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ANALYSIS BY CONFORMAL MAPPING OF STEADY FROZEN-LAYER CONFIGURATION INSIDE COLD RECTANGULAR CHANNELS CONTAINING WARM FLOWING LIQUIDS

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16. Abstract
A channel containing warm flowing liquid is cooled so that its walls are below the liquid freezing temperature. A frozen region forms in a shape such that the energy convected to the frozen interface by the warm liquid can be conducted through the frozen material to the walls. The shape of the frozen interface is determined by representing the frozen region by a rectangle in a potential plane, and this rectangle is mapped conformally into the physical plane. Either of two frozen regions can be obtained mathematically for a given value of the governing cooling parameter in a certain range, but one is physically unstable. For larger cooling than in this range, the channel will freeze completely.

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SUMMARY

Conformal mapping was applied to determine the shape of the solidified region that forms inside a cooled rectangular channel containing flowing liquid. The channel walls are at a uniform temperature below the freezing temperature of the liquid. The liquid supplies energy by convection to the frozen region interface, and the shape of the interface adjusts so that this energy can be conducted through the region to the cold walls. The interface shape is obtained by mapping the solidified region into a rectangle in the potential plane and then determining the conformal transformations between the potential and physical planes. For a given aspect ratio of the duct, the configuration of the frozen region and the heat flow through it depend on a single physical parameter involving the liquid heat-transfer coefficient, wall and liquid temperatures, and the freezing temperature. For a given value of this physical parameter in a certain range, it was mathematically found that there can be two shapes of the frozen region, although only one of these shapes would be physically stable. When the cooling parameter is above a certain value for each aspect ratio, the duct will freeze completely.

INTRODUCTION

The aim of this work is to study the two-dimensional solidification of warm liquids that flow through cooling channels of heat exchangers. Some compact heat exchangers utilize straight flow passages of a rectangular cross section, and in certain applications when warm liquids are being cooled, the conditions of temperature and flow can cause the channel walls to cool below the freezing temperature of the liquid. Consequently, some of the liquid freezes to form a frozen region between the cold walls and the flowing liquid. This provides an insulating region that reduces the amount of energy that can be

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transferred from the warm liquid to the walls. The heat flow through the frozen region must be computed to evaluate the system performance.

Little work has been done previously on freezing inside two-dimensional geometries. In reference 1, transient freezing inside a square prism was analyzed. The liquid was at the fusion temperature so that no convective energy was being supplied to the frozen interface. However, for the transient situation, energy is being released at the interface as a result of the heat of fusion, and this energy must be conducted away through the frozen region. Several references (refs. 2 to 5) present analyses of solidification in circular tubes or the gap between parallel plates, which are treated as one-dimensional situations at each axial location. The behavior in a circular tube is used later in the section RESULTS AND DISCUSSION to aid in the discussion of the present results.

The heat conduction solution in a two-dimensional region with known boundaries would ordinarily not present a great difficulty because numerical solutions of Laplace's equation can be utilized if an analytical solution is not available. In the present situation, however, the shape of the frozen region is unknown. The shape adjusts so that the energy supplied by convection from the flowing liquid to the frozen interface can pass by conduction through the solid to the cold walls. Thus, the frozen shape must be found to compute the heat being transferred.

The shape of the frozen region and the heat transfer through it is found by applying a conformal mapping technique that was developed in reference 6. The conformal mapping technique is feasible because the boundaries of the frozen region are at constant temperatures. The liquid - frozen-layer interface is at the equilibrium freezing point while the channel walls are at a specified constant temperature below the freezing point. As a result, the frozen region can be represented as a rectangle in the potential plane. The rectangle can be mapped conformally into the physical plane to determine the frozen configuration.

In a heat exchanger passage, the liquid temperature would be changing as it flows along the passage length. The present analysis is two-dimensional and, hence, does not include effects such as axial conduction. Usually, in a channel, the heat conduction in the axial direction is small compared with that transverse to the fluid flow direction. Hence, the present results can be applied locally along the channel length to compute the frozen-layer development in a channel.

SYMBOLS

A dimensionless half-width, a/γ

a half-width of long side of channel

B dimensionless half-width, b/γ

b	half-width of short side of channel
C_1, C_2, C_3	constants
c, d	intermediate parameters in mapping
F	elliptic integral of the first kind, $F(\varphi, k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}$
h	heat-transfer coefficient
K	complete elliptic integral of the first kind, $K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}$
k	thermal conductivity of solidified material
N	dimensionless outward normal distance, n/γ
n	outward normal distance from interface
Q	heat flow rate through frozen region per unit length of channel
q	heat flow per unit area and time
r	radius
s	length of frozen interface
T	dimensionless temperature, $(t - t_w)/(t_f - t_w)$
t	temperature; dummy integration variable
u	mapping variable, $\xi + i\eta$
W	analytic function, $-T + i\psi$
X, Y	dimensionless coordinates, x/γ, y/γ
x, y	Cartesian coordinates in physical plane
Z	dimensionless physical plane, $X + iY$
γ	length scale parameter, $\frac{k}{h_l} \frac{t_f - t_w}{t_l - t_f}$
ζ	quantity defined as $-\frac{\partial T}{\partial X} + i \frac{\partial T}{\partial Y}$
ξ, η	coordinates of u plane

θ argument of ζ

ψ imaginary part of W

ω complex quantity, $\ln|\zeta| + i\theta$

Subscripts:

f at freezing point

i inner

l liquid

o outer

s at frozen interface

w wall

ANALYSIS

Solution for General Case

The model of a typical cross section of a rectangular flow channel in which a steady-state frozen layer has formed is shown in figure 1. The warm liquid is at a constant bulk temperature t_l, and the local heat-transfer coefficient around the interface h_l is assumed constant. The channel walls are assumed to be held at a constant temperature t_w, which is below the liquid freezing temperature t_f. The warm liquid is supplying the heat flux $h_l(t_l - t_f)$ to the liquid - frozen-layer interface. The coordinates x_s and y_s of the interface are unknown and are dependent on the duct aspect ratio a/b, the freezing temperature t_f, the thermal conductivity k of the frozen material, and the parameters h_l, t_l, and t_w. The objectives of this analysis are to determine the coordinates x_s and y_s of the interface and the heat transfer Q to the channel walls per unit channel length.

Basically, the solution to the problem involves solving Laplace's heat conduction equation in the frozen region subject to the known boundary conditions. That is, in the frozen region

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0 \qquad (1)$$

On the interface (x_s, y_s),

$$\left.\begin{aligned} k \left.\frac{\partial t}{\partial n}\right|_{x_s, y_s} &= h_l(t_l - t_f) \\ t(x_s, y_s) &= t_f \end{aligned}\right\} \tag{2}$$

On the channel walls,

$$t = t_w = \text{constant} \tag{3}$$

Because of symmetry, only one quadrant of the channel cross section need be considered (fig. 2(a)). The symmetry provides the two conditions

$$\left.\begin{aligned} \left.\frac{\partial t}{\partial x}\right|_{x=a} &= 0 \\ \left.\frac{\partial t}{\partial y}\right|_{y=b} &= 0 \end{aligned}\right\} \tag{4}$$

The equations are now normalized by letting

$$T \equiv \frac{t - t_w}{t_f - t_w} \qquad N \equiv \frac{n}{\gamma}$$

$$X \equiv \frac{x}{\gamma} \qquad A \equiv \frac{a}{\gamma}$$

$$Y \equiv \frac{y}{\gamma} \qquad B \equiv \frac{b}{\gamma}$$

where

$$\gamma \equiv \frac{k(t_f - t_w)}{h_l(t_l - t_f)}$$

The normalized Laplace equation and boundary conditions take the form

$$\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = 0 \tag{5}$$

On the interface (X_s, Y_s) (fig. 2(b)),

$$\left.\begin{aligned} \left.\frac{\partial T}{\partial N}\right|_{X_s, Y_s} &= 1 \\ T(X_s, Y_s) &= 1 \end{aligned}\right\} \tag{6}$$

On the walls at $X = 0$, $0 \leq Y \leq B$ and at $0 \leq X \leq A$, $Y = 0$,

$$T = 0 \tag{7}$$

On the symmetry lines,

$$\left.\begin{aligned} \left.\frac{\partial T}{\partial X}\right|_{X=A} &= 0 \\ \left.\frac{\partial T}{\partial Y}\right|_{Y=B} &= 0 \end{aligned}\right\} \tag{8}$$

The conformal mapping method developed in reference 6 is applied herein. Let

$$W = -T + i\psi \tag{9}$$

where W is the complex potential having $-T$ and ψ as the real and imaginary parts. The functions $-T$ and ψ are conjugate harmonic functions of X and Y, and, therefore, W is an analytic function of the complex variable Z. The coordinates (X_s, Y_s) of the interface and the total heat transfer Q are found by representing the frozen region in the W plane and then finding a conformal transformation between the W and Z planes.

To find the relation between quantities in the W and Z planes, note that the derivative of an analytic function is independent of direction. Hence,

$$\frac{dW}{dZ} = -\frac{\partial T}{\partial X} + i\frac{\partial \psi}{\partial X} \tag{10}$$

The Cauchy-Riemann equations can be used to relate the derivatives of the real and imaginary parts of W, giving

$$\frac{\partial \psi}{\partial X} = \frac{\partial T}{\partial Y} \tag{11}$$

Using equation (11) to eliminate $\partial\psi/\partial X$ from equation (10) gives

$$\frac{dW}{dZ} = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y} \tag{12}$$

Now define ζ as

$$\zeta \equiv -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y} \tag{13}$$

so that equation (12) becomes

$$\frac{dW}{dZ} = \zeta$$

Integrating gives

$$Z = \int \frac{1}{\zeta}\, dW + \text{constant} \tag{14}$$

Equation (14) shows that the connection between Z and W involves an integration containing ζ; hence, ζ must be related to W before the integration can be performed. The general shapes of frozen region in the W and ζ planes are known from the physical conditions of the problem, as shown in the next section, and the relation between them is found by conformal mapping.

