NASA TECHNICAL NOTE

NASA TN D-5599

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NASA TN D-5599

DYNAMICS OF TWO SLOWLY ROTATING POINT-MASS VEHICLES CONNECTED BY A MASSLESS TETHER AND IN A CIRCULAR ORBIT

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JANUARY 1970

1. Report No.
NASA TN D-5599

2. Government Accession No.

3. Recipient's Catalog No.

4. Title and Subtitle
DYNAMICS OF TWO SLOWLY ROTATING POINT-MASS VEHICLES CONNECTED BY A MASSLESS TETHER AND IN A CIRCULAR ORBIT

5. Report Date
January 1970

6. Performing Organization Code

7. Author(s)
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8. Performing Organization Report No.
L-6584

9. Performing Organization Name and Address
NASA Langley Research Center
Hampton, Va. 23365

10. Work Unit No.
126-63-13-01-23

11. Contract or Grant No.

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration
Washington, D.C. 20546

13. Type of Report and Period Covered
Technical Note

14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

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17. Key Words Suggested by Author(s)
Tethered orbiting masses
Gravity-gradient stability

18. Distribution Statement
Unclassified – Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	27	$3.00

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151

DYNAMICS OF TWO SLOWLY ROTATING POINT-MASS VEHICLES CONNECTED BY A MASSLESS TETHER AND IN A CIRCULAR ORBIT

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SUMMARY

A study has been made of the dynamics of two point masses in circular orbit connected by a massless tether and rotating relative to the local vertical with angular rates of the same order of magnitude as the orbital angular velocity. Only motion in the orbital plane is considered. Equations of motion are developed which specify the behavior of the system during periods when the tether is in tension and slack. Results that allow prediction of the occurrence of a slack tether are obtained. It is also shown that the addition of damping, coupled with oscillations between a slack and taut tether, causes the system to seek closer alinement with the local vertical.

INTRODUCTION

Use has been made of a tether in Gemini flights for extravehicular maneuvers and as a means of providing artificial gravity. The tethered configuration (fig. 1) has also been proposed as a means of station keeping. (See ref. 1.) The characteristics of the motion of tethered bodies have been the subject of several theoretical studies in recent years. (See refs. 2 and 3.) These studies considered systems that were spinning with angular rates that were much larger than the orbital angular velocity.

If the system is spinning relative to the local vertical with an angular rate that is of the same order of magnitude as the orbital angular velocity, the gravity gradient between the two masses has a pronounced effect upon the dynamics of the system. This study is an analysis of the behavior of such a configuration.

Several simplifying assumptions are made in carrying out the study. The vehicles are taken to be point masses constrained to the orbital plane of the center of mass which is assumed to be in a circular orbit about the earth. Additional assumptions are that the tether is massless and can be characterized by a force constant, and any damping present in the system is viscous.

Equations of motions are developed which specify the behavior of the system during periods when the tether is in tension and slack. Results which allow prediction of the occurrence of slackness in the tether are presented. The study also shows that if damping is present and oscillations between a slack and taut tether occur, the motion of the masses relative to their center of mass is affected in such a way that the system becomes more nearly alined with the local vertical and no further slackness occurs. This result is desirable if gravity-gradient stabilization is an objective.

SYMBOLS

D	viscous damping coefficient
F	elliptic integral of first kind
$\hat{h}$	unit vector normal to orbital plane in direction of orbital angular momentum
I	moment of inertia of taut system about its center of mass, $m_1 l_1^2 + m_2 l_2^2$
$\hat{i},\hat{j},\hat{k}$	right-handed set of unit vectors in direction of local horizontal, local vertical, and normal to orbital plane of center of mass, respectively (see fig. 4)
K	constant depending upon initial conditions (defined in eq. (8))
k	spring constant of tether
L	Lagrangian
l	unstretched length of tether
l_1,l_2	unstretched lengths of tether from center of mass to m_1 and m_2
m_1,m_2	masses at end of tether
$n = r - l$	
P	period of oscillation of a simple spring assuming viscous damping
$\overline{R}$	vector from center of earth to center of mass

$\bar{R}_1, \bar{R}_2$ vectors from center of earth to m_1 and m_2

$\bar{r}_1, \bar{r}_2$ vectors from center of mass to m_1 and m_2

$\bar{r} = \bar{r}_1 - \bar{r}_2$

T tension in tether

t time

$\hat{U}_r, \hat{U}_\varphi$ unit vectors in radial and tangential directions, respectively

$\bar{V}$ inertial velocity of center of mass

$\bar{V}_1, \bar{V}_2$ inertial velocities of m_1 and m_2

$\bar{v}_1, \bar{v}_2$ velocities of m_1 and m_2 relative to center of mass

$\bar{v} = \bar{v}_1 - \bar{v}_2$

x,y,z components in $\hat{i}, \hat{j}, \hat{k}$ directions

ϵ fraction of radial velocity remaining after a slack-taut-slack transition

θ angle between reference line in orbital plane and local vertical

μ product of universal gravitational constant and mass of earth

φ angle between local vertical and $\bar{r}$

ψ angle defined as $\sin^{-1}\left(\sqrt{\frac{3}{K + \frac{3}{2}}} \sin \varphi\right)$

Subscripts:

B bound upon angular oscillation

E at an extremum

I	with respect to inertial space
s	value of quantity when tether becomes slack
T	value of a quantity when tether reaches full extension after having been slack
o	at initial time
1,2	refers to masses m_1 and m_2, respectively
r,φ	in radial and tangential directions, respectively

Dots over symbols denote derivatives with respect to time. A bar over a symbol denotes a vector whereas a caret over a symbol denotes a unit vector. When a vector quantity is written without a bar, the magnitude of the quantity is denoted.

THEORY AND ANALYSIS

Sets of equations are developed that specify the motion of the tethered system during periods when the tether is slack and taut, and a method for predicting whether an initially taut tether will become slack is derived.

Dynamics of the System During Taut Tether Periods

Consider the system shown in figure 1. Two point masses are connected by a massless tether and their center of mass is moving in a circular orbit with constant orbital angular velocity of magnitude $\dot{\theta}$. The tethered system is rotating (in the orbital plane) relative to the local vertical with angular rate $\dot{\varphi}$.

The equations of motion of the system are obtained by using the Lagrangian approach with the separation of the two masses r and the angle from the local vertical to the line connecting the masses φ serving as generalized coordinates. The Lagrangian is defined as

$$L = \text{Kinetic energy} - \text{Potential energy}$$

and the equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \qquad (1)$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = - D\dot{r} \tag{2}$$

where D is a viscous damping coefficient that is assumed to characterize the damping present in the system.

The Lagrangian may be written as

$$L = \frac{1}{2} m_1\left(\overline{V}_1\right)_I \cdot \left(\overline{V}_1\right)_I + \frac{1}{2} m_2\left(\overline{V}_2\right)_I \cdot \left(\overline{V}_2\right)_I + \frac{\mu m_1}{R_1} + \frac{\mu m_2}{R_2} - \frac{1}{2} k(r - l)^2 \qquad (r \geqq l) \tag{3}$$

In this expression $\left(\overline{V}_1\right)_I$ and $\left(\overline{V}_2\right)_I$ are the velocities of m_1 and m_2 relative to inertial space, R_1 and R_2 are the distances to m_1 and m_2 from the center of the earth, l is the unstretched length of the tether, μ is the universal gravitational constant times the mass of the earth, and k is the tether force constant. The quantities R_1, R_2, $\overline{V}_1$, and $\overline{V}_2$ may be written in terms of R, r, V, v, l, and φ as

$$R_1 = \sqrt{R^2 + r_1{}^2 + 2r_1 R \cos\varphi} = \sqrt{R^2 + \left(\frac{l_1}{l}\right)^2 r^2 + 2\left(\frac{l_1}{l}\right) rR \cos\varphi}$$

$$R_2 = \sqrt{R^2 + r_2{}^2 - 2r_2 R \cos\varphi} = \sqrt{R^2 + \left(\frac{l_2}{l}\right)^2 r^2 - 2\left(\frac{l_2}{l}\right) rR \cos\varphi}$$

$$\overline{V}_1 = \dot{\theta}\hat{h} \times \left(\overline{R} + \overline{r}_1\right) + \overline{v}_1 = \left(\dot{\theta}R \sin\varphi + \frac{l_1}{l}\dot{r}\right)\hat{U}_r + \left[\dot{\theta}R \cos\varphi + \frac{l_1}{l} r\left(\dot{\theta} + \dot{\varphi}\right)\right]\hat{U}_\varphi$$

$$\overline{V}_2 = \dot{\theta}\hat{h} \times \left(\overline{R} + \overline{r}_2\right) + \overline{v}_2 = \left(\dot{\theta}R \sin\varphi - \frac{l_2}{l}\dot{r}\right)\hat{U}_r + \left[\dot{\theta}R \cos\varphi - \frac{l_2}{l} r\left(\dot{\theta} + \dot{\varphi}\right)\right]\hat{U}_\varphi$$

