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ON FLOW PAST A CASCADE OF PARTIALLY CAVITATING CAMBERED BLADES

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NOTATION

A, B	Constants
a	Modulus of ζ_2
c	Chord length
C_L	Lift coefficient
C_D	Drag coefficient
d	Spacing between adjacent blades
i	$= \sqrt{-1}$
ℓ	Cavity length
p_1	Pressure at far upstream
p_c	Cavity pressure
s	Cavity length in the transformed plane
U_1	Flow velocity at far upstream
U_2	Flow velocity at far downstream
U_c	Cavity speed
u, v	Perturbed velocity components in the x- and y-direction respectively
w	$=(u - iv)/U_c$ complex velocity function
x, y	Cartesian coordinates of physical plane
y_f	Profile of the wetted blade surface
y_c	Cavity shape
α_1	Angle of attack
α_2	Flow angle with respect to x-direction at far downstream

γ	Stagger angle
ξ, η	Coordinates in the transformed plane
ζ	$= \xi + i\eta$
σ	Cavitation number
$R\ell$, Im	Denote the real and imaginary part of the designated quantity

INTRODUCTION

Many conditions of operation result in extensive cavitation within the rotors of turbomachineries. The flow past a supercavitating cascade of cambered blades has been discussed in some detail by Hsu (1969). The only available theoretical study on flow past a cascade of partially cavitating blades, that is, the cavity length is shorter than the chord length, is due to Wade (1963) who dealt with the problem of thin flat plate hydrofoils.

In the present report the problem of the flow of incompressible, inviscid fluid past a partially cavitating cascade of cambered blades is studied. The analysis is based on the assumptions that the flow is two-dimensional and that disturbances are small. With the aid of conformal transformation, it has been found possible to provide a simple method for calculating such complex flow problems.

MATHEMATICAL FORMULATIONS

Consider the flow schematically illustrated in Figure 1. The cascade consists of an infinite array of identical thin cambered blades having a stagger angle γ and an small angle of attack α_1. The mean chord length of each blade is c and the spacing of the blade in the direction of stagger angle is d.

The flow approaches the cascade with velocity U_1 and is turned by the aggregate effect of the cascade from its original horizontal direction to the direction α_2 and velocity U_2 at far downstream. The cavities of length $\ell < c$ are assumed to be thin and spring from the leading edge and terminate on the upper surface of each blade. As a first approximation the boundary conditions, applied along the x-axis, may be stated as follows:

On the wetted portion of the blade,

$$\frac{v}{U_c} = -\alpha_1 + \frac{dy_f}{dx} \qquad \begin{cases} \ell < x < c, \; y = 0^+ \\ 0 < x < c, \; y = 0^- \end{cases}$$

on the cavity,

$$\frac{u}{U_c} = 0 \qquad 0 < x < \ell, \; y = 0^+$$

at upstream infinity,

$$\frac{u}{U_c} = \frac{U_1}{U_c} - 1 = \frac{1}{\sqrt{1+\sigma}} - 1, \qquad \frac{v}{U_c} = 0$$

and the cavity closure condition,

$$\int_{\text{body}} \frac{dy}{dx} \, dx = 0$$

where

u, v = perturbed velocity component in the x- and y-axis respectively,

U_c = constant cavity speed = $\sqrt{1+\sigma}\, U_1$,

y_f = profile of the cambered blade,

σ = cavitation number = $\dfrac{p_1 - p_c}{\frac{1}{2}\rho U_1^{\,2}}$, a constant

p_1 = static pressure at upstream infinity,

p_c = constant cavity pressure, and

ρ = fluid density.

In order to find the perturbed velocity field

$$w = \frac{u}{U_c} - i\,\frac{v}{U_c}$$

it is expedient, first, to introduce a transformation function

$$z = x + iy = \frac{d}{2\pi}\left[e^{-i\gamma}\,\ell n\,\frac{1 - \dfrac{\zeta}{\zeta_1}}{1 - \dfrac{\zeta}{\zeta_2}} + e^{i\gamma}\,\ell n\,\frac{1 - \dfrac{\zeta}{\overline{\zeta_1}}}{1 - \dfrac{\zeta}{\overline{\zeta_2}}}\right] \qquad [1]$$