Conformal mapping transformations. - Refer now to figures 2(b) and 3. Since the wall $\widehat{6712}$ and the interface $\widehat{345}$ are isotherms, they become vertical lines in figure 3(a). No heat flows across boundaries $\widehat{23}$ and $\widehat{56}$; hence, these boundaries are along the direction of heat flow. The heat flow lines are normal to the isotherms, so these boundaries form the top and bottom of a rectangle in figure 3(a).

The frozen region in the temperature derivative plane is shown in figure 3(b). Because $\partial T/\partial N = 1$ on $\widehat{345}$, this boundary becomes a part of the unit circle in figure 3(b). Along $\widehat{123}$, $\partial T/\partial X = 0$, while along $\widehat{567}$, $\partial T/\partial Y = 0$ (fig. 2(b)), so that the frozen region maps into a quarter of the unit circle in figure 3(b). If a transformation between the regions in figures 3(a) and (b) can be found, the integration in equation (14) can be performed, and the line $\widehat{345}$ in figure 3(a) can be transformed into the physical plane to obtain the shape of the frozen interface. Several successive intermediate transformations are necessary to relate ζ and W. The derivation of the necessary transformations is now presented.

The region in the derivative plane ζ (fig. 3(b)) is mapped into the region of the ω plane shown in figure 3(c) by the following transformation:

$$\omega = \ln \zeta = \ln|\zeta| + i\theta \tag{15}$$

The generalized rectangular region of the ω plane is mapped into the upper half of the u-plane in the manner indicated by figures 3(c) and (d) by applying the Schwarz-Christoffel transformation

$$\frac{d\omega}{du} = \frac{C_1}{\sqrt{u^2 - 1}} \tag{16}$$

Integrating equation (16) gives

$$\omega = C_1 \ln\left(u + \sqrt{u^2 - 1}\right) + C_2$$

The constants $C_1 = -1/2$ and $C_2 = i\pi$ are chosen so that the points 3 and 5 of the u plane map into the points 3 and 5 of the ω plane such that the horizontal lines are located in the ω plane at $\theta = \pi$ and $\pi/2$. Therefore,

$$\omega = -\frac{1}{2} \ln\left(u + \sqrt{u^2 - 1}\right) + i\pi \tag{17}$$

Now the transformations relating ζ to W are completed by mapping the upper half of the u plane into the rectangle in the W plane by applying the Schwarz-Christoffel transformation in the following form:

$$\frac{dW}{du} = \frac{C_3}{\sqrt{(u^2 - 1)(u + c)(u - d)}} \tag{18}$$

The boundary of the rectangle in the W plane maps into the horizontal axis of the u plane shown in figure 3(d).

Integration into physical plane. - With the equations (15), (17), and (18) available, the integration in equation (14) can be carried out. The integration is performed through the use of the intermediate variable u

$$Z = \int \frac{1}{\zeta(u)} \frac{dW}{du} du + \text{constant} \tag{19}$$

To determine $\zeta(u)$, solve equation (15) for ζ to obtain $\zeta = e^{\omega}$ and substitute ω from equation (17) to obtain (note that $e^{i\pi} = -1$)

$$\zeta(u) = -\left(u + \sqrt{u^2 - 1}\right)^{-1/2} \tag{20}$$

Substitute equations (18) and (20) into equation (19) to obtain

$$Z = -C_3 \int \left[\frac{u + \sqrt{u^2 - 1}}{(u + 1)(u - 1)(u + c)(u - d)}\right]^{1/2} du + \text{constant} \tag{21}$$

The numerator of equation (21) can be expressed in the form

$$\left(u + \sqrt{u^2 - 1}\right)^{1/2} = \sqrt{\frac{u + 1}{2}} + \sqrt{\frac{u - 1}{2}}$$

as can be verified by squaring both sides. Then equation (21) becomes

$$Z = \frac{-C_3}{\sqrt{2}} \left[\int \frac{du}{\sqrt{(u - 1)(u + c)(u - d)}} + \int \frac{du}{\sqrt{(u + 1)(u + c)(u - d)}}\right] + \text{constant} \tag{22}$$

On the frozen interface, $u = \xi + i0$ for $-1 < \xi < 1$. Hence, along the interface,

$$Z_s - Z_5 = \frac{C_3}{\sqrt{2}} \left[\int_1^{\xi} \frac{d\xi}{\sqrt{(1 - \xi)(\xi + c)(d - \xi)}} + i \int_1^{\xi} \frac{d\xi}{\sqrt{(1 + \xi)(\xi + c)(d - \xi)}}\right]$$

$$\text{for } -1 < \xi < 1 \tag{23}$$

To find C_3, use the condition $Z_6 - Z_7 = iB$ in equation (22) evaluated along the real axis of the u-plane from $\xi = d$ to $\xi = \infty$ to obtain

$$iB = \frac{C_3}{\sqrt{2}}\left[\int_d^{\infty} \frac{d\xi}{\sqrt{(\xi - 1)(\xi + c)(\xi - d)}} + \int_d^{\infty} \frac{d\xi}{\sqrt{(\xi + 1)(\xi + c)(\xi - d)}}\right] \tag{24}$$

The integrals in equation (24) are complete elliptic integrals and can be written as

$$iB = \frac{C_3\sqrt{2}}{\sqrt{c + d}}\left[K\left(\sqrt{\frac{c + 1}{c + d}}\right) + K\left(\sqrt{\frac{c - 1}{c + d}}\right)\right] \tag{25}$$

By using equation (25), the C_3 is eliminated from equation (23) to give

$$Z_s - Z_5 = -iB\frac{\sqrt{c + d}}{2}\,\frac{\displaystyle\int_\xi^1 \frac{d\xi}{\sqrt{(1 - \xi)(\xi + c)(d - \xi)}} + i\int_\xi^1 \frac{d\xi}{\sqrt{(1 + \xi)(\xi + c)(d - \xi)}}}{K\left(\sqrt{\frac{c + 1}{c + d}}\right) + K\left(\sqrt{\frac{c - 1}{c + d}}\right)} \tag{26}$$

The Z_5 will be eliminated by first applying equation (26) at point 3, which is at $\xi = -1$, to give

$$Z_3 - Z_5 = -iB\frac{\sqrt{d+c}}{2}\,\frac{\displaystyle\int_{-1}^1 \frac{d\xi}{\sqrt{(1 - \xi)(\xi + c)(d - \xi)}} + i\int_{-1}^1 \frac{d\xi}{\sqrt{(1 + \xi)(\xi + c)(d - \xi)}}}{K\left(\sqrt{\frac{c + 1}{c + d}}\right) + K\left(\sqrt{\frac{c - 1}{c + d}}\right)} \tag{27}$$

Now, from figure 2(b), $Z_5 = A + iB - \mathcal{Re}(Z_3 - Z_5)$. By use of equation (27) to obtain $\mathcal{Re}(Z_3 - Z_5)$, Z_5 is eliminated from equation (26) to give

$$Z_s = A + iB - B\frac{\sqrt{c+d}}{2}\frac{\left[\int_{-1}^{1}\frac{d\xi}{\sqrt{(1+\xi)(\xi+c)(d-\xi)}} + \int_{\xi}^{1} i\frac{d\xi}{\sqrt{(1-\xi)(\xi+c)(d-\xi)}} - \frac{d\xi}{\sqrt{(1+\xi)(\xi+c)(d-\xi)}}\right]}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)}$$

$$\text{for } -1 < \xi < 1 \qquad (28)$$

To obtain X_s, take the real part of equation (28), which gives

$$\frac{X_s}{B} = \frac{A}{B} - \frac{\sqrt{c+d}}{2}\frac{\int_{-1}^{\xi}\frac{d\xi}{\sqrt{(1+\xi)(\xi+c)(d-\xi)}}}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \qquad \text{for } -1 < \xi < 1 \qquad (29)$$

Similarly, Y_s is found by taking the imaginary part of equation (28)

$$\frac{Y_s}{B} = 1 - \frac{\sqrt{c+d}}{2}\frac{\int_{\xi}^{1}\frac{d\xi}{\sqrt{(1-\xi)(\xi+c)(d-\xi)}}}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \qquad \text{for } -1 < \xi < 1 \qquad (30)$$

The integrals in the numerators of equations (29) and (30) can be transformed into elliptic integrals of the first kind to give

$$\frac{X_s}{B} = \frac{A}{B} - \frac{F\left[\sin^{-1}\sqrt{\frac{(c+d)(1+\xi)}{(1+d)(c+\xi)}}, \sqrt{\frac{1+d}{c+d}}\right]}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \qquad \text{for } -1 < \xi < 1 \qquad (31)$$

$$\frac{Y_s}{B} = 1 - \frac{F\left[\sin^{-1}\sqrt{\frac{(c+d)(1-\xi)}{(1+c)(d-\xi)}}, \sqrt{\frac{1+c}{c+d}}\right]}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \quad \text{for } -1 < \xi < 1 \tag{32}$$

The aspect ratio A/B must be expressed in terms of the quantities involved in the mapping. The length $A = Z_2 - Z_1$, where Z_2 and Z_1 correspond to $\xi = -c$ and $-\infty$. Then, from equation (21) (note that in this interval of ξ, $\sqrt{u^2 - 1} = \sqrt{e^{i\pi}|\xi+1|e^{i\pi}|\xi-1|} = e^{i\pi}\sqrt{\xi^2 - 1} = -\sqrt{\xi^2 - 1}$),

$$A = -C_3 \int_{-\infty}^{-c} \left[\frac{\xi - \sqrt{\xi^2 - 1}}{(\xi+1)(\xi-1)(\xi+c)(\xi-d)}\right]^{1/2} d\xi$$

Eliminate C_3 by using equation (25) to obtain

$$\frac{A}{B} = -i\frac{\sqrt{c+d}}{\sqrt{2}} \frac{\int_{-\infty}^{-c} \left[\frac{\xi - \sqrt{\xi^2 - 1}}{(\xi+1)(\xi-1)(\xi+c)(\xi-d)}\right]^{1/2} d\xi}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)}$$

Change the variable in the numerator by letting $\xi = -t$ to yield,

$$\frac{A}{B} = \frac{a}{b} = -\frac{\sqrt{c+d}}{\sqrt{2}} \frac{\int_{\infty}^{c} \left[\frac{t + \sqrt{t^2 - 1}}{(t+1)(t-1)(t-c)(t+d)}\right]^{1/2} dt}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} = \frac{K\left(\sqrt{\frac{d+1}{d+c}}\right) + K\left(\sqrt{\frac{d-1}{d+c}}\right)}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \tag{33}$$

Equation (33) relates the aspect ratio of the duct to the quantities c and d that are involved in the mapping.

