NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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Preface

The work described in this report was performed by the Engineering Mechanics Division of the Jet Propulsion Laboratory.

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Abstract

An analysis of the stability of circular rings under a uniformly distributed radial load is presented. It was found that, in general, the critical mode shapes are a combination of in- and out-of-plane displacements and that they occur at loads considerably below the classical (in-plane) critical load. The role of workless constraints was also studied. The analysis shows that workless constraints lead to an increase in the critical load.

The Stability of Circular Rings Under a Uniformly Distributed Radial Load

I. Introduction

A. Equilibrium Method

The criterion for stability of the deflected shape of an unrestrained circular ring acted on by a uniform distribution of radial forces was probably first established by Levy (Ref. 1) using the equilibrium method. An account of this work has been written by Biezeno and Grammel (Ref. 2). More recently, Boresi (Ref. 3) sought to refine the criteria using the energy method and a deflection pattern not bound by the classical assumptions of vanishing transverse normal and shear strain. In both analyses, the form of the disturbance from the original loaded configuration (the buckling mode) was limited to in-plane displacements without twist.

Most recently, the possibility of out-of-plane buckling modes was studied by Wah (Ref. 4). In effect, Wah's analysis was an extension of Timoshenko's (Ref. 5), which had been limited to in-plane disturbances. Wah follows Timoshenko's equilibrium method analysis and introduces an effective load arising from the circumferential normal strain into the equations of equilibrium proposed by Love (Ref. 6). The resulting equations are uncoupled with regard to in- and out-of-plane displacements and yield the classical criteria for the in-plane buckling modes and a much lower critical load for the out-of-plane modes. Wah's analysis was limited to rings with cross sections whose principal axes were oriented parallel to the radial and axial coordinate axis (Fig. 1).

Prior to the work of Wah, a general theory for bending and buckling of thin-walled open sections was proposed by Cheney (Ref. 7). The method of analysis was essentially the equilibrium method, in that a set of differential equations describing the nonaxisymmetric equilibrium the loaded circular state was derived and solved to yield the critical load. The cross section was not required to be symmetrical as in Wah's analysis, and the effect of warping of transverse cross sections was included by incorporating the warping function for straight bars. The equations describing the nonaxisymmetric equilibrium position were established by requiring that the first variation of the change in total potential energy vanish.

The method followed by Cheney can be shown to be equivalent to the energy method in principle. However, a number of simplifications were introduced by Cheney that might have an appreciable effect if one were to carry

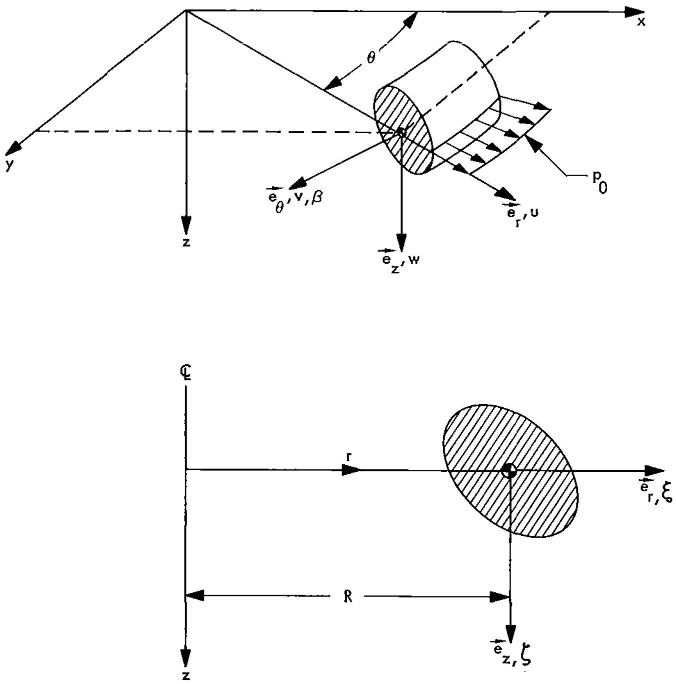

Fig. 1. Coordinate system

out a strict energy method analysis based on Cheney's assumptions for displacements. It is certainly not clear that the strain energy density expression used by Cheney is as adequate as that expressed in appropriate Lagrangian strain components for describing a deviation from the loaded circular state.

B. Energy Method

The object of the present study is the analysis of the stability of a ring acted upon by a constant (in magnitude and in direction—see Appendix D), uniformly distributed radial load by means of the energy method. This method was chosen because it readily allows the nonlinear behavior of the structure (prior to buckling) to be accounted for. Furthermore, it affords a conceptually superior framework in which to analyze the axisymmetric deviation mode. However, the analysis is not to be carried out in the strict sense. It is felt that, for the present, it is sufficient to explore the effect of a more general energy density expression and the role of the axisymmetric mode in a nonlinear analysis while neglecting the effect of warping. Admittedly, while looking for additional terms that might be significant, this one might overlook significant warping effects.

In addition, it is of interest to obtain the stability criteria for circular rings that are not free to deflect arbitrarily, but must obey certain kinematic restraints. For example, a ring used as a stiffener in a thin-walled shell of revolution subjected to external pressure might become unstable before the shell becomes unstable. The displacements of the ring will depend upon the relative stiffness of the shell in and out of the plane of the ring. A reasonable method might be to restrict the deflections, during buckling of the rings, to those perpendicular to the generator of the shell (Fig. 2). Thus, the problem to be studied is the stability of a ring acted upon by a uniform radial load and subjected to both in- and out-of-plane disturbances that may be constrained. However, the class of ring cross sections should not be restricted to those having principal axes parallel to the radial and axial coordinate axis.

The general stability criteria used here are those developed by Langhaar (Ref. 8) and applied by Boresi in analyzing the stability of in-plane displacements. For a circular ring acted upon by a uniformly distributed radial force acting through the line of centroids, there will exist, in general, both a uniform radial deflection (of the line of centroids) and a rotation of the cross section about the line of centroids. If the applied load acting over a segment of the ring remains constant in magnitude and is either constant in direction or continues to point through the original geometric center of the ring when the ring is deflected from its equilibrium position, a potential can be defined for the applied force and combined with the strain energy to define a total potential

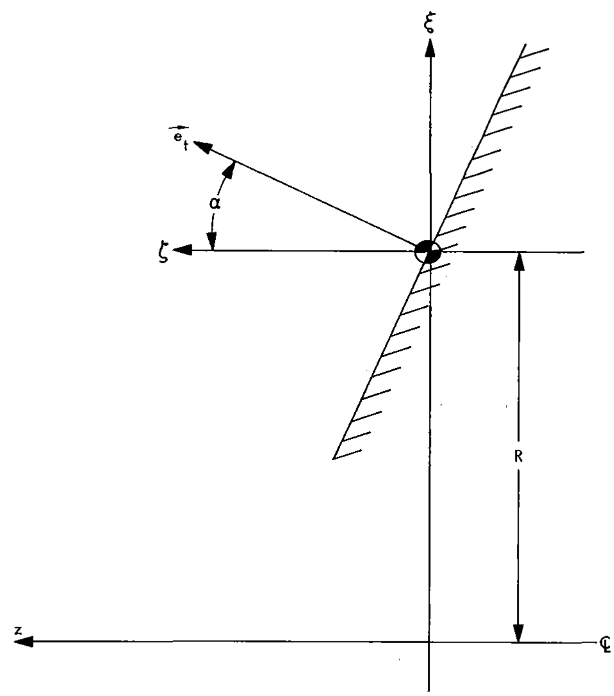

Fig. 2. Coordinate system for restraint of ring

energy. As shown in Ref. 8, the resulting conservative system is stable if the second variation of the total potential energy is positive definite. Furthermore, the system is unstable if the second variation is indefinite, negative definite, or negative semidefinite. The critical load is defined as that for which the second variation becomes positive semidefinite. In particular, Langhaar shows that the second variation is positive semidefinite if and only if the rank of the associated coefficient matrix becomes less than the order of the matrix. Hence, the determinant of the coefficient matrix must vanish when the second variation changes from positive definite to positive semidefinite. This is equivalent to stating that one of the stability coefficients (the coefficients of the generalized coordinates of the second variation of the total potential energy when expressed in canonical form) vanishes.

The energy criterion for stability is essentially that suggested by Temple and Bickley (Ref. 9). They argue that if a system of disturbances is considered applied to the ring, a practical condition for stability is the requirement that the strain energy acquired by the system during the disturbance be greater than the work done by the primary load (the distributed load) during the disturbance. The stability limit is reached when the additional strain energy is exactly equal to the work done by the uniform load.

In attempting to determine the value of load that would cause the second variation of the total strain energy to vanish, Langhaar (see Ref. 8) noted that Trefftz recognized that the existence of a minimum in the second variation was equivalent to requiring that the first variation of the second variation of the total strain energy vanish identically. This requirement leads to a set of homogeneous equations whose coefficient matrix is the associated coefficient matrix of the second variation. Thus, the requirement that a solution exist is that the determinant of the coefficient matrix vanish identically as before. It can be seen that this form of the stability criterion is described by Ziegler (Ref. 10) as the *dynamic* criterion. Clearly, the equation of motion for small disturbances superposed on an equilibrium configuration takes the same form as the equations suggested by Trefftz in the limiting case of vanishing frequency.

II. Description of the Displacements

As a preliminary step in evaluating the strain energy in the ring, the displacements must be expressed in terms of generalized variables that are considered a *reasonable* description of the expected behavior. The form that is used is similar to that used for straight beams and has the property that the (linearized) transverse normal and shear strains vanish at the centroid of the cross section. As pointed out in Ref. 8, the resultant stability criteria should lead to an upper limit to the true value (compared to that for a more general set of displacement variables and the same general criteria for stability) of the critical load, since the model is stiffer than the actual structure.

The method of analysis followed in this report ultimately requires that the expression for the total strain energy in the ring be expressible in terms of as few displacement variables as possible. However, the number of variables must be large enough to give a realistic description of the expected behavior. Because it is important to retain in-plane displacements with the same emphasis as the out-of-plane displacements when attempting a study of their interaction during buckling, the displacements of the ring are described in terms of the displacements of the centroidal axis and the (twist) rotation of the cross section about the centroidal axis.

In particular, let the displacement components of a ring be expressed in terms of generalized variables similar to those for straight beams, i.e., transverse cross sections translate and rotate as a rigid body. The effect of warping is not considered to be important. Thus, with coordinates (ξ, ζ) measured with respect to the centroid of the cross section (see Fig. 1), a point originally located at

$$\mathbf{r} = (R + \xi)\,\mathbf{e}_r + \zeta\mathbf{e}_z$$

will move to

$$\mathbf{r}' = \mathbf{r} + [U(\theta)\,\mathbf{e}_r + V(\theta)\,\mathbf{e}_\theta + W(\theta)\,\mathbf{e}_z] + [\beta(\theta)\mathbf{e}_\theta + K_r(\theta)\mathbf{e}_r + K_z(\theta)\mathbf{e}_z] \times (\xi\mathbf{e}_r + \zeta\mathbf{e}_z)$$

As a further condition, let the linearized transverse shear strain components vanish at the centroid. In accordance with this assumption, K_r and K_z are determined to be

$$K_r = \frac{W'}{R}, \qquad K_z = \frac{(V - U')}{R}, \qquad (\;)' = \frac{d(\;)}{d\theta}$$

Thus, the three displacement components u, v, and w are expressible in terms of the four generalized variables

U, V, W, and ( ); i.e.,

$$\left.\begin{aligned} u &= U + \zeta\beta \\ v &= -\frac{\xi}{R}(U') - \frac{\zeta W'}{R} + \frac{Vr}{R} \\ w &= W - \beta\xi \end{aligned}\right\} \tag{1}$$

The strain components consistent with these displacement forms are derived in Appendix A.

During the loading process, the uniform radial load p_0, acting through the centroid, is expected to produce a uniform radial displacement U_0 and a rotation β_0. Because the loading is axisymmetric, there will be no circumferential displacement component. The deviations from this uniform (unbuckled) configuration will be denoted by $\overline{U}, \overline{V}, \overline{W}$, and $\overline{\beta}$; i.e.,

$$\left.\begin{aligned} U(\theta) &= U_0 + \overline{U}(\theta) \\ \beta(\theta) &= \beta_0 + \overline{\beta}(\theta) \\ V(\theta), W(\theta) &= \overline{V}(\theta), \overline{W}(\theta) \end{aligned}\right\} \tag{2}$$

It will be assumed that the disturbance displacement components have the following form:

$$\left.\begin{aligned} \left(\frac{\overline{U}}{R}, \overline{\beta}\right) &= \sum_{n=0} (A_n, B_n) \cos n\theta \qquad (n \neq 1) \\ \frac{\overline{V}}{R} &= \sum_{n=2} (C_n) \sin n\theta \\ \frac{\overline{W}}{R} &= \sum_{n=2} (D_n) \cos n\theta \end{aligned}\right\} \tag{3}$$

At this stage, the configuration of the ring is described completely in terms of the variables U_0, β_0, A_n, B_n, C_n, and D_n.

The strain energy in a ring segment consistent with the assumed form of displacement is presented in Appendix B. When terms involving deviation components in the strain energy expression were expanded, only quadratic terms were retained. This restriction applies only to the order of the deviation components in any given term, because quadratic terms in the deviation components that contained U or β as a multiplicative factor have been retained. Furthermore, the circumferential strain has been assumed small enough so that $1 + e_{\theta\theta} \approx 1$.

The strain energy expression for a complete ring, expressed in terms of the displacement components of Eq. (3) and subject to the above limitation, is presented in Appendix C.

The expression for the work performed by the uniformly distributed radial load during a deviation from the unbuckled state is derived in Appendix D. It is shown that a contribution to the quadratic form for the total potential energy results only if the load is assumed to be centrally directed.

