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AN ACOUSTIC INSTABILITY DRIVEN BY
ABSORPTION OF RADIATION IN GASES

by

Michael J. Monsler

CSR T-69-4

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CENTER FOR SPACE RESEARCH
MASSACHUSETTS INSTITUTE OF TECHNOLOGY



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ERRATUM

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DRIVEN BY ABSORPTION OF RADIATION
IN GASES

by

MICHAEL MONSLER

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in June 1969 in partial fulfillment of the requirements for the degree
of Doctor of Philosophy.

ABSTRACT

The stability of a steady-state radiating gas with a nonuniform temperature distribution is investigated. It is found that an instability may arise due to the absorption of radiation by the fluctuations in the gas. The instability mechanism is critically dependent on the rate of change of opacity with temperature and a sufficiently large net radiative energy flux. If the opacity is a strongly increasing function of temperature, a small local increase in temperature in a fluctuation will cause a corresponding increase in the rate of absorption of the radiative flux. The ensuing heat released in the fluctuation will cause a further increase in temperature and opacity, creating an unstable situation. This can not happen in a uniform gas because if any local "hot spot" arises in a fluctuation it will be a net emitter instead of a net absorber and will radiate away its excess energy.

Two approaches have been undertaken to understand this stability problem. First, wave propagation in nonuniform radiating gases is examined and the criterion for amplification of acoustic waves traveling in the direction of the net heat flux is derived. Secondly an eigenvalue problem is solved for the complex frequencies of the natural oscillations of a nonuniform radiating gas, for the boundary conditions of black walls of constant temperature.

Critical values of the net radiative heat flux for neutral stability are calculated as a function of the other parameters. It is found that the lower normal modes of oscillation are most unstable and that the gas must be optically thick for the instability to occur. Boltzmann numbers which are either very large or very small compared to unity will be a stabilizing influence. Furthermore the instability can occur only if the opacity increases at least as the fourth power of the temperature. These stability criteria may differ for different boundary conditions, so that each problem should be tested for stability on an individual basis.

Thesis Supervisor: Jack L. Kerrebrock
Professor of Aeronautics and Astronautics

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I. INTRODUCTION

I.1 The Motivation for this Research

The trend in recent concepts for future propulsion and power systems is toward enormously greater operating temperatures to take advantage of increased exhaust velocities, thermodynamic efficiencies, and power densities. In these designs clever fluid mechanical or electromagnetic containment schemes are proposed to circumvent the limit on temperature imposed by materials problems. The extremely high temperatures that would be involved insure that thermal radiation in the gas or plasma will be an important energy transfer mechanism. For devices which will depend on radiative transfer as a key factor in the concept, the transfer of energy will be between different opaque regions of the gas, rather than between solid surfaces containing a nonparticipating gas.

Problems involving strong coupling between radiation and gas dynamics in internal flow have only been studied very recently compared to external flow problems such as nuclear blast waves and flow around reentry vehicles. An outstanding example of a proposed device with the above characteristics is the "light bulb" gaseous core nuclear rocket, in which enormous amounts of power would be transferred from a contained vortex of fissioning uranium gas to a propellant by thermal radiation. (See the latest review paper by McLafferty, Ref. 18, for a detailed description.) The regions of the gas which are predominantly emitting or absorbing must be quite opaque, and the intervening gas

quite transparent, in such a device. This implies enormously steep temperature gradients, perhaps of the order of ten-thousand degrees Kelvin per centimeter, and net radiative heat fluxes of the order of ten megawatts per square centimeter. There are reasons to believe that such an extreme state of nonequilibrium may not be attainable, due to instability of the predicted state with respect to small disturbances. That is, there may be conditions under which an infinitesimally small perturbation, such as a mere sound wave, may cause the state of the system to change spontaneously to a different stable state, which may be either steady or unsteady.

To see that steady-state radiative transfer in participating gases may not always be a stable physical process, it may be helpful to consider the analogy to combustion instabilities and detonation shock waves in combustible gases. Very simply, a combustible gas and an emitting-absorbing radiating gas are analogous in that a ready source of heat release is present, which may be of sufficient strength to determine the dynamics of the gas. The example of a detonation wave is familiar. When the temperature jump across a shock moving into a combustible gas is sufficient to ignite the gas, the subsequent heat release behind the shock will cause an acceleration and amplitude increase in the shock, which in turn increases the rate of heat release. This boot-strap effect may continue until a steady state is reached in which the heated gas has sonic velocity relative to the wave front. One may refer to Zeldovich, Ref. 19, for a review of detonation waves. The important observation here is that a shock wave in a gas which has an

opacity increasing with temperature may show the same effect when it absorbs radiation by an outside source. The temperature jump across the discontinuity will cause an increase in opacity behind the shock which in turn will lead to selective absorption and heat release there, amplifying the shock wave. The close analogy to the detonation wave is obvious. This type of radiation absorption shock wave has been observed experimentally by focusing a giant pulsed laser beam into a very small volume of air and watching the blast wave caused by this large deposition of energy in a short time. The laser radiation breaks down the gas and the resultant ionized gas becomes capable of absorbing the radiation. The radially expanding shock enclosing the laser induced plasma travels preferentially toward the source of radiation during the laser pulse, by the mechanism described above. These experiments are explained quite well in terms of a classical detonation wave model, reinforcing the analogy. Some selected references on this topic are 1) Ramsden and Savic (Ref. 20), 2) Razier (Ref. 21) and 3) Friedrich, Weigle, and Dougal (Ref. 22). The latter authors include an excellent bibliography on this subject.

The point of the above discussion is to give an appreciation for the phenomenon of a wave which is capable of growing in amplitude due solely to the heat released by the absorption of radiant energy. Three conditions must necessarily be satisfied for this to happen. There must be a source of radiation present, of higher effective radiating temperature than the medium supporting the wave; the wave must be sufficiently opaque for the absorption of radiation to be a significant

source of energy; and the opacity of the gas must be an increasing function of temperature so that a "runaway" effect is possible. These considerations lead to the following very important questions, which are central to this thesis. Are there any conditions under which an infinitesimally small disturbance, i.e. an ordinary fluctuation of the gas, can grow into one of these radiation driven shock waves? How is the criterion for wave growth dependent upon such properties as the flux of radiation or the rate of increase of opacity with temperature? Finally, under what conditions will a given radiative heat transfer problem be absolutely unstable due to this effect?

I.2 Previous Studies of the Effect of Radiation on Fluctuations in a Gas

The literature in this very specific field of study has a long history if not a large number of contributors. The first paper was authored by Stokes in 1851 and was entitled "An Examination of the Possible Effect of the Radiation of Heat on the Propagation of Sound", (Ref. 23). This title is practically the same as the titles of several contemporary papers. Stokes was interested in determining the extent of the dissipative effect of radiation on the acoustic speed, and he seemed to be the first to recognize that radiation cooling would tend to decrease the sound speed from the isentropic speed $a_s = \sqrt{\gamma RT}$ toward the isothermal speed $a_T = \sqrt{RT}$, where R is the gas constant, T the temperature, and γ the ratio of specific heats. A good commentary on this is provided by Lord Rayleigh in his classic, "The Theory of Sound" (Ref. 24).

The first actual devices utilizing the interaction of radiation and sound date from 1881, when Alexander Graham Bell (Ref. 25) did experiments on a device called a "photophone" in which a chopped beam of light entering a chamber of gas would produce audible sound. A more scientific analysis of this effect was published by Tyndall (Ref. 26) a year later.

The most comprehensive treatments of the behavior of waves in a uniform radiating gas are by Prokof'ev (Ref. 27) and Vincenti and Baldwin (Ref. 2). The Russian author's work is of greater generality in that it includes the effects of viscosity and heat conduction. The dispersion relation derived in this paper is for a gas of infinite extent however. The paper of Vincenti and Baldwin, though neglecting such molecular dissipative effects, is a more penetrating analysis of the effects of radiation on wave propagation. It treats the Problem of a semi-infinite expanse of uniform radiating gas, bounded by a plane radiating wall which may oscillate in temperature and displacement. This analysis is applicable for an arbitrary range of temperatures and optical depths, and for a grey gas. The authors find the fluctuating temperature and pressure field to be a superposition of a) sound waves modified by radiation and b) radiation induced waves which are essentially a kind of "heat wave" due to the diffusion of radiation. The goal of this paper was really to solve the most simple problem imaginable in radiation gas dynamics that would still illustrate the coupling effects between the radiation and the gas. This paper, along with the summary of the work presented in the textbook of Vincenti and

and Kruger (Ref. 3), constitute the minimum reading for anyone interested in research in this field. The paper by Lick (Ref. 27) also treats a uniform radiating gas in the half-space, adopting however the viewpoint of an impulsive piston problem and examining the resultant wave propagation.

The next interesting paper on this general subject is by Long and Vincenti, who examine acoustic waves in a confined gas, driven by a pulsed source of radiation. This analysis, a modern version of the work by Bell and Tyndall described earlier, is a theoretical treatment designed to delineate the important parameters preparatory to carrying out an experiment. Their proposed experiment will presumably study the standing acoustic waves set up in a tube by pulsed laser radiation. The similarities and differences between the paper by Long and Vincenti and the present work should be emphasized here. Both studies are concerned with standing acoustic waves in a confined gas, and make very nearly the same assumptions. Their analysis essentially seeks to determine resonance conditions and pressure response in a uniform gas for waves driven by a pulsating source of radiation at the boundaries. Our analysis treats the natural oscillations of a nonuniform gas. We find an instability, i.e. an exponential growth of these oscillations, driven by a steady net flux of radiation. This will undoubtedly become clearer as the problem is more fully explained in the following chapters.

I.3 The Method of Approach

We shall now describe how we propose to answer the questions asked in the first section concerning the growth of waves in a non-uniform radiating gas and how they may lead to an instability. The chapter following this introduction, Chapter II, is a brief review of the equations of radiation gas dynamics, which seeks to establish the notation and justify the simplifications introduced to make the equations manageable. Chapter III sets up a mathematical model of a steady-state radiative transfer cell, in which a net flux of radiation is allowed to pass from a hot wall to a cooler wall through a participating gas of arbitrary opacity. This is the idealized system which will be tested for its stability with respect to small perturbations. The equations of motion for these small perturbations, or fluctuations, are derived in Chapter IV, along with the boundary conditions that they must obey. These fluctuations may always be looked upon as consisting of two types of left and right running traveling waves. In Chapter V a "local" analysis is performed to determine the speeds and rates of amplification or evanescence of these traveling waves in nonuniform gases without regard to the effects of boundaries. The conditions for the growth of waves due to the absorption of radiation are given in this chapter. In order to determine what are the effects of boundaries on these growing waves the "local" wave propagation analysis must be contrasted to a boundary value problem. In Chapter VI the eigenvalue problem is solved for the normal modes of fluctuation, i.e. the natural oscillations, of the bounded non-

equilibrium radiating gas described in Chapter III. This analysis predicts that there may be a critical net radiative heat flux for a given gas in a given state, for which the gas will be absolutely unstable due to growth of fluctuations driven by absorption of radiation. Chapter VII examines the functional dependence of the criterion for this instability on the many parameters which characterize the radiating gas; and discusses the effects of the boundaries on this phenomenon. The above conclusion is of course restricted in generality due to some of the assumptions adopted in this very idealized model. Moreover it is unsubstantiated by experiment at the present time so its status must remain that of a prediction.

II. THE EQUATIONS OF RADIATION GAS DYNAMICS

II.1 Introduction

The coupling of the equation of radiative transfer with the equations of compressible gas dynamics results in a nonlinear integro-differential set of equations which are very difficult to solve in general. For any analytical study which seeks a solution to an idealized problem isolating the phenomenon of interest, simplifications of the general equations are necessary. In this chapter we discuss the approximations and compromises which have been made in order to reduce the complexity of the equations sufficiently. This approach will necessarily idealize much of the physics but it is appropriate to the study of the growth of fluctuations in a nonuniform thermally radiating gas, for which there is an imperfect theoretical understanding and no experimental data.

The emission and absorption of radiation in gases and plasmas is quite well understood at the present time and, given any specific situation, the spectra and energy distribution may be predicted and calculated with some confidence though also with some labor. The greatest remaining controversy, at least for a gas dynamicist, remains in the area of finding suitable approximations to the details of emission and absorption, in order to represent sufficiently accurately the exchange of energy between the gas and the radiation field with a minimum of calculational effort. The investigators interested in stellar radiation or discrimination of reentry vehicles will be interested in the minute details of the spectra, but those interested

in gas dynamics in the presence of thermal radiation may settle for an average of the absorption coefficient over photon frequency. Similarly, differences in opinion exist among people in this field as to the adequacy of replacing the integro-differential equation of radiative transfer with its moment equations. No attempt will be made here to mediate these disputes. (See Finkleman, Ref. 30, for a discussion of these points.) We shall be satisfied to use an approximation that seems useful, and to test the results to insure that no assumptions have been violated.

Excellent general references to this subject are the books by Vincenti and Kruger (Ref. 3), Zeldovich and Raizer (Ref. 4), and Sampson (Ref. 5).

II.2 Energy Transfer by Radiation

The basic equation of radiative transfer is usually expressed in terms of the specific intensity of radiation

$$I_\nu = I_\nu(x_i, t, l_i, \nu) \quad (2.1)$$

which is defined as the photon energy flux at the point x_i and time t in the frequency range ν to $\nu + d\nu$ and into the solid angle $d\Omega$ centered about the direction defined by the direction cosines l_i .

The radiant energy flux across a surface normal to the x_i axis would be then

$$l_i I_\nu d\Omega d\nu \quad (2.2)$$

The mass emission coefficient j_ν is defined such that the volumetric rate of emitted energy in the frequency range $(\nu, d\nu)$ and into the solid angle $d\Omega$ is

$$\rho j_\nu d\Omega d\nu \quad (2.3)$$

where ρ is the density of the gas. The mass absorption coefficient κ_ν is defined such that the volumetric rate of energy absorption by all processes in the gas except scattering in the range $(\nu, d\nu)$ and $d\Omega$ is

$$\rho \kappa_\nu I_\nu d\Omega d\nu \quad (2.4)$$

We neglect scattering of radiation in this formulation since for the relatively high densities of gas dynamic interest, and especially for an optically thick gas, this is a good approximation (see discussion in Sec. 5.4 of Sampson, 1965). The radiative transfer equation, which accounts for the continuity of photon energy, is then

$$\frac{1}{c} \frac{\partial I_\nu}{\partial t} + l_j \frac{\partial I_\nu}{\partial x_j} = \rho (j_\nu - \kappa_\nu I_\nu) \quad (2.5a)$$

The term accounting for the change of the specific intensity with time is usually extremely small in gas dynamic problems, since the limiting characteristic speed, being of the order of the speed of sound, is many orders of magnitude smaller than the speed of light, c . Then we have

$$l_j \frac{\partial I_\nu}{\partial x_j} = \rho (j_\nu - \kappa_\nu I_\nu) \quad (2.5b)$$

Once the above equation is solved for I_ν , other quantities of physical significance may be arrived at by taking moments of I_ν with respect to the direction cosines l_j . For example the energy density of radiation is

$$U_R = \frac{1}{c} \int_0^\infty d\nu \int_0^{4\pi} d\Omega I_\nu \quad (2.6)$$

The radiant heat flux vector and the radiation pressure tensor are

$$q_{Ri} = \int_0^\infty d\nu \int_0^{4\pi} d\Omega l_i I_\nu \quad (2.7)$$

$$p_{Rij} = \frac{1}{c} \int_0^\infty d\nu \int_0^{4\pi} d\Omega l_i l_j I_\nu \quad (2.8)$$

It would seem unnecessary to take higher moments, since they tend to have a less concrete physical meaning, and we shall not need them in the approximations we use for radiative transfer.

In this and the succeeding discussion we tend to follow the notation and treatment of Vincenti and Kruger (Ref. 3) and the report by P. Cheng (Ref. 7) rather than that of the older but very complete treatise of Chandrasekhar (Ref. 6).

II.3 Radiative Equilibrium

When there are no changes of the specific intensity in either space or time it will assume the equilibrium value given by statistical mechanics, often given the special name of the Planck function

$$B_\nu(T) = \frac{2h\nu^3/c^2}{[e^{h\nu/kT} - 1]} \quad (2.9)$$

where h is Planck's constant, k is Boltzmann's constant and T is the temperature of the matter with which the radiation is in equilibrium.

The frequency integrated equilibrium intensity is

$$B(T) = \int_0^\infty B_\nu d\nu = \frac{\sigma T^4}{\pi} \quad (2.10)$$

where

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} = 5.67 \times 10^{-8} \frac{\text{joules}}{\text{m}^2 \text{sec}^\circ \text{K}^4} \quad (2.11)$$

is the Stefan-Boltzmann constant. The equilibrium energy density, heat flux and radiation pressure follow from substitution of (2.10) into (2.6-8).

At equilibrium they are

$$U_R = \frac{4\sigma T^4}{c} \quad q_{Ri} = 0 \quad p_{Rij} = \frac{4\sigma T^4}{3c} \delta_{ij} \quad (2.12)$$

where δ_{ij} is the Kronecker delta. We note that the pressure tensor becomes isotropic at equilibrium and is equal to one-third of the energy density.

II.4 Local Thermodynamic Equilibrium

Under most conditions encountered in continuum gas dynamics, a well-defined temperature of the gas may be assumed because the molecular collision rates are high enough to keep the relative populations of the molecular energy states very near their equilibrium values. If the rates of population and depopulation by radiative emission and absorption are less than the collisional rates, and the collisional rates are near equilibrium, then a Boltzmann distribution of energy states at the local gas temperature may be assumed. This may be true even for great radiative nonequilibrium, for it is the gas that must be near equilibrium. Such a condition of local thermodynamic equilibrium exists in most cases where the length scale for the gradients of the thermodynamic field quantities is very much larger than the molecular mean free path. As we are mainly concerned with gases at high densities, we shall adopt the assumption of local thermodynamic equilibrium.

Under this assumption the source terms on the right hand side of the equation of radiative transfer become more simple. The mass emission coefficient j_ν is then

$$j_\nu = \left[(1 - e^{-h\nu/kT}) B_\nu + e^{-h\nu/kT} I_\nu \right] K_\nu \quad (2.13)$$

The first term describes spontaneous emission and the second describes stimulated emission. It is convenient to treat this second term as negative absorption and group it with the other absorption term.

Defining a new mass absorption coefficient (corrected for stimulated emission) as

$$\kappa'_\nu = \kappa_\nu (1 - e^{-h\nu/kT}) \quad (2.14)$$

and a volumetric absorption coefficient, or opacity, as

$$\alpha_\nu = \rho \kappa'_\nu \quad (2.15)$$

we can write the transfer equation, equation (2.5b), as

$$l_j \frac{\partial I_\nu}{\partial x_j} = \alpha_\nu (B_\nu - I_\nu) \quad (2.16)$$

II.5 Frequency Averages of the Opacity and the "Grey" Gas

It is easy to see that it is impossible to obtain a differential equation for the frequency averaged specific intensity

$$I = \int_0^\infty I_\nu d\nu \quad (2.17)$$

by merely integrating (2.16) over frequency, since integrals over the unknown I_ν appear, such as $\int \alpha_\nu I_\nu d\nu$. For the limiting case of an emission dominated gas $B_\nu \gg I_\nu$ and

$$l_j \frac{\partial I_\nu}{\partial x_j} \approx \alpha_\nu B_\nu \quad (2.18a)$$

so that, integrating over frequency and solid angle we have the optically thin approximation

$$\frac{\partial q_i}{\partial x_i} \approx 4\alpha_r \sigma T^4 \quad (2.18b)$$

The Planck frequency average of the opacity is then

$$\alpha_p = \frac{\pi}{\sigma T^4} \int_0^{\infty} \alpha_\nu B_\nu d\nu \quad (2.19)$$

For an optically thick gas $(B_\nu - I_\nu) \ll B_\nu$ so that we may expand in a Taylor expansion as $I_\nu \approx B_\nu + \frac{\partial B_\nu}{\partial T} dT + \dots$

Then

$$\frac{l_i}{\alpha_\nu} \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial x_i} \approx B_\nu - I_\nu \quad (2.20a)$$

If we multiply this by l_i and integrate over both frequency and solid angle (noting that the Planck function is isotropic) we get

$$q_i = -\frac{16}{3} \frac{\sigma T^3}{\alpha_R} \frac{\partial T}{\partial x_i} \quad (2.20b)$$

where the Rosseland frequency average of the opacity is

$$\frac{1}{\alpha_R} = \left(\frac{\pi}{4\sigma T^3} \right) \int_0^{\infty} \frac{1}{\alpha_\nu} \frac{dB_\nu}{dT} d\nu \quad (2.21)$$

It is apparent that these two average opacities may differ by a considerable margin, indicating that care must be exercised in their utilization for a particular problem, particularly when both optically thin and thick regions exist in the same problem.

In the approximation known as the "grey" gas one assumes, quite arbitrarily, that the equation of radiative transfer for some frequency averaged opacity is the same as it is for each infinitesimal frequency

interval, even though it is known that no such average can rationally be derived. Specifically, one replaces α_ν by a function of temperature and density $\alpha = \alpha(T, \rho) \neq \alpha(\nu, T, \rho)$ independent of frequency. Then one can integrate equation (2.16) over frequency to get

$$l_j \frac{\partial I}{\partial x_j} = \alpha \left(\frac{\sigma T^4}{\pi} - I \right) \quad (2.22)$$

This form for the transfer equation still retains the dependence on direction cosines however and can only be used in conjunction with the various moments of the specific intensity $I = I(x_i, t, l_j)$ with respect to the direction cosines l_j . In this respect we still must deal with integro-differential equations, which are quite inconvenient for perturbation calculations. A scheme to circumvent this problem is explained in the next section.

II.6 The Differential, or Moment Approximation

The equations directly relating the radiative properties of the system have been reduced to the following, where $I = I(x_i, t, l_j)$

$$l_j \frac{\partial I}{\partial x_j} = \alpha \left(\frac{\sigma T^4}{\pi} - I \right) \quad (2.22)$$

$$U_r = \frac{1}{c} \int_0^{4\pi} I d\Omega \quad (2.23a)$$

$$q_{Ri} = \int_0^{4\pi} l_i I d\Omega \quad (2.23b)$$

$$p_{\alpha ij} = \frac{1}{c} \int_0^{4\pi} l_i l_j I d\Omega \quad (2.23c)$$

Note that if the n^{th} moment of I is defined as

$$I_{ijk\dots n}^{(n)} = \int_0^{4\pi} l_i l_j l_k \dots l_n I d\Omega \quad (2.24)$$

we can write

$$U_R = \frac{I^{(0)}}{c} \quad q_{Ri} = I_i^{(1)} \quad p_{\alpha ij} = \frac{1}{c} I_{ij}^{(2)} \quad (2.25a,b,c)$$

The equation of radiative transfer, as written above, roughly corresponds to the Boltzmann equation of gasdynamics, with the magnitude of the velocities averaged over but with the directional dependence still intact. The usual accepted procedure in each case is to replace this equation with a set of equations which are moments of the parent equation to find sufficient relations between the moments of the dependent variable. The only difficult point then remaining is to justify cutting off this procedure after a certain number of moments, since all moments (an infinite number) are required to exactly represent the dynamics expressed in the original equation.

Let us proceed to sketch the method; a more extensive discussion and set of references may be found in Cheng (Ref. 7). We integrate (2.22) over solid angle, noting that

$$\int_0^{4\pi} d\Omega = 4\pi \quad \int_0^{4\pi} l_i d\Omega = 0 \quad \int_0^{4\pi} l_i l_j d\Omega = \frac{4\pi}{3} \delta_{ij} \quad (2.26a,b,c)$$

The zeroth moment equation is

$$\frac{\partial I_{ij}^{(1)}}{\partial x_j} = 4\alpha \sigma T^4 - \alpha I^{(0)} \quad (2.27)$$

The first moment equation is

$$\frac{\partial I_{ij}^{(2)}}{\partial x_j} = -\alpha I_i^{(1)} \quad (2.28)$$

One can proceed indefinitely but the usual approximation is to terminate the system of equations by expressing the highest moment as a linear combination of all lower moments. When this is done rigorously, through a spherical harmonic expansion, one gets merely that

$$I_{ij}^{(2)} = \frac{I^{(0)}}{3} \delta_{ij} \quad (2.29)$$

or in terms of physical quantities,

$$p_{Rij} = \frac{U_R}{3} \delta_{ij} \quad (2.30)$$

This means that in this approximation one finds that the pressure tensor and energy density are related in such a way as if the gas were in local radiative equilibrium. In practice this is nowhere near as restrictive an assumption as one might imagine, even in cases of such

large departures from radiative equilibrium as a beam of radiation in an optically thin gas. See, for example, Jischke (Ref. 8) for an accuracy test of this approximation procedure, based upon the above assumption. The assumption embodied in (2.30) is likely to be violated first in cases of high geometrical complexity, where large "shear" of the radiation field would be present. We shall not encounter such difficulties in one-dimensional problems.

One can, from the rigorous spherical harmonic expansion mentioned, express the specific intensity in a series of its moments, to the same approximation, as

$$I(l_j) = \frac{1}{4\pi} (I^{(0)} + 3I_j^{(1)}l_j) \quad (2.31)$$

or

$$I(l_j) = \frac{1}{4\pi} (cU_R + 3l_j q_{Rj})$$

It should be apparent by now to the uninitiated in radiation problems, that it is not correct to blithely replace the specific intensity I by the average over 4π steradians or by the net heat flux vector q_{Rj} , when they are in fact related in the above manner.