Heat flow through frozen region. - For completion of the solution, the heat flow through the frozen region needs to be determined. Also, the mapping parameters c and d need to be related to the basic physical variables in the problem, such as the convective heat-transfer coefficient, the liquid temperature, and the freezing temperature. The heat flow rate per unit channel length through the frozen region and into the cold wall is four times the value for one quadrant; that is,

$$\frac{Q}{4} = \int_{y=0}^{y=b} k \left.\frac{\partial t}{\partial x}\right|_{x=0} dy + \int_{x=0}^{x=a} k \left.\frac{\partial t}{\partial y}\right|_{y=0} dx$$

Normalizing gives

$$\frac{Q}{4k(t_f - t_w)} = \int_{Y=0}^{Y=B} \left.\frac{\partial T}{\partial X}\right|_{X=0} dY + \int_{X=0}^{X=A} \left.\frac{\partial T}{\partial Y}\right|_{Y=0} dY$$

Apply the Cauchy-Riemann conditions

$$-\frac{\partial T}{\partial X} = \frac{\partial \psi}{\partial Y}; \quad \frac{\partial T}{\partial Y} = \frac{\partial \psi}{\partial X}$$

to obtain

$$\frac{Q}{4k(t_f - t_w)} = -\int_{Y=0}^{Y=B} \left.\frac{\partial \psi}{\partial Y}\right|_{X=0} dY + \int_{X=0}^{X=A} \left.\frac{\partial \psi}{\partial X}\right|_{Y=0} dX$$

or

$$\frac{Q}{4k(t_f - t_w)} = -\psi(0, B) + \psi(A, 0) = (-i)\left[W(2) - W(6)\right] = (-i)\left[W(3) - W(5)\right]$$

The foregoing can be evaluated as

$$\frac{Q}{4k(t_f - t_w)} = -i \int_{\text{point 5}}^{\text{point 3}} \frac{dW}{du}\, du$$

which can also be written as

$$\frac{Q}{4k(t_f - t_w)} = i \int_{u=-1}^{1} \frac{dW}{du}\, du$$

Substitute dW/du from equation (18) and note that $u = \xi$ on the real axis

$$\frac{Q}{4k(t_f - t_w)} = iC_3 \int_{\xi=-1}^{1} \frac{d\xi}{\sqrt{(\xi^2 - 1)(\xi + c)(\xi - d)}} \tag{34}$$

The constant C_3 is conveniently found here by noting from figure 3(a) that $1 = W(6) - W(5)$, which can be written as

$$1 = \int_{u=1}^{d} \frac{dW}{du}\, du = C_3 \int_{\xi=1}^{d} \frac{d\xi}{\sqrt{(\xi^2 - 1)(\xi + c)(\xi - d)}}$$

Solve for C_3

$$C_3 = \frac{1}{\dfrac{1}{i} \displaystyle\int_{\xi=1}^{d} \frac{d\xi}{\sqrt{(\xi^2 - 1)(\xi + c)(d - \xi)}}} \tag{35}$$

Substitute equation (35) in (34) to obtain

$$\frac{Q}{4k(t_f - t_w)} = \frac{\displaystyle\int_{-1}^{1} \frac{d\xi}{\sqrt{(1 - \xi^2)(\xi + c)(d - \xi)}}}{\displaystyle\int_{1}^{d} \frac{d\xi}{\sqrt{(\xi^2 - 1)(\xi + c)(d - \xi)}}} \tag{36}$$

The integrals are complete elliptic integrals, and equation (36) can be put in the form

$$\frac{Q}{4k(t_f - t_w)} = \frac{K\left[\sqrt{\dfrac{2(c + d)}{(c + 1)(d + 1)}}\right]}{K\left[\sqrt{\dfrac{(c - 1)(d - 1)}{(c + 1)(d + 1)}}\right]} \tag{37}$$

Relation between physical conditions and mapping. - Finally, an expression is needed to relate the parameter

$$\frac{1}{B} = \frac{k}{h_l b} \frac{t_f - t_w}{t_l - t_f}$$

which contains the controllable variables, to the mapping quantities c and d. This relation can be found by noting that two expressions have been obtained for C_3, equations (25) and (35). Equating these two expressions gives

$$\frac{1}{B} = \frac{k}{h_l b} \frac{t_f - t_w}{t_l - t_f} = \sqrt{\frac{c + d}{2}} \frac{\displaystyle\int_1^d \frac{d\xi}{\sqrt{(\xi^2 - 1)(\xi + c)(d - \xi)}}}{K\left(\sqrt{\frac{c + 1}{c + d}}\right) + K\left(\sqrt{\frac{c - 1}{c + d}}\right)}$$

The numerator can be expressed as a standard elliptic integral giving

$$\frac{k}{h_l b} \frac{t_f - t_w}{t_l - t_f} = \frac{\sqrt{\frac{2(c + d)}{(c + 1)(d + 1)}} K\left[\sqrt{\frac{(c - 1)(d - 1)}{(c + 1)(d + 1)}}\right]}{K\left(\sqrt{\frac{c + 1}{c + d}}\right) + K\left(\sqrt{\frac{c - 1}{c + d}}\right)} \tag{38}$$

Evaluation of solution. - The solution to the problem consists of equations (31), (32), (33), (37), and (38). In using this solution, the aim is to calculate the total heat transfer Q and the coordinates x_s, y_s of the liquid - frozen-layer interface, given a known set of values for t_w, t_l, t_f, h_l, k, a, and b.

The first step is to evaluate equation (33) for a variety of c's and d's to find a set of pairs of c and d that results in the specified aspect ratio a/b. Then, each pair of c and d is used in equations (37) and (38) to determine the values of $Q/4k(t_f - t_w)$ and $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$. The c and d pair and the corresponding aspect ratio are used in equations (31) and (32) to obtain the coordinates of the liquid - frozen-layer interface. By use of the various c and d pairs for a fixed aspect ratio, a family of interface shapes can be plotted, and the values of $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$ and $Q/4k(t_f - t_w)$ are known for each shape. The results of these calculations are given in the section RESULTS AND DISCUSSION.

Limiting Cases

The previous analysis treated the general situation of a duct of any aspect ratio. Two limiting cases that are of interest and lead to simplified analytical results are shown in figure 4.

In figure 4(a), the frozen region is sufficiently thin so that a one-dimensional region is formed in the central region along the long side. The following sections will show that this condition corresponds to having the mapping quantities $c = 1$ and $d > 1$.

In figure 4(b), the frozen region is so thin that a one-dimensional region is formed in the central portion of each side. In this instance, the analysis will show that the mapping quantities have gone to the limits $c = 1$, $d = 1$.

Limiting case for $c = 1$, $d > 1$: layer one-dimensional in central region of long side. - With reference to figure 4(a), the frozen region becomes one-dimensional at the center of the long side. Increasing the aspect ratio will not change the shape of the frozen region but will only increase the extent of the one-dimensional region. Hence, a duct of infinite aspect ratio can be analyzed as shown in figure 5(a).

The geometry is nondimensionalized in the same manner as for figure 2(b), which yields figure 5(b). The only difference in the two situations is that now points 2 and 3 are both at infinity in the physical plane and, hence, also in the potential plane in figure 6(a). The subsequent mappings as shown in figures 6(b), (c), and (d) then correspond to those in figures 3(b), (c), and (d).