In these equations, $\dot{\theta}\hat{h}$ is the constant orbital angular velocity of the center of mass, R is the radius from the center of the earth to the center of mass, $\hat{U}_r$ is a unit vector from m_2 toward m_1, and $\hat{U}_\varphi$ is a unit vector defined as $\hat{U}_\varphi = \hat{h} \times \hat{U}_r$. Substitution of equation (3) into equations (1) and (2) yields the equations of motion of the tethered system:

$$r^2\ddot{\varphi} + 2r\dot{r}\left(\dot{\theta} + \dot{\varphi}\right) + \frac{3}{2} r^2\dot{\theta}^2 \sin 2\varphi = 0 \tag{4}$$

$$\ddot{r} + D\frac{l^2}{I}\dot{r} + r\dot{\theta}^2 - 3\dot{\theta}^2 r \cos^2\varphi - r\left(\dot{\theta} + \dot{\varphi}\right)^2 + \frac{kl^2}{I}(r - l) = 0 \qquad (r \geqq l) \tag{5}$$

where $I = m_1 l_1^2 + m_2 l_2^2$. The assumptions made in deriving these equations were the following:

(1) The center of mass remains in a stationary circular orbit

(2) The massless tether can be represented by a force constant, and any damping present is viscous

(3) In the expansion of $\left(R_1/R\right)^{-3}$ and $\left(R_2/R\right)^{-3}$, terms involving (r/R) raised to powers higher than 1 can be neglected.

The tension in the tether produced by the gravity gradient and the slow angular rates under study corresponds to extension of the tether that is insignificant relative to the tether length, and the tether can be regarded as of constant length. Under this condition, equations (4) and (5) simplify further to

$$\ddot{\varphi} = -\frac{3}{2}\dot{\theta}^2 \sin 2\varphi \tag{6a}$$

$$T = k(r - l) = \frac{I}{l}\left[\left(\dot{\theta} + \dot{\varphi}\right)^2 - \dot{\theta}^2\left(1 - 3\cos^2\varphi\right)\right] \qquad (T \geqq 0) \tag{6b}$$

This simplification is not valid, in general, immediately following the period when the tether again becomes fully extended after having been slack; in such cases the relative velocity between the two masses can produce tension that is considerably higher than that produced by the gravity gradient and the slow rotational rate. Analysis of the stretching motion during such a period will be presented in a later section of the paper.

Equation (6a) specifies the angular acceleration of the tethered system about its center of mass produced by torque resulting from the gravity gradient between the two masses. Solution of this equation specifies the angular orientation relative to the local vertical at arbitrary times. The expression

$$\ddot{\varphi} = -\frac{3}{2}\dot{\theta}^2 \sin 2\varphi$$

may be integrated once to yield

$$\dot{\varphi}^2 = \dot{\theta}^2\left(K + \frac{3}{2}\cos 2\varphi\right) \tag{7}$$

where K is a constant depending upon the initial angular velocity and orientation and clearly obeys the inequality

$$K = \left(\frac{\dot{\varphi}_o}{\dot{\theta}_o}\right)^2 - \frac{3}{2}\cos 2\varphi_o \geqq -\frac{3}{2} \tag{8}$$

By disregarding for the moment the possibility that the tether may lose tension and examining equations (6a) and (7), several statements can be made regarding the rigid-body motion. These statements are summarized as a function of the parameter K in table I.

TABLE I.- TYPES OF MOTION WHEN TENSION MAINTAINED

Value of K	Type of motion
$-\frac{3}{2}$	System remains alined with local vertical
$-\frac{3}{2} < K < \frac{3}{2}$	System oscillates relative to local vertical with amplitudes $-\sin^{-1}\left(\sqrt{\frac{K + \frac{3}{2}}{3}}\right) \leqq \varphi \leqq \sin^{-1}\left(\sqrt{\frac{K + \frac{3}{2}}{3}}\right)$
$\frac{3}{2}$	System approaches local horizontal and requires infinite time to get there
$> \frac{3}{2}$	System makes complete revolutions relative to local vertical

It should be emphasized that table I is applicable for the tethered system only while tension is maintained and must be modified somewhat to account for the occurrence of a slack tether.

Equation (7) may be integrated in closed form in terms of elliptic integrals of the first kind. The exact form of the result will depend upon the value of K.

The closed-form expressions are derived for the following cases:

Case I.- $K = -\frac{3}{2}$ $\quad (\varphi = 0,\pi;\ \dot{\varphi} = \ddot{\varphi} = 0)$

Case II.- $-\frac{3}{2} < K < \frac{3}{2}$

Case III.- $K = \frac{3}{2}$

Case IV.- $K > \frac{3}{2}$

For case I for which $K = -\frac{3}{2}$, inspection of equation (6a) reveals that the system tends to maintain this orientation because any small angular rate produced by an external source will be opposed by the torque created by the gravity gradient with resulting oscillations about the local vertical as seen in case II.

For case II where $-\frac{3}{2} < K < \frac{3}{2}$, by substituting the relation

$$\sin\psi = \sqrt{\frac{3}{K + \frac{3}{2}}}\sin\varphi$$

into equation (7) one obtains for the time elapsed during the rotation of the system from

$$\varphi_o = \sin^{-1}\left(\sqrt{\frac{K + \frac{3}{2}}{3}}\sin\psi_o\right) \quad \text{to} \quad \varphi = \sin^{-1}\left(\sqrt{\frac{K + \frac{3}{2}}{3}}\sin\psi\right)$$

$$\Delta t = \frac{1}{\sqrt{3}\dot{\theta}}\int_{\psi_o}^{\psi}\frac{d\psi}{\sqrt{1 - \frac{K + \frac{3}{2}}{3}\sin^2\psi}}$$

$$= \frac{1}{\sqrt{3}\dot{\theta}}\left[F\left(\sqrt{\frac{K + \frac{3}{2}}{3}}, \psi\right) - F\left(\sqrt{\frac{K + \frac{3}{2}}{3}}, \psi_o\right)\right]$$

where F is an elliptic integral of the first kind. This equation reveals that oscillations relative to the local vertical occur provided the tether is taut. The amplitude of the oscillations is

$$-\sin^{-1}\left(\sqrt{\frac{K + \frac{3}{2}}{3}}\right) \leqq \varphi \leqq \sin^{-1}\left(\sqrt{\frac{K + \frac{3}{2}}{3}}\right)$$

For case III where $K = \frac{3}{2}$, the integral becomes

$$\Delta t = \frac{1}{\sqrt{3}\dot{\theta}}\int_{\varphi_o}^{\varphi}\frac{d\varphi}{\cos\varphi} = \frac{1}{\sqrt{3}\dot{\theta}}\left[F(1,\varphi) - F(1,\varphi_o)\right]$$

$$= \frac{1}{\sqrt{3}\dot{\theta}}\log_e\left(\frac{\tan\varphi + \sec\varphi}{\tan\varphi_o + \sec\varphi_o}\right)$$

In this case an infinite amount of time would be required for the system to rotate from a particular orientation to $\varphi = \pi/2$ or $-\pi/2$. Also it can be seen from equation (6a) that if the system were at rest at $\varphi = \pm\pi/2$, it would tend to diverge from this point since any angular rate relative to the local vertical created by an external torque would lead to complete revolutions of the bodies about the local vertical as is shown in the following case.

For case IV, $K > \frac{3}{2}$. As can be seen from equation (7), rigid body motion is one of complete revolutions relative to the local vertical. The integral of equation (7) is

$$\Delta t = \frac{1}{\sqrt{K + \frac{3}{2}}\,\dot{\theta}} \int_{\varphi_o}^{\varphi} \frac{d\varphi}{\sqrt{1 - \frac{3 \sin^2\varphi}{K + \frac{3}{2}}}}$$

$$= \frac{1}{\sqrt{K + \frac{3}{2}}\,\dot{\theta}} \left[F\left(\sqrt{\frac{3}{K + \frac{3}{2}}}, \varphi\right) - F\left(\sqrt{\frac{3}{K + \frac{3}{2}}}, \varphi_o\right)\right]$$

Since each of these integrals expresses the orientation of the tethered system as a function of time only if the tether is taut, it is necessary to examine the expression for tension to determine when the tether may become slack. Substitution of equation (7) into equation (6b) yields the following expression for the tension in the tether:

$$T = \frac{I}{l}\,\dot{\theta}^2\left(K + \frac{3}{2} - 3 \sin^2\varphi \pm 2\sqrt{K + \frac{3}{2} - 3 \sin^2\varphi} + 3 \cos^2\varphi\right) \tag{9}$$

By finding the angular orientation at which T is minimum as a function of K and evaluating T at that point, one can determine whether slackness will occur. The extrema can be found by solving for the roots of $\frac{dT}{d\varphi}$. Forming this derivative gives

$$\frac{dT}{d\varphi} = -6\,\frac{I}{l}\,\dot{\theta}^2 \sin\varphi \cos\varphi\left[2 \pm \left(K + \frac{3}{2} - 3 \sin^2\varphi\right)^{-1/2}\right] \tag{10}$$

Therefore $\frac{dT}{d\varphi} = 0$ if $\varphi_E = 0,\ \pi/2,\ \pi,\ 3\pi/2$. In addition $\frac{dT}{d\varphi} = 0$ if $\dot{\varphi} < 0$ provided

$$2 - \left(K + \frac{3}{2} - 3 \sin^2\varphi_E\right)^{-1/2} = 0$$

The latter condition can only be satisfied for a certain range of initial conditions, that is, for a certain range of K. This statement is true because of the restriction that $\left|\sin \varphi_E\right| = \sqrt{\frac{K}{3} + \frac{5}{12}} \leqq 1$ and real. These restrictions can be written in terms of K as

$$-\frac{5}{4} \leqq K \leqq \frac{7}{4}$$

where the positive limit requires $|\sin \varphi_E| \leqq 1$ and the negative limit requires that $\sin \varphi_E$ be real.