III. Analysis of the Unbuckled Configuration

To apply the stability criteria, the form of the unbuckled displacement U_0 and β_0 vs p_0 must first be determined. Appendices C and D indicate that the total potential energy in a complete ring, acted upon by a uniformly distributed radial load that maintains its original orientation during an arbitrary deviation of displacement, is given by

$$\begin{aligned} \pi\overline{E}AR \Bigg\{ & \left(\frac{U_0}{R}\right)^2 \left(1 + \frac{I_{zz}}{AR^2}\right) - \frac{2I_{rz}}{AR^2}\frac{U_0\beta_0}{R} + \frac{I_{rr}}{AR^2}\beta_0^2 + \frac{2\lambda}{\overline{E}}\frac{U_0\beta_0^2}{R} \\ & + \frac{\lambda + G}{\overline{E}}\beta_0^4 + 2A_0\left[\frac{U_0}{R}\left(1 + \frac{I_{zz}}{AR^2}\right) - \frac{I_{rz}}{AR^2}\beta_0 + \frac{\lambda}{\overline{E}}\beta_0^2 - \frac{p_0R}{\overline{E}A}\right] \\ & + 2B_0\left[\beta_0\left(\frac{I_{rr}}{AR^2} + \frac{2\lambda}{\overline{E}}\frac{U_0}{R} + \frac{\lambda + \overline{E}}{\overline{E}}\beta_0^2\right) - \frac{I_{rz}}{AR^2}\frac{U_0}{R}\right] + \mathbf{q}_0^{T}\mathbf{K}_0\mathbf{q}_0 + \sum_{n=2}(\mathbf{q}_n^{T}\mathbf{K}_n\mathbf{q}_n)\Bigg\} \end{aligned} \tag{4}$$

The generalized coordinates $\mathbf{q}_0 = \{A_0, B_0\}^T$ represent the axisymmetric deviation components, whereas $\mathbf{q}_n = \{A_n, B_n, C_n, D_n\}^T$ represent the out-of-plane deviation components.

The equations describing the unbuckled equilibrium state result from the requirement that the coefficients of the independent deviation components A_0 and B_0 in Eq. (4) vanish identically; i.e.,

$$\frac{U_0}{R}\left(1+\frac{I_{zz}}{AR^2}\right)-\frac{I_{rz}}{AR^2}\beta_0+\frac{\lambda}{\bar{E}}\beta_0^2=\frac{p_0R}{\bar{E}A} \qquad (5a)$$

$$\beta_0\left(\frac{I_{rr}}{AR^2}+\frac{2\lambda}{\bar{E}}\frac{U_0}{R}+\frac{\lambda+\bar{E}}{\bar{E}}\beta_0^2\right)-\frac{I_{rz}}{AR^2}\frac{U_0}{R}=0 \qquad (5b)$$

In terms of the dimensionless variables

$$\bar{p}=\frac{U_0}{R}\frac{AR^2}{I_{rr}}, \qquad f=\beta_0\frac{AR^2}{I_{rz}}, \qquad \epsilon=\frac{\lambda+\bar{E}}{\bar{E}}\frac{I_{rz}^2}{AR^2I_{rr}}$$

Eqs. (5a) and (5b) can alternatively be written as

$$\bar{p}\left(1+\frac{I_{zz}}{AR^2}\right)-f\left(1-\frac{\lambda f}{\bar{E}}\right)\frac{I_{rz}^2}{AR^2I_{rr}}=\frac{p_0R^3}{\bar{E}I_{rr}} \qquad (6a)$$

$$f^3\frac{\lambda+\bar{E}}{\bar{E}}\frac{I_{rz}^2}{AR^2I_{rr}}+f\left(1+\frac{2\lambda\bar{p}}{\bar{E}}\right)-\bar{p}=0 \qquad (6b)$$

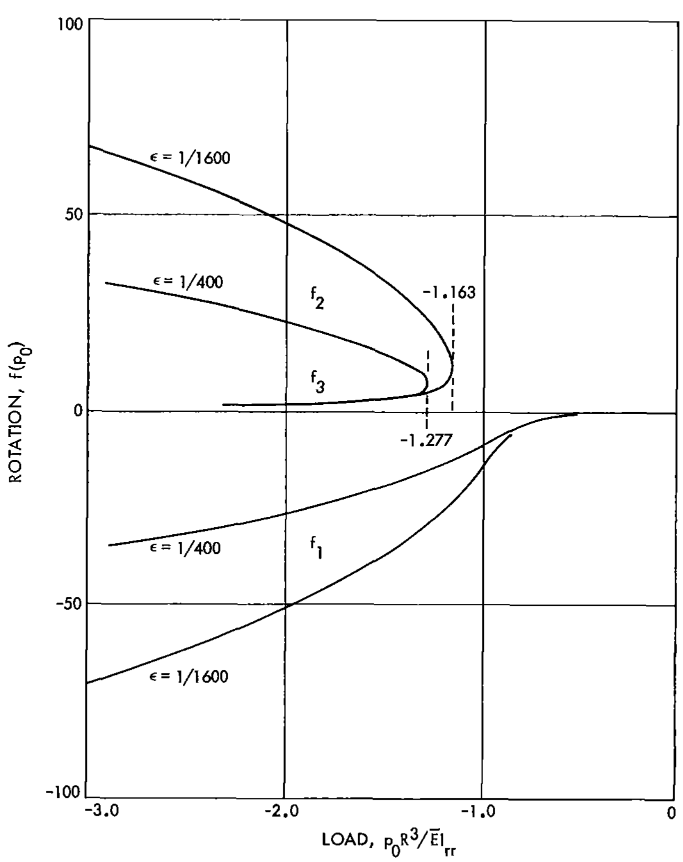

Fig. 3. Rotation of ring vs radial load

Furthermore, the displacement variable $\bar{p}$ may be eliminated between Eqs. (5a) and (6b) to give the following equation for $f(p_0)$:

$$f^2\frac{\lambda}{\bar{E}}\frac{I_{rz}^2}{AR^2I_{rr}}\left[f\left(\frac{\lambda+\bar{E}}{\lambda}-2\frac{\lambda}{\bar{E}}\right)+3\right]+f\left(1+\frac{2\lambda}{\bar{E}}\frac{p_0R^3}{\bar{E}I_{rr}}\right)-\frac{p_0R^3}{\bar{E}I_{rr}}=0 \qquad (7)$$

Because Eq. (7) is a cubic equation, it is expected that multiple roots will exist for a particular range of the parameters ϵ and p_0 (Ref. 11). In fact, as can be seen in Fig. 3, a single value of the rotation parameter f exists for a given value of load p_0 up to a particular value of load that depends upon the parameter ϵ. At loads greater than this value, the rotation parameter may exist at a small positive value f_3 or at the larger values f_1 and f_2 that are approximately equal in magnitude but opposite in sign. The corresponding load-displacement diagram is presented in Fig. 4.

The limiting form for $\epsilon=0$ and $I_{rz}=0$ can most conveniently be seen by examining Eqs. (5a) and (5b). In the limit $I_{rz}=0$, Eqs. (5a) and (5b) require

$$\frac{p_0R}{\bar{E}A}=\frac{\bar{p}\,I_{rr}}{AR^2}+\frac{\lambda}{\bar{E}}\beta_0^2$$

$$\beta_0\left[\beta_0^2+\frac{\bar{E}}{\lambda+\bar{E}}\frac{I_{rr}}{AR^2}\left(1+\frac{2\lambda\bar{p}}{\bar{E}}\right)\right]=0$$

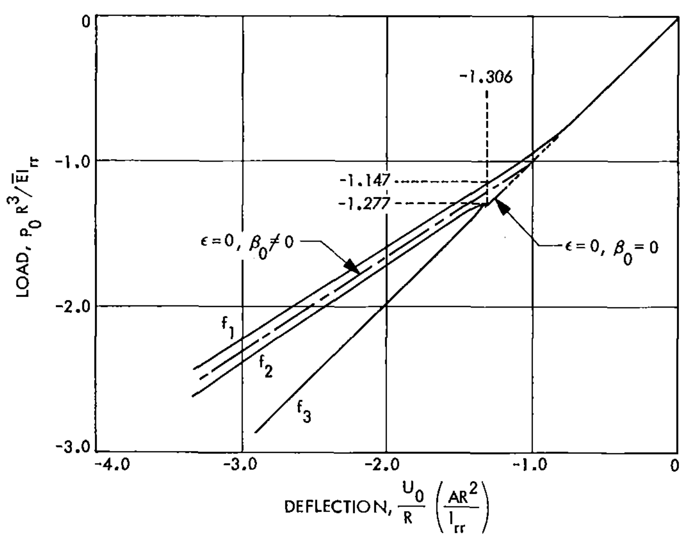

Fig. 4. Load-deflection diagram ($\nu = 1/3$, $\epsilon = 1/400$)

Thus, it follows that either

$$\beta_0 = 0$$

where

$$\frac{p_0 R^3}{\overline{E} I_{rr}} = \overline{p}$$

or

$$\beta_0^2 = -\frac{\overline{E}}{\lambda + \overline{E}} \frac{I_{rr}}{AR^2} \left(1 + \frac{2\lambda \overline{p}}{\overline{E}}\right)$$

where

$$\frac{p_0 R^3}{\overline{E} I_{rr}} = \overline{p} \frac{(1-2\nu)(1+\nu)}{(1-\nu)} - \nu$$

The latter choice is possible only if

$$1 + \frac{2\lambda \overline{p}}{\overline{E}} < 0$$

Note that the radial deflection $\overline{p}$ is given by

$$\overline{p} = \frac{p_0 R^3}{\overline{E} I_{rr}}$$

for either choice at

$$\overline{p} = -\frac{\overline{E}}{2\lambda} = -\frac{1}{2} \frac{1-\nu}{\nu}$$

These alternative solutions (see Figs. 4 and 5) are seen to form a forked branch extending from the point $\overline{p} = -\overline{E}/2\lambda = -1 (\nu = 1/3)$ and extending parallel to the curves f_1, f_2, and f_3 corresponding to $\epsilon \neq 0$. The point $\overline{p} = -\overline{E}/2\lambda$ will be seen to correspond to the limiting value of $\overline{p}$ where multiple values of the rotation parameter f appear as $\epsilon \to 0$.

To determine the behavior of the solution on the parameter ϵ, examine Eq. (6b) near the point $\overline{p}$ where multiple solutions for $f(\overline{p})$ appear. According to Burington (Ref. 11), the roots of $f(\overline{p})$ change character when

$$\frac{1}{4}\left(\frac{\overline{p}}{\epsilon}\right)^2 + \frac{1}{27}\left(\frac{1 + \frac{2\lambda \overline{p}}{\overline{E}}}{\epsilon}\right)^3 = 0 \tag{8}$$

is satisfied. As $-\overline{p}/\epsilon \geqslant 0$ for $\overline{p} \leqslant 0$, the roots of Eq. (6b) are

$$f = -2\left(-\frac{1}{3}\frac{1 + \frac{2\lambda \overline{p}}{\overline{E}}}{\epsilon}\right)^{1/2}, \quad \left(-\frac{1}{3}\frac{1 + \frac{2\lambda \overline{p}}{\overline{E}}}{\epsilon}\right)^{1/2}, \quad \left(-\frac{1}{3}\frac{1 + \frac{2\lambda \overline{p}}{\overline{E}}}{\epsilon}\right)^{1/2} \tag{9}$$

whenever Eq. (8) is satisfied.

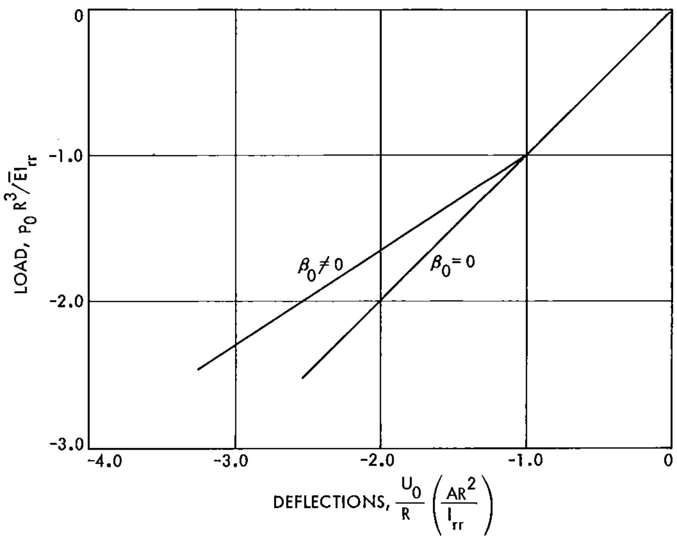

Fig. 5. Load-deflection diagram for symmetrical cross sections ($\nu = 1/3$)

Since $\epsilon \ll 1$, it is helpful to examine Eqs. (8) and (9) in the limit $\epsilon \to 0$. If the three terms in Eq. (6b) are to have the same order of magnitude, it is apparent that

$$1 + \frac{2\lambda \overline{p}}{\overline{E}} \to 0 \quad \text{as} \quad \epsilon \to 0$$

Hence, when

$$1 + \frac{2\lambda \overline{p}}{\overline{E}} = \delta(\epsilon)(a_0 + a_1 \delta + a_2 \delta^2 + \cdots)$$

is substituted into Eq. (8), it is apparent that $\delta = \epsilon^{1/3}$, and

$$\frac{a_0}{3} = -\left(\frac{1-\nu}{4\nu}\right)^{2/3}, \qquad \frac{a_1}{a_0} = -\frac{2}{3} a_0, \qquad \frac{a_2}{a_0} = \frac{1}{3} a_0^2$$

Furthermore, the roots of Eq. (6b) corresponding to

$$\frac{2\lambda \overline{p}}{\overline{E}} = -1 + a_0 \epsilon^{1/3}\left[1 - 2\epsilon^{1/3}\left(\frac{a_0}{3}\right) + 3\epsilon^{2/3}\left(\frac{a_0}{3}\right)^2 + \cdots\right] \tag{10}$$

are

$$f_1 = -2f_2$$

$$f_2 = f_3 = \frac{1}{\epsilon^{1/3}}\left(-\frac{a_0}{3}\right)^{1/2}\left[1 - \frac{a_0}{3}\epsilon^{1/3} + \left(\frac{a_0}{3}\right)^2 \epsilon^{2/3} + \cdots\right] \tag{11}$$

For the special case $\nu = 1/3$, $a_0 = -1.890$ and

$$\bar{p} = -1 - 1.890\,\epsilon^{1/3}(1 + 1.260\,\epsilon^{1/3} + 1.191\,\epsilon^{2/3} + \cdots)$$

$$f_2(\epsilon) = \frac{0.7938}{\epsilon^{1/3}}(1 + 0.6301\,\epsilon^{1/3} + 0.3970\,\epsilon^{2/3} + \cdots)$$

These relations have been evaluated for $\epsilon = 1/400$ and 1/1600; the results, together with the value of the loading parameter at which Eq. (8) is satisfied, are presented in Table 1.

