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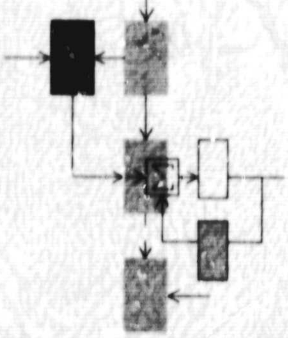

ON THE DESIGN OF A DIGITAL COMPUTER PROGRAM FOR THE DESIGN OF FEEDBACK COMPENSATORS IN TRANSFER FUNCTION FORM

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ABSTRACT

The purpose of this report is to indicate, in an informal manner, the possibility of constructing a digital computer program which can compute automatically the transfer function of a feedback compensator given the transfer function of the open loop plant and the statistical properties of any disturbance and/or measurement noise. The internal computations are based on the solution of the linear stochastic optimal control problem in state variable format. However, the answer is computed in the frequency domain because the feedback compensator is described in a partial fraction (Heavyside) expansion of the transfer function.

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1. Introduction

The design of linear time-invariant compensators for the control of linear systems represents the main problem in conventional servomechanism design. In classical servo design, the tools utilized were those of frequency domain techniques (Bode plots, Nyquist diagrams, Nichol's charts) and those of root locus plots. These time-tested design tools were developed during the period of analog computers and they necessarily did not utilize the capabilities of the digital computer for routine calculations.

On the other hand the theoretical and computational aspects of optimal control and estimation theory are tailored toward the use of the digital computer for designing control systems. This note illustrates that traditional problems in servomechanism design can indeed be attacked using the framework of optimal stochastic control. In addition, it is shown that it is possible to design a digital computer program which can specify the transfer function of the feedback compensator.

It is interesting to note that the proposed digital computer program accepts information about the problem of control in the form of transfer functions and specifies the answer in terms of transfer functions. However, all the internal calculations are carried out using state space concepts. Nonetheless, since the state-space calculations are internal to the program, the algorithm can be used effectively by control engineers that are "allergic" to state variables, optimal control, and Kalman filters.

The material contained in this report is well known to control theorists. However, a conscious effort to explain the potentialities of the theory to the practicing engineer for attacking conventional control problems has not been made. To overcome the communication barrier it is necessary to convince the practicing engineer — who is interested in results and techniques rather than theory — that the available theoretical and computational methods of modern control and estimation theory can be used in a consistent manner for servomechanism problems.

This report attempts to fill the "gap" by pointing out that a digital computer program can be written which "solves" a problem and gives an "answer" in the language that a practicing engineer can understand, namely, that of transfer functions. Furthermore, the proposed digital computer program* can be used to construct "root-locus" type experiments as one varies a parameter related to the gains allowed in the control system. The development of such digital computer programs can indeed free the engineer from time-consuming trial-and-error simulations and dog-work calculations; in this manner, the engineer can concentrate upon the hard part of the job: the modelling of a physical process, the development of reasonable specifications for performance, and the implementation of the control system.

2. The Main Problem

The main problem under consideration is the following. We are given the open-loop transfer function of a linear time-invariant plant G(s). We assume that G(s) is known and that it is given by

$$G(s) = \frac{\beta_{n-1} s^{n-1} + \ldots + \beta_1 s + \beta_o}{s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_1 s + \alpha_o} \tag{1}$$

We assume that:

1) the order, n, of G(s) is known,

2) the real scalars $\alpha_o, \alpha_1, \ldots, \alpha_{n-1}$ are known, and

3) the real scalars $\beta_o, \beta_1, \ldots, \beta_{n-1}$ are known.

Note that the tacit assumption that the plant G(s)** has at least one less zero than the number of poles has been made. This assumption is not critical and it can be relaxed. It is made solely for the purpose of clarity of presentation.

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* A detailed FORTRAN IV realization of all the proposed subroutines is currently under development.

**In general, G(s) will contain the true dynamics of the controlled plant, those of the actuator, and the sensor dynamics.

The control problem which we shall deal with in this note is that of designing a regulator. Perhaps this is best illustrated by examining Figure 1. In this configuration the reference input signal is set equal to zero (hence the term regulator). The plant G(s) is driven by the control u(t) which is corrupted by the additive white gaussian noise ξ (t). It is assumed that the plant output y(t) cannot be measured exactly. This is modelled by adding white gaussian measurement noise θ(t) to the output resulting in the data signal z(t). The problem is to design a single-input (z(t)) single-output (-u(t)) feedback compensator F(s) which accomplishes the required task of regulation, i.e., keeps the output y(t) near zero for all values of t.