It is necessary to determine whether these extrema are maxima, minima, or neither. If $\frac{d^2T}{d\varphi^2}$ evaluated at $\varphi = \varphi_E$ is negative, T is maximum; if $\frac{d^2T}{d\varphi^2}$ is positive at $\varphi = \varphi_E$, T is minimum. Upon applying this test, the results given in table II are obtained.

TABLE II.- ORIENTATIONS AT WHICH TENSION IS EXTREMAL

φ_E	Type of extremum
$\sin^{-1}\left[\pm\sqrt{\frac{K}{3} + \frac{5}{12}}\right]; \dot{\varphi}_E < 0; -\frac{5}{4} < K < \frac{7}{4}$	T is minimized
$0, \pi; \dot{\varphi}_E > 0$	T is maximized
$0, \pi; \dot{\varphi}_E < 0; -\frac{5}{4} < K$	T is maximized
$0, \pi; \dot{\varphi}_E < 0; -\frac{3}{2} < K < -\frac{5}{4}$	T is minimized
$\frac{\pi}{2}, \frac{3\pi}{2}; \dot{\varphi}_E > 0; K > \frac{3}{2}$	T is minimized
$\frac{\pi}{2}, \frac{3\pi}{2}; \dot{\varphi}_E < 0; K > \frac{7}{4}$	T is minimized
$\frac{\pi}{2}, \frac{3\pi}{2}; \dot{\varphi}_E < 0; \frac{3}{2} < K < \frac{7}{4}$	T is maximized

Figures 2(a) and 2(b) present the normalized tension as a function of the angle from the local vertical for several values of the parameter K. It is seen that the tether remains in tension if $K < 1$, if $K > \frac{3}{2}$ for $\dot{\varphi}$ positive, or if $K > \frac{11}{2}$ for $\dot{\varphi}$ negative. Note the consistency between figures 2(a) and 2(b) and table II as to the type of extrema. When tension is negative, the curves are dotted to emphasize that such portions of the curves do not represent the dynamics of the tethered system.

The angle at which the tension becomes zero as a function of K is shown in figures 3(a) and 3(b) for $\dot{\varphi}_o < 0$ and $\dot{\varphi}_o > 0$, respectively. The results from this figure can be used to modify table I as shown in table III.

TABLE III.- MODIFICATION OF TABLE I TO DENOTE CONDITIONS FOR OCCURRENCE OF SLACKNESS

Value of K	Type of motion (limited to orbital plane)
$K = -\frac{3}{2}$	System remains alined with local vertical and in tension
$-\frac{3}{2} < K \leqq 1$	System oscillates relative to local vertical with tension being maintained. $-\sin^{-1}\left(\sqrt{\frac{K + \frac{3}{2}}{3}}\right) \leqq \varphi \leqq \sin^{-1}\left(\sqrt{\frac{K + \frac{3}{2}}{3}}\right)$
$1 < K \leqq \frac{3}{2}$	Slackness occurs in tether
$K > \frac{3}{2}, \dot{\varphi} > 0$	Complete revolutions relative to local vertical occur with tension being maintained
$\frac{3}{2} < K < \frac{11}{2}, \dot{\varphi} < 0$	Slackness occurs in tether
$K > \frac{11}{2}, \dot{\varphi} < 0$	Complete revolutions occur relative to local vertical with tension being maintained

Slack Tether Period

If the tether is initially slack or becomes slack during rotation, the two masses will (again neglecting the effect of the tether) move independently of each other as long as their separation is less than the length of the tether. An outline of the development of the equations of motion as given in appendixes A and B of reference 4 is presented.

Consider the system as shown in figure 4. The Lagrangian associated with m_1 is given by

$$L_1 = \frac{1}{2} m_1\left(\overline{V}_1\right)_I \cdot \left(\overline{V}_1\right)_I + \frac{\mu m_1}{R_1} \tag{11}$$

The x-component of the equations of motion is

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{x}_1}\right) - \frac{\partial L_1}{\partial x_1} = 0 \tag{12}$$

with identical equations for y_1 and z_1. Performing the indicated operations and making use of the property that the orbital angular momentum is constant yields the equations of motion relative to the center of mass:

$$\left.\begin{aligned}
&\ddot{x}_1 - x_1\dot{\theta}^2 - 2\dot{y}_1\dot{\theta} - y_1\ddot{\theta} + \frac{\mu x_1}{R_1^3} = 0 \\
&\ddot{y}_1 - y_1\dot{\theta}^2 + 2\dot{x}_1\dot{\theta} + x_1\ddot{\theta} - \frac{\mu}{R^2} + \frac{\mu(y_1 + R)}{R_1^3} = 0 \\
&\ddot{z}_1 + \frac{\mu z_1}{R_1^3} = 0
\end{aligned}\right\} \qquad (13)$$

Specializing to the case for which the two masses are in the plane of the circular orbit of the center of mass, expanding $R_1 = \left[x_1^2 + (R + y_1)^2\right]^{1/2}$ in a binomial series, and retaining only first-order terms in x_1 and y_1 results in the following set of differential equations:

$$\ddot{x}_1 - 2\dot{\theta}\dot{y}_1 = 0$$

$$\ddot{y}_1 + 2\dot{\theta}\dot{x}_1 - 3\dot{\theta}^2 y_1 = 0$$

These approximate equations may be immediately integrated to yield the position and velocity of m_1 relative to the center of mass:

$$\left.\begin{aligned}
x_1 = {} & 2\left[-3(y_1)_s + \frac{2(\dot{x}_1)_s}{\dot{\theta}}\right]\sin\left[\dot{\theta}(t - t_s)\right] - \frac{2(\dot{y}_1)_s}{\dot{\theta}}\cos\left[\dot{\theta}(t - t_s)\right] \\
& - 3\left[\frac{(\dot{x}_1)_s}{\dot{\theta}} - 2(y_1)_s\right]\dot{\theta}(t - t_s) + (x_1)_s + \frac{2(\dot{y}_1)_s}{\dot{\theta}} \\
y_1 = {} & \left[-3(y_1)_s + \frac{2(\dot{x}_1)_s}{\dot{\theta}}\right]\cos\left[\dot{\theta}(t - t_s)\right] + \frac{(\dot{y}_1)_s}{\dot{\theta}}\sin\left[\dot{\theta}(t - t_s)\right] - 2\left[\frac{(\dot{x}_1)_s}{\dot{\theta}} - 2(y_1)_s\right]
\end{aligned}\right\} \qquad (14)$$

The equations specifying the position of m_2 are identical in form to these equations. Thus, equations depicting the motion of the two masses during the slack tether period have been presented.