Table 1. Solutions of Eq. (6b) at critical point for $\nu = 1/3$

Parameter ϵ	Rotation f_2	Deflection $\bar{p}$	Load $p_0R^3/\bar{E}I_{rr}$	
			$f = f_1$	$f = f_2$
1/400	6.392	−1.306	−1.147	−1.277
1/1600	9.816	−1.180	−1.092	−1.163

The load p_0 required at the given value of U_0/R to maintain the configuration f_2 and f_3 must be greater than that for f_1 (see Fig. 4). In particular, the values of the loading parameters corresponding to Eqs. (5b) and (8) being simultaneously satisfied are

$$\left.\frac{p_0R^3}{\bar{E}I_{rr}}\right|_{f=f_2,\,\nu=1/3} = -1 - 1.680\,\epsilon^{1/3} - 2.646\,\epsilon^{2/3} + \cdots$$

$$\left.\frac{p_0R^3}{\bar{E}I_{rr}}\right|_{f=f_1,\,\nu=1/3} = -1 - 1.050\,\epsilon^{1/3} - 0.265\,\epsilon^{2/3} + \cdots$$

In Eq. (4), the total strain energy should be examined as a function of both U_0 and p_0. If Eq. (4) is rewritten in terms of the dimensionless $\bar{p}$ and f variables, and if

$$\text{total strain energy} = \pi\bar{E}AR\left(\frac{I_{rr}}{AR^2}\right)^2 \bar{V}$$

where

$$\bar{V} = \bar{p}^2 + \frac{\bar{E}\,\epsilon}{\lambda + \bar{E}}\left[f^2\left(1 + \frac{2\lambda\bar{p}}{\bar{E}}\right) - 2\bar{p}f + \frac{\epsilon f^4}{2}\right]$$

it follows that $\bar{V} = \bar{V}(\bar{p}, f, \nu, \epsilon)$.

The function $\bar{V}(\bar{p},f)$ is determined once the solution of Eqs. (5a) and (6b) is obtained. Results of such a calculation are presented in Fig. 6 with $\bar{p} \propto U_0/R$ as the abscissa, and in Fig. 7 with p_0 as the abscissa. The combined presentation in Fig. 8 shows only the principal branch of the $\bar{V}$ diagram. In Figs. 6 and 7, it should be noted that the value of the strain energy, evaluated at the point where the roots of Eq. (5b) change character (as defined by Eqs. 10 and 11), is given by

$$\pi\bar{E}AR\left(\frac{I_{rr}}{AR^2}\right)^2\left[\bar{p}^2 + a_0^2\frac{1-\nu}{6}\epsilon^{2/3} + \cdots\right]$$

for the branch $f = f_2$, and

$$\pi\bar{E}AR\left(\frac{I_{rr}}{AR^2}\right)^2\left[\bar{p}^2 - \frac{4}{3}a_0^2\,(1-\nu)\,\epsilon^{2/3} + \cdots\right]$$

for the branch $f = f_1$. Specifically,

$$\bar{V}(f_1, \epsilon = 1/400) = 1.648$$

$$\bar{V}(f_2, \epsilon = 1/400) = 1.713$$

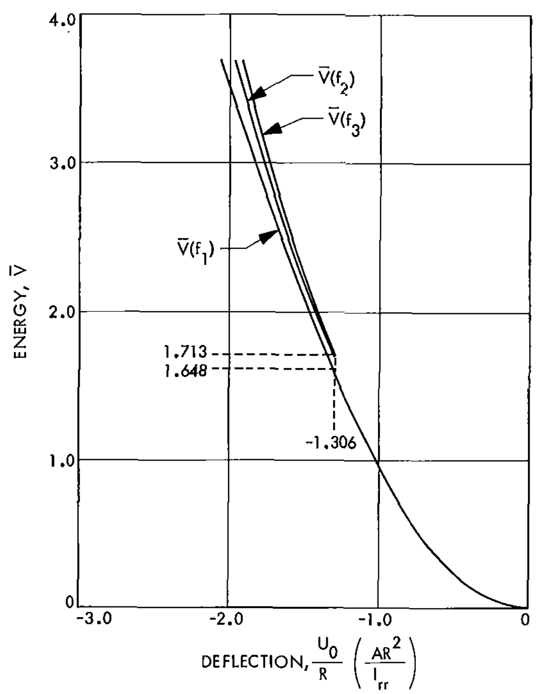

Fig. 6. Energy vs radial deflection ($\nu = 1/3$, $\epsilon = 1/400$)

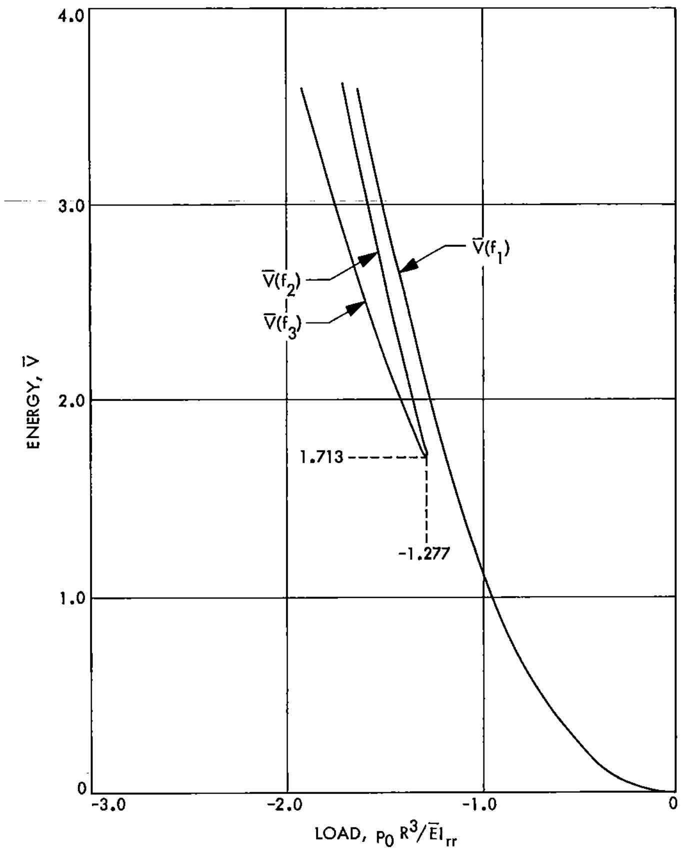

Fig. 7. Energy vs radial load ($\nu = 1/3$, $\epsilon = 1/400$)

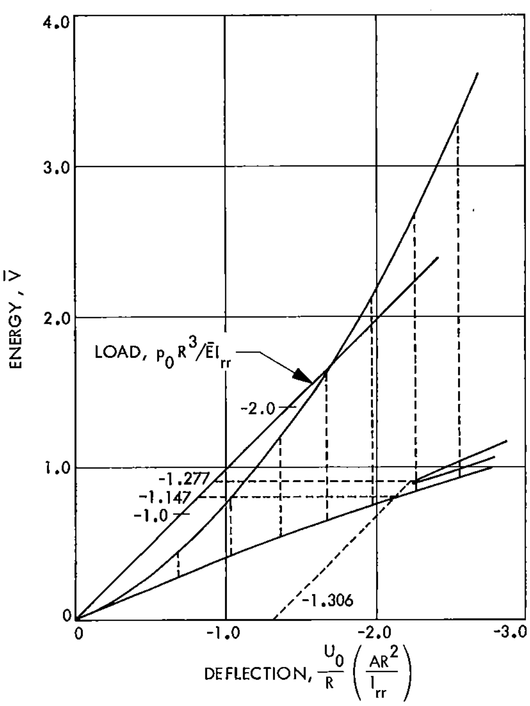

Fig. 8. Principal branch of the strain energy vs load-deflection curve

IV. Axisymmetric Instability of a Complete Ring

According to Langhaar (Ref. 8), the axisymmetric configuration specified by U_0 and β_0 vs p_0 is stable provided that the second variation of the total potential energy is positive definite; i.e.,

$$\mathbf{q}_0^T \mathbf{K}_0 \mathbf{q}_0 + \sum_{n=2} (\mathbf{q}_n^T \mathbf{K}_n \mathbf{q}_n) > 0 \tag{12}$$

This expression can be put in a more convenient form if definitions are given for a complete generalized displacement vector

$$\mathbf{q}^* = \begin{Bmatrix} \mathbf{q}_0 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{q}_n \end{Bmatrix}$$

and the corresponding coefficient matrix

$$\mathbf{K}^* = \begin{bmatrix} \mathbf{K}_0 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{K}_2 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{K}_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & \mathbf{K}_n \end{bmatrix}$$

Hence, the configuration specified by U_0 and β_0 is stable provided that

$$\mathbf{q}^{*T} \mathbf{K}^* \mathbf{q}^* > 0 \tag{13}$$

A critical state exists at the load that causes the determinant of the coefficient matrix $\mathbf{K}^*$ to vanish. This is equivalent to stating that one of the stability coefficients (the coefficients of the generalized coordinates when the quadratic form is written in canonical form, i.e., the sum of squares) vanishes identically. As

$$|\mathbf{K}^*| = |\mathbf{K}_0|\,|\mathbf{K}_2|\,|\mathbf{K}_3| \cdots |\mathbf{K}_n|$$

it follows that overall stability can be studied by investigating the individual stability of each mode.

This section of the report concerns an investigation of the properties of $|\mathbf{K}_0|$. The properties of $|\mathbf{K}_n|$ will be discussed in Section V.

In accordance with Appendix C, the determinant of $\mathbf{K}_0$ will vanish for $\bar{p}$ and f, satisfying

$$1 + \frac{2\lambda}{\bar{E}}\bar{p} + \epsilon\left[3f^2 - \frac{\bar{E}}{\lambda + \bar{E}}\left(1 - \frac{2\lambda f}{\bar{E}}\right)^2\right] = 0 \tag{14}$$

The coefficients have been simplified with the assumption that $I_{zz}/AR^2 \ll 1$. Note that if $\epsilon \ll 1$, the determinant of $\mathbf{K}_0$ changes from positive definite to negative definite at $(2\lambda/\bar{E})(\bar{p}) = -1$.

In the limit $\epsilon = 0$, we must investigate Eq. (14) for the two possible configurations that either

$$\beta_0 = 0, \qquad p \text{ arbitrary}$$

or

$$\beta_0^2 = -\frac{\bar{E}}{\lambda + \bar{E}}\frac{I_{rr}}{AR^2}\left(1 + \frac{2\lambda\bar{p}}{\bar{E}}\right), \qquad 1 + \frac{2\lambda\bar{p}}{\bar{E}} \leqslant 0$$

As can be easily verified, both choices reduce Eq. (14) to $1 + 2\lambda\bar{p}/\bar{E} = 0$. Hence, a ring of symmetrical cross section is unstable with respect to axisymmetric disturbances at

$$\frac{p_0 R^3}{\bar{E} I_{rr}} = -\frac{\bar{E}}{2\lambda}$$

where

$$\bar{p} = -\frac{\bar{E}}{2\lambda}$$

For $\epsilon \ll 1$, we expect that the simultaneous solution of Eqs. (14) and (6b) would be approximately

$$\bar{p} = -\frac{\bar{E}}{2\lambda}, \qquad f = O(\epsilon^{1/3})$$

The precise values of $\bar{p}$ and f, satisfying Eqs. (6b) and (14), can be obtained as the root of the following cubic equation by eliminating $\bar{p}$ from the set; i.e.,

$$4\frac{\lambda}{\bar{E}}f^3\left(1 - 2\frac{\lambda^2}{\bar{E}^2}\frac{\bar{E}}{\lambda + \bar{E}}\right) - 3f^2\left(1 - 4\frac{\lambda^2}{\bar{E}^2}\frac{\bar{E}}{\lambda + \bar{E}}\right)$$

$$-6\frac{\lambda f}{\bar{E}}\frac{\bar{E}}{\lambda + \bar{E}} - \frac{1}{\epsilon}\left(1 - \frac{\epsilon\bar{E}}{\lambda + \bar{E}}\right) = 0 \tag{15}$$

As can be verified (see Ref. 11), Eq. (15) possesses one real root, provided that

$$\frac{b^2}{4} + \frac{a^3}{27} > 0$$

where

$$a = \frac{3q - p^2}{3}, \qquad b = \frac{2p^3 - 9pq + 27r}{27}$$

When p, q, and r are identified as

$$p = -\frac{3}{4\nu}\frac{(1-\nu)(1-\nu-4\nu^2)}{(1+\nu)(1-2\nu)} = -3/4$$

for $\nu = 1/3$

$$q = -\frac{3}{2}\frac{(1-\nu)^2}{(1+\nu)(1-2\nu)} = -3/2$$

for $\nu = 1/3$

$$r = -\frac{(1-\nu)^2}{4\nu\epsilon}\frac{1-(1-\nu)\epsilon}{(1+\nu)(1-2\nu)} = -\frac{3}{4\epsilon}\left(1 - \frac{2\epsilon}{3}\right)$$

for $\nu = 1/3$

it follows that, for $\nu = 1/3$,

$$a = -\frac{27}{16}, \qquad b = -\frac{3}{4\epsilon}\left(1 - \frac{\epsilon}{8}\right)$$

and the criterion is easily satisfied for $\epsilon \ll 1$.

