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STATE TRANSITION MATRIX FOR INERTIAL NAVIGATION SYSTEMS

by

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March, 1969

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ABSTRACT

This report formulates and solves in state space notation the error equation for inertial navigation systems. The system is assumed to be moving at a constant celestial longitude rate. The state transition matrix is explicitly derived both for long-term and short-term operation. Examples are included to demonstrate the ease with which the state transition matrix can be used for error analysis.

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$g \sim$ gravity magnitude

$\varepsilon_N, \varepsilon_E, \varepsilon_D \sim$ error angles relating the computed or instrumented geographic frame to the geographic frame. These error angles result from positive rotations of the computed or instrument axes with respect to the geographic axes.

$\delta L \sim$ latitude error

$\delta\lambda \sim$ longitude error

Equation (1) is valid for a navigation system which is assumed to be moving at a constant celestial longitude rate (fixed base navigation is included as a special case). It has been assumed that the Coriolis compensations are supplied either from external information or without error.

2. Formulation in State Space Notation

Since this equation arises so frequently, it is advantageous to use state space methods to obtain a solution which is valid for an arbitrary forcing vector. This is accomplished by writing equation (1) as follows:

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}(t)\ \underline{u}(t) \tag{2}$$

where

$$\underline{x}(t) = \{\varepsilon_N, \varepsilon_E, \varepsilon_D, \delta L, \delta\lambda, \delta\dot{L}, \delta\dot{\lambda}\} \tag{3}$$

$$\underline{A} = \begin{bmatrix} 0 & -\dot{\lambda}\sin L & 0 & -\dot{\lambda}\sin L & 0 & 0 & \cos L \\ \dot{\lambda}\sin L & 0 & \dot{\lambda}\cos L & 0 & 0 & -1 & 0 \\ 0 & -\dot{\lambda}\cos L & 0 & -\dot{\lambda}\cos L & 0 & 0 & -\sin L \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{g}{r} & 0 & 0 & 0 & 0 & 0 \\ -\frac{g}{r}\sec L & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{u}(t)$ ∿ 7^{th} order forcing vector

$\underline{B}(t)$ ∿ 7 x q^{th} order matrix rotating the forcing vector to the state

superdot ∿ time differentiation

3. State Transition Matrix

The solution to equation (2) is given in terms of the state transition matrix, $\underline{\Phi}(t) = e^{\underline{A}t}$ as:

$$\underline{x}(t) = \underline{\Phi}(t-t_o)\underline{x}(t_o) + \int_{t_o}^{t} \underline{\Phi}(t-\sigma)\ \underline{B}(\sigma)\underline{u}(\sigma)d\sigma \qquad (4)$$

The state transition matrix satisfies the matrix differential equation:

$$\dot{\underline{\Phi}}(t-t_o) = \underline{A}\ \underline{\Phi}(t-t_o),$$

the initial condition:

$$\underline{\Phi}(0) = \underline{I},$$

where $\underline{I}$ is the identity matrix, and the composition law:

$$\underline{\Phi}(t) = \underline{\Phi}(t-t_o)\ \underline{\Phi}(t_o)$$

from which it follows that:

$$\underline{\Phi}^{-1}(t) = \underline{\Phi}(-t)$$

The transition matrix is found from the relationship:

$$\underline{\Phi}(t) = L^{-1}(\underline{I}S-\underline{A})^{-1} \tag{5}$$

where

$S \sim$ Laplace operator

$L^{-1} \sim$ inverse Laplace transformation

$()^{-1} \sim$ matrix inversion operator

Applying equation (5), the state transition matrix is found to be given by:

$$
\underline{I}(t) = \begin{bmatrix}
\cos\omega_s t & -\frac{\dot\lambda}{\omega_s}\sin L\sin\omega_s t & \frac{1}{2}\frac{\dot\lambda^2}{\omega_s{}^2}\sin 2L(\cos\omega_s t-\cos\dot\lambda t) & -\frac{\dot\lambda}{\omega_s}\sin L(\sin\omega_s t-\frac{\dot\lambda}{\omega_s}\sin\dot\lambda t) & 0 & 0 & \frac{\cos L}{\omega_s}\sin\omega_s t \\
\frac{\dot\lambda}{\omega_s}\sin L(\sin\omega_s t-\frac{\dot\lambda}{\omega_s}\sin\dot\lambda t) & \cos\omega_s t & \frac{\dot\lambda}{\omega_s}\cos L(\sin\omega_s t-\frac{\dot\lambda}{\omega_s}\sin\dot\lambda t) & -\frac{\dot\lambda^2}{\omega_s{}^2}(\cos\dot\lambda t-\cos\omega_s t) & 0 & -\frac{1}{\omega_s}\sin\omega_s t & 0 \\
\tan L(\cos\dot\lambda t-\cos\omega_s t) & -\sec L(\sin\dot\lambda t-\frac{\dot\lambda}{\omega_s}\sin^2 L\sin\omega_s t) & \cos\dot\lambda t & -\sec L(\sin\dot\lambda t-\frac{\dot\lambda}{\omega_s}\sin^2 L\sin\omega_s t) & 0 & 0 & -\frac{\sin L}{\omega_s}\sin\omega_s t \\
\sin L(\sin\dot\lambda t-\frac{\dot\lambda}{\omega_s}\sin\omega_s t) & \cos\dot\lambda t-\cos\omega_s t & \cos L(\sin\dot\lambda t-\frac{\dot\lambda}{\omega_s}\sin\omega_s t) & \cos\dot\lambda t & 0 & \frac{1}{\omega_s}\sin\omega_s t & 0 \\
\sec L(\cos\omega_s t-\cos^2 L-\sin^2 L\cos\dot\lambda t) & \tan L(\sin\dot\lambda t-\frac{\dot\lambda}{\omega_s}\sin\omega_s t) & \sin L(1-\cos\dot\lambda t) & \tan L(\sin\dot\lambda t-\frac{\dot\lambda}{\omega_s}\sin\omega_s t) & 1 & 0 & \frac{1}{\omega_s}\sin\omega_s t \\
\dot\lambda\sin L(\cos\dot\lambda t-\cos\omega_s t) & \omega_s\sin\omega_s t & \dot\lambda\cos L(\cos\dot\lambda t-\cos\omega_s t) & -\dot\lambda(\sin\dot\lambda t-\frac{\dot\lambda}{\omega_s}\sin\omega_s t) & 0 & \cos\omega_s t & 0 \\
-\omega_s\sec L(\sin\omega_s t-\frac{\dot\lambda}{\omega_s}\sin^2 L\sin\dot\lambda t) & \dot\lambda\tan L(\cos\dot\lambda t-\cos\omega_s t) & \dot\lambda\sin L(\sin\dot\lambda t-\frac{\dot\lambda}{\omega_s}\sin\omega_s t) & \dot\lambda\tan L(\cos\dot\lambda t-\cos\omega_s t) & 0 & 0 & \cos\omega_s t
\end{bmatrix} \quad (6)
$$

where

$\omega_s{}^2 = \frac{g}{r} \sim$ Schuler frequency squared.

4. State Transition Matrix for Short Sampling Times

The state space approach is used when optimal filtering techniques are applied to the inertial navigation system. In this situation, the state transition matrix is used to model the system's behavior over the sampling time, T. Thus small angle assumptions can be made in the above expression:

$$
\begin{aligned}
\cos \omega_s t &\cong 1, && e < 10\% \text{ for } T < 6 \text{ min} \\
&\cong 1 - \frac{\omega_s{}^2 T^2}{2}, && e < 10\% \text{ for } T < 16 \text{ min} \\
\sin \omega_s t &\cong \omega_s T, && e < 10\% \text{ for } T < 12 \text{ min} \\
&\cong \omega_s T - \frac{\omega_s{}^3 T^3}{6}, && e < 10\% \text{ for } T < 23 \text{ min} \\
\cos \dot{\lambda} t &\cong 1, && e < 10\% \text{ for } T < 100 \text{ min}, \ \dot{\lambda} = \omega_{ie} \\
\sin \dot{\lambda} t &\cong \dot{\lambda} T, && e < 10\% \text{ for } T < 190 \text{ min}, \ \dot{\lambda} = \omega_{ie}
\end{aligned}
$$