Behavior During Slack-Taut-Slack Transitions

After a period of time the slack tether will again become fully extended, and relative motion between the two bodies in the radial direction will not, in general, be zero. The resulting tension in the tether can be considerably higher than that produced by the gravity gradient between the two masses, and the assumption that the length of the tether remains constant should no longer be made. One must consider, therefore, the more complete radial equation of motion (eq. (5)):

$$\ddot{r} + D\frac{l^2}{I}\dot{r} + r\dot{\theta}^2 - 3\dot{\theta}^2 r\cos^2\varphi - r(\dot{\theta} + \dot{\varphi})^2 + k\frac{l^2}{I}(r - l) = 0 \qquad (r \geqq l)$$

By making the substitution $n = r - l$, this expression becomes

$$\ddot{n} + D\frac{l^2}{I}\dot{n} + \left\{k\frac{l^2}{I} - \left[(\dot{\theta} + \dot{\varphi})^2 - \dot{\theta}^2(1 - 3\cos^2\varphi)\right]\right\}n = \left[(\dot{\theta} + \dot{\varphi})^2 - \dot{\theta}^2(1 - 3\cos^2\varphi)\right]l$$

For the low angular rates considered in this study $\frac{kl^2}{I} >> (\dot{\theta} + \dot{\varphi})^2 - \dot{\theta}^2(1 - 3\cos^2\varphi)$. Thus,

$$\ddot{n} + \frac{Dl^2}{I}\dot{n} + \frac{kl^2}{I}n = \left[(\dot{\theta} + \dot{\varphi})^2 - \dot{\theta}^2(1 - 3\cos^2\varphi)\right]l = \frac{l^2}{I}T \tag{15}$$

The term to the right of the equality is the effect of the gravity gradient and the angular rotation upon the extensional motion of the tether. Since the effect of this term upon the tether length was neglected in the earlier analysis, it will also be neglected here. Then the equation of motion becomes a linear homogeneous differential equation with constant coefficients:

$$\ddot{n} + D\frac{l^2}{I}\dot{n} + \frac{kl^2}{I}n = 0$$

and has the solution

$$n = \frac{\dot{n}_T e^{-\frac{Dl^2}{2I}(t-t_T)}}{\sqrt{\frac{l^2}{I}\left(k - \frac{D^2 l^2}{4I}\right)}} \sin\left[\sqrt{\frac{l^2}{I}\left(k - \frac{D^2 l^2}{4I}\right)}(t - t_T)\right] \tag{16}$$

where $\dot{n}_T$ is the relative velocity between the two masses at the time the tether becomes fully extended after having been slack t_T.

The amount of damping required to remove a specified fraction of the relative velocity between the two masses during each slack-taut-slack transition may be determined from equation (16). Let the relations

$$\dot{n}\left(\frac{P}{2}\right) = -\epsilon\dot{n}_T \qquad (0 \leqq \epsilon \leqq 1)$$

and

$$n\left(\frac{P}{2}\right) = n_T = 0$$

specify the desired conditions at the end of each damping cycle where

$$P = \frac{2\pi}{\sqrt{\frac{l^2}{I}\left(k - \frac{D^2 l^2}{4I}\right)}}$$

Solving for the ratio of damping required to critical damping yields

$$\frac{D}{2\sqrt{\frac{kI}{l^2}}} = \sqrt{\frac{\left(\log_e \epsilon\right)^2}{\left[\pi^2 + \left(\log_e \epsilon\right)^2\right]}} \tag{17}$$

The percent of critical damping required as a function of ϵ as determined from equation (17) is shown in figure 5.

When initial conditions are such that the tether becomes slack, the motion is made up of periods of slackness alternating with short periods of tautness when the tether again becomes fully extended. A typical time history of the motion during the first slack period is presented in figure 6 for a case in which the center of mass is in synchronous orbit about the earth. It is evident from figure 6 that the relative velocity between the two masses is not zero when the tether again becomes fully extended and this condition results in a short period of tautness followed by another period of slackness.

After a series of alternations, with damping in the system, the relative velocity between the two masses at full extension essentially vanishes and the system again behaves as a rigid body. The number of alternations before this condition is reached can vary from 5 or 10 to more than a hundred depending upon the damping present and the initial conditions. The subsequent rigid-body motion can then persist indefinitely or slackness can recur depending upon the angular rate and orientation of the system when it becomes taut.

A computer program was written that describes the dynamics of the system in order to determine whether the final motion would be bounded rigid-body oscillations relative to the local vertical. Two tethered 930-kilogram masses were taken to be in a circular synchronous orbit, and solutions were obtained that covered the entire range of initial conditions for which slackness would occur. Time histories were computed with K_o, the parameter which defines the initial conditions, chosen at increments of 0.025. Other factors varied were damping (2 to 7 percent of critical damping), tether length (2500 and

10 000 meters), and the extensional rigidity ($333 \leqq kl \leqq 5540$ newtons) of the tether. Cases in which slackness did not occur and tumbling continued were not studied.

The results of this analysis are shown in figures 7(a), 7(b), and 7(c). The ordinate is the final amplitude of oscillation of the system relative to the local vertical; the abscissa K_o defines the initial conditions when coupled with the requirement that $\dot{\varphi}_o < 0$. By reference to figures 2(a) and 2(b), it is determined that the values of K_o in figure 7 correspond to large-amplitude oscillations or complete revolutions relative to the local vertical. The dashed line in the figures is the angular bound below which tension is maintained and above which slackness occurs for bounded oscillations and corresponds to the curve $K = 1$ in figure 2(a). With the exception of the absence of curves near $K_o = 5.5$ the figures show that the final steady-state condition is one involving bounded oscillations and no further slackness. (Final tension is instantaneously zero for cases with K_o near 1.)

Increased damping tends to smooth out the curves and generally leads to a smaller final amplitude of oscillation relative to the local vertical. Comparison of figures 7(a) and 7(b) indicates that the final amplitude of oscillation is insensitive to changes in the tether length over the range of variations studied. Comparison of figures 7(b) and 7(c) shows that the same statement is true for variation in the extensional rigidity of the tether.

No points are plotted in the vicinity of $K_o = 5.5$ because the system remained in such a condition that slackness would recur indefinitely. In all other cases studied, however, the final condition is one in which the system is closer to being alined with the local vertical than initially and no further slackness occurs. A more complete analysis considering the extensional motion produced (when the tether is in tension) by the slow variations in the gravity gradient and angular rate could show enhancement of this tendency toward gravity gradient stabilization.

It is of interest to observe the separation between the point masses during the slack periods to determine whether collisions between finite-sized vehicles might occur. The separation between masses for $\epsilon = 0.90$ and a 10 000-meter tether having a spring constant of 0.534 N/m was observed from the point of initial slackness until the rigid body mode was again reached, no further slackness occurring, for a range of initial values of K. The results, shown in table IV, are the number of times the separation was observed to be less than each indicated distance for a sample rate normalized to one observation every 8 minutes. The separation became less than 10 percent of the tether length for a number of cases. However, in no instance did the separation become as small as 1 percent of the tether length. Table IV was generated under the assumption that the relative velocity between masses was zero at the point of initial slackness.

TABLE IV.- NUMBER OF TIMES SEPARATION WAS LESS THAN THE INDICATED DISTANCES

[l = 10 000 meters]

K_o	Number of times separation was less than r/l of –			
	1.0	0.5	0.1	0.01
1.40	203	0	0	0
1.47	225	0	0	0
1.60	274	0	0	0
1.70	339	0	0	0
1.80	360	14	0	0
1.90	438	21	0	0
2.00	407	27	0	0
2.10	496	61	7	0
2.20	541	88	0	0
2.30	573	97	8	0
2.40	546	43	0	0
2.50	871	122	0	0
2.75	535	78	0	0
3.00	566	82	4	0
3.25	670	116	0	0
3.50	785	124	5	0
3.75	729	89	9	0
4.00	632	127	4	0
4.50	756	107	8	0

CONCLUDING REMARKS

Equations have been developed for a tethered system for the idealized case of a massless tether that allow prediction of the occurrence of slackness; furthermore, it is shown that the addition of damping, coupled with the occurrence of slackness, affects the motion of the masses relative to their center of mass in such a way that the system becomes more nearly alined with the local vertical and no further slackness occurs. In addition, the minimum separation between masses was greater than 1 percent of the tether length for all cases having zero relative velocity between masses at the point of initial slackness. A more complete analysis considering the slight extensional motion produced by the variation in the gradient and angular rate occurring while the system is in tension could show enhancement of the tendency toward gravity-gradient stabilization.

It should be pointed out that the assumption of a massless tether removes the effect of lateral cable oscillations; and it is under the conditions of low tension (slow rotation rates) that were studied that the amplitude of these oscillations can become large. Although minimization of the effect of these lateral oscillations could be accomplished by making the tether of light material, it is felt that the results presented in this report should be considered as qualitative in nature.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., August 13, 1969.

