_2 = 0$, the problem is trivial and therefore S_o is chosen as

$$S_o = L'\ell + G'g + H'h . \tag{5.30}$$

Further, since H is a constant (because h does not appear in F), there is no need to transform this variable. For this reason h is assumed to occur in S only in S_o.

With the aid of Eqs. (5.26), (5.28), (5.29), and (5.30), Eq. (5.27) becomes

$$
\begin{aligned}
&F_o\left(L' + \frac{\partial S_1}{\partial \ell} + \frac{\partial S_2}{\partial \ell} + \ldots\right) \\
&\quad + F_1\left(L' + \frac{\partial S_1}{\partial \ell} + \frac{\partial S_2}{\partial \ell} + \ldots, G' + \frac{\partial S_1}{\partial g} + \frac{\partial S_2}{\partial g} + \ldots, H', \ell, g\right) \\
&\quad = F_o^* + F_1^*\left(L', G', H', g + \frac{\partial S_1}{\partial G'} + \frac{\partial S_2}{\partial G'} + \ldots\right) \\
&\qquad + F_2^*\left(L', G', H', g + \frac{\partial S_1}{\partial G'} + \frac{\partial S_2}{\partial G'} + \ldots\right) + \ldots \; .
\end{aligned}
\tag{5.31}
$$

Expanding each term of Eq. (5.31) in Taylor's series through the second power of k_2 , the result is

$$
\begin{aligned}
&F_o(L') + \frac{\partial F_o}{\partial L'}\left(\frac{\partial S_1}{\partial \ell} + \frac{\partial S_2}{\partial \ell}\right) + \frac{1}{2}\,\frac{\partial^2 F_o}{\partial L'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^2 \\
&\quad + F_1(L', G', H', \ell, g) + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial \ell} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} + \ldots \\
&\quad = F_o^*(L') + F_1^*(L', G', H', g) + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'} \\
&\qquad + F_2^*(L', G', H', g) + \ldots \; .
\end{aligned}
\tag{5.32}
$$

The following equations are then obtained by equating terms of the same order.

Order 0:

$$
F_o(L') = F_o^*(L') \tag{5.33}
$$

Order 1:

$$
\frac{\partial F_o}{\partial L'}\frac{\partial S_1}{\partial \ell} + F_1(L', G', H', \ell, g) = F_1^*(L', G', H', g) \tag{5.34}
$$

Order 2:

$$
\begin{aligned}
&\frac{\partial F_o}{\partial L'}\frac{\partial S_2}{\partial \ell} + \frac{1}{2}\,\frac{\partial^2 F_o}{\partial L'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^2 + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial \ell} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} \\
&\quad = F_2^*(L', G', H', g) + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'}
\end{aligned}
\tag{5.35}
$$

This process may be continued to yield analogous equations for higher orders.

Equations (5.33), (5.34), and (5.35) can now be used to determine F_o^*, F_1^*, F_2^*, S_1, and S_2. From the first equation,

$$F_o^*(L') = F_o(L') = \frac{\mu^2}{2L'^2} . \tag{5.36}$$

Following the procedure given in Sec. III.2, Eq. (5.34) is solved by setting F_1^* equal to the portion of the left hand side of the equation that is independent of ℓ, i.e., F_1^* is chosen to be the portion of the Fourier series expansion for F_1 given by Eq. (5.24) that does not contain ℓ. Equivalently, F_1^* is equal to

$$F_1^* = F_{1s}(L',G',H',g)$$

where F_{1s} is known as the secular part of F_1 and is defined by

$$F_{1s}(L,G,H,g) = \frac{1}{2\pi} \int_o^{2\pi} F_1(L,G,H,\ell,g)\, d\ell .$$

But this integral in its present form can not be evaluated because, from Eq. (5.22), F_1 is explicitly a function of the true anomaly f rather than the mean anomaly ℓ. However, the differential relation between f and ℓ given by Eq. (2.6) can be used to change the variable of integration. With the aid of this relation, the expression defining F_{1s} becomes

$$F_{1s} = \frac{1}{2\pi} (1-e^2)^{-\frac{1}{2}} \int_o^{2\pi} (\frac{r}{a})^2 F_1\, df$$

$$= \frac{1}{2\pi} \frac{L}{G} \int_o^{2\pi} \frac{\mu^4 k_2}{L^6} [(-\frac{1}{2} + \frac{3}{2} \frac{H^2}{G^2})\frac{a}{r} + (\frac{3}{2} - \frac{3}{2} \frac{H^2}{G^2})\frac{a}{r} \cos (2g+2f)]df .$$

From Eqs. (2.1) and (2.2) and the definitions of the Delaunay variables,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{a}{r}\, df = \frac{1}{2\pi} \int_0^{2\pi} \frac{1+e \cos f}{1-e^2}\, df$$

$$= \frac{1}{2\pi} \frac{L^2}{G^2} (f + e \sin f)\Big|_0^{2\pi} = \frac{L^2}{G^2} .$$

Similarly,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{a}{r} \cos (2g+2f)\, df = 0 .$$

Thus,

$$F_{1s} = \frac{\mu^4 k_2}{L^3 G^3} \left(- \frac{1}{2} + \frac{3}{2} \frac{H^2}{G^2}\right)$$

and

$$F_1^* = \frac{\mu^4 k_2}{L'^3 G'^3} \left(- \frac{1}{2} + \frac{3}{2} \frac{H'^2}{G'^2}\right) . \qquad (5.37)$$

With this choice for F_1^*, the partial differential equation for S_1 is found to be

$$\frac{\partial S_1}{\partial \ell} = \frac{L'^2}{\mu^2} [F_1(L',G',H',\ell,g) - F_1^*(L',G',H')]$$

and $\frac{\partial S_1}{\partial \ell}$ is purely periodic in ℓ since

$$\frac{1}{2\pi} \int_0^{2\pi} (F_1 - F_1^*) d\ell = \frac{1}{2\pi} \int_0^{2\pi} F_1\, d\ell - F_{1s} = 0 .$$

The first order portion of the generating function is given by

$$S_1 = \frac{L'^3}{\mu^2} \int (F_1 - F_1^*) d\ell .$$

As before, F_1 is explicitly a function of f so that the differential relation between f and ℓ is employed in performing the integration. Then,

$$\int F_1 \, d\ell = \frac{L}{G} \int \left(\frac{r}{a}\right)^2 F_1 \, df$$

$$= \frac{L}{G} \cdot \frac{\mu^4 k_2}{L^6} \int \left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)\frac{a}{r} + \left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2}\right)\frac{a}{r} \cos (2g+2f)\right] df$$

$$= \frac{\mu^2 k_2}{G^3} \left\{\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right) (f + e \sin f)\right.$$

$$\left. + \left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2}\right)\left[\frac{1}{2} \sin (2g+2f) + \frac{e}{2} \sin (2g+f) + \frac{e}{6} \sin (2g+3f)\right]\right\}$$

and S_1 becomes

$$S_1 = \frac{\mu^2 k_2}{G'^3} \left\{\left(-\frac{1}{2} + \frac{3}{2}\frac{H'^2}{G'^2}\right) (f - \ell + e' \sin f)\right.$$
$$\left. + \left(\frac{3}{2} - \frac{3}{2}\frac{H'^2}{G'^2}\right) \left[\frac{1}{2} \sin (2g+2f) + \frac{e'}{2} \sin (2g+f) + \frac{e'}{6} \sin (2g+3f)\right]\right\} . \tag{5.38}$$

In Eq. (5.38),

$$e' \equiv \left(1 - \frac{G'^2}{L'^2}\right)^{\frac{1}{2}}$$

and f is a function of e' and ℓ that is determined from the equations

$$E - e' \sin E = \ell \tag{5.39}$$

$$\tan \frac{f}{2} = \left(\frac{1+e'}{1-e'}\right)^{\frac{1}{2}} \tan \frac{E}{2} . \tag{5.40}$$

Note that f as defined above is not the "true" true anomaly and E is not the true eccentric anomaly since both are given as functions of e' and ℓ rather than of e and ℓ .

The equations of transformation relating the primed and unprimed variables can now be obtained through first order by substituting the expressions for S_o and S_1 into Eqs. (5.26). For example,

$$L = L' + \frac{\partial S_1}{\partial \ell}$$

$$= L' + \frac{\mu^2 k_2}{L'^3}\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H'^2}{G'^2}\right)\left(\frac{a'^3}{r^3} - \frac{L'^3}{G'^3}\right)\right. \tag{5.41}$$

$$\left. + \left(\frac{3}{2} - \frac{3}{2}\frac{H'^2}{G'^2}\right)\frac{a'^3}{r^3}\cos(2g+2f)\right]$$

where

$$a' \equiv \frac{L'^2}{\mu}$$

and r is given by

$$r = \frac{a'(1-e'^2)}{1+e'\cos f} \tag{5.42}$$

with f a function of e' and ℓ as above.

Since $f(e',\ell)$ in Eq. (5.41) differs only slightly from the true anomaly $f(e,\ell)$, L differs from L' by short period terms of the first order in k_2 . The same is true of the differences between the remaining primed and unprimed variables. Thus, the first canonical transformation yields the first order short period terms in the solution.

In calculating the partial derivatives of S required above it should be recalled that S_1 given by Eq. (5.38) is an explicit function of the variable e' which in turn is a function of L' and G' . The following relations are therefore needed.

$$\frac{\partial}{\partial L'} = \frac{\partial e'}{\partial L'}\frac{\partial}{\partial e'} = \frac{G'^2}{e'L'^3}\frac{\partial}{\partial e'}$$

$$\frac{\partial}{\partial G'} = \frac{\partial e'}{\partial G'}\frac{\partial}{\partial e'} = -\frac{G'}{e'L'^2}\frac{\partial}{\partial e'}$$

The expressions for F_1^* and S_1 determined above can now be substituted into Eq. (5.35) which can then be solved for F_2^* and S_2. Brouwer was interested only in a first order solution for the periodic terms and did not calculate S_2 . The development of these second order short period terms, although very tedious, was carried out by Kozai (Ref. 30). Brouwer did, however, use the equation noted above to determine F_2^* . This was done for two reasons. The first is that F_2^* is required in the calculation of the secular terms in the solution through second order. The second reason is concerned with the determination of the first order long period terms and will be discussed shortly. The function F_2^* is equal to the terms on the left hand side of the equation that are independent of ℓ and is therefore given by

$$F_2^* = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2} \frac{\partial^2 F_0}{\partial L'^2} \left(\frac{\partial S_1}{\partial \ell} \right)^2 + \frac{\partial F_{1p}}{\partial L'} \frac{\partial S_1}{\partial \ell} + \frac{\partial F_1}{\partial G'} \frac{\partial S_1}{\partial g} \right] d\ell \qquad (5.43)$$

where F_{1p} designates the portion of F_1 that is purely periodic in ℓ , i.e.,

$$F_{1p} = F_1 - F_{1s} \; .$$

The process of performing the averaging with respect to ℓ can be simplified with the aid of the following relation from Tisserand (Ref. 9) or with Cayley's tables (Ref. 31). For $n \le -2$ and $m \ge 0$:

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^n \cos m f \, d\ell$$
$$= \left(\frac{e}{2}\right)^m (1-e^2)^{n+\frac{3}{2}} \sum_{\rho=0}^{-[\frac{n+m+3}{2}]} \binom{-n-2}{m+2\rho} \binom{m+2\rho}{\rho} \left(\frac{e}{2}\right)^{2\rho} .$$

In the previous equation, $[\frac{n+m+3}{2}]$ designates the integer portion of $\frac{n+m+3}{2}$ and

$$\binom{m}{n} \equiv \frac{\Gamma(m+1)}{\Gamma(n+1)\,\Gamma(m-n+1)}$$

where $\Gamma(\nu)$ is the Gamma-function. The final result is

$$F_2^* = \frac{\mu^6 k_2^2}{L'^{10}} \left[\frac{15}{32}\frac{L'^5}{G'^5}\left(1 - \frac{18}{5}\frac{H'^2}{G'^2} + \frac{H'^4}{G'^4}\right) + \frac{3}{8}\frac{L'^6}{G'^6}\left(1 - 6\frac{H'^2}{G'^2} + 9\frac{H'^4}{G'^4}\right) - \frac{15}{32}\frac{L'^7}{G'^7}\left(1 - 2\frac{H'^2}{G'^2} - 7\frac{H'^4}{G'^4}\right)\right] + \frac{\mu^6 k_2^2}{L'^{10}}\left[-\frac{3}{16}\left(\frac{L'^5}{G'^5} - \frac{L'^7}{G'^7}\right)\left(1 - 16\frac{H'^2}{G'^2} + 15\frac{H'^4}{G'^4}\right)\right]\cos 2g' \; . \tag{5.44}$$

Here, g has been replaced by g'. This is permissible because the error incurred is of third order.

At this point, the short period terms have been calculated through first order and the new Hamiltonian $F^*(L',G',H',g')$ has been determined through second order. The equations for the rates of change of the new variables are

$$\begin{aligned} \dot{L}' &= \frac{\partial F^*}{\partial \ell'} = 0 & \dot{\ell}' &= -\frac{\partial F^*}{\partial L'} \\ \dot{G}' &= \frac{\partial F^*}{\partial g'} & \dot{g}' &= -\frac{\partial F^*}{\partial G'} \\ \dot{H}' &= \frac{\partial F^*}{\partial h'} = 0 & \dot{h}' &= -\frac{\partial F^*}{\partial H'} \; . \end{aligned}$$

Note that all of the quantities appearing in F^* (through second order) except g' and G' are constants. Thus, the problem reduces essentially to solving the equations

$$\dot{G}' = \frac{\partial F^*}{\partial g'} \qquad \dot{g}' = -\frac{\partial F^*}{\partial G'} \; .$$

The solutions of the remaining differential equations can then be obtained by quadratures. However, the differential equations for G' and g' are so complex that an exact solution can not be obtained analytically. Fortunately, the new Hamiltonian is of such a form that the von Zeipel method can again be applied to complete the solution.

Long Period and Secular Terms

The new Hamiltonian through second order is

$$F^* = \frac{\mu^2}{2L'^2} + \frac{\mu^4 k_2}{L'^3 G'^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H'^2}{G'^2}\right) + F_2^*(L',G',H',g') .$$

As stated above, since L' and H' are constants, the system has been reduced to one degree of freedom. Therefore, it is seen that F_1^* is essentially a function of G' while F_2^* is periodic in g' and the Hamiltonian is of the form required for the elimination of g'' by the von Zeipel method.

The procedure, then, is to determine a new generating function $S^*(L'',G'',H'',\ell',g',h')$ describing a canonical transformation from the variables L',G',H',ℓ',g', and h' to the variables L'',G'',H'',ℓ'',g'', and h'' such that the new Hamiltonian F^{**} is a function of L'',G'', and H'' only.

The transformation is defined by the equations

$$\begin{aligned} L' &= \frac{\partial S^*}{\partial \ell'} & \ell'' &= \frac{\partial S^*}{\partial L''} \\ G' &= \frac{\partial S^*}{\partial g'} & g'' &= \frac{\partial S^*}{\partial G''} \\ H' &= \frac{\partial S^*}{\partial h'} & h'' &= \frac{\partial S^*}{\partial H''} \end{aligned} \qquad (5.45)$$

$$\begin{aligned} F^*(L',G',H',g') &= F_o^*(L') + F_1^*(L',G',H') + F_2^*(L',G',H',g') \\ &= F^{**}(L'',G'',H''). \end{aligned} \qquad (5.46)$$

The functions S^* and F^{**} are assumed to be given by

$$S^*(L'',G'',H'',\ell',g',h') = S_o^* + S_1^* + S_2^* + \ldots \qquad (5.47)$$

$$F^{**}(L'',G'',H'') = F_o^{**} + F_1^{**} + F_2^{**} + \ldots \tag{5.48}$$

where S_o^* is taken as

$$S_o^* = L''\ell' + G''g' + H''h' \tag{5.49}$$

and ℓ' and h' appear nowhere in S^* except in S_o^*. Using Eqs. (5.45), (5.47), (5.48), and (5.49), Eq. (5.46) becomes

$$\begin{aligned} F_o^*(L'') &+ F_1^*\left(L'',G'' + \frac{\partial S_1^*}{\partial g'} + \frac{\partial S_2^*}{\partial g'} + \ldots, H''\right) \\ &+ F_2^*\left(L'',G'' + \frac{\partial S_1^*}{\partial g'} + \frac{\partial S_2^*}{\partial g'} + \ldots, H'',g'\right) \\ &= F_o^{**} + F_1^{**}(L'',G'',H'') + F_2^{**}(L'',G'',H'') + \ldots . \end{aligned} \tag{5.50}$$

Expanding in Taylor's series through the second power in k_2 as before yields

$$\begin{aligned} F_o^*(L'') &+ F_1^*(L'',G'',H'') + \frac{\partial F_1^*}{\partial G''}\frac{\partial S_1^*}{\partial g'} + F_2^*(L'',G'',H'',g') + \ldots \\ &= F_o^{**}(L'') + F_1^{**}(L'',G'',H'') + F_2^{**}(L'',G'',H'') + \ldots . \end{aligned}$$

Equating terms of the same order gives

Order 0:

$$F_o^*(L'') = F_o^{**}(L'') \tag{5.51}$$

Order 1:

$$F_1^*(L'',G'',H'') = F_1^{**}(L'',G'',H'') \tag{5.52}$$

Order 2:

$$\frac{\partial F_1^*}{\partial G''}\frac{\partial S_1^*}{\partial g'} + F_2^*(L'',G'',H'',g') = F_2^{**}(L'',G'',H'') . \tag{5.53}$$

The zero and first order portions of F^{**} are obtained immediately from Eqs.(5.51) and (5.52) as

$$F_o^{**} = \frac{\mu^2}{2L''^2} \tag{5.54}$$

$$F_1^{**} = \frac{\mu^4 k_2}{L''^3 G''^3} \left(-\frac{1}{2} + \frac{3}{2}\frac{H''^2}{G''^2}\right) . \tag{5.55}$$

The quantity F_2^{**} is chosen as the part of F_2^{*} that is independent of g' . Thus, from Eq. (5.44),

$$\begin{aligned} F_2^{**} = \frac{\mu^6 k_2^2}{L''^{10}} \Big[& \frac{15}{32}\frac{L''^5}{G''^5}\left(1 - \frac{18}{5}\frac{H''^2}{G''^2} + \frac{H''^4}{G''^4}\right) + \frac{3}{8}\frac{L''^6}{G''^6}\left(1 - 6\frac{H''^2}{G''^2} + 9\frac{H''^4}{G''^4}\right) \\ & - \frac{15}{32}\frac{L''^7}{G''^7}\left(1 - 2\frac{H''^2}{G''^2} - 7\frac{H''^4}{G''^4}\right)\Big] . \end{aligned} \tag{5.56}$$

The equation for S_1^{*} becomes

$$\frac{\partial S_1^{*}}{\partial g'} = -\left(\frac{\partial F_1^{*}}{\partial G''}\right)^{-1} \left(F_2^{*} - F_2^{**}\right) \tag{5.57}$$

so that

$$S_1^{*} = \frac{\mu^2 k_2 G''}{L''^4}\left(\frac{L''^2}{G''^2} - \frac{L''^4}{G''^4}\right)\left[\frac{1}{16}\left(1 - 11\frac{H''^2}{G''^2}\right) - \frac{5}{2}\frac{H''^4}{G''^4}\left(1 - 5\frac{H''^2}{G''^2}\right)^{-1}\right]\sin 2g' . \tag{5.58}$$

Substituting Eqs. (5.49) and (5.58) into Eqs. (5.45) gives the transformation relations between the primed and double primed variables through first order. From the results the primed and double primed variables will be observed to differ by terms periodic in g' . Inasmuch as the period of g' is very nearly equal to the period of the argument of perigee g which is much longer than the time required for the satellite to complete a revolution, it is found that this transformation has produced the long period terms in the solutions.

Two important characteristics of these long period terms should be mentioned. First, as indicated by Eq. (5.57), these terms are obtained from F_2^*, the second order portion of the Hamiltonian generated by the first canonical transformation. The quantity F_2^* in turn results from the interaction of first order short period terms as revealed by Eq. (5.43). Thus, at least part of the first order long period terms arise from the interaction of the first order short period perturbations. It is also important to note that retaining the third, fourth, and fifth zonal harmonics in the disturbing function would lead to terms due to these harmonics in F_2^* and therefore would produce additional long period perturbations of the first order. Second, S_1^* can not be determined from Eq. (5.57) if $\frac{\partial F_1^*}{\partial G''} = 0$. In fact, S^* can not be generated by this approach if $\frac{\partial F_1^*}{\partial G''}$ is sufficiently small. This is the well-known problem of the "critical inclination", so called because

$$\frac{\partial F_1^*}{\partial G''} = 0$$

implies

$$1 - 5 \frac{H''^2}{G''^2} = 1 - 5 \cos^2 I'' = 0$$

or

$$I \approx \cos^{-1}\sqrt{\frac{1}{5}} = 63^\circ 26' .$$

For orbits near the critical inclination, special techniques must be used. Some of the attempts to handle this problem will be briefly reviewed in Sec. V.5.

