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# THE FULLY INDECOMPOSABLE MATRIX AND ITS ASSOCIATED BIPARTITE GRAPH -- AN INVESTIGATION OF COMBINATORIAL AND STRUCTURAL PROPERTIES

A Dissertation Presented to the
Faculty of the Department of Mathematics
of the University of Houston
in Partial Fulfillment of the
Requirements for the Degree
Doctor of Philosophy

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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HOUSTON, TEXAS

# THE FULLY INDECOMPOSABLE MATRIX AND ITS ASSOCIATED
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# ACKNOWLEDGEMENT

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# ABSTRACT

For any $n \times n$ (0,1) matrix $A$ , a correspondence is established between $A$ and a certain bigraph $G$ . Equivalences between various concepts pertaining to $A$ and $G$ , respectively, are demonstrated. It is shown that if $A$ is fully indecomposable, then $G$ is 2-connected. A method is developed for reducing a bigraph $G$ of a nearly decomposable matrix to a strictly smaller bigraph $G'$ , which is also associated with a nearly decomposable matrix $A'$ . This result is used to completely characterize the fully indecomposable matrices in terms of their bigraphs and leads to a lower-bound estimate for the permanent function on this class of matrices. Finally, the characterization theorem is shown to be relevant to the problems of (1) finding an upper-bound estimate for the permanent and (2) of determining the structure of the class of nearly decomposable matrices. Some partial results along these lines are given.

# I. INTRODUCTION

## 1. Definitions and Preliminaries

A nonnegative matrix is one whose entries are non-
negative real numbers. Essentially, this dissertation is an
investigation of combinatorial properties enjoyed by certain
classes of nonnegative matrices. The expression "combina-
torial properties," as used here, means roughly, those prop-
erties which pertain to the distribution of zeros among the
entries of A . A (0,1) matrix is one whose entries are
either 0 or 1 . One can readily see that the combinatorial
properties of an m×n nonnegative matrix A are completely
represented by an m×n (0,1) matrix A' whose zeros correspond
to the zeros of A and whose ones occur in the same locations
as the positive entries of A . In the light of this obser-
vation the theorems that follow are stated in terms of (0,1)
matrices when it is particularly convenient to do so. We
should keep in mind, however, that many of the results
extend immediately to nonnegative matrices. This section
closes with a few definitions.

A *permutation* matrix is an n×n (0,1) matrix having
precisely one 1 in each row and column. Observe that if P
is a permutation matrix and A is an arbitrary matrix,
multiplication of A on the left by P results in a matrix
A' , which is identical to A except that the ordering of

the rows of  A'  is a permutation of the row ordering of  A .
Multiplication on the right by  P  produces a similar result
with respect to the columns of  A .

We say that an n×m matrix  A  is *p-equivalent* to an
n×m matrix  B  if there is an n×n permutation matrix  P  and
an m×m permutation matrix  Q  such that  A = PBQ .  In the
case where  n = m  and  $Q = P^{-1} = P^T$ ,  we say that  A  is
*p-similar* to  B .

<u>Definition 1.1</u>:  An n×n matrix  A  is said to be
*partly decomposable* if it is p-equivalent to a matrix of
the form

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where  $C_{12}$  is an s×t zero submatrix with  s + t = n .  If
A  is not partly decomposable,  A  is said to be *fully inde-*
*composable*.  Similarly,  A  is said to be *reducible* or
*irreducible* when  A  is or is not p-similar to a matrix in
the form of  C .

The symbol  *FI*  always denotes "fully indecomposable"
in the sequel.

An n×n nonnegative matrix  A  is said to be *doubly*
*stochastic* if each of its row sums and column sums equals one.

If  A  is an n×n complex matrix, we define the *permanent* of  A  to be the complex valued function

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \ .$$

Chapter II, section 1 provides definitions of the *digraph* (directed graph) and the *bigraph* (bipartite graph) corresponding to a nonnegative matrix.

## 2. Historical Background

Among the most important early investigations involving combinatorial properties of nonnegative matrices was Frobenius' beautiful work [6], which extended the results of Perron [13] to irreducible nonnegative matrices.  The Perron-Frobenius theory links combinatorial properties of nonnegative matrices to their spectral properties in an especially fruitful way.  As a result, the concept of irreducibility has found applications in such diverse areas as the theory of stochastic matrices, numerical analysis, and partial differential equations.  For examples, see [2] and [19].

Apparently, the more general concept of an FI matrix did not attract attention until more recently.  In 1959, Marcus and Newman [10] made the significant discovery that if $\Omega_n$ is the set of n×n doubly stochastic matrices, and if $A\epsilon\Omega_n$ and

per A = min (per S) , then A is FI. (This is a partial
$S \epsilon \Omega_n$
result along the lines of the well-known Van der Waerden
conjecture concerning the minimum of the permanent function
on $\Omega_n$ .)

In 1965, Perfect and Mirsky [12] studied FI matrices
and discovered several properties relating them to doubly
stochastic matrices. Sinkhorn and Knopp [16], and Brualdi,
Parter, and Schneider [2], independently discovered a funda-
mental relationship between FI and doubly stochastic matrices
using totally different techniques. In their paper, Brualdi
et al. showed how FI matrices are related to the Perron-
Frobenius theory.

In two papers which appeared in 1969, Sinkhorn and
Knopp [15] and Sinkhorn [14] introduced the notion of a
nearly decomposable matrix, and presented a fundamental
theorem regarding the structure of these matrices. With this
result a powerful new tool became available for the study of
FI matrices. For convenience, we present their discoveries
below.

Definition 2.1: Let A be an FI matrix. If $a_{ij}$ is
a positive entry of A , then $a_{ij}$ is said to be *removable*
if the matrix A' , derived from A by replacing $a_{ij}$ with
a zero, is FI. If an FI matrix A has no removable entries,
then A is said to be *nearly decomposable*.

In the sequel, the abbreviation *ND* always denotes "nearly decomposable".

Theorem 2.1 (Sinkhorn and Knopp):  Let  A  be a non-negative n×n ND (0,1) matrix with  n > 1 .  Then permutation matrices  P  and  Q  and an integer  s > 1  exist such that PAQ corresponds to the matrix in figure 2.1, where each  $E_i$ has exactly one entry equal to  1 , and each  $A_i$  is ND.

With the aid of this theorem, Sinkhorn and Knopp [15] showed that if  A  is an FI matrix and if all of the nonzero summands of the function per A are identical, there is a unique positive matrix  B  of rank one such that  $b_{ij} = a_{ij}$  whenever  $a_{ij} > 0$ .  Subsequently, Sinkhorn used Theorem 2.1 to settle, in the affirmative, a conjecture of Marshall Hall concerning the behavior of the function per A on the set of n×n (0,1) matrices having precisely three ones in each column and each row.  In 1969, Minc [11] used Theorem 2.1 to obtain a very good lower-bound estimate for the permanent of FI (0,1) matrices.

$$
\begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 & 0 & E_1 \\
E_2 & A_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & E_3 & A_3 & \cdots & 0 & 0 & 0 \\
& & & \cdots & & & \\
0 & 0 & 0 & \cdots & A_{s-2} & 0 & 0 \\
0 & 0 & 0 & \cdots & E_{s-1} & A_{s-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & E_s & A_s
\end{pmatrix}
$$

Figure 2.1

After ND matrices were introduced and their importance was demonstrated, it was natural to investigate the properties of nearly reducible matrices. (If we replace the symbol "FI" by the word "irreducible" in Definition 2.1, we have the definition of a *nearly reducible* matrix.) Sinkhorn and Hedrick [17] established many of the important properties of these matrices.

In 1969, Hartfiel [9] studied nearly reducible matrices by examining associated directed graphs. This led to the following canonical form for a nearly reducible matrix.

Theorem 2.2 (Hartfiel): If A is an n×n nearly reducible (0,1) matrix with n > 1, then A is p-similar to the matrix of figure 2.1, where s > 1, $A_i$ is the 1×1 zero matrix for i = 1,2,$\cdots$,s-1, $A_s$ is an $n_s \times n_s$ nearly reducible submatrix, $E_1$ is the 1×1 matrix 1 for i = 2,3,$\cdots$,s-1, and $E_1$ and $E_s$ are $1 \times n_s$ and $n_s \times 1$ submatrices, respectively, each having precisely one nonzero entry.

Using a theorem discovered independently by Hartfiel and Hedrick which relates ND matrices to nearly reducible matrices, Hartfiel derived the following remarkable result.

<u>Theorem 2.3</u> (Hartfiel):   If   A   is an n×n ND (0,1) matrix with   n > 1 ,   then   A   is p-equivalent to the matrix of figure 2.1, where   s > 1 ,   $A_i$   and   $E_j$   are the 1×1 matrix 1   for   i = 1,···,s-1   and   j = 2,···,s-1 ,   $A_s$   is an $n_s \times n_s$ ND matrix, and   $E_1$   and   $E_s$   are 1×$n_s$ and $n_s$×1 submatrices, respectively, each having precisely one nonzero entry.

Theorem 2.3 is a considerable sharpening of Theorem 2.1, and we would like to acknowledge the helpful role it played during the investigations reported in this dissertation.

In May 1969, this author independently discovered the same lower bound as Minc, using Theorem 2.3.   Drs. Sinkhorn and Hartfiel, and Mr. Crosby, of the University of Houston, have also obtained this result independently.   A new proof of Minc's lower-bound estimate appears in Chapter III of this paper, using results developed herein.

### 3.   Discussion of Procedures and Goals

The remarkable results obtained by Hartfiel demonstrated the usefulness of studying digraphs to obtain information concerning irreducible matrices.   The power of this technique stems from two sources.

First, the digraph of an n×n (0,1) matrix  A  represents the entire p-similarity class of  A  (see ch. II, sec. 3, p. 23).  Certain symmetries of the similarity class are apparent from observing the graph, but cannot readily be recognized by inspecting a particular matrix representative of that class.