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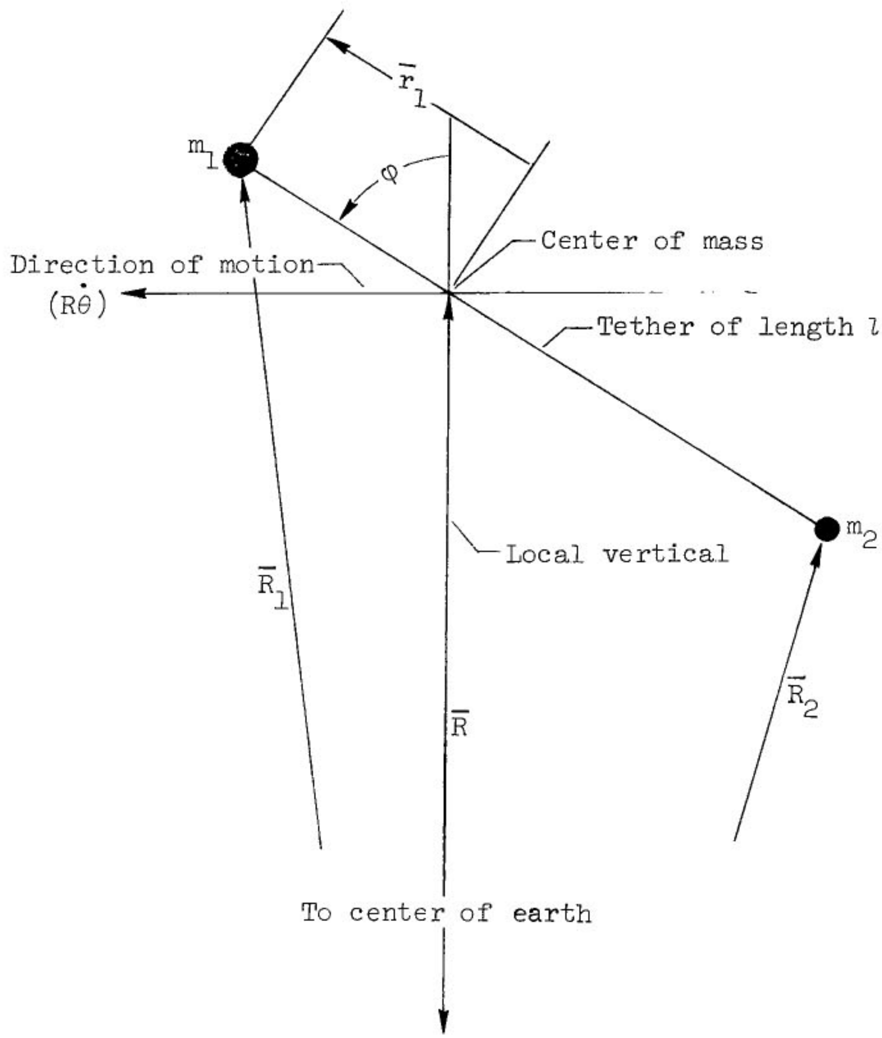
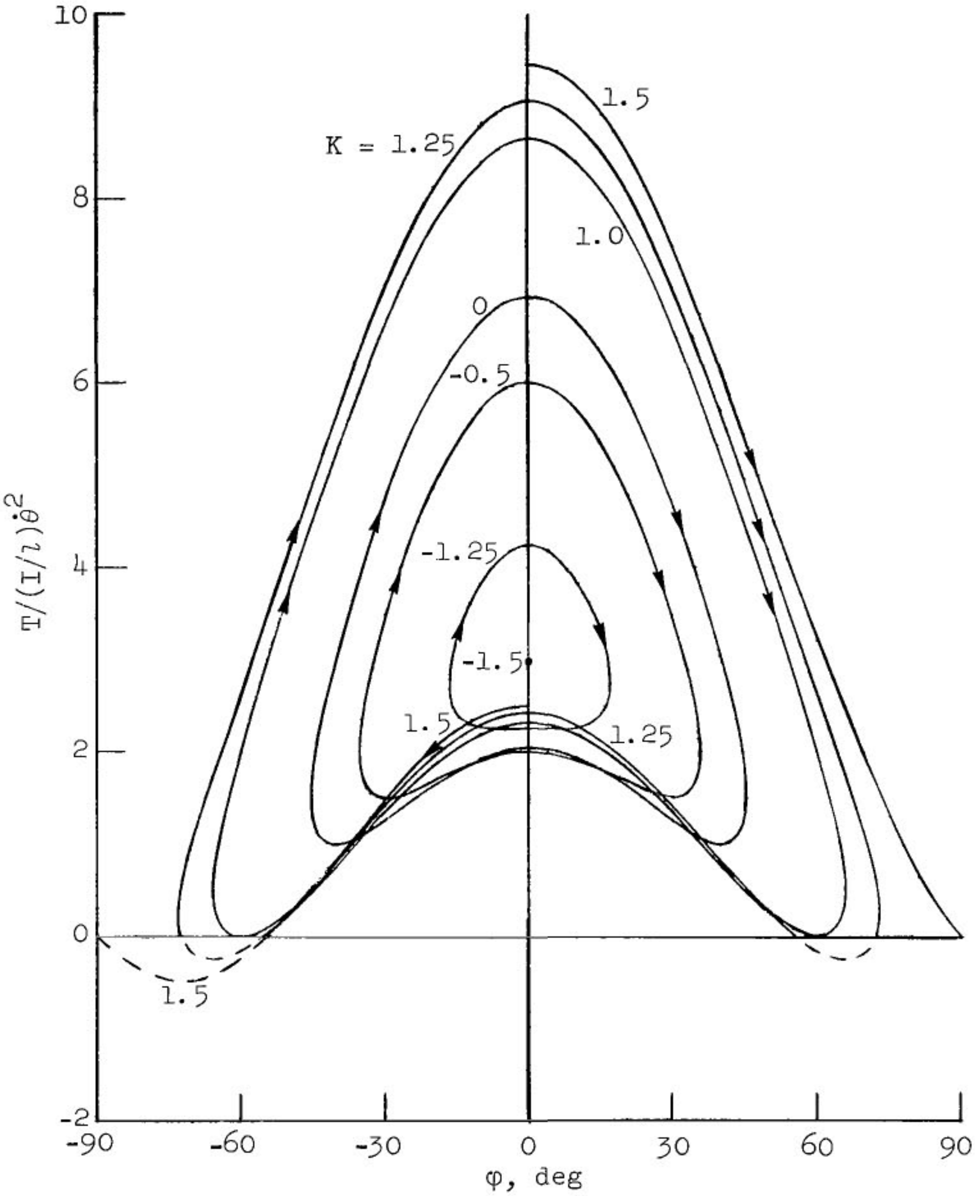
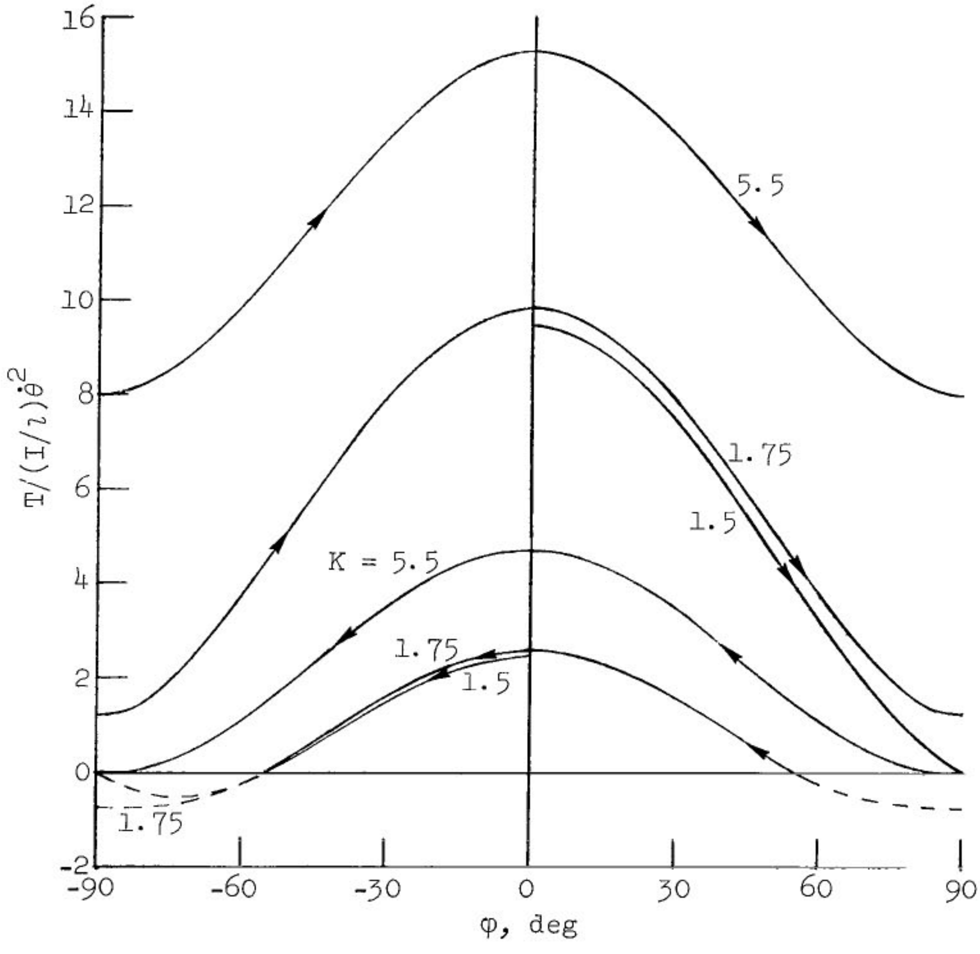


Figure 1.- Sketch of system.