which maps the multiple-connected region in the z-plane onto the ζ-plane. The function has branch points at $\zeta_1 = e^{i(\pi/2-\phi)}$ and $\zeta_2 = a^{i(\pi/2+\phi)}$ corresponding to up and down-stream infinity respectively in the physical z-plane. By crossing the branch cut $\widehat{\zeta_1\zeta_2}$, that is proceeding from one blade to the next one, the value of $\log(1-\zeta/\zeta_1)$ or $\log(1-\zeta/\zeta_2)$ changes by $2\pi i$ or $-2\pi i$. The leading and trailing edges of the blade are mapped to the origin and to infinity. The problem is now reduced to that of finding a harmonic function $w(\zeta)$ in the ζ-plane which satisfies the following boundary conditions:

$$\left.\begin{array}{llll}
\text{on the wetted surface} & & & \\
-\text{Im } w = -\alpha_1 + \dfrac{dy_f}{dx} & \left\{\begin{array}{l} -\infty < \xi < 0 \\ s < \xi < \infty \end{array}\right. & & \eta = 0 \\
\text{on the cavity,} & & & \\
\text{R}\ell \; w = 0 & 0 < \xi < s & & \eta = 0 \\
\text{at the upstream infinity, i.e., at } \zeta = \zeta_1, & & & \\
\text{R}\ell \; w = \dfrac{1}{\sqrt{1+\sigma}} - 1 & \text{Im } w = 0 & &
\end{array}\right\} \qquad [2]$$

and the closure condition,

$$-\text{Im} \oint_{\text{body}} w(\zeta) \frac{dz}{d\zeta} \, d\zeta = 0 \qquad [3]$$

where $\text{R}\ell$ and Im denote the real and imaginary part of the designated quantity respectively.

The cascade parameters are determined by the following equations

$$\tan \phi = \frac{a-1}{a+1} \tan \gamma \qquad [4]$$

$$\frac{c}{d} = \frac{1}{\pi} (\cos \gamma \, \ell n \, a + 2\phi \sin \gamma) \qquad [5]$$

$$\frac{\ell}{d} = \frac{1}{\pi} \left\{ \frac{\cos \gamma}{2} \ell n \frac{1-2s \sin \phi + s^2}{1 + 2(s/a)\sin \phi + (s/a)^2} \right.$$

$$\left. + \sin \gamma \left[\tan^{-1} \frac{s \cos \phi}{1 - s \sin \phi} - \tan^{-1} \frac{(s/a)\cos \phi}{1 + (s/a)\sin \phi} \right] \right\} \qquad [6]$$

The value of the cavity length s in the transformed plane, and the scaling parameters, a and ϕ can be computed once the geometrical characteristics of cascade c/d, ℓ/d and γ are given.

SOLUTION OF THE BOUNDARY VALUE PROBLEM

The boundary value problem in the transformed ζ-plane resembles closely that of flows past isolated partially cavitated hydrofoils and is a special case of the Riemann-Hilbert problem for a half plane (Muskhelishvili, 1953). The general solution may be shown to be of the form

$$w(\zeta) = - \frac{1}{\pi} H(\zeta) \int_{-\infty}^{\infty} \frac{iw(t)}{H(t)} \frac{dt}{t-\zeta} + P(\zeta) \, H(\zeta) \qquad [7]$$

where

$H(\zeta)$ is the fundamental solution which depends on flow behavior at edge points $\zeta = 0$ and $\zeta = s$,

$P(\zeta)$ is a rational function which may have isolated singularities only at edge points $\zeta = 0$ and $\zeta = s$ and at infinities.

The first term of Equation [7] is the particular solution which satisfies the mixed boundary conditions on the real ξ-axis while the second term is the general solution of the corresponding homogeneous problem.