We shall now replace the radiative transfer equation and moment definitions (2.22 and 2.23a,b,c) by equations (2.25a,b,c, 2.27, 2.28 and 2.29). Let us also make the new identification of the average flux as

$$G = I^{(0)} = cU_R \quad (2.32)$$

Then we have

$$\frac{\partial q_{Rj}}{\partial x_j} = \alpha (4\sigma T^4 - G) \quad (2.33)$$

$$\frac{\partial G}{\partial x_i} = -3\alpha q_i \quad (2.34)$$

$$p_R = \frac{U_R}{3} = \frac{G}{3c} = \left(\frac{p_{R11} + p_{R22} + p_{R33}}{3} \right) \quad (2.35)$$

where the scalar radiation pressure p_R is now assumed to be the average of its three purely normal components. Notice from equation (2.32) and (2.33) that the energy density $U_R = \frac{4\sigma T^4}{c}$ only in actual equilibrium, e.g. when the net volumetric heat absorption rate $\left(-\frac{\partial q_{Rj}}{\partial x_j}\right)$ vanishes. To repeat, in these equations the opacity is to be considered a thermodynamic variable, $\alpha = \alpha(T, p)$. It is simple to show that the pair of equations (2.33) and (2.34) go to the correct limits in the optically thin and thick extremes. Combine the two equations by differentiating (2.33) and eliminating the $\frac{\partial G}{\partial x_i}$ term. One gets

$$\frac{1}{\alpha} \frac{\partial}{\partial x_i} \left(\frac{1}{\alpha} \frac{\partial q_{Rj}}{\partial x_j} \right) - 3q_{Ri} = \frac{1}{\alpha} \frac{\partial}{\partial x_i} (4\sigma T^4)$$

In the optically thick case the opacity is extremely large and first term may be neglected leaving

$$q_{Ri} = -\frac{16}{3} \frac{\sigma T^3}{\alpha} \frac{\partial T}{\partial x_i}$$

which is the familiar diffusion approximation. In the optically thin case α is very small and the second term becomes negligible. The remaining terms can be integrated once to give the familiar thin, or emission-dominated limit

$$\frac{\partial q_{Ri}}{\partial x_i} = 4\alpha\sigma T^4$$

II.7 Radiative Boundary Conditions

The assignment of the proper bounding values for a problem in radiative transfer often involves some subtleties, particularly in making them consistent with one's approximations to the equation of radiative transfer. We discuss here the method of deriving boundary conditions suitable for the moment, or differential, approximation described in the previous section. Before immediately attempting to deal with the presence of a wall, we must break the net flux into two opposing streams of radiation. The geometry is given in Figure 1. We want to calculate the one-way heat flux in the gas through any imaginary surface. Let us take the surface for convenience to be the $i = 2, 3$ plane whose unit normal, k_i , is in the positive $i=1$ direction. We measure the angle Φ from this normal and Θ in the 2, 3 plane. If the rightward one-way flux is Q_{Ri}^+ and the leftward (positive to the left) flux is Q_{Ri}^- , then the net radiative flux in the $i=1$

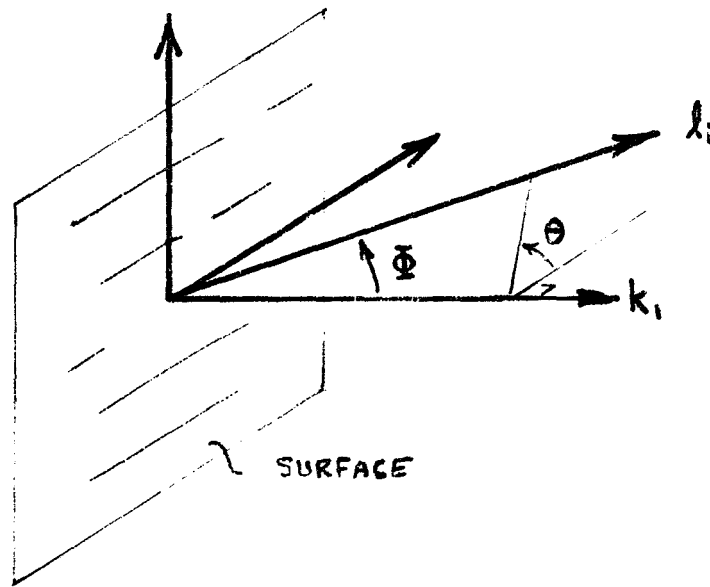


Fig. 2.1 Definition of Angles

direction is

$$q_{R_i} = q_{R_i} k_i = \int_0^{4\pi} I \cos \Phi d\Omega \quad (2.36)$$

$$= Q_{R_i}^+ - Q_{R_i}^- \quad (2.37)$$

where

$$d\Omega = \sin \Phi d\Phi d\theta$$

$$Q_{R_i}^+ = \int_0^{2\pi} \int_0^{\pi/2} I \cos \Phi \sin \Phi d\Phi d\theta \quad (2.38)$$

$$Q_{R_i}^- = \int_0^{2\pi} \int_{\pi/2}^{\pi} I (-\cos \Phi) \sin \Phi d\Phi d\theta \quad (2.39)$$

To calculate the one-way fluxes we must know the relation between the specific intensity as a function of the direction cosines l_i and the moment quantities. This is given by equation (2.31) as

$$I(l_i) = \frac{1}{4\pi} (G + 3q_{Rj} l_j)$$

Substituting this into (2.38) and using the relation for the l_i given here

$$l_1 = \cos \Phi \quad l_2 = \sin \Phi \cos \Theta \quad l_3 = \sin \Phi \sin \Theta$$

results in the following

$$Q_{R1}^+ = \int_0^{2\pi} \int_0^{\pi/2} [G + 3q_1 \cos \Phi] \frac{\cos \Phi \sin \Phi}{4\pi} d\Phi d\Theta$$

or,

$$Q_{R1}^+ = \frac{1}{4} [G + 2q_1] \quad (2.40)$$

then

$$Q_{R1}^- = \frac{1}{4} [G - 2q_1] \quad (2.41)$$

since

$$q_{R1} = Q_{R1}^+ - Q_{R1}^-$$

Now we can turn to the specific boundary conditions for a gas bounded by two ideal black walls. (See Fig. 2)

Each wall is at its own constant temperature $T_w(x=0) = T_{wa}$ and $T(x=L) = T_{wb}$. At the $x=0$ wall the frequency-integrated energy flux from the wall out into the gas is σT_{wa}^4 regardless of the behavior of the gas. The black wall absorbs all the radiation incident upon it,

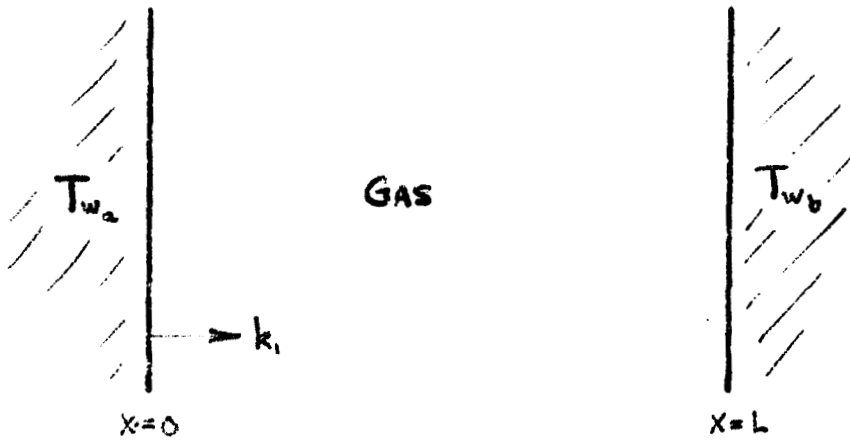


Fig. 2.2 Wall Geometry

of amount $Q_{R,1}^-$ from the gas but this is of no consequence since an outside agent keeps the wall at constant temperature. If we place our imaginary surface between the wall and the gas, and demand that the flux be continuous across it, we must arrive at the conclusion that all the flux emitted from the wall appears in the gas at $x=0+$. Thus we must have

$$\sigma T_{w_a}^4 = Q_{R,1}^+(x=0, t)$$

Similarly, at the other wall,

$$\sigma T_{w_b}^4 = Q_{R,1}^-(x=L, t)$$

Using equations (2.40) and (2.41) we may write the boundary conditions as

$$4\sigma T_{w_a}^4 = G(0, t) + 2q_R(0, t)$$

$$4\sigma T_{w_b}^4 = G(L, t) - 2q_R(L, t)$$

(2.42a, b)

These equations clearly exhibit the temperature slip at the walls due to the radiation. These results are of course for arbitrary opacity.

II.8 Equations of Ideal Radiation Gas Dynamics

The equations of inviscid ideal compressible gas dynamics do not change dramatically with the inclusion of the properties of radiation. As discussed in Zeldovich (Ref. 4) we merely replace the gas pressure and energy density by the total (gas plus radiation) quantities. The momentum of the photon gas is relativistically small and can be ignored. Since we have used the subscript notation $()_R$ for quantities associated with the radiation field, the gas quantities we shall leave unsubscripted. The continuity of mass equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0 \quad (2.44)$$

where ρ = gas mass density and u_j is a typical velocity component.

The three momentum equations have the form

$$\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j + p + p_R + \rho u_i u_j) = 0 \quad (2.45)$$

The volumetric energy density of the perfect plus photon gas may be written as $\frac{p}{\gamma-1} + 3p_R$ where γ is the ratio of specific heats for the gas alone. The energy equation may then be written in terms of these pressures as

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma-1} + 3p_R \right) + \frac{\partial}{\partial x_j} \left\{ u_j \left(\frac{\gamma p}{\gamma-1} + 4p_R + \rho \frac{u^2}{2} \right) + q_{Rj} \right\} = 0 \quad (2.46)$$

The equation of state of the gas is of course

$$p = \rho R T \quad (2.47)$$

The above equations constitute six equations in the eleven unknowns

$\rho, p, T, u_i, p_r, q_{rj}, \alpha$. They will be complete when the five radiative quantities α, p_r, q_{rj} are added.

It is apparent that the molecular transport effects of viscosity and heat conduction have been neglected. This is not to say they are never of importance, but only that they are best neglected in order to isolate the phenomenon of interest, namely the effects of radiation on the fluctuations of a gas not in equilibrium because of the radiation. Furthermore previous studies, e.g. Traugott (Ref. 9), have indicated that when a large radiation field is present the effects of molecular heat and momentum transfer are only apparent in such large discontinuities as normal shocks, which we disallow in the present analysis.

Chemical nonequilibrium also will be suppressed in the interests of simplicity. The fact that a gas will actually be a multiply-ionized plasma at very high temperatures will be rationalized with the statement that over the temperature range encountered in a specific problem the dense plasma is at a temperature sufficiently between successive ionization levels to be considered thermodynamically perfect. In order to account for the effects of ionization in an approximate manner, one can replace the equilibrium value of the ratio of specific heats γ with an effective adiabatic exponent which is somewhat closer to unity.

II.9 Summary

The equations of motion and state assembled in the preceeding discussion, in conjunction with a set of boundary conditions defining a specific situation, describe physical phenomena which are now reasonably well understood. An example of such a problem would be the structure of the precursor temperature gradient caused by the radiation from a strong shock front. However the fluctuations, or normal modes of oscillation of the system defined by these equations, are only understood for the case of a gas of infinite extent in equilibrium.

These equations are restated here for convenience, with slight algebraic condensation.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0 \quad (2.48)$$

$$\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} \left(p + \frac{G}{3c} + \rho u_i u_j \right) = 0 \quad (2.49)$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma-1} + \frac{G}{c} \right) + \frac{\partial}{\partial x_j} \left\{ u_j \left(\frac{\gamma p}{\gamma-1} + \frac{4G}{3c} + \frac{\rho u^2}{2} \right) + q_{Rj} \right\} = 0 \quad (2.50)$$

$$\frac{\partial q_{Rj}}{\partial x_j} = \alpha (4\sigma T^4 - G) \quad (2.51)$$

$$\frac{\partial G}{\partial x_i} = -3\alpha q_{Ri} \quad (2.52)$$

$$p = \rho RT \quad (2.53)$$

$$\alpha = \alpha(T, p) \quad (2.54)$$

The eleven equations in eleven unknowns form a uniquely solvable system of partial differential equations plus constitutive relations, which is third order in time and fourth order in space.

III. RADIATIVE HEAT TRANSFER IN A STATIONARY GAS OF ARBITRARY OPACITY

III.1 Introduction

In order to study the fluctuations in a radiating gas one must first have in mind a particular well defined physical system. In the following sections a specific steady-state radiative heat transfer problem is defined in which energy is transferred from a hot wall to a cooler wall by thermal radiation, through a gas which is an actively participating medium. The outstanding feature of this model is that it includes not only an arbitrary optical thickness of the gas, but an opacity that varies with temperature and pressure as well. In reality as well as in our model, the opacity in a gas may vary substantially between regions of differing temperature in a given situation, and may in fact be far more nonuniform than the temperature or density. This steady-state problem and its solution will be called the zeroth order solution.

III.2 The Model of a Physical Situation

The most simple model that can be proposed that will illustrate the physical phenomena in which we are interested seems to be that of a stationary perfect gas of arbitrary optical thickness contained between two solid black walls of different temperatures. A net flux of radiation is allowed to pass through the gas, causing gradients in the thermodynamic properties. The equations summarized in Sec. 2.8 become very simple for a steady-state planar geometry for which $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x}$.

$$u_o(x) = 0 \quad (3.1)$$

$$\frac{dp_o(x)}{dx} + \frac{1}{3c} \frac{dG_o(x)}{dx} = 0 \quad (3.2)$$

$$\frac{dq_o(x)}{dx} = 0 \quad (3.3)$$

$$G_o(x) = 4\sigma T_o^4(x) \quad (3.4)$$

$$\frac{dG_o(x)}{dx} = -3\alpha_o q_o \quad (3.5)$$

$$p_o = p_o R T_o \quad (3.6)$$

$$\alpha_o = \alpha_o(T_o, p_o) \quad (3.7)$$

First we note that if there is no time dependence and the gas is stationary, the net volumetric rate of heat exchange in the gas ($\frac{dq_o}{dx}$) must vanish. If we eliminate G_o from equations (3.2), (3.4) and (3.5), we get two equations for the temperature and the pressure.

$$\frac{dp_o}{dx} = \left(\frac{q_o}{c}\right) \alpha_o \quad (3.8)$$

$$\frac{dT_o}{dx} = -\frac{3}{16} \frac{q_o}{\sigma T_o^3} \alpha_o \quad (3.9)$$

Eliminating the opacity α_0 between these equations gives

$$\left(\frac{1}{p} \frac{dp_0}{dx}\right) = 4 \left[\frac{4\sigma T_0^4}{3p \cdot c} \right] \left(\frac{1}{T_0} \frac{dT_0}{dx} \right) \quad (3.10)$$

The expression in the square brackets is the ratio of the radiation pressure at equilibrium to the gas pressure. Let us estimate the magnitude of this ratio.

$$\frac{p_{\text{RAD.}}}{p_{\text{GAS.}}} = \left(\frac{4\sigma T^4}{3cp} \right) \approx \frac{10^{-4}}{(p_{\text{atm.}})} \left(\frac{T}{10^4 \text{ } ^\circ\text{K}} \right)^4 \quad (3.11)$$

Please see Fig. 3.1 for a plot of this nondimensional parameter as a function of its component dimensional quantities. Since this number is much smaller than unity at conditions like $T = 10^5 \text{ } ^\circ\text{K}$, $p = 100 \text{ atm.}$ and even at $T = 10^3 \text{ } ^\circ\text{K}$, $p = 10^{-6} \text{ atm.}$, we feel that we can ignore terms of this magnitude in comparison to terms of order unity in our model. If this is the case, our steady-state, or zeroth order equations become

$$u_0 = \frac{dp_0}{dx} = \frac{dq_0}{dx} = 0 \quad (3.12)$$

$$-\frac{1}{p_0} \frac{dp_0}{dx} = \frac{1}{T_0} \frac{dT_0}{dx} = -\frac{3}{16} \left(\frac{q_0}{\sigma T_0^4} \right) \alpha_0 \quad (3.13)$$

$$\alpha_0 = \alpha_0(T_0, p_0) \quad (3.14)$$

It is apparent from the form of equation (3.13) that one can get a differential equation for the temperature if a particular form for the

opacity $\alpha_o = \alpha_o(T_o, p_o)$ is taken. For convenience we chose a power law

$$\alpha_o = \alpha_a \left(\frac{T_o}{T_a}\right)^m \left(\frac{p_o}{p_a}\right)^n \quad (3.15)$$

so that the variations of opacity with temperature and pressure are represented by the constants

$$m = \frac{d \ln \alpha}{d \ln T}, \quad n = \frac{d \ln \alpha}{d \ln p} \quad (3.16a,b)$$

The ranges of values of m and n are discussed in Appendix A. The reference conditions denoted by the subscript $()_a$ are the gas conditions evaluated at the $x=0$ wall.

Since the pressure and net flux are constant, we have $p_o = p_a$ and $q_o = q_a$

Then we may write equation (3.13) as

$$-\left(\frac{T_o}{T_a}\right)^{3-m} \frac{d}{d(x/L)} \left(\frac{T_o}{T_a}\right) = \epsilon = \text{const.} \quad (3.17a)$$

where

$$\epsilon = \frac{3}{16} \left(\frac{q_a}{\sigma T_a^4}\right) \alpha_a L \quad (3.17b)$$

Solving, we get

$$\frac{T_o(x)}{T_a} = \begin{cases} \left[1 + (m-4)\epsilon \frac{x}{L}\right]^{\frac{1}{4-m}} & \text{if } m \neq 4 \\ \exp\left(-\epsilon \frac{x}{L}\right) & \text{if } m = 4 \end{cases} \quad (3.18a)$$

$$(3.18b)$$

It is important to notice that due to the boundary conditions on the radiation given in equation (2.43a,b), the gas temperature T_a at $x=0$ is not the wall temperature T_{wa} .

In fact, it is evident that the following relations must hold for the zeroth order solution,

$$\sigma T_{wa}^4 = \sigma T_a^4 + \frac{q_a}{2} \quad (3.19a)$$

$$\sigma T_{wb}^4 = \sigma T_b^4 - \frac{q_a}{2} \quad (3.19b)$$

$\begin{array}{ccc} \text{wall} & \leftarrow | \rightarrow & \text{gas} \\ \text{conditions} & & \text{conditions} \end{array}$

It will be convenient later on to have the temperature distribution in dimensionless form. Let

$$\xi = \frac{x}{L} \quad \eta(\xi) = \frac{T_0(\frac{x}{L})}{T_a} \quad (3.20)$$

then from equation (3.18)

$$\eta(\xi) = \begin{cases} [1 + (m-4)\epsilon\xi]^{\frac{1}{4-m}} & m \neq 4 \end{cases} \quad (3.21a)$$

$$\eta(\xi) = \begin{cases} \exp(-\epsilon\xi) & m = 4 \end{cases} \quad (3.21b)$$

The temperature distribution is plotted in Figure (3.2a-d) for various values of ϵ and m .

The temperature and other gradients are

$$-\frac{L}{T_0} \frac{dT_0}{dx} = \frac{L}{\rho_0} \frac{d\rho_0}{dx} = \frac{L}{4} \frac{1}{G_0} \frac{dG_0}{dx} = -\frac{d\eta}{d\xi} = \epsilon \eta^{m-3} \quad (3.22)$$

for any value of m .

All the steady-state or zeroth order quantities are expressible in terms of this distribution $\eta(\xi)$ as follows.

$$\frac{G_o}{G_a} = \eta^4(\xi), \quad \frac{p_o}{p_a} = \eta^{-1}, \quad \frac{T_o}{T_a} = \eta \quad (3.23a,b,c)$$

The quantity ϵ given in (3.17) which appears in the temperature distribution and is a measure of the steepness of the gradients in the gas (see 3.22) is actually a convenient combination of two more basic nondimensional quantities. We define the ratio of the net heat flux to twice the reference one-way equilibrium heat flux to be

$$Q = \left(\frac{q_o}{2\sigma T_a^4} \right) = \left[\frac{\sigma T_w^4}{\sigma T_a^4} - 1 \right] \quad (3.24)$$

The maximum range of Q is $0 \leq Q \leq 1$.

We also define an optical thickness to be the product of the reference opacity and the distance between the walls.

$$\tau = \alpha_a L \quad (3.25)$$

In terms of these quantities obviously

$$\epsilon = \frac{3}{8} Q \tau \quad (3.26)$$

To better appreciate the magnitude of these nondimensional parameters in terms of the physical quantities, see Figure (3.3). Since ϵ is a measure of the steepness of gradients, increasing either the net heat flux or the optical thickness will have an equivalent steepening effect on the gradients in the gas.

We can now calculate some of the characteristic temperature ratios of this zeroth order solution. The ratio of the gas temperatures at the walls is given by $\eta(\xi=1) = T_b/T_a$ or,

$$\eta(1) = \begin{cases} [1 + (m-4)\epsilon]^{\frac{1}{4-m}} & m \neq 4 \\ \exp(-\epsilon) & m = 4 \end{cases} \quad (3.27a)$$

$$(3.27b)$$

The ratio of the wall temperatures can be found from equations (3.19a,b) and is given by

$$\frac{T_{wb}}{T_{wa}} = \left[\frac{\sigma T_b^4 - q_0/2}{\sigma T_a^4 + q_0/2} \right]^{\frac{1}{4}} = \left[\frac{\eta^4(1) - Q}{1 + Q} \right]^{\frac{1}{4}} \quad (3.28)$$

III.3 The Maximum Allowable Radiative Heat Flux

Not all values of the variation of opacity with temperature, m , the heat flux parameter, Q , and the optical thickness, τ , lead to physically realizable temperature ratios. In other words, we can say that for a given variation of opacity with temperature, m , there is a maximum heat flux which can be passed through a stationary radiating gas. This maximum flux is given by that Q for which the wall temperature at the cold wall must be reduced to zero. From equation (3.28) this happens when $\sigma T_b^4 = \frac{q_0}{2}$ or, nondimensionally, when

$$\eta^4(1) = \frac{q_0}{2\sigma T_a^4} = Q_{max} \quad (3.29)$$

or,

$$Q_{max} = \begin{cases} [1 + (m-4)\epsilon]^{\frac{1}{1-\frac{m}{4}}} & m \neq 4 \\ \exp(-4\epsilon) & m = 4 \end{cases} \quad (3.30a)$$

$$(3.30b)$$

These last two equations can be thought of as defining a maximum parameter Q_m for any given m and τ . Unfortunately, we can only solve analytically for the inverse relationship $\tau = \tau(Q_m)$

Since $\epsilon = \frac{3}{8} Q \tau$, we solve (3.30a,b) for τ to get

$$\tau = \begin{cases} \left[\frac{1 - Q_m^{1-\frac{m}{4}}}{\frac{3}{2} (1-\frac{m}{4}) Q_m} \right] & m \neq 4 \\ \left[\frac{\ln Q_m^{-1}}{\frac{3}{2} Q_m} \right] & m = 4 \end{cases} \quad (3.31a)$$

$$(3.31b)$$

This maximum heat flux parameter $Q_{max} = Q_m(\tau, m)$ is plotted in Figure (3.4). The allowable range for Q then is $0 \leq Q \leq Q_m$. Notice $Q_m = 1$ only for $\tau = 0$, which is a completely transparent non-participating gas, and that Q_m decreases as τ increases.

III.4 Order of Solution

One may solve the complete zeroth order problem by the following sequence.

- a) Choose a constant pressure p_a , net radiative flux q_a , and a wall temperature T_{wa} , from which one can calculate $T_a = T_o(x=0)$.

b) Choose an optical thickness $\kappa_a L$ and variation of opacity with temperature w that best represents the physical situation of interest.

c) Calculate Q and ϵ and then the temperature distribution

$$\eta(\xi) = \frac{T_o}{T_a} \left(\frac{x}{L} \right) \quad \text{from which one can get the other wall temperature } T_{w_1}, \text{ and also the distributions } \rho_o(x) \text{ and } G_o(x).$$

Now that the model is defined and the zeroth order quantities are known, we can proceed to the derivation of the perturbation equations.

IV. THE PERTURBATION EQUATIONS FOR A STATIONARY RADIATING GAS

IV.1 Introduction

In this chapter we shall derive the equations of motion for the fluctuations of the radiating gas described in the last chapter. This is done by assuming that the fluid supports oscillations which are very small perturbations superimposed upon a state unchanging with time. The application of the usual linearized perturbation analysis will yield a set of differential equations which will be combined to form one partial differential equation in space and time of a mixed parabolic-hyperbolic type. The coefficients of this equation are complicated functions of the spatial coordinate due to the presence of the gradients described in the previous chapter. Some physically appropriate boundary conditions are discussed, as well as the whole question of normal modes and the proper solution of an initial value problem.

IV.2 The Perturbation Equations

The fluctuations obey the full unsteady gas dynamic equations as written in general in Sec. 2.9, however the remarks made earlier concerning the neglect of the radiation pressure, the adoption of a planar one-dimensional geometry, and a particular opacity function still apply. under these conditions, the unsteady equations are

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \quad (4.1)$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho u^2) = 0 \quad (4.2)$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\gamma-1}\right) + \frac{\partial}{\partial x}\left[u\left(\frac{\gamma p}{\gamma-1} + \rho \frac{u^2}{2}\right) + q\right] = 0 \quad (4.3)$$

$$\frac{\partial q}{\partial x} = \alpha (4\sigma T^4 - G) \quad (4.4)$$

$$\frac{\partial G}{\partial x} = -3\alpha q \quad (4.5)$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad (4.6)$$

$$\frac{d\alpha}{\alpha} = m \frac{dT}{T} + n \frac{dp}{p} \quad (4.7)$$

We now do a small perturbation analysis on equations (4.1-7) using an expansion like

$$\begin{aligned} \rho(x,t) &= \rho_0(x) + \rho_1(x,t) + \dots \\ T(x,t) &= T_0(x) + T_1(x,t) + \dots \quad \text{etc.} \end{aligned} \quad (4.8)$$

where $\left|\frac{\rho_1}{\rho_0}\right| \ll 1$, $\left|\frac{T_1}{T_0}\right| \ll 1$ etc.

and squared terms like ρ_1^2 , $\rho_1 T_1$ shall be dropped as higher order.

This will leave a set of first order equations linear in the perturbation variables. However the coefficients will vary with x due

to the zeroth order quantities like $T_0(x)$ and $\frac{dT_0(x)}{dx}$. The equations then become

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} + \frac{d\rho_0}{dx} u_1 = 0 \quad (4.9)$$

$$\rho_0 \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x} = 0 \quad (4.10)$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0 \frac{\partial u_1}{\partial x} + (\gamma - 1) \frac{\partial q_1}{\partial x} = 0 \quad (4.11)$$

$$\frac{\partial q_1}{\partial x} = \alpha_0 (16 \sigma T_0^3 T_1 - G_1) \quad (4.12)$$

$$\frac{\partial G_1}{\partial x} = -3(\alpha_0 q_1 + \alpha_1 q_0) \quad (4.13)$$

$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} \quad (4.14)$$

$$\frac{\alpha_1}{\alpha_0} = m \frac{T_1}{T_0} + n \frac{p_1}{p_0} \quad (4.15)$$

It seems advantageous to immediately use equations (4.14) and (4.15) to eliminate ρ_1 and α_1 from the set of equations, reducing the set to five equations in the five unknowns p_1 , T_1 , u_1 , q_1 , G_1 . They are

$$\frac{1}{p_0} \frac{\partial p_1}{\partial t} - \frac{1}{T_0} \frac{\partial T_1}{\partial t} + \frac{\partial u_1}{\partial x} + \left(\frac{1}{p_0} \frac{d\rho_0}{dx} \right) u_1 = 0 \quad (4.16)$$

$$\rho_0 \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x} = 0 \quad (4.17)$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0 \frac{\partial u_1}{\partial x} + (\gamma - 1) \frac{\partial q_1}{\partial x} = 0 \quad (4.18)$$

$$\frac{\partial q_1}{\partial x} - 16 \alpha_0 \sigma T_0^3 T_1 + \alpha_0 G_1 = 0 \quad (4.19)$$

$$\frac{1}{3} \frac{\partial G_1}{\partial x} + \alpha_0 q_1 + \left(\frac{m \alpha_0 q_0}{T_0} \right) T_1 + \left(\frac{n \alpha_0 q_0}{p_0} \right) p_1 = 0 \quad (4.20)$$

In fact we can write these equations in a very succinct manner if we let

$$\underline{v} = \{ p_1, T_1, u_1, q_1, G_1 \}$$

be a state vector the components of which are the perturbation variables.

Then the equations can be written quite symmetrically as

$$\underline{A} \frac{\partial}{\partial t} \underline{v} + \underline{B} \frac{\partial}{\partial x} \underline{v} + \underline{C} \underline{v} = 0 \quad (4.21)$$

where the matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ are

$$\begin{aligned}
 \underline{\underline{A}} &= \begin{bmatrix} \frac{1}{p_0} & -\frac{1}{T_0} & 0 & 0 & 0 \\ 0 & 0 & p_0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \underline{\underline{B}} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta p_0 & (\delta-1) & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} \\
 \underline{\underline{C}} &= \begin{bmatrix} 0 & 0 & \frac{1}{p_0} \frac{dp_0}{dx} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (-16\alpha_0 \sigma T_0^3) & 0 & 0 & \alpha_0 \\ (\frac{n\alpha_0 q_0}{p_0}) & (\frac{m\alpha_0 q_0}{T_0}) & 0 & \alpha_0 & 0 \end{bmatrix} & \underline{\underline{v}} &= \begin{Bmatrix} p_1 \\ T_1 \\ u_1 \\ q_1 \\ G_1 \end{Bmatrix}
 \end{aligned} \tag{4.22}$$

One reason to write the equations in this manner is that it is necessary in order to classify the system of equations. The reader may consult Courant and Hilbert (Ref. 10) for the details. The essence of the matter is to determine the roots λ_i of the following determinantal equation

$$|\underline{\underline{A}} - \lambda \underline{\underline{B}}| = 0 \tag{4.23}$$

If this determinant has only imaginary roots the system of equations is elliptical; if the roots are all real the system is hyperbolic; and if they are zero the system is parabolic. If various combinations

occur the system is of a mixed type and further classification subgroups are of limited utility.