In order to determine this single root, let

$$1 + \frac{2\lambda\bar{p}}{\bar{E}} = \delta\,(b_0 + \delta b_1 + \delta^2 b_2 + \cdots), \qquad \delta = \epsilon^{1/3}$$

$$f = \frac{1}{\delta}(f_0 + f_1\delta + f_2\delta^2 + \cdots)$$

The values f_i can be obtained by substituting the expansion for f into Eq. (15). A hierarchy of equations results that gives the following values:

$$\left.\begin{aligned} f_0^3 &= \frac{(1-\nu)^2/4\nu}{(1-2\nu)(1+\nu)} \\ f_1 &= f_0^3\,\frac{1-\nu-4\nu^2}{1-\nu} \\ \frac{f_2}{f_0^2} &= \frac{(1-\nu)^2/4\nu}{(1-2\nu)(1+\nu)} \end{aligned}\right\} \qquad (16)$$

The corresponding values of b_i are most conveniently obtained by substituting in the relation

$$\bar{p} = f\,\frac{1+\epsilon f^2}{1-\dfrac{2\lambda}{\bar{E}}f}$$

obtained by rewriting Eq. (6b). Then,

$$\bar{p} = -\frac{\bar{E}}{2\lambda}\left\{1+\frac{\delta}{f_0}\left(f_0^3+\frac{\bar{E}}{2\lambda}\right) + \frac{\delta^2}{f_0^2}\left[3f_1f_0^3-\left(f_1-\frac{\bar{E}}{2\lambda}\right)\left(f_0^3+\frac{\bar{E}}{2\lambda}\right)\right]+\cdots\right\}$$

so that

$$b_0 = -\frac{1-\nu}{f_0}\,\frac{3-3\nu-4\nu^2}{4\nu(1+\nu)(1-2\nu)}$$

$$b_1 = -\frac{(1-\nu)^3}{f_0^2}\,\frac{3-3\nu-8\nu^2}{8\nu^2(1-2\nu)^2(1+\nu)^2}$$

For $\nu = 1/3$, we find

$$f_0 = (3/4)^{1/3} = 0.9086, \qquad f_1 = 1/4, \qquad f_2 = f_0^5 = 0.6192$$

$$b_0 = -\frac{7}{4f_0} = -1.926, \qquad b_1 = -\frac{15}{8f_0^2} = -2.271$$

so that

$$\bar{p} = -1 - 1.926\,\delta - 2.271\,\delta^2 + \cdots$$

$$f = \frac{1}{\delta}(0.9086 + 0.25\,\delta + 0.6192\,\delta^2 + \cdots)$$

These relations have been evaluated for $\epsilon = 1/400$ and 1/1600; the results are presented in Table 2.

If the data of Table 1 are compared with those of Table 2, it is apparent that the following occurs: the axisymmetric deflection state becomes unstable at the value of the radial deflection $\bar{p}$ at which multiple roots appear and at the value of the rotation parameter f corresponding to the vertical tangency point on the f_1 and f_2 vs $\bar{p}$ diagram. This interpretation follows from the observation that the values of $\bar{p}$ agree within an error that is proportional to ϵ.

Table 2. Critical deflections for $\nu = 1/3$

Parameter ϵ	Critical deflection $\bar{p}_{crit}$	Critical rotation f_{crit}
1/400	−1.303	7.029
1/1600	−1.181	10.93

In summary, for a ring with a symmetrical cross section $\epsilon = 0$, the deflection states corresponding to that portion of the load-deflection curve (see Fig. 5) for which

$$-\frac{\bar{E}}{2\lambda} < \left(\frac{p_0R^3}{\bar{E}I_{rr}},\quad \frac{U_0}{R}\frac{AR^2}{I_{rr}}\right) \leqslant 0$$

are considered stable with respect to axisymmetric disturbances.

In the general case $\epsilon \neq 0$, the range of load over which the ring is stable with respect to axisymmetric disturbances is slightly larger than $|\bar{p}| \leqslant \bar{E}/2\lambda$. However, as the determinant of $\mathbf{K}_0$ changes sign at this critical load, it is concluded that the axisymmetric equilibrium configuration U_0 and β_0 vs p_0 is unstable for loads corresponding to (approximately) $\bar{p} = -\bar{E}/2\lambda$.

V. Out-of-Plane Stability of a Complete Ring

In this section, the stability of the equilibrium state specified by U_0 and β_0 vs p_0 will be discussed with respect to nonaxisymmetric deviations. In accordance with Eq. (13), the determinant of the overall coefficient matrix will vanish if any of the individual 4×4 $\mathbf{K}_n$ matrices possesses a vanishing determinant.

As is suggested in Appendix C, the analysis of $\mathbf{K}_n$ is most conveniently described in terms of the elements a_{ij}, where

$$[a_{ij}] = 2[k_{ij}]$$

In forming the determinant of $[a_{ij}]$, it is important to note that all coefficients a_{ij} are small (of the order of magnitude of $\bar{\epsilon}$ or less) except a_{11}, a_{13}, and a_{33}. Hence, in order to work with a well-conditioned matrix, it is necessary to essentially remove the third column of $|a_{ij}|$.

By appropriate multiplication and subtraction of the rows of $|a_{ij}|$, it follows that

$$|a_{ij}| = a_{33}^4 |C_{ij}|$$

where the coefficient matrix $[C_{ij}]$ is symmetric and defined as follows:

$$\left.\begin{aligned} C_{11} &= a_{11} - \frac{a_{13}^2}{a_{33}}, & C_{22} &= a_{22} - \frac{a_{23}^2}{a_{33}} \\ C_{12} &= a_{12} - \frac{a_{23}\,a_{13}}{a_{33}}, & C_{23} &= a_{24} - \frac{a_{23}\,a_{34}}{a_{33}} \\ C_{13} &= a_{14} - \frac{a_{34}\,a_{13}}{a_{33}}, & C_{33} &= a_{44} - \frac{a_{34}^2}{a_{33}} \end{aligned}\right\} \qquad (17)$$

The significance of removing the third column of $|a_{ij}|$ is apparent if it is assumed that the criterion that the quadratic form changes from positive definite to positive semidefinite is equivalent to requiring that it possess a relative minimum. Hence, the requirement

$$\delta\,(\mathbf{q}_n^{\mathrm{T}}\,\mathbf{K}_n\,\mathbf{q}_n) = 0$$

is equivalent to

$$\mathbf{K}_n\,\mathbf{q}_n = 0$$

or

$$[a_{ij}]\,\mathbf{q}_n = 0$$

possessing a nontrivial solution. As the first and third of this set of four equations is

$$A_n + nC_n = 0$$

to order $\bar{\epsilon}$, it is apparent that the deviation mode shape is nearly inextensional in the circumferential direction.

Since a_{33} cannot vanish in the range of interest, the stability criterion reduces to

$$|C_{ij}| = 0 \qquad (18)$$

The elements C_{ij} can be expressed simply, since it has been noted in Sections III and IV that the range of interest of $\bar{p}$ and f corresponds to

$$\bar{p} = O(1), \qquad f = \frac{1}{\epsilon^{1/3}}\,O(1)$$

Thus, since $\bar{\epsilon} \ll 1$, the elements C_{ij} become

$$C_{11} = \bar{\epsilon}(n^2-1)^2\left(\bar{I}_{zz} + \frac{\bar{p}}{n^2}\frac{\lambda+\bar{E}}{\bar{E}}\right) + O(\bar{\epsilon}^{4/3}\,\bar{I}_{rz}^{2/3}) + O(\bar{\epsilon}^2)$$

$$C_{12} = \bar{\epsilon}(n^2-1)\,\bar{I}_{rz} + O(\bar{\epsilon}^{5/3}\,\bar{I}_{rz}^{1/3}) + \bar{I}_{rz}\,O(\bar{\epsilon}^2)$$

$$C_{13} = n^2\bar{\epsilon}(n^2-1)\,\bar{I}_{rz} + O(\bar{\epsilon}^{5/3}\,\bar{I}_{rz}^{1/3}) + \bar{I}_{rz}\,O(\bar{\epsilon}^2)$$

$$C_{22} = \bar{\epsilon}\left(1 + \frac{2\lambda\bar{p}}{\bar{E}} + n^2\frac{GJ}{\bar{E}I_{rr}}\right) + O(\bar{\epsilon}^{4/3}\,\bar{I}_{rz}^{2/3}) + O(\bar{\epsilon}^2)$$

$$C_{23} = n^2\bar{\epsilon}\left(1 + \frac{GJ}{\bar{E}I_{rr}}\right) + O(\bar{\epsilon}^{5/3}\,\bar{I}_{rz}^{4/3}) + O(\bar{\epsilon}^2)$$

$$C_{33} = n^2\bar{\epsilon}\left(n^2 + \frac{GJ}{\bar{E}I_{rr}} + \bar{p}\frac{\lambda+\bar{E}}{\bar{E}}\right) + O(\bar{\epsilon}^{4/3}\,\bar{I}_{rz}^{2/3}) + O(\bar{\epsilon}^2)$$

If all but the leading terms in the coefficient matrix C_{ij} are omitted, the stability requirement of Eq. (18) becomes

$$\begin{aligned} &\left(\bar{I}_{zz} + \frac{\dfrac{\bar{p}}{n^2}}{1-\nu}\right)\Bigg\{2\nu\left(\frac{\bar{p}}{1-\nu}\right)^2 \\ &\quad + \left(\frac{\bar{p}}{1-\nu}\right)\left[1 + n^2\frac{GJ}{\bar{E}I_{rr}} + 2\nu\left(n^2 + \frac{GJ}{\bar{E}I_{rr}}\right)\right] \\ &\quad + (n^2-1)^2\frac{GJ}{\bar{E}I_{rr}}\Bigg\} \\ &\quad - \bar{I}_{rz}^2\left[(n^2-1)^2\frac{GJ}{\bar{E}I_{rr}} + \frac{\bar{p}}{1-\nu}(1+2\nu n^2)\right] = 0 \end{aligned} \qquad (19)$$

For $\bar{I}_{rz} = 0$, the critical deflection parameter $\bar{p}_{crit}$ is given by

$$\bar{p}_{crit} = -(1-\nu)n^2\bar{I}_{zz} \qquad (20)$$

and the two additional roots of the quadratic equation

$$\begin{aligned} 2\nu\left(\frac{\bar{p}}{1-\nu}\right)^2 + \left(\frac{\bar{p}}{1-\nu}\right)\left[1 + n^2\frac{GJ}{\bar{E}I_{rr}} + 2\nu\left(n^2 + \frac{GJ}{\bar{E}I_{rr}}\right)\right] \\ + (n^2-1)^2\frac{GJ}{\bar{E}I_{rr}} = 0 \end{aligned} \qquad (21)$$

It should be noted that the mode shape corresponding to Eq. (20) is an inextensional, in-plane displacement, whereas the remaining roots correspond to out-of-plane displacements. This uncoupling of the buckling modes occurs for $\bar{I}_{rz} = 0$ only.

For $\bar{I}_{rz} \neq 0$, Eq. (19) is equivalent to the following cubic equation:

$$\frac{2\nu}{n^2}\left(\frac{\bar{p}}{1-\nu}\right)^3 + \left(\frac{\bar{p}}{1-\nu}\right)^2 \times \left\{2\nu \bar{I}_{zz} + \frac{1}{n^2}\left[1 + n^2\frac{GJ}{\bar{E}I_{rr}} + 2\nu\left(n^2 + \frac{GJ}{\bar{E}I_{rr}}\right)\right]\right\} + \left(\frac{\bar{p}}{1-\nu}\right)\left[(n^2-1)^2\frac{GJ}{n^2\bar{E}I_{rr}} + (\bar{I}_{zz} - I^2_{rz})(1+2\nu n^2) + \bar{I}_{zz}\frac{GJ}{\bar{E}I_{rr}}(n^2+2\nu)\right] + (\bar{I}_{zz} - \bar{I}^2_{rz})(n^2-1)^2\frac{GJ}{\bar{E}I_{rr}} = 0 \tag{22}$$

It should be noted that the effect of $\bar{I}_{rz} \neq 0$ is reflected in the parameter $(\bar{I}_{zz} - \bar{I}^2_{rz})$ or $(1 - I^2_{rz}/I_{rr}I_{zz})$. This is also a familiar property of the linear analysis of ring structures. Furthermore, Biezeno and Grammel (see Ref. 2) indicate that this parameter is always positive.

The roots of Eq. (22) have been obtained for $\nu = 1/3$, $1/4 \leqslant \bar{I}_{zz} \leqslant 3.0$, $0 \leqslant \bar{I}_{rz} \leqslant 0.8\,\bar{I}_{zz}$, $n = 2$, and $n = 3$. In all cases, the least root corresponding to $n = 3$ has a numerical value greater than the least root corresponding to $n = 2$. The remaining roots may overlap. The values of the least roots corresponding to $n = 2$ are presented in Fig. 9. It should be noted that the abscissa for any curve $\bar{I}_z$ (constant) cannot reach zero without causing the parameter $\bar{I}_{zz} - \bar{I}^2_{rz}$ to vanish.

