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TITLE- Optimal Control for a Rocket in a Three-Dimensional Central Force Field

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ABSTRACT

A fuel optimum solution for a rocket in a drag-free central force field is presented. The solution is restricted to powered flights in which the maximum position change is small compared with the distance from the center of the force field. This condition is satisfied by most powered maneuvers currently being utilized in space trajectories.

First, the solution of the problem in a uniform force field is obtained using Pontryagin's Principle in the Optimal Control theory. A perturbation technique is then used to solve the central force field problem. The introduction of the perturbation technique makes the differential equations integrable, but at the same time imposes the position change restriction.

Because of the lengthy form of the integrals obtained, a numerical method is recommended for evaluating the integration constants.

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TECHNICAL MEMORANDUM

NASW-417

1.0 INTRODUCTION

This memorandum presents a restricted solution of the optimum thrust orientation program for a rocket in a drag-free, three-dimensional central force field. The solution is analytically obtained using the theory of optimal control. The restriction on the applicability of the solution is that the maximum distance between any two points on the flight path must be small when compared with the distance from the center of the force field. This restriction exists because of the utilization of a perturbation technique which is introduced in order to make the equations integrable. Since most space trajectories being considered currently consist of long coasting arcs and short powered flight arcs, this restricted solution has numerous applications in physical problems. Powered lunar descent trajectories and transfers between trajectories are typical problems in which the solution may be used.

The general, unrestricted solution of this problem has not yet been obtained. Only a few of the integrals of the Euler-Lagrange differential equations for the problem are known[1]. The optimum thrust program for a rocket in a two dimensional uniform force field was first obtained using the calculus of variations by Lawden in 1957[2]. A summary and some additional results can be found in Reference [3]. Leitmann later treated the problem again using the theory of optimal control[4,5]. The second section of this memo presents a brief summary of Pontryagin's theory of optimal control[4,5]. Using the theory, the optimal solution for a rocket in a three-dimensional uniform force field is obtained in Section 3. The problem in a central force field is formulated in Section 4. It is then shown that under the restriction described above, the system can be considered as a neighboring system of the uniform force field problem. A perturbation technique is used to simplify the differential equations based on the already integrated uniform force field system as the parent system. The perturbed system is integrated and these integrals are combined with the integrals for the parent system to furnish the restricted solution to the central force field problem.

Two things concerning the solution should be noted. First, the solution obtained from Pontryagin's Maximum Principle has only satisfied some necessity tests for being an optimum; its optimality is not guaranteed. Some theorems on the existence of optimal controls for linear systems have been established [4], but they do not apply to the problem being considered. Second, if the solution to the parent system was in fact an optimal solution, the optimum may no longer exist in the perturbed system. However, if it does exist, then the solution to the perturbed system is a good representation of the optimal solution. Without the sufficiency test and the existence proof, one can only rely on numerical comparison to determine the validity of the solution for each individual problem.

A numerical targeting scheme is being developed to evaluate the integration constants for given boundary conditions. A discussion of the scheme is given in the last section. Details of the numerical method and illustrative applications are not included in this memorandum. It is intended to apply the solution to obtain a fuel optimum thrust orientation program for the powered descent phase of the lunar landing mission.

2.0 OPTIMAL CONTROL FOR A DYNAMICAL SYSTEM

The problem of finding an optimal control for a dynamical system is briefly described and formulated in this section.

2.1 The State and the Control

Consider a dynamical system characterized by a set of n variables $x_1, x_2, \ldots, x_n$. These variables are functions of time of class C^2, and they define the state of the system at any instant of time. If a vector in an n-dimensional Euclidean space E^n is defined as

$$\vec{x} = (x_1, \ldots, x_n),$$

the state of the system at any instant of time may be represented by a point in the space which will be referred to as the state space.

The behavior of the system is governed by a set of differential equations called the state equations given by

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u_1, \ldots u_m)^*$$

$$i = 1, 2, \ldots, n$$

where $u_1, u_2, \ldots, u_m$ are co ?ol variables. The vector $\vec{u} \triangleq (u_1, u_2, \ldots, u_m)$, which lies in an m-dimensional Euclidean space E^m, will be called the control. A control is admissable if it satisfies the two properties

i) $\vec{u} = \vec{u}(t)$ is a piecewise continuous function in some finite interval $t \epsilon [t^o, t^f]$.

ii) $\vec{u} \epsilon U \forall t \epsilon [t^o, t^f]$ and $U \subset E^m$ is a prescribed, bounded set.

The functions f_i are assumed to be continuous in all arguments, $\frac{\partial f_i}{\partial x_j}$ exist and are continuous for $i,j=1,2,\ldots,n$.

2.2 The Cost Variable

The cost to transfer the system from a state $\vec{x}(t_1)$ to another state $\vec{x}(t_2)$ is given by the integral

$$\int_{t^o}^{t} f_o(\vec{x},\vec{u})dt, \ \vec{u} \epsilon U \text{ and } t \epsilon [t^o, t^f].$$

*Time may appear explicitly in the equation as a state variable.

Note that the cost is dependent on the path of the transfer and the path is dependent on the control $\vec{u}$. Introducing a cost variable x_o defined by

$$x_o(t) = \int_{t^o}^{t} f_o(\vec{x}(\tau), \vec{u}(\tau))d\tau,$$

one has

$$\dot{x}_o = f_o(\vec{x}, \vec{u})$$

and

$$x_o(t^o) = 0.$$

Let the state space E^n be augmented to E^{n+1} by including the cost variable x_o. A point in the augmented state space is then given by

$$\vec{x} = (x_o, x_1, \ldots, x_n).$$

This vector in E^{n+1} will be referred as the state of the system. The state equations can now be written in the vector form

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u}) \qquad (1)$$

where $\vec{f} = (f_o, f_1, \ldots, f_n)$ is also a vector in E^{n+1}. Equation (1) is assumed to have a unique solution for every given initial condition $\vec{x}(t^o) = \vec{x}^o$ with a given control $\vec{u}(t)$. Note that x_o does not appear explicitly in $\vec{f}$.

2.3 The Initial and The Terminal Manifolds

The state of the system is prescribed at $t = t^o$ and $t = t^f > t^o$ by some end conditions

$$\begin{cases} W_r(\vec{x}(t^o)) = 0, & r=1, \ldots, p \leq n \\ W_s(\vec{x}(t^f)) = 0, & s = 1, \ldots, q \leq n, \end{cases} \tag{2}$$

in which $t^f - t^o$ is not specified and the cost variable x_o does not appear explicitly. The equations $W_r(\vec{x}(t^o)) = 0$ define an (n-p)-dimensional manifold, called the initial manifold, in the state space E^n. The equations $W_s = 0$ define an (n-q)-dimensional manifold, called the terminal manifold, in the E^n. Since $x_o(t^o) = 0$, the initial manifold is unchanged when E^n is augmented to E^{n+1}. On the other hand, since $x_o(t^f)$ is not restricted by the end conditions, the terminal manifold becomes (n+1-q)-dimensional in E^{n+1}. Note that if p=q=n, the initial and the terminal manifolds each reduce to a single point in the state space E^n.

2.4 The Optimal Control and the Pontryagin's Maximum Principle

A trajectory of the system is a solution curve of equation (1) in E^{n+1} whose projection in E^n initiates in the initial manifold and terminates in the terminal manifold (see Figure 1).

An optimal control $\vec{u}^*(t)$, $t^o \leq t \leq t^f$, is a control $\vec{u} \in U$ whose resulting trajectory minimizes the total cost $x_o(t^f)$. The minimum value of the cost $x_o(t^f)$ is unique but the optimal control and the corresponding optimal trajectory may not be unique. In other words, there may exist more than one control within U that results in different trajectories with the same minimum cost.

A neccessary condition for a control $\vec{u} \in U$ to be optimal is given by a maximum principle established by Pontryagin. The maximum principle can also be used to find the optimal control (or controls) if it exists. Before stating the principle, some additional terms must be introduced. An adjoint variable $\vec{\lambda}(t) = (\lambda_o, \lambda_1, \ldots, \lambda_n)$ is a vector in E^{n+1} and its components satisfy the adjoint equations

$$\lambda_j = - \sum_{i=o}^{n} \frac{\partial f_i(\vec{x}, \vec{u})}{\partial x_j} \lambda_i \quad j=0,1. \ldots, n. \tag{3}$$

A function H is formed by taking the scalar product of $\vec{\lambda}$ and $\vec{f}$

$$H(\vec{x}, \vec{\lambda}, \vec{u}) = \vec{\lambda} \cdot \vec{f}. \tag{4}$$

Using the function H, the state equation (1) and the adjoint equation (3) can be rewritten as

$$\begin{cases} \dot{x}_j = \dfrac{\partial H(\vec{x},\vec{\lambda},\vec{u})}{\partial \lambda_j} \\ \dot{\lambda}_j = - \dfrac{\partial H(\vec{x},\vec{\lambda},\vec{u})}{\partial x_j} \qquad j=0,1,\ldots, n. \end{cases} \tag{5}$$

For the problem given above, Pontryagin's Maximum

Principle states that if $\vec{u}^*(t)$, $t^\circ \le t \le t^f$, is an optimal control, which results in an optimal trajectory $x^*(t)$, then there exists a nonzero continuous function $\vec{\lambda}(t)$, satisfying the adjoint equations, such that

$$\left.\begin{array}{ll} \text{i)} & \sup\limits_{\vec{u}\in U} H(\vec{\lambda}(t), \vec{x}^*(t), \vec{u}) = H(\vec{\lambda}(t), \vec{x}^*(t), \vec{u}^*(t)), \\ \text{ii)} & H\left(\vec{\lambda}(t), \vec{x}^*(t), \vec{u}^*(t)\right) = 0, \\ \text{iii)} & \lambda_o(t) = \text{constant} \le 0, \end{array}\right\} \text{for all } t\in[t^\circ, t^f]$$

iv) The projection of $\vec{\lambda}(t)$ in the state space E^n is normal to the initial manifold at $t=t^\circ$ and is normal to the terminal manifold at $t=t^f$.