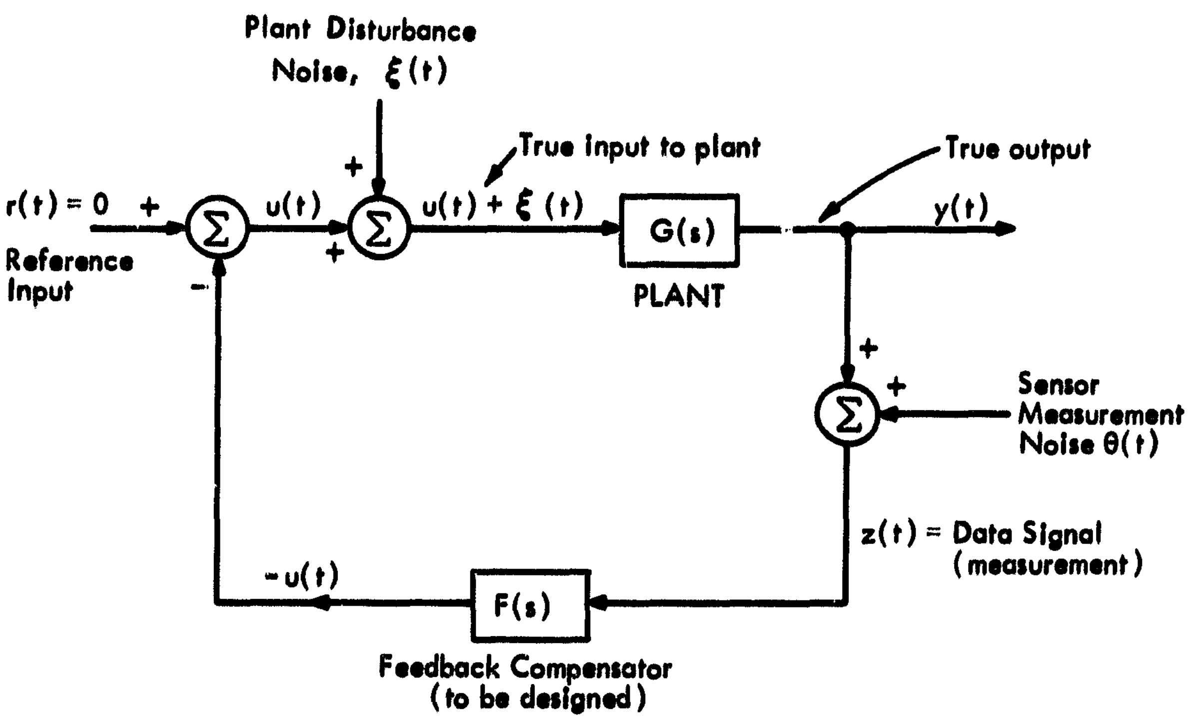

Fig. 1 Block Diagram of Feedback Control System

What we wish to develop is a digital computer program which specifies the transfer function F(s) of the feedback compensator. The procedure which we shall suggest determines the transfer function F(s) in a Heavyside partial function expansion, i.e., F(s) is specified by

$$F(s) = \frac{\rho_1}{s - \mu_1} + \frac{\rho_2}{s - \mu_2} + \dots + \frac{\rho_m}{s - \mu_m} \tag{2}$$

where:

(a) m is the order of F(s),

(b) $\mu_1, \mu_2, \ldots, \mu_m$ are the poles (real or complex) of F(s),

(c) $\rho_1, \rho_2, \ldots, \rho_m$ are the residues (real or complex) at the poles of F(s).

Thus the digital computer program will compute the scalars m, $\mu_1, \mu_2, \ldots, \mu_m$, and $\rho_1, \rho_2, \ldots, \rho_m$. If desired, it can print F(s) as indicated in Eq. (2).

We shall approach the problem by demanding that the feedback compensator F(s) be optimal (in a precise sense to be defined later on); clearly, the compensator F(s) will depend not only on the performance index but also on the plant dynamics G(s), as well as the statistical properties of the plant disturbance noise ξ (t) and the sensor (measurement) noise $\theta(t)$.

Loosely speaking, we desire to construct a digital computer program of the type illustrated in Figure 2 which accepts the available data and "prints out" F(s). Hence, we must specify the detailed internal computations to be performed by the digital computer.

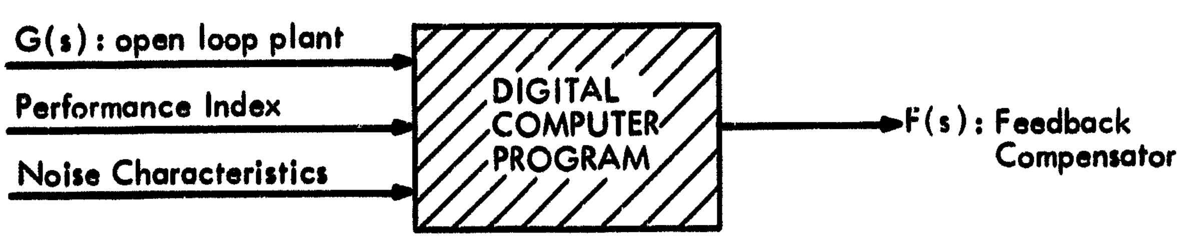

Fig. 2 Structure of the Digital Computer Program to be Designed

3. The Index of Performance

The criterion of goodness which we shall use to define optimality is a quadratic one and is given by

$$J = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} [q\,y^2(t) + u^2(t)]\,dt \qquad (3)$$

In the above integral, q is a positive real weighting constant. Thus, the numerical specification of q defines completely the index of performance which is to be used. The value of q will affect the coefficients of the feedback compensator F(s). Hence, q will be an "input" to the digital computer program of Figure 2. The variation of q will yield root locus type plots as we shall see later on.

4. Noise Statistics

It is reasonable to expect that the feedback compensator F(s) will depend upon the level of the two noise processes $\xi(t)$ and $\theta(t)$ of Figure 1, because a good "regulator" must filter out the effects of noise in the system. For this reason, it is necessary to specify the noise characteristics.

Plant Disturbance Noise: We assume that $\xi(t)$ is a stationary, ergodic, gaussian white noise process with zero mean, i.e.,

$$E\{\xi(t)\} = 0 \qquad \text{for all } t \qquad (4)$$

and known variance

$$E\{\xi(t)\xi(\tau)\} = \Xi\,\delta(t-\tau)\ ;\ \Xi > 0 \qquad (5)$$

We remark that $\xi(t)$ is used to model the following types of uncertainty:

1) errors in modelling the plant by G(s), including the effects of parameter variations of G(s) about their nominal values;

2) errors, due to actuators, in the translation of the control signal u(t) into physical signals;

3) additional disturbance forces acting on the plant.