If one calculates the determinant (4.23) for the coefficients given by (4.22) the result is

$$\lambda^2 \left(\lambda^2 - \frac{1}{\gamma RT_0} \right) = 0 \quad (4.24)$$

so that

$$\lambda = \pm 0, \pm \frac{1}{\sqrt{\gamma RT_0}} \quad (4.25)$$

These λ 's define the characteristic directions and they appear here in pairs. The second set of characteristics are for the familiar acoustic mode, and this mode consists of oscillatory waves in the usual sense since the characteristics are real and therefore hyperbolic. The first set of characteristics, which are zero, define the "radiation induced waves" named by Vincenti and Baldwin in Ref. 2. These are not waves in the proper sense since the mode is parabolic, hence they are like thermal or heat "waves", which are non-oscillatory propagating disturbances. Both modes have left and right running components. The main equations are of the mixed hyperbolic-parabolic type then and the fluctuations are apparently a mix of the familiar acoustic waves and thermal waves. A discussion of the propagation speeds and damping or amplification rates of these waves will be given in the next chapter.

At this point it is desirable to nondimensionalize the equations and make use of the zeroth order distribution $\eta(\xi)$. We repeat the following results from the last section

$$\eta(\xi) = \frac{T_0}{T_a} = \left(\frac{p_0}{p_a}\right)^{-1} = \left(\frac{G_0}{G_a}\right)^{1/4} \quad (4.26)$$

and

$$-\frac{1}{T_0} \frac{dT_0}{d\xi} = \frac{1}{p_0} \frac{dp_0}{d\xi} = \frac{1}{4G_0} \frac{dG_0}{d\xi} = -\frac{d\eta}{d\xi} = \epsilon \eta^{m-3} \quad (4.27)$$

where

$$\begin{aligned} \eta(\xi) &= \left[1 + (m-4)\epsilon\xi\right]^{\frac{1}{4-m}} & m \neq 4 \\ &= \exp(-\epsilon\xi) & m = 4 \end{aligned} \quad (4.28)$$

and p_0 and q_0 are constants. Remember that the convenient nondimensional parameter ϵ consists of a product of two more basic ones

$$\epsilon = \frac{3}{8} Q \tau \quad (4.29)$$

where

$$Q = \frac{q_0}{2\sigma T_a^4}$$

Ratio of net heat flux
to one-way equilibrium
heat flux (4.30)

and

$$\tau = \alpha_a L$$

Ratio of distance between
walls to photon mean free (4.31)
path

The other important nondimensional parameter is the Boltzmann number

$$N_b = \left(\frac{\gamma}{\gamma-1}\right) \frac{p_a \sqrt{RT_a}}{\sigma T_a^4}$$

Ratio of characteristic
gas enthalpy flux to
the one-way equilibrium
heat flux (4.32)

See Figure (4.1) for a plot of N_b in terms of physical quantities.

Now we can express the equations in dimensionless form in terms of the state vector

$$\underline{v} = \left\{ \left(\frac{p_i}{p_a} \right), \left(\frac{T_i}{T_a} \right), \frac{u_i}{\sqrt{RT_a}}, \frac{q_i}{\sigma T_a^4}, \frac{G_i}{\sigma T_a^4} \right\} \quad (4.33)$$

and the dimensionless time $\hat{t} = \frac{t \sqrt{RT_a}}{L}$ and length $\hat{x} = \frac{x}{L}$ in the same symbolic form

$$\underline{A} \frac{\partial}{\partial \hat{t}} \underline{v} + \underline{B} \frac{\partial}{\partial \hat{x}} \underline{v} + \underline{C} \underline{v} = 0 \quad (4.34)$$

where

$$\underline{A} = \begin{bmatrix} 1 & -\eta^{-1} & 0 & 0 & 0 \\ 0 & 0 & \eta^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \underline{B} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & \left(\frac{\delta}{N_s} \right) & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 0 & 0 & \left(\frac{3}{16} Q \tau \eta^{m-3} \right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (-16 \tau \eta^{m+3}) & 0 & 0 & (\tau \eta^m) \\ (n Q \tau \eta^m) & (m Q \tau \eta^{m-1}) & 0 & (\tau \eta^m) & 0 \end{bmatrix}$$

(4.35)

Note that the uniform gas limit is approached when $Q \rightarrow 0$, $\eta(\xi) \rightarrow 1.0$.

Since matrix $\underline{A}$ has three columns which have at least one nonzero element, the system is third order in time. Similarly, since $\underline{B}$ has four columns having at least one nonzero element the system is fourth order in the spatial coordinate ξ and will require four boundary conditions. We attend to this in the next section.

IV.3 Boundary Conditions

Since the system of perturbation equations we have derived is a fourth order system we need four conditions at the walls to complete the spatial description of the problem. Two of these restrictions are gasdynamic in nature and two are radiative. The first set of boundary conditions require that there be no flow through the wall, i.e. that the fluid velocity at the wall vanish. This is true of the zeroth order solution u_0 and must be true of the perturbation solution.

$$u, (\xi = 0) = u, (\xi = 1) = 0 \quad (4.36)$$

The second set of boundary conditions may be derived from the radiative conditions discussed in Sec. 2.7. These state that for all time we must have, from (2.43a,b)

$$4\sigma T_{w_a}^4 = 4\sigma T^4(0,t) - \frac{1}{\alpha} \frac{\partial q}{\partial x}(0,t) + 2q(0,t)$$

$$4\sigma T_{w_b}^4 = 4\sigma T^4(1,t) - \frac{1}{\alpha} \frac{\partial q}{\partial x}(1,t) - 2q(1,t)$$

Now our model is for black walls that maintain a constant temperature

with time. When we perturb the above equations, and subtract off the zeroth order or steady state components, a fluctuation boundary condition will remain. For example, at the $\xi=0$ wall, since $T_{\text{wall}} \neq T_w(t)$ and $T_{\text{gas}}(\xi=0, t) = T_a + T_1(0, t)$ we have

$$4\sigma T_a^4 = 4\sigma T_a^4 + 16\sigma T_a^3 T_1(0, t) - \cancel{\frac{1}{\alpha_0} \frac{dq_0}{dx}}^0 + \cancel{\frac{\alpha_1}{\alpha_0} \frac{dq_0}{dx}}^0 - \frac{1}{\alpha_0} \frac{\partial q_1(0, t)}{\partial t} + 2q_0 + 2q_1(0, t)$$

After subtracting off the zeroth order relation we are left with

$$0 = 16\sigma T_a^3 T_1(0, t) - \frac{1}{\alpha_0} \frac{\partial q_1(0, t)}{\partial t} + 2q_1(0, t) \quad (4.37a)$$

Similarly we have for the other wall ($\xi=1$)

$$0 = 16\sigma T_b^3 T_1(1, t) - \frac{1}{\alpha_b} \frac{\partial q_1(1, t)}{\partial t} - 2q_1(1, t) \quad (4.37b)$$

The equations may be interpreted as follows. The perfectly opaque wall absorbs and immediately reemits all radiation incident on it so as to keep a constant temperature. When the gas temperature fluctuates next to the wall the radiative flux must fluctuate in such a way as to maintain continuity of energy flux through our imaginary plane separating the gas from the wall. For example, instantaneously a small gas temperature decrease near the wall will result in an increased radiative flux from the wall to the gas. The gas and the boundary may exchange energy during a portion of a cycle of oscillation. If a time average is taken however, it is not clear whether there can be a net

power flow from the walls to the gas, even though variations of wall temperature and displacement are prohibited. It would appear that the only conclusive way to demonstrate the importance of the role played by the boundaries is to solve the boundary value problem for the allowed oscillations.

The chosen wall boundary conditions are summarized now in dimensionless form

$$\begin{aligned}\frac{u_1(0)}{\sqrt{RT_w}} &= 0 \\ \frac{u_1(1)}{\sqrt{RT_w}} &= 0\end{aligned}\tag{4.38a,b}$$

$$\begin{aligned}16 \frac{T_1(0)}{T_w} - \frac{1}{2} \frac{\partial}{\partial \xi} \left(\frac{q_1(0)}{\sigma T_w^4} \right) + 2 \frac{q_1(0)}{\sigma T_w^4} &= 0 \\ 16 \eta^{(1)} \frac{T_1(1)}{T_w} - \frac{1}{2} \eta^{(1)} \frac{\partial}{\partial \xi} \left(\frac{q_1(1)}{\sigma T_w^4} \right) - 2 \frac{q_1(1)}{\sigma T_w^4} &= 0\end{aligned}\tag{4.39 a,b}$$

Before continuing it is well to note an alternate way of writing the radiative boundary conditions. If we perturb equations (2.42a,b) directly these boundary conditions become simply that

$$\begin{aligned}\left(\frac{G_1(0)}{\sigma T_w^4} \right) + 2 \left(\frac{q_1(0)}{\sigma T_w^4} \right) &= 0 \\ \text{and} & \\ \left(\frac{G_1(1)}{\sigma T_w^4} \right) - 2 \left(\frac{q_1(1)}{\sigma T_w^4} \right) &= 0\end{aligned}\tag{4.40a,b}$$

The choice of whether to use statements (4.39a,b) or (4.40a,b),

which are related through equation (4.12), depends only on mathematical preference or ease of physical interpretation, for they are entirely equivalent.

IV.4 Normal Modes

Fortunately our set of partial differential equations has the property of having coefficients which do not depend upon time, since the coefficients depend only on the steady-state zeroth order solution. When this is the case it is more convenient to work with each of the normal modes of the system rather than the general time response. We can determine the normal modes by separating the perturbation solution into a product of spatial and time dependent terms, for example

$$\frac{p(\xi, \hat{t})}{p_n} = p(\xi, \Omega) e^{-i\Omega \hat{t}} \quad (4.41)$$

where the separation parameter is in this case the nondimensional complex frequency $\Omega = \Omega_r + i\Omega_i$. It will turn out that this representation will be possible only for particular values of Ω once the boundary conditions are satisfied, though there will in general be an infinite number of these characteristic values. Each characteristic value and its corresponding solution constitute a normal mode. Once the normal modes $p_n(\xi, \Omega_n)$ are found one solves the initial value problem by superimposing all the normal modes at time $\hat{t} = 0$ in the correct combination to form the initial specified distribution $p(\xi, \hat{t} = 0)$, where

$$\frac{p_1(\xi, \hat{t})}{p_a} = p(\xi, \hat{t}) = \sum_k A_k p_k(\xi, \Omega_k) e^{-i\Omega_k \hat{t}} \quad (4.42)$$

It is readily apparent that if any one of these modes has a frequency which has a positive imaginary part the behavior of the perturbation for large time will be exponential growth to arbitrarily large values. This is an absolute instability in the linear sense. Marginal, or neutral stability occurs for the k^{th} mode when $\Omega_k = 0$. It should be clear at this point that to study the stability of a system it is not necessary to solve the entire initial value problem; rather it is sufficient to find the characteristic frequencies as a function of the important system parameters. If any have positive imaginary parts it is impossible to find physically correct system behavior for all time, since one cannot then construct a superposition of modes that is asymptotically finite for large times.

The notation from now on will be that a variable without subscripts will be the dimensionless perturbation after the time has been separated out, so that

$$p(\xi, \Omega) = \frac{p_1(\xi, \Omega)}{p_a} \quad u(\xi, \Omega) = \frac{u_1}{\gamma R T_a} \quad (4.43)$$

Once equation (4.42) has been substituted into our system of equations and the exponential time factor cancelled out only the spatial derivatives survive.

Our first order system of equations for the characteristic vector

$$\underline{v} = \{ p, T, u, q, G \} \quad (4.44)$$

may then be written as

$$\underline{B} \frac{d}{dz} \underline{v} + (\underline{C} - i\Omega \underline{A}) \underline{v} = 0 \quad (4.45)$$

where $\underline{A}$, $\underline{B}$, and $\underline{C}$ are given in (4.35).

This is the form most convenient for numerical integration, but not the most convenient for approximate analytical solutions, for which it is desirable to have a higher order equation in one unknown.

IV.5 The Consolidation to One Dependent Variable

The system of equations for p, T, u, q, G given by (4.45) with the coefficients (4.35) are repeated below for convenience. We have introduced the prime notation for derivatives with respect to the independent variable ($' = \frac{d}{dz}$). The equations are

$$u' + \epsilon \eta^{m-3} u + i\Omega \eta^{-1} T - i\Omega p = 0 \quad (4.46)$$

$$p' - i\Omega \eta^{-1} u = 0 \quad (4.47)$$

$$\delta u' + \frac{\delta}{N_0} q' - i\Omega p = 0 \quad (4.48)$$

$$q' - 16\tau \eta^{m+3} T + \tau \eta^m G = 0 \quad (4.49)$$

$$G' + 3\tau \eta^m q + 16m\epsilon \eta^{m-1} T + 16n\eta^m \epsilon p = 0 \quad (4.50)$$

The boundary conditions, from Sec. 4.3 are

$$u(0) = 0 \quad (4.51)$$

$$u(1) = 0 \quad (4.52)$$

$$G(0) + 2q(0) = 0 \quad (4.53)$$

$$G(1) - 2q(1) = 0 \quad (4.54)$$

We now wish to express these equations in terms of one fourth order differential equation in one independent variable. We choose the pressure perturbation $p(\xi, \Omega)$ since it may have a preferred physical significance. All other dependent variables can then be expressed in terms of the pressure perturbation and its derivatives. If the equation for the pressure perturbation $p(\xi, \Omega)$ is as follows

$$a p^{iv} + b p''' + c p'' + d p' + e p = 0 \quad (4.55)$$

the variable coefficients $a(\eta(\xi), \Omega)$, $b(\eta(\xi), \Omega)$ etc. are

$$a = 1 - F \eta^{-m-4} \quad (4.56a)$$

$$b = \epsilon \left\{ \eta^{m-3} + (2m-11) \eta^{m-4} - 3(m-1)F \eta^{-8} \right\} \quad (4.56b)$$

54.

$$c = \Omega^2 \eta^{-1} + F \left(3\tau^2 \eta^{m-4} - \frac{\Omega^2}{8} \eta^{-m-5} \right) - \epsilon^2 \left[4\eta^{2m-7} + (m-7)\eta^{2m-8} + F(m^2 + m - 9)\eta^{m-12} \right] \quad (4.56c)$$

$$d = \epsilon \left[\Omega^2 (2m + n - 8) \eta^{m-5} - 3F \left(\frac{m\Omega^2}{8} \eta^{-9} + \tau^2 \eta^{2m-8} \right) \right] + \epsilon^3 \left[2(m-2)\eta^{3m-11} - (m-3)\eta^{3m-12} + 21F\eta^{2m-16} \right] \quad (4.56d)$$

$$e = \frac{\Omega^2 F}{8} \left[3\tau^2 \eta^{m-5} - \epsilon^2 m(m+4)\eta^{m-13} \right] \quad (4.56e)$$

where $F = \frac{i\Omega N_0}{16\tau}$

(4.57)

is a convenient grouping.

The other dependent variables bear the following relationships to the pressure perturbations

$$u = -\frac{i\eta p'}{\Omega} \quad (4.58)$$

$$T = \frac{\eta^2 p''}{\Omega^2} + \frac{\epsilon}{\Omega^2} [\eta^{m-1} - \eta^{m-2}] p' + \eta p \quad (4.59)$$

$$\begin{aligned} \left(\frac{1}{16}\right)G &= \frac{1}{\Omega^2} [\eta^5 - F\eta^{1-m}] p'' + \frac{\epsilon}{\Omega^2} [\eta^{m+2} - \eta^{m+1} + F\eta^{-3}] p' \\ &\quad + \left[\eta^4 - \frac{F}{8} \eta^{-m} \right] p \end{aligned} \quad (4.60)$$

$$\begin{aligned}
\left(-\frac{3\tau}{16}\right) q &= \frac{1}{\Omega^2} \left[\eta^{5-m} - F \eta^{1-2m} \right] p''' \\
&+ \frac{\epsilon}{\Omega^2} \left[\eta^2 + (m-6) \eta - (m-2) F \eta^{m-3} \right] p'' \\
&+ \left\{ \eta^{4-m} - \frac{F}{8} \eta^{-2m} + \frac{\epsilon^2}{\Omega^2} \left[-2\eta^{m-2} + \eta^{m-3} + 3F \eta^{-7} \right] \right\} p' \\
&+ \epsilon \left[m+n-4 - \frac{mF}{8} \eta^{m-4} \right] p
\end{aligned}
\tag{4.61}$$

Using these relationships it is possible to give the boundary conditions (4.51) (4.54) in terms of the pressure perturbation.

Defining $\eta_1 = \eta(\xi=1)$, and noting $\eta(\xi=0)=1$, we have, for $\Omega \neq 0$

$$p'(0) = 0 \tag{4.62}$$

$$p'(1) = 0 \tag{4.63}$$

$$\begin{aligned}
(1-F) p'''(0) &+ \left[(m-5 - (m-2)F) \epsilon - \frac{3\tau}{2} (1-F) \right] p''(0) \\
&+ \left[\left(1 - \frac{F}{8}\right) \Omega^2 - \frac{3\tau F}{2} \epsilon + \epsilon^2 (3F-1) \right] p'(0) \\
&+ \Omega^2 \left[\epsilon \left(m+n-4 - \frac{mF}{8}\right) - \frac{3\tau}{2} \left(1 - \frac{F}{8}\right) \right] p(0) = 0
\end{aligned}
\tag{4.64}$$

$$\begin{aligned}
& (\eta_i^{5-m} - F \eta_i^{1-2m}) p'''(1) + \left\{ \epsilon (\eta_i^2 + (m-6)\eta_i - (m-2)F \eta_i^{-m-3}) \right. \\
& \quad \left. + \frac{3\tau}{2} (\eta_i^5 - F \eta_i^{1-m}) \right\} p''(1) \\
& + \left\{ \Omega^2 \left(\eta_i^{4m} - \frac{F}{8} \eta_i^{-2m} \right) + \epsilon^2 \left(-2\eta_i^{m-2} + \eta_i^{m-3} + 3F \eta_i^{-7} \right) \right. \\
& \quad \left. + \frac{3\tau\epsilon}{2} \left(\eta_i^{m+2} - \eta_i^{m+1} + F \eta_i^{-3} \right) \right\} p'(1) \\
& + \Omega^2 \left\{ \epsilon \left(m+n-4 - \frac{mF}{8} \eta_i^{-m-4} \right) + \frac{3\tau}{2} \left(\eta_i^4 - \frac{F}{8} \eta_i^{-m} \right) \right\} p(1) = 0
\end{aligned}$$

(4.65)

IV.6 Summation

The specification of a well defined boundary value problem is now complete. There are no restrictions whatsoever on the Boltzmann number or optical depth in the above formulation, a power-law variation of opacity with temperature and pressure is allowed, and extremely large net radiative fluxes can be accommodated. This model is more general in the latter two respects than the earlier treatment of Vincenti and Baldwin (Ref. 2), and needless to say our formulation reduces to theirs in the limit $m=n=q=0$, no net heat flux and constant opacity.

An important fact of physical significance is immediately evident from the above equations. In the limit of no gradients, $\epsilon=0$, all

terms involving the variations of opacity with temperature and pressure vanish as well, indicating that no matter how dramatically the opacity varies with temperature a temperature fluctuation in a uniform gas will be completely unaffected. This is the first indication that a net radiative flux is an essential ingredient in the growth of a fluctuation, as sort of a "potential source of power input" to the fluctuation. We must wait until we have a solution however to show that the second essential ingredient, the variation of opacity, controls the extent to which the oscillation will grow or damp out by exchanging energy with the fluid.

V. A "LOCAL" ANALYSIS OF THE EFFECT OF A NET RADIATIVE FLUX ON
WAVE PROPAGATION

V.1 Introduction

One of the more profitable techniques used in scientific analysis is to pursue the analogies between a new problem under study and the well understood classical solutions to more simple problems. If one understands forced harmonic oscillators for example, and pursues the analogy with care, he can explain a considerable body of physical phenomena. The difficulty of course always lies in the dissimilarities between problems, but it may be a profitable approach nevertheless.

The present problem bears a great resemblance to vibration problems in strings or beams with varying properties. The analogy need not be carried into a comparison of equations however, for the following remarks will make clear the similarities. When one is confronted with a partial differential equation of the type encountered in this study or in the study of vibrations and sound in general, the approach is often quite standard. One first separates the time and space variables to reduce the problem to an ordinary differential equation. The problem is then to solve a boundary value problem, which is trivial if the properties are constant between boundaries. The solutions are then linear combinations of exponentials and there exists an algebraic relationship between frequency and wave number called the dispersion relation. The imposition of boundary conditions restricts the allowed wave numbers to a denumerably infinite set of particular values, so that the allowed frequencies are then also

calculable. In solving such problems one can usually determine the normal modes of a beam, say, for a number of different boundary conditions, such as for a pinned end and then for a cantilevered end. The striking observation about such problems is that the spectra of allowed frequencies of oscillation are quite different for the lowest frequencies but approach each other for larger frequencies and wave numbers. Apparently the smaller the wave length of the disturbance compared to the distance between the boundaries, the less effect the boundaries have on the frequency spectrum. (One may consult the treatise by Morse and Ingard, Ref. 11, for detailed discussions and examples.) If this is the case then one ought to be able to learn something about wave propagation at higher frequencies without solving the boundary value problem, depending instead solely on the study of the dispersion relation. This is all well known and is used constantly in problems of this type; the procedure is reviewed only to contrast it to the case where the oscillating medium has properties which vary with the spatial variable. There is danger in carrying over the idea of a dispersion relation and applying it to wave propagation in inhomogeneous media with careless abandon, as is so often done in the literature. To point out these differences, a simple example shall be examined in greater detail.

V.2 An Illustrative Example of Wave Propagation in Inhomogeneous Media

In a planar geometry the essential mathematical difference between wave propagation in homogeneous and inhomogeneous media is that the

differential equations have constant and varying coefficients respectively. The principal effect of varying coefficients in the wave equations is that for a wave of one frequency, the wave number of the wave is coupled to the amplitude so that there is no longer a simple relationship between frequency and wave number. A simple example will clarify this. Suppose we have the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2} = 0 \quad (5.1)$$

on the interval $0 \leq x \leq L$

We can separate out the time dependence and study only the n^{th} normal mode if we let

$$y = \sum_n u_n(x, \omega_n) e^{-i\omega_n t} \quad (5.2)$$

Then we have an ordinary differential equation for $u_n(x, \omega_n)$

$$\frac{d^2 u_n}{dx^2} + \frac{\omega_n^2}{a^2} u_n = 0 \quad (5.3)$$

If the coefficient is a constant, one ordinarily studies the Fourier representation of $u_n(x)$ by assuming that, for example

$$u_n(x, \omega_n) = \int_{-\infty}^{+\infty} U_n(k, \omega_n) e^{ikx} \frac{dk}{2\pi} \quad (5.4a)$$

and

$$U_n(k, \omega_n) = \int_0^L u_n(x, \omega_n) e^{-ikx} dx \quad (5.4b)$$

Let us multiply Equation (5.3) by e^{-ikx} and integrate over x . We have then

$$\int_0^L \left[-k^2 + \frac{\omega_n^2}{a^2} \right] u_n(x, \omega_n) e^{-ikx} dx = 0 \quad (5.5)$$

Noting that the content of the square bracket is not a function of x and using equation (5.4b) leads to the statement

$$\left[-k^2 + \frac{\omega_n^2}{a^2} \right] U_n(k, \omega_n) = 0 \quad (5.6)$$

Since U_n will in general not be zero, the contents of the brackets must vanish and

$$a^2 k^2 = \omega_n^2 \quad (5.7)$$

is the desired dispersion relation $k = k(\omega_n)$. Roots of this equation give left and right running waves with constant phase velocity a .

Contrast with this situation the case of varying coefficients.

For example let

$$\frac{1}{a^2} = \left(1 + \frac{\delta x}{L} \right), \quad \delta \text{ const.} \quad (5.8)$$

Then equation (5.5) becomes

$$0 = \int_0^L \left[-k^2 + \frac{\omega_n^2}{a_0^2} \left(1 + \frac{\delta x}{L} \right) \right] u_n(x, \omega_n) e^{-ikx} dx \quad (5.9)$$

It is apparent that one may not rigorously manipulate this integral to isolate a dispersion relationship as in (5.6). The k, ω and $U_n(k)$ are interrelated in a more complicated functional relationship

$k = k(\omega_n, U_n(k))$. For this simple example this relationship can be determined, though it is not very useful.

We make use of the Fourier Transform identity that if the transform of $f(x)$ is $F(k)$ the transform of $xf(x)$ is $i \frac{dF}{dk}$. See Morse and Feshbach (Ref. 12) or any similar work. Then equation (5.9) becomes

$$\left(-k^2 + \frac{\omega_n^2}{a_0^2}\right)U_n + \frac{i\omega_n^2 \delta}{a_0^2 L} \frac{dU_n}{dk} = 0 \quad (5.10)$$

The proper analog to the dispersion relation then is

$$k^2 = \frac{\omega_n^2}{a_0^2} \left[1 + \frac{i\delta}{kL} \left(\frac{k}{U_n} \frac{dU_n}{dk} \right) \right] \quad (5.11)$$

which may be interpreted as saying that the relationship between frequency and wavenumber depends on the slope of the Fourier amplitude in wavenumber space. This slope cannot be known until one has the actual solution to the differential equation (5.3) and associated boundary conditions. But this conflicts with our hope for determining the frequency-wavenumber relationship for large wavenumbers which experience would indicate to be insensitive to boundary conditions.

Before attempting to resolve this dilemma we must revise our criterion for when the wavenumber or frequency spectrum should be insensitive to the boundary conditions. For a homogeneous gas this

occurs when the wavelength λ is small compared to the length L between boundaries

$$\lambda = \frac{2\pi}{k} \ll L \quad (5.12)$$

In an inhomogeneous medium a new length scale appears, which is the characteristic length for change of properties of the medium. For example, if the density varies with distance, the length scale $\left| \frac{1}{\rho} \frac{d\rho}{dx} \right|^{-1}$ may be smaller than the distance L between the boundaries. In this case it is the proper length scale to which we must compare the disturbance wavelength.

$$\lambda = \frac{2\pi}{k} \ll \left| \frac{1}{\rho} \frac{d\rho}{dx} \right|^{-1} \quad (5.13)$$

This can be a more restrictive criterion if the medium properties change appreciably. In any case it is the most restrictive condition that gives the proper criterion for when wavenumbers are "large enough" for boundary conditions to have little effect.

We return to the attempt to find a way to extract wavenumber versus frequency information from a wave equation with varying coefficients. The crux of the matter is that the individual Fourier components are no longer each a solution to the differential equation. The eigenfunction solutions are no longer harmonic. However, it is felt that for small gradients in the medium and therefore for slowly varying coefficients, the solutions should not be too different from pure sinusoids. We want to investigate here the conditions under

which the actual eigenfunctions can be approximated by harmonic functions of slowly varying amplitude and phase.

To do this we assume a trial eigenfunction solution of the form

$$\phi_n = C(k, \omega_n) e^{+ikx} \quad (5.14)$$

where the wavenumber is now complex.

$$k = k_r + ik_i \quad (5.15)$$

There is an infinity of these eigenfunctions which must be summed if one is to satisfy various boundary conditions.

$$u(x, \omega_n) = \int_{-\infty}^{+\infty} \phi_n \frac{dk}{2\pi} \quad (5.16)$$

The idea is to find out if there are any values of k_r and k_i for which these are approximate solutions in some sense. Let us start with the differential equation with varying coefficients

$$\frac{d^2 u_n}{dx^2} + \frac{\omega_n^2}{a_0^2} \left(1 + \delta \frac{x}{L}\right) u_n = 0 \quad (5.17)$$

If we substitute our trial solution ϕ_n for u_n it will not identically satisfy the equation since it is only an approximation to the actual eigenfunction. Let this error be the function $E(x, \omega_n, k)$

$$\frac{d^2 \phi_n}{dx^2} + \frac{\omega_n^2}{a_0^2} \left(1 + \delta \frac{x}{L}\right) \phi_n = E(x, \omega_n, k) \neq 0 \quad (5.18)$$

Using (5.14) we can write

$$C(k, \omega_n) \left[-k^2 + \frac{\omega_n^2}{a^2} \left(1 + s \frac{x}{L} \right) \right] e^{ikx} = E(x, \omega_n, k) \quad (5.19)$$

Now the idea is to choose k in such a way that the average error

$$\bar{E}(\omega_n, k) = \int_a^b E(x, \omega_n, k) r(x, k) \frac{dx}{L} \quad (5.20)$$

where $r(x, k)$ is some weighting function, vanishes over some interval $a \leq x \leq b$. Several clarifications may be necessary here.