With points 2 and 3 at the same location in figure 6(d), the parameter c becomes 1. Equation (26) for coordinates on the frozen interface then becomes (note that $K(0) = \pi/2$)

$$\frac{Z_s(\xi) - Z_5}{B} = \frac{\sqrt{1+d}}{2} \, \frac{\displaystyle\int_{\xi}^{1} \left[\frac{1}{(1+\xi)\sqrt{d-\xi}} - i\,\frac{1}{\sqrt{(1-\xi)(\xi+1)(d-\xi)}}\right] d\xi}{K\left(\sqrt{\dfrac{2}{1+d}}\right) + \dfrac{\pi}{2}} \qquad -1 < \xi < 1 \tag{39}$$

The distance $Z_5 - Z_6$ will be found to relate the interface to the coordinates of the duct. Using equation (22) with $c = 1$ and noting that u_5 and u_6 correspond to $\xi = 1$ and $\xi = d$ in figure 6(d) result in

$$Z_5 - Z_6 = -\frac{C_3}{\sqrt{2}\, i} \int_{\xi=d}^{1} \left[\frac{1}{\sqrt{(\xi - 1)(\xi + 1)(d - \xi)}} + \frac{1}{(\xi + 1)\sqrt{d - \xi}}\right] d\xi \qquad 1 < \xi < d \tag{40}$$

From equation (25),

$$\frac{C_3}{i} = \frac{B\sqrt{\frac{1+d}{2}}}{K\left(\sqrt{\frac{2}{1+d}}\right) + \frac{\pi}{2}} \tag{41}$$

Eliminate C_3/i from equation (40) by using equation (41) and combine the result with equation (39) to eliminate Z_5. These operations give

$$\frac{Z_s(\xi) - Z_6}{B} = \frac{Z_s(\xi) - iB}{B}$$

$$= \frac{\int_\xi^1 \left[\frac{1}{(1+\xi)\sqrt{d-\xi}} - \frac{i}{\sqrt{(1-\xi)(\xi+1)(d-\xi)}}\right] d\xi + \int_1^d \left[\frac{1}{\sqrt{(\xi-1)(\xi+1)(d-\xi)}} + \frac{1}{(\xi+1)\sqrt{d-\xi}}\right] d\xi}{\frac{2}{\sqrt{1+d}}\left[K\left(\sqrt{\frac{2}{1+d}}\right) + \frac{\pi}{2}\right]}$$

Separate the real and imaginary parts to obtain

$$\frac{X_s}{B} = \frac{\int_\xi^1 \frac{d\xi}{(1+\xi)\sqrt{d-\xi}} + \int_1^d \left[\frac{1}{\sqrt{(\xi-1)(\xi+1)(d-\xi)}} + \frac{1}{(\xi+1)\sqrt{d-\xi}}\right] d\xi}{\frac{2}{\sqrt{1+d}}\left[K\left(\sqrt{\frac{2}{1+d}}\right) + \frac{\pi}{2}\right]}$$

$$\frac{Y_s}{B} = 1 - \frac{\int_\xi^1 \frac{d\xi}{\sqrt{(1-\xi)(\xi+1)(d-\xi)}}}{\frac{2}{\sqrt{1+d}}\left[K\left(\sqrt{\frac{2}{1+d}}\right) + \frac{\pi}{2}\right]}$$

The integrals in the numerator of X_s/B can either be carried out or expressed as an elliptic integral, and the integral in the numerator of the Y_s/B expression can be expressed as an elliptic integral. These operations yield the expressions

$$\frac{X_s}{B} = \frac{\ln\left[\frac{\xi+1}{2}\left(\frac{\sqrt{d+1} - \sqrt{d-1}}{\sqrt{d+1} - \sqrt{d-\xi}}\right)^2\right] + \ln\frac{\sqrt{d+1} + \sqrt{d-1}}{\sqrt{d+1} - \sqrt{d-1}} + 2K\left(\sqrt{\frac{d-1}{d+1}}\right)}{2K\left(\sqrt{\frac{2}{1+d}}\right) + \pi} \qquad -1 < \xi < 1 \tag{42}$$

$$\frac{Y_s}{B} = 1 - \frac{F\left[\sin^{-1}\sqrt{\frac{(1+d)(1-\xi)}{2(d-\xi)}}, \sqrt{\frac{2}{1+d}}\right]}{K\left(\sqrt{\frac{2}{1+d}}\right) + \frac{\pi}{2}} \qquad -1 < \xi < 1 \tag{43}$$

Equation (36) can be used to compute the heat flow by letting $c = 1$. Also, the integral in the numerator is evaluated from ξ_a to 1, where ξ_a is the value of ξ in equation (42) when $X_s/B = A/B$. This cuts off the infinite aspect ratio configuration in figure 5(b) so that it extends only to the size of the finite aspect ratio duct. The heat flow is then

$$\frac{Q}{4k(t_f - t_w)} = \frac{\displaystyle\int_{\xi_a}^{1} \frac{d\xi}{(\xi+1)\sqrt{(1-\xi)(d-\xi)}}}{\displaystyle\int_{1}^{d} \frac{d\xi}{(\xi+1)\sqrt{(\xi-1)(d-\xi)}}}$$

Both integrals can be evaluated analytically, and after simplification, the heat flow becomes

$$\frac{Q}{4k(t_f - t_w)} = \frac{1}{\pi}\ln\left[\frac{2\sqrt{2(1+d)}\sqrt{\xi_a^2 - (d+1)\xi_a + d} + 4(d+1)}{(1+\xi_a)(d-1)} - \frac{d+3}{d-1}\right] \tag{44}$$

Using equation (38) with $c = 1$ gives

$$\frac{k}{h_l b}\frac{t_f - t_w}{t_l - t_f} = \frac{1}{B} = \frac{1}{\frac{2}{\pi}K\left(\sqrt{\frac{2}{1+d}}\right) + 1} \tag{45}$$

Frozen interface profiles are evaluated by choosing a value of d. The physical parameter 1/B is found from equation (45). The corresponding heat flow and coordinates of the frozen region are found from equations (44), (42), and (43).

Limiting case where $c = 1$ and $d = 1$: frozen layer in corner region. - For a very thin frozen layer, the thickness is constant in the central portion of both sides of the duct away from the corners, as shown in figure 4(b). The configuration can then be found by joining four of the corner regions shown in figure 7(a).

As in the preceding analyses, the quantities are nondimensionalized to yield figure 7(b). The geometry is symmetric about the line $\widehat{14}$. Since the pairs of points 2, 3 and 5, 6 are at infinity, the mapping in the potential plane consists of two constant temperature lines extending to infinity. The mappings, as shown in figure 8, are similar to those in figure 3. In the u plane, the pairs 5, 6 and 2, 3 will each be at a single point that can be fixed at (1, 0) and (-1, 0). Thus, the corner solution is a limiting case of the general solution for the situation when $c = 1$ and $d = 1$. Use equation (45) to eliminate B from equation (39) and then let $d = 1$ to obtain (note t is a dummy variable of integration)

$$Z_s(\xi) - Z_5 = \frac{\sqrt{2}}{\pi}\int_{\xi}^{1}\left[\frac{1}{(1+t)\sqrt{1-t}} - i\frac{1}{(1-t)\sqrt{1+t}}\right]dt$$

At Z_4, $\xi = 0$, so

$$Z_4 - Z_5 = \frac{\sqrt{2}}{\pi}\int_{0}^{1}\left[\frac{1}{(1+t)\sqrt{1-t}} - i\frac{1}{(1-t)\sqrt{1+t}}\right]dt$$

Then

$$Z_s - Z_4 = -\frac{\sqrt{2}}{\pi}\int_{0}^{\xi}\left[\frac{1}{(1+t)\sqrt{1-t}} - i\frac{1}{(1-t)\sqrt{1+t}}\right]dt \tag{46}$$

Equation (46) is integrated to yield

$$Z_s - Z_4 = -\frac{1}{\pi}\left(\ln\frac{\dfrac{\sqrt{2}-\sqrt{1-\xi}}{\sqrt{2}+\sqrt{1-\xi}}}{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}} + i\ln\frac{\dfrac{\sqrt{2}-\sqrt{1+\xi}}{\sqrt{2}+\sqrt{1+\xi}}}{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}\right) \tag{47}$$

The real part of $Z_s - Z_4$ must approach $X_5 - X_4$ as $\xi \rightarrow 1$, which gives

$$X_5 - X_4 = \frac{1}{\pi}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \tag{48}$$

At a location far from the corner, the interface is flat and parallel to the wall. A heat balance yields

$$\frac{k(t_f - t_w)}{x_5} = h_l(t_l - t_f)$$

so that

$$\frac{x_5}{\gamma} = X_5 = 1$$

Substituting into equation (48) and noting the symmetry of figure 7(b) gives

$$X_4 = Y_4 = 1 + \frac{1}{\pi}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \tag{49}$$

Now, the final expressions for the coordinates of the interface can be written. From symmetry, only the portion of the profile between Z_4 and Z_5 need be considered. Using equation (49) and the real and imaginary parts of equation (47) gives

$$\frac{X_s}{B} = \frac{x_s}{b} = \frac{1}{B}\left[1 + \frac{1}{\pi}\ln\left(\frac{\sqrt{2}+\sqrt{1-\xi}}{\sqrt{2}-\sqrt{1-\xi}}\right)\right] \qquad 0 \leq \xi \leq 1 \tag{50}$$

$$\frac{Y_s}{B} = \frac{y_s}{b} = \frac{1}{B}\left[1 + \frac{1}{\pi}\ln\left(\frac{\sqrt{2} + \sqrt{1+\xi}}{\sqrt{2} - \sqrt{1+\xi}}\right)\right] \qquad 0 \leq \xi \leq 1 \tag{51}$$

It is worth noting that the normalized coordinates X_s, Y_s of the interface yield a single interface curve that is independent of any parameter. The dimensionless coordinates x_s/b and y_s/b depend only on the physical parameter B.