Sensor Noise: We assume that $\theta(t)$ is a stationary, ergodic, gaussian white noise process with zero mean, i.e.,

$$E\{\theta(t)\} = 0 \qquad \text{for all } t \qquad (6)$$

and known variance

$$E\{\theta(t)\theta(\tau)\} = \Theta\,\delta(t-\tau)\ ;\ \Theta > 0 \qquad (7)$$

Furthermore, we assume that $\xi(t)$ and $\theta(t)$ are statistically independent (uncorrelated) for all t, i.e.,

$$E\{\xi(t)\theta(\tau)\} = 0 \qquad \text{for all } t \text{ and } \tau \tag{8}$$

We remark that $\theta(t)$ is used to model the following types of uncertainty:

1) errors in measuring $y(t)$ by sensing instruments;

2) errors in the realization of $F(s)$ using physical components;

3) additional unavoidable disturbances acting on the compensator $F(s)$.

5. Plant Characteristics

We have indicated that the plant $G(s)$ is a linear time-invariant one with the known transfer function (1). In addition, we shall assume that $G(s)$ does not have any common poles and zeroes (or that $G(s)$ is completely controllable and observable).

6. Further Refinements on the Computer Program

Using the above assumptions one can be more specific about the computer program of Figure 2 in the sense of specifying the numbers which are introduced into it. The required input data is illustrated in Figure 3.

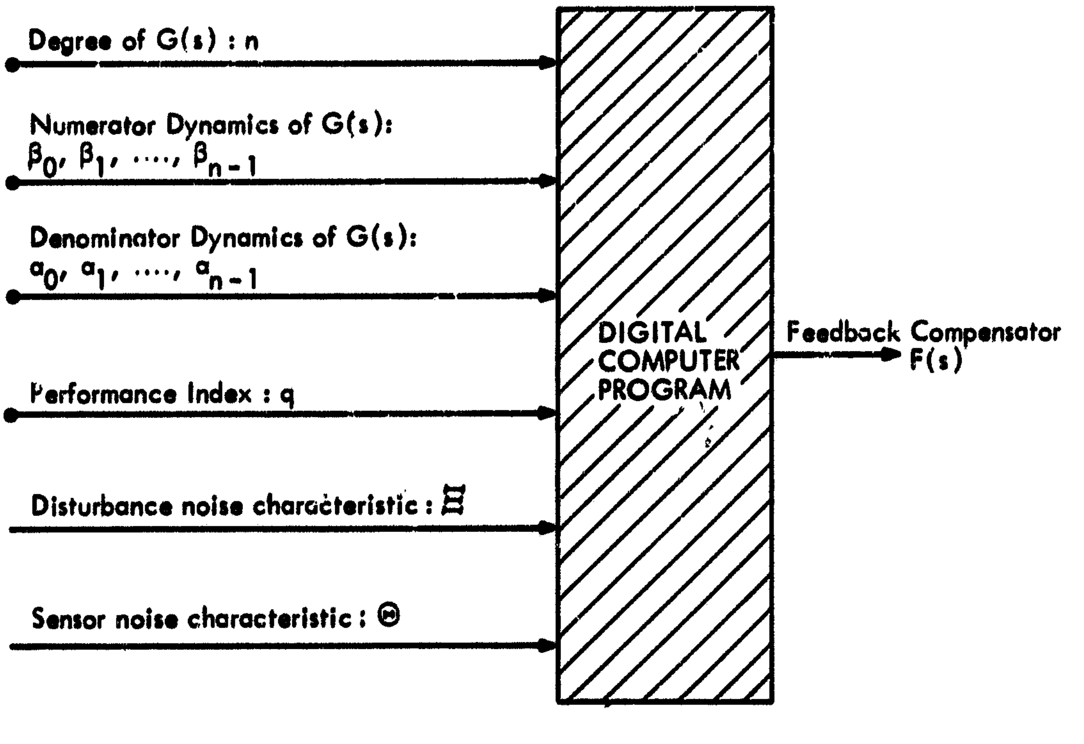

Fig. 3 More Detailed Specification of the Input Data to the Digital Computer Program

Thus, the input data to the digital computer program involves the specification of $4 + 2(n-1)$ numbers, where n is the order of $G(s)$.

7. Internal Structure of the Computer Program

To understand the equations that the digital computer program must solve, we shall now present the steps involved and the types of theoretical results which are utilized.

Step 1: The computer transforms the ransfer function $G(s)$ into the so-called "standard c llable" state variable representation.

Step 2: The computer evaluates the solution to a deterministic optimal control problem. This involves the determination of the "steady state" solution of a matrix Riccati differential equation.

Step 3: The computer evaluates the "steady state" Kalman-Bucy filter for the problem and uses the "separation theorem" to construct the optimal control. Once more this involves the determination of the "steady state" solution of a matrix Riccati differential equation. At this step the feedback compensator, $F(s)$, has been obtained in state variable form.

Step 4: The computer transforms the compensator state variable description into its transfer function equivalent, $F(s)$.

The detailed equations that must be solved are presented in the remainder of this note. Comments on additional options that can be attached to this basic algorithm to make it even more useful for computer-aided design will also be included.

8. From the Transfer Function G(s) to the Standard Controllable State Variable Representation

In this section we describe in detail how one transforms the given plant transfer function $G(s)$

$$G(s) = \frac{\beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \ldots + \beta_1 s + \beta_o}{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_o} \tag{9}$$

into the standard controllable state variable representation

$$\left.\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}\,\underline{x}(t) + \underline{b}\,u(t) && (10) \\ y(t) &= \underline{c}'\,\underline{x}(t) && (11) \end{aligned}\right\}$$

In Eqs. (10), (11), $\underline{A}$ is an nxn constant matrix, $\underline{b}$ is a constant vector with n components, and $\underline{c}$ is a constant vector with n components. Note that n is the order (= number of poles) of G(s).