Second, the classic theorem (Theorem II.3.1), which characterizes irreducibility of a matrix in terms of the strong connectivity of its digraph, is very useful in studying the combinatorial properties of nearly reducible matrices.

In view of the preceding observations, this author believes that the bigraph is a natural object to study in order to obtain information about the combinatorial properties of an FI matrix.  Bigraphs are chosen in this setting because the bigraph of a (0,1) matrix  A  represents the entire p-equivalence class of  A  (ch. II, sec. 3, p. 22).

As far as this author knew, no useful theorem existed which characterized FI matrices in terms of their bigraphs in a sense analagous to Theorem II.3.1.  With Theorem II.4.14 this characterization is accomplished.  Further, to demonstrate the effectiveness of our method, it was felt that we should be able to obtain a result similar in usefulness to Hartfiel's Theorem for ND matrices, using solely the intrinsic properties of FI matrices and their associated bigraphs.  We achieved

this result in Theorem II.4.15. The remaining goals we sought to accomplish were a characterization of ND matrices or their bigraphs in terms of some easily observable parameters, and the discovery of some good upper- and lower-bound estimates for the permanent of an n×n (0,1) FI matrix (in particular, ND matrix).

The extent of success of this latter portion of our program is revealed in Chapter III, and is discussed in the summary.

We close this chapter with a short section describing some notation and conventions for internal referencing.

### 4. Notation and Conventions

The following conventions have been adopted in this dissertation for referencing theorems, corollaries, lemmas, and definitions. If the theorem referenced is stated in the same chapter as the reference, the reference is in the form "Theorem i.j". This refers to the theorem numbered i.j which appears in section i, where i and j are Arabic numerals.

If the theorem referenced is stated in a different chapter than the reference, the reference reads "Theorem c.i.j", where c is the Roman numeral corresponding to the chapter in which the theorem appears, and i and j are as in the previous case.

The set theoretic notation used corresponds to the notation in [7] with the exception that if $S$ is a finite set, $|S|$ denotes the number of elements in $S$ .

The following special notation is frequently used. Suppose $a_{ij}$ is an entry of an m×n matrix $A$ . Then the *matrix* $E_{ij}$ *corresponding to the number* $a_{ij}$ is the m×n matrix having the entry 1 in the i,j*th* position and 0 elsewhere. When the notation $E_{ij}$ is used with an entry $a_{ij}$ of an m×n (0,1) matrix $A$ in the sequel, it is understood without confusion that $E_{ij}$ is the m×n (0,1) matrix corresponding to $a_{ij}$ .

## II.   THE STRUCTURE OF BIGRAPHS CORRESPONDING

## TO FULLY INDECOMPOSABLE MATRICES

### 1.   Definitions and Preliminary Material from Graph

### Theory

Since comprehensive treatises on graph theory have become available only recently, and since the notation and definitions vary widely among different authors, a complete list is included of the graph-theoretic definitions in this dissertation.

Definition 1.1:  A *graph*  G  consists of a nonempty set V(G) whose elements are called *vertices* of  G , together with a set E(G) consisting of unordered pairs of vertices, with the condition that if  v  is in V(G), then {v,v} is not in E(G). The elements of E(G) are called the *edges* of  G .  It is customary to visualize or to illustrate a graph by representing its vertices as points and its edges by lines connecting the vertices (see fig. 1.1).  For notational economy, we will write uv to denote an edge {u,v}.  The symbol  G  will always represent a graph.

The definition of a graph which we have adopted is a restricted one in a sense.  Some authors define E(G) differently, to admit edges whose end points coincide (*loops*), and to allow a pair of vertices to be connected by more than one edge (*multiple edges*).
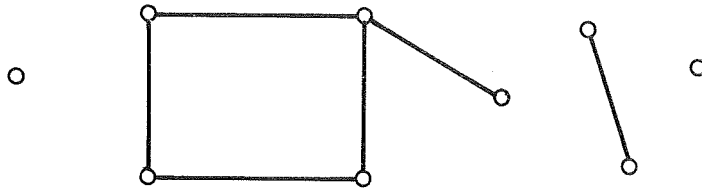
Figure 1.1

If uv is an edge of G , then u and v are said to be its *end points*. In this case, the vertices u and v are said to be *adjacent*. The edge uv and the vertex u (or the vertex v ) are *incident*. We say that two edges e and f of G are *adjacent* if they are both incident to a vertex v .

The *valence* of a vertex v in G , denoted *val(v)*, is a nonnegative integer representing the number of edges of G which are incident to v . Vertices with one, two, or three incident edges are called *monovalent, divalent,* and *trivalent,* respectively. A vertex which is not divalent is called a *node*.

The notion of a subgraph of  G  is now introduced,
and some important special subgraphs and graphs related to
G  are defined.

A *subgraph* of  G  is a graph  H  such that  $V(H) \subset V(G)$
and  $E(H) \subset E(G)$ .  If  $V(H) \neq V(G)$  or  $E(H) \neq E(G)$ ,  H
is said to be a *proper* subgraph of  G .

If  $R \subset E(G)$ ,  the *subgraph induced by*  R  is the
graph [R] whose edge set is  R ,  and whose vertex set con-
sists of every member of V(G) which is an end point of an
edge in  R .  Similarly, if  $S \subset V(G)$ ,  the *subgraph induced
by*  S  is the graph [S] whose vertex set is  S ,  and whose
edges are all members of E(G) with both end points in  S .

H  is a *spanning* subgraph of  G  if  $V(H) = V(G)$ .

If  $S \subset V(G)$ ,  then  $G' = G - S$  is the subgraph of
G  induced by the vertices V(G)-S.  If  S  consists of a
single vertex  v ,  we simply write  G' = G - v .

If  $R \subset E(G)$ ,  then  $G' = G - R$  is the subgraph of
G  whose vertex set is  V(G) and whose edge set is  E(G)-R.
If  R  consists of a single edge  e ,  we write  G' = G - e .

If  G  and  D  are graphs, then the graph  $G = G \cup D$
is the graph with edge set  $E(G) \cup E(D)$  and vertex set
$V(G) \cup V(D)$ .

The important concept of a path in G is now introduced.

Suppose $v_i$ (i = 0,1,$\cdots$,n) are in V(G) and $e_j$ (j = 1,$\cdots$,n) are in E(G). Let P(n) = $v_0, e_1, v_1, \cdots, e_n, v_n$ be a sequence whose terms are alternately vertices and edges of G , and where $e_i$ is incident to both $v_{i-1}$ and $v_i$ for i = 1,$\cdots$,n . P(n) is called a *path* in G with *end points* $v_0$ and $v_n$ . The vertices $v_1, \cdots, v_{n-1}$ are called the *interior vertices* of P(n). Since the graph G does not have multiple edges, P(n) is completely determined by its subsequence of vertices, and we will denote P(n) by $v_0 v_1 \cdots v_n$. The positive integer n , corresponding to the number of edges contained in P(n) is called the *length* of the path P(n). A path $v_0 \cdots v_n$ is said to be a *simple path* if i $\neq$ j implies that $v_i \neq v_j$ for 0 $\leqq$ i, j $\leqq$ n . A path is *closed* if $v_0 = v_n$ . A *cycle* is a closed path which is simple except that $v_0 = v_n$ .

There are several important concepts which are defined in terms of a path: A graph G is said to be *connected* if every pair of vertices u and v in G are the end points of some path in G . In a connected graph G , the *distance* between any two vertices v and w of G [which we denote by d(u,v)] is defined to be the length of the shortest path in G having u and v as end points. If H is a subgraph of G and u and v are in V(H), the *distance in* H

between u and v [denoted $d_H(u,v)$] is the length of the shortest path *in* $H$ with u and v as end points. Note that $d_H(u,v)$ may not be the same as $d_G(u,v)$ when u and v are in V(H). In general, $d_G(u,v) \leqq d_H(u,v)$ . Again, suppose H is a subgraph of G . A path P of G is said to *avoid* H if none of the vertices of P is in V(H). It follows that if P avoids H , none of the edges of P is in E(H). If H is a subgraph of G , H is a *maximal connected subgraph* if H is connected and is not properly contained in any other connected subgraph of G .

If G is a graph, a maximal connected subgraph is called a *component* of G . The components clearly form a unique (disjoint) partition of G .

The following concepts are of special importance in this paper, and will be used frequently in the sequel.

A vertex v of a connected graph G is said to be an *articulation point* if the graph G-v has two or more components; v is said to *separate* G .

Definition 1.2: A connected graph G is said to be 2-connected if it does not contain an articulation ponit.

A graph G is said to be *regular* if every vertex of G has the same valence. In this case, the valence of G is defined to be the valence of any vertex and is denoted by *val(G).*

Definition 1.3:  A *k-factor* of  G  is a regular, spanning subgraph  F  with  val(F) = k .

Of particular interest in this dissertation are the 1-factors of a graph  G .  Theorem 1.1 is a characterization of 1-factors which will be useful in the sequel, and is an immediate consequence of the following definitions:  A subset R  of E(G) is called *independent* if every vertex of  G  is incident to at most one edge in  R .  A subset  R  of E(G) is said to *cover* a subset  S  of V(G) if every vertex of  S  is incident to at least one edge in  R .

Theorem 1.1:  The subgraph  F  is a 1-factor of  G  if and only if E(F) is an independent set of edges which covers V(G).

In the literature of graph theory, 1-factors are sometimes referred to as "perfect matchings".

If  F'  is a set of independent edges of  G ,  and if there is a set of edges  $S \subset E(G) - F'$  such that F = [F'] ∪ [S]  is a 1-factor of  G ,  we call  F  an *extension of*  F'  *to a 1-factor of*  G .

A *digraph*, or *directed graph*, is a graph as defined in Definition 1.1, except that an edge is an ordered pair of vertices (u,v), and it is admissible to have an edge (v,v).