2.5 The Transversality Condition

Condition (iv) of the principle is also called the transversality condition. Let $\vec{n}^\circ$ be a vector in E^n tangent to the initial manifold at $x^*(t^\circ)$. $\vec{n}^\circ = (n_1^\circ, \dots n^\circ{}_n)$ must satisfy

$$\sum_{j=1}^{n} \left.\frac{\partial W_r(\vec{x})}{\partial x_j}\right|_{\vec{x}=\vec{x}^*(t^\circ)} n_j^\circ = 0 \qquad r=1, \dots, p \tag{6.1}$$

With equation (6.1), p of the n components of $\vec{n}^\circ$ may be expressed in linear combinations of the remaining (n-p) components, which are arbitrary. The condition that $\vec{\lambda}(t)$ must be normal to the initial manifold can now be expressed by

$$\vec{\lambda}(t^{\circ}) \cdot \vec{\eta}^{\circ} = 0. \tag{6.2}$$

Since (n-p) of the n components in $\vec{\eta}^{\circ}$ are arbitrary, the coefficients of these arbitrary components in (6.2) must vanish, yielding (n-p) conditions on $\vec{x}$ and $\vec{\lambda}$ at $t = t^{\circ}$. Similarly, (n-q) conditions are given at the terminal manifold by

$$\sum_{d=1}^{n} \left. \frac{\partial W_s(x)}{\partial x_j} \right|_{\vec{x}=\vec{y}^{*}(t^f)} \eta_j^f \qquad s=1, \ldots, q \tag{7.1}$$

and

$$\vec{\lambda}(t^f) \cdot \vec{\eta}^f = 0. \tag{7.2}$$

Finding the optimal control $\vec{u}^*$ and optimal trajectory $\vec{x}^*$ involves integrating the (n+1) state equations and the (n+1) adjoint equations. Excluding $\lambda_o(t^{\circ})$ and $x_o(t^{\circ})$, a total of 2n integration constants arise from the integrals. These constants are evaluated using the (p+q) end conditions and the (2n-p-q) transversality conditions. $\lambda_o(t^{\circ}) \triangleq \lambda_o^{\circ}$ is unrestricted and $x_o(t^{\circ}) = x_o^{\circ} \triangleq 0$ by definition.

2.6 The Bang-Bang Control

If some of the components of the control $\vec{u}$ appear linearly in the state equations, that is, if

$$\dot{\vec{x}} = F(\vec{x},\vec{u}^n)\vec{u}^\ell + \vec{f}'(\vec{x},\vec{u}^n)$$

where

$$\vec{u}^\ell = (u_1, u_2, \ldots, u_v),$$

$$\vec{u}^n = (u_{v+1}, u_{v+2}, \ldots, u_m),$$

$\vec{f}'$ is a vector function nonlinear in $\vec{u}^n$, and F is a (n+1)x(m-r) matrix, then the function H assumes the form

$$H(\vec{x},\lambda,\vec{u}) = \vec{\lambda}\cdot F\vec{u}^\ell + \vec{\lambda}\cdot\vec{f}' \tag{8}$$

The first term of H written in component form is

$$\vec{\lambda}\cdot F\vec{u}^\ell = \sum_{k=1}^{v} \sigma_k u_k$$

where

$$\sigma_k = \sum_{j=o}^{n} \lambda_j F_{jk}, \qquad k=1,2, \ldots, v.$$

If each contr.l which appears linearly is constrained by

$$u_k^{min} \leq u_k \leq u_k^{max} \qquad k = 1,2, \ldots, v$$

then the condition that $H(\vec{\lambda}, \vec{x}*, \vec{u}*)$ is the supremum of $H(\vec{\lambda}, \vec{x}*, \vec{u})\ \forall\ \vec{u} \in U$ implies

$$u_k^*(t) = u_k^{min} \text{ if } \sigma_k(t) < o$$

$$u_k^*(t) = u_k^{max} \text{ if } \sigma_k(t) > o$$

σ_k is called the <u>switching function</u> for the k^{th} control u_k; whenever $\sigma_k(t)$ changes sign, u_k^* switches from u_k^{min} to the u_k^{max} or conversely. If $\sigma_k(t)$ is continuous and it crosses zero only a finite number of times in $[t^o, t^f]$, then its corresponding control variable u_k is called a <u>bang-bang control</u> [4], or simply, u_k is bang-bang. A bang-bang control takes on only its limiting values through the entire time interval $[t^o, t^f]$.

2.7 Non-integral cost variable

The cost of transfer in some dynamical systems is given by

$$\text{cost} = \int_{t^o}^{t^f} f_o(\vec{x},\vec{u})dt + G(\vec{x}(t^f)). \qquad (9)$$

It contains a non-integral term G in addition to the integral. In order to apply the maximum principle, one may convert the cost into an integral form by introducing two new variables x_{n+1} and u_{m+1} given by

$$x_{n+1} = G(\vec{x})$$

$$u_{m+1} = \dot{x}_{n+1}.$$

Consequently, the cost becomes

$$\text{cost} = \int_{t^o}^{t^f} (f_o + u_{m+1})dt.$$

The maximum principle can then be applied. It was shown [4] that the new variables x_{n+1} and u_{m+1} do not enter in either the H function or the transversality conditions; therefore, conditions i), ii) and iii) in the principle can be directly applied disregarding the new variables. The new adjoint variable λ_{n+1} corresponding to x_{n+1} was shown to satisfy

$$\lambda_{n+1} = -\lambda_o = \text{constant} \leq o.$$

Hence, λ_{n+1} can also be disregarded. The only necessary modification caused by increasing the dimension of the state space is in the terminal transversality condition. When condition (iv) of the maximum principle is applied to the new system, one gets instead of (7.1) and (7.2) the following

$$\sum_{j=1}^{n} \frac{\partial W_s(\vec{x})}{\partial x_j} \Bigg|_{\vec{x}=\vec{x}^*(t^f)} n_j^f = 0, \quad s=1, \ldots, q \tag{10.1}$$

$$-\lambda_o \sum_{j=1}^{n} \frac{\partial G(\vec{x})}{\partial x_j} n_j^f + \sum_{j=1}^{n} \lambda_j n_j^f = 0 \tag{10.2}$$

In summary, systems having nonintegral cost variables given by (9) can be treated the same as systems having integral cost variables except that (10.1) and (10.2) are used in place of (7.1) and (7.2).

3.0 ROCKET IN A UNIFORM FORCE FIELD

The optimal control for a rocket in a drag-free two-dimensional uniform force field has been found [5]. The solution of the same problem in a three-dimensional uniform force field is obtained here. The solution is then used as the basis of perturbation from which the restricted solution for a rocket in a central force field will be found.

3.1 State Equations

A rocket is placed in a uniform force field. Let x, y, and z denote the components of the position vector of the rocket with reference to an inertially fixed rectangular coordinate frame. The frame is chosen such that the field force is in the negative y-direction (Figure 2). The gravitational force applied to the rocket is denoted by mg_o, where m is the mass of the rocket and g is a constant. Let β denote the fuel mass flow rate of the propulsion system and c denote the effective exhaust velocity. The magnitude of the thrust vector $\vec{T}$ is given by

$$|\vec{T}| = c\beta.$$

The direction of the thrust vector is characterized by the two angles θ and ϕ ; θ is the angle between $\vec{T}$ and the xy-plane and ϕ is the angle between the x-axis and the projection of $\vec{T}$ in the xy-plane (Figure 2). Defining u, v and w to be the components of the velocity vector in the same reference frame, one can write the equations of motion in the following form

$$\dot{x} - u = 0,$$

$$\dot{y} - v = 0,$$

$$\dot{z} - w = 0,$$

$$\dot{u} - (c\beta \cos\theta \cos\phi)/m = 0,$$

$$\dot{v} + g - (c\beta \cos\theta \sin\phi)/m = 0,$$

$$\dot{w} - (c\beta \sin\theta)/m = 0.$$

The total mass moving with the rocket decreases as fuel is expelled through the exhaust. The relationship between the mass and the fuel flow rate is given by the differential equation

$$\dot{m} + \beta = 0.$$

The state of the system is defined by the seven variables $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = u$, $x_5 = v$, $x_6 = w$ and $x_7 = m$. For generality, let the cost to transfer the system from t^o to t be of the form of equation (9), i.e.,