The standard controllable representation for G(s) is specified completely as follows:

Subroutine 1A Given the number n and the coefficients of the denominator polynomial of G(s), $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, the elements a_{ij}, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$ are constructed as follows:

$$a_{ij} = \begin{cases} 0 & \text{for } i = 1,2,\ldots,n-1 \text{ and } j \neq i+1 \\ 1 & \text{for } i = 1,2,\ldots,n-1 \text{ and } j = i+1 \\ -\alpha_0 & \text{for } i = n,\ j = 1 \\ -\alpha_1 & \text{for } i = n,\ j = 2 \\ \cdots & \\ -\alpha_{n-1} & \text{for } i = n,\ j = n \end{cases} \tag{12}$$

Thus, the $\underline{A}$ matrix is given by

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \ldots & -\alpha_{n-1} \end{bmatrix} \tag{13}$$

Subroutine 1B The components $b_1, \ldots, b_n$ of the column vector $\underline{b}$ are computed by

$$b_i = \begin{cases} 0 & \text{for } i = 1,2,\ldots,n-1 \\ 1 & \text{for } i = n \end{cases} \tag{14}$$

Thus, $\underline{b}$ is given by

$$\underline{b} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix} \tag{15}$$

<u>Subroutine 1C</u> The components $c_1, c_2, \ldots, c_n$ of the vector $\underline{c}$ are given by

$$\begin{aligned} c_1 &= \beta_0 \\ c_2 &= \beta_1 \\ c_3 &= \beta_2 \\ &\cdots\cdots \\ c_n &= \beta_{n-1} \end{aligned} \tag{16}$$

Thus, the row vector $\underline{c}'$ is

$$\underline{c}' = [\beta_{n-1}, \ldots, \beta_1, \beta_0] \tag{17}$$

The structure of the overall subprogram required for the evaluation of the matrix $\underline{A}$ and the vectors $\underline{b}$ and $\underline{c}$ is shown in Figure 4.

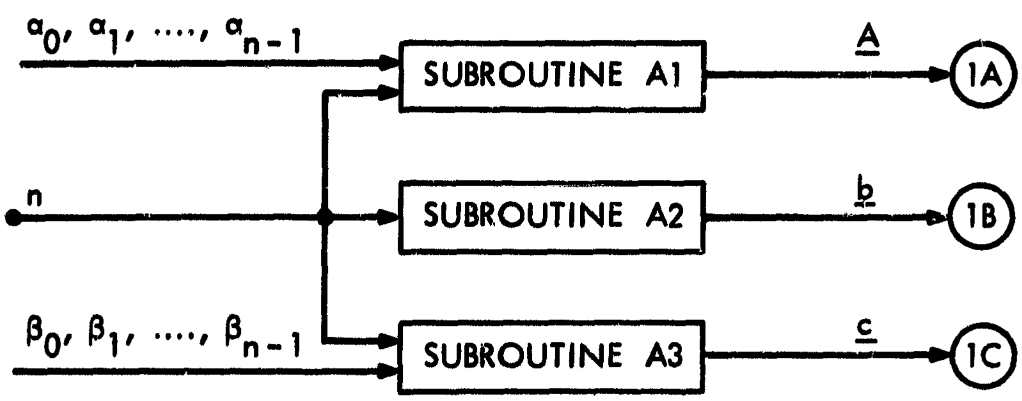

Fig. 4 Transformation of G(s) Parameters to $\underline{A}$, $\underline{b}$, $\underline{c}$ Matrix and Vectors

9. The Riccati Subroutine

A key subroutine in the overall digital computer program is one which computes a positive definite matrix solution of a matrix Riccati algebraic equation. The best description of this subroutine is as follows: consider the algebraic matrix equation

$$\underline{O} = \underline{X}\,\underline{M}_1 + \underline{M}'_1\underline{X} - \underline{X}\,\underline{M}_2\underline{X} + \underline{M}_3 \tag{18}$$

where $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ are known constant nxn matrices and $\underline{X}$ is a constant nxn matrix (the unknown). If, in addition,

$$\underline{M}_2 = \underline{M}'_2 = \text{positive semidefinite} \tag{19}$$

$$\underline{M}_3 = \underline{M}'_3 = \text{positive semidefinite} \tag{20}$$

then it is possible to compute (see references (1) to (4)) the unique positive definite symmetric solution of (18). For this reason, we shall visualize the algorithm as shown in Figure 5.

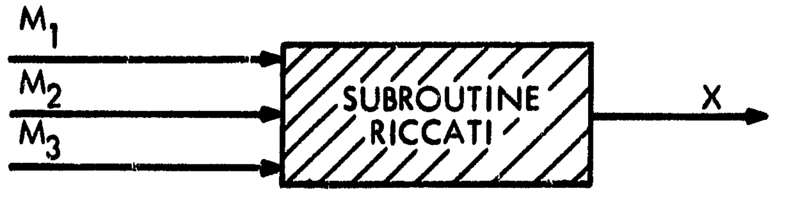

Fig. 5 Input-Output Characteristics of Subroutine Riccati

10. The Solution of the Deterministic Linear Regulator Problem

Once the state variable representation of the plant has been obtained (i.e., $\underline{A}$, $\underline{b}$, $\underline{c}$ have been computed), and the cost functional (2) has been specified by means of the positive constant q, then the computer can carry out the following matrix computations

1) Find the nxn solution matrix $\hat{\underline{K}}$ of the matrix Riccati equation

$$\underline{K}\,\underline{A} + \underline{A}'\underline{K} - \underline{K}\,\underline{b}\,\underline{b}'\,\underline{K} + q\,\underline{c}\,\underline{c}' = \underline{0} \tag{21}$$

2) Find the column vector $\underline{g}$ given by

$$\underline{g}' = \underline{b}'\hat{\underline{K}} \tag{22}$$

or, since $\hat{\underline{K}}$ is a symmetric positive definite matrix,

$$\underline{g} = \hat{\underline{K}}\,\underline{b} \tag{23}$$

Figure 6 illustrates how this is to be accomplished using the Riccati subroutine.