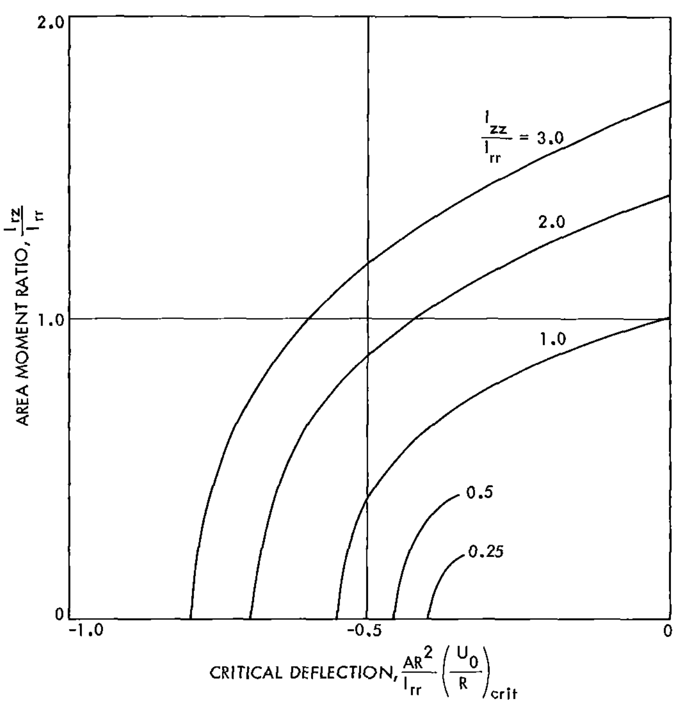

Fig. 9. Critical deflection vs I_{rz} for an unrestrained ring ($\nu = 1/3$, $n = 2$)

VI. Buckling of a Restrained Ring

Since the free ring is the exception rather than the rule in structural idealizations, it is important to investigate the effects of restricting the disturbance mode shape. In particular, let the radial and axial displacements of the centroidal axis satisfy the condition that there be no component of disturbance displacement along an axis inclined at an angle α to the z-axis (see Fig. 2); i.e.,

$$\boldsymbol{\delta} \cdot \mathbf{e}_t = 0$$

where

$$\mathbf{e}_t = \mathbf{e}_z \cos\alpha + \mathbf{e}_r \sin\alpha$$

$$\boldsymbol{\delta} = \overline{U}\,\mathbf{e}_r + \overline{W}\,\mathbf{e}_z$$

In terms of the disturbance parameters, this condition requires that

$$A_n \sin\alpha + D_n \cos\alpha = 0$$

Because D_n is now dependent upon A_n, the criterion for stability derived in the previous section must be modified. If the analysis is restricted to forces that maintain their original line of action after buckling, the only modification that is necessary to the quadratic form is the elimination of D_n as a variable. In this way, the holonomic character of the system is retained.

With regard to the nth mode, the substitution of

$$D_n = -A_n \tan\alpha$$

changes the quadratic form in the total potential energy expression of Eq. (4) to

$$\{A_n, B_n, C_n\}\,\mathbf{K}^{(\alpha)}_n \begin{Bmatrix} A_n \\ B_n \\ C_n \end{Bmatrix}$$

where $\mathbf{K}^{(\alpha)}_n = [k^{(\alpha)}_{ij}]$, and

$$k^{(\alpha)}_{11} = k_{11} - 2k_{14}\tan\alpha + k_{44}\tan^2\alpha$$

$$k^{(\alpha)}_{22} = k_{22} \qquad k^{(\alpha)}_{33} = k_{33}$$

$$k^{(\alpha)}_{12} = k_{12} - k_{24}\tan\alpha = k^{(\alpha)}_{21}$$

$$k^{(\alpha)}_{13} = k_{13} - k_{34}\tan\alpha = k^{(\alpha)}_{31}$$

$$k^{(\alpha)}_{23} = k_{23} = k^{(\alpha)}_{32}$$

The variable C_n is essentially eliminated by removing the third column. In terms of the coefficient C_{ij} defined in Eq. (17), the determinant of $\mathbf{K}_n^{(\alpha)}$ can be shown to vanish for

$$|B_{ij}| = 0 \tag{23}$$

where

$$\left.\begin{aligned} B_{11} &= C_{11} - 2C_{13}\tan\alpha + C_{33}\tan^2\alpha \\ B_{12} &= C_{12} - C_{23}\tan\alpha = B_{21} \\ B_{22} &= C_{22} \end{aligned}\right\} \tag{24}$$

For $\alpha = 0$ and $I_{rz} \neq 0$, the criterion for stability is that $\bar{p}$ satisfy

$$C_{11}C_{22} - C_{12}^2 = 0 \tag{25a}$$

Taking only the leading terms in the expression for the elements C_{ij}, we obtain the following quadratic equation for $\bar{p}$:

$$\frac{2\nu}{n^2}\left(\frac{\bar{p}}{1-\nu}\right)^2 + \frac{\dfrac{\bar{p}}{n^2}}{1-\nu}\left(1 + n^2\frac{GJ}{\overline{E}I_{rr}} + 2\nu n^2\bar{I}_{zz}\right) + (\bar{I}_{zz} - \bar{I}_{rz}^2) + n^2\bar{I}_{zz}\frac{GJ}{\overline{E}I_{rr}} = 0 \tag{25b}$$

In the general case $\alpha \neq 0$ and $I_{rz} \neq 0$, Eq. (23) requires that

$$(C_{11}C_{22} - C_{12}^2) - 2\tan\alpha\,(C_{13}C_{22} - C_{12}C_{23}) + (C_{22}C_{33} - C_{23}^2)\tan^2\alpha = 0 \tag{26a}$$

Taking only the leading terms in the expressions for the elements C_{ij}, we obtain the following quadratic equation for $\bar{p}$:

$$\begin{aligned} &\frac{2\nu}{n^2}\left(\frac{\bar{p}}{1-\nu}\right)^2\left[1 + \frac{n^4\tan^2\alpha}{(n^2-1)^2}\right] + \frac{\dfrac{\bar{p}}{n^2}}{1-\nu}\left\{1 + n^2\frac{GJ}{\overline{E}I_{rr}} + 2\nu n^2 I_{zz} - \frac{4\nu n^4\bar{I}_{rz}}{n^2-1}\tan\alpha \right. \\ &\qquad \left. + \frac{n^4\tan^2\alpha}{(n^2-1)^2}\left[1 + n^2\frac{GJ}{\overline{E}I_{rr}} + 2\nu\left(n^2 + \frac{GJ}{\overline{E}I_{rr}}\right)\right]\right\} \\ &\qquad + \left\{\bar{I}_{zz} - \bar{I}_{rz}^2 + n^2\frac{GJ}{\overline{E}I_{rr}}(\bar{I}_{zz} - 2\bar{I}_{rz}\tan\alpha + \tan^2\alpha)\right\} = 0 \end{aligned} \tag{26b}$$

Note that the above analysis presupposes that $\tan\alpha = O(1)$, since A_n is used as a variable. For restraint conditions corresponding to α at approximately 90 deg, D_n would be a more convenient variable, and the analysis would proceed in a manner analogous to that above.

The roots of Eq. (26b) have been obtained for $\nu = 1/3$, $1/4 \leqslant \bar{I}_{zz} \leqslant 3.0$, $-0.8\bar{I}_z \leqslant \bar{I}_{rz} \leqslant 0.8\bar{I}_z$, $n = 2$, and $\alpha = 0$, 15, 30, 45, and 60 deg. The least of the two roots is shown plotted in Figs. 10–14. A summary of the results for a symmetrical, restrained ring is presented in Fig. 15. Note that the effect of restraint for unsymmetrical cross sections $\bar{I}_{rz} \neq 0$ is a general increase of the critical deflection parameter.

VII. Discussion

To interpret the significance of the various stability criteria established in Section VI, the load-deflection diagrams should be reviewed (see Figs. 3, 4, and 5). For a ring with a symmetrical cross section, the radial deflection U_0 is directly proportional to the radial load p_0, and the deflection increases with no rotation of the cross section $\beta_0 = 0$. At the load corresponding to a radial deflection $\bar{p} = -\overline{E}/2\lambda$, a multiple-valued deflection configuration appears with no jump in the rotation. For values of load corresponding to a radial deflection greater than $-\overline{E}/2\lambda$, three different rotations exist for a given radial deflection, namely,

$$\beta_0 = 0, \qquad \pm\left[\frac{-\overline{E}}{\lambda + \overline{E}}\,\frac{I_{rr}}{AR^2}\left(1 + \frac{2\lambda\bar{p}}{\overline{E}}\right)\right]^{1/2}$$

As shown in Section IV, the point at which this multiple-valued configuration appears is unstable with respect to axisymmetric disturbances. Hence, it is unlikely that a ring of symmetrical cross section could be loaded above this point.

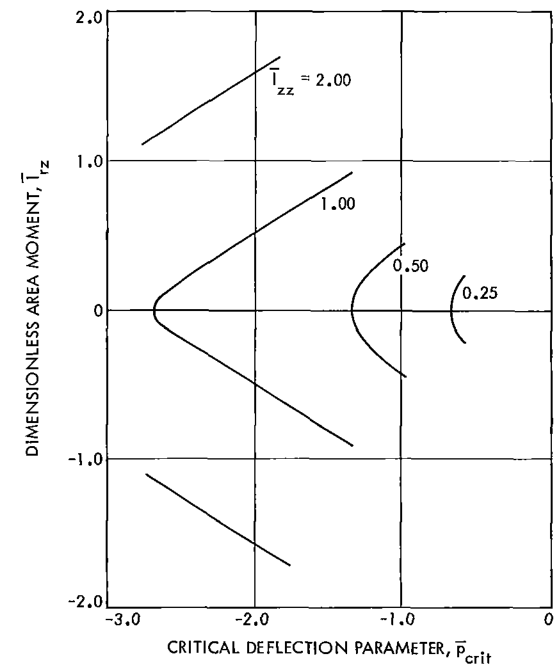

Fig. 10. Critical deflections for a restrained ring ($\alpha = 0$ deg, $n = 2$)

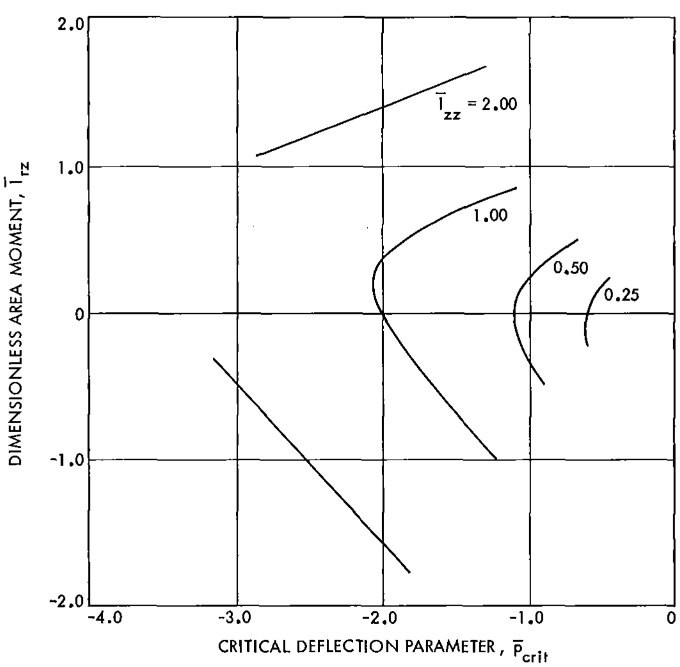

Fig. 11. Critical deflections for a restrained ring ($\alpha = 15$ deg, $n = 2$)

For a ring with an unsymmetrical cross section, the radial deflection U_0 is always accompanied by a rotation β_0 for all load values. If the ring were loaded by a constrained, radial deflection device, the radial deflection and the rotation would be single-valued functions of the load up to a load value corresponding to a radial deflection slightly greater than $\bar{p} = -\bar{E}/2\lambda$. At this point, three values of rotation can exist for the same radial deflection. Hence, there may be a jump in the rotation and a consequent increase in the radial load required to hold the ring in the new position. The value of the rotation that appears at this point is opposite in sign to that which accompanied the radial deflection from the unloaded state. As the radial deflection increases from this point, two values of the rotation parameter increase in magnitude with the load (the f_1 and f_2 branch), whereas the third value tends to decrease in magnitude with the load.

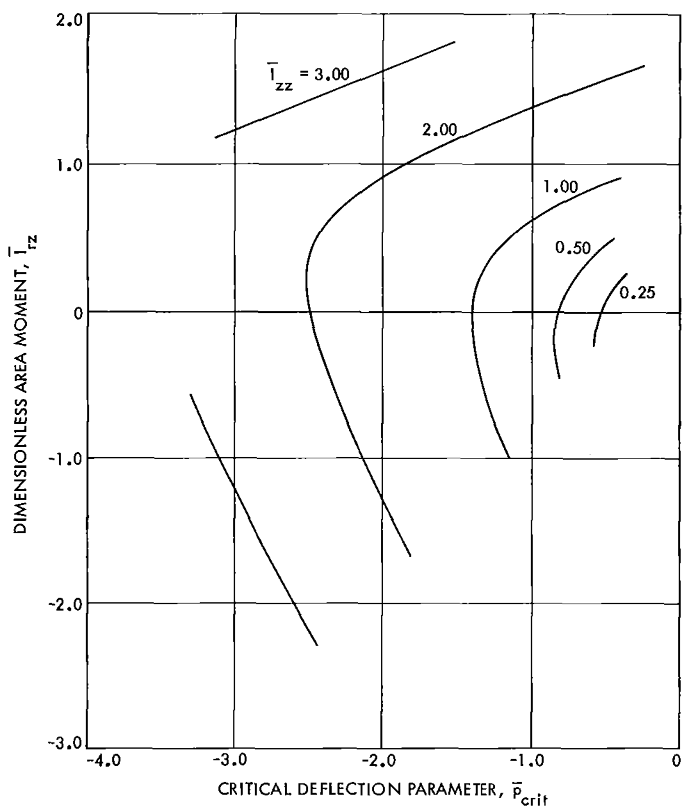

Fig. 12. Critical deflections for a restrained ring ($\alpha = 30$ deg, $n = 2$)

As shown in Section IV, the determinant of K_0 changes from positive to negative definite at a load corresponding to $|\bar{p}|$ slightly in excess of $\bar{E}/2\lambda$. Hence, one must conclude that an axisymmetric deflected state cannot exist for loads in excess of approximately $-\bar{E}/2\lambda$. The classical critical load is, therefore, impossible to achieve.