In accordance with the linearized formulation it is required that $w(\zeta)$ satisfies the following conditions (Geurst,1959):

$$\left.\begin{aligned} w(\zeta) &= \zeta^{-\frac{1}{2}} && \text{at } \zeta = 0 \\ w(\zeta) &= (\zeta-s)^{-\frac{1}{2}} && \text{at } \zeta = s \\ w(\zeta) &\text{ is finite} && \text{at } \zeta = \pm\infty \quad \text{(Kutta condition)} \end{aligned}\right\} \qquad [8]$$

The functions $H(\zeta)$ and $P(\zeta)$ therefore takes the form

$$\left.\begin{aligned} H(\zeta) &= \sqrt{\zeta(\zeta-s)} \\ P(\zeta) &= \frac{A\zeta+B}{\zeta(\zeta-s)} \end{aligned}\right\} \qquad [9]$$

With A and B real constant. The final form of the general solution is given by

$$w(\zeta) = \frac{\sqrt{\zeta(\zeta-s)}}{\pi}\left[\int_{-\infty}^{0} \frac{-\alpha_1 + \frac{dy_f}{dx}(t)}{\sqrt{t(t-s)}}\,\frac{dt}{t-\zeta} + \int_{s}^{\infty} \frac{-\alpha_1 + \frac{dy_f}{dx}(t)}{\sqrt{t(t-s)}}\,\frac{dt}{t-\zeta}\right] + \frac{A\zeta+B}{\sqrt{\zeta(\zeta-s)}} \qquad [10]$$

which should, in addition, satisfy

at $\zeta = \zeta_1$ (upstream infinity)

$$R\ell\, w = \frac{1}{\sqrt{1+\sigma}} - 1 \qquad [11]$$

$$\mathrm{Im}\, w = 0 \qquad [12]$$

and the closure condition,

$$-\,\mathrm{Im}\oint_{\text{body}} w(\zeta)\,\frac{dz}{d\zeta}\,d\zeta = \mathrm{Im}\oint_{\substack{\text{about } \zeta_1 \\ \text{and } \zeta_2}} w(\zeta)\,\frac{dz}{d\zeta}\,d\zeta$$

$$= -d\, R\ell\left[e^{-i\gamma}\left(1 - \frac{1}{\sqrt{1+\sigma}} + w(\zeta_2)\right)\right] = 0 \qquad [13]$$

Equations [11], [12] and [13] determine, uniquely, the constants A, B and σ.

For flat-plate blades, i.e. $dy_f/dx = 0$, the complex velocity is of the form

$$w(\zeta) = \frac{A\zeta+B}{\sqrt{\zeta(\zeta-s)}} - i\alpha_1$$

which, together with boundary and closure conditions Equations [11], [12] and [13] may be shown to yield a solution identical to that given by Wade (1963).

The lift and drag, experienced by the blade as a result of fluid flow may be shown to a first order of approximation, to be

$$C_L = \frac{\text{lift}}{\frac{1}{2}\rho U_c{}^2 c} = -\frac{2}{c} R\ell \oint w(z)dz = -\frac{2}{c} R\ell \oint w(\zeta) \frac{dz}{d\zeta} d\zeta$$

$$= \frac{2}{\cos\gamma} \frac{d}{c} \operatorname{Im} w(\zeta_2) \qquad [14]$$

$$C_D = \frac{\text{drag}}{\frac{1}{2}\rho U_c{}^2 c} = \frac{1}{c} \operatorname{Im} \int_{\text{blade}} [w(z)]^2 \, dz$$

$$= \frac{1}{c} \operatorname{Im} \left\{ \int_{-\infty}^{0} [w(\zeta)]^2 \frac{dz}{d\zeta} d\zeta + \int_{0}^{s} [w(\zeta)]^2 \frac{dz}{d\zeta} dz \right\} \qquad [15]$$

The cavity shape can similarly be approximated as:

$$y_c = \int_0^x \frac{dy_c}{dx}\, dx = -\mathrm{Im} \int_0^x w(x, 0^+)\, dx$$

$$= -\mathrm{Im} \int_0^{x(\xi,0)} w(\zeta)\, \frac{dz}{d\zeta}\, d\zeta \qquad [16]$$

The exit flow velocity U_2/U_c and angle α_2 are approximately given by:

$$\frac{U_2}{U_c} = \left\{ \left[\frac{1}{\sqrt{1+\sigma}} + R\ell\ w(\zeta_2) \right]^2 + \left[-\mathrm{Im}\ w(\zeta_2) \right]^2 \right\}^{\frac{1}{2}} \qquad [17]$$

$$\alpha_2 = \sin^{-1} \left[\frac{-\mathrm{Im}\ w(\zeta_2)}{U_2/U_c} \right] \doteq -\mathrm{Im}\ w(\zeta_2)$$

$$= -\frac{1}{2}\, C_L \left(\frac{c}{d} \right) \cos\gamma \qquad [18]$$

CONCLUDING REMARKS

This study gives the linearized solution for the flow past a straight cascade of partially cavitating cambered blades. From the analysis, it is possible to determine the lift and drag coefficients, cavitation number, cavity shape and downstream flow conditions for any given specific cascade geometry, blade shape, cavity length and initial inflow conditions.