The solution is nearly complete, the first order short period and first order long period terms having been found; only the first and second order secular terms remain. These are determined easily with the aid of the Hamiltonian

$$\begin{aligned} F^{**} = {} & \frac{\mu^2}{2L''^2} + \frac{\mu^4 k_2}{L''^3 G''^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H''^2}{G''^2}\right) \\ & + \frac{\mu^6 k_2^2}{L''^{10}}\left[\frac{15}{32}\frac{L''^5}{G''^5}\left(1 - \frac{18}{5}\frac{H''^2}{G''^2} + \frac{H''^4}{G''^4}\right) + \frac{3}{8}\frac{L''^6}{G''^6}\left(1 - 6\frac{H''^2}{G''^2} + 9\frac{H''^4}{G''^4}\right)\right. \\ & \left. - \frac{15}{32}\frac{L''^7}{G''^7}\left(1 - 2\frac{H''^2}{G''^2} - 7\frac{H''^4}{G''^4}\right)\right] . \end{aligned}$$

Because ℓ'' , g'' , and h'' do not appear in F^{**} , the canonical equations for the double primed variables can be integrated to yield

$$L'' = \text{cons.} = L'$$

$$G'' = \text{cons.}$$

$$H'' = \text{cons.} = H$$

$$\begin{aligned} \ell'' = {} & \ell_o'' + n_o t \left\{1 + 3\frac{\mu^2 k_2}{L' G''^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2}\right)\right. \\ & + \frac{\mu^4 k_2^2}{L'^8}\left[\frac{75}{32}\frac{L'^5}{G''^5} + \frac{3}{2}\frac{L'^6}{G''^6} - \frac{45}{32}\frac{L'^7}{G''^7}\right. \\ & + \left(-\frac{135}{16}\frac{L'^5}{G''^5} - 9\frac{L'^6}{G''^6} + \frac{45}{16}\frac{L'^7}{G''^7}\right)\frac{H^2}{G''^2} \\ & \left.\left. + \left(\frac{75}{32}\frac{L'^5}{G''^5} + \frac{27}{2}\frac{L'^6}{G''^6} + \frac{315}{32}\frac{L'^7}{G''^7}\right)\frac{H^4}{G''^4}\right]\right\} \end{aligned} \tag{5.59}$$

$$\begin{aligned} g'' = {} & g_o'' + n_o t \left\{3\frac{\mu^2 k_2}{G''^4}\left(-\frac{1}{2} + \frac{5}{2}\frac{H^2}{G''^2}\right)\right. \\ & + \frac{\mu^4 k_2^2}{L'^8}\left[\frac{75}{32}\frac{L'^6}{G''^6} + \frac{9}{4}\frac{L'^7}{G''^7} - \frac{105}{32}\frac{L'^8}{G''^8}\right. \\ & + \left(-\frac{189}{16}\frac{L'^6}{G''^6} - 18\frac{L'^7}{G''^7} + \frac{135}{16}\frac{L'^8}{G''^8}\right)\frac{H^2}{G''^2} \\ & \left.\left. + \left(\frac{135}{32}\frac{L'^6}{G''^6} + \frac{135}{4}\frac{L'^7}{G''^7} + \frac{1155}{32}\frac{L'^8}{G''^8}\right)\frac{H^4}{G''^4}\right]\right\} \end{aligned} \tag{5.60}$$

$$\begin{aligned} h'' &= h_o'' + n_o t \{- 3 \frac{\mu^2 k_2}{G''^4} \frac{H}{G''} \\ &+ \frac{\mu^4 k_2^2}{L'^8} [(\frac{27}{8} \frac{L'^6}{G''^6} + \frac{9}{2} \frac{L'^7}{G''^7} - \frac{15}{8} \frac{L'^8}{G''^8}) \frac{H}{G''} \\ &+ (- \frac{15}{8} \frac{L'^6}{G''^6} - \frac{27}{2} \frac{L'^7}{G''^7} - \frac{105}{8} \frac{L'^8}{G''^8}) \frac{H^3}{G''^3}]\} \end{aligned} \tag{5.61}$$

where

$$n_o \equiv \frac{\mu^2}{L'^3} \quad .$$

The constants $L', G'', H, \ell_o'', g_o''$, and h_o'' are calculated by inverting the solutions at the initial time (see Ref. 32).

It is found that, as is true of the results obtained by applying Kaula's method, the solutions consist of short period, long period, and secular terms. For convenience, Brouwer also gives the solutions for the modified Keplerian variables a , e , I , ℓ , g , and h and briefly discusses the procedures for computation.

Added Comments on the Secular Solution

Brouwer has given the expression for the secular rates in ℓ , g , and h through second order in k_2 . However, it is easily shown that the secular terms in the solution for ℓ can be calculated only through first order using the equations in Brouwer's paper. This occurs because n_o which appears in the term of order zero in ℓ'' (Eq. (5.59)) is a function of L' and this quantity can be calculated only through first order. Then, the second order error in L' produces a second order error in n_o which leads in turn to a second order secular error in ℓ . There are at least two ways to remedy this situation. One method would be to extend the solution to include the second order short period terms in L as recommended in Ref. 33. A simpler approach, however, is given by Breakwell and Vagners (Ref. 34) and will be presented here.

From Eq. (5.59), Brouwer's expression for the secular rate in the mean anomaly is

$$\dot{\ell}'' = n_o[1 + 3\frac{\mu^2 k_2}{L'G''^3}(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2}) + k_2^2\ \delta_2(L',G'',H)] \qquad (5.62)$$

with

$$n_o = \frac{\mu^2}{L'^3} .$$

Consider now the original Hamiltonian $F(L,G,H,\ell,g)$. Since it is independent of the time, F is a constant whose value can be determined from the known initial conditions. Also, the canonical transformations have been chosen so that

$$F^{**} = F^{*} = F .$$

Hence, F^{**} is a known constant. Define the constant $\hat{a}$ by

$$\frac{\mu}{2\hat{a}} = F^{**} = \frac{\mu^2}{2L'^2}[1 + 2\frac{\mu^2 k_2}{L'G''^3}(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2}) + \frac{2L'^2}{\mu^2}F_2^{**}] + O(k_2^3). \qquad (5.63)$$

Then, define $\hat{n}$ by

$$\begin{aligned}
\hat{n} &= \left(\frac{\mu}{\hat{a}^3}\right)^{\frac{1}{2}} = \frac{\sqrt{8}}{\mu}\left(\frac{\mu}{2\hat{a}}\right)^{3/2} \\
&= \frac{\mu^2}{L'^3}[1 + 2\frac{\mu^2 k_2}{L'G''^3}(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2}) + \frac{2L'^2}{\mu^2}F_2^{**}]^{3/2} + O(k_2^3) \\
&= n_o[1 + 3\frac{\mu^2 k_2}{L'G''^3}(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2})] \qquad (5.64) \\
&\quad + n_o[\frac{3L'^2}{\mu^2}F_2^{**} + \frac{3}{2}\frac{\mu^4 k_2^2}{L'^2 G''^6}(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2})^2] + O(k_2^3) \\
&\equiv n_o[1 + 3\frac{\mu^2 k_2}{L'G''^3}(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2})] - k_2^2 n_2(L',G'',H) + O(k_2^3) .
\end{aligned}$$

From Eq. (5.64),

$$n_o[1 + 3 \frac{\mu^2 k_2}{L'G''^3} (- \frac{1}{2} + \frac{3}{2} \frac{H^2}{G''^2})] = \hat{n} + k_2{}^2 n_2 + O(k_2{}^3)$$

so that the expression for $\dot{\ell}''$ can be written as

$$\dot{\ell}'' = \hat{n} + k_2{}^2 (n_2 + n_o \delta_2) + O(k_2{}^3) . \tag{5.65}$$

The value of $\hat{n}$ can be calculated exactly and L' first appears in the second order term. Therefore, the secular rate in ℓ is given through second order by the previous equation even though L' is determined only through first order.

Several additional works applying the von Zeipel method to the artificial Earth satellite problem should be noted. As mentioned previously, Kozai has obtained a complete second order solution (Ref. 30). Lyddane (Ref. 35) has formulated the problem in terms of Poincaré variables and has generated solutions valid for small eccentricities and inclinations. Further, Giacaglia (Ref. 36) has extended the solutions to include the effects of any zonal harmonic and Garfinkel (Ref. 37) has derived expressions for the tesseral harmonic perturbations. A solution incorporating the effects of atmospheric drag is given by Brouwer and Hori (Ref. 38).

This concludes the discussion of Brouwer's solution of the artificial Earth satellite problem. A comparison of these results with those obtained in Sec. V.2 follows.

V.4 Comparison of the Solutions

The Secular Terms

Through first order, the secular terms in g and h in Brouwer's solutions can be obtained by integrating the equations

$$\dot{g}'' = -\frac{\partial F_1^{**}}{\partial G''} \qquad \dot{h}'' = -\frac{\partial F_1^{**}}{\partial H''}$$

where F_1^{**} from Eq. (5.55) is a function of the constants L'' , G'' , and H'' . But from Eqs. (5.37) and (5.52),

$$F_1^{**} = F_1^{*}(L'',G'',H'') = F_{1s}(L'',G'',H'')$$

or, from Eqs. (5.21) and (5.25),

$$F_1^{*} = R_s(L'',G'',H'')$$

where R_s is the portion of the disturbing function that is independent of ℓ and g . The equations for the first order secular rates become

$$\dot{g}'' = -\frac{\partial R_s}{\partial G''} \qquad \dot{h}'' = -\frac{\partial R_s}{\partial H''} .$$

In terms of Keplerian elements, the partial derivatives in the previous equations are

$$\begin{aligned}\frac{\partial}{\partial G} R_s(a,e,I) &= \frac{\partial R_s}{\partial e}\frac{\partial e}{\partial G} + \frac{\partial R_s}{\partial I}\frac{\partial I}{\partial G} \\ &= -\frac{G}{eL^2}\frac{\partial R_s}{\partial e} + \frac{\cot I}{G}\frac{\partial R_s}{\partial I} \\ &= -\frac{\sqrt{1-e^2}}{n\,a^2\,e}\frac{\partial R_s}{\partial e} + \frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R_s}{\partial I}\end{aligned}$$

and

$$\frac{\partial}{\partial H} R_s(a,e,I) = \frac{\partial R_s}{\partial I}\frac{\partial I}{\partial H} = -\frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R_s}{\partial I}$$

so that, considering R_s to be a function of a'' , e'' , and I'' , the secular rates are given by

$$\begin{aligned}\dot{g}'' &= \dot{\omega}'' = \frac{\sqrt{1-e''^2}}{n''a''^2e''}\,\frac{\partial R_s}{\partial e''} - \frac{\cot I''}{n''a''^2\sqrt{1-e''^2}}\,\frac{\partial R_s}{\partial I''} \\ \dot{h}'' &= \dot{\Omega}'' = \frac{1}{n''a''^2\sqrt{1-e''^2}\,\sin I''}\,\frac{\partial R_s}{\partial I''} .\end{aligned} \tag{5.66}$$

Since R is periodic in g and ℓ ,

$$\frac{\partial R_s}{\partial e} = \left(\frac{\partial R}{\partial e}\right)_s \qquad \frac{\partial R_s}{\partial I} = \left(\frac{\partial R}{\partial I}\right)_s \tag{5.67}$$

As before, the subscript s indicates that only terms independent of ℓ and g are included.