Like graphs, digraphs are often displayed geometrically as a network of points joined by lines. In this case, however, each line is given an arrow to indicate the orientation of the edge it represents (fig. 1.2).
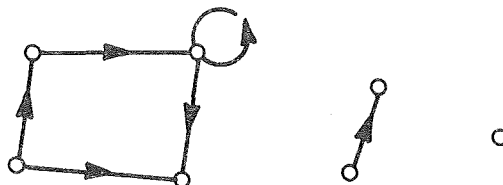
Figure 1.2

Note that in a digraph, an edge uv is distinct from the edge vu.

We now define several special graphs for which it is convenient to have a standard notation and terminology.

An *edge graph* is a graph consisting of a single edge together with its end points. A *vertex graph* is a graph consisting of a single vertex with no edges. The *complete n-graph* $K_n$ is the graph with n vertices and an edge joining every pair of vertices.

We now define a bigraph, an object of central interest in this dissertation.

Definition 1.4:   A *bigraph*, or *bipartite graph*, is a graph  G  whose vertex set can be written as the disjoint partition  $V(G) = S \cup T$ ,  such that each edge of  G  has one end point in  S  and the other in  T .

G  is called an *(m,n) bigraph* if  $|S| = m$  and  $|T| = n$ . Bigraphs are sometimes called bicolorable graphs, and it is often illuminating and economical to refer to the vertices in S  as "green" and the vertices in  T  as "blue". We will not hesitate to use this convention whenever it will clarify or shorten our exposition.

Observe that the edge set of the bigraph  G  can be considered as a subset of S×T.   This practice is also frequently useful in discussing bigraphs [4], and is used at least once in what follows.

Notice that if $v_0 v_1 \cdots v_n$ is a path in a bigraph  G , the vertices are alternately blue and green.

The *complete bigraph*  $K_{m,n}$  is the graph with  m  green vertices and  n  blue vertices such that each green vertex is joined by an edge to every blue vertex.

A *tree* is a connected graph containing no cycles.  More generally, a *forest* is a graph having no cycles.  Hence, the connected components of a forest are trees.

We end this section with some well-known graph-theoretical results which are needed later.

Theorem 1.2 (König):  A graph  G  is a bigraph if and only if  G  does not contain a cycle of odd length.  (For a proof see [18], p. 68.)

Theorem 1.3 (Whitney):  Let  G  be a 2-connected graph.  Suppose  K  is a 2-connected proper subgraph of  G  containing at least one edge.  Then we can write  G  as  H ∪ L , where  H  is a 2-connected proper subgraph of  G  containing  K , and  L  is a simple path in  G  that avoids  H  except for its distinct end points which are in V(H).  (For a proof see [18], p. 85.)

Theorem 1.4:  Every tree has at least two monovalent vertices.  (For a proof see [18], p. 19.)

2.  Nonnegative Matrices — Definitions and Preliminaries

The following definitions and Theorems 2.1, 2.2, and 2.3 are essentially as they appear in [15].

A *diagonal* of an n×n square matrix  A  is a collection of  n  entries of  A , precisely one of which appears in each row and column of  A . If  $\sigma \epsilon S_n$  (the symmetric group of degree  n ), the diagonal  $a_{1\sigma(1)}, a_{2\sigma(2)}, \cdots, a_{n\sigma(n)}$  is said to correspond to  $\sigma$ . It is clear that this relation establishes a 1-to-1 correspondence between the permutations in  $S_n$  and the diagonals of  A . A *diagonal product* of  A  is the product of the entries in a diagonal of  A . We should observe here that the diagonal products of  A  are precisely the terms of the permanent function defined in the introduction.

A diagonal  d  of  A  is said to be *positive* if every entry of  A  in  d  is positive.

A nonnegative m×n matrix  A  is *chainable* if for every pair of positive entries  $a_{i_1 j_1}$  and  $a_{i_k j_k}$  there is a sequence of positive entries  $a_{i_1 j_1}, \cdots, a_{i_k j_k}$  where, for  $r = 1, \cdots, k-1$ , either  $i_r = i_{r+1}$  or  $j_r = j_{r+1}$ . This sequence is called a *chain* with end points  $a_{i_1 j_1}$  and  $a_{i_k j_k}$ .

Recall that an n×n nonnegative matrix  A  is said to be doubly stochastic if every row sum and column sum equals 1.

A nonnegative square matrix  A  has *doubly stochastic pattern* if there is a doubly stochastic matrix  B  such that  $a_{ij} = 0$  if and only if  $b_{ij} = 0$ .

A nonnegative matrix  A  is said to have *support* if every positive entry lies on a positive diagonal.

Theorem 2.1:  If  A  is an n×n FI matrix, then any (n-1)×(n-1) submatrix contains a positive diagonal of (n-1) entries.

Theorem 2.2:  An n×n nonnegative matrix  A  has doubly stochastic pattern if and only if  A  has support.

Theorem 2.3:  An n×n nonnegative matrix  A  is FI if and only if  A  is chainable and has support.

The following theorems are immediate consequences of the definitions.

Theorem 2.4:  An n×n nonnegative matrix  A  is FI if and only if  A  does not contain an s×t zero submatrix with s + t = n .

Theorem 2.5:  If  A  is an FI matrix, every row and every column contains at least two positive entries.

3.  Correspondences Between Matrices and Graphs

In this section the fundamental relationship between a nonnegative matrix and a bigraph is defined.  After the basic definition, a number of theorems are listed which reveal a useful correspondence between some analagous concepts in graph theory and matrix theory.  Many of these theorems follow directly from preceding definitions.

Definition 3.1: Let A be an m×n (0,1) matrix. We associate a bigraph G with A in the following manner: Let $V(G) = S \cup T$ where S is a set of m vertices $\{v_1, v_2, \cdots, v_m\}$, and T is a set of n vertices $\{w_1, w_2, \cdots, w_n\}$. We let the vertex $v_i$ correspond to the $i$th row of A for $i = 1, \cdots, m$, and $w_j$ correspond to the $j$th column of A for $j = 1, \cdots, n$. For each i and j, where $i = 1, \cdots, m$ and $j = 1, \cdots, n$, there is an edge $v_i w_j$ in E(G) if and only if the entry $a_{ij}$ of A is positive. G is called the *bigraph corresponding to* A, or simply the *bigraph of* A. We call A *a matrix representative of* G.

The last terminology becomes more meaningful when we observe that the graph G can be made to correspond to any (0,1) matrix in the p-equivalence class of A. This follows from the fact that permutations of rows and columns of A correspond to a renumbering of the vertices in S and T, respectively. Thus, A is but one representative of the p-equivalence class corresponding to G. These facts and the invariance of many of the combinatorial properties of matrices under p-equivalence transformations, make bigraphs a powerful tool in studying these properties. These comments will become clearer as the theory unfolds.

Definition 3.2: Let A be an n×n (0,1) matrix. We associate a digraph G with A in the following manner: Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. For any pair i and j where $1 \leq i$ and $j \leq n$, there is an edge $v_i v_j$ in E(G) if and only if the entry $a_{ij}$ of A is positive. We call G *the digraph of* A.

Notice that the digraph G of an n×n (0,1) matrix A represents the entire p-similarity class of A in a manner analagous to the representation of a p-equivalence class by a single bigraph.

If uv is an edge of a digraph, u is called the *initial vertex* and v the *terminal vertex* of uv. By a *directed path* P in a digraph G, we mean a sequence $v_0, e_1, v_1, \cdots, e_n, v_n$, whose entries are alternately vertices and edges of G such that $v_{i-1}$ is the initial vertex and $v_i$ the terminal vertex of $e_i$ for $i = 1, \cdots, n$. The vertices $v_0$ and $v_n$ are called the *initial vertex* and the *terminal vertex*, respectively, of the directed path P.

A digraph G is said to be *strongly connected* if for every pair u and v of vertices of G, there is a directed path P with initial vertex u and terminal vertex v, and there is a directed path P' with initial vertex v and terminal vertex u.

The following well-known result gives a complete characterization of irreducible matrices in terms of their digraphs, and is an indispensable tool in the study of irreducible and nearly reducible matrices. (See [9], and [1], p. 123.)

Theorem 3.1: An n×n (0,1) matrix A is irreducible if and only if its digraph G is strongly connected.

This result with many of its applications appears in [19].

We now note some interesting correspondences between analagous concepts pertaining to nonnegative matrices and their bigraphs.

From the above definitions we see that rows and columns of A correspond to green and blue vertices, respectively, of its bigraph G , and that the positive entries of A correspond to the edges of G .

Theorem 3.2: Let G be the bigraph of an n×n nonnegative matrix A . If G is connected, then A is chainable. If A is FI, then G is connected.

Proof: Suppose G is connected, and let $a_{i_1 j_1}$ and $a_{i_k j_k}$ be any two positive entries in A . Then $v_{i_1} w_{j_1}$ and $v_{i_k} w_{j_k}$ are edges of G . There is a path

$w_{j_1} v_{i_2} w_{j_2}, \cdots, w_{j_{k-1}} v_{i_k}$ joining the vertices $w_{j_1}$ and $v_{i_k}$

by the connectedness of G . Then $a_{i_1 j_1} a_{i_2 j_1}, \cdots, a_{i_k j_{k-1}} a_{i_k j_k}$,

which corresponds to the sequence of adjacent edges

$v_{i_1} w_{j_1}, w_{j_1} v_{i_2}, \cdots, w_{j_{k-1}} v_{i_k}, v_{i_k} w_{j_k}$, is the necessary chain

satisfying the chainability condition for A .

Conversely, suppose A is FI. By Theorem 2.5, every

row and every column of A contains at least two positive

entries. This is equivalent to saying that every vertex of

G has at least two incident edges. If $v_{i_1}$ and $w_{j_k}$ are

any two vertices in G , there is an edge $v_{i_1} w_{j_1}$ incident

to $v_{i_1}$ and an edge $v_{i_k} w_{j_k}$ incident to $w_{j_k}$ . These two

edges correspond to positive entries $a_{i_1 j_1}$ and $a_{i_k j_k}$ of

A , and there is a chain $a_{i_1 j_1}, \cdots, a_{i_k j_k}$ with end points

$a_{i_1 j_1}$ and $a_{i_k j_k}$ . Clearly, extraneous entries can be

eliminated from the chain to yield a chain with the property

that if $i_r = i_{r+1}$ , then $i_{r+1} \neq i_{r+2}$ and $j_{r+1} = j_{r+2}$ ,

and if $j_r = j_{r+1}$ , then $j_{r+1} \neq j_{r+2}$ and $i_{r+1} = i_{r+2}$ ,

for $r = 1, \cdots, k-2$ . But such a chain corresponds to a path

in G with end points $v_{i_1}$ and $w_{j_k}$ ; hence, G is

connected.