As indicated by the experiments of Wade and Acosta (1966) the flow is unstable as the cavity length-chord ratio is closed to unity; the analysis is, therefore, meanless in this flow regime. It is also needless to say that the present analysis is limited to cases in which the disturbance, caused by the presence of the blades, is very small - an inherent restriction in the linear approximation. However in practice, the blades are, generally, quite thin; consequently the linearized results are valuable as a guide in the design of turbomachines and other applications.

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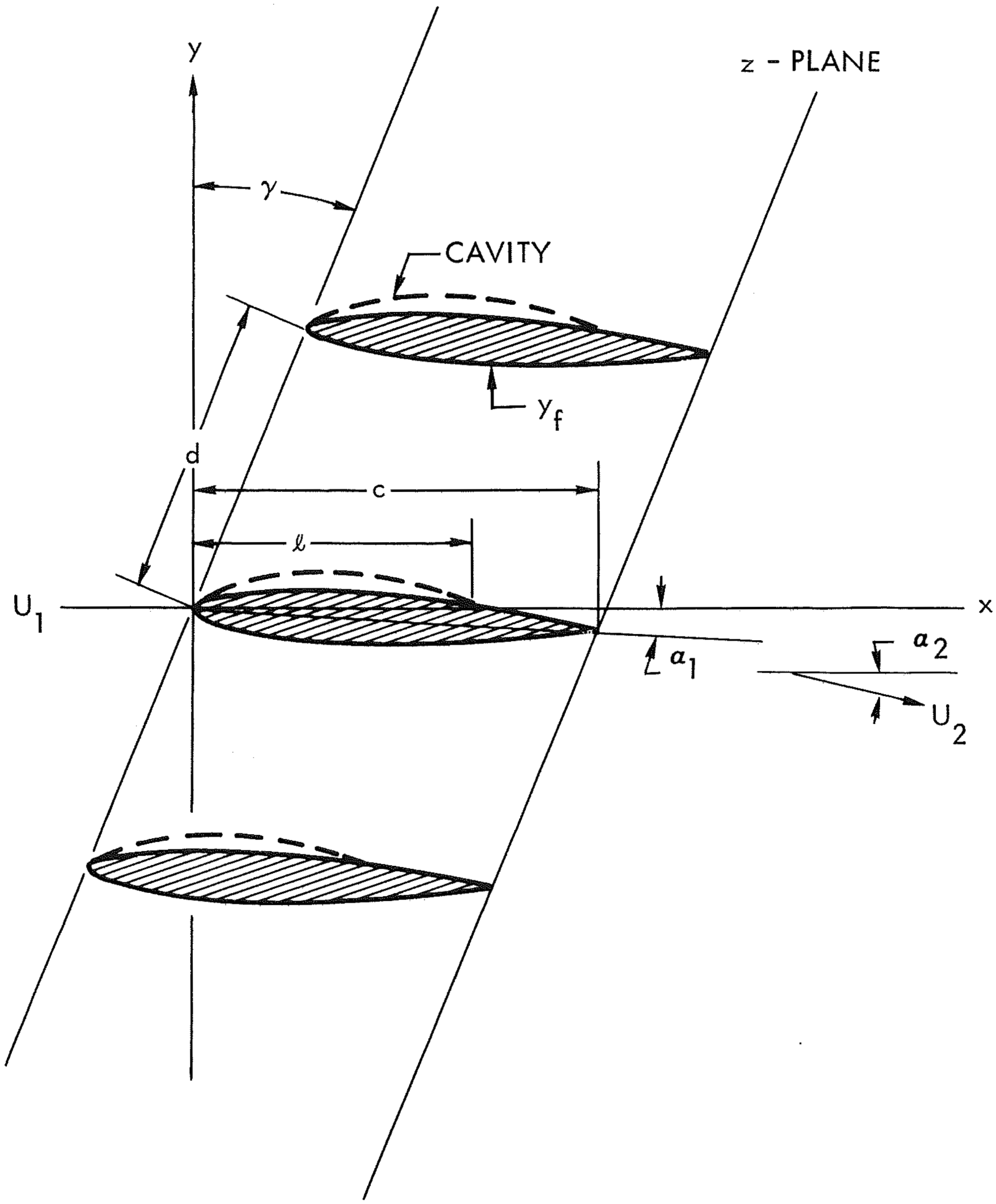