$$\text{cost} = x_o(t) + G(\vec{x}(t))$$

$$= \int_{t^o}^{t} f_o(\vec{x}(t), \vec{u}(t))dt + G(\vec{x}(t))$$

The augmented state space is an E^8 containing vectors $x = (x_o, x_1, \ldots, x_7)$. If the cost is defined as the total fuel consumption, then one has

$$f_o \equiv 0, \qquad x_o \equiv 0$$

and

$$G(\vec{x}(t)) = x_7(t^o) - x_7(t).$$

Since $x_o \equiv 0$, there is no need to bring λ_o into the adjoint equations (although λ_o will appear in the transversality condition). The fuel flow rate and the two steering angles are the control variables of the system; however, it is more convenient to introduce the following alternative definition. Let $\vec{u} = (u_1, u_2, u_3, u_4)$ be given by

$$u_1 = \beta,$$

$$u_2 = \cos\theta \cos\phi,$$

$$u_3 = \cos\theta \sin\phi,$$

$$u_4 = \sin\theta,$$

with the constraints

$$|u_2|, \ |u_3|, \ |u_4| \leq 1,$$

and

$$u_2^2 + u_3^2 + u_4^2 = 1,$$

the unit vector (u_2, u_3, u_4) represents the direction of the thrust. Another constraint is introduced by assuming that the fuel flow rate u_1 is throttlable between an upper limit u_1^{max} and a lower limit $u_1^{min} \geq 0$, i.e.,:

$$o \leq u_1^{min} \leq u_1 \leq u_1^{max}$$

These two constraints define the admissable set $U \subset E^4$ for the control $\vec{u}$. With the definition of $\vec{x}$ and $\vec{u}$, the state equations become

$$\dot{x}_1 = x_4, \tag{11.1}$$

$$\dot{x}_2 = x_5, \tag{11.2}$$

$$\dot{x}_3 = x_6, \tag{11.3}$$

$$\dot{x}_4 = \frac{c}{x_7} u_1 u_2, \tag{11.4}$$

$$\dot{x}_5 = \frac{c}{x_7} u_1 u_3 - g_o, \tag{11.5}$$

$$\dot{x}_6 = \frac{c}{x_7} u_1 u_4, \tag{11.6}$$

and

$$\dot{x}_7 = -u_1. \tag{11.7}$$

3.2 End Conditions

Let the position and velocity of the rocket be given at $t = t^o = 0$ and at $t = t^f > 0$. Let the mass of the rocket be given at $t = 0$. These are represented by the following equations

$$x_1(0) = x(0) = x^o, \tag{12.1}$$

$$x_2(0) = y(0) = y^o, \tag{12.2}$$

$$x_3(0) = z(0) = z^o, \tag{12.3}$$

$$x_4(0) = u(0) = u^o, \tag{12.a4}$$

$$x_5(0) = v(0) = v^o, \tag{12.5}$$

$$x_6(0) = w(0) = w^o, \tag{12.6}$$

$$x_7(0) = m(0) = m^o, \tag{12.7}$$

$$x_1(t^f) = x(t^f) = x^f, \tag{12.8}$$

$$x_2(t^f) = y(t^f) = y^f, \tag{12.9}$$

$$x_3(t^f) = z(t^f) = z^f, \tag{12.10}$$

$$x_4(t^f) = u(t^f) = u^f, \tag{12.11}$$

$$x_5(t^f) = v(t^f) = v^f, \tag{12.12}$$

$$x_6(t^f) = w(t^f) = w^f. \tag{12.13}$$

Equations (12.1) to (12.7) define a fixed point in the state space, equations (12.8) to (12.13) define a one-dimensional terminal manifold. Consequently, the initial transversality condition, given (6.1) and (6.2), will be trivially satisfied.

Because the cost variable is a non-integral type, the terminal transversality condition is given by Equations (10.1) and (10.2). Substituting the end conditions (12.8) to (12.13) into (10.1) yields

$$\eta_i^f = 0 \qquad i = 1,2,\ldots, 6.$$

Consequently, the tangent vector at the terminal manifold is given by

$$\vec{\eta}^f = (0, \ldots, 0, \eta_7^f).$$

The terminal transversality condition (10.2) then becomes

$$(\lambda_o + \lambda_7)\eta_7^f \Big|_{t=t^f} = 0.$$

Since η_7^f is arbitrary, one has

$$\lambda_7(t^f) = -\lambda_o. \qquad (12.14)$$

The maximum principle stated that $\lambda_o \leq o$; hence (12.14) does not fix the value of $\lambda_7(t^f)$. In general, both $\lambda_o = o$ and $\lambda_o = -1 < o$ should be tested.

3.3 The H Function and the Adjoint Equation

To find the optimal control $\vec{u}^* \epsilon U$ for the system, the adjoint vector $\vec{\lambda} = (\lambda_o, \lambda_1, \ldots, \lambda_n)$ is introduced. The components of the adjoint vector satisfy the adjoint equations given by (3) which become

$$\dot{\lambda}_1 = 0, \qquad (13.1)$$

$$\dot{\lambda}_2 = 0, \qquad (13.2)$$

$$\dot{\lambda}_3 = 0, \qquad (13.3)$$

$$\dot{\lambda}_4 = -\lambda_1, \qquad (13.4)$$

$$\dot{\lambda}_5 = -\lambda_2, \qquad (13.5)$$

$$\dot{\lambda}_6 = -\lambda_3, \qquad (13.6)$$

$$\dot{\lambda}_7 = \frac{cu_1}{x_7^2}(\lambda_4 u_2 + \lambda_5 u_3 + \lambda_6 u_4) \qquad (13.7)$$

Forming $\vec{\lambda} \cdot \vec{f}$ yields the H function

$$H(\vec{x},\vec{\lambda}, \vec{u}) = \lambda_1 x_4 + \lambda_2 x_5 + \lambda_3 x_6 - \lambda_5 g_o$$

$$+ [\frac{c}{x_7}(\lambda_4 u_2 + \lambda_5 u_3 + \lambda_6 u_4) - \lambda_7]u_1 \qquad (14)$$

Recall that u_1 satisfies $u_1^{min} \leq u_1 \leq u_1^{max}$ and it is seen to appear linearly in H ; hence, u_1 is bang-bang provided that its switching function

$$\sigma_1 = \frac{c}{x_7} (\lambda_4 u_2 + \lambda_5 u_3 + \lambda_6 u_4) - \lambda_7 \qquad (15)$$

vanishes only a finite number of times in $[t^o, t^f]$.

3.4 Integrals of the System

The integrals of equations (12.1) through (12.6) are easily obtained

$$\lambda_1 = C_1, \qquad (16.1)$$

$$\lambda_2 = C_2, \qquad (16.2)$$

$$\lambda_3 = C_3, \qquad (16.3)$$

$$\lambda_4 = C_4 - C_1 t, \qquad (16.4)$$

$$\lambda_5 = C_5 - C_2 t, \qquad (16.5)$$

$$\lambda_6 = C_6 - C_3 t, \qquad (16.6)$$

where the C_i's are the integration constants. In view of the continuity requirement on $\vec{\lambda}$, C_1 to C_6 remain unchanged in $[t^o, t^f]$ despite the possible discontinuities in u_1 caused by switchings.

In equation (14), the term $(\lambda_4 u_2 + \lambda_5 u_3 + \lambda_6 u_4)$ represents the scalar product of the vector $(\lambda_4, \lambda_5, \lambda_6)$ and the unit vector (u_2, u_3, u_4). Since $\frac{cu_1}{x_7}$ is a non-negative quantity, the product

$$\frac{cu_1}{x_7} (\lambda_4 u_2 + \lambda_5 u_3 + \lambda_6 u_4)$$

attains its maximum when the unit vector (u_2, u_3, u_4) is co-directional with $(\lambda_4, \lambda_5, \lambda_6)$. Consequently, this becomes a necessary condition for H to be the supremum as was required in the maximum principle. The co-directional requirement implies that

$$u_2 = \lambda_4 / \lambda, \tag{16.7}$$

$$u_2 = \lambda_5 / \lambda, \tag{16.8}$$

$$u_4 = \lambda_6 / \lambda \tag{16.9}$$

where $\lambda = \sqrt{\lambda_4^2 + \lambda_5^2 + \lambda_6^2} \geq 0$ is the magnitude of $(\lambda_4, \lambda_5, \lambda_6)$. Converting the control variables back to the steering angles, one has

$$\tan \phi = u_3/u_2 = \lambda_5/\lambda_4 = \frac{C_5 - C_2 t}{C_4 - C_1 t}$$

and

$$\tan\theta = u_4/\sqrt{u_2^2 + u_3^2} = \lambda_6/\sqrt{\lambda_4^2 + \lambda_5^2}$$

$$= \frac{C_6 - C_3 t}{\sqrt{(C_4 - C_1 t)^2 + (C_5 - C_2 t)^2}}$$

Two things should be noted at this point. First, the direction controls $\phi(t)$ and $\theta(t)$ are obtained independently from the burning rate u_1. Second, if $\theta \equiv 0$ (a planar problem), the result reduces to the bi-linear tangent steering law obtained by Lawden [2].