A relation is needed for heat flow through the frozen layer. Because the walls that form the corner are semi-infinite, the total heat flow through each wall depends on the total distance along the wall from the corner. Since the frozen layer configuration is symmetrical about the corner angle bisector, only one wall need be considered. The heat flow through the vertical wall can be found by using equations (36) and (37). When $c = 1$ and $d = 1$, the denominator of equation (36) gives

$$\lim_{\substack{c\to 1\\ d\to 1}} \int_1^d \frac{d\xi}{\sqrt{(\xi^2 - 1)(\xi + c)(d - \xi)}} = \lim_{\substack{c\to 1\\ d\to 1}} \frac{2}{\sqrt{(d+1)(c+1)}}\, K\left[\sqrt{\frac{(c-1)(d-1)}{(c+1)(d+1)}}\right] = K(0) = \frac{\pi}{2}$$

Then, the normalized heat flow through the vertical wall from the corner to an arbitrary location is given by

$$\frac{Q}{4k(t_f - t_w)} = \frac{\displaystyle\int_0^\xi \frac{d\xi}{1 - \xi^2}}{\dfrac{\pi}{2}} = \frac{1}{\pi}\ln\left(\frac{1+\xi}{1-\xi}\right) \qquad 0 \leq \xi \leq 1 \tag{52}$$

To find the value of $Q/4k(t_f - t_w)$ for a quadrant of a rectangular duct, equation (52) is applied to each of the two sides that represent the duct walls, giving

$$\frac{Q}{4k(t_f - t_w)} = \frac{1}{\pi}\ln\left(\frac{1+\xi_a}{1-\xi_a}\right) + \frac{1}{\pi}\ln\left(\frac{1+\xi_b}{1-\xi_b}\right) = \frac{1}{\pi}\ln\left(\frac{1+\xi_a}{1-\xi_a}\cdot\frac{1+\xi_b}{1-\xi_b}\right) \tag{53}$$

where ξ_a and ξ_b are found from equation (51) applied along each of the sides

$$\frac{A}{B} = \frac{1}{B}\left(1 + \frac{1}{\pi}\ln\frac{\sqrt{2} + \sqrt{1 + \xi_a}}{\sqrt{2} - \sqrt{1 + \xi_a}}\right)$$

$$1 = \frac{1}{B}\left(1 + \frac{1}{\pi}\ln\frac{\sqrt{2} + \sqrt{1 + \xi_b}}{\sqrt{2} - \sqrt{1 + \xi_b}}\right)$$

Evaluation of Heat Flow From Interface Length

As an alternative to obtaining the heat flow by the analytical expressions given previously, a simple graphical method utilizing the interface curves can be used. If s is the entire length of the interface, then

$$Q = h_l s(t_l - t_f)$$

or, in dimensionless form,

$$\frac{Q}{4k(t_f - t_w)} = \frac{h_l b}{k}\,\frac{s}{4b}\,\frac{t_l - t_f}{t_f - t_w} = \frac{1}{4}\left(\frac{s}{b}\right)B \tag{54}$$

For one quadrant of the duct, the interface length is $s/4$ and can be measured graphically. This measurement is then divided by the length of the vertical side and multiplied by the parameter B to give the dimensionless heat flow.

Some of the analytical expressions such as equation (53) are difficult to evaluate accurately in some instances. For aspect ratios larger than about 4, the ξ_a and ξ_b can become close to 1 so that the natural log term is very sensitive to small inaccuracies in ξ_a and ξ_b. In this instance, equation (54) is especially useful.

For a very thin layer, the length of the interface is closely approximated by $4a + 4b - 8\gamma$, where γ is the frozen layer thickness when the layer is very thin relative to the duct dimensions. Substitute into equation (54) to obtain for thin layers

$$\frac{Q}{4k(t_f - t_w)} = \frac{1}{4}\left(4\frac{a}{b} + 4 - 8\frac{\gamma}{b}\right)B = \left(\frac{a}{b} + 1 - \frac{2}{B}\right)B \tag{55}$$

RESULTS AND DISCUSSION

In the ANALYSIS were derived relations that can be used to evaluate the coordinates of the frozen interface and the heat flow through the frozen region as a function of the imposed physical conditions. This evaluation can be made for each aspect ratio. The desired quantities are interrelated by means of the mapping quantities c and d. The general analysis for $c > 1$, $d > 1$ was used first to evaluate frozen profiles by starting with large values of c and then decreasing c toward 1. The large values of c and d correspond to thick frozen regions. For a fixed aspect ratio, both the c and d decrease as the frozen region becomes thinner. When c is very close to 1, some of the elliptic integrals become difficult to evaluate accurately. The limiting solution for $c = 1$ and $d > 1$ was then used by starting with the smallest value of d used in the general solution. Profiles of the interface were evaluated for several smaller d values until d was very close to 1. As d approaches 1, it is convenient to use the limiting case for $c = 1$, $d = 1$ applicable for thin frozen layers.

The contours of the frozen region for aspect ratios $a/b = 1, 2, 3, 4$, and 5 are given in figures 9(a) to (e). The curves in each plot are for various values of the controllable physical cooling parameter $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$. The dimensionless heat flow as a function of this physical parameter is given in figure 10 on both rectangular and semilogarithmic coordinates, the latter to better reveal the details for thick frozen regions.

As a typical set of frozen configurations, examine figure 9(d) for $a/b = 4$. For thin layers, corresponding to a small value of the cooling parameter, the layer is of constant thickness except close to the corner. As the cooling parameter is increased within the range of $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$ up to about 0.4, the frozen thickness increases fairly uniformly around the duct. Then, as the cooling parameter is further increased by a very small amount in the neighborhood of 0.5, the thickness along the short side increases substantially while that on the long side remains almost constant. For thick frozen regions, the interface approaches a circular shape.

One of the interesting aspects of the profiles in figure 9 is that for each aspect ratio there is a maximum value of $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$. In the case of $a/b = 4$, for example, the maximum value is about 0.5. For a larger cooling parameter than this maximum, no equilibrium profiles are possible, and the channel would freeze completely.

The fact that there is a maximum $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$ for each a/b leads to the most interesting feature of figure 9. For each value of the cooling parameter below the maximum, there are mathematically two possible frozen regions. One set, shown in figure 9 as solid curves, corresponds to thin and moderately thick frozen regions, while the dashed set is for very thick regions. Since the heat flow, as given by equation (54), is proportional to the interface length, the heat flow will also be a double-valued function of the physical cooling parameter. This function is shown in figure 10. As the channel aspect ratio increases, the heat flow (and interface profiles) are very sensitive to the

$(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$ near its maximum value. As will be shown later in this section, the dashed set represents unstable contours that will not exist in actual physical situations.

It is convenient to examine the simpler one-dimensional situation of freezing inside a cylindrical tube to discuss more easily the stability of the dashed and solid profiles. As shown in figure 11, if the inside radius of the frozen region is r_i, the energy convected to the interface is $Q_{conv} = h_l(t_l - t_f)2\pi r_i$. This expression is equated to the conduction through the frozen region to give

$$h_l(t_l - t_f)2\pi r_i = 2\pi k \frac{t_f - t_w}{\ln \frac{r_o}{r_i}} \tag{56}$$

Rearranging yields

$$\frac{k}{h_l r_o} \frac{t_f - t_w}{t_l - t_f} = \frac{r_i}{r_o} \ln \frac{r_o}{r_i} \tag{57}$$

Equation (57) is plotted in figure 11, and the double-valued nature of r_i/r_o for various values of the physical parameter $(k/h_l r_o)\left[(t_f - t_w)/(t_l - t_f)\right]$ is evident. The maximum value of the cooling parameter is found by differentiating equation (57) and setting it equal to zero

$$\frac{d\left(\frac{k}{h_l r_o} \frac{t_f - t_w}{t_l - t_f}\right)}{d\left(\frac{r_i}{r_o}\right)} = 0 = -1 + \ln \frac{r_o}{r_i}$$

This operation gives $r_i/r_o = 1/e$, which is then substituted into equation (57) to give the maximum value of the cooling parameter as

$$\left.\frac{k}{h_l r_o} \frac{t_f - t_w}{t_l - t_f}\right|_{max} = \frac{1}{e}$$

For a cooling parameter larger than $1/e$, there is no condition where the convective heating from the warm liquid can be as large as the conduction away from the frozen interface; for this condition, the tube will freeze solid.

The curve in figure 11 represents a state of equality between the convective heat supplied to the frozen layer and the heat conducted through the frozen layer. For every value of $(k/h_l r_o)\left[(t_f - t_w)/(t_l - t_f)\right] < 1/e$, two frozen-layer thicknesses are possible. The question then arises as to whether both layer thicknesses can exist as states of stable equilibrium. The answer is found by examining changes that occur when each equality state is perturbed slightly; that is, the equality radius ratio r_i/r_o of the frozen layer is increased or decreased slightly and then examined to see if the radius returns to its original value.

The stability analysis is begun by noting that Q_{conv} changes linearly with r_i and the conduction through the layer Q_{cond} varies as $\left[\ln(r_o/r_i)\right]^{-1}$. The difference $Q_{conv} - Q_{cond}$ is from equation (56)

$$Q_{conv} - Q_{cond} = 2\pi r_o\left(\frac{r_i}{r_o}\right)h_l(t_l - t_f) - 2\pi k \frac{t_f - t_w}{\ln\left(\frac{r_o}{r_i}\right)} \tag{58}$$

At equilibrium, $Q_{conv} - Q_{cond} = 0$ at every r_i. Now, examine the small change $d(Q_{conv} - Q_{cond})$ that occurs when a small disturbance causes a change dr_i in the location of the liquid - frozen-layer interface. To do this, write

$$d(Q_{conv} - Q_{cond}) = \frac{d}{d\left(\frac{r_i}{r_o}\right)}(Q_{conv} - Q_{cond})d\left(\frac{r_i}{r_o}\right)$$

Differentiating equation (58) with respect to r_i/r_o results in

$$\frac{d}{d\left(\frac{r_i}{r_o}\right)}(Q_{conv} - Q_{cond}) = 2\pi r_o h_l(t_l - t_f)\left[1 - \frac{k}{r_o h_l}\left(\frac{t_f - t_w}{t_l - t_f}\right)\frac{1}{\left(\frac{r_i}{r_o}\right)\left(\ln\frac{r_i}{r_o}\right)^2}\right]$$

Since the interface is being perturbed from equilibrium, equation (57) is used to simplify the expression in brackets to obtain

$$\frac{d}{d\left(\frac{r_i}{r_o}\right)}(Q_{conv} - Q_{cond}) = 2\pi r_o h_l (t_l - t_f)\left[1 + \frac{1}{\ln\left(\frac{r_i}{r_o}\right)}\right] \tag{59}$$

When $r_i/r_o > 1/e$, the derivative is negative, and, for $r_i/r_o < 1/e$, the derivative is positive.