FIGURE 1 - DEFINITION SKETCH

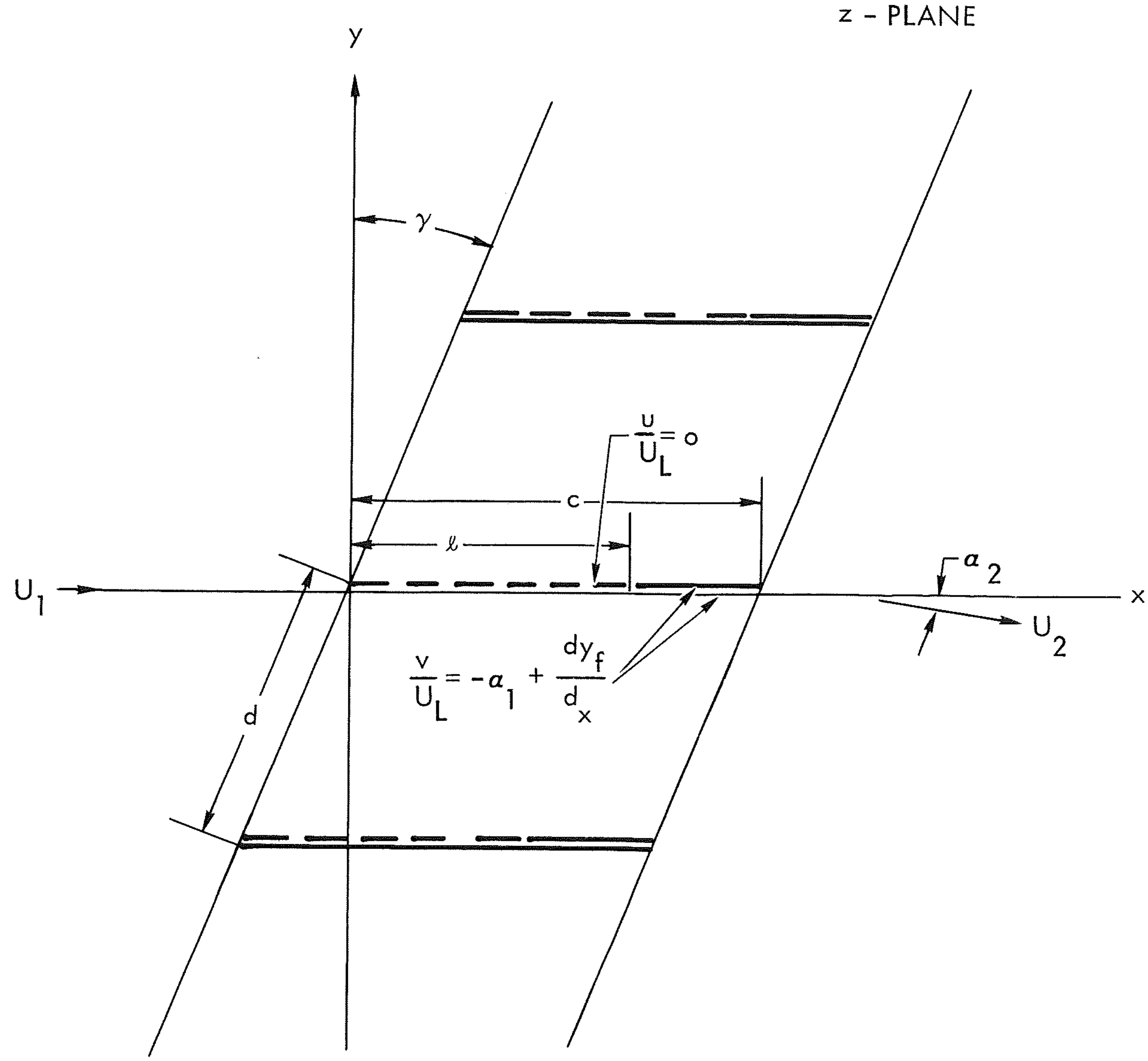

FIGURE 2 - LINEARIZED BOUNDRY - VALUE PROBLEM IN z - PLANE