1) There is no approximation which is obviously the "best".

Different choices for the weighting function $r(x, k)$ will result in different values for k .

2) It is unreasonable to expect to get a good approximation over a very large interval when one has only one unknown parameter to vary for a best fit. It seems reasonable to require that the average error vanish over one wavelength, but not over the entire distance between boundaries.

3) A reasonable weighting function for such an average is the trial function itself (or rather its complex conjugate). This allows one to take advantage of orthogonality properties and greatly simplifies the integrals.

4) More elaborate procedures for finding approximate eigenvalues are extant of course (such as the Galerkin method), but our method is fine for a local approximation. See Collatz (Ref. 13) for the details of such methods.

Let us use the complex conjugate of the trial function Φ_n^* as $r(x, k)$ and try it out on the equation (5.19)

$$CC^* \int_a^b \left[-k^2 + \frac{\omega_n^2}{a_0^2} \left(1 + S \frac{x}{L} \right) \right] e^{i(k-k^*)x} \frac{dx}{L} = \bar{E}(\omega_n, k)$$

Integrating and requiring that $\bar{E} = 0$ gives

$$k^2 = \frac{\omega_n^2}{a_0^2} \left(1 + \frac{S}{2} \left(\frac{b-a}{L} \right) \right)$$

Let the length of the interval $(b-a)$ be $\frac{\lambda}{2\pi} = \frac{1}{|k|}$

Then

$$k^2 = \frac{\omega_n^2}{a_0^2} \left(1 + \frac{S}{2|k|L} \right) \quad (5.21)$$

It is apparent that this reduces to the dispersion relation for a homogeneous medium for either negligible gradient $S \rightarrow 0$ or of large wavenumbers ($kL \gg 1$). This reinforces the intuitive arguments given earlier in the section about the propagation of short wavelength disturbances in inhomogeneous media.

This method deserves its name as a "local" analysis because it is an accurate description only over a distance of approximately a wavelength. We will use it in the following sections to find the effect of a net heat flux on the propagation of waves in radiating gas. This local analysis is of course useful over a larger domain of wavenumbers if the gradients are not too steep. A very useful approximation is described in the next section which simplifies the coefficients in the

differential equation for the waves so that they are more tractable for the application of our local analysis.

V.3 The Small Gradient Approximation

The ordinary differential equation with varying coefficients which describes the spatial behavior of the acoustic waves and the radiatively induced heat waves in a radiating gas is given by equations (4.55) and (4.56a-e). This one is quite a bit more complicated than our example in the previous section in that it is of fourth order and has very unwieldy coefficients. That the equation is of higher order will cause us little additional difficulty, and the coefficients may be simplified by judicious approximation without sacrificing too much generality. The equation for the pressure perturbation is of the form

$$a p^{iv} + b p''' + c p'' + d p' + e p = 0 \quad (5.22)$$

where $(\quad)' = \frac{d(\quad)}{d\xi}$, $\xi = \frac{x}{L}$ and the coefficients are functions of the zeroth order distribution $\eta(\xi)$. Let us examine this distribution, $\eta = \frac{T_0}{T_a}(\frac{x}{L})$ the dimensionless temperature function, more closely.

From (3.21) we have

$$\eta(\xi) = \frac{T_0(\xi)}{T_a} = \begin{cases} [1 + (m-4)\epsilon\xi]^{\frac{1}{4-m}} & m \neq 4 \\ \exp(-\epsilon\xi) & m = 4 \end{cases} \quad (5.23)$$

The temperature gradient is then

$$\frac{L}{T_0} \frac{dT_0}{dx} = \frac{1}{T_0} \frac{dT_0}{d\xi} = \frac{d\eta}{d\xi} = -\epsilon \eta^{m-3} \quad (\text{any } m) \quad (5.24)$$

Let us assume that the change of temperature over the length L is small compared to the magnitude of the temperature. This is equivalent to assuming that

$$\epsilon \ll 1$$

so that, expanding η and $\frac{d\eta}{d\xi}$ in powers of ϵ we have

$$\eta(\xi) \approx 1 - \epsilon \xi + O(\epsilon^2) \quad (5.25)$$

$$\frac{d\eta}{d\xi} = \frac{1}{T_0} \frac{dT_0}{d\xi} \approx -\epsilon + O(\epsilon^2)$$

We shall adopt this simplifying assumption of the small gradient for this local analysis since it is extremely useful and is not as restrictive as it may seem. To see this note that

$$\epsilon = \frac{3}{8} Q \tau \quad (5.26)$$

where

$$Q = \frac{q_0}{2\sigma T_e^4} \quad \text{and} \quad \tau = \alpha_e L \quad (5.27)$$

Since we want to leave the problem unrestricted as to optical depth τ , the physical limitation of the restriction $\epsilon \ll 1$ is best summarized

by

$$Q \ll \frac{8}{3\tau} \quad (5.28)$$

or, in physical variables

$$q_0 \ll \frac{16}{3} \frac{\sigma T_a^4}{\alpha_a L} \quad (5.29)$$

To appreciate the physical implications of this restriction note that for a characteristic temperature of $T_a = 80,000^\circ\text{K}$ and an opacity α_a such that there are a hundred photon mean free paths between the walls ($\tau = 100$) we must restrict ourselves to a net radiative heat flux of somewhat less than ten megawatts per square centimeter. This amount of departure from equilibrium should be sufficient to illustrate the effects of a net flux on the wave propagation in the gas, and yet allow a substantial algebraic simplification.

To introduce the small gradient approximation into the differential equation we shall expand all the η terms, keeping only first order terms in ϵ . (Note that this is distinct from a "linear gradient" approximation which would keep all terms of first order in ξ and be inconsistent.) When we encounter powers of η the following expansion is appropriate

$$\eta^k(\xi) \approx 1 - k\epsilon\xi + O(\epsilon^2) \quad (5.30)$$

for any value of the variation of opacity w . To first order in ϵ the temperature distribution does not depend upon w . Carrying out this expansion on the five coefficients of the differential equation given

in (4.56a-e), results in the following more simple differential equation for the pressure perturbation $p(\xi, \Omega)$

$$\begin{aligned}
 & [1 - F(1 + (m+4)\epsilon\xi)] p^{iv} \\
 & + [(2m-10) - F(3m-3)] \epsilon p''' \\
 & + \left\{ \Omega^2(1 + \epsilon\xi) - F \left[\frac{\Omega^2}{8} (1 - (3m-5)\epsilon\xi) - 3\tau^2(1 - (m-4)\epsilon\xi) \right] \right\} p'' \\
 & + \left[\Omega^2(2m+n-8) - 3F \left(\frac{m\Omega^2}{8} + \tau^2 \right) \right] \epsilon p' \\
 & + \left[3 \frac{\tau^2 \Omega^2}{8} F(1 - (m-5)\epsilon\xi) \right] p = 0
 \end{aligned}
 \tag{5.31}$$

where $F = i \frac{\Omega N_B}{16\tau}$

The boundary conditions (4.62 - 4.65) may be expanded in the same manner if one desires a complete solution to the problem in this approximation. Our goal however is merely to find the wavenumber-frequency relationship by a local analysis, for which only the above equation is needed.

V.4 The Local Analysis Applied to Waves in a Radiating Gas

The simple example discussed in Sec. 5.2 of wave propagation in the presence of gradients incorporated all the essential concepts of a local analysis. We may then apply the same techniques to the more complicated wave equation for the pressure perturbation (5.31) with a minimum of additional discussion. Let us define a complex dimensionless wavenumber which bears the following relationship to the physical wavenumber

$$K = K_r + iK_i = (k_r + ik_i)L \quad (5.32)$$

In a local analysis a suitable trial eigenfunction would be

$$\tilde{p} = P(\Omega) e^{ikx} = P(\Omega) e^{iK\xi} \quad (5.33)$$

The procedure to be followed is to substitute $\tilde{p}$ into the differential equation (5.31), multiply by the complex conjugate $\tilde{p}^*$, integrate over a wavelength and require that the real and imaginary parts of K be chosen so as to make the entire expression vanish. The result of this is, that

$$\int_{-\frac{1}{2|K|}}^{+\frac{1}{2|K|}} \left\{ K^4 \left[1 - F(1 + (m-4)\epsilon\zeta) \right] \right. \\ \left. - iK^3 \epsilon \left[2m-10 - F(3m-3) \right] \right. \\ \left. - K^2 \left[\Omega^2(1+\epsilon\zeta) - F\left[\frac{\Omega^2}{8}(1-(3m-5)\epsilon\zeta) - 3\tau^2(1-(m-4)\epsilon\zeta) \right] \right] \right\} d\zeta = 0$$

(equation continued on next page)

(equation continued from previous page)

$$\begin{aligned}
& + iK\epsilon \left[\Omega^2 (2m+n-8) - 3F \left(\frac{m\Omega^2}{8} + \tau^2 \right) \right] \\
& + \frac{3\tau^2\Omega^2 F}{8} \left(1 - (m-5)\epsilon \right) \} d\zeta = 0
\end{aligned} \tag{5.34}$$

Carrying out the integrations is trivial and our modified "dispersion" relation between the wavenumber $K=kL$ and frequency $\Omega = \frac{\omega L}{\sqrt{RT_0}}$ is

$$\begin{aligned}
& K^4 \left[1 - F \left(1 + \frac{(m-4)\epsilon}{2|K|} \right) \right] - iK^3 \epsilon \left[2m-10 - 3F(m-1) \right] \\
& - K^2 \left[\Omega^2 \left(1 + \frac{\epsilon}{2|K|} \right) - F \left[\frac{\Omega^2}{8} \left(1 - \frac{(3m-5)\epsilon}{2|K|} - 3\tau^2 \left(1 - \frac{(m-4)\epsilon}{2|K|} \right) \right) \right] \right] \\
& + iK\epsilon \left[\Omega^2 (2m+n-8) - 3F \left(\frac{m\Omega^2}{8} + \tau^2 \right) \right] \\
& + \frac{3\tau^2\Omega F}{8} \left(1 - \frac{(m-5)\epsilon}{2|K|} \right) = 0
\end{aligned} \tag{5.35}$$

The term $\frac{\epsilon}{|K|}$ appears prominently, and is to be interpreted physically as the ratio of disturbance wavelength to the characteristic length of the gradients in the gas. To see this note that

$$\epsilon = \left| \frac{L}{T_0} \frac{dT_0}{dx} \right|_{x=0} ; \quad |K| = \frac{2\pi L}{\lambda}$$

so that

$$\left| \frac{\epsilon}{K} \right| = \left| \left(\frac{\lambda}{2\pi} \right) \frac{dT_0}{T_0 dx} \right|_{x=0} \tag{5.36}$$

Also since $\epsilon = \frac{3}{8} Q \tau = \frac{3}{16} \frac{q}{\sigma T_a^4} \alpha_a L$ one can

make the ratio $\left| \frac{\epsilon}{K} \right| = \left(\frac{3}{32\pi} \right) \left(\frac{q}{\sigma T_a^4} \right) (\alpha_a \lambda)$ in order

to interpret $\left| \frac{\epsilon}{K} \right|$ in terms of the net radiative flux and the optical depth of the wave $\alpha_a \lambda$ but this does not seem to be as instructive.

In any event $\left| \frac{\epsilon}{K} \right|$ is the correct small expansion parameter in a local analysis of this type, for it becomes small either for smaller gradients or smaller wavelengths.

Let us write Equation (5.35) in a different manner, so as to isolate the terms proportional to $\left| \frac{\epsilon}{K} \right|$.

$$\begin{aligned} & \left\{ K^4 (1-F) - K^2 \left[\Omega^2 - F \left(\frac{\Omega^2}{8} - 3\tau^2 \right) \right] + \frac{3\tau^2 \Omega^2 F}{8} \right\} \\ & - \left| \frac{\epsilon}{K} \right| \left\{ i K^3 |K| \left[2m-10 - 3F(m-1) \right] \right. \\ & \quad \left. - i K |K| \left[\Omega^2 (2m+n-8) - 3F \left(\frac{m\Omega^2}{8} + \tau^2 \right) \right] \right. \\ & \quad \left. + K^2 \left[\frac{\Omega^2}{2} + F \left(\frac{\Omega^2}{8} \left(\frac{3m-5}{2} \right) - 3\tau^2 \frac{(m-4)}{2} \right) \right] \right. \\ & \quad \left. + \frac{3\tau^2 \Omega^2 F}{8} \left(\frac{m-5}{2} \right) \right\} = 0 \end{aligned}$$

(5.37)

where, again $F = \frac{i\Omega N_s}{16\tau}$ is just a convenient grouping.

To gain some physical insight into all this symbolism we need to investigate the various pertinent limits of the nondimensional parameters. This is done in the following section.

V.5 Interpretation of Results in Various Limiting Cases

First it is necessary to understand wave propagation in a homogeneous radiating gas. For the limit of no gradients $\left|\frac{\epsilon}{K}\right| = 0$ and (5.37) degenerates to

$$K^4 (1-F) - K^2 \left[\Omega^2 \left(1 - \frac{F}{Y}\right) + 3\tau^2 F \right] + 3\tau^2 \frac{\Omega^2}{Y} F = 0 \quad (5.38)$$

with $F = \frac{i\Omega N_s}{16\tau}$

The four roots of this polynomial may be found exactly to be

$$K = \pm \left[\frac{H \pm \left[H^2 - 12 \frac{\tau^2 \Omega^2 F (1-F)}{Y} \right]^{\frac{1}{2}}}{2(1-F)} \right]^{\frac{1}{2}} \quad (5.39)$$

where $H = \Omega^2 \left(1 - \frac{F}{Y}\right) + 3\tau^2 F$

The pair of wavenumbers corresponding to the inside plus sign are for right and left running modified acoustic waves and the pair corresponding to the inside negative sign are for right and left running

radiative induced waves. It is often more instructive to rearrange equation (5.38), after substituting in $i\frac{\Omega N_b}{16\tau} = F$, so that the equation is in the following form

$$K^2(K^2 - \Omega^2) - i\frac{\Omega N_b}{16\tau}(K^2 + 3\tau^2)\left(K^2 - \frac{\Omega^2}{8}\right) = 0 \quad (5.40)$$

where $K = kL$, $\Omega = \frac{\omega L}{\sqrt{RT_e}}$

and the ratio of the phase velocity of a wave to the reference isothermal speed of sound would be

$$\frac{\Omega}{K} = \frac{\omega}{k\sqrt{RT_e}}$$

For very large Boltzmann numbers $N_b \gg 1$, corresponding to low temperatures, the second term of the equation dominates and the roots become independent of N_b . The roots are then

$$K = \pm \frac{\Omega}{\sqrt{8}}, \quad \pm i\sqrt{3}\tau \quad (5.41)$$

The first mode is the acoustic mode with phase velocity equal to the isentropic speed of sound $\sqrt{\gamma RT_e}$. This mode has zero rate of decay

$K_i = 0$. The second mode is the radiative induced mode, which is nonoscillatory and has a rate of decay proportional to optical depth.

For very small Boltzmann numbers, $N_b \ll 1$, which correspond to very high temperatures and therefore a high efficiency for radiative energy transfer, the left hand terms of Equation (5.40) are dominant and the roots

$$K = \pm \Omega, \pm 0 \quad (5.42)$$

are again independent of N_0 . The first mode is the acoustic mode with a phase velocity equal to the isothermal speed of sound. Acoustic waves will travel at the isothermal speed because large radiative transfer efficiency prevents the occurrence of temperature differences in the gas. The radiation induced mode degenerates to zero phase velocity and zero damping because temperature cannot fluctuate in this limit.

Wave propagation in uniform radiating gases is the topic of Vincenti and Baldwin's original paper (Ref. 2) and one should consult this work for curves of damping rates and phase velocities. Alternatively, the more concise discussion in the text of Vincenti and Kruger (Ref. 3) is a well written summary of this work.

Our main task here is to understand the effects of a net flux of radiation in an inhomogeneous gas however, so we shall now try to extract some information from the equation which includes the effects of gradients, equation (5.37). Since we can calculate exactly the wavenumbers for no gradient $\epsilon = 0$ from expression (5.39), let us do an expansion in the small parameter $\frac{\epsilon}{|K|}$ so as to determine the first order change in the wavenumber due to the effects of gradients. Let

$$K = K_0 + \frac{\epsilon}{|K_0|} K_1 + O\left(\frac{\epsilon^2}{K_0^2}\right) \quad (5.43)$$

where $\epsilon/|K_0| \ll 1$

and substitute this into (5.37), keeping only terms of first order in

$$\epsilon/|K_0|.$$

The result is that

$$K_1 = \frac{1}{2|K_0|} \left\{ \frac{K_0}{2} \left[\Omega^2 + F \left(\frac{\Omega^2}{8} (3m-5) - 3\tau^2 (m-4) \right) \right] \right. \\ \left. + \frac{3}{2} \frac{(m-5) \tau^2 \Omega^2 F}{8K_0} + i|K_0| \left[K_0^2 (2(m-5) - 3F(m-1)) \right. \right. \\ \left. \left. - \Omega^2 (2m+n-8) + 3F \left(\frac{m\Omega^2}{8} + \tau^2 \right) \right] \right\} \\ \left/ \left[2K_0^2 (1-F) - \Omega^2 \left(1 - \frac{F}{8} \right) - 3\tau^2 F \right] \right. \quad (5.44)$$

Again we must look at the limiting behavior to realize any physical sense from this. One interesting case is for a very high temperature gas. This limit is the Small Boltzmann Number limit.

A. The Small Boltzmann Number Limit

In this case N_g is considered small enough so that we may neglect N_g^2 in comparison to N_g . If we make this approximation in equation (5.39) we get the four zeroth order wavenumbers

$$K_{0,1,2} = \pm \left(\frac{i+1}{8} \right) \sqrt{\frac{3\Omega N_g \tau}{8}} \quad (5.45)$$

and

$$K_{0,3,4} = \pm \left[\Omega + \left(\frac{\gamma-1}{2\gamma} \right) i \frac{N_s}{16\tau} (3\tau^2 + \Omega^2) \right] \quad (5.46)$$

The corresponding corrections due to the presence of the net radiative flux, K_1 , are

$$K_{1,2} = \mp \frac{1}{4} \quad (5.47)$$

$$K_{1,3,4} = \pm \frac{1}{4} - i \left(1 + \frac{\eta}{2} \right) \quad (5.48)$$

In evaluating the above K_1 the terms proportional to N_s have been dropped since, when they are multiplied by the other small parameter $\epsilon/|K_0|$, the resulting term $\frac{\epsilon N_s}{|K_0|}$ would be quite negligible. Combine the K_0 and K_1 as in $K = K_0 + \frac{\epsilon}{|K_0|} K_1$ we have for the wavenumbers

$$\begin{array}{l} \text{Radiation} \\ \text{Induced} \\ \text{Mode} \end{array} \quad K_{1,2} = \pm \left(\frac{i+1}{8} \right) \sqrt{\frac{3\Omega N_s \tau}{\gamma}} \mp 2\epsilon \sqrt{\frac{\gamma}{6\Omega N_s \tau}} \quad (5.49)$$

$$\begin{array}{l} \text{Modified} \\ \text{Acoustic} \\ \text{Mode} \end{array} \quad K_{3,4} = \pm \Omega \pm \left(\frac{\gamma-1}{2\gamma} \right) \left(\frac{i N_s}{16\tau} \right) (3\tau^2 + \Omega^2) \quad (5.50)$$

$$\pm \frac{\epsilon}{4\Omega} - i \left(1 + \frac{\eta}{2} \right) \frac{\epsilon}{\Omega}$$

In order to interpret these correctly remember that the wave has the form $\exp(iK_r \xi - K_i \xi)$. Hence the right running waves are represented by the top signs and the left running waves by the lower

signs, for each mode. We see that the real part of the wavenumber of the radiation induced mode is decreased by the influence of the gradients and the damping factor is unaffected. (Our small N_s approximation is not as good for this mode as for the other because $|K_o|$ is proportional to $\sqrt{N_s}$ and gets very small, which restricts ϵ to very small values if $|\frac{\epsilon}{K_o}|$ is also to be less than unity.)

The modified acoustic mode is the most interesting one here.

The real part of the wavenumber

$$K_r = \pm \Omega \left(1 + \frac{\epsilon}{4\Omega^2} \right) \quad (5.51)$$

is increased by the effect of the gradients and the imaginary part

$$K_i = \pm \left(\frac{\gamma-1}{2\gamma} \right) \frac{N_s}{16\tau} (3\tau^2 + \Omega^2) - \left(1 + \frac{n}{2} \right) \frac{\epsilon}{\Omega} \quad (5.52)$$

has a correction term for the effect of gradients which does not change sign. The effect of this term is to decrease the damping term of the right running wave and increase the damping term of the left running wave. Evidently the effect of the net heat flux on the right running wave, in the limit of small Boltzmann number, may even be a net amplification, if the second term is larger in magnitude than the first. The criterion for net growth of this wave is then

$$\left(1 + \frac{n}{2} \right) \frac{\epsilon}{\Omega} > \left(\frac{\gamma-1}{\gamma} \right) \frac{N_s}{32\tau} (3\tau^2 + \Omega^2) \quad (5.53)$$

Let us write this in terms of the net heat flux parameter $Q = \frac{q}{2\sigma T_e^4}$

since $\epsilon = \frac{3}{8} Q \tau$

as

$$Q > \left(\frac{\gamma-1}{2\gamma}\right) \frac{N_s \Omega}{(2+n)} \left(1 + \frac{\Omega^2}{3\tau^2}\right) \quad (5.54)$$

Thus the critical net heat flux for growth of this wave is more easily reached for smaller Boltzmann numbers, lower frequency waves, greater optical thickness and greater variations of opacity with pressure. In physical variables, this criterion for growth of the wave is

$$q_* > \frac{\omega L p}{(2+n)} \left(1 + \frac{1}{3} \frac{\omega^2}{\alpha^2 R T_*}\right) \quad (5.55)$$

The term $\frac{\alpha \sqrt{RT}}{\omega}$ is often called a Bouger number in radiation gas dynamics. For acoustic waves the Bouger number is the ratio of the wavelength of an acoustic wave to a photon mean free path, hence it measures the optical depth of the wave itself. Obviously the larger Bouger numbers corresponding to opaque waves favor growth of this wave.

The physical interpretation of this criterion is clear even if one cannot arrive at equation (5.55) by a more simple "back of the envelope" calculation. One can draw analogies to the studies done by people interested in combustion stability (e.g. Chu, B.T., Refs. 16 and 17) which arrive at conclusions such as this: If in a uniform gas a plane source of energy of strength q , in units of power per unit area, is turned on at $t=0$, the influence will propagate into the gas at the

signal velocity $v = \frac{\gamma-1}{2\gamma} \frac{q}{p}$. If the source of heat is so intense that this speed is greater than or equal to the speed of sound then a shock will form, just as if the expanding heated gas were a piston. In our case, though a net heat flux is present everywhere, the heat absorbed by the gas is zero everywhere $\left(\frac{\partial q_0}{\partial x} = 0\right)$. Hence it is quite likely that the criterion for the growth of a pressure disturbance given in (5.54) or (5.55) is a comparison between the speed at which a volume element needs to expand due to the absorption of radiant energy in the fluctuation and the speed of the disturbance itself. If the energy excess in the fluctuation cannot escape fast enough the fluctuation must grow in amplitude. Unfortunately no more "exact" explanation can be offered at the present time.

It is an interesting observation that the variation of opacity with temperature κ does not appear in this criterion for wave growth, whereas the variation with pressure μ does appear. Apparently this is a consequence of the small Boltzmann number limit, in which the high radiative transfer causes disturbances to be quite isothermal, thereby minimizing temperature fluctuation effects.

Now let us turn to the low temperature limit.

B. The Large Boltzmann Number Limit

For this case we shall proceed as in the opposite limit, however, here we shall do an expansion in the small parameter $N_b^{-1} \ll 1$. Using this approximation and neglecting all terms of order N_b^{-2} in equation (5.39) gives for the four zeroth order wavenumbers

$$K_{0,1,2} = \pm i\sqrt{3}\tau \left[1 + \frac{i24\tau^3\gamma}{\Omega N_s} \left(\frac{\Omega^2 + 3\tau^2}{\Omega^2 + 3\gamma\tau^2} \right) \right] \quad (5.56)$$

$$K_{0,3,4} = \pm \frac{\Omega}{\sqrt{\gamma}} \left[1 + \frac{i8\tau(\gamma-1)\Omega}{N_s(\Omega^2 + 3\gamma\tau^2)} \right] \quad (5.57)$$

The corresponding corrections due to the effects of the gradients are

$$K_{1,1,2} = \left\{ \pm \frac{1}{4} \left[\frac{2m\Omega^2}{\gamma} - 3\tau^2(m-4) + \frac{3}{2}i \left[\frac{m\Omega^2}{\gamma} + (4m-3)\tau^2 \right] \right] \right\} \quad (5.58)$$

$$\left[\frac{\Omega^2}{\gamma} + 3\tau^2 \right]$$

$$K_{1,3,4} = \left\{ \mp \frac{1}{4} \left[\frac{\Omega^2}{\gamma} (3m-5) - 3\tau^2 \right] - \frac{3}{2}i \left[\frac{\Omega^2}{\gamma} + \tau^2 \right] \right\} \quad (5.59)$$

$$\left[\frac{\Omega^2}{\gamma} + 3\tau^2 \right]$$

Again, in the above equations (5.58) and (5.59) the terms corresponding to N_s^{-1} have not been included because when K_1 is to be multiplied by $\frac{\epsilon}{|K_0|}$ it will result in the negligible product $\frac{\epsilon}{|K_0|N_s}$.

The total wavenumber is $K = K_0 + \frac{\epsilon K_1}{|K_0|}$ and equations (5.56 - 5.59) may be combined to form these expressions for K . There seems to be no need to write these out. A closer examination will reveal that the corrections to the wavenumber have the effect of increasing the damping for left and right running radiation induced waves (mode 1,2)

and for the left running modified acoustic wave K_4 . The interesting wave is again the right running acoustic wave, which travels parallel to the net radiative flux. The real part of the full wavenumber for this case is

$$K_{3r} = + \frac{\Omega}{\sqrt{\gamma}} \left[1 - \frac{\epsilon}{4} \frac{\gamma}{\Omega^2} \frac{\left(\frac{\Omega^2}{\gamma} (3m-5) - 3\tau^2 \right)}{\left(\frac{\Omega^2}{\gamma} + 3\tau^2 \right)} \right] \quad (5.60)$$

The factor of $\sqrt{\gamma}$ in this expression indicates that the wave travels at predominately the isentropic speed of sound, since the isothermal speed of sound is the reference speed used in normalizing the frequency

$$\Omega = \frac{\omega L}{\sqrt{RT_a}}. \quad \text{The imaginary part of } K_3 \text{ is}$$

$$K_{3i} = \frac{+ 8\tau \Omega^2 (\gamma-1)}{\sqrt{\gamma} N_B (\Omega^2 + 3\gamma\tau^2)} - \frac{3}{2} \frac{\epsilon \sqrt{\gamma}}{\Omega} \left(\frac{\Omega^2 + \gamma\tau^2}{\Omega^2 + 3\gamma\tau^2} \right) \quad (5.61)$$

This right running wave will grow in the direction of its propagation when the net radiative flux is large enough so that the second term dominates. Since $\epsilon = \frac{3}{8} Q\tau$ this happens for

$$Q > \frac{2\gamma(\gamma-1) \left(\frac{8}{3}\right)^2 \Omega^3}{N_B (\Omega^2 + \gamma\tau^2)} \quad (5.62)$$

In physical quantities this is

$$q > \frac{\left[\frac{16}{3} (\gamma-1) \sigma T_a^4 \right]^2 \left(\frac{\omega L}{\sqrt{RT_a}} \right)}{\rho \sqrt{RT_a} \left(1 + \frac{\alpha_a^2 \gamma RT_a}{\omega^2} \right)} \quad (5.63)$$

In this large Boltzmann number approximation the critical net radiative flux is smaller for larger Boltzmann numbers, larger optical depths, and smaller frequencies. This criterion is very easily exceeded, for instance the critical heat flux for air at standard temperature and pressure, and for an optically thin wave of a kilocycle per second frequency, would be about one watt per square meter. This is very misleading however because the actual amplification rate calculated from (5.61) is so small for such small temperatures and radiative fluxes that the waves would actually be damped out by such effects as molecular heat conduction and viscosity, which have been omitted. The results for small Boltzmann number do not suffer from this criticism however because in that case both the equilibrium and net radiative transfer can be quite large.