Consider first the region $1 > r_i/r_o > 1/e$, which is along the solid portion of the curve in figure 11 and for which $\left[d/d(r_i/r_o)\right](Q_{conv} - Q_{cond})$ is negative. If r_i/r_o is increased by $d(r_i/r_o)$, Q_{conv} increases because of the area increase at the interface, and Q_{cond} also increases because of the decreased frozen-layer thickness. However, $d(Q_{conv} - Q_{cond})$ is negative in this region, which means that as r_i/r_o increased, Q_{conv} increased less than Q_{cond} increased. That is, the cooling by conduction increased more than the heat supplied by convection. Hence, refreezing occurs, and the original radius is restored. With similar reasoning, when r_i/r_o is decreased by $d(r_i/r_o)$, Q_{conv} decreases because of the area decrease, and Q_{cond} decreases because of the increased frozen-layer thickness. However, because $d(Q_{conv} - Q_{cond})$ is positive, Q_{conv} dominates Q_{cond}, and the added increment $d(r_i/r_o)$ of frozen layer is melted to restore the original radius. This establishes that the solid portion of the curve in figure 11 gives states that are in stable equilibrium.

Now, consider the region $0 < r_i/r_o < 1/e$, which is along the dashed portion of the curve in figure 11 and where $\left[d/d(r_i/r_o)\right](Q_{conv} - Q_{cond})$ is positive. When the perturbation $d(r_i/r_o)$ is positive, the $d(Q_{conv} - Q_{cond})$ is also positive; consequently, Q_{conv} increases more than Q_{cond}. The frozen layer is unstable and will melt until the stable equilibrium radius on the solid portion of the curve is reached (path a in fig. 11). When $d(r_i/r_o)$ is negative, $d(Q_{conv} - Q_{cond})$ is negative; therefore, the increase in Q_{cond} dominates, and freezing occurs until all the liquid is frozen (path b in fig. 11). Thus, the dashed portion of the curve in figure 11 represents an unstable equilibrium region that will not actually exist in reality.

In summary, for the circular tube case for the situation where a flow of warm liquid is maintained, the only stable configurations are when

$$\frac{k}{h_l r_o}\,\frac{t_f - t_w}{t_l - t_f} < \frac{1}{e} \qquad \text{and} \qquad \frac{r_i}{r_o} > \frac{1}{e}$$

For $(k/h_l r_o)\left[(t_f - t_w)/(t_l - t_f)\right] > 1/e$, no equilibrium states exist, and the tube will freeze completely. The tube can also freeze completely for $(k/h_l r_o)\left[(t_f - t_w)/(t_l - t_f)\right] < 1/e$ if by some means the r_i/r_o falls below the dashed curve in figure 11.

This same type of reasoning applies for the rectangular ducts. The maximum values of $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$, for which frozen regions can be maintained with flowing liquid, can be found from figure 10. The values ranged from about 0.4 to 0.5 as the aspect ratio is increased from 1 to 5. The solid profiles in figure 9 are the stable profiles. The dashed profiles are admissible in the sense of a mathematical equality of heat flows but are unstable physically.

CONCLUDING REMARKS

A conformal mapping technique was utilized to determine two-dimensional frozen-layer configurations for solidification in a cooled rectangular channel carrying flowing warm liquid. The frozen region was mapped onto a potential plane $W = -T + i\psi$, where the negative of the dimensionless temperature is the potential function for heat flow. The shape of the frozen region is found from the integral relating the physical plane to the potential plane

$$Z = \int \frac{1}{\zeta(W)} dW$$

where $\zeta = -(\partial T/\partial X) + i(\partial T/\partial Y)$. The functional relation between ζ and W, necessary to carry out the integral, was found by conformal mapping.

The configuration of the frozen region for each duct aspect ratio depended on a single physical cooling parameter $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$. For each aspect ratio, there was a maximum value of the cooling parameter above which the duct would freeze completely. The interesting feature found was that below this maximum, two possible mathematical solutions exist for each value of the cooling, one was a thin frozen region and the other a thick region. Stability considerations revealed, however, that the thick region configuration was unstable. As a consequence, only thin frozen region configurations are physically possible and, then, only when the value of $(k/h_l b)\left[(t_f - t_w)/(t_l - t_f)\right]$ is below a certain magnitude for each aspect ratio.

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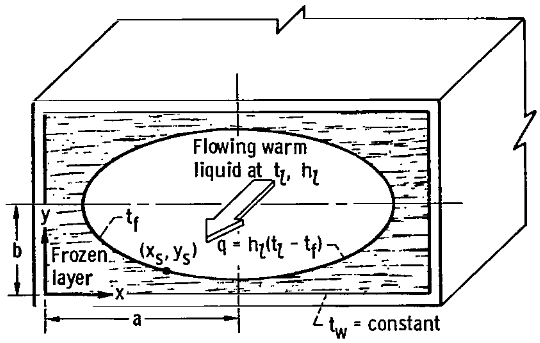
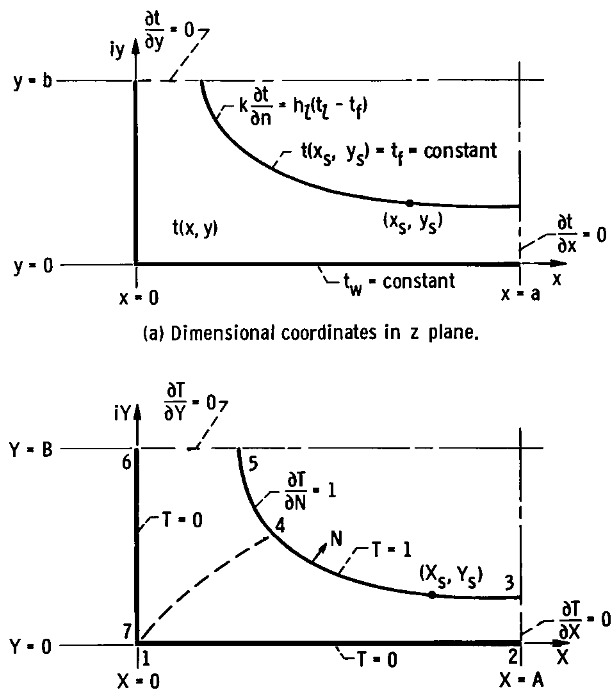


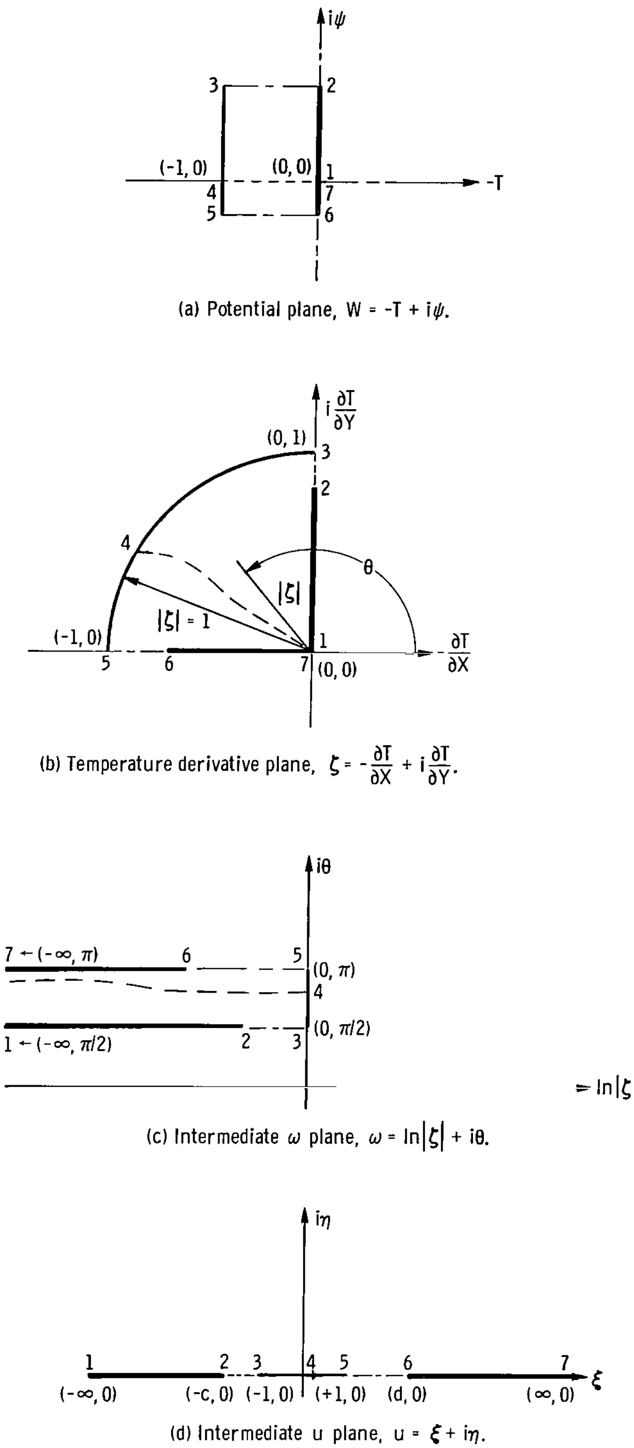
Figure 1. - Cooled rectangular channel containing warm flowing liquid, part of which has frozen onto walls.