Theorem 3.3: Let G be the bigraph of an n×n non-negative matrix A. Then the entries of any positive diagonal of A correspond to the edges of a 1-factor of G and the edges of any 1-factor of G correspond to the entries of a positive diagonal of A.

Proof: Let F be a set of edges of G and d the corresponding positive entries of A. Then [F] is a 1-factor of G if and only if F is a spanning independent set, if and only if each vertex of G appears exactly once as an end point of some edge in F, if and only if each row and column contains exactly one entry of d, if and only if d is a (positive) diagonal.

Corollary 3.4: If A is an n×n (0,1) matrix and G its bigraph, then the number of distinct 1-factors of G = per A.

Theorem 3.5: If A is a nonnegative matrix and G its bigraph, then A has doubly stochastic pattern if and only if every edge of G is contained in a 1-factor of G.

Proof: Apply Theorems 2.2 and 3.3.

When every edge of a graph G is contained in a 1-factor of G, we say that G has *support*.

Suppose $G$ is an $(m,n)$ bigraph with $V(G) = S \cup T$. Suppose further that $U$ and $V$ are nonempty subsets of $S$ and $T$, respectively. If, in addition, the subgraph induced by $U \cup V$ has no edges (equivalently $(U \times V) \cap E(G) = \emptyset$), we say that $U \cup V$ is an *independent* subset of $V(G)$. Note that this definition ensures that an independent subset of $V(G)$ always contains both green and blue vertices.

Theorem 3.6: Let $A$ be an $n \times n$ nonnegative matrix with bigraph $G$. Then $A$ has an $s \times t$ zero submatrix if and only if $G$ has an independent set of edges $U \cup V$ with $|U| = s$ and $|V| = t$.

Proof: This follows from the definitions.

Corollary 3.7: An $n \times n$ nonnegative matrix $A$ is FI if and only if its bigraph $G$ does not have an independent set of edges $U \cup V$ with $|U| = s$ and $|V| = t$ such that $s + t = n$.

Proof: Apply Theorem 2.4.

4. Main Results on Bigraphs of Fully Indecomposable Matrices

In this section we develop the most significant results in this dissertation. The most important of these are Theorems 4.9, 4.10, and 4.14.

Theorems 4.9 and 4.10 constitute a new tool which is particularly applicable to inductive-type arguments involving ND matrices. It is hoped that further applications of this result will be obtained in the future.

Theorem 4.14 characterizes FI matrices in terms of the structure of their bigraphs. Like the analagous well-known result which characterizes irreducible matrices in terms of the strong connectivity of their directed graphs (Theorem 3.1), it is a useful tool in determining when a matrix is FI. Furthermore, its power in revealing important properties of FI matrices is demonstrated by the fact that it yields Hartfiel's theorem, and by the applications which follow in Chapter III.

We begin our development by determining the structure of 2-connected bigraphs corresponding to square matrices via a classical theorem of Hassler Whitney (Theorem 1.3). The structure of bigraphs of FI matrices (Theorem 4.14 below) is developed independently of this material, but the consequences of Whitney's theorem illuminated much of this author's research, and we feel that it provides the proper setting for the presentation of our results.

Theorem 4.1: Let A be an n×n FI matrix. Then the bigraph G of A is 2-connected.

<u>Proof</u>: Since the theorem is obvious for $n = 1$, we can assume $n \geq 2$. $G$ is an $(n,n)$ bigraph with $V(G) = S \cup T$. By Corollary 3.7, if $A$ is FI, then there do not exist non-empty subsets $U \subset S$ and $V \subset T$ with $|U| + |V| = n$ and $(U \times V) \cap E(G) = \emptyset$.

Our method of proving the theorem is to assume that $A$ is FI, that $G$ is not 2-connected, and show that this leads to a contradiction.

Since $G$ is not 2-connected, there is an articulation point $x$ in $V(G)$, so that $G-x$ has $k$ connected components $(k > 1)$. Let $P$ be the vertex set of any single component of $G-x$ and $Q$ the union of the vertex sets of the remaining components. Thus, $P$ and $Q$ are nonempty and $|P \cup Q| = 2n - 1$, since $|V(G)| = 2n$. We observe that $(S \cap P) \cup (S \cap Q) \cup (T \cap P) \cup (T \cap Q)$ is a disjoint partition of $V(G)-x$. Hence

$$(1) \quad |(S \cap P)| + |(S \cap Q)| + |(T \cap P)| + |(T \cap Q)| = 2n - 1$$

Let $U = S \cap P$ and $V = T \cap Q$. Likewise, let $U' = S \cap Q$ and $V' = T \cap P$. We now show that $(U \times V) \cap E(G) = (U' \times V') \cap E(G) = \emptyset$. For, suppose $(U \times V) \cap E(G) \neq \emptyset$. Then there would be an edge $uv$ of $G$ with

$u \neq x$, $v \neq x$, $u$ in $P$, and $v$ in $Q$. This contradicts the fact that $P$ is the vertex set of a component of $G-x$. Hence, $(U \times V) \cap E(G) = \emptyset$ and, similarly, $(U' \times V') \cap E(G) = \emptyset$.

Now, if $|U| + |V| \leq n - 1$ and $|U'| + |V'| \leq n - 1$, then $|U| + |V| + |U'| + |V'| \leq 2n - 2$, which contradicts equation (1). Hence, either $|U| + |V| \geq n$ or $|U'| + |V'| \geq n$. Suppose $|U| + |V| \geq n$. Then $U$ and $V$ are both nonempty. To see this, we assume $U = \emptyset$. Then $|V| \geq n$, and, since $V \subset T$ and $|T| = n$, we have $V = T$. Therefore, $T \subset Q$. But since $U$ is empty, $P \subset T$, and it follows that $P \subset Q$. This is impossible, so $U \neq 0$. Similarly, $V \neq \emptyset$.

We have constructed nonempty subsets $U$ and $V$ of $S$ and $T$, respectively, with $(U \times V) \cap E(G) = \emptyset$ and $|U| + |V| \geq n$. If $|U| + |V| > n$, since $n \geq 2$, we can remove a vertex from one of the two sets and still have a nonempty pair of sets satisfying the above properties. By induction, it follows that we can assume $|U| + |V| = n$. But this is a contradiction, since it implies that $A$ is partly decomposable, by Corollary 3.7. Hence, $G$ is 2-connected.

At this point it is natural to ask whether the FI matrices are characterized by the 2-connectivity of their bigraphs. The following example shows a 2-connected (n,n) bigraph G with its representative partly decomposable matrix A, and provides a negative answer to this question.

G:

3   3'      1

        2

2'          1'

        6'

6           4

5'   5      4'

A:

|    | 1' | 2' | 3' | 4' | 5' | 6' |
|----|----|----|----|----|----|----|
| 1  | 1  | 0  | 1  | 0  | 0  | 0  |
| 2  | 1  | 1  | 0  | 0  | 0  | 0  |
| 3  | 0  | 1  | 1  | 0  | 0  | 0  |
| 4  | 1  | 0  | 0  | 1  | 0  | 1  |
| 5  | 0  | 0  | 0  | 1  | 1  | 0  |
| 6  | 0  | 1  | 0  | 0  | 1  | 1  |
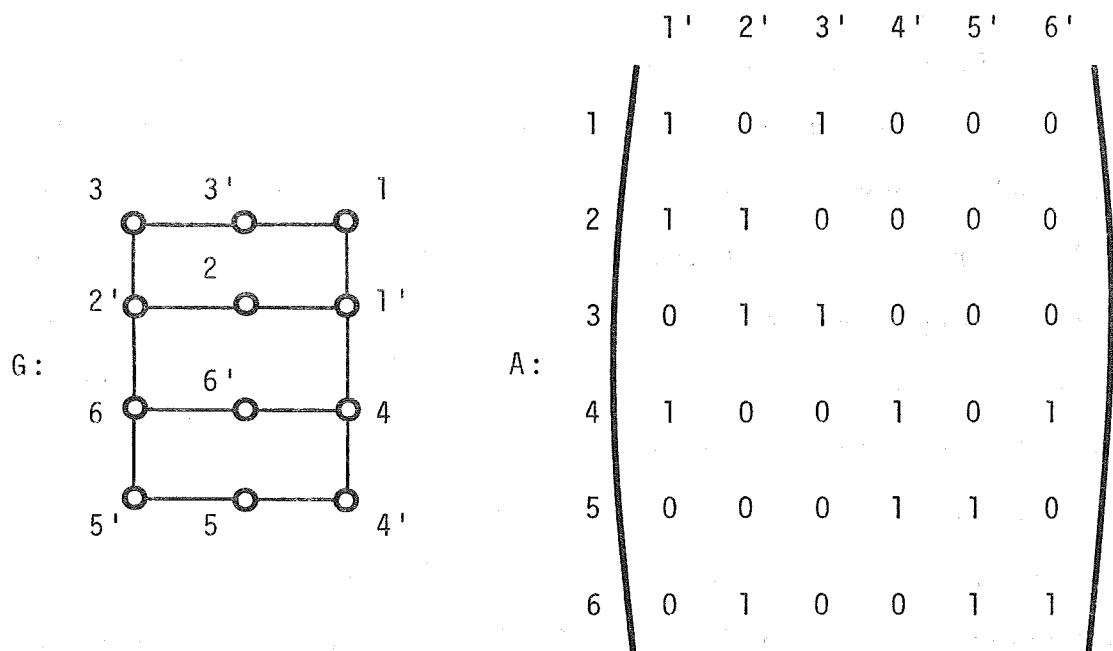
Theorem 4.14 tells us which 2-connected bigraphs correspond to FI matrices.