Now, Brouwer's double primed variables differ from true values by first order. Also, Ingram's $\bar{a}$, $\bar{e}$, and $\bar{I}$ differ from the true values of a , e , and I by first order. Thus, $\bar{a}$, $\bar{e}$, and $\bar{I}$ differ from a'' , e'' , and I'' , respectively, by terms of the first order. Equations (5.66) will therefore still be accurate through first order if the double primed variables are replaced by barred variables. Finally, Brouwer's expressions for the secular rates in g and h become

$$\begin{aligned}\dot{g}'' &= \dot{\omega}'' = \frac{\sqrt{1-\bar{e}^2}}{\bar{n}\,\bar{a}^2\,\bar{e}}\left(\frac{\partial R}{\partial \bar{e}}\right)_s - \frac{\cot \bar{I}}{\bar{n}\,\bar{a}^2\sqrt{1-\bar{e}^2}}\left(\frac{\partial R}{\partial \bar{I}}\right)_s \\ \dot{h}'' &= \dot{\Omega}'' = \frac{1}{\bar{n}\,\bar{a}^2\sqrt{1-\bar{e}^2}\,\sin \bar{I}}\left(\frac{\partial R}{\partial \bar{I}}\right)_s .\end{aligned}$$

Comparing these results with Lagrange's planetary equations (noting that Ingram's disturbing function is the same as Brouwer's through first order), it is seen that the first order secular rates are equal to the terms on the right hand sides of the equations for $\dot{\omega}$ and $\dot{\Omega}$ that are independent of ω and ℓ (or M) treated as being functions of the constants $\bar{a}$, $\bar{e}$, and $\bar{I}$. This is precisely the way Ingram's secular rates $\dot{\omega}_s$ and $\dot{\Omega}_s$ were determined. Then, through first order,

$$\dot{g}'' = \dot{\omega}_s \qquad \dot{h}'' = \dot{\Omega}_s$$

and the first order secular terms in ω and Ω in the two solutions are equivalent.

Consider next the secular rate in the mean anomaly. It has been shown that Brouwer's solution for this quantity can be written in the form

$$\dot{\ell}'' = \hat{n} + k_2^{\,2}(n_2 + n_o\,\delta_2) + O(k_2^{\,3})$$

where

$$\hat{n} \equiv \frac{\sqrt{8}}{\mu}(F^{**})^{3/2} = \frac{\sqrt{8}}{\mu}\,F^{3/2}$$

$$= \frac{\sqrt{8}}{\mu}\left(\frac{\mu}{2a} + R\right)^{3/2}$$

with R being defined by Eq. (5.20). From Eqs. (5.12) and (5.13), the secular rate in Ingram's solution is

$$n_o \equiv \frac{\sqrt{8}}{\mu}\left(\frac{\mu}{2a_o}\right)^{3/2} = \frac{\sqrt{8}}{\mu}\left(\frac{\mu}{2a} + R\right)^{3/2}$$

where the disturbing functions in the previous equations are the same through first order (note that this n_o is not the same as Brouwer's). Thus, both methods produce the same secular term in the mean anomaly through first order.

Finally, since the solutions for L , G , and H given by Brouwer contain no secular terms through first order, there will be no secular terms in the corresponding solutions for a , e , and I through first order, again in complete agreement with Ingram's results.

It is therefore concluded that the secular rates determined by the two methods are equivalent through first order. However, Brouwer also includes the second order secular rates in ℓ , g , and h . Thus, as far as secular terms are concerned, Brouwer's results should be more accurate than those given by Ingram.

The Short Period Terms

The comparison of the short period terms proceeds in a manner similar to the analysis given above for the secular terms. That is, the calculation of the short period terms by the von Zeipel method is shown to be equivalent through first order to the technique used by Ingram.

The first order short period terms in Brouwer's solutions, designated by $\Delta(\ )_{sp}$ below, can be written as follows:

$$\Delta L_{sp} = \frac{\partial S_1}{\partial \ell} = \frac{L'^3}{\mu^2} F_{1p} = \frac{1}{n_o} \int \frac{\partial F_{1p}}{\partial \ell} d\ell \tag{5.68}$$

$$\begin{aligned} \Delta G_{sp} &= \frac{\partial S_1}{\partial g} = \frac{\partial}{\partial g} \int \frac{\partial S_1}{\partial \ell} d\ell = \frac{\partial}{\partial g} \int \frac{L'^3}{\mu^2} F_{1p} d\ell \\ &= \frac{1}{n_o} \int \frac{\partial F_{1p}}{\partial g} d\ell \end{aligned} \tag{5.69}$$

$$\Delta H_{sp} = 0 \tag{5.70}$$

$$\begin{aligned} \Delta \ell_{sp} &= - \frac{\partial S_1}{\partial L'} = - \frac{\partial}{\partial L'} \int \frac{L'^3}{\mu^2} F_{1p} d\ell \\ &= - \int \left(\frac{3}{L'} \frac{L'^3}{\mu^2} F_{1p} + \frac{L'^3}{\mu^2} \frac{\partial F_{1p}}{\partial L'}\right) d\ell \\ &= - \frac{3}{L'} \int \Delta L_{sp} d\ell - \frac{1}{n_o} \int \frac{\partial F_{1p}}{\partial L'} d\ell \end{aligned} \tag{5.71}$$

$$\begin{aligned} \Delta g_{sp} &= - \frac{\partial S_1}{\partial G'} = - \frac{\partial}{\partial G'} \int \frac{L'^3}{\mu^2} F_{1p} d\ell \\ &= - \frac{1}{n_o} \int \frac{\partial F_{1p}}{\partial G'} d\ell \end{aligned} \tag{5.72}$$

$$\Delta h_{sp} = -\frac{\partial S_1}{\partial H'} = -\frac{\partial}{\partial H'} \int \frac{L'^3}{\mu^2} F_{1p} \, d\ell \qquad (5.73)$$

$$= -\frac{1}{n_o} \int \frac{\partial F_{1p}}{\partial H'} \, d\ell \, .$$

In the previous equations F_{1p} , the portion of F_1 that is purely periodic in ℓ , is taken to be a function of L' , G' , H' , ℓ , and g .

Consider, for example, the calculation of the short period perturbations in the semi-major axis. From Brouwer's solutions and the definition of the Delaunay variable L ,

$$a = \frac{L^2}{\mu} = \frac{1}{\mu}(L' + \Delta L_{sp})^2 + 0(k_2{}^2)$$

$$= \frac{L'^2}{\mu} + \frac{2L'}{\mu} \Delta L_{sp} + 0(k_2{}^2) \, .$$

Ignoring second order terms,

$$a = a' + \frac{2L'}{\mu} \Delta L_{sp} = a' + \Delta a_{sp} \, .$$

Now, from Eq. (5.68) and the definition of F_1 ,

$$\frac{2L'}{\mu} \Delta L_{sp} = \frac{2L'}{\mu n_o} \int \frac{\partial F_{1p}}{\partial \ell} \, d\ell = \frac{2L'}{\mu n_o} \int \frac{\partial R_p}{\partial \ell} \, d\ell$$

$$= \frac{1}{n_o} \int \frac{2}{n_o a'} \left(\frac{\partial R}{\partial \ell}\right)_p d\ell$$

and therefore

$$\Delta a_{sp} = \frac{1}{n_o} \int \frac{2}{n_o a'} \left(\frac{\partial R}{\partial \ell}\right)_p d\ell \, . \qquad (5.74)$$

The quantity R_p , which can be taken to be a function of a' , e' , I' , ℓ , and g , is the portion of the disturbing function that is purely periodic in ℓ and is given by

$$R_p(a',e',I',g,\ell) = R(a',e',I',g,\ell) - R_s(a',e',I') .$$

Through first order, a', e', and I' in Eq. (5.74) can be replaced by $\bar{a}$, $\bar{e}$, and $\bar{I}$, respectively. Then,

$$\begin{aligned}\Delta a_{sp} &= \frac{1}{\bar{n}} \int \frac{2}{\bar{n}\,\bar{a}} \left(\frac{\partial R}{\partial \ell}\right)_p d\ell \\ &= \frac{1}{\bar{n}} \int \frac{2}{\bar{n}\,\bar{a}} \left[\frac{\partial}{\partial M} R(\bar{a},\bar{e},\bar{I},\omega,M)\right]_p dM .\end{aligned}$$

But

$$\frac{dM}{\bar{n}} = \frac{dM}{\hat{n}} + O(k_2{}^2) = dt + O(k_2{}^2)$$

so that, still through first order,

$$\Delta a_{sp} = \int \frac{2}{\bar{n}\,\bar{a}} \left(\frac{\partial R}{\partial M}\right)_p dt$$

where the integration is performed by holding all variables constant except M which is given by

$$M = \ell''_o + \hat{n}t = \bar{M} + \hat{n}t + O(k_2) .$$

The variable ω in the result can be replaced by

$$\omega = \bar{\omega} + \dot{\omega}_s t$$

with an error no larger than second order. Thus, the short period terms in Brouwer's solution for the semi-major axis can be obtained by integrating with respect to time the portion of the right hand side of the equation for $\dot{a}$ that is purely periodic in ℓ with a, e, and I replaced by $\bar{a}$, $\bar{e}$, and $\bar{I}$, respectively, ω treated as a constant, and M being given by

$$M = \overline{M} + \hat{n}t .$$

Also, ω is approximated by

$$\omega = \overline{\omega} + \dot{\omega}_s t$$

in the result. This process is equivalent to the procedure used by Ingram through first order since his allowing ω to vary secularly during the integration introduces terms of the second order.

As a second example, consider Eq. (5.71) giving the short period terms in the mean anomaly. Introducing a' and R_p as above, the equation becomes

$$\Delta\ell_{sp} = -\frac{3\mu}{2L'^2}\int \Delta a_{sp}\, d\ell - \frac{1}{n_o}\int \frac{\partial R_p}{\partial L'}\, d\ell .$$

But

$$\frac{3\mu}{2L'^2} = \frac{3}{2a'}$$

and

$$\frac{\partial}{\partial L'} R_p(a',e',I',g,\ell) = \frac{\partial R_p}{\partial a'}\frac{\partial a'}{\partial L'} + \frac{\partial R_p}{\partial e'}\frac{\partial e'}{\partial L'}$$

$$= \frac{2}{n_o a'}\left(\frac{\partial R}{\partial a'}\right)_p + \frac{1-e'^2}{n_o a'^2 e'}\left(\frac{\partial R}{\partial e'}\right)_p$$

so that

$$\Delta\ell_{sp} = -\frac{3n_o}{2a'}\cdot\frac{1}{n_o}\int \Delta a_{sp}\, d\ell$$

$$-\frac{1}{n_o}\int\left[\frac{2}{n_o a'}\left(\frac{\partial R}{\partial a'}\right)_p + \frac{1-e'^2}{n_o a'^2 e'}\left(\frac{\partial R}{\partial e'}\right)_p\right] d\ell .$$

Changing from primed variables to barred variables, replacing n_o in some cases with $\hat{n}$, and using M in place of ℓ to designate the mean anomaly, the result through first order is

$$\Delta M_{sp} = -\frac{3\bar{n}}{2\bar{a}} \cdot \frac{1}{\hat{n}} \int \Delta a_{sp}\, dM$$
$$- \frac{1}{\hat{n}} \int \left[\frac{2}{\bar{n}\,\bar{a}} \left(\frac{\partial R}{\partial \bar{a}} \right)_p + \frac{1-\bar{e}^2}{\bar{n}\,\bar{a}^2\,\bar{e}} \left(\frac{\partial R}{\partial \bar{e}} \right)_p \right] d\ell \, .$$

Introducing t as the variable of integration with the aid of the relation

$$M = \bar{M} + \hat{n}t + O(k_2)$$

gives

$$\Delta M_{sp} = \int \frac{\partial \bar{n}}{\partial \bar{a}}\, \Delta a_{sp}\, dt - \int \left[\frac{2}{\bar{n}\,\bar{a}} \left(\frac{\partial R}{\partial \bar{a}} \right)_p + \frac{1-\bar{e}^2}{\bar{n}\,\bar{a}^2\,\bar{e}} \left(\frac{\partial R}{\partial \bar{e}} \right)_p \right] dt \, .$$

It is clear that integrating this equation as before and letting

$$\omega = \bar{\omega} + \dot{\omega}_s t$$

in the result will yield short period terms through first order that are equal to those derived by Ingram.