Strictly speaking, the actual value of the radial load corresponding to the load parameter p_0 must be found

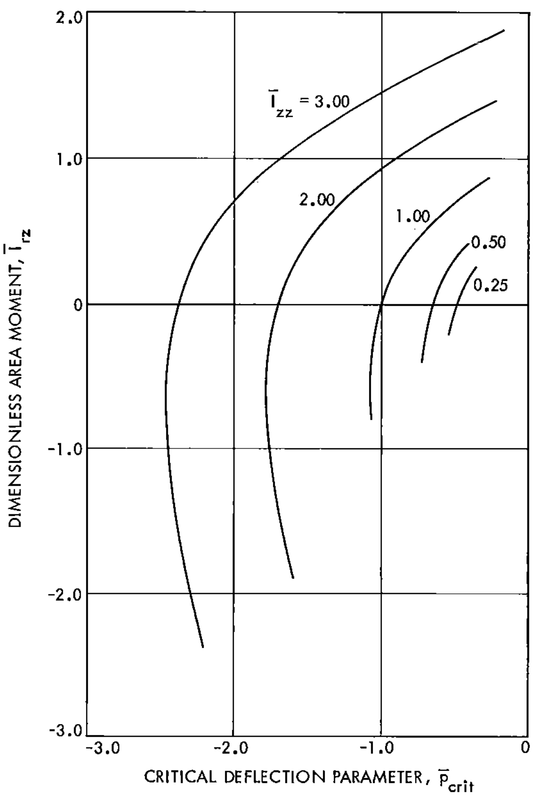

Fig. 13. Critical deflections for a restrained ring (α = 45 deg, n = 2)

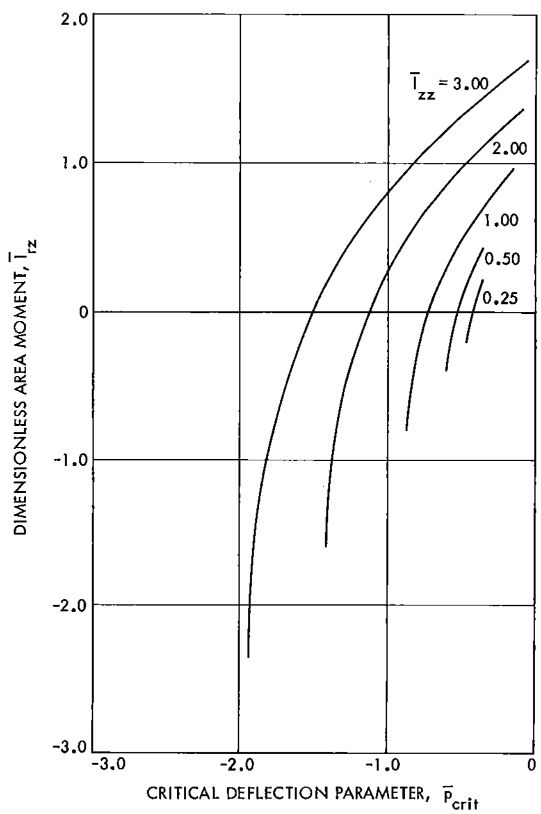

Fig. 14. Critical deflections for a restrained ring (α = 60 deg, n = 2)

by considering the length of the centroidal axis in the deformed configuration. The form of the energy principle used defines the distributed force per unit length of centroidal axis in the undeformed state as p_0 and the *correspondence rule* indicates that p_0 is also the force per unit length in the deformed configuration. Hence, the total load acting on the ring in the deformed state is equal to the total load acting on the ring in the undeformed state multiplied by the ratio of the undeformed periphery to the deformed periphery. Considering that the linearized circumferential normal strain $e_{\theta\theta}$ was neglected in comparison with unity in computing the total strain energy, there is essentially no distinction between either the periphery or the total load in either the deformed or the undeformed state.

The results of the stability calculations made in Section V apply only to very thin rings. In particular, Eqs. 19–22 were written with the assumption that only the leading terms in the coefficients C_{ij} need be retained. This implies, in general, that terms of order $\bar{\epsilon}^{1/3} = (I_{rr}/AR^2)^{1/3}$ were negligible in comparison with unity for $\bar{I}_{rz} = I_{rz}/I_{rr} = O(1)$. For $\bar{I}_{rz} = 0$, the assumption is that $O(\bar{\epsilon})$ is negligible in comparison with unity. Hence, if improved accuracy is warranted for $\bar{I}_{rz} \neq 0$, the technique employed in Section V must be replaced by a search technique wherein successive values of $\bar{p}$ and the corresponding values of f_1 would be substituted into $|C_{ij}|$ until $|C_{ij}|$ vanished.

This procedure would require the simultaneous solution of Eq. (6b) (or a table of values of f vs $\bar{p}$) for each step. It is only for the limiting case described in Section V for $\bar{\epsilon} \ll 1$ that the terms in $|C_{ij}|$ do not involve f_1; hence, simple cubic and quadratic equations, Eqs. (21) and (22), result. An analogous procedure would also have to be undertaken for the constrained ring, as the same approximations were made in Section VI as in Section V regarding $\bar{\epsilon} \ll 1$.

Furthermore, the radial load parameter $\bar{p}$, in the discussion above, was considered to be equal to the loading parameter $p_0R^3/\bar{E}I_{rr}$ of Eq. (6a). Since the value of f_1, corresponding to the critical value of $\bar{p}$, has been shown to be near $O(1)/\epsilon^{1/3}$ (see Section III), it follows from Eq. (6a) that taking $\bar{p} \approx p_0R^3/\bar{E}I_{rr}$ also involves regarding $\epsilon^{1/3}$ as negligible in comparison with unity. Hence, as in the evaluation of $|C_{ij}|$, the explicit evaluation of

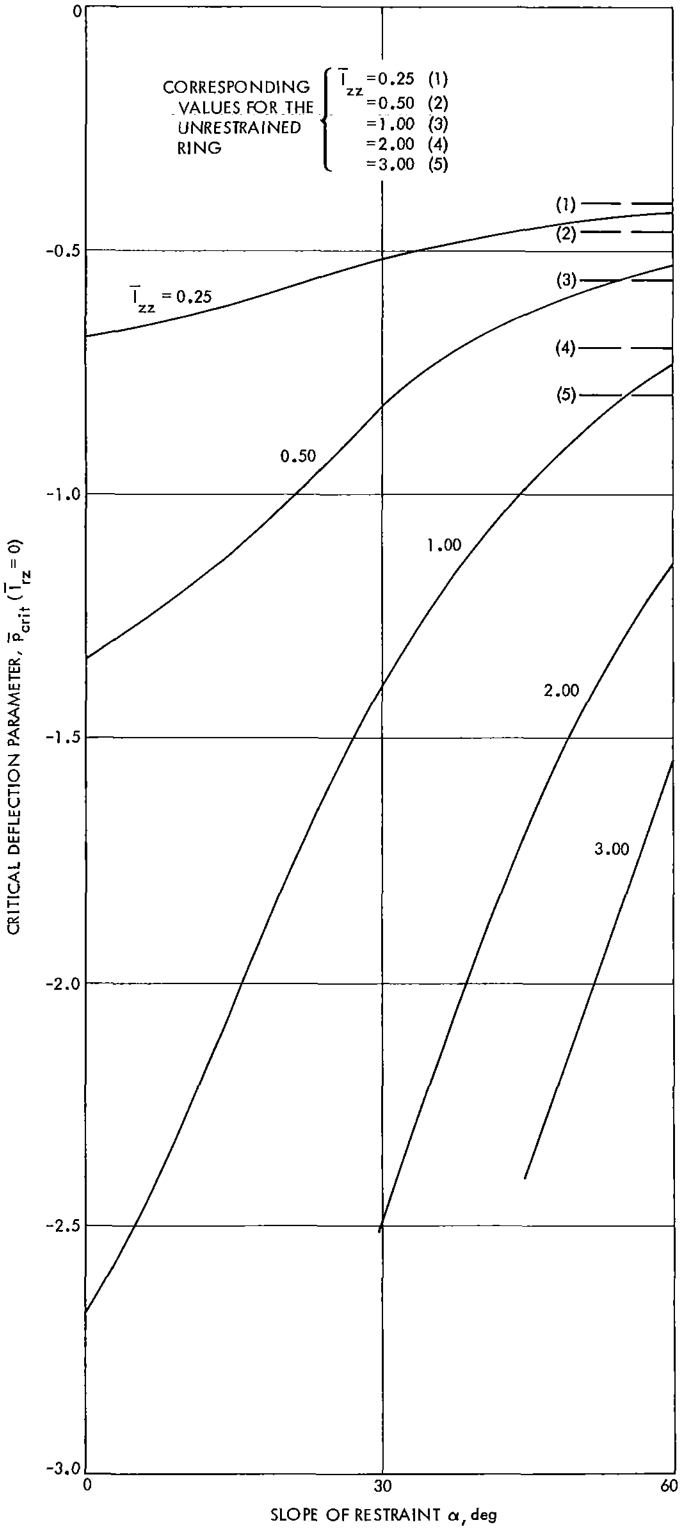

Fig. 15. Summary of critical deflections for a symmetrical, restrained ring

f vs $\bar{p}$ is only required when the assumption $\epsilon^{1/3} \ll 1$ is not warranted.

The results for the nonaxisymmetric modes based upon Eq. (19) indicate that this analysis does not recover the classical result achieved by Timoshenko (see Ref. 5). The classical result is interpreted here to mean the critical load corresponding to in-plane displacements only. To obtain such a mode, one must examine the limiting case $\bar{I}_{rz} = 0$. Following Eq. (20), and noting that $\bar{p} = p_0R^3/\bar{E}I_{rr}$ for $I_{rz} = 0$, and $\beta_0 = 0$, we see that

$$-\left(\frac{p_0R^3}{EI_{zz}}\right)_{crit} = \frac{4(1-\nu)^2}{(1+\nu)(1-2\nu)} \qquad n = 2, \nu \text{ arbitrary}$$

$$= 4.00 \qquad (\nu = 1/3)$$

$$= 3.60 \qquad (\nu = 1/4)$$

The minimum value of this expression is 3.56 and occurs for $\nu = 1/5$. This formula is essentially the result of setting $C_{11} = 0$.

When comparing the minimum value with the classical value (3.00), one should note that the result reported by Boresi, as well as Wempner and Kesti (see Ref. 12), was that the classical value corresponds to a hydrostatic type of loading in which the load at a point on the ring remains normal to the centroidal axis during buckling. The type of loading studied here (see Appendix D) is that identified by Wempner and Kesti as constant—the orientation of the loading does not change during buckling. The value of the critical load for constant loading was found to be 4.00 by Wempner and Kesti using the equilibrium method. Note that the identical result is obtained here for $\nu = 1/3$, and that Wah obtained the value 3.00.

The results of calculations based upon Eq. (21) for nonaxisymmetric buckling of symmetrical sections might be qualitatively compared with the results of Cheney and Wah. The intention here is that the dependence of the out-of-plane buckling loads on the postbuckling behavior of the loading be noted rather than that any quantitative comparison be made. As can be seen in Table 3, the results given for the constant load are consistently about half the values obtained by Cheney and Wah for the hydrostatic type of loading.

Salient characteristics of the numerical results of this analysis are presented in the concluding statements that follow.

Table 3. Comparison of critical load for symmetrical sections ($I_{rz} = 0$)

Area moment I_{zz}	Results	
	Present ($\nu = 1/3$)	Ref. 4
0.25	−0.5948	−1.467
0.50	−0.6800	−1.558
1.00	−0.8258	−1.688
2.00	−1.043	−1.842
3.00	−1.194	−1.929

The decreasing slope of the curves $\bar{I}_z$ (constant) should not necessarily be interpreted as a destabilizing effect due to $\bar{I}_{rz} \neq 0$ (see Fig. 9). A more instructive cross plot might be made for a given cross section showing the variation of the critical load with inclination of the principal axis. In fact, such a cross plot would probably be similarly inclined, but not as sharply because the parameter $\bar{I}_z$ changes with inclination. The same is true for the restrained ring. With the latter, however, the lack of symmetry of the results with respect to the parameter $\bar{I}_{rz}$ indicates that there is a preferred direction for a given cross section having a greater critical load.

Finally, it should be noted that the effect of restraint (see Fig. 15) for symmetrical cross sections is true in general; that is, the critical load for the restrained ring is greater than the critical load for the corresponding unrestrained ring. Thus, the behavior of curves for $\bar{I}_{rz}$ (constant) is qualitatively similar to that of the curves of Fig. 15 for all values of the parameters studied in Figs. 10–14.

Appendix A

Strain Components in Circular Polar Coordinates

Let $d\mathbf{R} = \mathbf{e}_r\, dr + \mathbf{e}_\theta\, rd_\theta + \mathbf{e}_z\, dz$ be a differential line segment in an elastic material that becomes $d\mathbf{R}'$ when the material is deformed with the displacement field

$$\boldsymbol{\delta}(r,\theta,z) = u(r,\theta,z)\,\mathbf{e}_r + v(r,\theta,z)\mathbf{e}_\theta + w(r,\theta,z)\mathbf{e}_z$$

As

$$\begin{aligned} d\mathbf{R}' &= d\mathbf{R} + d\boldsymbol{\delta} \\ &= \mathbf{e}_r[(1+u_r)dr + (u_\theta - v)d\theta + u_z\, dz] \\ &\quad + \mathbf{e}_\theta\left[v_r\, dr + rd\theta\left(1 + \frac{v_\theta}{r} + \frac{u}{r}\right) + v_z\, d_z\right] \\ &\quad + \mathbf{e}_z[w_r\, dr + w_\theta\, d\theta + (1+w_z)dz] \end{aligned}$$

and defining the components of the Lagrangian (Green's) strain tensor as

$$\frac{(d\mathbf{R}')^2 - (d\mathbf{R})^2}{2(d\mathbf{R})^2} = l^2 E_{rr} + m^2 E_{\theta\theta} + n^2 E_{zz} + 2lm\, E_{r\theta} + 2ln\, E_{rz} + 2mn\, E_{\theta z}$$

it follows that

$$E_{rr} = u_r + \frac{1}{2}(u_r^2 + v_r^2 + w_r^2)$$

$$E_{\theta\theta} = \frac{1}{r}(u + v_\theta) + \frac{1}{2r^2}[(u_\theta - v)^2 + (u + v_\theta)^2 + w_\theta^2]$$

$$E_{zz} = w_z + \frac{1}{2}(u_z^2 + v_z^2 + w_z^2)$$

$$2E_{r\theta} = v_r + \frac{1}{r}(u_\theta - v) + \frac{1}{r}[u_r(u_\theta - v) + v_r(u + v_\theta) + w_r w_\theta]$$

$$2E_{rz} = u_z + w_r + (u_r u_z + v_r v_z + w_r w_z)$$

$$2E_{\theta z} = v_z + \frac{1}{r}w_\theta + \frac{1}{r}[u_z(u_\theta - v) + v_z(u + v_\theta) + w_\theta w_z]$$

where l, m, and n are the direction cosines of the line segment $d\mathbf{R}$, i.e., $d\mathbf{R} = |\,d\mathbf{R}\,|(l\mathbf{e}_r + m\mathbf{e}_\theta + n\mathbf{e}_z)$.