Theorem 4.2: If $G$ is a 2-connected graph which is not a single vertex, then we can write $G$ as a union of subgraphs: $G = P_0 \cup P_1 \cup \cdots \cup P_k$ (denote the subgraph $P_0 \cup P_1 \cup \cdots \cup P_j$ by $B_j$ for $j = 0,1,\cdots,k$), where $P_0$ is an edge graph, and $P_i$ is a simple path which avoids $B_{i-1}$, except for its distinct end points which are contained in $V(B_{j-1})$.

Proof: We use induction on the number of edges in $G$. If $G$ has one edge, we write $G = P_0$, and the theorem is trivially true. Suppose $G$ has $n > 1$ edges, and the theorem holds for all 2-connected graphs with fewer than $n$ edges. Pick an edge $e$ in $G$. Then $[e]$ is a proper 2-connected subgraph and, by Theorem 1.3 (Whitney), $G = H \cup L$ where $H$ is a 2-connected proper subgraph of $G$ which contains $e$, and $L$ is a simple path which avoids $H$, except for its distinct end points which are in $V(H)$. Then $H$ is a proper subgraph of $G$, and hence, $E(H)$ is a proper subset of $E(G)$. Otherwise, there would be isolated vertices in $G$, and $G$ would not be 2-connected. (In fact, $G$ would not even be connected.) Therefore the induction hypothesis applies, and $H = P_0 \cup P_1 \cup \cdots \cup P_{k'}$. Then we can write $G = P_0 \cup \cdots \cup P_{k'} \cup L$, and the theorem is proved.

Note that the representation $G = P_0 \cup P_1 \cup \cdots \cup P_k$ is not unique. We will call $P_0 \cup P_1 \cup \cdots \cup P_k$ a *path decomposition* of $G$. In Chapter III, we see that for bigraphs corresponding to FI matrices, the integer $k$ is unique and depends on $G$ alone, and not on the particular path decomposition.

Next we investigate path decompositions for 2-connected bigraphs corresponding to $n \times n$ nonnegative matrices.

Suppose $A$ is an $n \times n$ nonnegative matrix whose $(n,n)$ bigraph $G$ is 2-connected. Then $G = P_0 \cup P_1 \cup \cdots \cup P_k$. For a given $i$, where $1 \leq i \leq k$, $P_i$ is a path with end points, say, $u$ and $v$, whose length $L$ is either odd or even. Assume $L$ is odd. Then $d_{B_{i-1}}(u,v)$ must be odd (equivalently, $u$ and $v$ are differently colored vertices) or the adjunction of $P_i$ would introduce a cycle of odd length into $G$, contradicting Theorem 1.2. Similarly, if $P_i$ is of even length, then $d_{B_{i-1}}(u,v)$ must be even (equivalently, $u$ and $v$ have the same color).

Furthermore, if $P_i$ has odd length, then the number of vertices of each color added to $V(B_{i-1})$ in passing to $V(B_i)$ is the same. If, however, $P_i$ has even length and its end points are, for example green, then the adjunction of $P_i$ adds $r$ additional green vertices and $r+1$ additional blue vertices, where $r \geq 0$. Since $P_0$ is a $(1,1)$ bigraph, the

even paths of any path decomposition of G must balance properly to achieve an equal number of green and blue vertices for G . The following is a summation of the preceding observations. (As always $B_i$ denotes $P_0 \cup P_1 \cup \cdots \cup P_i$ for $0 \leqq i \leqq k$ .)

Theorem 4.3: Let A be an n×n nonnegative matrix and suppose its bigraph G is 2-connected. Then

(1) $G = P_0 \cup P_1 \cup \cdots \cup P_k$ , where $P_0$ is an edge graph, $P_1$ is a simple path of odd length which avoids $P_0$ except for its end points, which coincide with those of $P_0$ .

(2) For $2 \leqq i \leqq k$ , $P_i$ is a simple path which avoids $B_{i-1}$ except for its end points u and v , which are in $V(B_{i-1})$ , where

 (a) $P_i$ is of odd length, and $d_{B_{i-1}}(u,v)$ is odd, or

 (b) $P_i$ is of even length, and $d_{B_{i-1}}(u,v)$ is even.

(3) The number of even paths among the $P_i$ $(2 \leqq i \leqq k)$ is 0 or an even number, exactly half of which have green end points, the other half blue end points.

Before presenting our main results, the following lemmas are proved:

Lemma 4.4: If A is an n×n FI matrix with n ≥ 2 , and if $a_{ij}$ is any positive entry of A , then the n×n matrix A' = A - $a_{ij}E_{ij}$ is chainable.

Proof: The bigraph of a matrix is connected if the matrix if FI by Theorem 3.2. Let G be the bigraph corresponding to A . Then there is an e in E(G) corresponding to $a_{ij}$ , and G' = G - e is the bigraph of A' . If G' is not connected, the end points of e , each with a valence at least equal to 2 , are articulation points in G , so G is not 2-connected. This contradicts the fact that A is FI. Hence, G' is connected and, therefore, A' is chainable.

Lemma 4.5: If A is an n×n ND matrix with n ≥ 2 , and $a_{ij}$ is any positive entry of A , then the matrix A' = A - $a_{ij}E_{ij}$ contains a positive diagonal.

Proof: The lemma is clearly true for n = 2 . Suppose n > 2 and the lemma holds for k×k ND matrices when k < n . By Theorem 2.3, every ND matrix has a positive diagonal. Using Theorem I.2.1 we bring A into the canonical form stated therein. If $a_{ij}$ occurs as the positive element of one of the $E_i$ (1 ≤ i ≤ s), then A' still has a positive

diagonal passing through $A_i$ $(i \leq i \leq s)$. If $a_{ij}$ occurs in one of the $A_i$, for example $A_{i'}$, and if $A_{i'}$ is not 1×1, we invoke the induction hypothesis to establish that $A_{i'}$ still has a positive diagonal after the replacement of $a_{ij}$ by 0. Hence, A' still has a positive diagonal passing through the $A_i$ $(1 \leq i \leq s)$. If $A_{i'}$ is 1×1, Theorem 2.1 allows us to conclude that there is a positive diagonal passing through $E_{i'}$ and $E_j$, where $j = i' + 1 \pmod{s}$.

Lemma 4.6 (Hartfiel): If A is an n×n ND (0,1) matrix with $n \geq 3$, then its bigraph G cannot contain a cycle of length 4.

Proof: Observe that it suffices to show that A does not contain a 2×2 positive submatrix.

We bring A into the canonical form of Theorem I.2.1, and agree to adopt the notation and terminology of that theorem. Some additional notation is required.

The square submatrices $A_k$ are said to be $n_k \times n_k$ for $k = 1, \cdots, s$. The single positive entry in the submatrix $E_k$ is denoted by $a_{i_k j_k}$ for $k = 1, \cdots, s$. By $A_k(i,j)$ is meant the $(n_k - 1) \times (n_k - 1)$ submatrix formed by striking out the row and column of $A_k$ corresponding, respectively, to the $i$th row and $j$th column of A, for $k = 1, \cdots, s$.

In accordance with these definitions, $i_k$ and $j_k$ , respectively, denote the row and column of $A$ which contains the unique positive entry of the submatrix $E_k$ .

We now show that none of the $A_k$ can be a 2×2 submatrix. Suppose for some $k'$ $(1 \le k' \le s)$, $A_{k'}$ is a 2×2 submatrix. Then $A_{k'}$ is positive, since it is ND. Without loss of generality, we assume that $i_{k'}$, $i_{k'}+1$, $j_{k'+1}$ , and $j_{k'+1}+1$ index, respectively, the rows and columns of $A$ which determine the submatrix $A_{k'}$ . But then $a_{i_{k'} j_{k'+1}}$ is removable. For, by Lemma 4.4,

$$A' = A - a_{i_{k'} j_{k'+1}} E_{i_{k'} j_{k'+1}}$$ is chainable, and it remains

to be shown that $A'$ has support. Suppose $a$ is any positive entry of $A'$ which lies on a diagonal $d$ of $A$ , where $a_{i_{k'} j_{k'+1}}$ is an entry of $d$ . There are two cases to consider:

(1) $a = a_{i_{k'}+1, j_{k'+1}+1}$ or $a = a_{i_k j_k}$ for some $k$ , $1 \le k \le s$ .

(2) $a$ is an entry of $A_k$ for some $k \ne k'$ , $1 \le k \le s$ .

In the first case, there is a positive diagonal $d_k$ in each of $A_k(i_k, j_{k+1})$ for $k = 1, \cdots, s$ (again, $k+1$ denotes addition mod $s$), by Theorem 2.1. But then the entries of all the $d_k$'s plus the entries $a_{i_k j_k}$ $(k = 1, \cdots, s)$, and the

single entry $a_{i_{k'}+1, j_{k'+1}+1}$ form a positive diagonal of $A'$

containing a . In the second case, if a is an entry of,

say, $A_{k_0}$ , then $A_{k_0}$ has a positive diagonal $d_0$

containing a , and the entries of $d_0$ together with the

entries of some diagonal of each $A_k$ , ($k \neq k_0$, $k \neq k'$,

$1 \leq k \leq s$), plus the entries $a_{i_{k'}, j_{k'+1}+1}$ and

$a_{i_{k'}+1, j_{k'+1}}$ of $A_{k'}$ form a positive diagonal of $A$

containing a . In any case, $A'$ has support and is,

therefore, ND. This contradiction proves that none of the

$A_k$ is 2×2 ($k = 1, \cdots, s$).

It follows that the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

is the only 3×3 ND (0,1) matrix (up to p-equivalence). Hence

it is clear that the lemma holds for all r×r ND (0,1) matrices

with r = 3 , and assume it holds whenever $r \geq 3$ and r < n .