Applying the above procedure to the short period terms in the other variables will lead to the same conclusion. That is, the short period perturbations given by Brouwer and Ingram are equivalent through first order. However, Ingram's solutions are in the form of truncated power series whereas Brouwer's are in closed form. Thus, even though the results are the same, Brouwer's short period solutions are probably more efficient from a computational standpoint.

The Long Period Terms

Since Brouwer's S_1^* is a function of L'' , G'' , H'' , and g' , all of the variables except L and H will contain long period terms. Therefore, of the variables employed in Sec. V.2, only the semi-major axis will not contain long period terms of the first order in k_2. However, from Eqs. (5.14) through (5.19), none of the first order solutions derived by Ingram's approach have long period terms. This fact is not surprising when it is recalled that the first order long period portions of Brouwer's solutions result from the interactions of the first order short period perturbations; such interactions are not taken into account in Ingram's approach.

Summary

The following conclusions about the solutions presented in Secs. V.2 and V.3 have been reached.

1. The secular terms in Ingram's solutions are equivalent to the first order secular terms in Brouwer's solution. However, Brouwer's secular terms are expected to be more accurate since they are given through second order.
2. The short period terms in the two solutions are equivalent through first order. Brouwer's solution is probably computationally more efficient because it is in closed form.
3. First order long period terms are absent from Ingram's solutions.

One additional comment must be made about the relation between the short period terms in the two solutions. It can be verified that the short period terms in the solution obtained in Sec. V.2 are purely periodic in the mean anomaly (i.e., the average of the short period terms with respect to M is zero) while the short period terms obtained by Brouwer have a non-zero average value. This occurs because changing the variable of integration from ℓ to f in the process of evaluating S_1 produces a term in S_1

that is a function of L' , G' , H' , and g only. However, the resulting terms in Brouwer's short period solutions are canceled by like terms arising in the long period solutions so that the net result is short period terms in Brouwer's solution that are purely periodic in ℓ and equal to the corresponding terms in Ingram's solution through first order.

V.5 Critical Inclination and Resonance

As indicated in Sec. III.3, the phenomena of resonance can lead to difficulties in the development of general perturbations solutions. In artificial Earth satellite theories, this problem is encountered in the study of motion near the critical inclination where, as shown in the previous section, a small divisor appears in the long period terms in Brouwer's solution, and in the study of the motions of synchronous and repeating groundtrack satellites when the tesseral harmonics are included in the Earth's potential.

The mathematical and physical implications of resonance are clearly demonstrated in Brown's discussion (Ref. 21) of the motion of a simple pendulum. Furthermore, this elementary example is important because the equations describing the resonance motions of artificial Earth satellites often reduce to the simple pendulum equation (see Refs. 39, 40, and 41). These reasons provide the motivation for the following analysis of the pendulum problem.

Let the pendulum consist of a particle attached to one end of a weightless rod, the other end of the rod being connected to a horizontal axis about which the rod rotates without friction in a vertical plane. If x is the angle between the rod and a vertical line drawn in the direction of the uniform gravitational acceleration, the equation of motion can be put in the form

$$\ddot{x} + \alpha^2 \sin x = 0 .$$

Multiplying this equation by $\dot{x}$ and integrating the result gives the energy equation

$$\dot{x}^2 = C + 2\alpha^2 \cos x \tag{5.75}$$

where C is a constant that is determined from the initial conditions. An examination of the consequences of assigning various values to the energy constant C leads to the separation of the problem into three cases. They are:

1. If $C > 2\alpha^2$, $\dot{x}^2$ is non-zero for all values of x. Thus, the velocity never changes sign and $|x|$ always increases (i.e., the pendulum makes complete revolutions about its support). The resulting motion is called circulation.
2. If $C = 2\alpha^2$, the motion is called asymptotic. It will be shown that the pendulum approaches the unstable equilibrium point $x = \pm\pi$ as $t \to \pm\infty$.
3. If $-2\alpha^2 < C < 2\alpha^2$, there are two values of x in the range $-\pi \leq x < \pi$ for which $\dot{x}$ becomes zero. The maximum value of $|x|$ is bounded and the pendulum experiences an oscillatory motion (called libration) about the stable equilibrium point $x = 0$.

Each of these cases will now be considered in detail.

<u>Case 1</u>: Circulatory Motion

With $C > 2\alpha^2$, Eq. (5.75) yields

$$\begin{aligned}\dot{x} &= (C + 2\alpha^2 \cos x)^{\frac{1}{2}} \\ &= \sqrt{C + 2\alpha^2}\ \left(1 - k^2 \sin^2 \frac{x}{2}\right)^{\frac{1}{2}}\end{aligned} \tag{5.76}$$

where

$$k^2 = \frac{4\alpha^2}{C+2\alpha^2} < 1 . \tag{5.77}$$

The solution of Eq. (5.76) can be represented by an elliptic integral of the first kind in the form

$$t - t_0 = \frac{2}{\sqrt{C+2a^2}} \int_0^{\frac{x}{2}} \frac{d\tau}{(1-k^2 \sin^2 \tau)^{\frac{1}{2}}} . \tag{5.78}$$

A new constant n is introduced by the equation

$$\frac{2\pi}{n} = \int_0^{2\pi} \frac{dx}{(C+2\alpha^2 \cos x)^{\frac{1}{2}}} = \frac{4K}{\sqrt{C+2\alpha^2}} \tag{5.79}$$

where K is the complete elliptic integral of the first kind. Using Eq. (5.79) to write C in Eq. (5.78) in terms of n yields

$$\frac{1}{2}\sqrt{C+2\alpha^2}\ (t-t_0) = \frac{K}{\pi}(nt+\beta) = \int_0^{\frac{x}{2}} \frac{d\tau}{(1-k^2 \sin^2 \tau)^{\frac{1}{2}}} .$$

Therefore, the solution of Eq. (5.76) becomes

$$\sin \frac{x}{2} = \mathrm{sn}\, \frac{K}{\pi}(nt+\beta)$$

or

$$x = 2\ \mathrm{am}\, \frac{K}{\pi}(nt+\beta) . \tag{5.80}$$

The function am is the amplitude of the elliptic integral of the first kind. For the purposes of this discussion, the solution will be written as

$$x = nt + \beta + 4 \sum_{s=1}^{\infty} \frac{q^s}{s(1+q^{2s})} \sin s(nt+\beta) \qquad (5.81)$$

(see Ref. 42). The coefficients of the trigonometric terms in this equation can be expressed as power series in $\frac{\alpha^2}{n^2}$ by an iterative technique which is outlined below.

The first step in the evaluation of the coefficients involves using the series expansion of q in powers of k^2 (Ref. 42) to write each coefficient as a power series in k^2. For example, for $s = 1$ the result is

$$\frac{4q}{1+q^2} = \frac{k^2}{4} \left[1 + 8\left(\frac{k^2}{16}\right) + 83\left(\frac{k^2}{16}\right)^2 + \ldots \right] . \qquad (5.82)$$

Next, the modulus k must be expressed in terms of α^2 and n^2. From Eqs. (5.77) and (5.79),

$$k^2 = \frac{\alpha^2}{n^2} \cdot \frac{\pi^2}{K^2} .$$

But the quantity $\frac{\pi^2}{K^2}$ also can be expanded into a power series in k^2. Thus, k^2 can be obtained as a power series in $\frac{\alpha^2}{n^2}$ by solving the equation

$$k^2 = \frac{4\alpha^2}{n^2} \left(1 - \frac{1}{2} k^2 - \frac{3}{32} k^4 - \ldots\right)$$

by iteration. Substituting the result into Eq. (5.82) gives

$$\frac{4q}{1+q^2} = \frac{\alpha^2}{n^2} - \frac{5}{16} \frac{\alpha^6}{n^6} - \ldots .$$

Continuing the evaluation of the coefficients finally leads to the following form of the solution for circulatory motion.

$$x = nt + \beta + (\frac{\alpha^2}{n^2} - \frac{5}{16}\frac{\alpha^6}{n^6} - \ldots)\sin(nt+\beta)$$
$$+ (\frac{1}{8}\frac{\alpha^4}{n^4} - \frac{35}{16}\frac{\alpha^8}{n^8} - \ldots)\sin 2(nt+\beta) \qquad (5.83)$$
$$+ (\frac{1}{48}\frac{\alpha^6}{n^6} - \ldots)\sin 3(nt+\beta) + \ldots$$

The two constants of integration in the solution are n and β. Note that the form of this solution closely resembles the forms of the solutions for Ω, ω, and ℓ in the Earth satellite problem.

Small divisors are encountered in this solution if the value of C is allowed to approach $2\alpha^2$ since k^2 will approach unity and the complete elliptic integral of the first kind will become unbounded. Equation (5.79) indicates that n will approach zero under these circumstances and therefore the coefficients of the trigonometric terms in the solution will become very large. The appearance of a small divisor in the previous solution leads directly to consideration of the problem with $C = 2\alpha^2$.

Case 2: Asymptotic Motion ($C = 2\alpha^2$)

For asymptotic motion, the expression for $\dot{x}$ becomes

$$\dot{x} = \sqrt{2\alpha^2}\ (1 + \cos x)^{\frac{1}{2}} = 2\alpha\ \left|\cos\frac{x}{2}\right| . \qquad (5.84)$$

Since $\dot{x}$ and $\ddot{x}$ both become zero at $x = \pm\pi$, the value of $|x|$ cannot exceed π and it follows that

$$\left|\cos\frac{x}{2}\right| = \cos\frac{x}{2} .$$

The solution of Eq. (5.84) is

$$t - t_o = \frac{1}{\alpha} \int_o^{\frac{x}{2}} \frac{d\tau}{\cos \tau}$$

$$= \frac{1}{\alpha} \ln \tan \left(\frac{\pi}{4} + \frac{x}{4}\right)$$

or

$$x = 4 \tan^{-1} [\exp (\alpha t+\gamma)] - \pi . \tag{5.85}$$

Clearly, the pendulum approaches the unstable equilibrium point $x = \pm\pi$ as $t \to \pm\infty$. The form of the solution is quite different from the form obtained for circulatory motion.