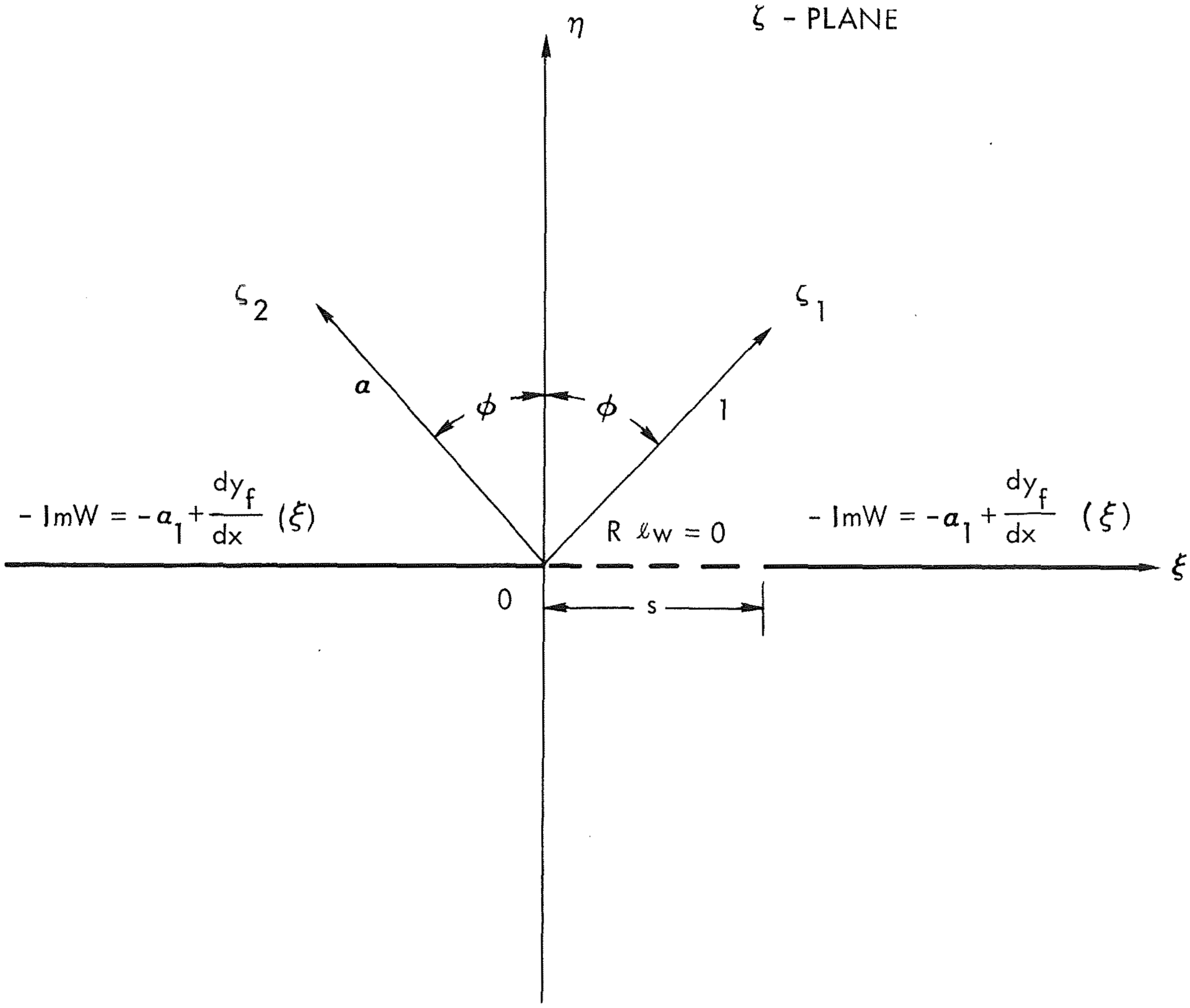

FIGURE 3 - LINEARIZED BOUNDARY - VALUE PROBLEM IN THE TRANSFORMED ζ - PLANE

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