Now, suppose A contains a 2×2 positive submatrix $A'$ . Then,

by the induction hypothesis, $A'$ is not contained in any of

the $A_k$ , for $1 \leq k \leq s$ . Clearly, it must be that s = 2

and A corresponds to the canonical form

$$\begin{pmatrix} A_1 & E_1 \\ E_2 & A_2 \end{pmatrix}$$

It is also clear that $a_{i_1 j_1}$ and $a_{i_2 j_2}$ must be entries of $A'$, so that $i_1$ and $i_2$, and $j_1$ and $j_2$ are the rows and columns, respectively, which determine $A'$. At least one of the $A_k$, say $A_2$, is larger than $1 \times 1$, and hence is at least $3 \times 3$. Then the positive entry $a_{i_2 j_1}$ is removable. For by Lemma 4.4, $A'' = A - a_{i_2 j_1} E_{i_2 j_1}$ is chainable. We show that $A''$ has support in the following manner: suppose $a$ is a positive entry of $A''$ which lies on a diagonal $d$ of $A$ containing $a_{i_2 j_1}$. Again we consider two cases: (1) $a$ is an entry of $A_1$, (2) $a$ is an entry of $A_2$. In the first case, let $d_1$ be any positive diagonal of $A_1$ which contains $a$, and let $d_2$ be any positive diagonal in the submatrix formed from $A_2$ by replacing $a_{i_2 j_1}$ by zero (such a diagonal exists by Lemma 4.5). Then the entries of $d_1$, combined with the entries of $d_2$ form a positive diagonal of $A''$ containing $a$. In the second case, the entries of $d$ contained in $A_2$ (other than $a_{i_2 j_1}$), plus the entries $a_{i_1 j_1}$ and $a_{i_2 j_2}$ form a positive diagonal of $A''$ containing $a$, when $A_1$ is $1 \times 1$. When $A_1$ is larger than $1 \times 1$, we must add the entries of some diagonal in $A_1(i_1, j_2)$. At any rate, $A''$ has support.

This completes the proof of the lemma.

Lemma 4.7: If A is an n×n ND matrix with n ≥ 3 , then there are at least three rows and three columns of A having precisely two positive entries.

Proof: We use induction on n . The lemma is clearly true for the 3×3 ND matrix (see proof of Lemma 4.6). Suppose n > 3 . We put A in the canonical form of Theorem I.2.1. If every $A_i$ , where $1 \leq i \leq s$ , is 1×1, the lemma is clearly true. If some $A_{i'}$ is not 1×1, it is at least 3×3 and satisfies the induction hypothesis. It is clear that $A_{i'}$ contributes at least two rows and two columns of A which contain precisely two positive entries since $E_{i_0}$ and $E_{i_k}$ [$k = i_0 + 1$ (mod s)] contain only one positive entry each. The lemma follows.

Definition 4.1: If $P = v_0 v_1 \cdots v_n$ is a simple path in a graph G , then the edges, if any, of G (other than $v_0 v_1$) which are incident to the end point $v_0$ are called the *edges of attachment of* $v_0$ *with respect to* P . The *edges of attachment of the end point* $v_n$ are defined similarly. Henceforth, when there is no danger of confusion, the qualifying phrase "with respect to P " is suppressed.

Definition 4.2: Let G be an (n,n) bigraph, and suppose v is a divalent vertex of G . Let a and b be the unique vertices which are adjacent to v . We form a new entity G' by eliminating the vertex v and the edges

av and vb from G , and identifying the vertices a and b
to form a single vertex v' (see fig. 4.1). We call G'
*the contraction of G with respect to* v .

Notice that in general G' can contain loops or mul-
tiple edges and, therefore, not be a graph according to our
definition. Fortunately, in the interesting case of bigraphs
of ND matrices, G' is a graph.
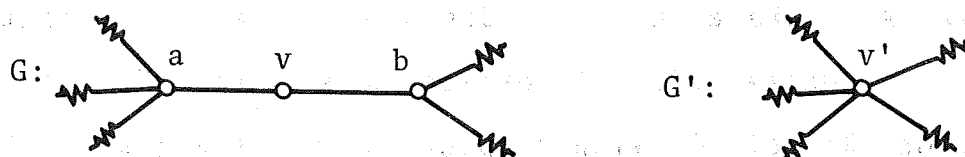


Figure 4.1

Definition 4.3: If G is the bigraph of an FI matrix
A , the edge e is said to be *removable* if G-e is the
bigraph of an FI matrix A' .

An immediate consequence of the above definition is
that if G is the bigraph of an FI matrix A , an edge e
of G is removable if and only if the positive entry $a_{ij}$
of A , which corresponds to e , is removable.

<u>Lemma 4.8</u>: Let  A  be an n×n FI matrix. Suppose $a_{ij}$ is an entry of  A  which is not removable, and  $a_{st}$  an entry which is removable. Then  $a_{ij}$  is not removable in $A - a_{st}E_{st}$.

<u>Proof</u>:  $a_{ij}$  is the only positive element in an s×t submatrix of  A  (and, hence, of $A - a_{st}E_{st}$) with  s + t = n .

Recall that a node  w  is a vertex which is not divalent. In the case where  w  is a node of the bigraph of an FI matrix, Theorem 2.5 tells us that  val(w) $\geqq$ 3 .

<u>Theorem 4.9</u>:  Let  G  be the bigraph of an n×n ND (0,1) matrix  A , where  n $\geqq$ 3 . Suppose  G  has a divalent vertex  v  whose adjacent vertices are nodes. Then the contraction  G'  of  G  with respect to  v  is a graph. Furthermore,  G'  is the bigraph of an (n-1)×(n-1) ND (0,1) matrix A' , and per A = per A' .

<u>Proof</u>:  Let  a  and  b  be the distinct nodes in V(G) which are adjacent to  v . Then  a  is not adjacent to  b . Let $\{e_1, \cdots, e_q\}$ and $\{f_1, \cdots, f_r\}$ denote the edges of attachment of  a  and  b , respectively. We will refer to these edges by this terminology whether we are considering them as being either in E(G) or in E(G'). (See Figure 4.2.) Observe

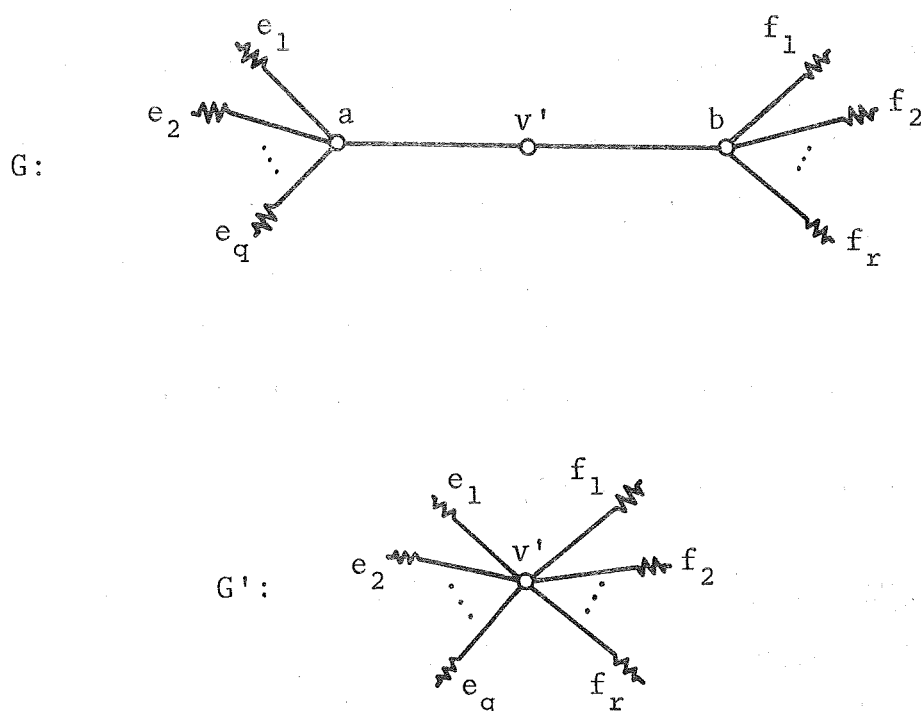Figure 4.2

that since a and b are nodes, $q \geqq 2$ and $r \geqq 2$. We will divide the proof into four parts:

(1) G' is a graph.

(2) There is a 1-1 correspondence between the 1-factors of G and G', respectively, such that if the 1-factor F of G corresponds to the 1-factor F' of G', then the edges of F in E(G') coincide with E(F').

(3)   G'   corresponds to an  (n-1)×(n-1) FI (0,1) matrix
      A' ,  and  Per A = Per A' .

(4)   A'  is ND, or equivalently, no edge of  G'  is
      removable.  We will demonstrate this by assuming
      that there is an edge  e  in E(G') such that G'-e
      is the bigraph of an FI matrix, and showing that,
      in this case, G-e is also the bigraph of an FI
      matrix, which contradicts the fact that  A  was ND.

The proofs of (1) through (4) follow:

Proof of (1):  It is clear that  G'  is a graph if and
only if it has no loops or multiple edges.  Now,  G'  has a
loop if and only if  a  and  b  are adjacent.  In this case,
the cycle avba has length  3 ,  contradicting Theorem 1.2.
Also,  G'  has multiple edges if and only if there is a vertex
w ≠ v ,  such that  w  is adjacent to both  a  and  b .  In
this case the cycle avbwa has length  4 ,  which contradicts
Lemma 4.6.  It follows that  G'  is a graph.

Proof of (2):  Suppose  F  is a 1-factor of  G .  Let
F' = E(F) ∩ E(G') .  It is clear that  F'  is an independent
set of edges which covers every vertex of  G'  except,
possibly,  v' .  We have either that av is in E(F) and vb is
not, or vb is in E(F) and av is not.  In the former case,
b  is covered by an edge of attachment  $f_i$, ε E(F) .  But

then $f_{i'} \in F'$ and $f_{i'}$ covers $v'$ . Similarly, in the remaining case, there is an $e_{j'} \in F'$ which covers $v'$ .