Case 3: Libration

The pendulum librates when $|C| < 2\alpha^2$. As before, the equation of motion is

$$\dot{x} = (C + 2\alpha^2 \cos x)^{\frac{1}{2}} .$$

This equation indicates that the velocity $\dot{x}$ becomes zero when

$$\cos x = - \frac{C}{2\alpha^2} . \tag{5.86}$$

If the values of x satisfying Eq. (5.86) are denoted by $\pm\phi$, the motion of the pendulum is bounded by $x = \pm\phi(|\phi| < \pi)$ and, in addition,

$$C = -2\alpha^2 \cos \phi .$$

The solution of the equation of motion can then be written as

$$\begin{aligned} t + \text{cons.} &= \int \frac{dx}{(-2\alpha^2 \cos \phi + 2\alpha^2 \cos x)^{\frac{1}{2}}} \\ &= \frac{1}{2\alpha} \int \frac{dx}{(\sin^2 \frac{\phi}{2} - \sin^2 \frac{x}{2})^{\frac{1}{2}}} . \end{aligned} \tag{5.87}$$

The integral on the right hand side of the previous equation can be transformed into an elliptic integral of the first kind by introducing the new variable ψ defined by

$$\sin \frac{\phi}{2} \sin \psi = \sin \frac{x}{2} .$$

Equation (5.87) becomes

$$t + \text{cons.} = \frac{1}{\alpha} \int \frac{d\psi}{(1 - \sin^2 \frac{\phi}{2} \sin^2 \psi)^{\frac{1}{2}}}$$

and the solution is

$$\sin \psi = \text{sn}\ \alpha(t + \text{cons.})$$

or

$$x = 2 \sin^{-1} k \,[\text{sn}\ \alpha(t + \text{cons.})] \tag{5.88}$$

with $k = \sin \frac{\phi}{2}$.

From Ref. 42,

$$\text{sn}\ \alpha(t+\text{cons.}) = \frac{2\pi}{kK} \sum_{s=0}^{\infty} \frac{q^{s+\frac{1}{2}}}{1-q^{2s+1}} \sin\left[(2s+1) \frac{\pi\alpha}{2K} (t+\text{cons.})\right] .$$

Defining the constant p by the equation

$$\frac{\pi}{p} = \int_0^{\pi} \frac{d\psi}{\alpha(1-k^2 \sin^2 \psi)^{\frac{1}{2}}} = \frac{2K}{\alpha}$$

yields

$$p = \frac{\pi\alpha}{2K} \tag{5.89}$$

and therefore

$$\text{sn}\ \alpha(t+\text{cons.}) = \frac{2\pi}{kK}\sum_{s=0}^{\infty}\frac{q^{s+\frac{1}{2}}}{1-q^{2s+1}}\sin(2s+1)(pt+\delta)$$

where δ is a constant. The quantities q and K in this equation can be written as power series in k^2. Substituting the resulting expression for $\text{sn}\ \alpha(t + \text{cons.})$ into the series expansion for the $\sin^{-1}$ in Eq. (5.88) gives the solution for librational motion in the form

$$x = k\,a(k^2)\sin(pt+\delta) + k\,b(k^2)\sin 3(pt+\delta) + \ldots \tag{5.90}$$

where the coefficients a and b are power series in k^2. Note that, from Eq. (5.89), the frequency p is a function of the amplitude of oscillation ϕ through K.

The previous discussion has shown that a complete description of the motion of a simple pendulum requires three distinct solutions, only one of which is valid for a given set of initial conditions. The regions in which the various solutions apply can be represented graphically by considering the level curves in the phase plane (Fig. 25). Each curve is obtained by plotting, for a given C, values of x and $\dot{x}$ satisfying the energy equation

$$\dot{x}^2 = C + 2\alpha^2\cos x\ .$$

Circulation occurs in the region in which $C > 2\alpha^2$. Since a level curve represents a path of the system in phase space, it is evident from the figure that $\dot{x}$ will oscillate about some mean value and $|x|$ will continuously increase in this region. The contours $C = 2\alpha^2$ enclose the libration region. Note that both x and $\dot{x}$ are bounded in librational motion and that the asymptotic solution separates the regions of libration and circulation.

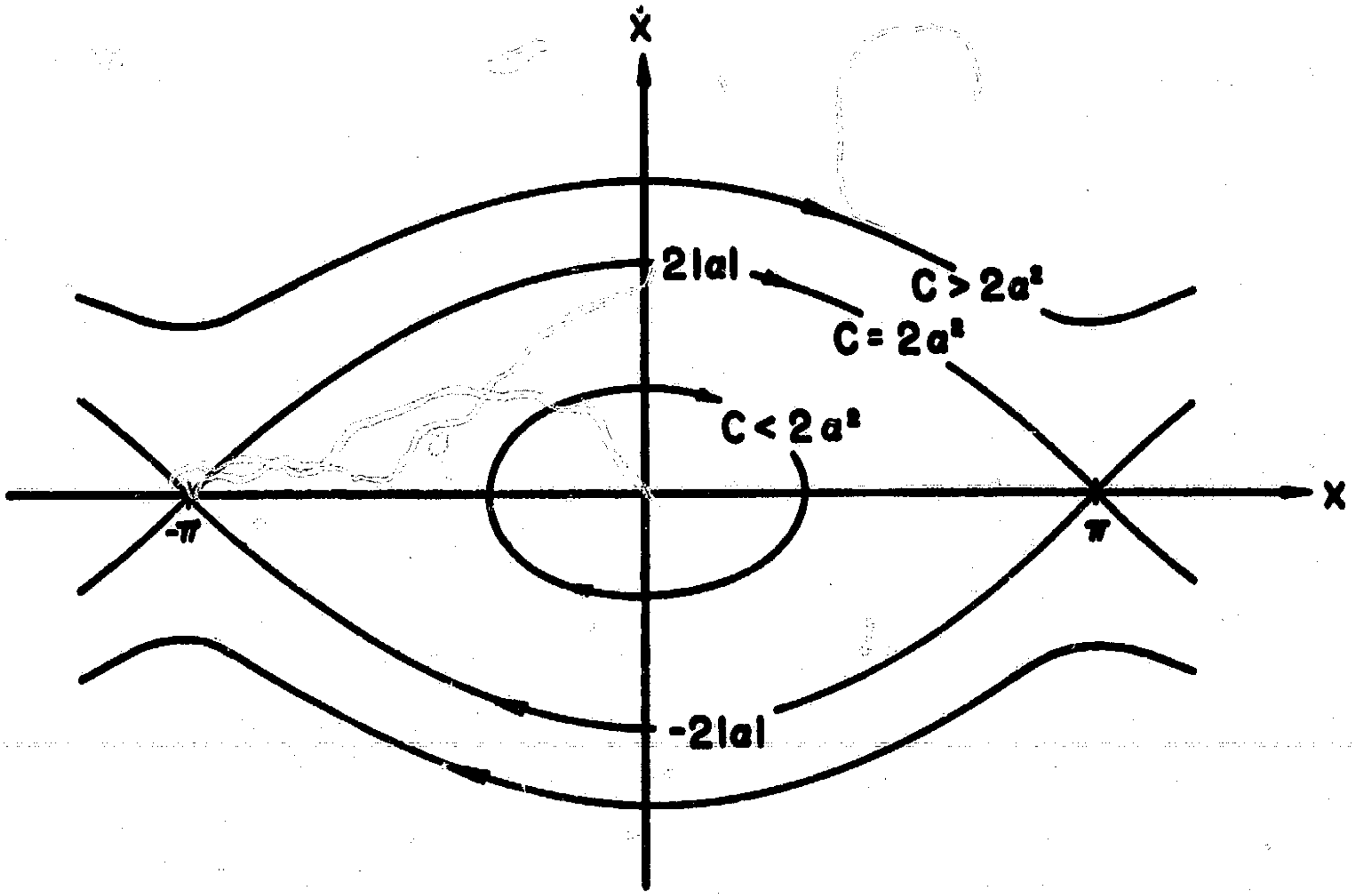

FIGURE 25. Level Curves for the Simple Pendulum

The analogy between the motion of a simple pendulum and the motion of an artificial Earth satellite can be demonstrated by considering the level curves of the Hamiltonian obtained in Brouwer's solution by the elimination of the short period terms (Ref. 20). Dropping primes, this Hamiltonian through second order is

$$F^*(L,G,H,g) = F_o^*(L) + \frac{\mu^4 k_2}{L^3 G^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right) + k_2^2 P(L,G,H) + k_2^2 \overline{Q}(L,G,H) \cos 2g$$

where L and H are constants. The general characteristics of the level curves of F^* are approximated by the level curves of the function $\tilde{F}$ defined by

$$\tilde{F} = \frac{1}{G^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right) + k_2\, Q(G) \cos 2g \ . \qquad (5.91)$$

In this equation, the function $\overline{Q}(L,G,H)$ is denoted by $Q(G)$ for convenience. Since

$$\frac{\partial \tilde{F}}{\partial G} = \frac{3}{2G^4}\left(1 - 5\frac{H^2}{G^2}\right) + k_2 \frac{dQ}{dG} \cos 2g \ ,$$

$\tilde{F}$ has an approximate extremum with respect to G at $\frac{H^2}{G^2} = \frac{1}{5}$ (i.e., at the critical inclination). This extremum is a minimum because

$$\left.\frac{\partial^2 \tilde{F}}{\partial G^2}\right|_{\frac{H^2}{G^2} = \frac{1}{5}} \approx \frac{3}{G^5} > 0 \ .$$

Also,

$$\frac{\partial \tilde{F}}{\partial g} = -2k_2\, Q(G) \sin 2g$$

so that $\tilde{F}$ is extremized with respect to g at $g = \frac{n\pi}{2}$ $(n = 0,\pm1,\pm2,\ldots)$. It is found by direct calculation that $Q(G) > 0$ in the vicinity of the critical inclination when the effects of the second and fourth zonal harmonics of the Earth are included in the analysis. Therefore, the maxima of $\tilde{F}$ with respect to g occur at $g = m\pi$ and the minima occur at $g = (2m+1)\frac{\pi}{2}$ $(m = 0, \pm1,\pm2,\ldots)$.

The properties of $\tilde{F}$ enumerated above indicate that the surface described by Eq. (5.91) when $\tilde{F}$, G, and g are plotted along the axes of an orthogonal Cartesian coordinate system will have absolute minima at $G = \sqrt{5}\,H$, $g = \ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ and saddles at $G = \sqrt{5}\,H$, $g = \ldots, -\pi, 0, \pi, \ldots$. The level curves of $\tilde{F}$ are sketched in Fig. 26. Clearly, these curves are very similar to those obtained in the analysis of the simple pendulum problem. The non-resonant solutions discussed in Secs. V.2 and V.3 are valid outside of the contours passing through $(G,g) = (\sqrt{5}\,H, 0)$ and therefore represent circulatory motion (i.e., $|g|$ increases with time). Within these contours, the satellite experiences libration and special techniques must be used to develop a solution.