Some general remarks may be made about the steering angles. The adjoint variables λ_4, λ_5, and λ_6 are linear functions of time; each may vanish at most once in the interval $[t^\circ, t^f]$. The angles θ and ϕ defined by these adjoint variables are continuous functions of time except for the two special cases:

i) $\lambda_4(t_1) = \lambda_5(t_1) = 0$ for some $t_1 \epsilon [t^\circ, t^f]$

ii) $\lambda_4(t_1) = \lambda_5(t_1) = \lambda_6(t_1) = 0$ for some $t_1 \epsilon [t^\circ, t^f]$.

For the first case, ϕ will have a jump of π radians at $t = t_1$. For the second case, $\lambda(t_1) = 0$ and both σ and ϕ will each have a jump of π radians at $t = t_1$.

As has been shown previously, u_1 is bang-bang if the switching function σ_1 vanishes only a finite number of times. To study the behavior of σ_1, the obtained integrals are used to rewrite equation (15) as

$$\sigma_1(t) = \frac{c\lambda}{x_7} - \lambda_7$$

Differentiating σ_1 once and using equations (11.7) and (13.7), one has

$$\dot{\sigma}_1(t) = \frac{c\dot{\lambda}}{x_7} = \frac{c(At - E)}{x_7(t)\ \lambda(t)}$$

where

$$A = (C_1^2 + C_2^2 + C_3^2) \geq o$$

and

$$E = C_1C_4 + C_2C_5 + C_3C_6.$$

In this equation, c is a positive constant and neither x_7 (the mass of the rocket) nor $\lambda(t)$ change their signs in $[t^o,t^f]$. Hence, three possibilities exist concerning the behavior of σ_1 and $\dot{\sigma}_1$

i) $\dot{\sigma}_1 \equiv o$ and $\sigma \equiv o$ for all $t \epsilon [t^o, t^f]$. This arises when $C_1 = C_2 = C_3 = 0$ and $\sigma_1(t^o) = 0$. The H function reduces to $H = -\lambda_5 g$.

The condition $H = 0$ for optimality implies $\lambda_5 = 0$. Consequently, the steering angles reduce to

$$\tan \phi = 0$$

and

$$\tan \theta = C_6/C_4 = \text{Constant}$$

which represent a constant horizontal direction. Because of the conditions imposed on the integration constants, this solution cannot satisfy the prescribed end conditions in general.

ii) $\dot{\sigma}_1 \equiv 0$ and $\sigma_1 = \text{Constant} \neq 0$ for all $t \, \epsilon [t^o, t^f]$.

This case also requires $C_1 = C_2 = C_3 = 0$. The corresponding H function becomes

$$H = \sigma_1 u_1 - \lambda_5 g_o = 0.$$

Consequently

$$\sigma_1 = C_5 g_o / u_1.$$

If $\sigma_1 > o$, one has $u_1 = u_1^{max}$ for $t\epsilon[t^o, f^f]$. If $\sigma_1 < 0$, one has $u_1 = u_1^{min}$ for $t\epsilon[t^o, t^f]$. Either case results in

$$\tan \phi = C_5/C_4 = \text{Constant}$$

and

$$\tan \theta = C_6/\sqrt{C_4^2 + C_5^2} = \text{Constant.}$$

This solution is again unacceptable in the general case for the same reason given in i).

iii) $A \neq 0$ and $\sigma_1(t) \neq 0$ in $[t^o, t^f]$.

This is the general form of the switching function. The numerator of $\dot{\sigma}_1$, $(At-E)$, is a linear function of time; it may vanish at most once and this can occur only when $E>0$. When $(At-E) = 0$, one of the following is true

a) $\dot{\sigma}_1 = \dot{\lambda} = 0$, $\lambda \neq 0$

This implies that σ_1 has an extremum. At this point

$$\ddot{\sigma}_1(t) = \frac{cA}{x_7 \lambda} > 0$$

hence the extremum is a minimum. Furthermore, since $\dot{\sigma}_1$ does not vanish elsewhere, σ_1 is monotonic on both sides of the extremum thus making the minimum a global minimum.

b) $\dot{\lambda}=\lambda=0$

This implies that $\dot{\sigma}_1$ has a jump from a negative value to a positive value resulting in a cusp in the shape of $\dot{\sigma}_1$. Again, since $\dot{\sigma}_1$ does not vanish elsewhere, the cusp is a global minimum.

In conclusion, the fuel rate control u_1 is bang-bang, its switching function σ_1 is either a strictly monotonic function or a function that is strictly monotonic on both sides of a global minimum. The resulting switching sequence is one of the following:

max — min — max

max — min

min — max

max —

min — .

Since u_1 is a piecewise constant function, its switching from one value to another does not affect the form of the integrated results. In the subsequent manipulations,

$$u_1 = \text{Constant } (= u_1^{max} \text{ or } u_1^{min}) \tag{16.10}$$

will be used to obtain the remaining integrals of the system. The integrals so obtained are valid everywhere in $[t^{o},t^{f}]$ but the integration constants will generally change their values at a switching point to maintain the continuity of the solution.

Integrating equation (11.7), one has

$$x_7 = C_7 - u_1 t. \qquad (16.11)$$

Letting

$$t = \frac{1}{u_1}(C_7 - x_7)$$

and

$$dt = -\frac{dx_7}{u_1},$$

equation (13.7) becomes

$$\frac{d\lambda_7}{dx_7} = \frac{-c}{u_1 {x_7}^2}\sqrt{Ax_7^2 + Bx_7 + D}$$

in which

$$A = (C_1^2 + C_2^2 + C_3^2) > 0$$

$$B = 2[C_1(C_4u_1 - C_1C_7) + C_2(C_5u_1 - C_2C_7) + C_3(C_6u_1 - C_3C_7)]$$

$$D = [(C_4u_1 - C_1C_7)^2 + (C_5u_1 - C_2C_7)^2 + (C_6u_1 - C_3C_7)^2] > 0.$$

Defining

$$Q = 4AD - B^2 = 4\Big\{[(C_4u_1 - C_1C_7)\,C_2 + (C_5u_1 - C_2C_7)C_3]^2 + [(C_4u_1 - C_1C_7)\,C_3 + (C_6u_1 - C_3C_7)C_1]^2 + [(C_5u_1 - C_2C_7)\,C_1 + (C_4u_1 - C_1C_7)C_2]^2\Big\} > 0,$$

the integral of the above differential equation is

$$\lambda_7 = \frac{c}{u_1}\left[\frac{\sqrt{Ax_7^2+Bx_7+D}}{x_7} - \sqrt{A}\,\sinh^{-1}\frac{2Ax_7+B}{\sqrt{Q}} - \frac{B}{2\sqrt{D}}\sinh^{-1}\frac{Bx_7+2D}{x_7\sqrt{Q}}\right] + C_8. \quad (16.12)$$

Since $x_7 = C_7 - u_1 t$ is now a known function of time, the integral(16.12) is well defined.

The integrals of equations (11.4), (11.5) and (11.6) are obtained similarly as

$$x_4 = -c[(C_4u_1 - C_1C_7)\; S_1\;(x_7) + C_1\; S_2(x_7)] + C_9, \quad (16.13)$$

$$x_5 = -c[(C_5u_1 - C_2C_7)\; S_1\;(x_7) + C_2\; S_2(x_7)] + C_{11} - g_o t + C_{10}, \quad (16.14)$$

and

$$x_6 = -c[(C_6u_1 - C_3C_7)\; S_1\;(x_7) + C_3\; S_2(x_7)] + C_{11} \quad (16.15)$$

in which

$$S_1(x_7) = \frac{-1}{\sqrt{D}}\;\sinh^{-1}\frac{Bx_7+2D}{x_7\sqrt{Q}}$$

$$S_2(x_7) = \frac{1}{\sqrt{A}} \sinh^{-1} \frac{2Ax_7 + B}{\sqrt{Q}} .$$

The integrals of equations (11.7) to (11.9) all involve the integrals of S_1 and S_2, which are obtained below.

$$\int S_1(x_7)\, dt = \frac{1}{\sqrt{D}} \int \sinh^{-1}\left(\frac{B}{Q} + \frac{2D}{x_7\sqrt{Q}}\right) dt.$$

Letting

$$\frac{2D}{x_7\sqrt{Q}} = s ,$$

one has

$$dt = \frac{-dx_7}{u_1} = \frac{-2D}{u_1\sqrt{Q}}\, d\left(\frac{1}{s}\right)$$

and the above integral is carried out by parts

$$\int S_1 dt = \frac{2\sqrt{D}}{u_1\sqrt{Q}} \int \left[\sinh^{-1}\left(\frac{B}{\sqrt{Q}} + s\right)\right] d\left(\frac{1}{s}\right)$$

$$= \frac{2\sqrt{D}}{u_1\sqrt{Q}} \left\{\frac{1}{s}\sinh^{-1}\left(\frac{B}{\sqrt{Q}} + s\right) - \int \frac{1}{s}\, d\left[\sinh^{-1}\left(\frac{B}{\sqrt{Q}} + s\right)\right]\right\} .$$

The integral part of the last expression can be computed he

$$\int \frac{ds}{s\sqrt{1 + \left(\frac{B}{\sqrt{Q}} + s\right)^2}} = \int \frac{ds}{s\sqrt{s^2 + \frac{2B}{\sqrt{Q}}s + \left(1 + \frac{B^2}{Q}\right)}}$$

$$= -\frac{1}{\sqrt{1+B^2/Q}} \sinh^{-1}\left[\frac{\left(1 + \frac{B^2}{Q}\right) + \frac{B}{\sqrt{Q}}s}{s}\right].$$