(a) Dimensional coordinates in z plane.

(b) Normalized coordinates in Z plane.

Figure 2. - Frozen layer in dimensional and normalized coordinates of physical plane.



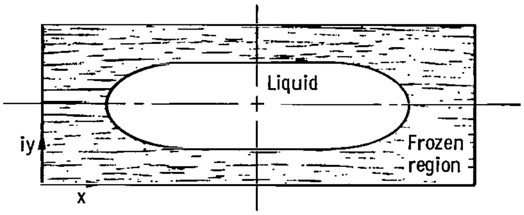
(a) Potential plane, $W = -T + i\psi$.

(b) Temperature derivative plane, $\zeta = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}$.

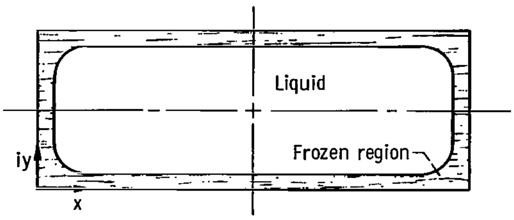
(c) Intermediate ω plane, $\omega = \ln|\zeta| + i\theta$.

(d) Intermediate u plane, $u = \xi + i\eta$.

Figure 3. - Transformation planes for general case: $c > 1$, $d > 1$.

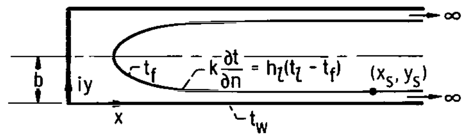


(a) Layer is one-dimensional in central region of long sides, $c \to 1$, $d > 1$.

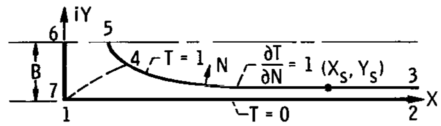


(b) Layer is one-dimensional in central region of long and short sides, $c \to 1$, $d \to 1$.

Figure 4. - Limiting cases where portions of frozen region are one-dimensional.

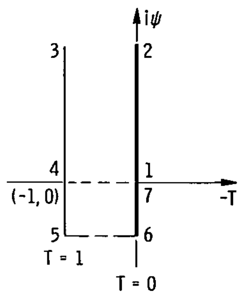


(a) Dimensional coordinates in z plane.

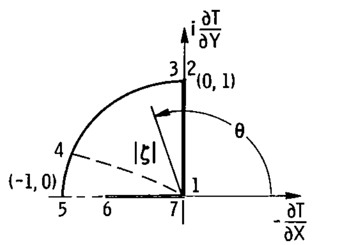


(b) Normalized coordinates in Z plane.

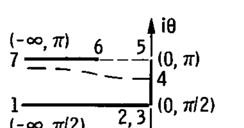
Figure 5. - Limiting case of frozen layer one dimensional along long side of channel.



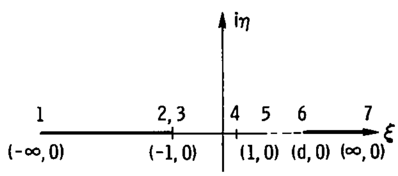
(a) Potential plane, $W = -T + i\psi$.



(b) Temperature derivative plane, $\zeta = -(\partial T/\partial X) + i(\partial T/\partial Y)$.

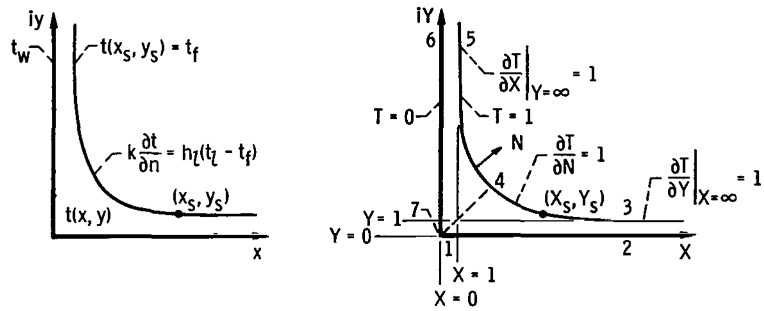


(c) First intermediate plane, $\omega = \ln|\zeta| + i\theta$.



(d) Second intermediate plane, $u = \xi + i\eta$.

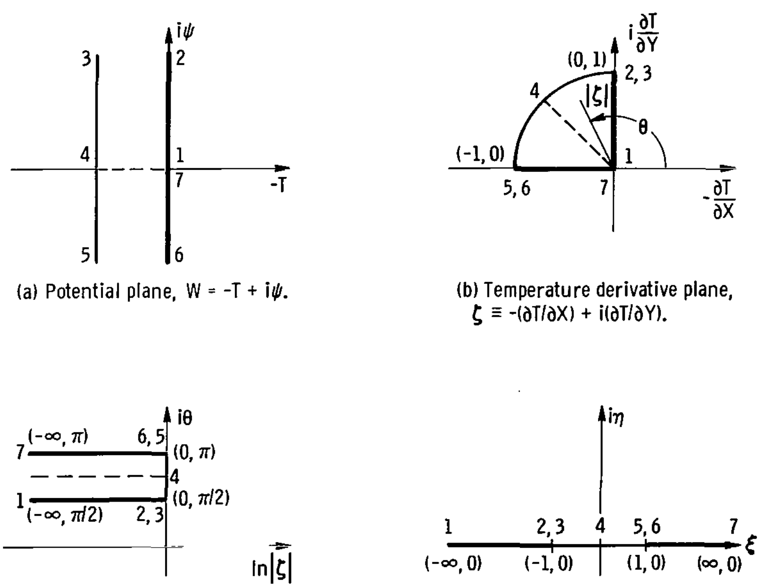
Figure 6. - Transformation planes for infinite aspect ratio duct.



(a) Dimensional coordinates.

(b) Normalized coordinates.

Figure 7. - Limiting case of very thin frozen layer in channel corners.



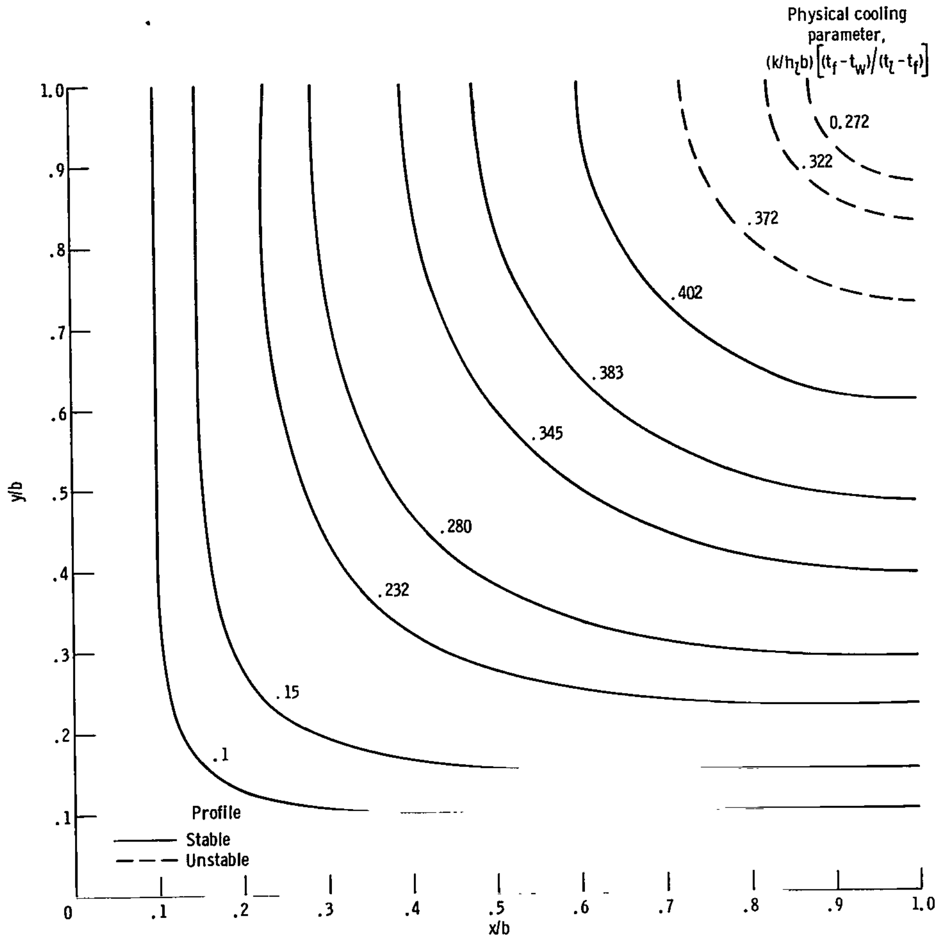
(a) Potential plane, $W = -T + i\psi$.

(b) Temperature derivative plane, $\zeta \equiv -(\partial T/\partial X) + i(\partial T/\partial Y)$.