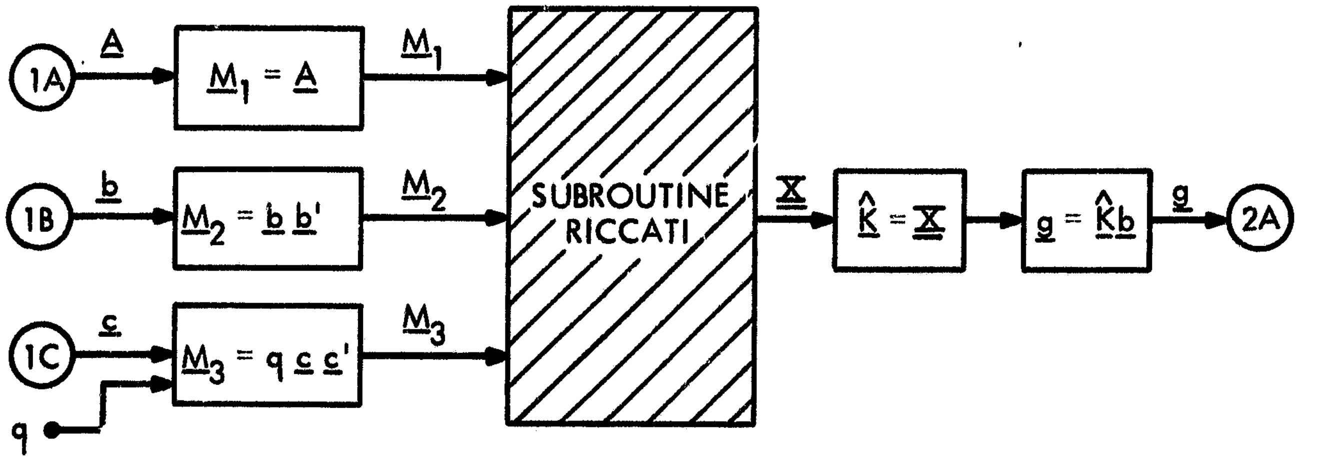

Fig. 6 Computation of Vector $\underline{g}$ Given $\underline{A}$, $\underline{b}$, $\underline{c}$ and q

11. Construction of the Steady State Compensator in State Variable Form

It is possible to generate continuously in time estimates of the state vector $\underline{x}(t)$ from the observed data $z(t)$. For any input $u(t)$, the steady state Kalman-Bucy filter will indeed generate a state estimate $\hat{\underline{x}}(t)$ of $\underline{x}(t)$ according to the following vector differential equation

$$\frac{d}{dt}\hat{\underline{x}}(t) = \left[\underline{A} - \frac{1}{\Theta}\hat{\underline{\Sigma}}\,\underline{c}\,\underline{c}'\right]\hat{\underline{x}}(t) + \underline{b}\,u(t) + \frac{1}{\Theta}\hat{\underline{\Sigma}}\,\underline{c}\,z(t) \tag{24}$$

However, for optimal stochastic control the separation theorem requires that

$$u(t) = -\underline{g}'\hat{\underline{x}}(t) \tag{25}$$

where $\hat{\underline{x}}(t)$ is the Kalman-Bucy filter state estimate and $\underline{g}$ is the vector defined by Eq. (23). Substituting (25) into (24) we obtain

$$\left.\begin{aligned} \frac{d}{dt}\hat{\underline{x}}(t) &= \left[\underline{A} - \frac{1}{\Theta}\hat{\underline{\Sigma}}\,\underline{c}\,\underline{c}' - \underline{b}\,\underline{g}'\right]\hat{\underline{x}}(t) + \frac{1}{\Theta}\hat{\underline{\Sigma}}\,\underline{c}\,z(t) & (26)\\ -u(t) &= \underline{g}'\hat{\underline{x}}(t) & (27) \end{aligned}\right\}$$

In the above equations the nxn symmetric and positive definite matrix $\hat{\underline{\Sigma}}$ is the solution of the matrix Riccati algebraic equation

$$\underline{\Sigma}\,\underline{A} + \underline{A}'\underline{\Sigma} - \frac{1}{\Theta}\underline{\Sigma}\,\underline{c}\,\underline{c}'\underline{\Sigma} + \Xi\,\underline{b}\,\underline{b}' = \underline{0} \tag{28}$$

Let

$$\tilde{\underline{A}} \triangleq \underline{A} - \frac{1}{\Theta}\hat{\underline{\Sigma}}\,\underline{c}\,\underline{c}' - \underline{b}\,\underline{g}' \tag{29}$$

$$\tilde{\underline{b}} \triangleq \frac{1}{\Theta}\hat{\underline{\Sigma}}\,\underline{c} \tag{30}$$

$$\tilde{\underline{c}} \triangleq \underline{g} \tag{31}$$

Using this notation, eqs. (26) and (27) take the form

$$\left.\begin{aligned} \frac{d}{dt}\hat{\underline{x}}(t) &= \tilde{\underline{A}}\,\hat{\underline{x}}(t) + \tilde{\underline{b}}\,z(t)\\ -u(t) &= \tilde{\underline{c}}'\hat{\underline{x}}(t) \end{aligned}\right\} \tag{32}$$

which is the state variable representation of the sought-for compensator F(s) — see Figure 1 — whose input is the (data) signal $z(t)$ and its output is the (negative control) signal $-u(t)$. The computation of the matrix $\tilde{\underline{A}}$ and of the vectors $\tilde{\underline{b}}$ and $\tilde{\underline{c}}$ is shown in Figure 7.