The material in this section has been presented to illustrate the physical and mathematical significance of the phenomena of resonance and no attempt has been made to investigate methods that can be used to generate general perturbations solutions describing resonant motion. A discussion of a formal solution of the resonance problem may be found in a paper by Garfinkel (Ref. 41). Resonant motions of Earth satellites have been studied by many authors. Solutions for the motion in the vicinity of the critical inclination have been developed by Hori (Ref. 43) and Garfinkel (Ref. 40) and extensive studies of the critical inclination problem for the case of small eccentricity have been

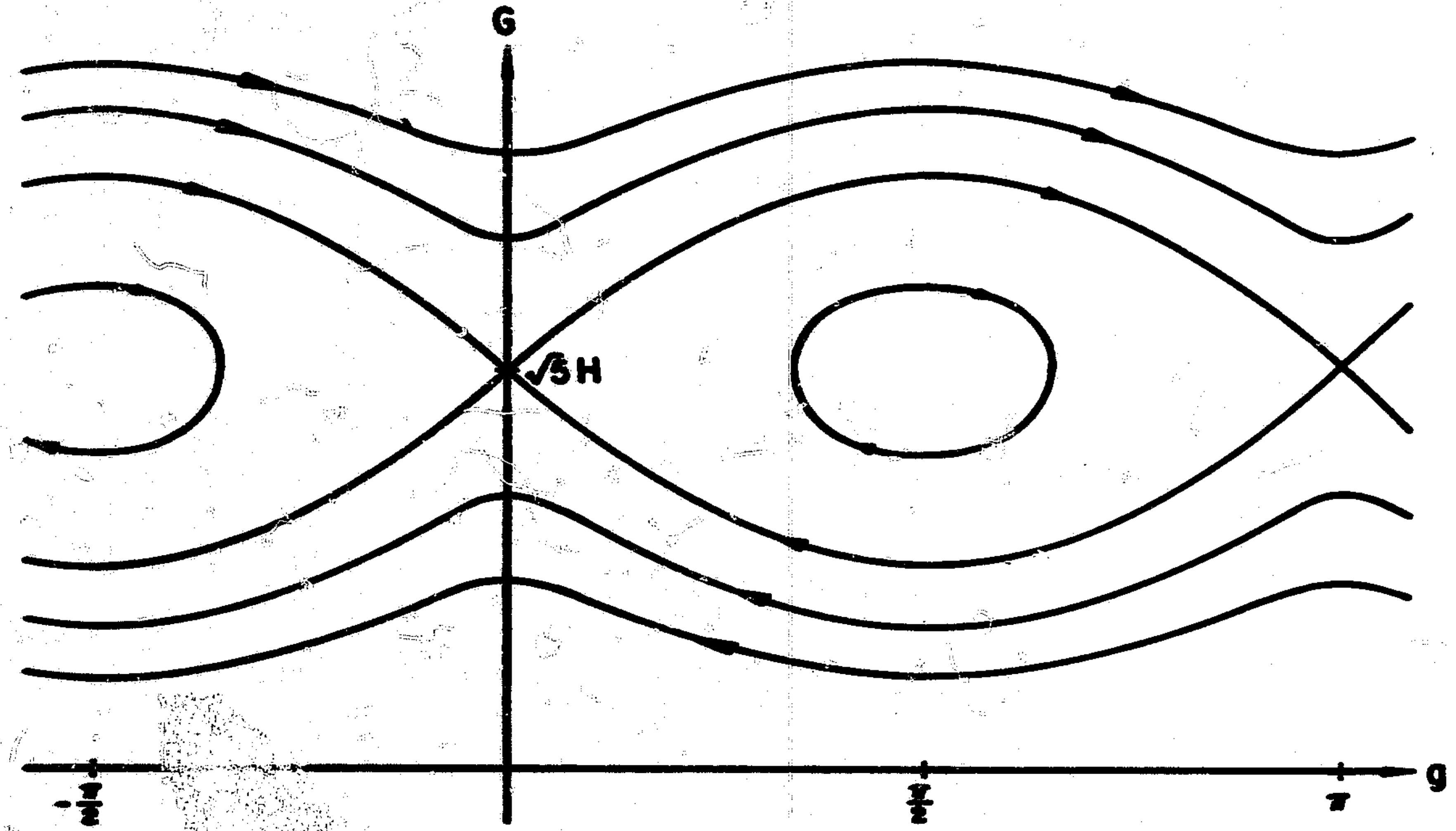

FIGURE 26. Level Curves for $\tilde{F}$

made by Izsak (Ref. 44) and Aoki (Refs. 45 and 46). The effects of resonance with the tesseral harmonics of the Earth's gravitational field on the motions of synchronous and repeating groundtrack satellites are analyzed in Refs. 6 and 39, respectively. Vagners (Ref. 25) investigates the problem of resonance with the tesseral harmonics by first determining the solutions for circular orbits and then perturbing these solutions with the eccentricity as a small parameter.

CHAPTER VI

COMMENTS ON THE LUNAR SATELLITE PROBLEM

The success achieved in the application of the von Zeipel method to the artificial Earth satellite problem encouraged several investigators to employ the technique in attempts to develop theories describing the motions of artificial Lunar satellites (Refs. 47, 48, 49, 50, and 25). All of these theories were generated before the knowledge of the Moon's gravitational field obtained from studies of the trajectories of spacecraft in the Lunar Orbiter series was available. Since this field has been found to be more complex than had been anticipated, possibly requiring harmonics through eleventh order in the potential for an adequate description (Ref. 51), the models used in the analyses lacked the completeness required to yield results of the desired accuracy. More important to the discussion here, however, is the failure of the von Zeipel method to provide a complete analytical solution to the problem even for the simplified mathematical models used in the studies cited above.

The basic difference between the Earth and Lunar satellite problems is the relative magnitudes of the perturbing terms. As the discussion in Sec. V.1 indicated, the dominant perturbations experienced by a near Earth satellite are due to the second zonal harmonic J_{20}, all other perturbing terms being of second order in this quantity. For the Moon, however, the higher harmonics in the central body potential as well as the terms arising from the third body effect of the Earth are of the same order as J_{20}. Thus, the orders of magnitude associated with the terms in the Hamiltonian for a Lunar satellite are quite different from those in the Earth satellite problem.

Still, some progress toward an analytical solution can be made. With the aid of the von Zeipel method, the short period perturbations (periodic in the mean anomaly) and medium period perturbations (terms with periods of approximately one month that result from the tesseral harmonics and the third body potential) can be determined (Ref. 25). At this point, as in the solution of the Earth satellite problem, the new Hamiltonian is essentially a function of the Delaunay variables G and g. Unfortunately, it is not in the form required for the calculation of the long period terms by the von Zeipel method. In fact, according to Kozai (Ref. 47), Giacaglia (Ref. 48), and Oesterwinter (Ref. 49), the reduced problem can not be solved by any method of successive approximations. Giacaglia succeeds in developing an approximate solution by using a special technique. Without exception, the authors of the other theories either compute level curves or recommend solving the second order system by numerical integration. The latter process is expected to be very efficient because the short and medium period terms have been eliminated and integration steps on the order of a day may be possible (Ref. 50).

Therefore, at the present time, most of the available Lunar satellite theories are semi-analytical in form. Since only the expressions for the short and medium period perturbations have been obtained analytically, the characteristics of the long term motions of Lunar satellites are not well known.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

A brief review of some topics from celestial mechanics that are pertinent to the study of the artificial Earth satellite problem was given in Chapter I. Two general perturbations techniques were then introduced and used to generate approximate solutions for the motion of a nonlinear spring. Some of the conclusions arising from a comparison of these approximate solutions with numerically integrated trajectories were:

1. The primary sources of error in the approximate solutions are the neglected higher order secular terms. Ignoring these terms not only produces a secular error in the angular variable, but also produces a large error in the generalized momentum over long time periods by causing a phase shift between the approximate and the numerically integrated trajectories.
2. If the time interval is short enough that the phase error is small and if the perturbing forces are sufficiently small, the errors in the first order solutions can be approximated by the magnitudes of the second order terms.

The application of the general perturbations methods to the example problem mentioned above helped to prepare for the more difficult analysis of artificial satellite motion. Chapter V was devoted to the formulation of the artificial Earth satellite problem and to a discussion of the material of Refs. 5 and 7 concerning the determination of the effects of the

Earth's oblateness on the motion of a satellite. The following well-known characteristics of the solutions were noted.

1. For non-resonant motion, the solutions for the osculating elements a , e , and I consist of periodic terms with short and long periods, while the solutions for the quantities Ω , ω , and M include in addition terms that are linear in time.
2. The second order terms in the Hamiltonian yield first order long period terms in the solutions.
3. The appearance of a small divisor indicates that the series solution is not valid near the critical inclination so that other techniques must be employed to obtain a solution in this region.
4. Brouwer's solution for the secular rate in the mean anomaly must be modified to yield the correct value through second order when using the theory to predict forward from an initial point.

A comparison of Brouwer's solution for the effects of oblateness with those derived by using Ingram's approach was also given in this chapter. The important conclusions resulting from this analysis are:

1. Brouwer's solution is probably the most efficient computationally since it is in closed form in the mean anomaly while Ingram's is in the form of a series in the angular variables ω and M with coefficients that are infinite series in the eccentricity.
2. The short period and secular terms in the two solutions are equivalent through first order. Brouwer's secular terms are expected to be more accurate, however, since they are given through second order.
3. The mean motion used by Ingram is accurate only through first order in J_{20} .

4. Ingram's solutions do not contain the first order long period terms which arise from the interactions of the first order short period oblateness effects.

Finally, the possibility of applying the von Zeipel method to the Lunar satellite problem was briefly considered in Chapter VI. It was mentioned that this technique has been used to obtain analytical representations of the short and medium period motions, but that at present the long period motion must be determined from the level curves or by numerical integration.

Recommendations for Further Study

Many relevant topics have not been considered or have been mentioned only briefly. Some of the areas that merit further study are:

1. Methods for including effects that have been neglected here, such as atmospheric drag, in the solutions.
2. Procedures for studying motion in resonance regions.
3. Techniques for deriving error bounds on the solutions (see Ref. 34, for example). It would be desirable to determine the time intervals over which the accuracy of the approximate solutions is adequate by methods other than direct comparison with numerically integrated trajectories.
4. Comparison of the first order long period terms in the solution derived by Kaula (Ref. 6) with those developed by Brouwer (Ref. 5), Giacaglia (Ref. 36), and Garfinkel (Ref. 37) using the von Zeipel method.
5. A more extensive comparison of Ingram's solutions for artificial Earth satellite motion with those obtained with the von Zeipel

method. Only the solutions for the oblateness effects have been discussed here. Additional work has shown that Ingram's solutions for the first order long period effects of zonal harmonics beyond the second are not complete. The completeness of the first order long period solutions describing the effects of the tesseral harmonics and the third body remains to be determined.