V.6 Remarks on Prediction of Stability Based Upon Results of
the "Local" Analysis.

The goal of this chapter was to give an appreciation for the behavior of the individual waves which can be supported by a non-uniform radiating gas. Certain conclusions have been reached concerning this behavior as a function of the important nondimensional parameters. Briefly, the important results can be summed up as follows. Both the left and right running radiation induced waves, or heat waves, are evanescent for all values of the parameters. The same seems to be true for the left running acoustic wave, which is going against the net radiative flux. The acoustic wave propagating in the same direction as the net flux may grow in amplitude exponentially for either very small or very large Boltzmann numbers. This growth is favored for those waves which have the lowest frequencies and which are most opaque. The criterion for growth of these disturbances does not depend on the variation of opacity with temperature, although it does depend on the variation of opacity with pressure in the case of small Boltzmann numbers.

The conditions for growth of a wave given in the last section should not be construed as stability criteria. Certainly a necessary condition for instability to occur is that some disturbance be capable of growth, but this is not sufficient. The question of overall stability is intimately concerned with the boundary conditions. It might seem desirable to avoid solving a boundary value problem by trying to calculate the amplitude changes of a wave in one "round

trip" between the walls from a local analysis, in conjunction with the reflection conditions at the boundaries. This is very difficult to do for higher order systems with nontrivial boundary conditions, and is not likely to yield reliable conclusions about overall stability.

It is implicit in the local analysis that the individual component waves are singled out for study, and hence decoupled from the others.

The two interacting factors which tend to couple the waves are 1) the presence of inhomogeneities and 2) the boundary conditions. Our fourth order system has two wave modes, and in spite of the fact that the behavior of the acoustic waves dominates, the other wave is always present. For example if the energy and momentum of the waves are conserved upon reflection from a boundary, it is only the sum of these quantities for the two modes which is conserved. The individual wave amplitudes may change greatly in reflection. These reflection conditions depend on the gradients of the properties of the media at the boundary and are very difficult to calculate for more than one wave mode. The general conclusion from this is that even if waves of short wavelength may be analyzed independently of the presence of the boundaries, the determination of overall stability would require knowledge of reflection conditions at the boundaries. But since this information seems unattainable for coupled wave modes in inhomogeneous media, it is necessary to solve the differential equation properly subject to the full boundary conditions. We then have an eigenvalue problem for the actual frequencies and damping of growth rates of the fluctuations

allowed by the boundaries. This is the proper method of inferring the stability or instability of the system.

VI. THE ABSOLUTE STABILITY OF A NONUNIFORM RADIATING GAS BOUNDED BY BLACK CONSTANT TEMPERATURE WALLS

VI.1 Introduction

The task of this chapter is to solve the eigenvalue problem which was set up in Chapter IV. The determination of the normal modes of oscillation of a nonuniform radiating gas bounded by constant temperature black walls has been reduced to the solution of a fourth order ordinary differential equation in space with variable coefficients and associated boundary conditions. This equation is given by equations (4.55) and (4.56a-d) and the boundary conditions by equations (4.62) - (4.65). The complete solutions to this differential equation and boundary conditions give the shapes of the oscillations at a given time and consist of a superposition of all the coupled waves in the gas. However, for the determination of stability it is not necessary to actually determine these complete solutions, i.e. to calculate the arbitrary constants in the general solution. All the necessary information is contained in the complex eigenvalues obtained from the condition of solvability for these constants in the general solution. We now proceed to find the approximate general solution, stopping briefly to describe the method to be used in constructing a physically sound approximation.

VI.2 The WKB Approximation

The most difficult factor in solving the differential equation (4.55) is the variation of the coefficients with the spacial coordinate.

The Wentzel-Kramers-Brillouin (WKB) approximation is a frequently used technique in quantum mechanics to solve the Schroedinger equation with a variable potential. The time dependent Schroedinger equation is a wave equation with complex coefficients which vary with the spatial coordinate and is analogous to the equation of interest here. The essential differences lie in the eigenvalues, which are complex in our problem and are purely real for the Schroedinger equation where they represent energy states. In a sense the eigenvalues for the fluctuations in a gas also represent allowed states of the oscillating system. One may consult any good text on Quantum Mechanics such as Slater (Ref. 14) or Dicke and Wittke (Ref. 15) for a discussion of the application of the WKB method to Schroedingers Equation.

The WKB approximation is applicable whenever the wavelength does not vary appreciably over a region comparable to the wavelength itself. This means that we must have

$$\left| \frac{d\lambda(x)}{dx} \right| \ll 1 \quad (6.1)$$

or, in terms of wavenumber,

$$\left| \frac{1}{k^2} \frac{dk}{dx} \right| \ll 1 \quad (6.2)$$

We know that this will probably be a reasonable approximation for small gradients since the wavelength is constant when the gradient vanishes. It is important to differentiate between this approximation, which is applicable when the variation in wavelength is small over a distance

comparable to a wavelength, and the short wavelength approximation used in the "local" analysis of the previous chapter, which required the wavelength to be small compared to the characteristic length of the gradient.

$$\lambda \ll \left| \frac{1}{T} \frac{dT}{dx} \right|^{-1} \quad (6.3)$$

We will see later that these criteria are in fact related, as expected, but, this cannot be assumed a priori and must be checked after the solution is obtained using the WKB approximation.

The actual application of the method proceeds as follows. One makes the change of variables

$$p(x) = e^{\phi(x)} \quad (6.4)$$

because the solutions are expected to be "like" exponentials for wavelengths small compared to the characteristic lengths of the gradient. We expect $\phi' = \frac{d\phi}{dx}$ to be like the wavenumber k and ϕ'' to be like the variation of wavenumber with distance. The essence of the WKB approximation is that

$$\left| \frac{\phi''}{\phi'^2} \right| \ll 1 \quad (6.5)$$

and that ϕ'' may therefore be neglected compared to ϕ'^2 . For instance, the successive derivatives of equation (6.4) are

Of course, all higher derivatives, such as ϕ''' , are also neglected. From this point on it is best to illustrate the method in actual application.

VI.3 The General Solution using the WKB Method

The equation for the pressure perturbation that is to be solved can be represented by the form

$$a(\xi)p'''' + b(\xi)p''' + c(\xi)p'' + d(\xi)p' + e(\xi)p = 0 \quad (6.6)$$

where all five coefficients are functions of ξ , and are complex.

The coefficient $a(\xi)$ is well behaved so we need not consider singular behavior. To apply the WKB Method we substitute for p and its derivatives as follows.

$$p(\xi, \Omega) = e^{\phi(\xi, \Omega)} \quad (6.7a)$$

$$p' = \phi' p \quad (6.7b)$$

$$p'' = (\phi'' + \phi'^2) p \approx \phi'^2 p \quad (6.7c)$$

$$p''' = (\phi''' + 3\phi''\phi' + \phi'^3) p \approx \phi'^3 p \quad (6.7d)$$

$$p'''' = (\phi'''' + 4\phi'''\phi' + 3\phi''^2 + 6\phi'^2\phi'' + \phi'^4) p \approx \phi'^4 p \quad (6.7e)$$

since

$$\phi'' \ll \phi'^2$$

(6.8)

When we substitute equations (6.7a-e) into (6.6) we get an equation for

$$a \phi'^4 + b \phi'^3 + c \phi'^2 + d \phi' + e = 0 \quad (6.9)$$

The effect of this transformation from p to ϕ and of neglecting all second derivatives or higher is to transform the problem from one of finding the solution to a fourth order differential equation to one of finding the four roots of a fourth degree polynomial and solving the resultant four first order differential equations. Suppose we are able to factor equation (6.9) so that we have the four factors

$$\phi'_i(\xi, \Omega) \quad i=1,4 \quad (6.10)$$

These four functions are integrated to get

$$\phi_i(\xi) = \int \phi'_i d\xi + \ln C_i \quad i=1,4 \quad (6.11)$$

Since our original fourth order differential equation is a linear one we superimpose these four linearly independent solutions to form the general solution for the pressure perturbation in the following manner

$$p(\xi, \Omega) = \sum_{i=1}^4 C_i(\Omega) e^{\phi_i(\xi, \Omega)} \quad (6.12)$$

where the C_i are the four arbitrary constants of integration.

The general solution represented by equation (6.12) is the eigenfunction for one normal mode corresponding to a value of the complex eigenvalue Ω and constants $C_i(\Omega)$ which are to be determined from the

boundary conditions. There are likely to an infinite number of allowed Ω 's forming a complete discrete spectrum. Perhaps our notation should include indices on each $\mathfrak{p}$ and its corresponding Ω but there should be no need to complicate the notation once this is firmly in mind.

Each normal mode is required to satisfy the boundary conditions independently, but since the boundary conditions are homogeneous we will only be able to solve for three of the C_i in terms of a fourth. It is impossible to determine the absolute magnitude of each normal mode from the boundary conditions, for this latter information is to be derived from a superposition of all the normal modes to satisfy a prescribed initial condition. Since we are primarily interested in ascertaining stability however, we only need to concern ourselves with the individual normal modes, and in particular the eigenvalue corresponding to each mode that is selected out by the boundary conditions.

The next step is to actually factor equation (6.9) and integrate the four resulting factors, the $\phi'_i(\xi, \Omega)$. A return look at the complexity of the coefficients $a(\xi)$, $b(\xi)$, etc. given in equations (4.56a-e) is enough to convince one that it is a hopeless task analytically. Several approximate techniques were applied to determine the factors of this polynomial, for example expansions in powers of a small Boltzmann number, but these approximations seemed to be of limited utility. Instead the ϕ'_i were found numerically by a root finding program and integrated numerically as well. This is easily done in practice and allows us to avoid imposing further restrictions

on the solution beyond the original WKB assumptions. The details of the numerical procedure are best left for later, when the entire problem is set up. Now the procedure for finding the eigenvalues from the general solution and boundary conditions shall be described.

VI.4 Determination of Eigenvalues

We now require that the general solution (6.12) satisfy the boundary conditions described in Sec. 4.3 and written out in equations (4.62 - 4.66). There are five arbitrary complex constants, four C_i and Ω , and since we have four boundary conditions for the fourth order differential equation, four of the five unknowns may now be specified. Note that the boundary conditions are formulated as a homogeneous set of equations which are linear combinations of the dependent variable and its derivatives, all evaluated at the boundaries.

Let us proceed immediately to the determination of the eigenvalues. The first step is to substitute the general solution (6.12) directly into the boundary conditions given in equations (4.62 - 4.66), being careful to observe the assumptions of the WKB approximation regarding the negligible magnitude of the second (and higher) derivatives of ϕ compared to the terms including the first derivative. The results of this substitution are these four equations

$$\sum_{i=1}^4 C_i \{ \phi'_i(0) \} = 0 \quad (6.13)$$

$$\sum_{i=1}^4 C_i \{ \phi'_i(1) \} e^{\phi_i(1)} = 0 \quad (6.14)$$

$$\begin{aligned}
\sum_{i=1}^4 C_i \left\{ (1-F) \phi_i^{'3}(0) + \left[(m-5 - (m-2)F) \epsilon - \frac{3\tau}{2} (1-F) \right] \phi_i^{'2}(0) \right. \\
+ \left[\left(1 - \frac{F}{8}\right) \Omega^2 - \frac{3\tau F}{2} \epsilon + \epsilon^2 (3F-1) \right] \phi_i'(0) \\
\left. + \Omega^2 \left[\epsilon \left(m+n-4 - \frac{mF}{8}\right) - \frac{3\tau}{2} \left(1 - \frac{F}{8}\right) \right] \right\} = 0
\end{aligned} \tag{6.15}$$

$$\begin{aligned}
\sum_{i=1}^4 C_i \left\{ \left[\eta_i^{5-m} - F \eta_i^{1-2m} \right] \phi_i^{'3}(1) + \left[\epsilon (\eta_i^2 + (m-6)\eta_i - (m-2)F \eta_i^{m-3}) \right. \right. \\
+ \frac{3\tau}{2} (\eta_i^5 - F \eta_i^{1-m}) \left. \right] \phi_i^{'2}(1) \\
+ \left[\Omega^2 \left(\eta_i^{4-m} - \frac{F}{8} \eta_i^{-2m} \right) + \epsilon^2 \left(-2 \eta_i^{m-2} + \eta_i^{m-3} + 3F \eta_i^{-7} \right) \right. \\
+ \frac{3\tau \epsilon}{2} \left(\eta_i^{m+2} - \eta_i^{m+1} + F \eta_i^{-3} \right) \left. \right] \phi_i'(1) \\
\left. + \Omega^2 \left[\epsilon \left(m+n-4 - \frac{mF}{8} \eta_i^{-m-4}\right) + \frac{3\tau}{2} \left(\eta_i^4 - \frac{F}{8} \eta_i^{-m} \right) \right] \right\} e^{\phi_i(1)} = 0
\end{aligned} \tag{6.16}$$

where $\eta_i = \eta(\xi=1)$, $\epsilon = \frac{3}{8} Q \tau$, $F = \frac{i \Omega N_s}{16 \tau}$.

The above set of four equations for the four unknowns $C_i(\Omega)$ may be written as

$$\sum_{i=1}^4 D_{ji}(\Omega) C_i(\Omega) = 0 \quad j=1,4 \tag{6.17}$$

where the sixteen D_{ji} are known because the ϕ_i can be calculated as

described in Sec. 6.3. In order for this set of linear homogeneous algebraic equations for C_i to have a nontrivial solution the 4×4 determinant of the coefficients D_{ji} must vanish

$$|D_{ji}(\Omega)| = 0 \quad (6.18)$$

This complex determinant is a transcendental function of the complex variable $\Omega = \Omega_r + i\Omega_i$ and has in general an infinity of roots. The Ω 's for which both the real and imaginary parts of the determinant vanish are the eigenvalues for which we are searching. This determinant, for which we shall simply use the symbol D , is a function not only of Ω , but also of the nondimensional parameters which characterize a given problem

$$D = |D_{ji}| = D(\Omega_r + i\Omega_i; N_s, \tau, m, n, \gamma, Q) \quad (6.19)$$

Once the parameters defining a problem for a particular gas are specified $(N_s, \tau, m, n, \gamma)$ the roots of (6.19) define an infinity of relations for the root locus of the eigenvalues as a function of the amount of net radiative flux through the gas.

Thus
$$\Omega_s = \Omega_{s,r} + i\Omega_{s,i} = \Omega_s(Q)$$

for the s^{th} normal mode.

These roots must be found numerically since the elements of the determinant are so large and unwieldy; several attempts at analytical approximation proved fruitless.

The root locus $\Omega_s = \Omega_s(a)$ for the normal mode may be displayed as in the schematic diagram.

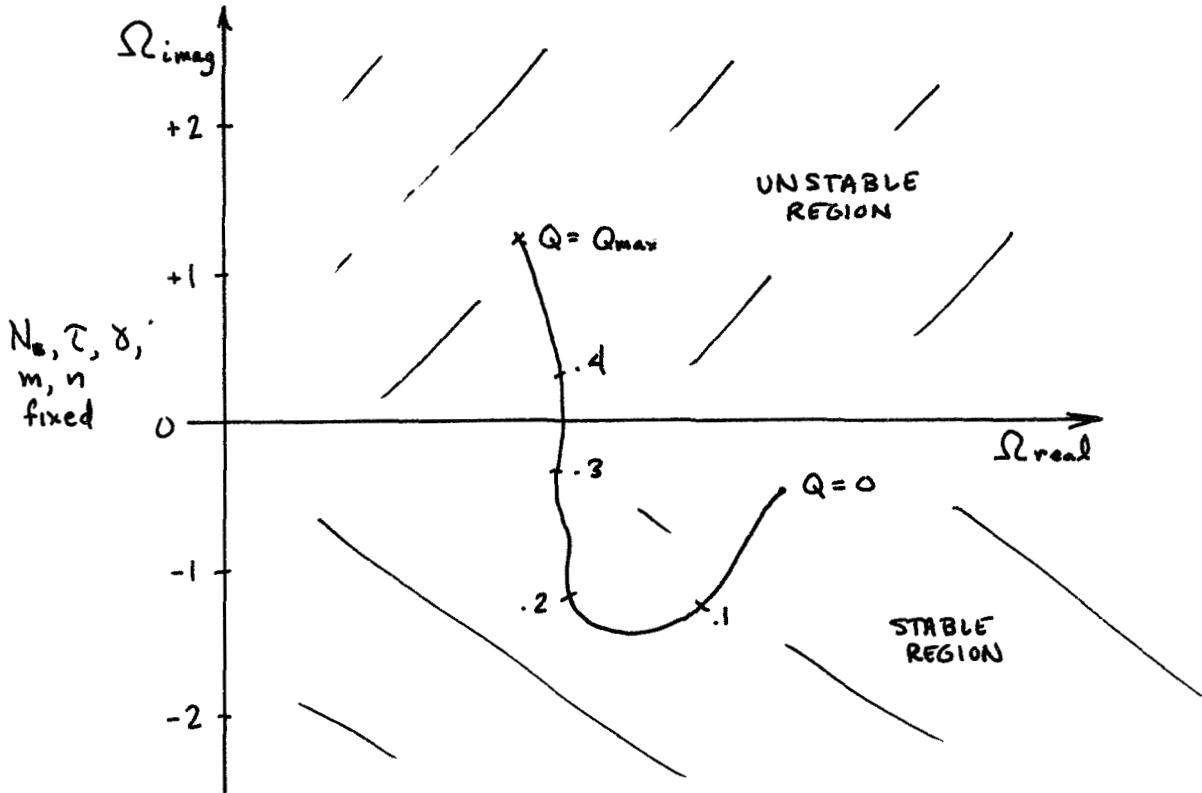


Figure VI.1 Schematic Root Locus

This hypothetical root locus diagram illustrates the kind of information desired from a numerical search for the roots of the determinant discussed above. Since the harmonic time factor for each normal mode was assumed to be $\exp(-i\Omega_r \hat{t} + \Omega_i \hat{t})$ it is apparent that if the imaginary part of Ω_s is ever positive the s^{th} normal mode will have an amplitude that will grow exponentially with time. Hence the upper half plane of our diagram represents the unstable region. The horizontal axis therefore represents the locus of marginal

stability, which means the normal mode is purely oscillatory and suffers neither damping or growth. The critical net radiative flux, which is the largest net radiative flux for which the system is stable to small perturbations, is that Q for which $\Omega_i = 0$ for any normal mode. A number of such root loci have been calculated for a range of values of the nondimensional parameters which characterize a given problem.

VI.5 The Numerical Calculation of the Eigenvalues

Although the details of the computer program designed to calculate the eigenvalues have been relegated to an appendix, a brief outline of the numerical procedure used to obtain the results shall be given here. In order to calculate one complex eigenvalue Ω , for fixed values of all the other parameters, one must

- a) Calculate the zeroth order distribution $\eta(\xi)$.
- b) Calculate the complex coefficients, equations (4.56a-d) for the differential equation (4.55)
- c) Factor the polynomial for the four $\phi'_i(\xi)$. For each $\phi'_i(\xi)$ we must repeat steps (a-c) for enough values of ξ in the interval $0 \leq \xi \leq 1$ to represent these functions of ξ to reasonable accuracy.
- d) Integrate the four $\phi'_i(\xi)$ numerically over the interval in order to obtain the $\phi_i(1)$.
- e) Calculate the boundary condition expressions from equations (4.62 - 5), substituting in the calculated $\phi'_i(0)$, $\phi'_i(1)$, and $\phi_i(1)$.
- f) Multiply out the 4×4 complex determinant (6.19) and separate into real and imaginary parts.

This provides two real equations in two real unknowns $D_{\text{real}}(\Omega_r, \Omega_i) = 0$ and $D_{\text{imag}}(\Omega_r, \Omega_i) = 0$

g) Find the simultaneous roots of these two algebraic nonlinear equations, by a method of successive approximations such as the Newton-Raphson procedure. (See Nielsen, Ref. 28). Extremely accurate guesses for the roots are required to numerically determine the roots for such transcendental equations, as opposed to the ease of finding the roots of polynomials.

The above described procedure is followed to find a root for each normal mode of the infinite set of normal modes. To give an idea of the sensitivity of the procedure, it is possible in some cases to make a guess for a root that is within one percent of the actual value and have the computer still not converge to the desired root.

Now one would like ideally to find an alternate procedure to directly calculate the locus of $\Omega_i = 0$, i.e. the locus of neutral stability, for a fixed frequency Ω_r . For instance it would be desirable to have curves of optical depth τ vs. net heat flux parameter Q for which the normal mode of frequency Ω_r is neutrally stable. This was attempted and yielded no results because guesses for the location of these roots could not be made close enough so that the calculation would converge to the actual roots. (For further reading of difficulties of this nature in stability analyses, see the thesis by Callen, Ref. 29).

A more systematic procedure for finding the eigenvalues is to trace out a root locus, $\Omega = \Omega(Q)$ for all the other parameters

fixed, as described in Sec. 6.4. This procedure has the advantage of starting with easily calculated values of the eigenvalues at $Q = 0$ (those for a uniform gas). From then on one makes small increments in Q and follows the steps (a-g) given earlier in the section, using the eigenvalue at Q as the guess for the eigenvalue at $Q + \Delta Q$. In this way one can trace out the root locus for the entire range

$0 \leq Q \leq Q_{\max}$. This is a very laborious procedure since increments of a few percent in Q must be taken to trace out a curve for values of Q over three or four decades. On the average seven to eight minutes of calculating time on a very fast digital computer is required per root locus, since all manipulations must be done in double precision numbers and complex variables. Unfortunately, only a small fraction of the cases attempted successfully yielded stability information. The results that have been obtained for these eigenvalues are presented and discussed in the following chapter.

VII. RESULTS AND CONCLUSIONS

VII. The Stability of a Normal Mode: Some Typical Root Loci

In previous chapters we have set up an eigenvalue problem to determine the complex frequencies of the natural oscillations of a non-equilibrium radiating gas bounded by black constant temperature walls. The eigenvalues are determined by finding the roots of a determinant numerically. The locus of some of these roots has been found for the variation of the net radiative flux parameter, holding all other parameters constant. An example of such a root locus plot of $\Omega = \Omega(Q)$ is given in Figure (7.1). The plot was calculated for fixed values of the following nondimensional parameters.

$$\text{Boltzmann Number} \quad N_b = \frac{\gamma}{\gamma-1} \left| \frac{p \sqrt{RT}}{\sigma T^4} \right| = 0.1$$

$$\text{Optical Depth} \quad \tau = \alpha L = 100.$$

$$\text{Variation of Opacity with Temperature} \quad m = 6$$

$$\text{Variation of Opacity with Pressure} \quad n = 1$$

$$\text{Ratio of Specific Heats} \quad \gamma = 1.667$$

$$\text{Mode Number} \quad s = 6$$

This case will be referred to often as a reference case because the nondimensional parameters are fairly representative of the physical regime of interest in a device such as a gas core nuclear rocket. The result in Figure (7.1) is for the 6th normal mode, as indicated by the notation $s = 6$. On all of the root locus plots the real part of the frequency is the abscissa and the imaginary part the ordinate. The values of the parameter Q are marked off on the locus. It is

important to note that the time dependence was taken to be $\exp(-i\Omega\hat{t})$
 $= \exp(-i\Omega_r\hat{t} + \Omega_i\hat{t})$ so that when the imaginary part becomes positive
 the normal mode will grow exponentially with time, causing an
 instability. Conversely, all points in the lower half plane
 represent stable points. The frequency $\Omega = \frac{\omega L}{\sqrt{RT_e}}$ is normalized
 by a characteristic acoustic frequency, so that when Ω_i is of order
 unity the growth rate is on the order of an e-fold per cycle of
 oscillation, which is quite large.

The values of the net radiative flux parameter Q for which the
 imaginary part of the frequency vanishes is called the critical net
 flux, for any larger value of Q than the critical value Q_c will cause
 the gas to become unstable. For each root locus the value of Q
 increases monotonically from zero to the maximum value $Q_m = Q_{max}(m, \tau)$
 unless the iterative procedure diverged at some point causing the
 root locus to terminate prematurely. In this case only a portion of
 the locus can be given. The case given in Figure (7.1) for example
 becomes unstable for a critical heat flux parameter of $Q_c = 0.010$ and
 terminates at $Q = .0176$, short of the maximum at $Q_m = 0.047$. These
 maximum heat fluxes were given in Figure (3.5) as a function of m and τ .

Let us examine a second case. These parameters were held fixed
 for the calculation

$$\begin{array}{lll} \gamma = 1.667 & m = 8 & n = 1 \\ N_s = 0.1 & \tau = 100 & S = 6 \end{array}$$

This case is identical to the previous one except for the fact that
 the opacity varies as the eight power of the temperature ($m = 8$)

instead of the sixth power. This case is plotted in Figure (7.2). The scales on the curves have been changed to reflect the fact that the root locus could be carried to larger growth rates. The root locus enters the unstable region for a critical net heat flux of $Q_c = .0064$ and terminates at $Q = .057$ because of a divergence in the iterations, preventing the continuation of the locus until $Q_m = .078$. Now let us examine an entirely different set of parameters to continue the illustration of these diagrams. The following case is a stable one, for which the parameters are

$$\begin{array}{lll} \gamma = 1.667 & m = 6 & n = 1 \\ N_b = 10^{-4} & \tau = 100 & S = 14 \end{array}$$

This case, for the higher frequency of the 14th normal mode, is for a very small Boltzmann number corresponding to extremely high temperature. The root locus is given in Figure (7.3). The root locus proceeds directly from $Q = 0$ to $Q = Q_{max}$ without terminating prematurely and never enters the top half plane at all. In fact this mode gets more and more strongly damped with increasing heat flux.

Finally, a root locus is given in Figure (7.4) which appears quite different in shape than the others. This case is for the quantities

$$\begin{array}{lll} \gamma = 1.667 & m = 8 & n = 1 \\ N_b = 1.0 & \tau = 100 & S = 6 \end{array}$$

Unfortunately the root locus calculation diverged just after the crossing of the neutral stability axis, but at least we were able to arrive at the critical net heat flux parameter $Q_c = .048$ for this

case. An obvious distinguishing feature is the "bump" in the curve. This sudden change toward greater damping followed a rapid change toward the unstable region is not clear. Attempts to explain this based on relative rates of damping or growth of component waves have proved unrewarding. Considering the great complexity of the transcendental determinant possessing these root loci however, we are satisfied to get information on stability and growth rates at the present time. Asking physical significance for every wiggle in the root locus must await better understanding of the overall phenomena.

These examples of root loci are indicative of the entire set of results obtained so far. A table of values is provided for the cases which gave unequivocal stability information in Table (&.1). The large majority of the cases that were run failed in some manner before achieving useful results, leaving gaps in the set of parameters for which stability information is desired. These attempts were not recorded.