We have shown that $[F']$ is a 1-factor for $G'$ which is uniquely determined by $F$ and has the same edges as $F$ in $E(G')$ . We write $[F'] = \pi(F)$ . On the other hand, if $F'$ is any 1-factor of $G'$ , $E(F')$ contains either an edge of attachment $e_{s'}$ of $a$ or an edge of attachment $f_{t'}$ of $b$ . Say $e_s \in E(F')$ . Then $f_i \notin E(F')$ for $1 \leq i \leq r$ , so that $F = F' \cup [vb]$ is a 1-factor of $G$ having the same edges in $E(G')$ as $F'$ . Clearly, $\pi(F) = F'$ . This establishes the desired 1-1 correspondence.

Proof of (3): Observe that $G'$ is connected since $G$ is connected. If $e \in E(G')$ , there is a 1-factor $F$ of $G$ such that $e \in E(F)$ . But then using (2), $e \in E(\pi(F))$ , and it follows that any edge of $G'$ is contained in a 1-factor of $G'$ . $G'$ is an $(n-1, n-1)$ bigraph, and we let $A'$ be any $(n-1) \times (n-1)$, $(0,1)$ matrix representative of $G'$ . Then $A'$ is FI by Theorems 2.3, 3.2, and 3.3. Furthermore, it follows immediately from (2) that per $A = $ per $A'$ .

Proof of (4): Assume that $e$ is a removable edge of $G'$ . Then $G'-e$ is connected and has support. We know that $G-e$ is connected, otherwise $G$ could not be 2-connected. We will now show that $e$ is removable from $G$ by demonstrating that $G-e$ has support.

Consider first the edges av and bv of G-e . Notice that there still remains in E(G'-e) at least one edge of attachment $e_{i'}$ of a . (This is because $q \geqq 2$ and $r \geqq 2$ .) Now, $e_{i'} \varepsilon E(G'-e)$ so there is a 1-factor F' of G'-e which contains $e_{i'}$ and contains no $f_i$ , where $1 \leqq i \leqq r$ . Then F' $\cup$ [vb] is a 1-factor for G-e which contains vb. In a similar manner, we find a 1-factor of G-e containing av.

It remains to show that if f is an arbitrary edge of G-e other than av or vb, then f is contained in a 1-factor of G-e. But this is trivial, for there is a 1-factor F' of G'-e containing f . F' contains an edge of attachment $e_{i'}$ of a or $f_{i'}$ of b . In the former case F' $\cup$ [vb] and in the latter F' $\cup$ [av] are 1-factors of G-e containing f . Hence, G-e has support and therefore, is FI. This completes the proof of the theorem.

Theorem 4.9 has shown us that if A is a special type of ND (0,1) matrix, we can associate with the bigraph G of A a strictly smaller bigraph G' , which also corresponds to an ND (0,1) matrix. We need a similar result for the remaining ND matrices, and this is provided by the following.

Theorem 4.10: Let G be the bigraph of an n×n ND (0,1) matrix A , where n ≥ 3 . Suppose G does not have a divalent vertex whose adjacent vertices are nodes. G does have a divalent vertex v , by Lemma 4.7. At least one of the two vertices adjacent to v , say, w , is also divalent. We denote the unique vertices adjacent to v and w , respectively, by a and b (fig. 4.3). Then the
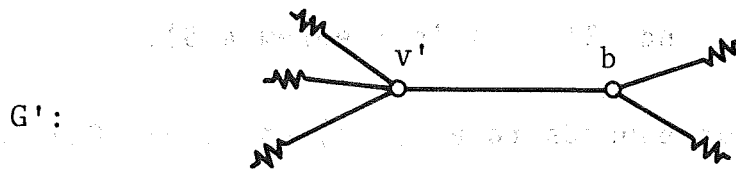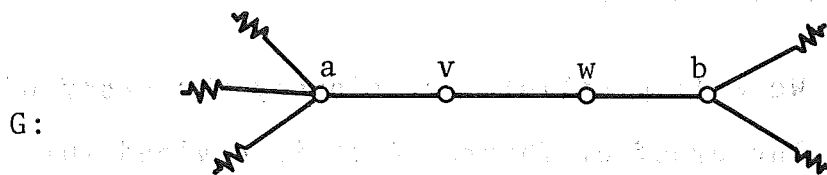


Figure 4.3

contraction  G'  of  G  with respect to  v  is a graph.
Furthermore, one of the following is true:

   (1)   G'  is the bigraph of an (n-1)×(n-1) ND (0,1)
         matrix  A' ,  or

   (2)   The edge v'b is removable from  G'  and, in this
         case, the graph  G" = G' - v'b  is the bigraph of
         an (n-1)×(n-1) ND (0,1) matrix  A" .

In either case  per A = per A' .

   Proof:  We will parallel very closely the proof of
Theorem 4.9.  The proof of Theorem 4.10 is divided into five
parts:

   (1')  G'  is a graph.

   (2')  There is a 1-1 correspondence between the 1-factors
         of  G  and  G'  (as in Theorem 4.9).

   (3')  G'  corresponds to an (n-1)×(n-1) FI (0,1) matrix
         A'  with  per A = per A' .

   (4')  No edge of  G' ,  other than v'b, is removable.

   (5')  Either  G'  or  G" = G' - v'b  corresponds to an
         ND (0,1) matrix.

Proof of (1'):  We let a,v and  w  correspond to the

vertices a,v. and  b ,  respectively, in Theorem 4.9.  The

proof is then identical to the proof of part (1) of that

theorem.

Proof of (2'):  Let  F  be a 1-factor of  G .  Then

either vw is in E(F) or av and wb are in E(F).  In the former

case,  F' = [E(F) ∩ E(G')] ,  and in the latter case,

F' = [E(F) ∩ E(G')] ∪ [v'b]  is a 1-factor for  G'  which

has precisely the same edges in E(G') as  F  does.  We write

F' = π(F) .  Now if  F'  is an arbitrary 1-factor of  G' ,

either  v'b ε E(F')  or not.  In the former case

F = F' ∪ [av] ∪ [wb] ,  and in the latter,  F = F' ∪ [vw]

is the unique 1-factor of  G  such that  π(F) = F' .

Proof of (3'):  This proof is identical to the proof

of part (3) of Theorem 4.9.

Proof of (4'):  The edges of attachment of  a  in  G

and of  v'  in  G'  are identical.  Also,  b  has the same

edges of attachment in  G'  as in  G .  Suppose there is an

edge  e ≠ v'b  in  G'  which is removable.  Let  f  be an

arbitrary edge of G-e.  If  f = av  or  f = wb ,  we let  F'

be a 1-factor of G'-e containing v'b.  Then

$F' \cup [av] \cup [wb]$ is a 1-factor of $G-e$ containing av and wb. If $f = vw$, there is a 1-factor $F'$ of $G'-e$ containing edges of attachment of $v'$ and of $b$, respectively, in $G'-e$. Then $F' \cup [vw]$ is a 1-factor of $G-e$ containing vw. In the remaining case, $f$ does not lie on the path avwb, and $f$ occurs in $G'-e$. There is a 1-factor $F'$ of $G'-e$ containing $f$, which contains $v'b$ or does not. In the former case, $F = F' \cup [av] \cup [wb]$, and in the latter case, $F = F' \cup [vw]$ is the 1-factor of $G-e$ which contains $f$. Now $G-e$ is connected, for otherwise an end point of $e$ would be an articulation point of $G$, which contradicts the 2-connectedness of $G$.

Proof of (5'): $G'$ corresponds to an $(n-1) \times (n-1)$ FI (0,1) matrix $A'$. We consider two cases:

(a) $u'v$ is not a removable edge. In this case $A'$ is ND.

(b) $u'v$ is removable. By Lemma 4.8, no edge of $G'-u'v$ is removable, and if $u'v$ corresponds to the positive entry $a_{ij}$ of $A'$, then $A'' = A' - a_{ij}E_{ij}$ is ND.

This completes the proof of the theorem.

The following graphs illustrate that both of the conditions treated by Theorem 4.10 can exist.
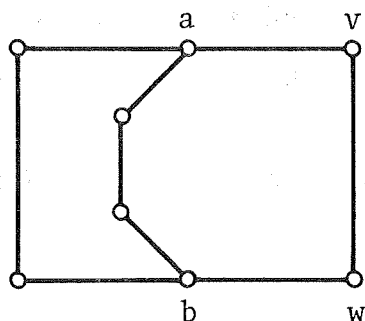
(1) G:
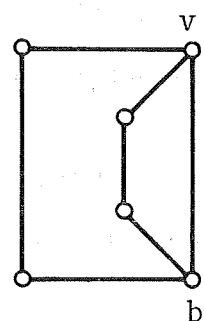


In this example we see that  G  is ND, and v'b is removable from  G' .
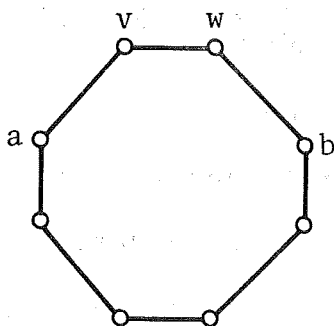
(2) G:
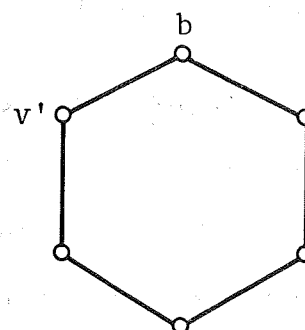


Here,  G  and  G'  are ND.

We now use the inductive tools developed in the preceding theorems to prove our main result.

We first need a few lemmas:

Lemma 4.11:  If $n \geq 2$, and $G$ is the bigraph of an FI $n \times n$ (0,1) matrix $A$, and if $v$ is a green vertex and $w$ a blue vertex of $G$, then $G-\{v,w\}$ contains a 1-factor.

Proof:  This lemma is a direct consequence of Theorem 2.1.