Substituting x_7 back into the results, one has the integral

$$\int S_1(x_7)dt = -\frac{x_7}{u_1} S_1(x_7) + \frac{1}{u_1} S_2(x_7).$$

The integral of S_2 is obtained similarly, with the result

$$\int S_2(x_7)dt = -\frac{1}{u_1}\left(x_7 + \frac{B}{2A}\right) S_2(x_7) + \frac{1}{u_1 A}\sqrt{Ax_7^2 + Bx_7 + D} .$$

Using the above results, equations (11.7) to (11.9) are integrated as

$$x_1 = \frac{c}{u_1}(C_4u_1 - C_1C_7)(x_7S_1 - S_2) + C_1(x_7 + \frac{B}{2A})S_2$$

$$- \frac{C_1}{A}\sqrt{Ax_7^2 + Bx_7 + D} + C_9t + C_{12}, \qquad (16.16)$$

$$x_2 = \frac{c}{u_1}(C_5u_1 - C_2C_7)(x_7S_1 - S_2) + C_2(x_7 + \frac{B}{2A})S_2$$

$$- \frac{C_2}{A}\sqrt{Ax_7^2 + Bx_7 + D} - \frac{g_o t^2}{2} + C_{10}t + C_{13} \qquad (16.17)$$

and

$$x_3 = \frac{c}{u_1}(C_6u_1 - C_3C_7)(x_7S_1 - S_2) - C_3(x_7 + \frac{B}{2A})S_2$$

$$- \frac{C_3}{A}\sqrt{Ax_7^2 + Bx_7 + D} + C_{11}t + C_{14} \qquad (16.18)$$

Equations (16.1) to (16.8) give the solutions for the state variable $\vec{x}$, the adjoint variable $\vec{\lambda}$ and the control $\vec{u}$. A total of fourteen integration constants have been introduced in the solutions. These constants and the total transfer time $(t^f - t^o)$ are the fifteen unknown constants to be evaluated in the integrals. The end conditions and the terminal transversality condition given by equations (12.1) to (12.14) provide fourteen relationships. In equation (12.14), since no evident reason was found to reject either $\lambda_o = 0$ or $\lambda_o = -1$, both cases are to be considered. The condition $H = 0$ gives the fifteenth relationship, which now reduces to

$$-C_8u_1 - C_5g_o + C_9 + C_{10} + C_{11} = 0. \qquad (17)$$

Because of the switching property of u_1, these fifteen relationships cannot be solved algebraically to determine the fifteen unknown constants. The first six constants have been shown to remain unchanged at a switching point. The remaining constants (C_7 to C_{14}) will, in general, change their values at a switching point. The continuity of $\vec{x}$ and λ_7 at the switching points are to be used to determine the changes. The switching times, if they exist, are given by $\sigma_1(t) = 0$. In view of the lengthy form of the integrals and the difficulties in evaluating these constants, a numerical scheme would be more desirable when dealing with a specific problem.

4.0 A ROCKET IN A CENTRAL FORCE FIELD

The problem of a rocket in a drag-free central force field is encountered repeatedly in the design of space trajectories. Since the general optimum fuel solution for this problem has not been completely integrated, a restricted solution is presented here in closed form. If the set of state equations for a rocket in a central force field is compared with the set for a rocket in a uniform force field, one sees that the differences between them become very small if the maximum position change of the rocket during the flight remains small when compared to the distance from the center of the central force field. This suggests that under such circumstances the system with a central force field can be considered as a perturbed system of a parent system having a uniform force field. When the solution to the parent system is known, the differential equations for the perturbed system may be simplified which sometimes makes them integrable, as in this case. The solution so obtained is restricted to powered flights having small position changes as described above.

4.1 The State Equations

Consider a rocket placed in a central force field (Figure 3). The center of the force field is located at 0. An inertial reference frame is chosen with its origin located on the surface of the central force body of radius R_o. The y-axis points along the radius that passes through the origin, the x and z-axes are tangent to the surface. If the same assumptions are made about the rocket as those in Section 3.1, the equations of motion for this system are

$$\dot{x} - u = 0,$$

$$\dot{y} - v = 0,$$

$$\dot{z} - w = 0,$$

$$\dot{u} - g_o\left(\frac{R_o}{R_o+h}\right)^2 \sin\tau \sin\rho - (c\beta \cos\theta \cos\phi)/m = 0,$$

$$\dot{v} - g\left(\frac{R_o}{R_o+h}\right)^2 \cos\tau - (c\beta \cos\theta \sin\phi)/m = 0,$$

$$\dot{w} - g_o\left(\frac{R_o}{R_o+h}\right)^2 \sin\tau \cos\rho - (c\beta \sin\theta)/m = 0$$

in which h is the altitude, g_o is the gravitational constant at the surface of the central force body, τ is the central angle measured from the y-axis to the rocket and ρ is the angle between the xy-plane and the plane containing the y-axis and the rocket. Defining

$$A = R_o \tau$$

$$\varepsilon = 1/R_o$$

and introducing the new variables

$$X = A \cos \rho$$

$$Z = A \sin \rho,$$

the above equations are expanded in a power series of ε. When terms involving second or higher powers of ε are omitted from these equations, one obtains the following:

$$\dot{X} + \varepsilon (\dot{X}h + X\dot{h}) - u = 0,$$

$$\dot{h} - \varepsilon (X\dot{X} + Z\dot{Z}) - v = 0,$$

$$\dot{Z} + \varepsilon (\dot{Z}h + Z\dot{h}) - w = 0,$$

$$\dot{u} - \varepsilon(c\beta \;\; \cos \theta \cos \phi)/m + \varepsilon g_o X = 0,$$

$$\dot{v} + g_o - \varepsilon(c\beta \;\; \cos \theta \sin \phi)/m - 2\varepsilon g_o h = 0,$$

$$\dot{w} - (c\beta \;\; \sin \theta)/m + \varepsilon g_o Z = 0.$$

If assumptions concerning the burning rate are made similar to those in Section 3.1, the mass flow rate equation remains unchanged

$$\dot{m} + \beta = 0.$$

Defining $y_1 = X$, $y_2 = h$, $y_3 = Z$, $y_4 = u$, $y_5 = v$, $y_6 = w$ and and $y_7 = m$, one may denote the state of the system by $\vec{y} \in E^7$. More than one derivative appears in each of the first three equations of motion. In order to convert these equations to the forms

$$\dot{y}_i = f_i(\vec{y}),$$

the first three equations are solved simultaneously for $\dot{X}$, $\dot{h}$ and $\dot{Z}$. The results, retaining only linear terms of ε, are

$$\dot{X} = u + \varepsilon hv,$$

$$\dot{h} = v + \varepsilon(Xu + hv + Zw)$$

$$\dot{Z} = w - \varepsilon Zv.$$

The control of the system is denoted by $\vec{v} = (v_1, v_2, v_3, v_4)$, in which

$$v_1 = \beta,$$

$$v_2 = \cos\theta \cos\phi,$$

$$v_3 = \cos\theta \sin\phi,$$

and

$$v_4 = \sin\theta.$$

The control must lie within the admissable set defined by

$$u_1^{max} \geq v_1 \geq u_1^{min},$$

$$|v_2|, |v_3|, |v_4| \leq 1$$

and

$$v_2^2 + v_3^2 + v_4^2 = 1.$$

With these definitions, the state equations of the perturbed system becomes

$$\dot{y}_1 = y_4 - \varepsilon(y_1 y_5 + y_2 y_4) \tag{18.1}$$

$$\dot{y}_2 = y_5 + \varepsilon(y_1 y_4 + y_3 y_6), \tag{18.2}$$

$$\dot{y}_3 = y_6 - \varepsilon(y_3 y_5 + y_2 y_6), \tag{18.3}$$

$$\dot{y}_4 = \frac{c}{y_7} v_1 v_3 - \varepsilon g_o y_1, \tag{18.4}$$

$$\dot{y}_5 = \frac{c}{y_7} v_1 v_3 - g_o + 2\varepsilon g_o y_2, \tag{18.5}$$

$$\dot{y}_6 = \frac{c}{y_7} v_1 v_4 - \varepsilon g_o y_3 \tag{18.6}$$

and

$$\dot{y}_7 = -v_1. \tag{18.7}$$

It can easily be seen that when $\varepsilon \to 0$, these equations approach the state equations of the parent system given by (11.1) to (11.7). The solution for y is therefore assumed to approach the solution of $\dot{x}$ when $\varepsilon \to 0$.