VII.2 Loci of Neutral Stability

It is difficult to assimilate all of the effects on stability of changing the many parameters which characterize a radiative heat transfer problem by examining a large number of root locus diagrams. A most convenient and useful way to display stability information is to describe the boundary of the stable region in parameter space. This boundary is defined by the locus of $\Omega_i = 0$ and is referred to as the locus of neutral stability, without regard to the other parameters being varied along the locus. Since all normal modes are stable for

no gradients or net radiative heat flux in the gas, the most interesting questions are the following. What is the largest net flux of radiation for which the gas is stable? How does this critical flux vary as a function of the other parameters? Are there any values of the other parameters for which the gas is stable for any possible value of the net heat flux? The following results answer these questions in a restricted domain of the governing parameters. A complete parametric study, however desirable, is impractical due to the numerous important parameters (m , τ , N_s , Ω and Q at the least) and the large amount of computing labor required per data point. Moreover, a large part of the parameter space remains uninvestigated due to the inability of the chosen computational scheme to converge to an answer for certain ranges of these parameters. These failings of the method aside, much can be learned from the results which are available since they span the regions of greatest physical interest quite well.

In order to reduce the number of parameters to a manageable level, two which have been left arbitrary up until now will be specified. The ratio of specific heats has been taken as that for a monatomic gas, $\gamma = 1.667$. For the extremely high temperatures of interest in the radiation dominated problems discussed here, a gas is likely to be completely dissociated and often singly or doubly ionized. In any event the equations do not seem to be very sensitive to this parameter compared to quantities which may vary over orders of magnitude. Secondly, the index of the variation of opacity with pressure has been given the value of $n = 1$ since many gases of

engineering interest have opacities which vary with the first power of the pressure. Again the parameter $\mathfrak{N}$ just does not vary in nature over as wide a range as other parameters so we shall take the above representative value. In the derivation of the equations in previous chapters it was just as easy to carry these arbitrary parameters along to be able to see if they entered in the final formulation in a crucial way, but this does not seem to be the case. Let us now determine the effects of the remaining parameters on the critical (neutrally stable) net radiative flux.

A. The Effect of the variation of $\mathfrak{M}$

We have assumed an opacity which varies as $\alpha \sim T^{\mathfrak{M}}$ because an opacity which is an increasing function of temperature is an essential ingredient in the explanations of the cause of instability. To test this we return to the first case discussed in Sec. 7.1, the root locus of which was given in Figure (7.1), and compare to it cases of the same $N_8 = 0.1$, $\tau = 100$, $S = 6$ but with different values of $\mathfrak{M}$. We select the following values from Table (7.1)

$\tau = 100$					
$N_8 = 0.1$					
$S = 6$					
$\mathfrak{M}$	10	8	6	5	4
Q_c	.0046	.0064	.01	.0146	STABLE

This information is more clearly displayed in Figure (7.5). At these values of τ and N_8 the sixth normal mode is stable for $\mathfrak{M}$ less than or equal to four. The critical net radiative flux for stability decreases with increasing $\mathfrak{M}$, until the largest stable flux at $\mathfrak{M} = 10$

is less than half that at $m = 6$.

Another example can be given at a higher Boltzmann number; for example consider these cases

$$\tau = 100 \quad N_g = 1.0 \quad S = 6$$

m	10	8	6	5
Q_c	.054	.048	.0371	stable

These points are plotted in Figure (7.6). For this case the gas is stable for opacities which vary with the fifth power of the temperature or less. At this Boltzmann number the critical net radiative flux is not very sensitive to m , though it does increase some with increasing m .

These examples seem to confirm that an opacity which increases sufficiently strongly with temperature is a necessary condition for an instability to occur. Cases which were run for values of the index m less than or equal to four were always found to be stable. For instance, a constant opacity, $m = 0$, seems absolutely stable.

B. The Effect of the Variation of τ

Let us return to the reference case given in Figure (7.1) and investigate the effect on the stability of changing the optical thickness, keeping all other parameters fixed. The numbers held constant in this comparison are N_g , m , S , and of course γ and η . The following values of the critical radiative flux may be taken from Table (7.1).

$$N_s = 0.1, \quad m = 6, \quad S = 6$$

τ	10	50	100	150
Q_c	stable	stable	.01	.0115

This is not a clear trend at all. It seems to indicate that as τ is increased or decreased from a value near 10^2 the gas becomes stable or less unstable. There is not enough data to determine the shape of the stability boundary for these parameters or to guess why there should be a minimum critical heat flux for $50 < \tau < 150$

Let us look at another comparison, for a larger Boltzmann number N_s and index m but for the same mode

$$N_s = 1 \quad m = 8, \quad S = 6$$

τ	5	10	30	50	100
Q_c	St.	St.	St.	.071	.0487

This case exhibits a monotonic trend with increasing optical depth. The critical flux decreases with increasing τ after the gas becomes unstable, which seems reasonable.

For a third comparison we examine a small Boltzmann number case, and for the second normal mode.

$$N_s = .01 \quad m = 8 \quad S = 2$$

τ	5	10	50	100
Q_c	.21	St.	.0645	.056

For τ larger than ten the trend is the same as for the previous case but the neutrally stable point for $\tau = 5$ is confusing. This is the smallest value of the optical thickness for which a neutrally stable point was ever found, test cases run at $\tau = 1, .1$ having proved stable. In general, larger values of τ than 100 would not converge properly.

It is difficult to draw conclusions from these examples since no patterns seem to emerge other than that alternating regions of stability and instability may be possible. A stability diagram of the general shape of the one sketched in Figure (7.5) would explain the numerical results but there is danger in extrapolating too far based on inconclusive results.

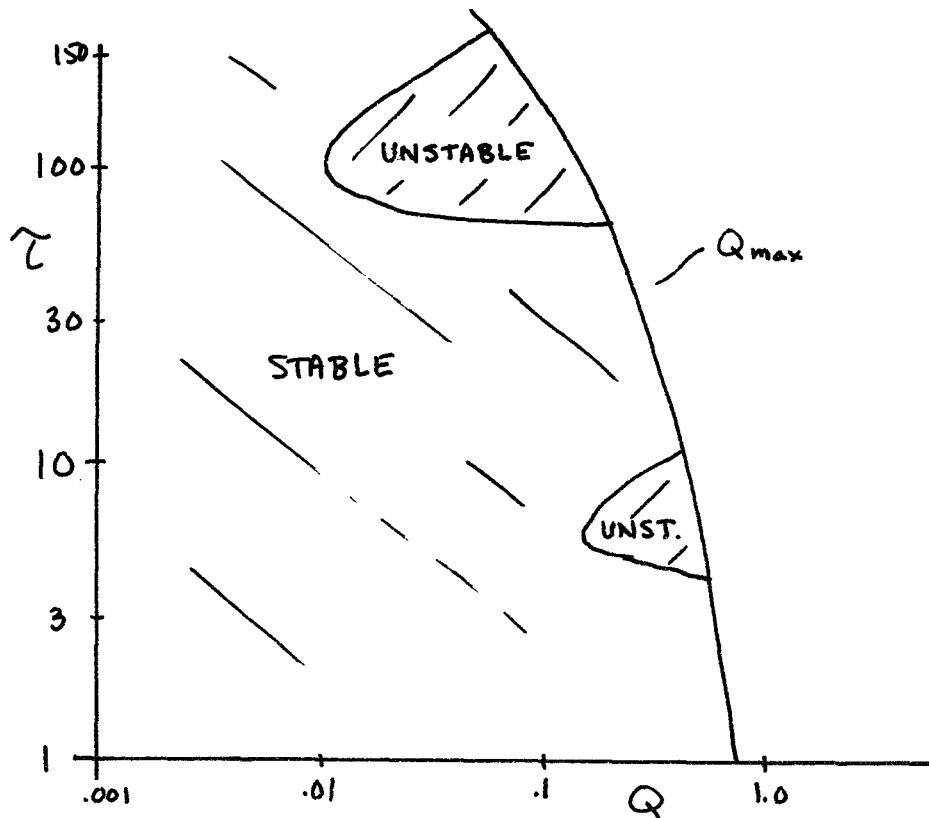


Fig. 7.5

We can say this: The gas must be optically thick for an instability to occur, and the instability can occur for smaller critical net radiative fluxes as the gas gets more opaque.

C. The Effect of the Variation of N_b

Again we return to our reference example which is the case of the sixth normal mode, for which $m = 6$, $N_b = 0.1$, $\tau = 100$. This case has a critical net flux of $Q_c = .01$. We would like to compare this to cases of larger and smaller Boltzmann number

$$m = 6, \quad \tau = 100, \quad S = 6$$

N_b	.01	.05	.1	1.0
Q_c	St.	St.	.01	.0371

There seems to be a minimum critical heat flux for some Boltzmann number, because the mode is stable for smaller values and has a greater critical flux for larger values of this number. This is exactly what one would expect physically, for reasons that are easily understood. First there is little reason to expect a radiation driven instability for very low temperatures for which $N_b \gg 1$ since radiation energy transport is then negligible compared to molecular energy transport. Secondly, for very high temperatures the radiative transport dominates and becomes such an efficient mechanism that it prevents temperature fluctuations from arising in the gas. This tendency toward isothermal disturbances for $N_b \ll 1$ would kill off an instability which depends upon opacity-temperature fluctuations.

One may see from Table (7.1) that all cases that were run for $N_s = 10^{-4}$ and many of those at $N_s = 10^{-2}$ were stable. Runs for Boltzmann numbers larger than unity were attempted but would not successfully converge to the root locus.

In summary, it seems reasonable to expect that an instability caused by the absorption of radiation will only exist for a moderate range of Boltzmann numbers, given very roughly by $10^{-3} \lesssim N_s \lesssim 1$

D. The Dependence on the Mode Number S

It has previously been explained that the stability of a system depends only on the most unstable mode, for if any mode is unstable it will not be possible to construct a bounded solution to an initial value problem. Hence we want to look for the mode which has the smallest critical net radiative flux parameter Q_c . For many stability problems the frequency eigenvalue which becomes unstable first is $\Omega_r = 0$, which is called a stationary, or nonoscillatory instability. In our problems no values of the other parameters could be found for which $\Omega = \Omega_r + i\Omega_i = 0$, but this is not surprising since we have been looking for an oscillatory instability. In fact our approach of finding a general solution by the WKB method limits us to looking for solutions for the higher frequencies and shorter wavelengths which satisfy the WKB assumptions. The quality of our answers is definitely better for higher mode numbers.

Let us examine this dependence for a few specific cases. First we shall look at our reference case

$$N_s = 0.1, \quad \tau = 100, \quad m = 6$$

S	2	6	10
Q_c	St.	.010	.0344

The sixth mode is unstable at a lower critical radiative flux than the tenth mode, but the second mode is stable. The results for the second modes are quite suspect however since the WKB assumption that a wavelength ought not vary significantly over its own length was either pushed to the limit, or violated, for these cases.

A demonstration of the conclusion that the lower modes become unstable for lower critical fluxes is provided by the following matrix of results taken from Table (7.1). The critical values of the flux Q_c are given in the boxes

S = 6	S = 10	
.01	.0344	$N_s = 0.1, \tau = 100, m = 6$
.0371	St.	$= 1.0, \quad = 100, \quad = 6$
.0064	.016	$= 0.1 \quad = 100, \quad = 8$
.0487	.054	$= 1.0, \quad = 100, \quad = 8$

The reason that the sixth mode was chosen for study was simply that it provided the compromise of being the lowest mode which satisfied the WKB assumption for most of the cases. This limitation of not being able to study the lowest modes is a major drawback to the WKB method of solving this stability problem.

VII.3 Further Discussion of Results

The study of the effects of changing the four major parameters M , τ , N_s and Ω_r on the neutral stability of the radiating gas is now complete. A more thorough parametric investigation of this problem will have to await the application of a more powerful method, for the limitations described in the last sections prevent obtaining a great deal more stability information than that which is presented here. In spite of the obvious lack of generality, the important features of this instability have been uncovered over a range of parameters which corresponds roughly to the region of most physical interest. In fact it is likely that enough is now known to attempt a direct numerical integration of the perturbation equations, for this attack seems to be practical only when the broad features of a stability problem have been investigated by more approximate analysis.

Now that we have some understanding of the boundary value problem which has just been solved for the frequency eigenvalues, it is interesting to compare the results for the critical heat flux for neutral stability to the heat flux criterion for growth of acoustic waves which was derived in Chapter V. Recall that the "local" analysis for wave propagation predicts that the acoustic wave traveling in the same direction as the heat flux will amplify exponentially with distance if

$$Q > \left(\frac{\gamma-1}{2\gamma}\right) \frac{N_s \Omega_r}{(2+n)} \left(1 + \frac{\Omega_r^2}{3\tau^2}\right) \quad (7.1)$$

This criterion is for small Boltzmann numbers, $N_B \ll 1$, and is equation (5.54) of Chapter V. The question arises as to whether or not this simple expression can reliably predict system stability. The hope was that it would obviate the need to solve difficult boundary value problems for every different set of boundary conditions.

We must examine the dependence of the simple expression above on the various parameters. First, there is no dependence on the variation of opacity with temperature in equation (7.1), especially the indication that there is absolute stability for $m < 4$, as in the eigenvalue problem. The dependence on optical depth seems reasonable in that the criterion becomes more stringent as the gas becomes optically thin. The dependence on frequency seems reasonable also since larger frequencies imply a larger flux necessary for instability in the eigenvalue problem also. The most obvious shortcoming of equation (7.1) in matching the more rigorous stability analysis is the dependence on the Boltzmann number. The simple expression predicts that a smaller radiative flux is necessary for the growth of acoustic waves as the Boltzmann number is decreased, directly conflicting with the results of the eigenvalue problem which indicate absolute stability for small enough Boltzmann numbers. For extremely small Boltzmann numbers such that disturbances become isothermal, pressure waves may still amplify as they are driven by the radiation. However, for this limit the instability mechanism involving opacity and temperature fluctuations is suppressed, so that the gas is stable for any net radiative flux.

It is instructive to make a quantitative comparison between the criteria. The "local" analysis was done under the assumption of small wavelength compared to the gradient length so we shall compare its prediction for the critical flux to the actual critical flux results for the sixth mode, which is a high enough mode to satisfy the WKB analysis.

We choose the conditions of $\gamma = \frac{5}{3}$, $\tau = 100$, $n = 1$ and $\Omega_p = 6\pi$ for which equation (7.1) gives

$$Q \gtrsim \left(\frac{2\pi}{5}\right) N_s \quad (7.2)$$

as the criterion for growth of acoustic waves.

Some data from Table 7.1 is given here for the critical fluxes.

	$m = 6$	$m = 8$
$N_s = 1.0$	.0371	.0487
$N_s = 0.1$	.01	.0064
$N_s = .01$	St.	St.

The comparison between the values calculated from equation (7.2) and those given in the table is poor. Although the criterion for the growth of waves in a nonuniform radiating gas and the stability criterion for such a gas bounded by walls are certainly related, the former does not seem to have the correct functional dependence to adequately represent the latter criterion for stability estimates.

VII.4 Conclusions

We set out to investigate the possibility of the occurrence of an instability driven by the absorption of radiation in a gas. The

broad features of an instability mechanism were evident from some physical reasoning which depended on analogy to combustion instabilities, detonation waves and radiation shock waves. The factors which were thought to be necessary and important in affecting stability have been largely confirmed by the results presented in the previous sections. We now attempt to distill the important observations and conclusions from this research.

1) An instability may arise in a nonuniform radiating gas enclosed by black constant temperature walls. The onset of this instability depends on the values of the various nondimensional parameters which govern the problem, in a complex manner. There is no reason to believe that the instability will only occur for the particular boundary conditions chosen in the present analysis, but on the other hand it is quite likely that the stability criteria will differ in detail for different boundary conditions. In any case this analysis shows that such an instability indeed exists, subject to all the many restrictions and assumptions of this analysis.

2) The instability will occur only for moderately small Boltzmann numbers, $N_b = \frac{\gamma}{\gamma-1} \frac{pRT}{\sigma T^4}$, since either very large or very small Boltzmann numbers will enhance stability. In physical terms this means that the temperature must be high enough for radiation transfer to be an important energy transfer mechanism, but not so large as to have the effect of preventing temperature fluctuations from occurring at all. To stabilize an engineering device such as the gas core nuclear rocket, one would want to operate at as high a temperature and low a pressure as possible.

3) The instability will occur only for those conditions under which the opacity varies more strongly than the fourth power of the temperature. That is, if $\kappa \sim T^m$ where κ is the opacity and T the temperature, then an instability is possible only for m larger than four. The stability criterion is not a particularly strong function of m , as long as it is larger than the cut off value of four.

Although it was guessed that the opacity must be a sufficiently increasing function of temperature, only the stability analysis could provide this minimum value. In extrapolating this information to the design of an engineering device, we can say that stability ought to be assured if one can tailor the gas composition in such a way as to have the opacity a constant or a decreasing function of temperature over the temperature range of interest. This generalization may be invalid for different boundary conditions of course, but the above described effect is probably one that is intrinsic to the radiating gas and not the boundaries.

4) An instability may arise only for a gas which is optically thick. The trend of the effect of optical thickness on neutral stability was not conclusively established, for it did not always seem to be the case that more opaque gases were more unstable. The effect of optical thickness on stability seems to be quite complex and not very amenable to generalization. An optically thin gas however, was never found to be unstable. No unambiguous trend toward greater or lesser stability could be established based on comparisons between the average photon wavelength, disturbance wavelength, and distance between the walls.

5) The lowest frequency normal modes will be most unstable, or will become unstable first as the net radiative heat flux is increased. The instability is an oscillatory one; the natural oscillations of the system will simply grow everywhere exponentially in time once the critical heat flux is exceeded. This conclusion is only valid of course for amplitudes of fluctuation small enough to satisfy our assumptions of linearity. It would be purely speculation to guess what new final state this system would evolve to after it goes unstable. Two alternatives seem possible; either a steady two-dimensional convection cell could arise or perhaps an unsteady configuration involving radiatively driven shocks could occur. To determine this would require a non-linear analysis however, so this topic is reserved for future research.

6) The criterion for growth of individual acoustic waves which was derived by a "local" analysis of wave propagation in nonuniform radiating gases does not seem to be a reliable indicator of the onset of instability for a gas bounded by black constant temperature walls. The most serious shortcoming of the approximate criterion is its inability to represent the correct dependence of the stability criterion on the Boltzmann number of the variation of opacity with temperature. The conclusion is that even for disturbances of short wavelength compared to the distance between the walls the boundary conditions are an important factor in determining stability. This is contrary to our hope that the important features of any instability would be dominated by propagation characteristics of the wave rather

than the reflection characteristics. These effects are apparently strongly enough coupled to require treating each different possible set of boundary conditions as an individual basis.

These conclusions essentially answer all the questions posed in the introduction, yet many other questions remain unanswered. For example the influence of the optical depth on the locus of neutral stability is not understood. It was not possible to devise a numerical scheme to successfully calculate the locus of neutral stability as a function of optical depth using the methods we have chosen in this thesis. The first recommendation for future research would be to undertake a direct numerical integration to substantiate the results of this thesis and to provide information over a larger range of the parameters.

Other suggestions for future analysis are obvious. The same type of problem should be solved for other boundary conditions, such as a gas bounded by transparent walls and heated by an outside source of radiation. The model should be extended to be able to determine the effects of a flow velocity on this radiation driven instability. Perhaps it would be valuable to model the ionized gas as a two-fluid model and at the same time include a more realistic absorption coefficient for the gas. Another possibility is a reorientation of the model to examine the stability of a confined gas heated by a steady-state laser source. This may suggest a specific experimental test for this instability. Finally, more thought ought to be given as to whether such a radiation driven acoustic instability can happen in nature, specifically in stellar atmospheres.

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TABLE 7.1

m	$\gamma = 1.667$	τ	$n = 1$		
	N_0		s	Q_c	Q_m
10	1	100	6	.054	.112
10	.1	100	6	.0046	.112
8	1	100	6	.0487	.078
8	1	100	10	.054	.078
8	1	50	6	.071	.108
8	1	30	10	stable	.138
8	1	30	6	stable	.135
8	1	10	6	stable	.2255
8	1	10	6	stable	.2237
8	1	5	6	stable	.2933
8	.1	100	2	.0385	.078
8	.1	100	10	.0159	.078
8	.1	100	10	.0159	.078
8	.1	100	6	.0064	.078
8	.01	100	6	stable	.078
8	.01	100	2	.056	.0718
8	.01	50	2	.0645	.1063
8	.01	50	6	stable	.1063
8	.01	30	10	stable	.135
8	.01	30	6	stable	.135
8	.01	10	2	stable	.2237
8	.01	10	6	stable	.2237

TABLE 7.1 (Continued)

$\gamma = 1.667$			$n = 1$		
m	N_s	τ	s	Q_c	Q_m
8	.01	5	2	.21	.293
8	.01	5	6	stable	.2933
6	1	100	6	.0371	.0475
6	1	100	6	.0375	.0475
6	1	100	30	stable	.0475
6	1	100	14	stable	.0475
6	1	10	14	stable	.0475
6	1	10	6	stable	.171
6	.1	100	2	stable	.0475
6	.1	100	10	.0344	.0475
6	.1	100	6	.01	.0475
6	.1	10	6	stable	.171
6	.1	150	6	.0115	.0369
6	.1	50	6	stable	.072
6	.05	100	6	stable	.0475
6	.01	100	2	stable	.0475
6	.01	100	6	stable	.0475
6	.01	100	14	stable	.0475
6	.01	50	2	stable	.07
6	.01	10	2	stable	.171
6	.01	10	14	stable	.171
6	.01	5	2	.198	.256

TABLE 7.1 (Concluded)

$\gamma = 1.667$			$n = 1$		
m	N_0	τ	s	Q_c	Q_m
6	.0001	100	14	stable	.0475
6	.0001	.1	14	stable	.879
6	.0001	100	2	stable	.0475
6	.0001	1	14	stable	.52
5	1	100	6	stable	.0345
5	.1	100	6	.0146	.0345
4	1	100	30	stable	.0245
0	1	100	14	stable	.00667