In terms of the generalized coordinates U, V, W, and β, the strain components become

$$E_{rr} = \frac{1}{2}\left[\beta^2 + \left(\frac{V - U'}{R}\right)^2\right]$$

$$E_{\theta\theta} = e_{\theta\theta}\left(1 + \frac{e_{\theta\theta}}{2}\right) + \frac{1}{2r^2}\left\{\left[\frac{r}{R}(U' - V) + \zeta\left(\beta' + \frac{W'}{R}\right)\right]^2 + (W' - \xi\beta')^2\right\}$$

$$re_{\theta\theta} = u + v_\theta = U + \zeta\beta + \frac{r}{R}V' - \frac{\xi}{R}U'' - \frac{\zeta}{R}W''$$

$$E_{zz} = \frac{1}{2}\left[\beta^2 + \left(\frac{W'}{R}\right)^2\right]$$

$$2E_{r\theta} = \frac{\zeta}{r}\left(\beta' + \frac{W'}{R}\right) - \frac{\beta}{r}(W' - \xi\beta') + e_{\theta\theta}\frac{V - U'}{R}$$

$$2E_{rz} = -\frac{W'}{R^2}(V - U')$$

$$2E_{\theta z} = -\frac{1}{r}\left(\beta' + \frac{W'}{R}\right)(\xi - \beta\zeta) + \beta\frac{U' - V}{R} - e_{\theta\theta}\frac{W'}{R}$$

Appendix B
The Strain Energy in a Ring Segment

The strain energy $\theta_1 \leqslant \theta \leqslant \theta_2$ in a ring segment can be written, following Fung (Ref. 13), as the integral of the strain energy density $\rho_0 W$ over the volume of the ring in its undeformed configuration, i.e.,

$$\int_{V_0} \rho_0 W(E_{ij}) dV_0$$

where

$$S_{ij} = \frac{\partial(\rho_0 W)}{\partial E_{ij}}$$

are the components of the Kirchhoff stress tensor. For a linearly elastic, isotropic material,

$$S_{ij} = \lambda E_{kk}\,\delta_{ij} + 2G\,E_{ij}$$

where λ and G are the Lamé constants of the material, it then follows that the strain energy density takes the form

$$\rho_0 W(E_{ij}) = \frac{\overline{E}}{2}\left(E^2_{rr} + E^2_{\theta\theta} + E^2_{zz}\right)$$
$$+ \lambda\left(E_{rr}E_{\theta\theta} + E_{zz}E_{\theta\theta} + E_{rr}E_{zz}\right)$$
$$+ G\left(E^2_{r\theta} + E^2_{rz} + E^2_{\theta z} + E^2_{\theta r} + E^2_{zr} + E^2_{z\theta}\right)$$

where $\overline{E} = \lambda + 2G$.

In terms of the generalized coordinates U, V, W, and β, the individual terms in the strain energy expression become

$$\int_A r\,dA\left(E^2_{rr} + E^2_{zz}\right) = \frac{RA\beta^2}{2}\left[\beta^2 + \left(\frac{V-U'}{R}\right)^2 + \left(\frac{W'}{R}\right)^2 + \cdots\right]$$

$$\int_A r\,dA\,e^2_{\theta\theta} = AR\left(\frac{U+V'}{R}\right)^2 + \frac{I_{zz}}{R}\left(\frac{U+U''}{R}\right)^2 + \frac{I_{rr}}{R}\left(\beta - \frac{W''}{R}\right)^2 - \frac{2I_{rz}}{R}\left(\frac{U+U''}{R}\right)\left(\beta - \frac{W''}{R}\right)$$

$$\int_A r\,dA\,E^2_{\theta\theta} = \int_A r\,dA\,e^2_{\theta\theta} + RA^*\left(\frac{W'}{R}\right)^2\left(\frac{U}{R} - \beta\,\frac{I_{rz} + I^*_{rz}}{A^*R^2}\right)$$
$$+ 2\frac{I_{zz}}{R}\frac{W'\beta'}{R}\left(\frac{U}{R}\frac{I_{zz} + I^*_{zz}}{I_{zz}} - \beta\frac{I^*_{rz}}{I_{zz}}\right) + AU\left(\frac{U'-V}{R}\right)^2$$
$$+ \frac{I_{zz}}{R}\beta'^2\left(\frac{U}{R}\frac{I^*_{zz}}{I_{zz}} + \frac{\beta}{R}\frac{I^*_{rrz}}{I_{zz}}\right)$$
$$+ 2\frac{I_{rr}}{R}\left(\beta' + \frac{W'}{R}\right)\left(\frac{V-U'}{R}\right)\left(\frac{U}{R}\frac{I_{rz}}{I_{rr}} - \beta\right)$$
$$+ \frac{I_{rr}}{R}\left(\beta' + \frac{W'}{R}\right)^2\left(\frac{I^*_{rr}}{I_{rr}}\frac{U}{R} + \beta\frac{I^*_{rrr}}{R I_{rr}}\right)$$

$$\int_A r\,dA\,E_{\theta\theta}\left(E_{rr} + E_{zz}\right) = AR\beta^2\left[\frac{U+V'}{R} + \frac{1}{2}\left(\frac{U'-V}{R}\right)^2\right.$$
$$\left. + \frac{J}{2AR^2}\left(\beta' + \frac{W'}{R}\right)^2 + \frac{1}{2}\left(\frac{W'}{R}\right)^2\right] + \frac{AR}{2}\frac{U+V'}{R}\left[\left(\frac{W'}{R}\right)^2\right.$$
$$\left. + \left(\frac{V-U'}{R}\right)^2\right]$$

$$\int_A r\,dA\,E_{rr}E_{zz} = \frac{1}{4}AR\beta^2\left[\beta^2 + \left(\frac{W'}{R}\right)^2 + \left(\frac{V-U'}{R}\right)^2\right]$$

$$4\int_A r\,dA\,(E_{\theta r}^2 + E_{\theta z}^2) = \frac{J}{R}\left(\beta' + \frac{W'}{R}\right)^2 + AR\left[\left(\frac{W'}{R}\right)^2 + \left(\frac{V-U'}{R}\right)^2\right]$$

$$\times\left[\left(\frac{U}{R}\right)^2\left(1+\frac{I_{zz}}{AR^2}\right) + \beta^2\left(1+\frac{I_{rr}}{AR^2}\right) - \frac{2U\beta}{R}\frac{I_{rz}}{AR^2}\right]$$

$$-\frac{2}{R}\left(\beta' + \frac{W'}{R}\right)\left\{\frac{V-U'}{R}I_{rr}\left[\frac{U}{R}\frac{I_{rz}+\beta I_{zz}}{I_{rr}} - \beta\left(1+\beta\frac{I_{rz}}{I_{rr}}\right)\right]\right.$$

$$\left.+\frac{W'}{R}I_{zz}\left[\frac{U}{R}\left(1-\beta\frac{I_{rz}}{I_{zz}}\right) - \beta\frac{I_{rz}-\beta I_{rr}}{I_{zz}}\right]\right\}$$

where terms in U', V, and W' of order higher than quadratic have been discarded, and $e_{\theta\theta}$ and β^2 have been neglected in comparison with unity. Note that higher-order terms have been retained if they contain U or β as a factor. The following area properties have been used:

$$\int_A (\xi, \zeta)\,dA = 0$$

$$\int_A \frac{dA}{1+\frac{\xi}{R}} = A\left(1+\frac{I_{zz}}{AR^2}\right), \qquad \int_A \frac{(\xi,\zeta)\,dA}{1+\frac{\xi}{R}} = -\frac{1}{R}(I_{zz}, I_{rz})$$

$$\int_A \frac{(\xi^2, \xi\zeta, \zeta^2)\,dA}{1+\frac{\xi}{R}} = (I_{zz}, I_{rz}, I_{rr})$$

$$J = I_{rr} + I_{zz}, \qquad J^* = I_{rr}^* + I_{zz}^*$$

$$\int_A \frac{(\xi^2, \xi\zeta, \zeta^2)\,dA}{\left(1+\frac{\xi}{R}\right)^2} = (I_{zz}^*, I_{rz}^*, I_{rr}^*)$$

$$\int_A \frac{dA}{\left(1+\frac{\xi}{R}\right)^2} = A^* = A\left(1+\frac{2I_{zz}+I_{zz}^*}{AR^2}\right)$$

$$\int_A \frac{\zeta\,dA}{\left(1+\frac{\xi}{R}\right)^2} = -\frac{I_{rz}+I_{rz}^*}{R}$$

$$\int_A \frac{\xi\,dA}{\left(1+\frac{\xi}{R}\right)^2} = -\frac{I_{zz}+I_{zz}^*}{R}$$

$$\int_A \frac{(\xi^3, \xi^2\zeta, \zeta^3)}{\left(1+\frac{\xi}{R}\right)^2}\,dA = (I_{zzz}^*, I_{rrz}^*, I_{rrr}^*)$$

Appendix C

Strain Energy Relations for a Complete Ring

If the form for the displacement components given in Eqs. (2) and (3) are substituted into the individual strain energy contributions of Appendix B, the following is obtained:

$$\int_{\theta=0}^{2\pi} \int_A r\,d\theta\,dA\,E^2_{rr} + E^2_{zz}) = \pi A R \beta_0^2 \left\{ \beta_0^2 + 4\beta_0 B_0 + 6B_0^2 + \sum_{n=2} \left[3B_n^2 + \frac{n^2}{2} D_n^2 + \frac{1}{2}(C_n + nA_n)^2 \right] \right\}$$

$$= 2 \int_0^{2\pi} \int_A r\,d\theta\,dA\,E_{rr} E_{zz}$$

$$\int_{\theta=0}^{2\pi} \int_A r\,d\theta\,dA\,e^2_{\theta\theta} =$$

$$2\pi A R \left[\left(\frac{U_0}{R} + A_0 \right)^2 \left(1 + \frac{I_{zz}}{AR^2} \right) - \frac{2I_{rz}}{AR^2} \left(\frac{U_0}{R} + A_0 \right) (\beta_0 + B_0) + \frac{I_{rr}}{AR^2} (\beta_0 + B_0)^2 \right]$$

$$+ \pi A R \sum_{n=2} \left[(A_n + nC_n)^2 + \frac{I_{zz}}{AR^2} (n^2 - 1)^2 A_n^2 + \frac{I_{rr}}{AR^2} (B_n + n^2 D_n)^2 + \frac{2I_{rz}}{AR^2} (n^2 - 1) A_n (B_n + n^2 D_n) \right]$$

$$\int_{\theta=0}^{2\pi} \int_A r\,d\theta\,dA\,E^2_{\theta\theta} = \int_{\theta=0}^{2\pi} \int_A r\,d\theta\,dA\,e^2_{\theta\theta} + \pi A R \frac{U_0}{R} \sum_{n=2} \left[(C_n + nA_n)^2 + \frac{A^*}{A} n^2 D_n^2 \left(1 - \frac{\beta_0 R}{U_0} \frac{I_{rz} + I^*_{rz}}{A^* R^2} \right) \right.$$

$$+ \frac{2I_{zz}}{AR^2} n^2 B_n D_n \left(1 + \frac{I^*_{zz}}{I_{zz}} - \frac{\beta_0 R}{U_0} \frac{I^*_{rz}}{I_{zz}} \right) + \frac{I_{zz}}{AR^2} n^2 B_n^2 \left(\frac{I^*_{zz}}{I_{zz}} + \frac{\beta_0 R}{U_0} \frac{I^*_{rrz}}{R I_{zz}} \right)$$

$$\left. - \frac{2I_{rr}}{AR^2} n (B_n + D_n)(C_n + nA_n) \left(\frac{I_{rz}}{I_{rr}} - \frac{\beta_0 R}{U_0} \right) + \frac{I_{rr}}{AR^2} n^2 (B_n + D_n)^2 \left(\frac{I^*_{rr}}{I_{rr}} + \frac{\beta_0 R}{U_0} \frac{I^*_{rrr}}{R I_{rr}} \right) \right]$$

$$\int_{\theta=0}^{2\pi} \int_A r\,d\theta\,dA\,E_{\theta\theta} (E_{rr} + E_{zz}) = \pi A R \left\{ \frac{2U_0}{R} (\beta_0 + B_0)^2 + 2A_0 (\beta_0^2 + 2\beta_0 B_0) \right.$$

$$+ \frac{U_0}{2R} \sum_{n=2} [n^2 D_n^2 + (C_n + nA_n)^2 + 2B_n^2] + 2\beta_0 \sum_{n=2} B_n (A_n + nC_n)$$

$$\left. + \frac{\beta_0^2}{2} \sum_{n=2} \left[(nA_n + C_n)^2 + \frac{Jn^2}{AR^2} (B_n + D_n)^2 + n^2 D_n^2 \right] \right\}$$

$$\int_{\theta=0}^{2\pi} \int_A r\,d\theta\,dA\,(E^2_{\theta r} + E^2_{\theta z}) =$$

$$\sum_{n=2} \left\{ \frac{\pi J n^2}{4R} (B_n + D_n)^2 + \frac{\pi n}{2R} (B_n + D_n) \left[I_{rr} (C_n + nA_n) \left(\frac{U_0}{R} \frac{I_{rz} + \beta_0 I_{zz}}{I_{rr}} - \beta_0 \frac{I_{rr} + \beta_0 I_{rz}}{I_{rr}} \right) \right. \right.$$

$$\left. - I_{zz} n D_n \left(\frac{U_0}{R} \frac{I_{zz} - \beta_0 I_{rz}}{I_{zz}} - \beta_0 \frac{I_{rz} - \beta_0 I_{rr}}{I_{zz}} \right) \right] + \frac{\pi A R}{4} \left[n^2 D_n^2 + (C_n + nA_n)^2 \right]$$

$$\left. \times \left[\left(\frac{U_0}{R} \right)^2 \left(1 + \frac{I_{zz}}{AR^2} \right) + \beta_0^2 \left(1 + \frac{I_{rr}}{AR^2} \right) - \frac{2U_0 \beta_0}{R} \frac{I_{rz}}{AR^2} \right] \right\}$$