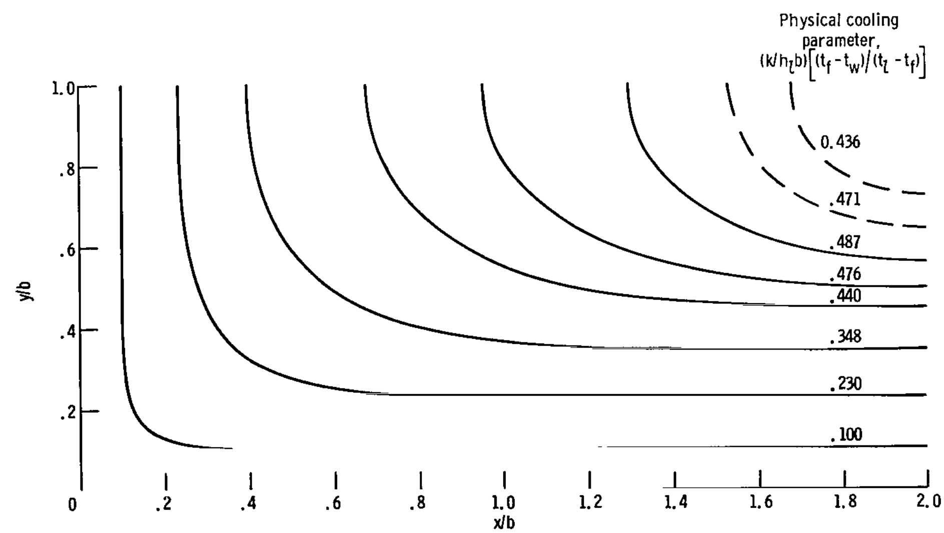
(c) Intermediate ω plane, $\omega = \ln|\zeta| + i\theta$.

(d) Intermediate u plane, $u = \xi + i\eta$.

Figure 8. - Transformation planes for corner solution.

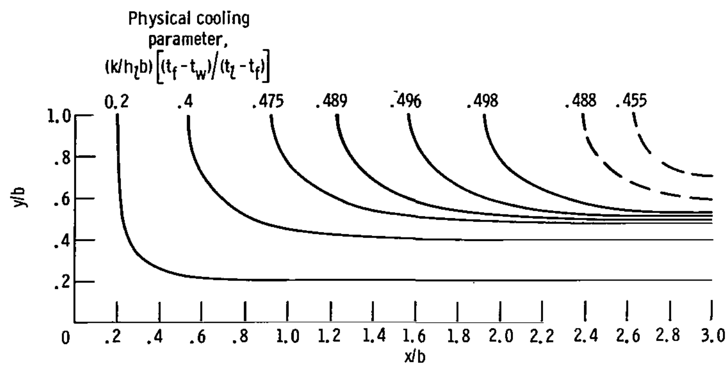


(a) Duct aspect ratio, 1.

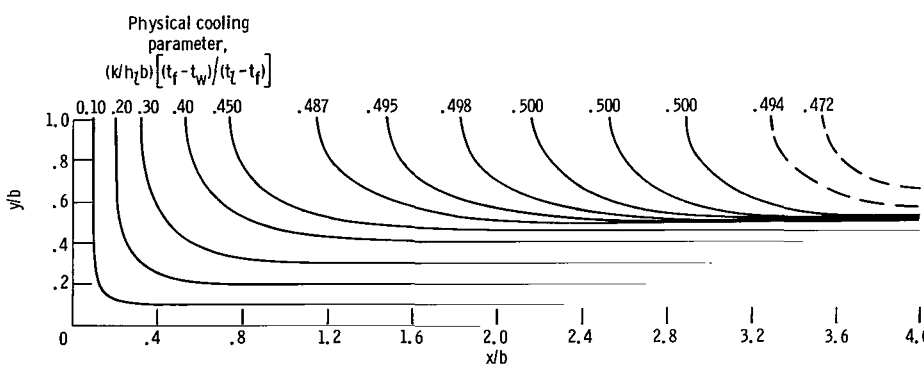


(b) Duct aspect ratio, 2.

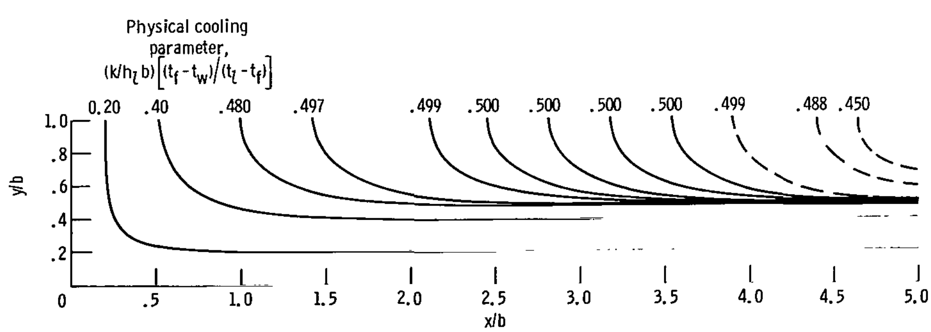
Figure 9. - Frozen layer profiles.



(c) Duct aspect ratio, 3.

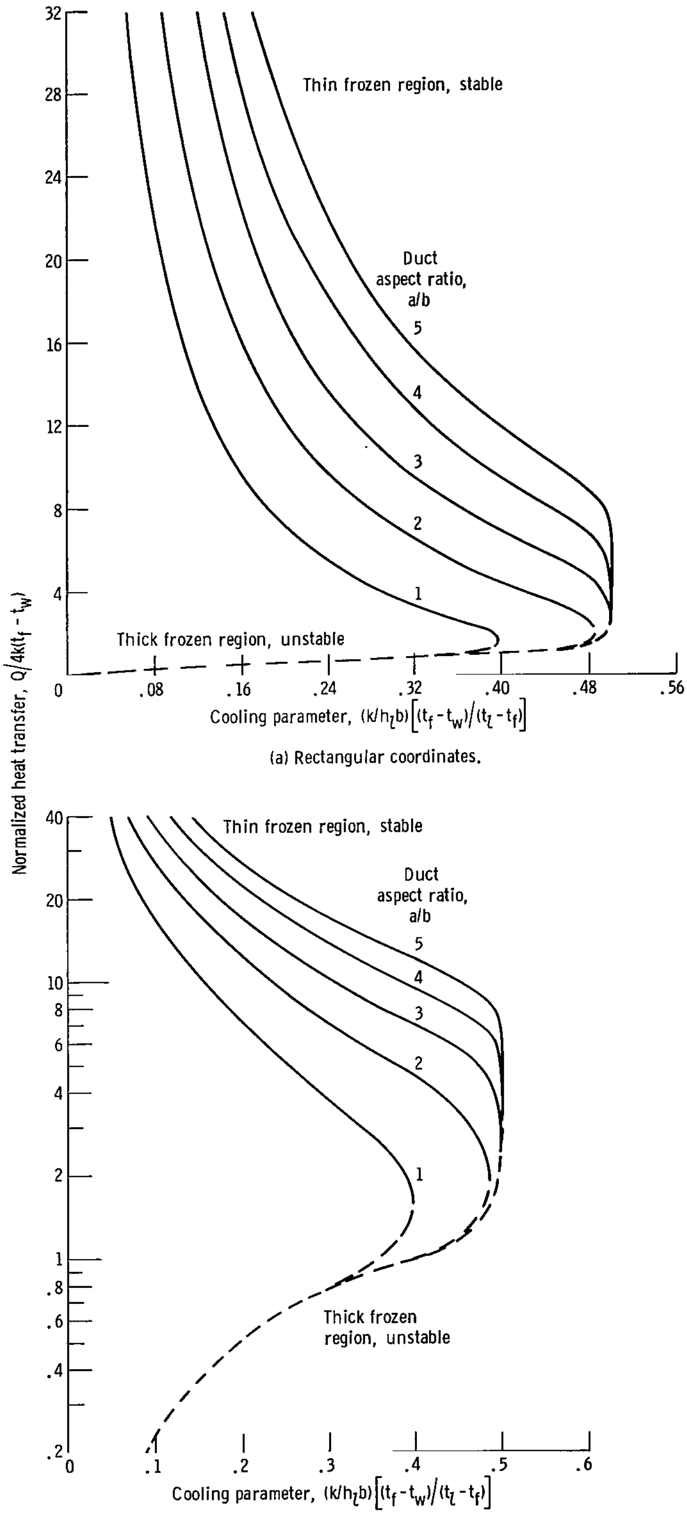


(d) Duct aspect ratio, 4.



(e) Duct aspect ratio, 5.

Figure 9. - Concluded.



(a) Rectangular coordinates.

(b) Semilogarithmic coordinates.

Figure 10. - Relation between total heat transfer and parameter containing imposed conditions.

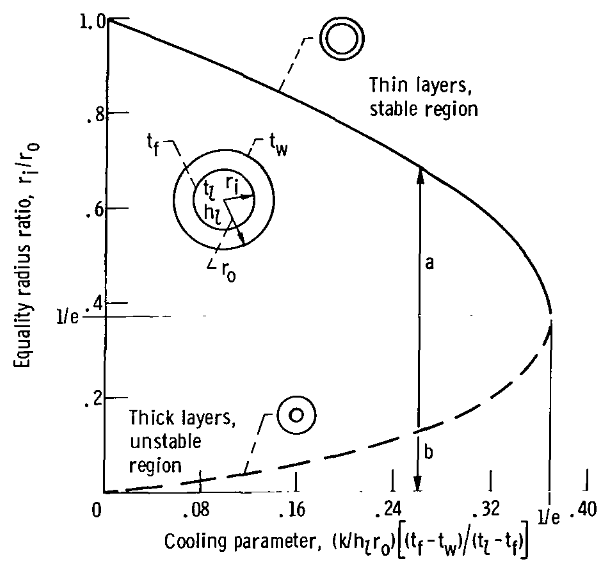


Figure 11. - State of equality between heat convection from liquid and heat conduction in frozen layer inside a circular tube.

NASA-Langley, 1970 —— 33 E-5104