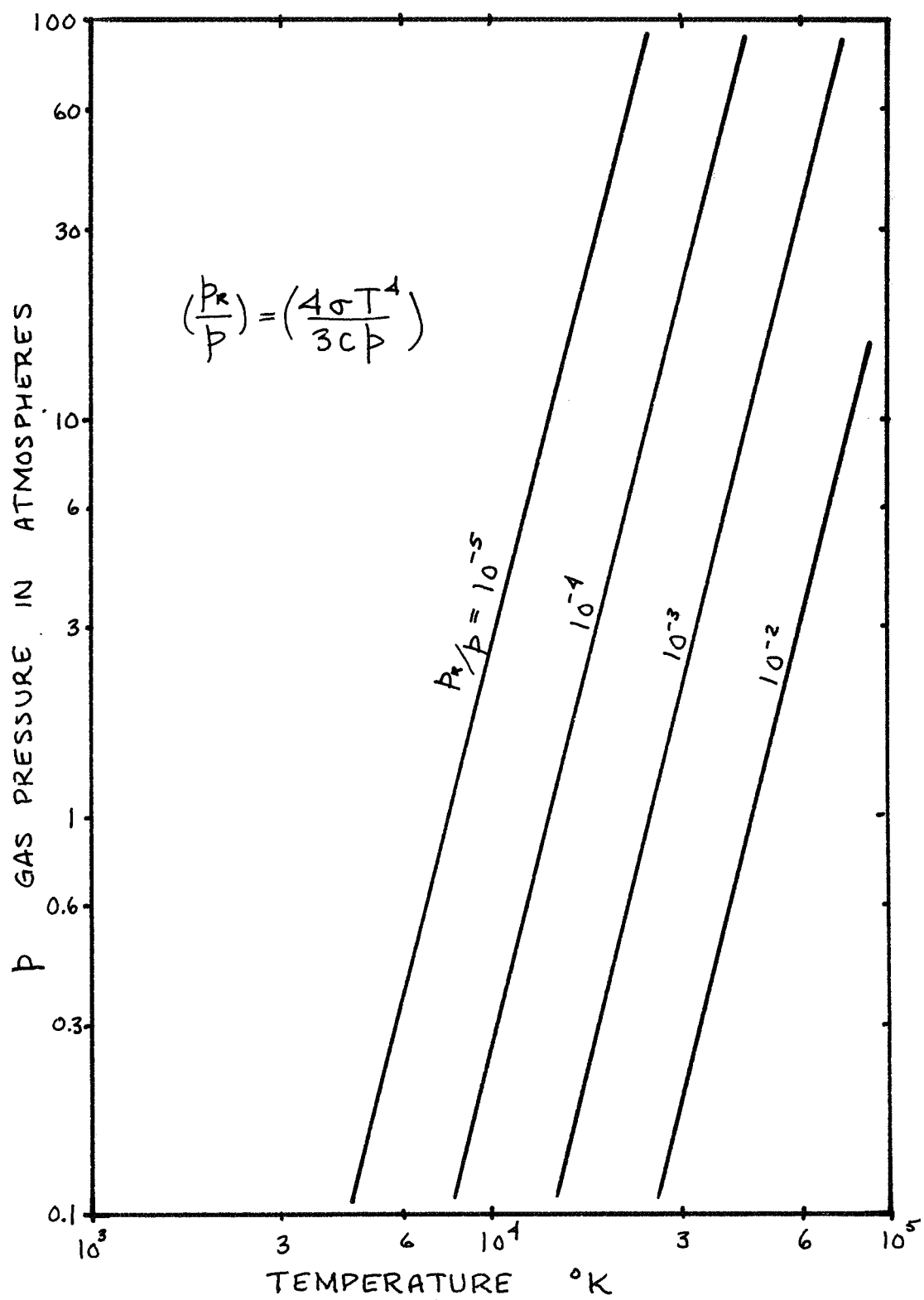


FIGURE 3.1 RADIATION TO GAS PRESSURE RATIO

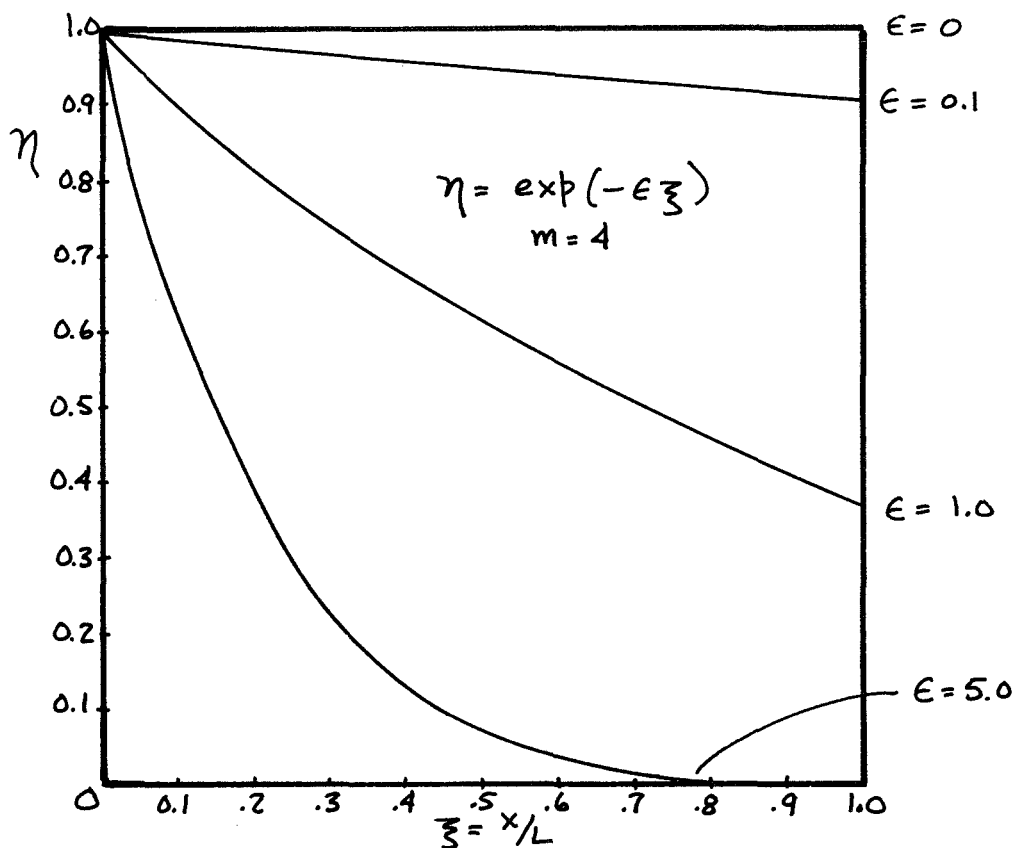


FIGURE 3.2a NONDIMENSIONAL TEMPERATURE DISTRIBUTION

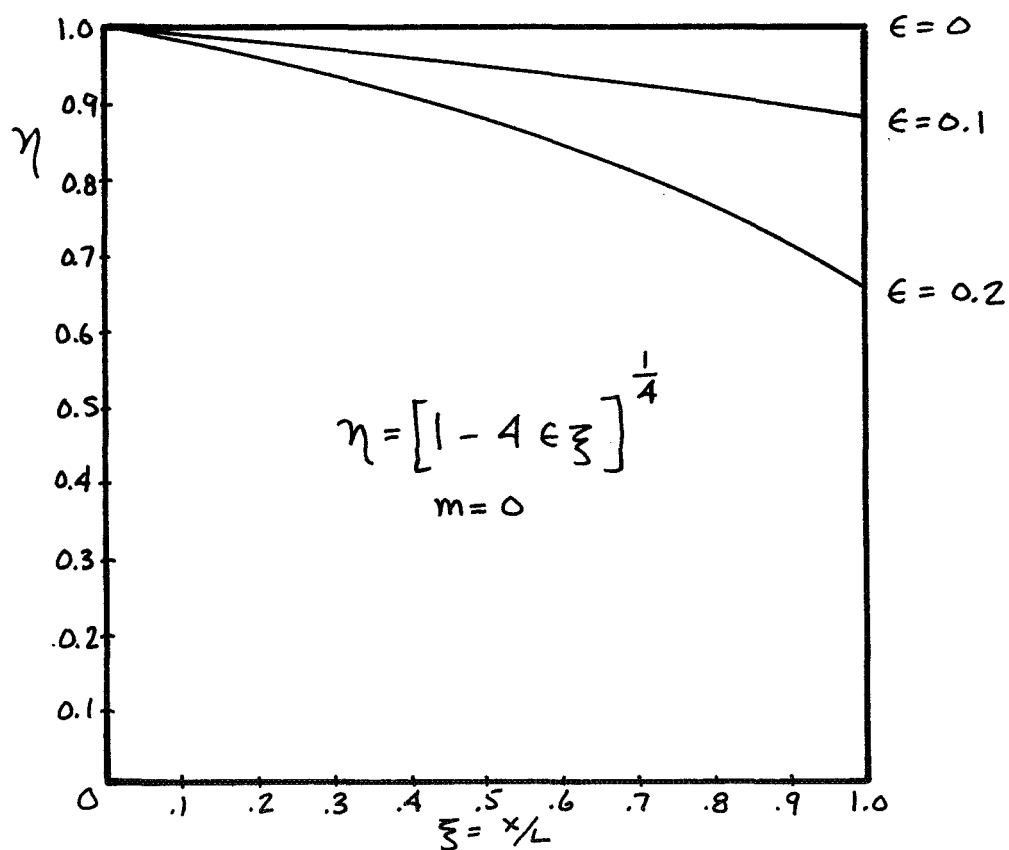


FIGURE 3.2b NONDIMENSIONAL TEMPERATURE DISTRIBUTION

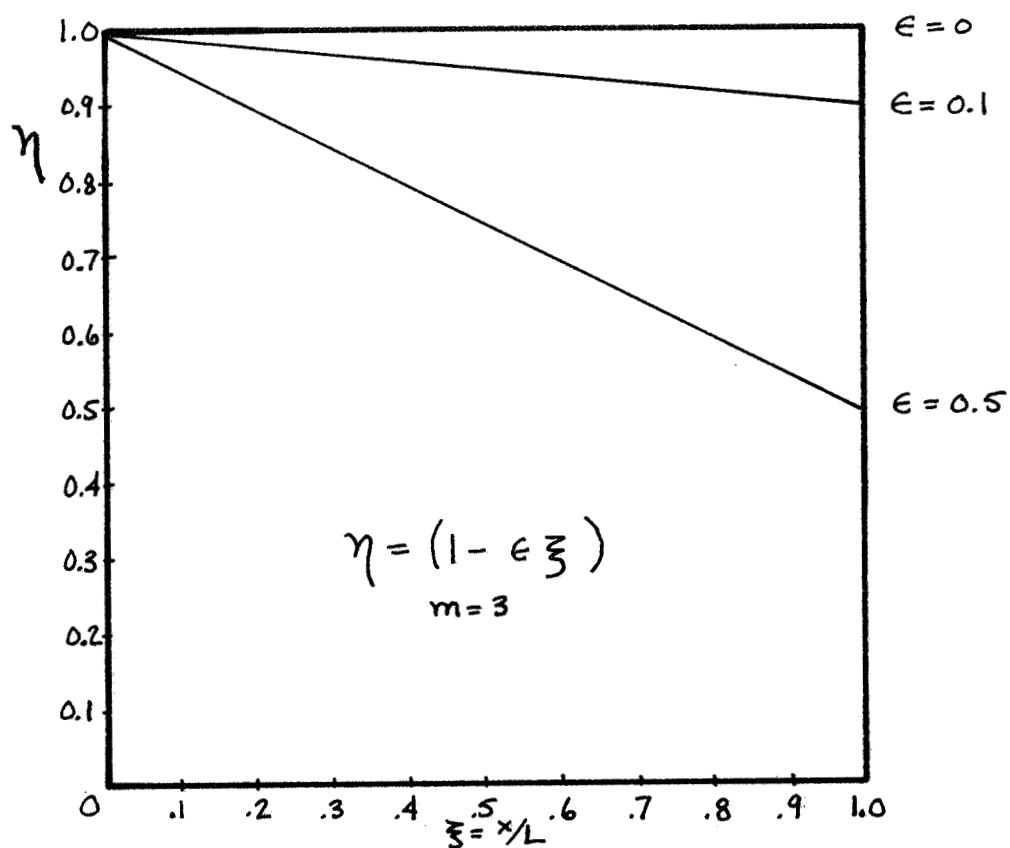


FIGURE 3.2c NONDIMENSIONAL TEMPERATURE DISTRIBUTION

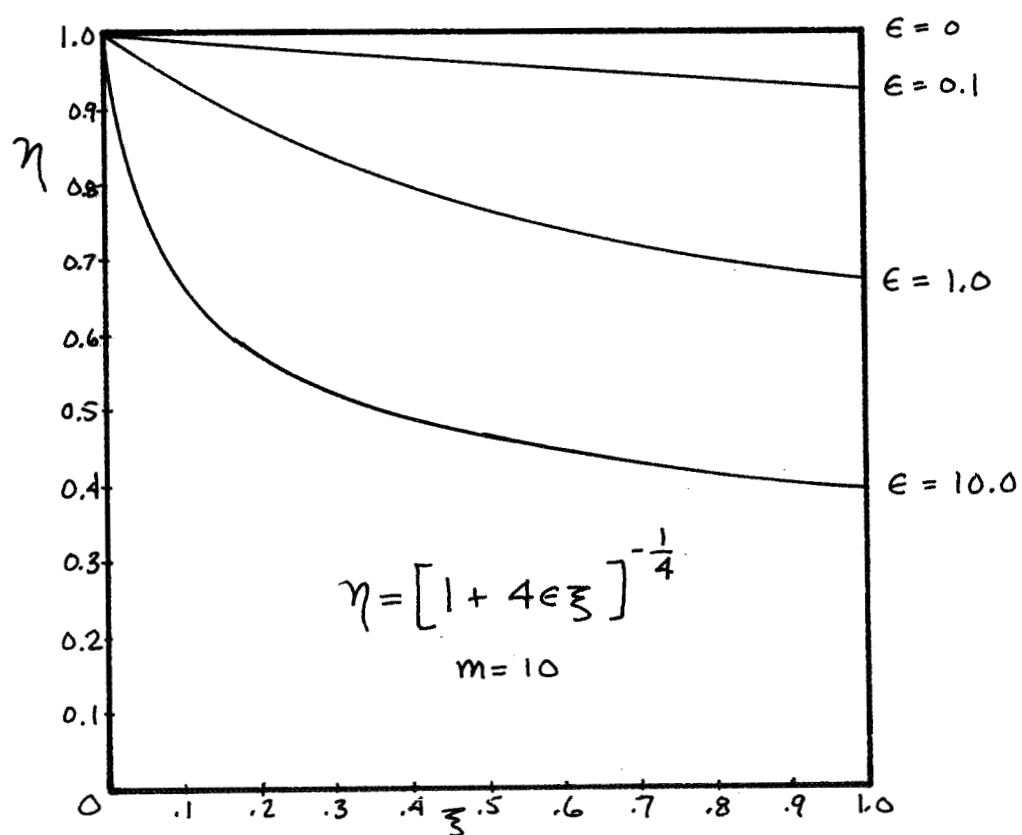


FIGURE 3.2d NONDIMENSIONAL TEMPERATURE DISTRIBUTION

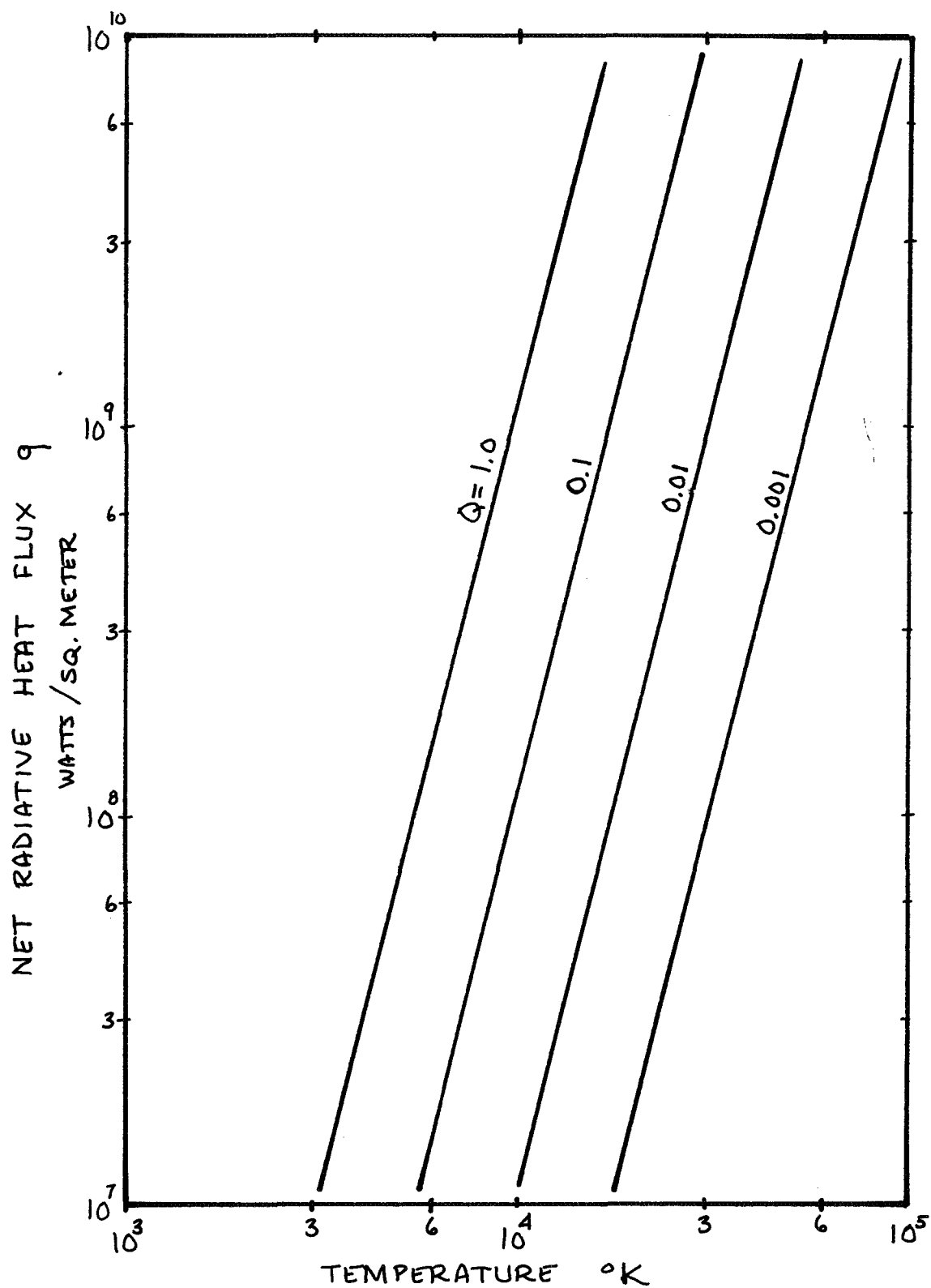


FIGURE 3.3 HEAT FLUX PARAMETER Q IN TERMS OF PHYSICAL PROPERTIES

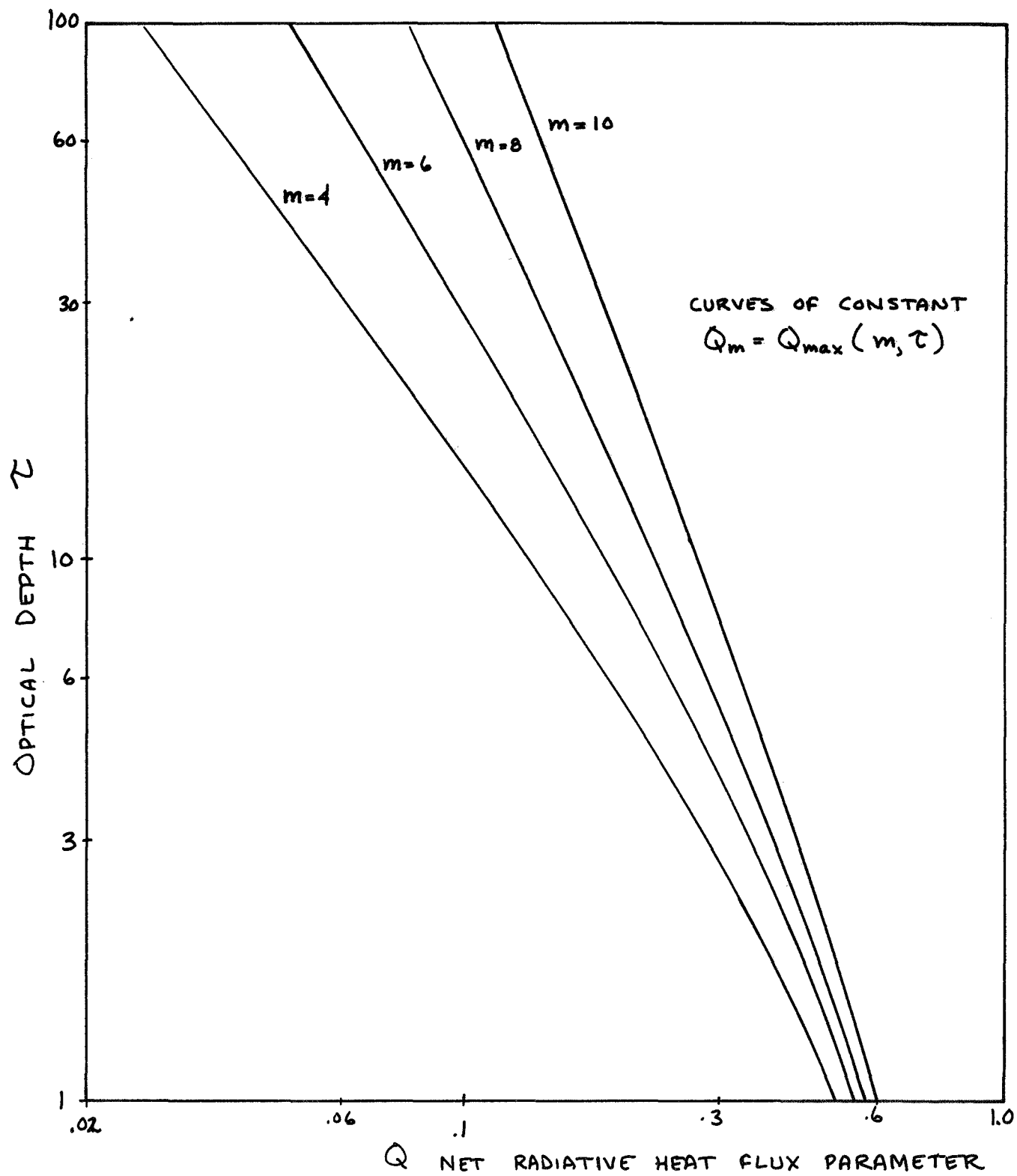


FIGURE 3.4 MAXIMUM ALLOWABLE RADIATIVE HEAT FLUX

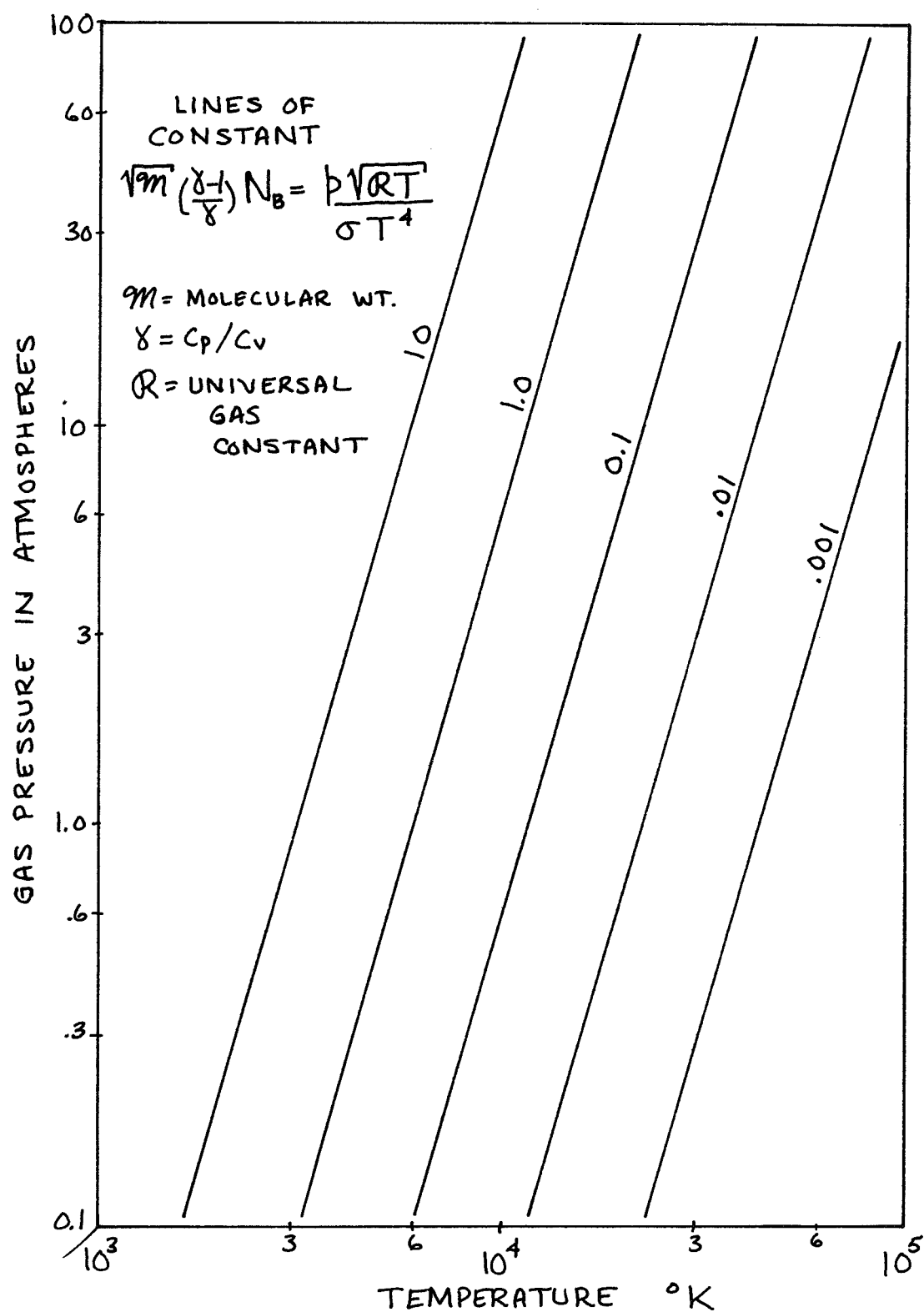


FIGURE 4.1 BOLTZMANN NUMBER N_B IN
 TERMS OF PHYSICAL QUANTITIES

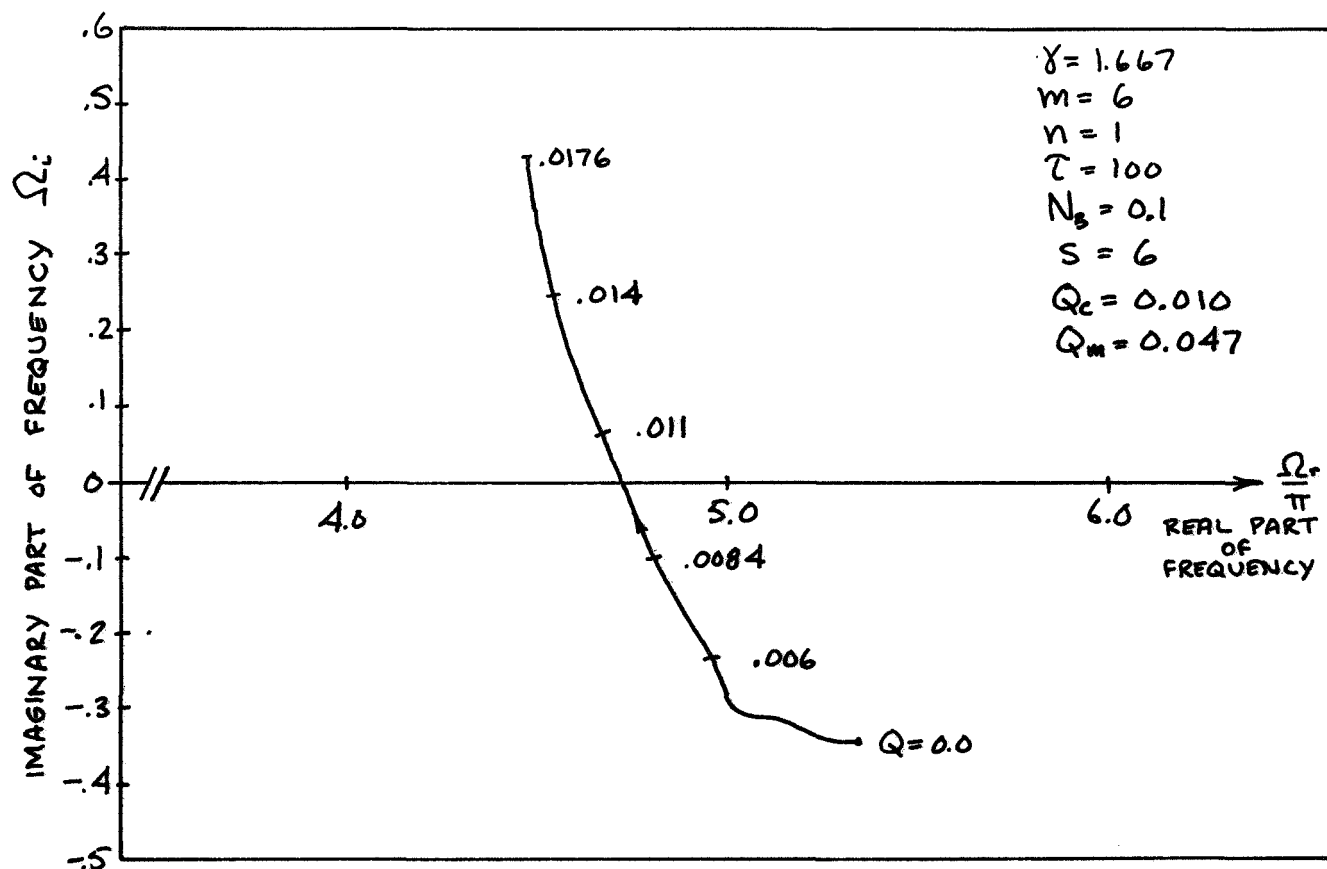


FIGURE 7.1 ROOT LOCUS

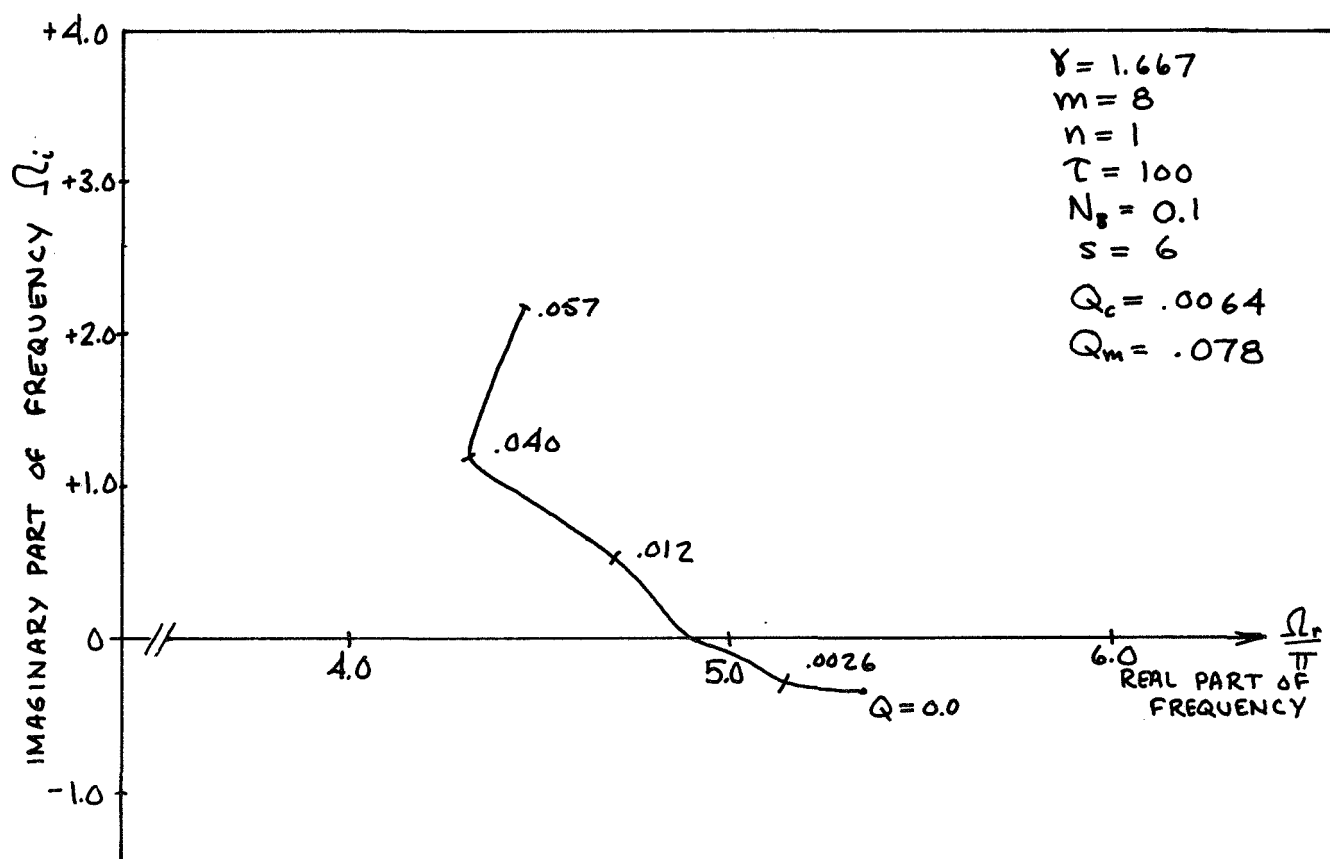


FIGURE 7.2 ROOT LOCUS

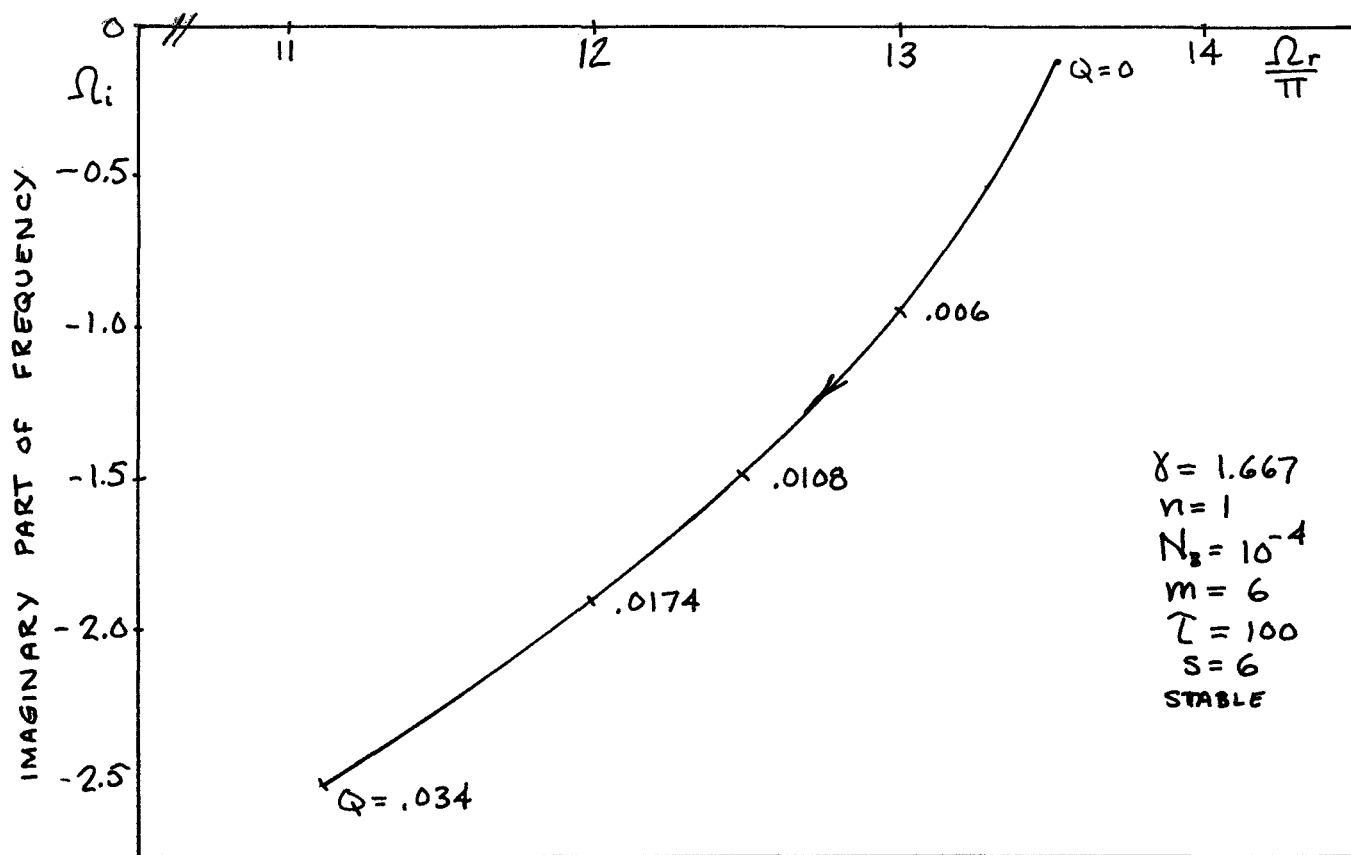


FIGURE 7.3 ROOT LOCUS

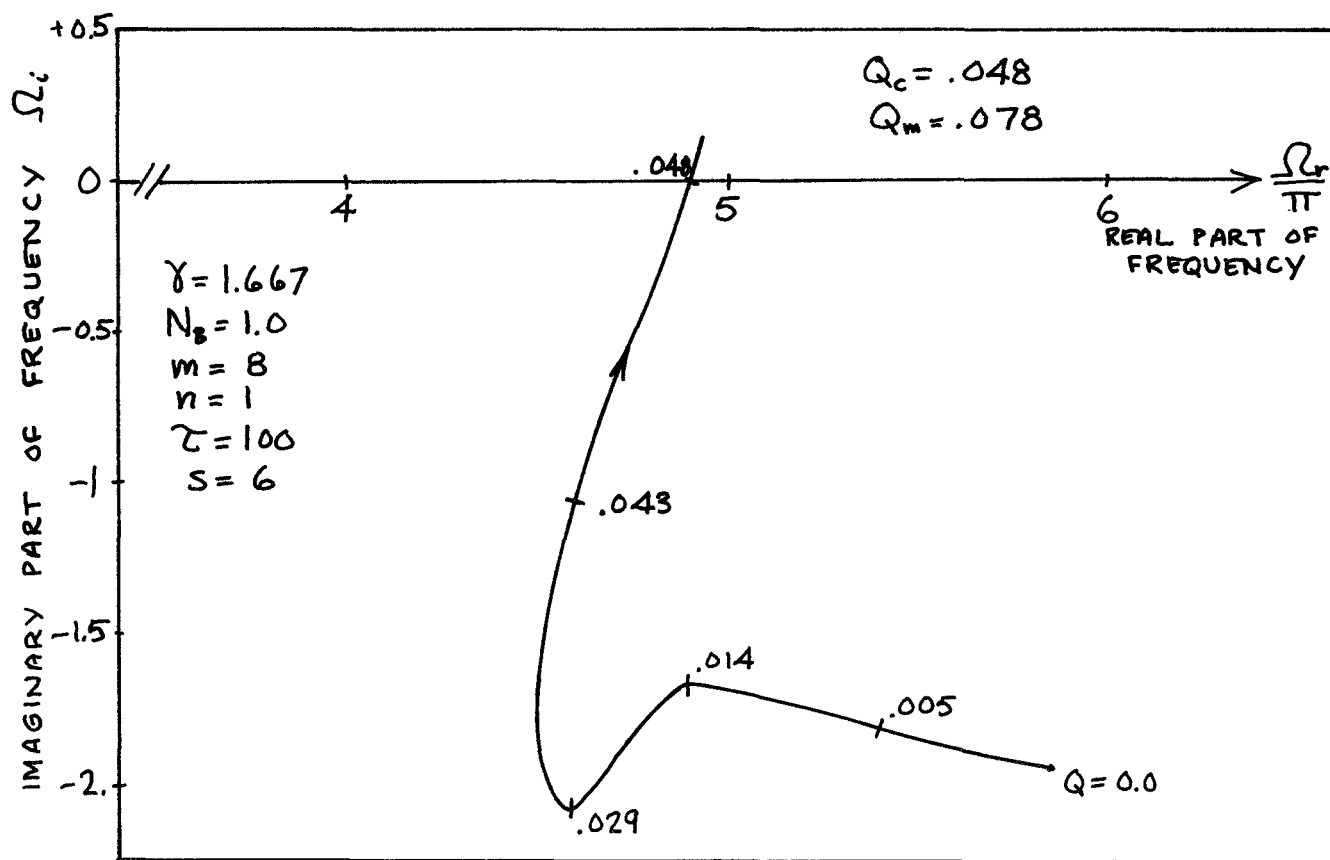


FIGURE 7.4 ROOT LOCUS

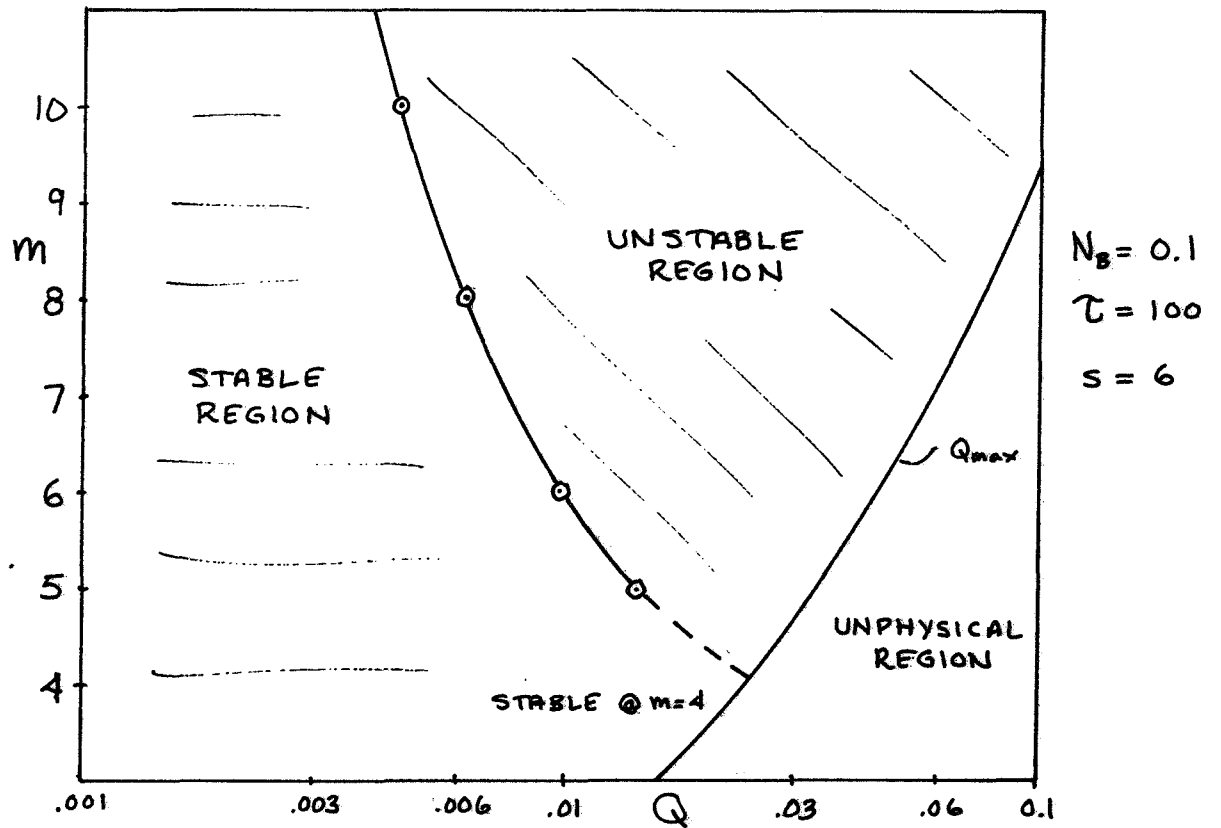


FIGURE 7.5 NEUTRAL STABILITY LOCUS

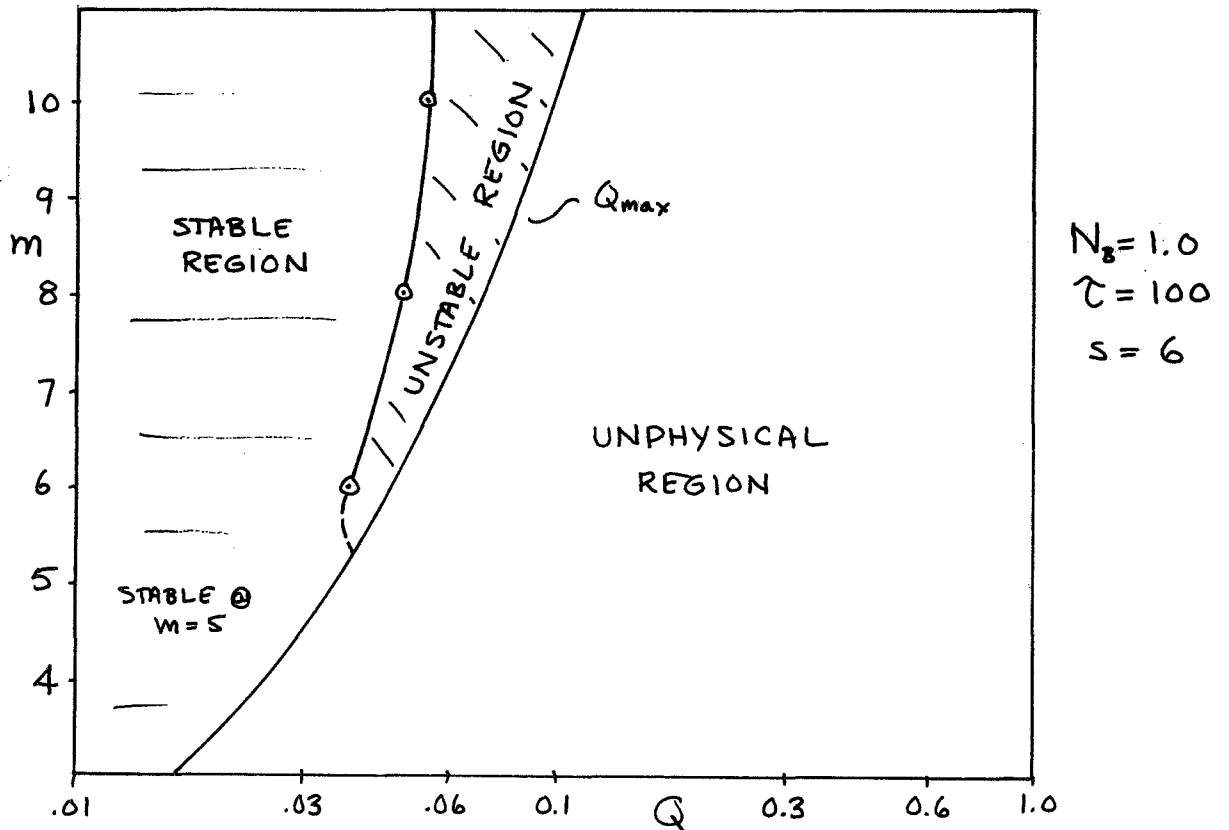


FIGURE 7.6 NEUTRAL STABILITY LOCUS

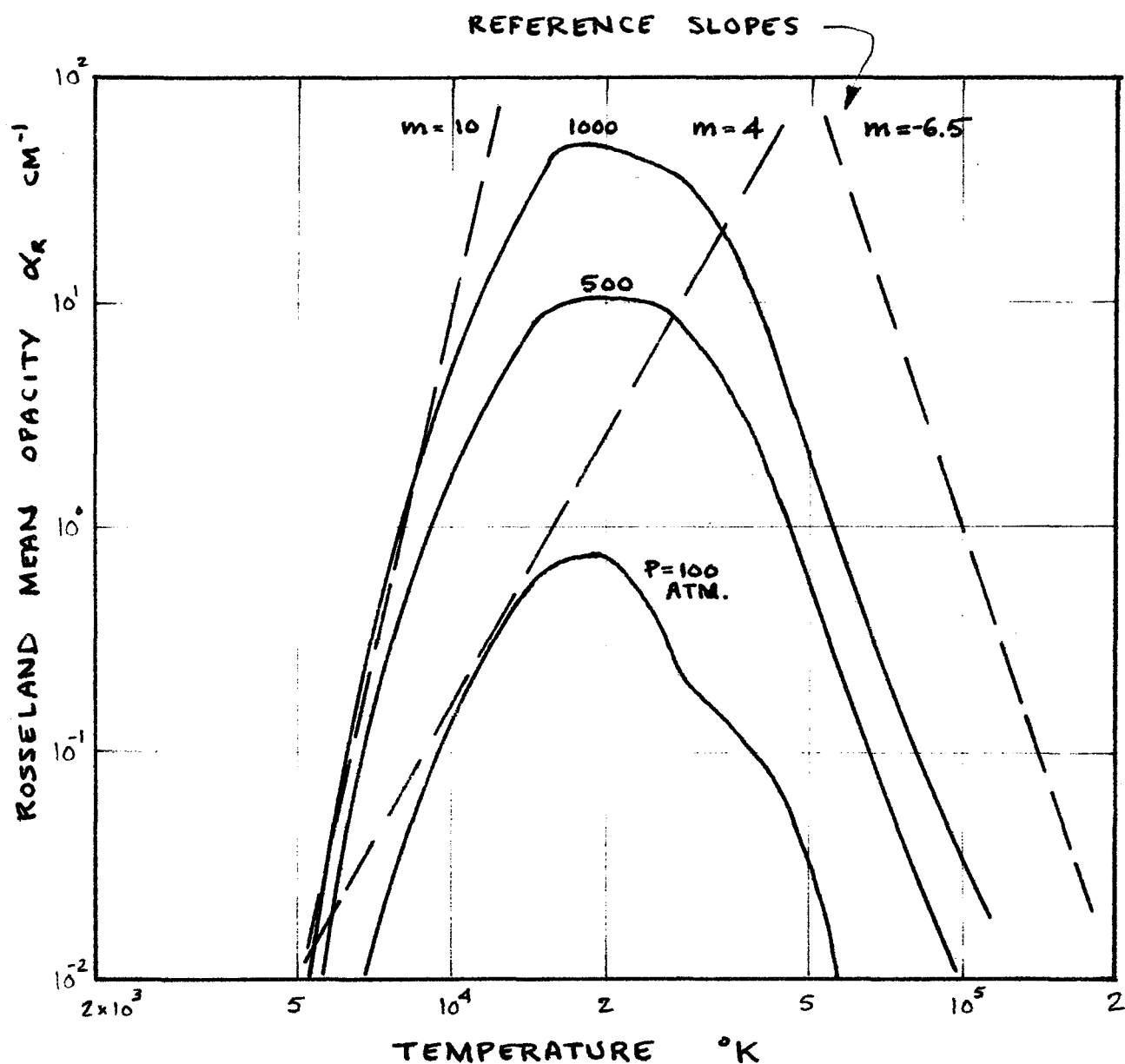


FIGURE A.1 EFFECT OF TEMPERATURE ON THE
ROSSELAND MEAN OPACITY OF HYDROGEN

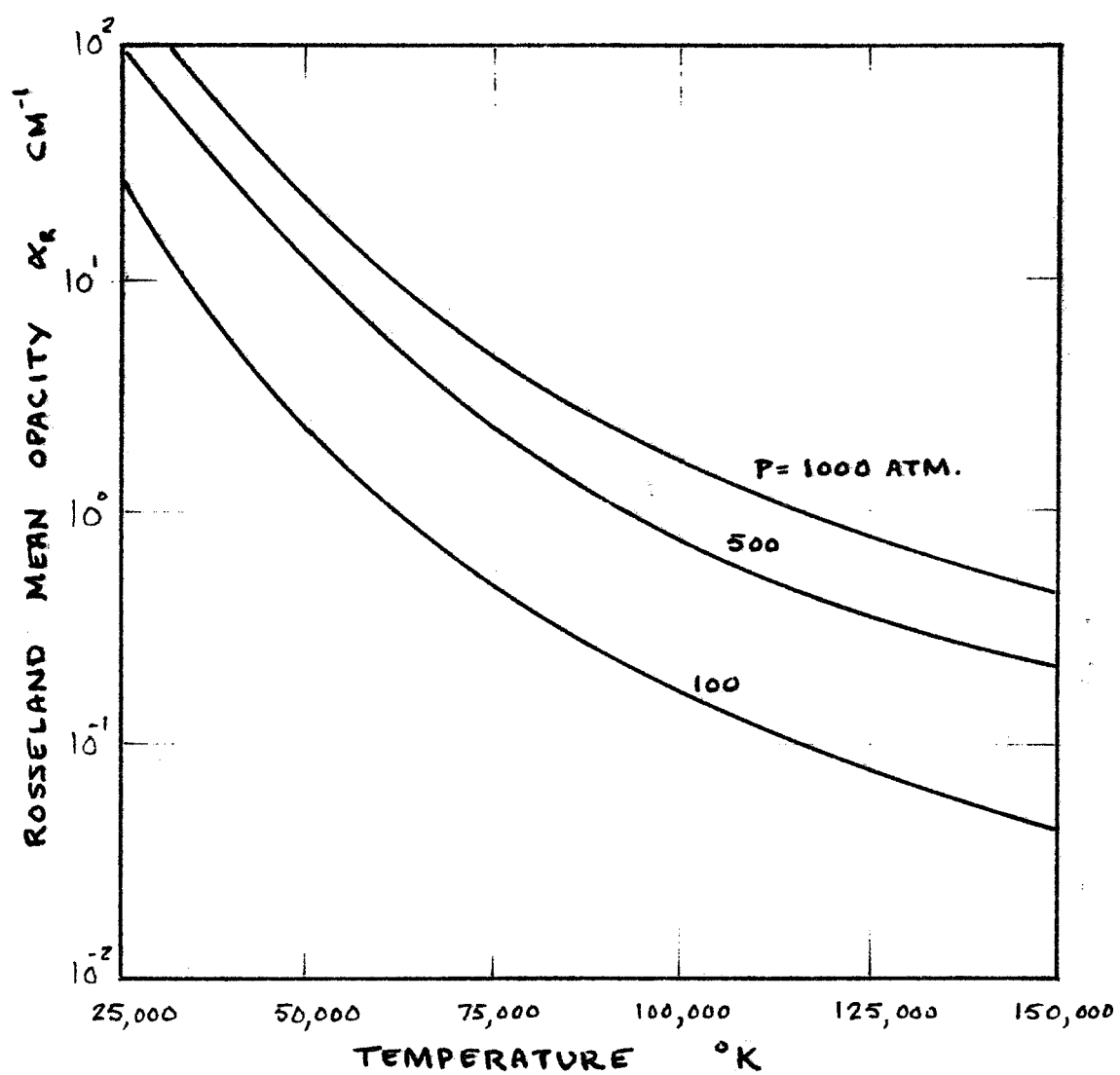


FIGURE A.2 EFFECT OF TEMPERATURE ON THE
ROSSELAND MEAN OPACITY OF URANIUM

APPENDIX A - TYPICAL ROSSELAND MEAN OPACITIES

In this report many references are made to the opacity of the gas without regard to the specification of the particular processes by which gases absorb radiation. This has been done purposely to keep the analysis of the fluctuations as general as possible. We would like our analysis to apply to situations as diverse as a multiply ionized uranium gas, a hydrogen gas seeded with submicron tungsten particles, a pure alkali metal gas, or an undissociated molecular gas such as CO_2 . In each case the details of the physics of absorption may be different, but the gross features of the energy transfer between the radiation and the gas may be represented over some range of temperature and pressure by a simple approximate expression. We have assumed that a volumetric absorption coefficient, or opacity, which is an average over all photon energies or frequencies is a sufficiently accurate description of this coupling. Any gas which is modeled by using an opacity which is independent of frequency is referred to as a grey gas. In a gas which is optically thick, a condition which we find to be necessary for an instability to occur, a specific grey gas opacity is appropriate which is the following weighted average over photon frequency.

$$\alpha_{\kappa}(T, p) = \left(\frac{4\sigma T^3}{\pi} \right) \int_0^{\infty} \frac{1}{\alpha_{\nu}(T, p, \nu)} \frac{dB_{\nu}}{dT} d\nu \quad (\text{A.1})$$

This is the Rosseland mean opacity.

In addition to using a grey gas opacity such as the Rosseland mean in our analysis, we assume a particular function of temperature and pressure, as follows

$$\alpha(T, p) = \alpha_a \left(\frac{T}{T_a}\right)^m \left(\frac{p}{p_a}\right)^n \quad (\text{A.2})$$

where the subscript ()_a represents the quantity evaluated at some reference condition. We would like to estimate the range of values of the exponents m and n which seem physically reasonable by comparison to calculated Rosseland opacities of actual gases. The Rosseland opacity for hydrogen is given as a function of temperature and pressure by Krascella (Ref. 32). We have reproduced Figure 22 of that reference as our Figure A.1. First notice that the value of n required to fit the pressure dependence is about $n = \frac{5}{3}$. The value of m is a strong function of the temperature chosen, for the slope of the curves varies greatly. Since

$$m = \frac{d \ln \alpha}{d \ln T} \quad (\text{A.3})$$