4.2 The H Function and the Adjoint Equations

Denoting the adjoint variable of the system by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_7)$, the H function reads

$$H(\vec{y}, \vec{v}, \vec{\alpha}) = \vec{\alpha}\cdot\dot{\vec{y}}$$

$$= \alpha_1 y_4 + \alpha_2 y_5 + \alpha_3 y_6 - \alpha_5 g_o$$

$$+\varepsilon\left[\frac{c}{y_7}(\alpha_4 v_2 + \alpha_5 v_3 + \alpha_6 v_4) - \alpha_7\right]v_1$$

$$- \varepsilon[\alpha_1(y_1 y_5 + y_2 y_4) - \alpha_2(y_1 y_4 + y_3 y_6)$$

$$+ \alpha_3(y_3 y_5 + y_2 y_6) + \alpha_4 g_o y_1$$

$$- 2\alpha_5 g_o y_2 + \alpha_6 g_o y_3]. \qquad (19)$$

The adjoint equations are

$$\dot{\alpha}_1 = \varepsilon(\alpha_1 y_5 - \alpha_2 y_4 + \alpha_4 g_o), \qquad (20.1)$$

$$\dot{\alpha}_2 = \varepsilon(\alpha_1 y_4 + \alpha_3 y_6 - 2\alpha_5 g_o), \qquad (20.2)$$

$$\dot{\alpha}_3 = \varepsilon(-\alpha_2 y_6 + \alpha_3 y_5 + \alpha_6 g_o), \qquad (20.3)$$

$$\dot{\alpha}_4 = -\alpha_1 + \varepsilon(\alpha_1 y_2 - \alpha_2 y_1), \qquad (20.4)$$

$$\dot{\alpha}_5 = -\alpha_2 + \varepsilon(\alpha_1 y_1 + \alpha_3 y_3), \qquad (20.5)$$

$$\dot{\alpha}_6 = -\alpha_3 + \varepsilon(-\alpha_2 y_3 + \alpha_3 y_2) \qquad (20.6)$$

$$\dot{\alpha}_7 = \frac{c v_1}{y_7^2}(\alpha_4 u_2 + \alpha_5 u_3 + \alpha_6 u_4). \qquad (20.7)$$

These equations approach the adjoint equations of the parent system when $\varepsilon \to 0$.

4.3 Equations of First Variation

Let the functions bearing an asterisk denote the solutions to the parent system. The solutions for the perturbed system are assumed to be expandable into power series in ε about $\varepsilon = 0$, with the expansions converging to the parent solutions when $\varepsilon \to 0$. When terms including second or higher orders of ε are dropped, the assumed solutions for the perturbed system have

$$y_i = x_i^* + \varepsilon \bar{x}_i, \; i=1, \ldots, 7$$

$$\alpha_i = \lambda_i^* + \varepsilon \bar{\lambda}_i, \; i=1, \ldots, 7$$

and

$$v_i = u_i^* + \varepsilon \bar{u}_i, \; i=1, \ldots, 4$$

where the barred quantities are unknown functions of time. Substituting these assumed solutions into equations (18.1) to (18.7) and eliminating the parent state equations, as well as terms of second or higher order in ε, one has

$$\dot{\bar{x}}_1 = \bar{x}_4 - (x_2^* x_4^* + x_1^* x_5^*), \qquad (21.1)$$

$$\dot{\bar{x}}_2 = \bar{x}_5 + (x_1^* x_4^* + x_3^* x_6^*), \qquad (21.2)$$

$$\dot{\bar{x}}_3 = \bar{x}_6 - (x_2^* x_6^* + x_3^* x_5^*), \qquad (21.3)$$

$$\dot{\bar{x}}_4 = \frac{c}{x_7^*}\left(\frac{-\bar{x}_7 u_1^* u_2^*}{x_7^*} + \bar{u}_1 u_2^* + \bar{u}_2 u_1^*\right) - g_o x_1^*, \qquad (21.4)$$

$$\dot{\bar{x}}_5 = \frac{c}{x_7^*}\left(\frac{-\bar{x}_7 u_1^* u_3^*}{x_7^*} + \bar{u}_1 u_3^* + \bar{u}_3 u_1\right) + 2g_o x_2^*, \qquad (21.5)$$

$$\dot{\bar{x}}_6 = \frac{c}{x_7^*}\left(\frac{-\bar{x}_7 u_1^* u_4^*}{x_7^*} + \bar{u}_1 u_4^* + \bar{u}_4 u_1^*\right) - g_o x_3^*, \qquad (21.6)$$

$$\dot{\bar{x}}_7 = -\bar{u}_1. \qquad (21.7)$$

Similarly, the adjoint equations become

$$\dot{\bar{\lambda}}_1 = \lambda_1^* x_5^* - \lambda_2^* x_4^* + \lambda_4^* g_o, \tag{22.1}$$

$$\dot{\bar{\lambda}}_2 = \lambda_1^* x_4^* + \lambda_3^* x_6^* - 2\lambda_5^* g_o, \tag{22.2}$$

$$\dot{\bar{\lambda}}_3 = -\lambda_2^* x_6^* + \lambda_3^* x_5^* + \lambda_6^* g_o, \tag{22.3}$$

$$\dot{\bar{\lambda}}_4 = -\bar{\lambda}_1 + \lambda_1^* x_2^* - \lambda_2^* x_1^*, \tag{22.4}$$

$$\dot{\bar{\lambda}}_5 = -\bar{\lambda}_2 + \lambda_1^* x_1^* + \lambda_3^* x_3^*, \tag{22.5}$$

$$\dot{\bar{\lambda}}_6 = -\bar{\lambda}_3 - \lambda_2^* x_3^* + \lambda_3^* x_2^*, \tag{22.6}$$

and

$$\dot{\bar{\lambda}}_7 = \frac{c}{(x_7^*)^2}[u_1^*(\bar{\lambda}_4 u_2^* + \lambda_4^* \bar{u}_2 + \bar{\lambda}_5^* u_3^* + \lambda_5^* \bar{u}_3^* + \bar{\lambda}_6 u_4^* + \lambda_6^* \bar{u}_4)$$

$$- 2u_1^* \lambda^* \frac{\bar{x}_7}{x_7^*} + \bar{u}_1 \lambda^*] \tag{22.7}$$

in which

$$\lambda^* = \sqrt{(\lambda_4^*)^2 + (\lambda_5^*)^2 + (\lambda_6^*)^2}.$$

4.4 Integrals of the Perturbed System

The first six adjoint equations in (22) can be directly integrated as

$$\bar{\lambda}_1 = C_1x_2^* - C_2x_1^* + (C_4t - \frac{C_1t^2}{2}) + \bar{C}_2, \qquad (23.1)$$

$$\bar{\lambda}_2 = C_1x_1^* + C_3x_3^* + 2g_o(C_5t - \frac{C_2t^2}{2}) + \bar{C}_2, \qquad (23.2)$$

$$\bar{\lambda}_3 = -C_2x_3^* + C_3x_2^* + g_o(C_6t - \frac{C_3t^2}{2}) + \bar{C}_3, \qquad (23.3)$$

$$\bar{\lambda}_4 = -g_o(\frac{C_4t^2}{2} - \frac{C_1t^3}{3}) + \bar{C}_1t + \bar{C}_4, \qquad (23.4)$$

$$\bar{\lambda}_5 = -2g_o(\frac{C_5t^2}{2} - \frac{C_2t^3}{3}) + \bar{C}_2t + \bar{C}_5, \qquad (23.5)$$

and

$$\bar{\lambda}_6 = -g_o\left(\frac{C_6 t^2}{2} - \frac{C_3 t^2}{3}\right) + \bar{C}_3 t + \bar{C}_6 \qquad (23.6)$$

in which the $\bar{C}_i$'s are integration constants. In order to integrate the remaining equations, the control $\vec{v}$ for the perturbed system must be determined. The maximum principle is applied to the H function, given by (19), to find the optimal control. Knowing that $v_1>0$, the supremacy of H implies that the vector (v_2, v_3, v_4) must be co-directional with the vector $(\alpha_4, \alpha_5, \alpha_6)$, i.e.,

$$v_2 = \frac{\alpha_4}{\alpha},$$

$$v_3 = \frac{\alpha_5}{\alpha}$$

and

$$v_4 = \frac{\alpha_6}{\alpha}$$

in which

$$\alpha = \sqrt{\alpha_4^2 + \alpha_5^2 + \alpha_6^2}.$$

Substituting the assumed solutions into these expressions and eliminating terms involving higher orders of ε yields

$$\bar{u}_2 = \frac{1}{\lambda^*}[\bar{\lambda}_4 - \lambda_4^* \Lambda], \qquad (23.7)$$

$$\bar{u}_3 = \frac{1}{\lambda^*}[\bar{\lambda}_5 - \lambda_5^* \Lambda], \qquad (23.8)$$

and

$$u_4 = \frac{1}{\lambda^*}[\bar{\lambda}_6 - \lambda_6^* \Lambda] \qquad (23.9)$$

in which

$$\Lambda = \frac{1}{(\lambda^*)^2}(\lambda_4^*\bar{\lambda}_4 + \lambda_5^*\bar{\lambda}_5 + \lambda_6^*\bar{\lambda}_6).$$

Note that the λ_i's and $\bar{\lambda}_i$'s appearing in the $\bar{u}_i$'s are all known functions of time. The control for the perturbed system may be expressed in terms of the steering angles, which are

$$\tan\phi = \frac{\lambda_5^*}{\lambda_4^*}\left[1 + \varepsilon\left(\frac{\bar{\lambda}_5}{\lambda_5^*} - \frac{\bar{\lambda}_4}{\lambda_4^*}\right)\right],$$

$$\tan\theta = \frac{\lambda_5^*}{\sqrt{(\lambda_4^*)^2 + (\lambda_5^*)^2}}\left[1 + \varepsilon\left(\frac{\bar{\lambda}_6}{\lambda_6^*} - \frac{\lambda_4^*\bar{\lambda}_4^* + \lambda_5^*\bar{\lambda}_5^*}{(\lambda_5^*)^2 + (\lambda_5^*)^2}\right)\right].$$

Note that when $\varepsilon \to 0$, the steering angles converge to the parent solutions. The fuel rate control v_1 appears linearly in the H function. The switching function for v_1 is identical in form to σ_1 of the parent system; hence, its behavior neighbors that of σ_1. Consequently, v_1 is bang-bang, and it has the same switching sequence as u_1 except that the switching times are slightly perturbed. Knowing that v_1 is a piecewise constant function and it coincides with u_1 except near a switching time, one can set $\bar{u}_1 = 0$ and integrate equations (21.4) to (21.7), the right hand sides of which are now known functions of time. These results may then be used to integrate the remaining equations. A total of fourteen integration constants, $\bar{C}_1$ to $\bar{C}_{14}$, will be introduced in the procedure. Because of the lengthy form of the integrands, it is difficult to obtain these integrals in closed form. However, they are all expressible in quadratures and can be numerically evaluated quite easily.