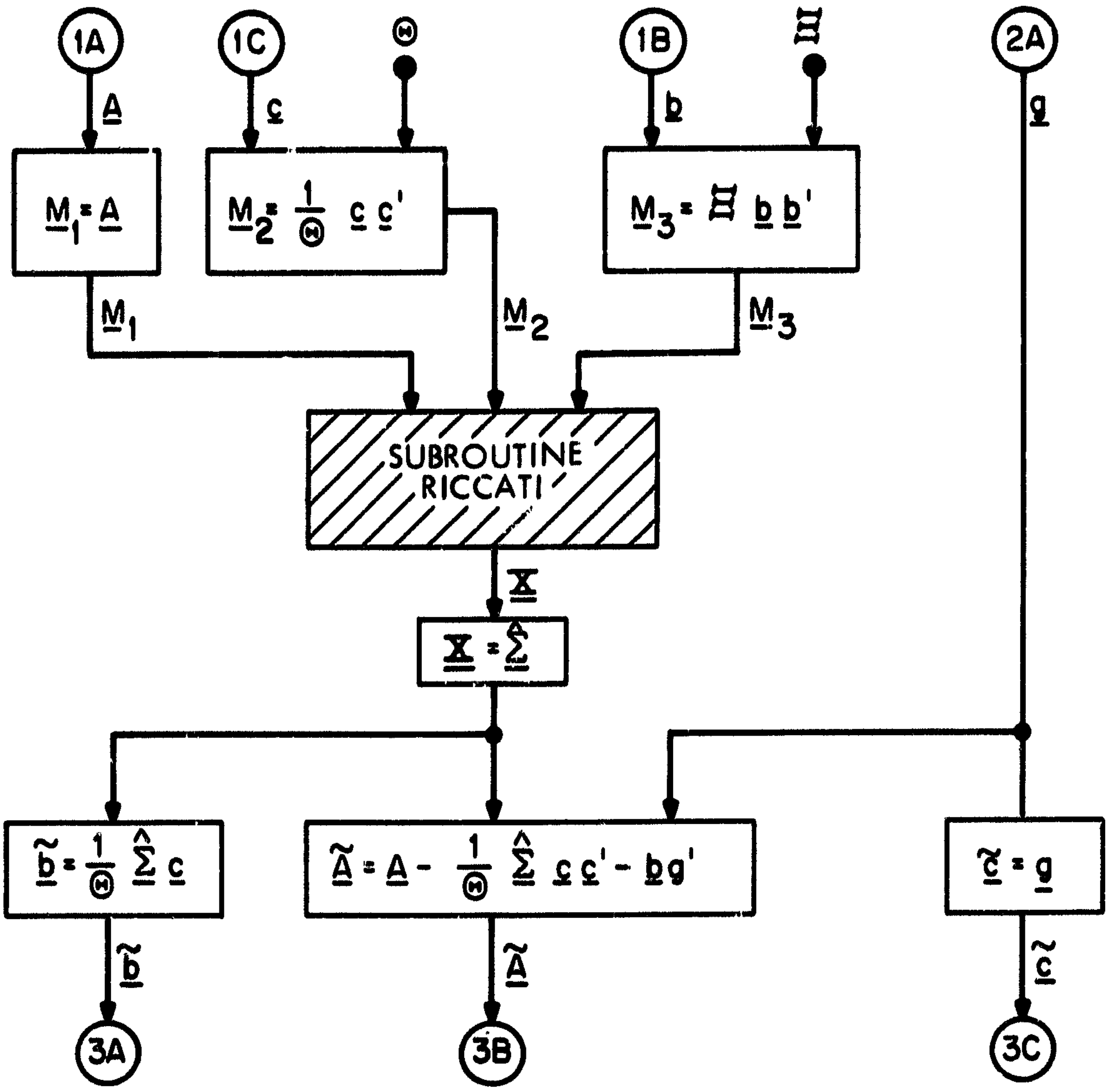

Fig. 7 Computation of $\tilde{A}$, $\tilde{b}$, and $\tilde{c}$ Given A, b, c, g and the Noise Variances Θ and Ξ

12. From the State Variable to Transfer Function Description of the Compensator

The specification of the matrix $\underline{\tilde{A}}$ and of the column vectors $\underline{\tilde{b}}$ and $\underline{\tilde{c}}$ specifies the compensator F(s) in state variable form according to Equation (32). The next step is to translate this information into an input-output description.

In general starting with the state variable description of the compensator

$$\frac{d}{dt}\hat{\underline{x}}(t) = \underline{\tilde{A}}\hat{\underline{x}}(t) + \underline{\tilde{b}}\, z(t)\ ; \quad -u(t) = \underline{\tilde{c}}'\hat{\underline{x}}(t) \tag{33}$$

one can readily write down the transfer function

$$F(s) = \frac{\mathcal{L}\{-u(t)\}}{\mathcal{L}\{z(t)\}} \tag{34}$$

as follows

$$F(s) = \underline{\tilde{c}}'(s\underline{I} - \underline{\tilde{A}})^{-1}\underline{\tilde{b}} \tag{35}$$

where "s" is the complex frequency domain variable. In order for the computer to evaluate F(s) directly it must carry out the <u>nonnumerical</u> inversion of the matrix $(s\underline{I} - \underline{\tilde{A}})$. It is possible to do that* directly although it is a nontrivial programming task.

An alternate way of computing F(s) is to carry out a similarity transformation on the system (33). This requires the evaluation of the eigenvalues and eigenvectors of $\underline{\tilde{A}}$, a task which can be performed using standard computer library subroutines. We shall briefly outline this procedure.