lines of constant slope on such a log-log plot are lines of constant m . Some reference slopes are superimposed on the graph which indicate that the range of m for temperatures of $5 \times 10^3 < T < 10^5$ °Kelvin and pressures of $1 < p < 10^3$ atmospheres is approximately $-10 < m < +20$. These large variations of opacity with temperature seem to be characteristic of Rosseland mean opacities for continuum absorption processes. The hydrogen opacity was determined from bound-free and

free-free transitions of molecular and atomic hydrogen including positive and negative hydrogen ions, and from the bound-bound Lyman- α transition. Details may be obtained from the original report, Ref. 32.

In this same report the opacity of uranium gas is calculated from the line and bound-free transitions from the ground and excited states of the neutral and singly, doubly and triply ionized atoms. This would be typical of the nuclear fuel in a gaseous core nuclear rocket. In Figure A.2 we have given the information which was presented in Figure 9 of Ref. 32. The value of η which is appropriate is almost exactly unity. The possible values of m are all negative since the opacity is everywhere a decreasing function of temperature. The range of m is approximately $-10 < m < -2$.

These two examples are given because they are indicative of the gases which may be used in the first engineering devices. Opacity curves for other gases, such as air, are not really different in character from the one given for hydrogen. (See for example the paper of Armstrong et al., Ref. 33, for the Rosseland opacity of high temperature air.) It would seem that the values of m and η which are used in this report, $0 < m < 10$ and $\eta = 1$, can be quite realistic for actual gases.

SYMBOL TABLE

A, B, C	matrix coefficients in state equation
a_s	isentropic speed of sound
a_r	isothermal speed of sound
$B(T)$	frequency integrated Planck function
B_p	equilibrium specific intensity of radiation, the Planck function
C, C_i	arbitrary complex constants
c	speed of light
D	characteristic determinant
$\bar{E}(\omega, k)$	average error
G	zeroth moment of the specific intensity, i.e. the intensity averaged over all directions
h	Planck's constant
I	frequency averaged specific intensity
I_p	specific intensity of radiation
$I^{(n)}$	n^{th} moment of the specific intensity with respect to the direction cosines
j_s	mass emission coefficient
K	dimensionless wavenumber
k	wavenumber
k	Boltzmann's constant
k_i	a unit normal vector
L	distance between walls
l_i	direction cosine
m	index in opacity - temperature power law

N_B	Boltzmann number
n	index in opacity-pressure power law
p	gas pressure
p_{rji}	radiation pressure tensor
q_{ri}, q	net radiative heat flux vector
Q	non dim. net radiative heat flux parameter
Q_R	one way radiative flux
R	gas constant
$r(x, k)$	weighting function
s	number of normal mode
T	temperature
U_n	Fourier amplitude of normal mode
U_R	energy density of radiation
u_i	gas velocity component
u_n	normal mode amplitude
v	signal velocity of heat wave
$\underline{v}$	state vector
y	wave amplitude
α	grey gas opacity
α_p	Planck mean opacity
α_R	Rosseland mean opacity
α_ν	spectral opacity or volumetric absorption coefficient
γ	ratio of specific heats
δ	small parameter of wave equation

δ_{ij}	Kronecker delta
ϵ	nondimensional gradient steepness paramter
η	nondimensional temperature distribution
κ_ν	spectral mass absorption coefficient
λ	wavelength of disturbance
λ_i	characteristic root
ξ	dimensionless distance
ρ	gas density
σ	Stefan-Boltzmann constant
τ	optical thickness
ϕ	trial eigenfunction, WKB function
Ω	solid angle, complex frequency eigenvalue
ω	circular frequency of oscillation
$()_a$	evaluated at $x = 0$
$()_b$	evaluated at $x = L$
$()_c$	critical value for neutral stability
$()_m$	maximum value of quantity
$()_o$	zeroth order solution quantity
$()_r$	radiation quantity
$()_w$	wall condition
$()_\nu$	frequency dependent quantity
$()_1$	first order solution quantity
$()^\wedge$	a nondimensionalized quantity
$()^*$	complex conjugate
$()^\sim$	a trial eigenfunction