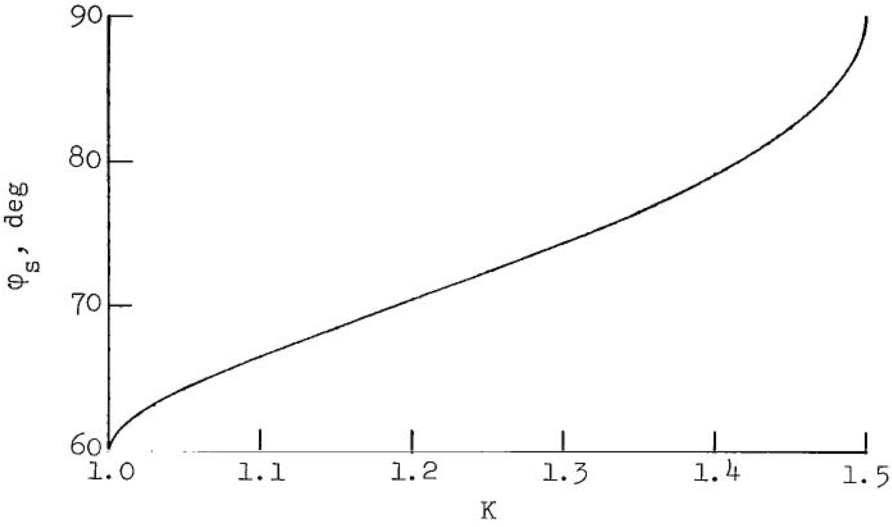
(a) Bounded oscillations.

Figure 2.- Normalized tension as a function of angle from local vertical for specified values of K. The dashed portions of the curves depict the motion of a rigid body under compression and do not, therefore, represent the motion of the tethered system.

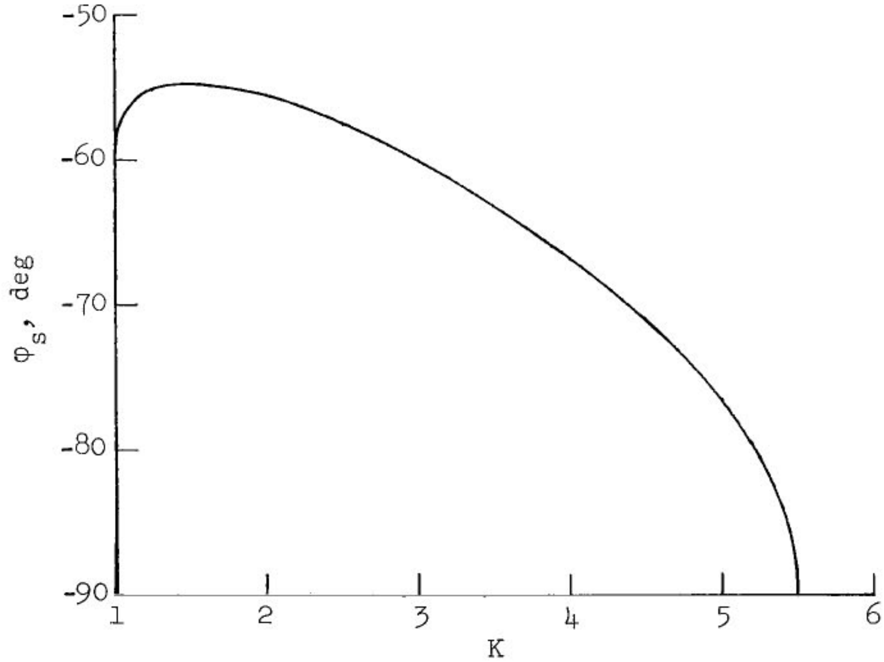


(b) Complete revolutions.

Figure 2.- Concluded.



(a) $\dot{\Phi}_0 > 0$.



(b) $\dot{\Phi}_0 < 0$. $\Phi_s(\dot{\Phi}_0 < 0) < \Phi_0 < \Phi_s(\dot{\Phi}_0 > 0)$.

Figure 3.- Angle from local vertical at which the tension becomes zero as a function of the parameter K.

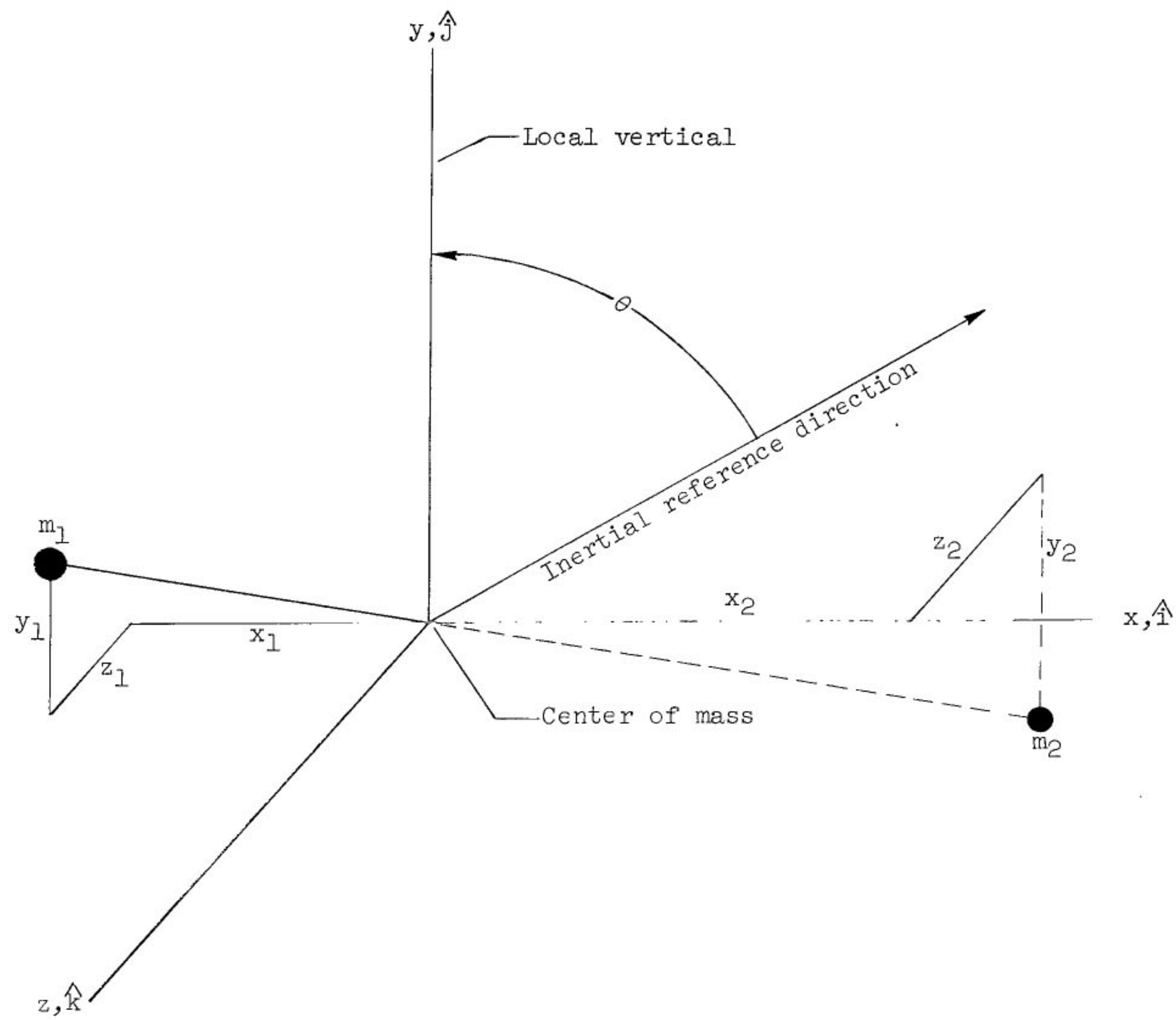


Figure 4.- Location of m_1 and m_2 as seen from their center of mass in a coordinate system rotating with the orbital angular velocity.