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the nxn matrix $\underline{\tilde{A}}$ and let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ be the associated eigenvectors. Let $\underline{M}$

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* Such a digital computer program is currently under investigation.

$$\underline{M} = \begin{bmatrix} \mu_1 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mu_n \end{bmatrix} \tag{36}$$

be the diagonal matrix of the eigenvalues of $\underline{\tilde{A}}$. Since the eigenvalues and eigenvectors are related by

$$\underline{\tilde{A}}\,\underline{v}_k = \mu_k \underline{v}_k \quad ; \quad k = 1, 2, \dots, n \tag{37}$$

it follows that

$$\underline{\tilde{A}}\,\underline{V} = \underline{V}\,\underline{M} \tag{38}$$

where the nxn matrix $\underline{V}$ is constructed by letting its column vectors be the computed eigenvectors, i.e.,

$$\underline{V} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \tag{39}$$

From (26) we obtain

$$\underline{M} = \underline{V}^{-1}\underline{\tilde{A}}\,\underline{V} \tag{40}$$

Now let $\underline{w}(t)$ be a column vector related to $\hat{\underline{x}}(t)$ by

$$\underline{w}(t) = \underline{V}^{-1}\hat{\underline{x}}(t) \tag{41}$$

or

$$\hat{\underline{x}}(t) = \underline{V}\,\underline{w}(t) \tag{42}$$

From Eqs. (33) and (36) to (42) we conclude that

$$\left.\begin{aligned} \frac{d}{dt}\underline{w}(t) &= \underline{M}\,\underline{w}(t) + \underline{V}^{-1}\underline{\tilde{b}}\,z(t) \\ -u(t) &= \underline{\tilde{c}}'\,\underline{V}\,\underline{w}(t) \end{aligned}\right\} \tag{43}$$

If we let

$$\underline{q} \triangleq \underline{V}^{-1}\underline{\tilde{b}} \tag{44}$$

$$\underline{p}' = \underline{\tilde{c}}'\,\underline{V} \quad \text{or} \quad \underline{p} = \underline{V}'\underline{\tilde{c}} \tag{45}$$

then Eq. (43) reduces to

$$\left.\begin{aligned} \frac{d}{dt}\underline{w}(t) &= \underline{M}\,\underline{w}(t) + \underline{q}\,z(t) \\ -u(t) &= \underline{p}'\underline{w}(t) \end{aligned}\right\} \tag{46}$$

Now the transfer function of the compensator F(s) is

$$F(s) = \underline{p}'\,(s\underline{I} - \underline{M})^{-1}\,\underline{q} \tag{47}$$

However, the fact that $(s\underline{I} - \underline{M})$ is now a diagonal matrix, namely

$$s\underline{I} - \underline{M} = \begin{bmatrix} s-\mu_1 & 0 & \dots & 0 \\ 0 & s-\mu_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s-\mu_n \end{bmatrix} \tag{48}$$

can be utilized to compute its inverse analytically

$$(s\underline{I} - \underline{M})^{-1} = \begin{bmatrix} \frac{1}{s-\mu_1} & 0 & \dots & 0 \\ 0 & \frac{1}{s-\mu_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{s-\mu_n} \end{bmatrix} \tag{49}$$

The transfer function F(s) can now be evaluated analytically

$$F(s) = \sum_{k=1}^{n} \frac{q_k p_k}{s-\mu_k} = \sum_{k=1}^{n} \frac{\rho_k}{s-\mu_k} \tag{50}$$

where $q_1, q_2, \dots, q_n$ are the components of the vector $\underline{q}$ and $p_1, p_2, \dots, p_n$ are the components of the vector $\underline{p}$, and where

$$\rho_k \triangleq q_k p_k \qquad k = 1, 2, \dots, n \tag{51}$$

We note that

(a) $\mu_1, \mu_2, \dots, \mu_n$ are the poles of the compensator F(s).

(b) $\rho_1, \rho_2, \dots, \rho_n$ are the residues at the corresponding poles of F(s).

(c) $F(s)$ according to Eq. (50) is specified, as desired, in a partial fraction expansion.

This reasoning can now be used to construct a digital computer program, as shown in Figure 8, which computes the poles μ_k and the residues ρ_k of the compensator $F(s)$.

13. Discussion

It should be evident that one can make three types of "root locus" studies upon the poles $\mu_1, \mu_2, \ldots, \mu_n$ of the feedback compensator. These can be carried out by the digital computer program by

a) Varying Ξ (the variance of the plant disturbance noise) from 0 to $+\infty$ while holding Θ and q constant. The resultant changes in the poles of $F(s)$ will provide the engineer with an idea of the dependence (or sensitivity) of $F(s)$ upon the magnitude of the input noise power.

b) Varying Θ (the variance of the sensor measurement noise) from ϵ, $\epsilon > 0$, to $+\infty$ while holding Ξ and q constant. The resultant changes in the poles of $F(s)$ will enable the engineer to deduce the effects of the measurement accuracy upon the compensator.

c) Varying q (the weighting constant in the performance index (2)) from ϵ, $\epsilon > 0$, to $+\infty$. The effect of increasing q is to introduce more gain in the system and, in essence, decrease the time constants of the closed loop system. In other words, the larger the q the faster the system nulls out any nonzero values of the output.