Hence, the strain energy in a complete ring can be written as a sum of contributions from each order n of the deviation displacements. For reference, it is convenient to write the strain energy in the following form:

$$\text{Strain energy} = \pi\overline{E}AR\left\{\left(\frac{U_0}{R}\right)^2\left(1+\frac{I_{zz}}{AR^2}\right) - \frac{2I_{rz}}{AR^2}\frac{U_0\beta_0}{R} + \frac{I_{rr}}{AR^2}\beta_0^2 \right.$$

$$+ \frac{2\lambda}{\overline{E}}\frac{U_0\beta_0^2}{R} + \frac{\lambda+G}{\overline{E}}\beta_0^4 + 2A_0\left[\frac{\lambda}{\overline{E}}\beta_0^2 + \frac{U_0}{R}\left(1+\frac{I_{zz}}{AR^2}\right) - \frac{I_{rz}}{AR^2}\beta_0\right]$$

$$+ 2B_0\left[\beta_0\left(\frac{I_{rr}}{AR^2} + \frac{2\lambda}{\overline{E}}\frac{U_0}{R} + \frac{\lambda+\overline{E}}{\overline{E}}\beta_0^2\right) - \frac{I_{rz}}{AR^2}\frac{U_0}{R}\right]$$

$$\left. + \mathbf{q}_0^{\mathrm{T}}\,\mathbf{K}_0\,\mathbf{q}_0 + \sum_{n=2}(\mathbf{q}_n^{\mathrm{T}}\,\mathbf{K}_n\,\mathbf{q}_n)\right\}$$

The vectors $\mathbf{q}_0$ and $\mathbf{q}_n$ and the matrix $\mathbf{K}_0$ are defined as follows:

$$\mathbf{q}_0 = \begin{Bmatrix} A_0 \\ B_0 \end{Bmatrix}, \qquad \mathbf{q}_n = \begin{Bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{Bmatrix}$$

$$\mathbf{K}_0 = \begin{bmatrix} \left(1+\frac{I_{zz}}{AR^2}\right) & \left(\frac{2\lambda}{\overline{E}}\beta_0 - \frac{I_{rz}}{AR^2}\right) \\ \left(\frac{2\lambda}{\overline{E}}\beta_0 - \frac{I_{rz}}{AR^2}\right) & \left(\frac{2\lambda}{\overline{E}}\frac{U_0}{R} + \frac{I_{rr}}{AR^2} + 3\frac{\lambda+\overline{E}}{\overline{E}}\beta_0^2\right) \end{bmatrix}$$

The 4×4 matrix $\mathbf{K}_n$ is most conveniently represented as

$$\mathbf{K}_n = [k_{ij}], \qquad 2k_{ij} = a_{ij}$$

where the elements a_{ij} are

$$a_{11} = 1 + \bar{\epsilon}\left[(n^2-1)^2\bar{I}_{zz} + \bar{p}\,n^2\frac{\lambda+\overline{E}}{\overline{E}}\right] + \frac{G}{\overline{E}}\bar{p}^2n^2\bar{\epsilon}^2 + 2n^2\bar{\epsilon}^2\bar{I}^2_{rz}\left(\frac{\lambda+G}{\overline{E}}f^2 - \bar{\epsilon}\bar{p}f\frac{G}{\overline{E}}\right)$$

$$a_{12} = \bar{\epsilon}n^2\bar{I}_{rz}\left[\frac{n^2-1}{n^2} + \frac{2\lambda}{\overline{E}}\frac{f}{n^2} - \bar{\epsilon}\frac{\lambda+G}{\overline{E}}(\bar{p}-f) + \bar{\epsilon}^2 f\frac{G}{\overline{E}}(\bar{p}\,\bar{I}_z - f\,\bar{I}^2_{rz})\right]$$

$$a_{13} = n\left[1 + \bar{\epsilon}\bar{p}\frac{\lambda+\overline{E}}{\overline{E}} + \frac{G}{\overline{E}}\bar{\epsilon}^2\bar{p}^2 + 2\bar{\epsilon}^2f^2\bar{I}^2_{rz}\frac{\lambda+G}{\overline{E}} - 2\bar{\epsilon}^3\frac{G}{\overline{E}}\bar{I}^2_{rz}f\bar{p}\right]$$

$$a_{14} = n^2\bar{I}_{rz}\bar{\epsilon}\left[n^2 - 1 - \bar{\epsilon}(\bar{p}-f)\frac{\lambda+G}{\overline{E}} + \bar{\epsilon}^2f\frac{G}{\overline{E}}(\bar{p}\,\bar{I}_z - f\,\bar{I}^2_{rz})\right]$$

$$a_{22} = \bar{\epsilon}\left[1 + \frac{2\lambda\bar{p}}{\overline{E}} + n^2\frac{GJ}{\overline{E}I_{rr}} + \bar{\epsilon}n^2\left(\bar{p}\frac{J^*}{I_{rr}} + f\,\bar{I}_{rz}\frac{I^*_{rrz}+I^*_{rrr}}{RI_{rr}}\right) + 3\frac{\lambda+\overline{E}}{\overline{E}}\bar{\epsilon}f^2\bar{I}^2_{rz} + \bar{\epsilon}^2\frac{\lambda J}{\overline{E}I_{rr}}n^2f^2\bar{I}^2_{rz}\right]$$

$$a_{23} = \bar{\epsilon}n\,\bar{I}_{rz}\left[\frac{2\lambda f}{\overline{E}} - \bar{\epsilon}\frac{\lambda+G}{\overline{E}}(\bar{p}-f) + \bar{\epsilon}^2\frac{fG}{\overline{E}}(\bar{p}\,\bar{I}_z - f\,\bar{I}^2_{rz})\right]$$

$$a_{24} = n^2\bar{\epsilon}\left\{1 + \frac{GJ}{\bar{E}I_{rr}} + \bar{\epsilon}\left(\bar{p}\,\bar{I}_z\,\frac{\lambda + G}{\bar{E}} + \bar{p}\,\frac{J^*}{I_{rr}} + f\bar{I}_{rz}\,\frac{I^*_{rrr} - R\bar{I}^*_{rz}}{RI_{rr}} + \frac{G}{\bar{E}}\,f\,\bar{I}^2_{rz}\right)\right.$$

$$\left. + \bar{\epsilon}^2\,\bar{I}^2_{rz}\left[\frac{\lambda}{\bar{E}}\,\frac{Jf^2}{I_{rr}} + \frac{Gf}{\bar{E}}\,(\bar{p} - f)\right]\right\}$$

$$a_{33} = n^2 + \bar{\epsilon}\,\bar{p}\,\frac{\lambda + \bar{E}}{\bar{E}} + 2\bar{\epsilon}^2 f^2\,\bar{I}^2_{rz}\,\frac{\lambda + G}{\bar{E}} + \bar{\epsilon}^2\,\frac{G\bar{p}^2}{\bar{E}} - 2\bar{\epsilon}^3\,\frac{G}{\bar{E}}\,\bar{p}\,f\,\bar{I}^2_{rz}$$

$$a_{34} = n\,\bar{\epsilon}^2\,\bar{I}_{rz}\left[\frac{\lambda + G}{\bar{E}}\,(f - \bar{p}) + \bar{\epsilon}\,\frac{fG}{\bar{E}}\,(\bar{I}_z\,\bar{p} - f\,\bar{I}^2_{rz})\right]$$

$$a_{44} = n^2\,\bar{\epsilon}\left[n^2 + \frac{GJ}{\bar{E}I_{rr}} + \bar{p}\,\frac{\lambda + \bar{E}}{\bar{E}} + \bar{p}\bar{\epsilon}\left(\frac{I^*_{rr}}{I_{rr}} - \frac{2G}{\bar{E}}\,\bar{I}_{zz} + \bar{p}\,\frac{G}{\bar{E}}\right)\right.$$

$$\left. + \bar{\epsilon}\,f\,\bar{I}_{rz}\left(2\,\frac{\lambda + G}{\bar{E}}\,f\,\bar{I}_{rz} - \frac{\lambda\bar{I}_{rz}}{\bar{E}} + \frac{I^*_{rrr} - RI^*_{rz}}{RI_{rr}}\right) + \bar{\epsilon}^2 f^2\,\bar{I}^2_{rz}\left(\frac{\lambda J}{\bar{E}I_{rr}} - \frac{2G}{\bar{E}}\right)\right]$$

where

$$\bar{\epsilon} = \frac{I_{rr}}{AR^2}, \qquad \epsilon = \frac{\bar{\epsilon}\,\bar{I}^2_{rz}}{1 - \nu}$$

$$\bar{I}_z = \frac{I_{zz}}{I_{rr}}, \qquad \bar{I}_{rz} = \frac{I_{rz}}{I_{rr}}$$

and

$$\bar{p} = \frac{U_0}{R}\,\frac{AR^2}{I_{rr}}, \qquad f = \beta_0\,\frac{AR^2}{I_{rz}}, \qquad \epsilon = \frac{\lambda + \bar{E}}{\bar{E}}\,\frac{I^2_{rz}}{AR^2\,I_{rr}}$$

Appendix D

The Work Expression

The work performed by a uniformly distributed radial load during a deviation from the unbuckled state depends upon the orientation of the forces acting on a line segment during the displacement. The most realistic distinction seems to be that between (1) a force that retains both its magnitude and direction, and (2) a force that is constant in magnitude but whose line of action passes through the undeformed center of the ring.

In general, a differential force $d\mathbf{F}$ acting on a line segment $R\,d\theta$ of the line of centroids will perform work dW_e during a displacement $\mathbf{r}_2 - \mathbf{r}_1$ (Fig. D-1) where

$$dW_e = \int_{\mathbf{r}=\mathbf{r}_1}^{\mathbf{r}_2} d\mathbf{F} \cdot d\mathbf{r}$$

For a constant load, $d\mathbf{F}_1 = p_0 R d\theta \, \mathbf{e}_r^{(0)}$

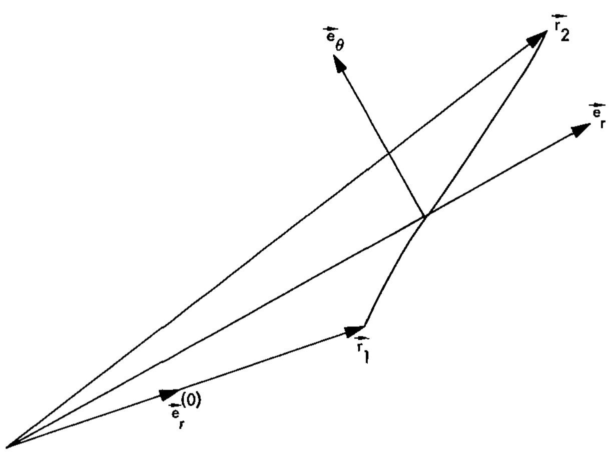

Fig. D-1. Sign convention for work expression

For a centrally directed load, $d\mathbf{F}_2 = p_0 R d\theta \, \mathbf{e}_r$

If $d\mathbf{r} = \mathbf{e}_r dr + \mathbf{e}_\theta \, r d\theta + \mathbf{e}_z \, dz$, the differential work expressions become

$$dW_e^{(1)} = p_0 \, R d\theta \, \mathbf{e}_r^{(0)} \, (\mathbf{r}_2 - \mathbf{r}_1)$$

$$dW_e^{(2)} = p_0 \, R d\theta \, (r_2 - r_1)$$

Furthermore, if $\mathbf{r}_2 - \mathbf{r}_1 = \overline{U}\mathbf{e}_r^{(0)} + \overline{V}\mathbf{e}_\theta^{(0)} + \overline{W}\mathbf{e}_z^{(0)}$, it follows that

$$dW_e^{(1)} = p_0 \, R d\theta \, \overline{U}$$

Also, with $r_2^2 = (r_1 + U)^2 + \overline{V}^2 + \overline{W}^2$,

then

$$r_2 - r_1 = \overline{U} + \frac{1}{2r_1}(\overline{V}^2 + \overline{W}^2) + \cdots$$

and

$$dW_e^{(2)} = p_0 \, R d\theta \left[\overline{U} + \frac{1}{2r_1}(\overline{V}^2 + \overline{W}^2) + \cdots \right]$$

Finally, using the deviation displacement forms of Eq. (3), we find that the total work expressions become

$$W_e^{(1)} = 2\pi p_0 \, R^2 A_0$$

$$W_e^{(2)} = 2\pi p_0 \, R^2 A_0 + \frac{\pi p_0 R^2}{2} \sum_{n=2} (C_n^2 + D_n^2)$$

Nomenclature

A area of cross section

A_n Fourier coefficient of radial deflection

a_{ij} two times the elements of the stiffness matrix

B_{ij} elements of a coefficient matrix

B_n Fourier coefficient of tangential rotation

C_{ij} elements of a coefficient matrix

C_n Fourier coefficient of tangential deflection

D_n Fourier coefficient of axial deflection

E Young's modulus of material

$\bar{E}$ Lamé constant of material plus twice the shear modulus of material

f nondimensional rotation

G shear modulus of material

I_{ij} area moment

$\bar{I}$ dimensionless area moment

J area polar moment

$\mathbf{K}_n$ stiffness matrices

k_{ij} elements of the stiffness matrices

n Fourier wave number

p_0 intensity of distributed radial load

$\bar{p}$ nondimensional radial deflection

$\mathbf{q}_n$ generalized displacement vector

v radial coordinate

R radial coordinate to centroid of section

U radial component of displacement

V tangential component of displacement

W axial component of displacement

z axial coordinate

α constraint parameter

β tangential rotation component

ϵ dimensionless parameter

$\bar{\epsilon}$ slenderness ratio

ζ centroidal axial coordinate

θ tangential coordinate

κ_r radial rotation component

κ_z axial rotation component

λ Lamé constant of material

ν Poisson's ratio

ξ centroidal radial coordinate

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