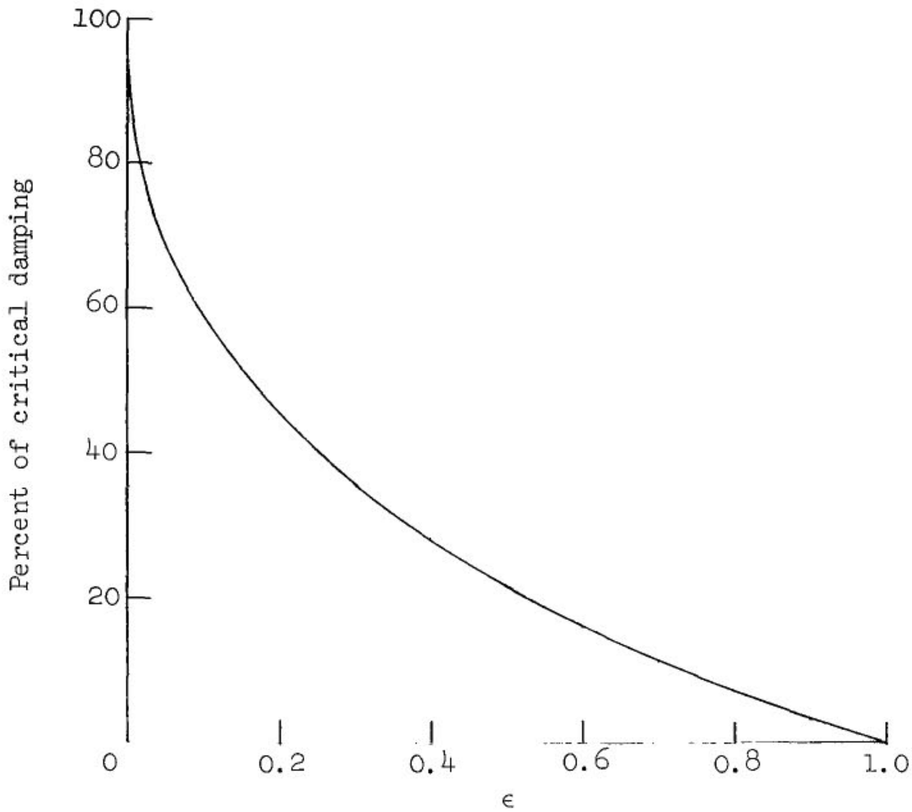


Figure 5.- Percent of the critical damping required to reduce the relative velocity between the two masses to fraction ε of its original value in one slack-taut-slack transition.

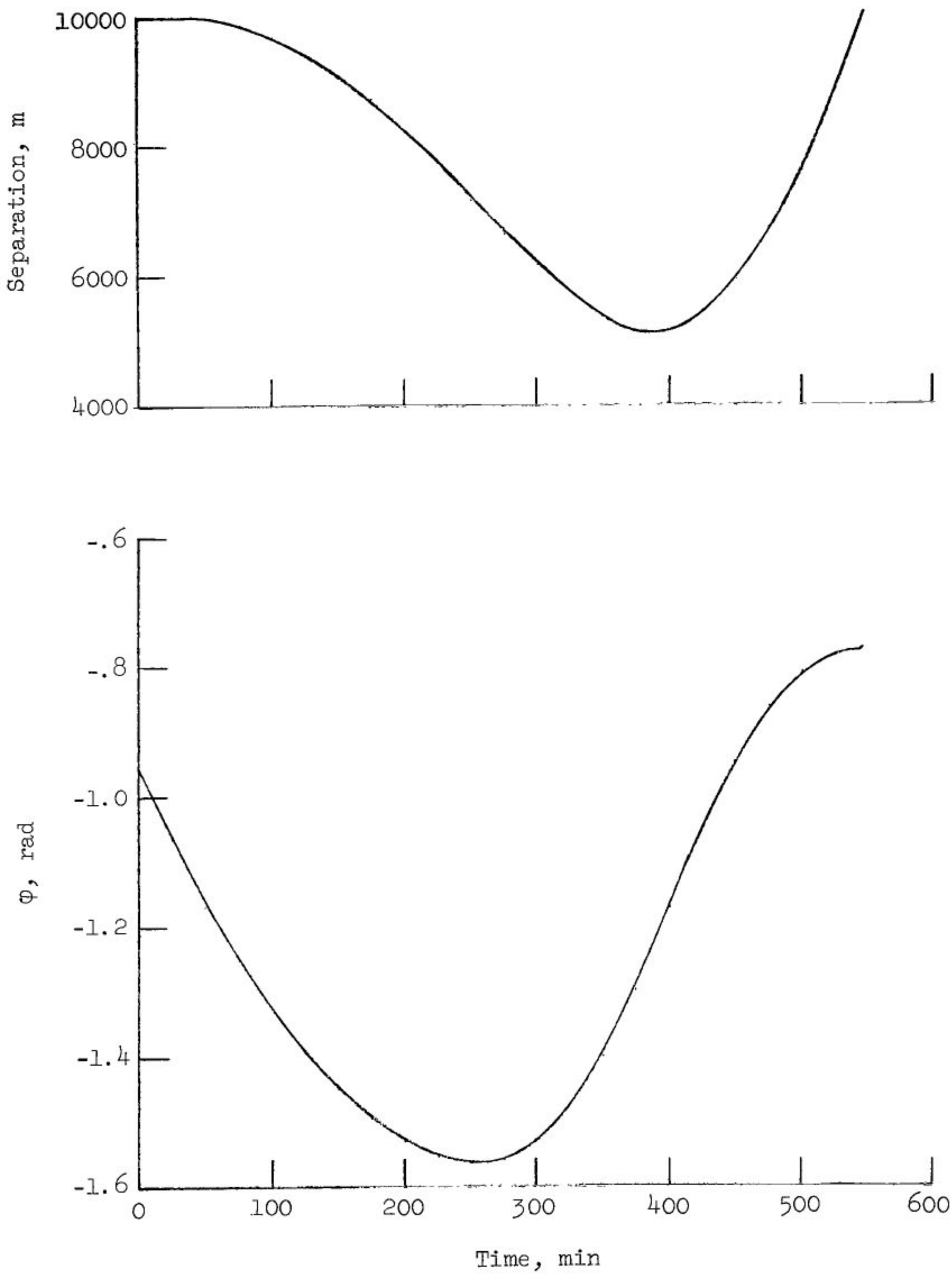
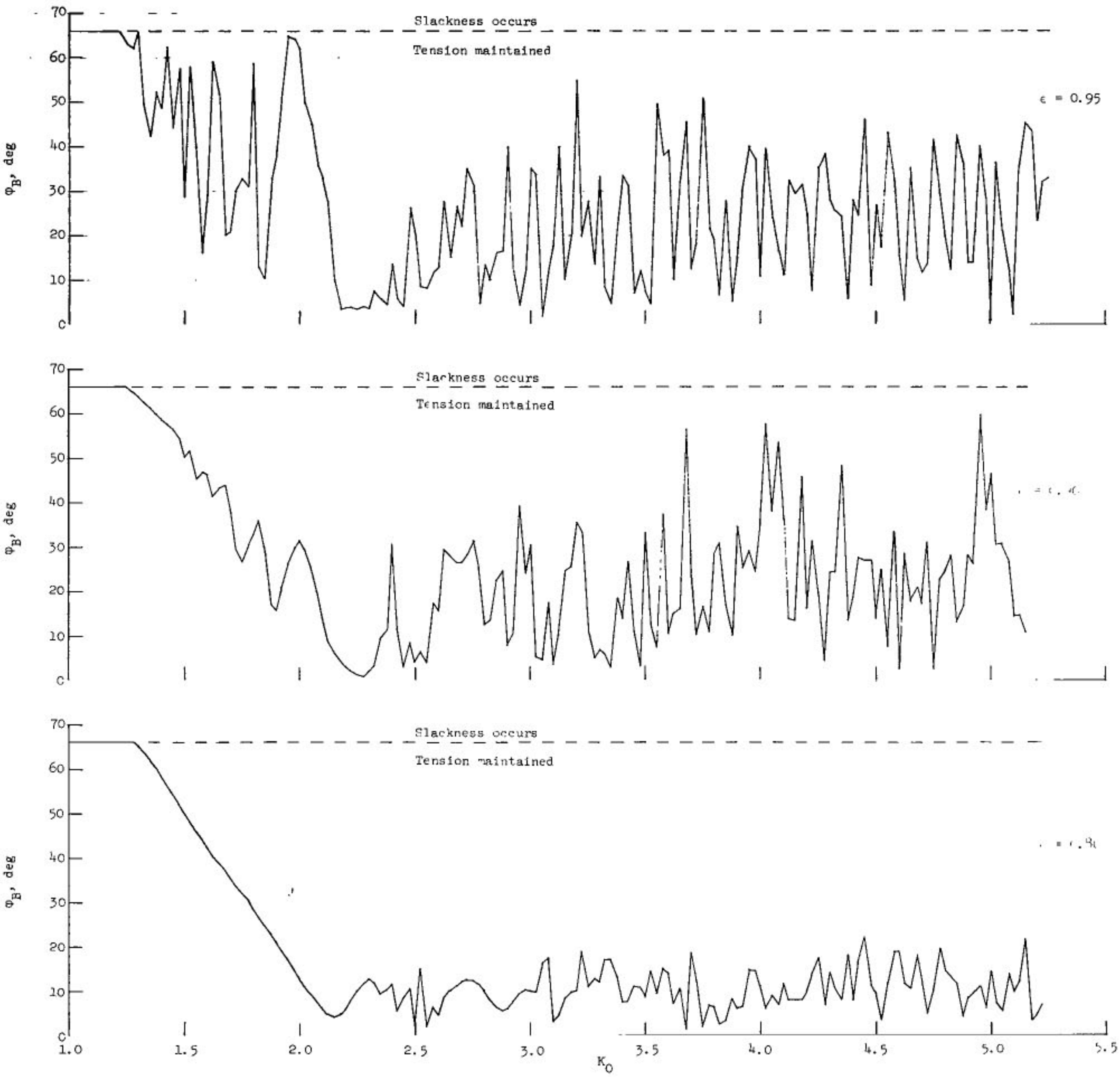
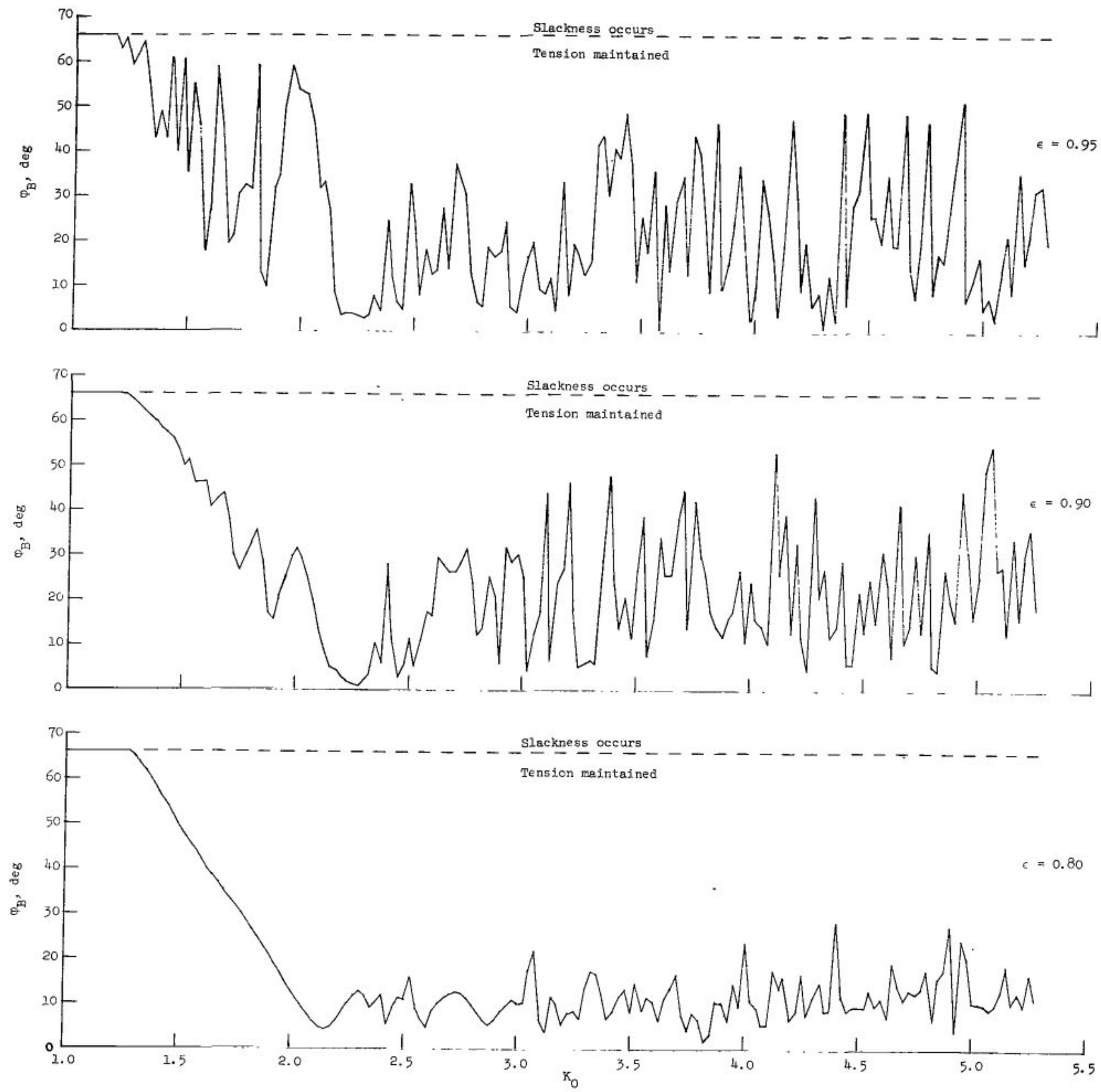