where e is the maximum error associated with the approximation. Thus for update times of less than six minutes (T < 6 min), the following state transition matrix should give adequate results:

$$
\underline{\Phi}(t) = \begin{bmatrix}
1 & -\dot{\lambda}t \sin L & 0 & -\dot{\lambda}t \sin L & 0 & 0 & t \cos L \\
\dot{\lambda}t \sin L & 1 & \dot{\lambda}t \cos L & 0 & 0 & -t & 0 \\
0 & -\dot{\lambda}t \cos L & 1 & -\dot{\lambda}t \cos L & 0 & 0 & -t \sin L \\
0 & 0 & 0 & 1 & 0 & t & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & t \\
0 & \omega_s{}^2 t & 0 & 0 & 0 & 1 & 0 \\
-\omega_s{}^2 t \sec L & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \qquad (7)
$$

5. Examples

5.1 Initial Condition Errors

The solution for the initial condition errors is made by inspection of equations (4) and (6). Thus:

$$\underline{x}(t) = \underline{\Phi}(t)\ \underline{x}(0)$$

where $\underline{\Phi}(t)$ is given by equation (6) and $\underline{x}(t)$ is given by equation (3).

5.2 Accelerometer Bias Errors

If we take accelerometer bias to be the sole source of error, then it can be shown that:*

$$\underline{F}(t) = \underline{F} = \{0,\ 0,\ 0,\ (u)f_N,\ (u)f_E\} \ .$$

where $(u)f_N$ and $(u)f_E$ are the north and east accelerometer bias', respectively. Thus, in state space notation,

$$\underline{B}(t)\ \underline{u}(t) = \underline{B}\ \underline{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\sec L}{r} \end{bmatrix} \begin{bmatrix} (u)f_N \\ (u)f_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{r}(u)f_N \\ \frac{\sec L}{r}(u)f_E \end{bmatrix}$$

*Britting, K.R.; "Analysis of Local Vertical Inertial Navigation Systems," M.S.L. Report RE-52, February, 1969.

Thus, from equation (4), the error response to accelerometer bias starting at $t_o = 0$, is given by:

$$\underline{x}(t) = \int_o^t \underline{\Phi}(t-\sigma)\ \underline{B}\ \underline{u}\ d\sigma$$

or:

$$\underline{x}(t) = \begin{bmatrix} \frac{(u)f_E}{g} \int_o^t \sin \omega_s(t-\sigma)\ d\sigma \\ -\frac{(u)f_N}{g} \int_o^t \sin \omega_s(t-\sigma)\ d\sigma \\ -\tan L\ \frac{(u)f_E}{g} \int_o^t \sin \omega_s(t-\sigma)\ d\sigma \\ \frac{(u)f_N}{g} \int_o^t \sin \omega_s(t-\sigma)\ d\sigma \\ \sec L\ \frac{(u)f_E}{g} \int_o^t \sin \omega_s(t-\sigma)\ d\sigma \\ \frac{(u)f_N}{r} \int_o^t \cos \omega_s(t-\sigma)\ d\sigma \\ \sec L\ \frac{(u)f_E}{r} \int_o^t \cos \omega_s(t-\sigma)\ d\sigma \end{bmatrix}$$

Integration yields the result:

$$\underline{x}(t) = \begin{bmatrix} \varepsilon_N \\ \varepsilon_E \\ \varepsilon_D \\ \delta L \\ \delta\lambda \\ \delta\dot{L} \\ \delta\dot{\lambda} \end{bmatrix} = \begin{bmatrix} (1 - \cos\omega_s t)\ (u) f_E/g \\ -(1 - \cos\omega_s t)\ (u) f_N/g \\ -(1 - \cos\omega_s t)\ \tan L\ (u) f_E/g \\ (1 - \cos\omega_s t)\ (u) f_N/g \\ (1 - \cos\omega_s t)\ \sec L\ (u) f_E/g \\ \sin\omega_s t\ (u) f_N/r\omega_s \\ \sin\omega_s t\ \sec L\ (u) f_E/r\omega_s \end{bmatrix}$$

Thus it is seen that this method yields results very quickly and efficiently compared with solving equation (1) via Cramer's Rule.