4.5 End Conditions and Transversality Conditions

The end conditions for the perturbed variables are dependent on the choice of the coordinate system and the selection of the boundary conditions for the parent system. Let the coordinate system be chosen as that shown in Figure 3, with the y-axis containing the given initial position of the rocket. Furthermore, let the initial and final position as well as the velocity vectors be prescribed in the same coordinates for both the parent system and the perturbed system. The following end conditions result:

$$\bar{x}_i^o = \bar{x}_i(0) = 0, \qquad i=1,\ldots,6, \qquad (24.1)$$

$$\bar{x}_1^f = \bar{x}_1(t^f) = -x_1^f x_2^f, \tag{24.2}$$

$$\bar{x}_1^f = \bar{x}_2(t^f) = 1/2\,[(x_1^f)^2 + (x_3^f)^2], \tag{24.3}$$

$$\bar{x}_3^f = x_3(t^f) = -x_2^f x_3^f, \tag{24.4}$$

and

$$\bar{x}_i^f = \bar{x}_i(t^f) = 0, \qquad i=4,5,6 \tag{24.5}$$

If the initial mass of the rocket is assumed to be the same as in the parent system, one has

$$\bar{x}_7^o = \bar{x}_7(0) = 0. \tag{24.6}$$

Using the cost function

$$G(\vec{y}(t)) = y_7(t^o) - y_7(t),$$

the terminal transversality condition for the perturbed system is

$$[\alpha_o + \alpha_7]_{t=t^f} = 0$$

in which $\alpha_o = \lambda_o$ is equal to either zero or minus one. Let the total flight time for the perturbed system be assumed as

$$t^f = t^{f*} + \varepsilon \, \bar{t}^f$$

in which t^{f*} is the flight time for the parent system. The first order Taylor's expansion of the transversality condition about t^{f*} is

$$\lambda_o + \lambda_7^*(t^{f*}) + \varepsilon[\dot{\lambda}_7^*(t^{f*}) \, \bar{t}^f + \bar{\lambda}_7(t^{f*})] = 0.$$

Knowing that $\lambda_7^*(t^{f*}) = \lambda_o$ yields

$$\bar{t}^f = -\bar{\lambda}_7(t^{f*})/\dot{\lambda}_7^*(t^{f*}) \qquad (24.7)$$

Both $\bar{\lambda}_7(t^{f*})$ and $\dot{\lambda}_7^*$ (t^{f*}) contain the integration constants and t^{f*}. The computed value for $\bar{t}^f$ indicates the change in required burning time for the perturbed system.

Substituting the assumed solutions into equation (19) and eliminating the parent H function, the condition $H = 0$ reduces to

$$\lambda_1^*(\bar{x}_4 - x_2^* x_4^* - x_1^* x_5^*) + \lambda_2^*(\bar{x}_5 + x_1^* x_4^* + x_3^* x_6^*)$$

$$+ \lambda_3^*(\bar{x}_6 - x_2^* x_6^* - x_3^* x_5^*) + \frac{c\lambda^*}{x_7^*}\left(\frac{-u_1^*\bar{x}_7}{x_7^*} + \bar{u}_1\right)$$

$$+ \frac{cu_1^*}{x_7^*}(\lambda_4^*\bar{u}_2 + \lambda_5^*\bar{u}_3 + \lambda_6^*\bar{u}_4) - \lambda_4^* g_o x_1^* + 2\lambda_5^* g_o x_2^*$$

$$- \lambda_6^* g_o x_3^* - \lambda_7^*\bar{u}_1 + \bar{\lambda}_1 x_4^* + \bar{\lambda}_2 x_5^* + \bar{\lambda}_3 x_6^*$$

$$+ \frac{cu_1^*}{x_7^*}(\bar{\lambda}_4 u_2^* + \bar{\lambda}_5 u_3^* + \bar{\lambda}_6 \bar{u}_4) - \bar{\lambda}_5 g_o - \bar{\lambda}_7 u_1^* = 0.$$

Using the integrals (23.1) to (23.9) and evaluating the above equation at t=0, one has

$$g_o(-C_4 x_1^o + 2C_5 x_2^o - C_6 x_3^o - \bar{C}_5) + \bar{C}_1 x_4^o + \bar{C}_2 x_5^o + \bar{C}_3 x_6^o$$

$$- \bar{\lambda}_7^o u_1^o + \frac{cu_1^o}{x_7^o}\left[\frac{C_4\bar{C}_4 + C_5\bar{C}_5 + C_6\bar{C}_6}{\sqrt{C_4^2 + C_5^2 + C_6^2}}\right] = 0. \qquad (24.8)$$

Equations (24.1) to (24.8) provide fifteen equations to solve for the fourteen integration constants, $\bar{C}_1$ to $\bar{C}_{14}$, and the change in flight time $\bar{t}^f$.

5.0 NUMERICAL METHOD

In the uniform force field problem, the determination of the constants involves the solving of 15 coupled nonlinear simultaneous equations. It is impractical to do it analytically. With the assistance of computers, a numerical scheme is much more appealing. If a numerical method is to be used, a different approach may be taken to simplify the process. Instead of using the analytical form of the solutions and solving for all the constants, the equations of motion are integrated numerically with the given initial conditions. The integrated optimal control is used in the numerical integration and the unknown integration constants in the control are used as parameters which are numerically varied to find the flight path that satisfies the prescribed terminal position and velocity. This method eliminates the process of solving a set of simultaneous equations. Note that the solutions are analytically obtained from the control theory, i.e., only the determination of the integration constants involves numerical methods. This is the basic difference between the numerical method used here and other methods of numerically minimizing the total fuel consumption.

After the constants in the parent solution have been found, the constants in the perturbed system may be obtained in a similar fashion.

6.0 SUMMARY

The fuel optimum solution of a rocket in a three-dimensional uniform force field was formulated using the optimal control theory, and then integrated analytically. This solution is a generalization of a known two-dimensional solution. It was then shown that if the maximum position change is small, the same problem in a central force field can be considered as a neighboring system of the uniform force field system. A perturbation technique was introduced to find the solution for the central force field system using the uniform force field problem as the parent system. The optimal control for the perturbed system was integrated in closed form. Other differential equations were shown to be integrable in quadratures.

Application of the solution to a specific problem involves the evaluation of the integration constants for given initial and terminal conditions. Because of the lengthy form of these integrals, an analytical method is impractical. An iterative scheme is currently being developed to numerically evaluate the constants in the integrals. The basic difference between this method and other methods of numerically minimizing the total fuel consumption is that the solutions are obtained analytically from the control theory; only the determination of integration constants involves numerical methods.

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Attachments
References
Figures 1-3