Figure 6.- Time histories during a slack period of the separation between the point masses and the orientation, relative to the local vertical, of a line connecting them. $K_0 = 1.75$; $l = 10\ 000$ m.



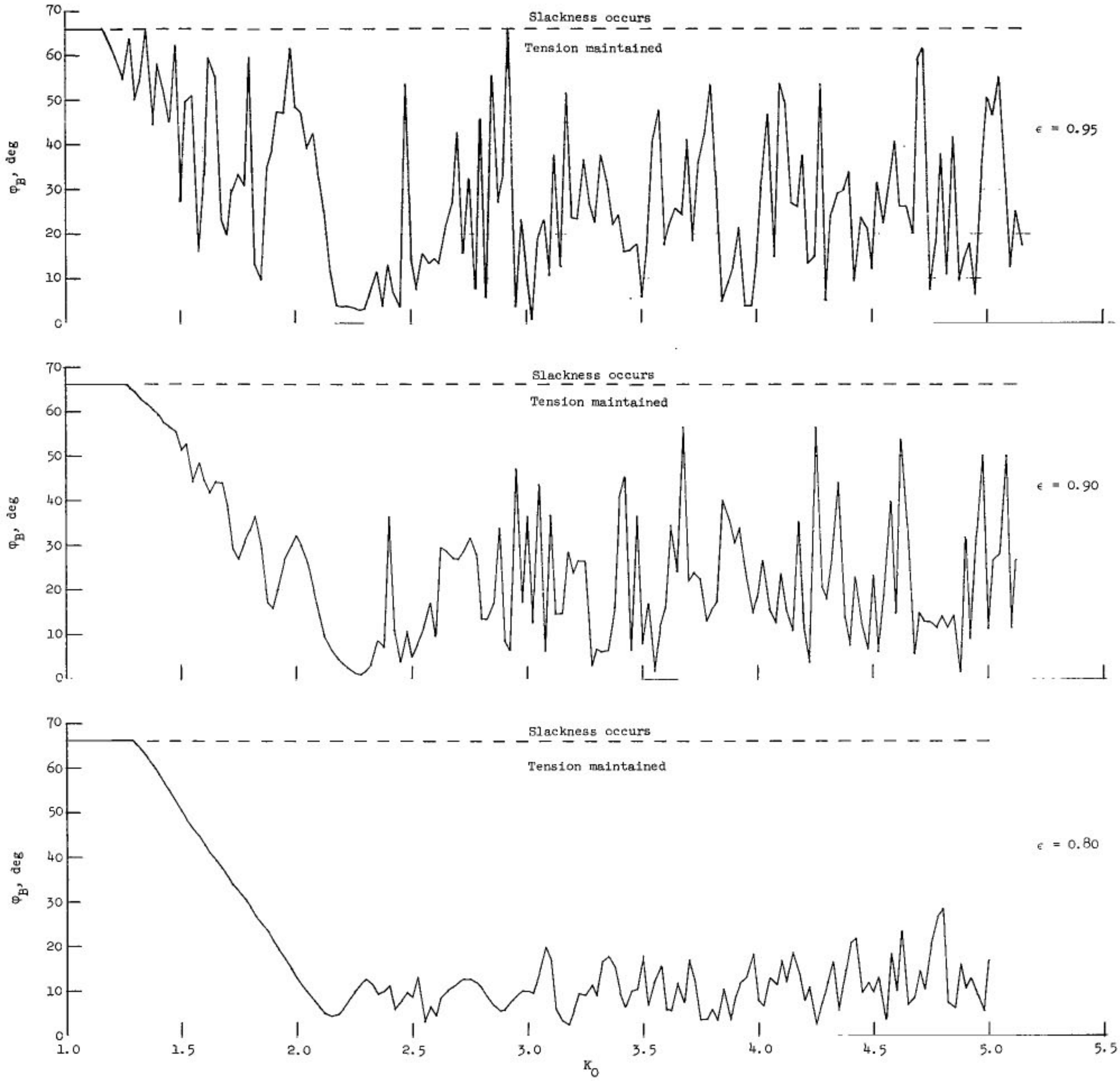
(a) $k = 0.534$ N/m; $l = 10\ 000$ m.

Figure 7.- Angular bound of oscillations of final configuration as a function of K_0 where slackness occurred in the initial damped system.



(b) k = 2.136 N/m; l = 2500 m.

Figure 7.- Continued.



(c) k = 0.1335 N/m; l = 2500 m.

Figure 7.- Concluded.

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