6. The question of the validity of using Ingram's solutions to predict Lunar satellite motions. The short period and intermediate period terms could be compared with results obtained by applying the von Zeipel method. The regions of the motion and time intervals for which the long period and secular terms yield accurate results might be determined by an analysis of the level curves describing the long term motion of the system. Indications are that these solutions will give accurate results over long periods of time only for very restricted sets of initial conditions.

APPENDIX A

Derivations of $\frac{df}{dM}$ and $\frac{df}{de}$

From Eq. (2.3) the relation between the true anomaly f and the eccentric anomaly E is

$$\tan\frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan\frac{E}{2} \,. \tag{A-1}$$

Taking the total differential of both sides of this equation yields

$$\frac{1}{2}\sec^2\frac{f}{2}\,df = \tan\frac{E}{2}\,d\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \sec^2\frac{E}{2}\,dE \,. \tag{A-2}$$

Considering f to be a function of eccentricity and eccentric anomaly, and E to be a function of eccentricity and mean anomaly,

$$\begin{aligned} df &= \frac{\partial f}{\partial e}\,de + \frac{\partial f}{\partial E}\,dE \\ &= \frac{\partial f}{\partial e}\,de + \frac{\partial f}{\partial E}\left(\frac{\partial E}{\partial e}\,de + \frac{\partial E}{\partial M}\,dM\right) \end{aligned}$$

or

$$df = \left(\frac{\partial f}{\partial e} + \frac{\partial f}{\partial E}\frac{\partial E}{\partial e}\right)de + \frac{\partial f}{\partial E}\frac{\partial E}{\partial M}\,dM \,. \tag{A-3}$$

But from Eq. (A-2),

$$\begin{aligned} df = {} & \left[\frac{2\tan\frac{E}{2}}{\sec^2\frac{f}{2}}\,d\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} + \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \frac{\sec^2\frac{E}{2}}{\sec^2\frac{f}{2}}\frac{\partial E}{\partial e}\right] de \\ & + \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \frac{\sec^2\frac{E}{2}}{\sec^2\frac{f}{2}}\frac{\partial E}{\partial M}\,dM \,. \end{aligned} \tag{A-4}$$

Thus,

$$\frac{df}{dM} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \frac{\sec^2 \frac{E}{2}}{\sec^2 \frac{f}{2}} \frac{\partial E}{\partial M} . \tag{A-5}$$

Now,

$$\sec^2 \frac{E}{2} = 1 + \tan^2 \frac{E}{2} = 1 + \frac{1-e}{1+e} \tan^2 \frac{f}{2}$$

$$= \frac{2(1+e \cos f)}{(1+e)(1+\cos f)}$$

and

$$\sec^2 \frac{f}{2} = 1 + \tan^2 \frac{f}{2} = \frac{2}{1+\cos f} .$$

Eq. (A-5) becomes

$$\frac{df}{dM} = \frac{1+e \cos f}{\sqrt{1-e^2}} \frac{\partial E}{\partial M} = \frac{a}{r}\sqrt{1-e^2} \frac{\partial E}{\partial M} . \tag{A-6}$$

From Kepler's equation (Eq. (2.4)),

$$dM = (1-e \cos E)dE - \sin E \, de$$

or

$$dE = \frac{dM}{1-e \cos E} + \frac{\sin E}{1-e \cos E} de$$

so that, using Eq. (2.1),

$$\frac{\partial E}{\partial M} = \frac{1}{1-e \cos E} = \frac{a}{r} .$$

Therefore, the expression for $\frac{df}{dM}$ becomes

$$\frac{df}{dM} = \frac{a^2}{r^2}\sqrt{1-e^2}\ .$$

For the second derivation, from Eq. (A-1),

$$\frac{1}{2}\sec^2\frac{f}{2}\ \frac{\partial f}{\partial e} = \frac{1}{2}\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}\frac{2}{(1-e)^2}\ \tan\frac{E}{2}$$

$$+\ \frac{1}{2}\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\sec^2\frac{E}{2}\ \frac{\partial E}{\partial e}\ . \qquad \text{(A-7)}$$

From Kepler's equation,

$$\frac{\partial E}{\partial e} - \sin E - e\cos E\,\frac{\partial E}{\partial e} = 0$$

or

$$\frac{\partial E}{\partial e} = \frac{\sin E}{1-e\cos E} = \frac{a}{r}\sin E$$

$$= 2\,\frac{a}{r}\sin\frac{E}{2}\cos\frac{E}{2}\ .$$

Eq. (A-7) becomes

$$\sec^2\frac{f}{2}\ \frac{\partial f}{\partial e} = \frac{2}{1-e^2}\tan\frac{f}{2} + \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\frac{2a}{r}\tan\frac{E}{2}$$

$$= 2\left(\frac{L^2}{G^2}+\frac{a}{r}\right)\tan\frac{f}{2}\ .$$

Finally,

$$\frac{\partial f}{\partial e} = \left(\frac{L^2}{G^2}+\frac{a}{r}\right)\sin f\ .$$

APPENDIX B

Legendre Polynomials and Associated Legendre Functions

The following relations were taken from Ref. 52. The Legendre polynomials may be computed from the equation

$$P_\ell(\nu) = \sum_{n=0}^{m} (-1)^n \frac{(2\ell-2n)!}{2^\ell n!(\ell-n)!(\ell-2n)!} \nu^{\ell-2n}$$

where m is $\frac{\ell}{2}$ or $\frac{\ell-1}{2}$, whichever is an integer, or from Rodrigue's formula

$$P_\ell(\nu) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{d\nu^\ell} (\nu^2-1)^\ell .$$

Also, they may be calculated recursively from the expression

$$(\ell+1)P_{\ell+1}(\nu) - (2\ell+1)\nu P_\ell(\nu) + \ell P_{\ell-1}(\nu) = 0 .$$

The first six polynomials are given below.

$P_o(\nu) = 1$ $\qquad$ $P_3(\nu) = \frac{1}{2}(5\nu^3-3\nu)$

$P_1(\nu) = \nu$ $\qquad$ $P_4(\nu) = \frac{1}{8}(35\nu^4-30\nu^2+3)$

$P_2(\nu) = \frac{1}{2}(3\nu^2-1)$ $\qquad$ $P_5(\nu) = \frac{1}{8}(63\nu^5-70\nu^3+15\nu)$

The associated Legendre functions are defined by

$$P_\ell^m(\nu) = (1-\nu^2)^{m/2} \frac{d^m}{d\nu^m} P_\ell(\nu)$$

with $m > 0$.

APPENDIX C

Inclination and Eccentricity Functions

The following expressions for the functions $F_{\ell mp}(I)$ and $G_{\ell pq}(e)$ appearing in Eq. (2.25) are taken from Ref. 3.

1. Inclination Function

$$F_{\ell mp}(I) = \sum_{t} \frac{(2\ell-2t)!}{t!(\ell-t)!(\ell-m-2t)!2^{2\ell-2t}} \sin^{\ell-m-2t} I$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^{s} I \sum_{c} \binom{\ell-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k}$$

In this equation, k is the integer part of $\frac{\ell-m}{2}$, t is summed from zero to the lesser of p or k, and c is summed over all values making the binomial coefficients non-zero. The binomial coefficients are defined by

$$\binom{m}{s} = \begin{cases} \frac{m!}{s!(m-s)!} & m \geq s \\ 0 & m < s \end{cases}$$

2. Eccentricity Function

$$G_{\ell pq}(e) = (-1)^{|q|} (1+\beta^2)^{\ell} \beta^{|q|} \sum_{k=0}^{\infty} P_{\ell pqk} Q_{\ell pqk} \beta^{2k}$$

The quantities β, $P_{\ell pqk}$, and $Q_{\ell pqk}$ are defined as follows:

$$\beta = \frac{e}{1 + \sqrt{1-e^2}}$$

$$P_{\ell pqk} = \sum_{r=0}^{h} \binom{2p'-2\ell}{h-r} \frac{(-1)^r}{r!} \left[\frac{(\ell-2p'+q')e}{2\beta}\right]^r$$

$$h = \begin{cases} k+q' & q' > 0 \\ k & q' < 0 \end{cases}$$

$$Q_{\ell pqk} = \sum_{r=0}^{h} \binom{-2p'}{h-r} \frac{1}{r!} \left[\frac{(\ell-2p'+q')e}{2}\right]^r$$

$$h = \begin{cases} k & q' > 0 \\ k-q' & q' < 0 \, . \end{cases}$$

In the previous equations,

$$p' = \begin{cases} p & p \le \frac{\ell}{2} \\ \ell-p & p > \frac{\ell}{2} \end{cases} \qquad q' = \begin{cases} q & p \le \frac{\ell}{2} \\ -q & p > \frac{\ell}{2} \, . \end{cases}$$

APPENDIX D

Spherical Harmonic Coefficients

Many different forms of the expansion of the potential of an arbitrary body in terms of spherical harmonics can be found in the literature, each form having its own set of coefficients. The table below gives the relations between the coefficients used in several of the references cited in this thesis.

Brouwer (Ref. 5)	$-\frac{2k_2}{a_e^2}$	$\frac{A_{3.0}}{a_e^3}$	$\frac{8k_4}{3a_e^4}$	$\frac{A_{5.0}}{a_e^5}$		
Garfinkel (Ref. 37)	$-J_2$	$-J_3$	$-J_4$	$-J_5$	$-J_{\ell m}\cos m\alpha_{\ell m}$	$J_{\ell m}\sin m\alpha_{\ell m}$
Giacaglia (Ref. 35)	$-J_2$	$-J_3$	$-J_4$	$-J_5$		
Ingram (Ref. 7)	J_{20}	J_{30}	J_{40}	J_{50}	$J_{\ell m}\cos m\lambda_{\ell m}$	$J_{\ell m}\sin m\lambda_{\ell m}$
Kaula (Ref. 3)	C_2	C_3	C_4	C_5	$C_{\ell m}$	$S_{\ell m}$
Kozai (Ref. 14)	$-\frac{2A_2}{3a_e^2}$	$\frac{A_3}{a_e^3}$	$\frac{8A_4}{35a_e^4}$			

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VITA

Claude Earl Hildebrand, Jr. was born at [redacted], on [redacted] [redacted] the son of Claude and Margaret Hildebrand. He graduated from the Comfort Public Schools in May of 1960 and entered the University of Texas at Austin in September of that year. Upon receiving the Bachelor of Science degree in aerospace engineering in January of 1965 he began his graduate study. He has held summer employment at the U.S. Naval Ordnance Laboratory in Silver Spring, Maryland, the Space and Information Systems Division of North American Aviation at Downey, California, and at the NASA Manned Spacecraft Center in Houston, Texas, and for the past two years has been a teaching assistant in the Department of Aerospace Engineering at the University of Texas at Austin. He is married to the former Mary Ann Seidensticker of Comfort, Texas, and is a member of Sigma Gamma Tau, Tau Beta Pi, and Sigma Xi.

Permanent address: P. O. Box 162
Comfort, Texas

This thesis was typed by Mrs. Mollie Schluter.