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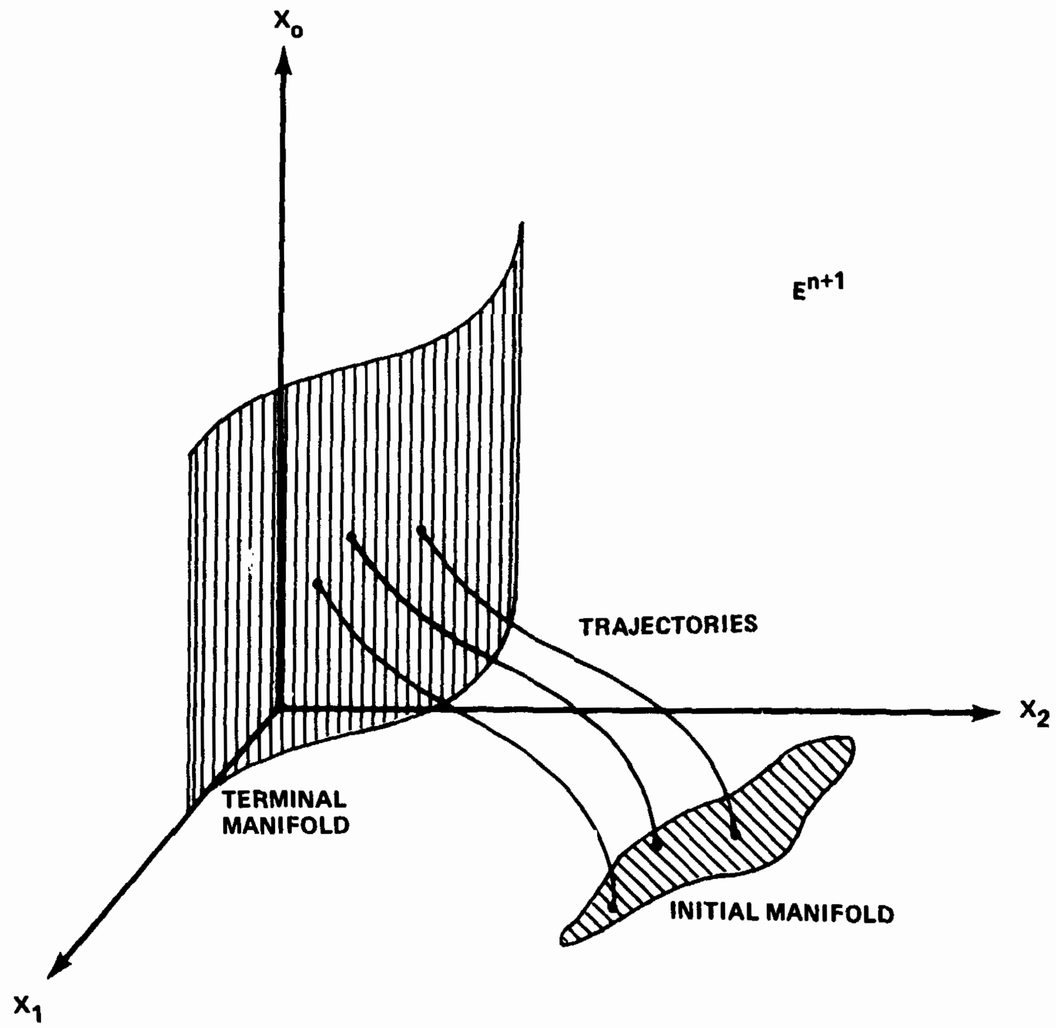

FIGURE 1 - TRAJECTORIES IN THE AUGMENTED STATE SPACE

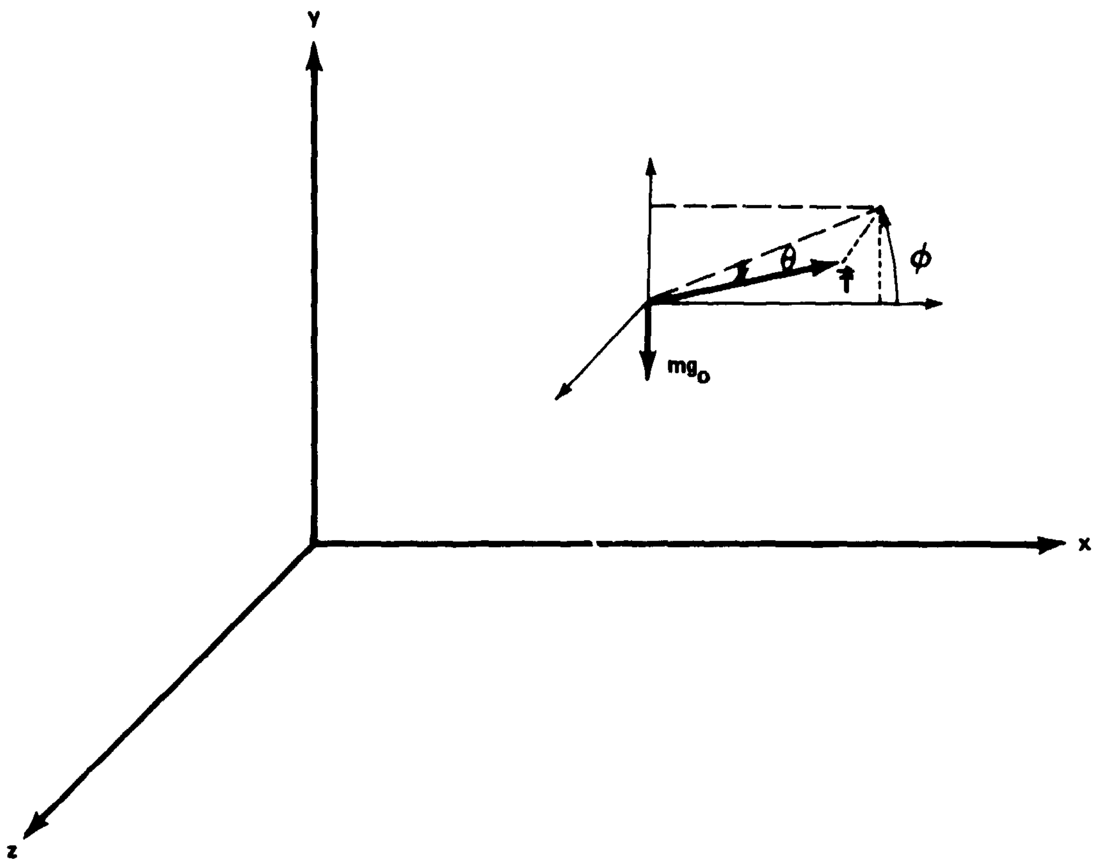

FIGURE 2 - A ROCKET IN A UNIFORM FORCE FIELD

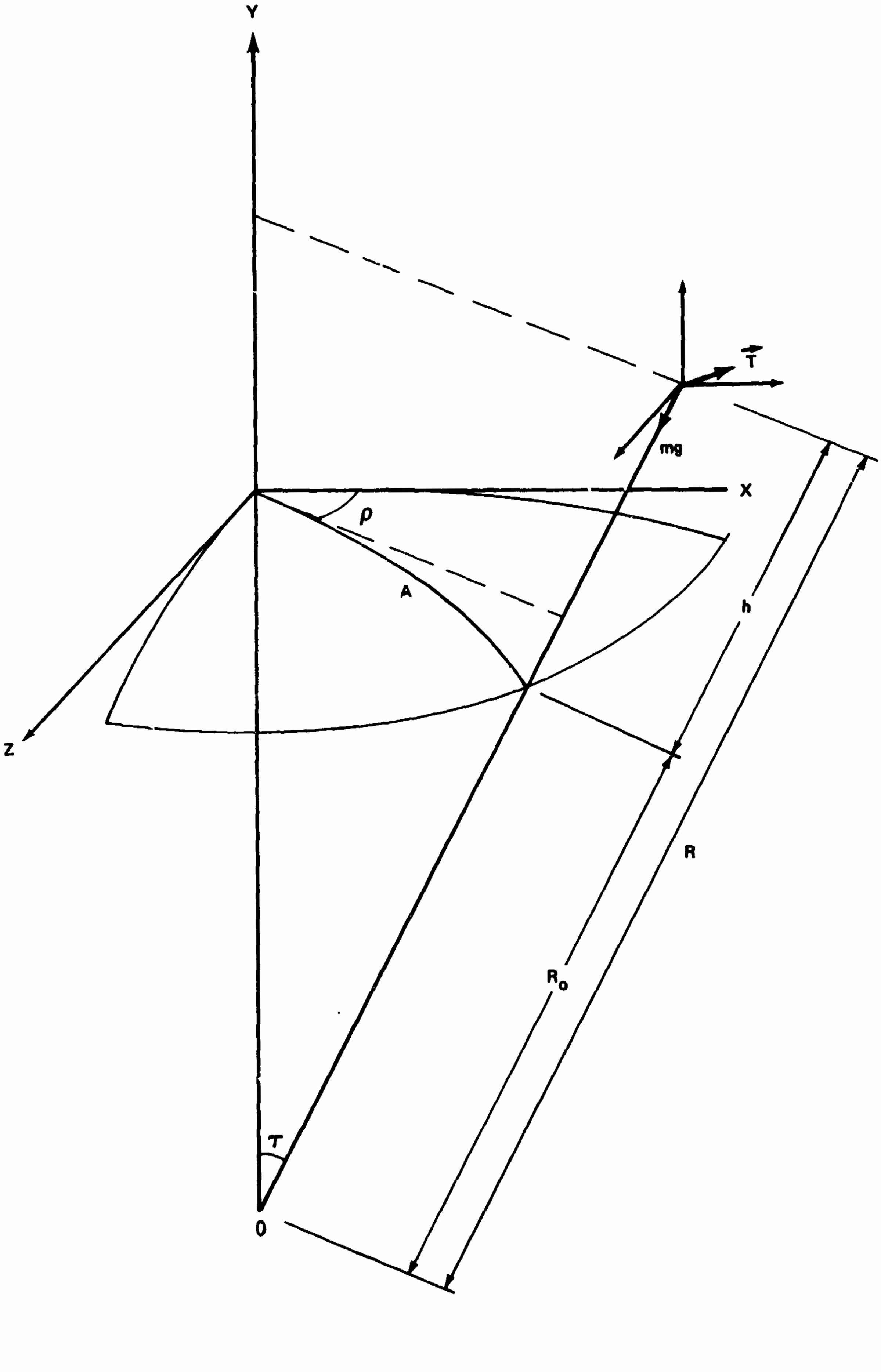

FIGURE 3 - A ROCKET IN A CENTRAL FORCE FIELD