If in addition on monitors the residues $\rho_1, \rho_2, \ldots, \rho_n$ of the compensator $F(s)$, one can use this information to perhaps simplify the compensator. For example, suppose that we deal with a 5th order system so that

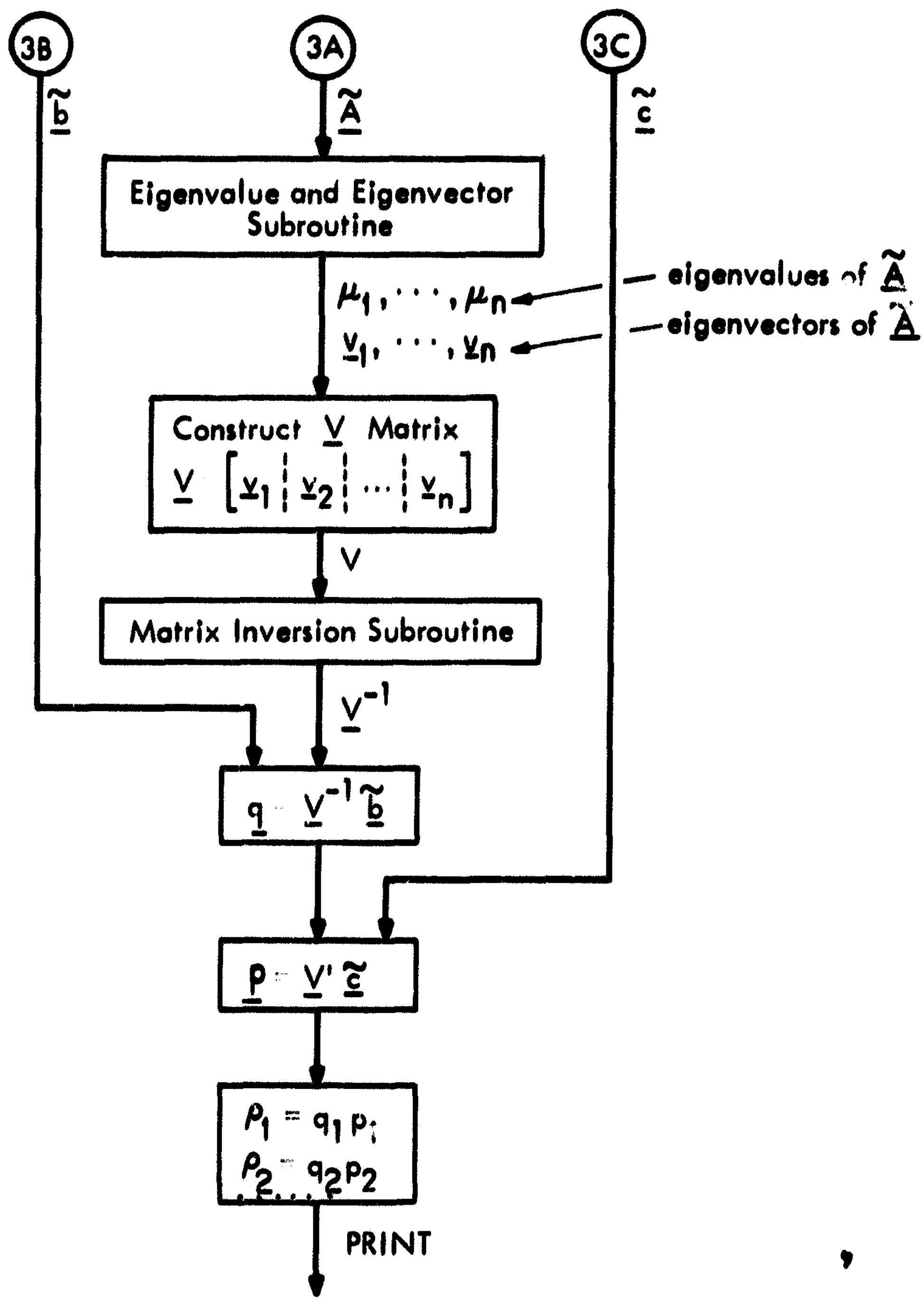
$$F(s) = \frac{\rho_1}{s-\mu_1} + \frac{\rho_2}{s-\mu_2} + \frac{\rho_3}{s-\mu_3} + \frac{\rho_4}{s-\mu_4} + \frac{\rho_5}{s-\mu_5}$$

and suppose that

$$|\rho_1| \ll |\rho_3| < |\rho_2| < |\rho_5|$$

and

$$|\rho_4| \ll |\rho_3| < |\rho_2| < |\rho_5|$$



$$F(s) = \frac{\rho_1}{s - \mu_1} + \cdots\cdots\cdots + \frac{\rho_n}{s - \mu_n}$$

Fig. 8 Computation of the Compensator F(s) in Partial Fraction Expansion Form From the $\tilde{A}$, $\tilde{b}$, and $\tilde{c}$ Matrices

This implies that if we use the simpler compensator

$$F(s) = \frac{\rho_2}{s - \mu_2} + \frac{\rho_3}{s - \mu_3} + \frac{\rho_5}{s - \mu_5}$$

we may expect a good (but suboptimal) performance of our closed loop system. Thus this method of computing the truly optimal solution may also be used as a guide for simplified designs.

REFERENCES

1. Athans, M. and Levine, W. S., "On the Numerical Solution of the Matrix Riccati Differential Equation Using a Runge-Kutta Scheme", M.I.T. Electronic Systems Laboratory Report, ESL-R-276, July, 1966.

2. Levis, A. H., and Athans, M., "Sampled Data Control of High-Speed Trains", M.I.T. Electronic Systems Laboratory Report ESL -R-339, January, 1968.

3. Kalman, R. E., and Englar, T., "ASP-The Automatic Synthesis Program (Program C)", NASA Contractor Report CR-475, June, 1966.

4. Potter, J. E., "Matrix Quadratic Solutions", SIAM Journal of Applied Mathematics, Vol. 14, No. 3, May, 1966, pp. 496-501.

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13. ABSTRACT

The purpose of this report is to indicate, in an informal manner, the possibility of constructing a digital computer program which can compute automatically the transfer function of a feedback compensator given the transfer function of the open loop plant and the statistical properties of any disturbance and/or measurement noise. The internal computations are based on the solution of the linear stochastic optimal control problem in state variable format. However, the answer is computed in the frequency domain because the feedback compensator is described in a partial fraction (Heavyside) expansion of the transfer function.

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