Lemma 4.12:  Suppose the graph $G$ has a path decomposition $G = P_0 \cup \cdots \cup P_k$ (as usual, $B_j = P_0 \cup \cdots \cup P_j$ for $j = 0, \cdots, k$) such that $P_0$ is an edge graph and $P_i$ ($i = 1, \cdots, k$) is a simple path which avoids $B_{i-1}$, except for its distinct end points, which are contained in $V(B_{i-1})$. Then if $v \in V(G)$, $v$ is either an end point of $P_0$ or, otherwise, an interior vertex of some $P_i$ ($i = 1, \cdots, k$).

Proof:  We use induction on $k$. The lemma is trivially true if $k = 0$. Suppose $k > 1$ and the lemma is true for $B_{k-1}$. Then the lemma holds by the induction hypothesis for those vertices (including the end points of $P_k$) which are in $V(B_{k-1})$. The remaining vertices of $G$ are interior to $P_k$.

Lemma 4.13:  Under the hypothesis of Lemma 4.12, each vertex of  G  is interior to at most one of the  $P_i$  (i = 1,···,k).

Proof:  This follows trivially from induction on  k .

Theorem 4.14:  Let  G  be the bigraph of an n×n (0,1) matrix  A .  Then  A  is FI if and only if there is a path decomposition  $G = P_0 \cup P_1 \cup \cdots \cup P_k$  ($B_j$  as usual denotes  $P_0 \cup P_1 \cup \cdots \cup P_j$  for  j = 0,···,k) satisfying the following:

(1)  $P_0$  is an edge graph.

(2)  $P_i$  (i = 1,2,···,k) is a simple path of *odd length*, which avoids  $B_{i-1}$  except for its end vertices  u  and  v ;  and  $d_{B_{i-1}}(u,v)$  is an odd positive integer.

We will henceforth refer to any path decomposition for G  which satisfies the hypothesis of this theorem as a *good path decomposition* for  G .

Proof:  Let  $G = P_0 \cup P_1 \cup \cdots \cup P_k$  be a good path decomposition for  G .  We use induction on  k  to prove (a) that  G  is connected and (b) that  G  has support.  If k = 0 , it is clear that  G  satisfies (a) and (b).  Clearly, $G = B_{k-1} \cup P_k$  is connected.  Now let  $P_k = v_0 v_1 \cdots v_s$ ,

where  s  is odd.  Partition $E(P_k)$ into the two sets
$X = \{v_0v_1, v_2v_3, \cdots, v_{s-1}v_s\}$  and
$Y = \{v_1v_2, v_3v_4, \cdots, v_{s-2}v_{s-1}\}$ .  Suppose  f  is any edge in
G .  Then  f  is in one of the disjoint sets $E(B_{k-1})$,  X ,
or  Y .  If  $f \in E(B_{k-1})$ ,  there is a 1-factor  F'  of
$B_{k-1}$  with  $f \in F'$ .  But then  $F' \cup [Y]$  is a 1-factor of
G  containing  f .  If  $f \in Y$ ,  we observe that  $B_{k-1}$  has
some 1-factor  F'  since it is FI, and  $F' \cup [Y]$  yields the
desired 1-factor of  G .  If  $f \in X$ ,  Lemma 4.11 tells us
that  $G-\{v_0,v_s\}$  contains a 1-factor  F' ,  and then  $F' \cup [X]$
is the desired 1-factor of  G .  Since  f  was chosen
arbitrarily,  G  has support.  In any event,  G  satisfies
(a) and (b), and it follows that  A  is FI.

We now prove the converse with the aid of the preceding
theorems.  Suppose  A  is FI.  Observe that if  n = 1  or
n = 2 ,  G  is an edge graph or a 4-cycle, respectively.  The
theorem is clearly true for these cases so we may assume
$n \geq 3$ .  We will use induction on  N ,  the number of edges
in E(G).  By Theorem 2.5  $N \geq 6$ .  If  N = 6  then  A  is 3×3
and no element is removable.  Then  A  must be p-equivalent
to the 3×3 ND matrix shown in the proof of Lemma 4.6.  In
this case,  G  is a 6-cycle, and obviously satisfies the
theorem.

Suppose then that $N > 6$, the theorem holds for any $K$, when $K$ is a bigraph of an FI matrix with $|E(K)| = M$ and $M < N$. We distinguish three cases:

Case I: $G$ has a removable edge $e$.

$G-e$ satisfies the induction hypothesis and we write $G - e = P_0 \cup P_1 \cup \cdots \cup P_s$. But then $G = P_0 \cup \cdots \cup P_s \cup [e]$ satisfies the theorem.

Case II: $G$ has no removable edges, and $G$ has a divalent vertex $v$ which is adjacent to two nodes.

We can apply Theorem 4.9. We adopt the terminology and notation of that theorem and refer to Figure 4.2 in the following:

$G'$ satisfies the induction hypothesis, so we can write $G' = P_0 \cup \cdots \cup P_s$. By Lemmas 4.12 and 4.13, $v'$ is either an end point of $P_0$, or interior to some $P_i$ ($1 \leqq i \leqq s$). In the first instance, we can assume without loss of generality that $P_0$ is an edge of attachment $f_{i'}$ of $b$. Also, $v'$ is an end point of $P_1$. We denote the other end point of $P_1$ by $u$. If the edge of $P_1$, which is incident to $v'$, is an edge of attachment $e_{i'}$ of $a$, we let $P_1' = P_1 \cup [avb]$, and identify $v'$ and $a$ so that $P_1'$ is a simple path of odd length with end points $u$ and $b$.

Then $P_0 \cup P_1' \cup P_2 \cup \cdots \cup P_s$ is a good path decomposition

for $G$ . If the edge of $P_1$ , which is incident to $v'$ ,

is some edge of attachment $f_{i''}$ of $b$ other than $f_{i'} = P_0$ ,

we observe that the edges of attachment $e_i$ $(i = 1, \cdots, r)$

of $a$ correspond to the edges of distinct paths $P_{j_i}$

$(i = 1, \cdots, r)$ which have end point $v'$ . Let $P_{j_{i'}}$ be the

one having the smallest index $j_{i'}$ , with end points $v'$

and, say, $u$ . Let $P_{j_{i'}}'$ be the path $P_{j_{i'}} \cup [avb]$ , with

$b$ incident to $f_{i''}$ . Then, $P_{j_{i'}}'$ is a simple path of $G$

of odd length with end points $b$ and $u$ , and

$P_0 \cup \cdots \cup P_{j_{i'}-1} \cup P_{j_{i'}}' \cup P_{j_{i'}+1} \cup \cdots \cup P_s$ is a good path

decomposition for $G$ . The remaining possibility in Case II

is that $v'$ is an interior vertex to some $P_{i'}$ , where

$1 \leqq i' \leqq k$ . In this case, two edges of $P_{i'}$ are incident

to $v'$ . We must distinguish two cases:

(a) Both of the edges of $P_i$ incident to $v'$

occur among the edges of attachment either of $a$

or of $b$ . Without loss of generality, we can

assume that they are edges of attachment of $a$ .

In this case, all the edges of attachment of $b$

are end edges of distinct paths $P_{i_j}$ , where

$j = 1, \cdots, r$ . Let $P_{i_{j'}}$ be the path with the

smallest index, and suppose its end point, other

than $v'$ , to be $u$ . We form the simple path

of odd length $P'_{i_{j'}} = P_{i_{j'}} \cup [avb]$ with end points $u$ and $a$, by making $b$ incident to the edge of $P_{i_{j'}}$ which was incident to $v'$ in $G'$. Then

$P_0 \cup \cdots \cup P_{i_{j'}-1} \cup P'_{i_{j'}} \cup P_{i_{j'}+1} \cdots \cup P_s$ is a good path decomposition for $G$.

(b) One of the edges of $P_i$ incident to $v'$ is an edge of attachment $e$ of $a$ and the other is an edge of attachment $f$ of $b$. In this case, we merely set $P'_i = P_i \cup [avb]$, where $a$ is incident to $e$ and $b$ is incident to $f$. Then $P_0 \cup \cdots \cup P_{i-1} \cup P'_i \cup P_{i+1} \cup \cdots \cup P_s$ is a good path decomposition for $G$.

This completes the proof of Case II. There is one remaining possibility.

Case III: $G$ has no removable edges, and every divalent vertex $v$ has an adjacent divalent vertex $w$.

In this case, we can apply Theorem 4.10. We adopt the terminology of this theorem, and refer to Figure 4.3.

If $G'$ is ND, then $G'$ satisfies the induction hypothesis and we have a good path decomposition $G' = P_0 \cup \cdots \cup P_s$. Now the edge $v'b$ occurs in one of the

$P_i$ , say, in $P_{i_1}$ . In this case, we simple insert the
path avw into $P_{i_1}$ to form the path $Q_{i_1}$ . Then
$P_0 \cup \cdots \cup P_{i_1-1} \cup Q_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_s$ is a good path
decomposition for G .

If G' is not ND, G" is, and we can write
G" = $P_0 \cup \cdots \cup P_s$ ; then G = $P_0 \cup \cdots \cup P_s \cup$ [avwb] is
a good path decomposition for G . This completes the proof
of the theorem.

The following result is implied immediately by the
preceding theorem.

Theorem 4.15: If G is the bigraph of an n×n (0,1) ND
matrix A , then in the decomposition G = $P_0 \cup P_1 \cup \cdots \cup P_k$
of Theorem 4.14, $P_i$ has length $\geqq$ 3 for i = 1,$\cdots$,k .

Proof: Say $P_{i'}$ has length 1 , where 1 < i' $\leqq$ k .
Then G' = $P_0 \cup \cdots \cup P_{i'-1} \cup P_{i'+1} \cup \cdots \cup P_k$ corresponds
to an FI matrix by the proof of the first part of Theorem 4.14.
But then the edge $E(P_{i'})$ is removable, contradicting the fact
that G was ND.

We show below that this last result is equivalent to
Hartfiel's Theorem (Theorem I.2.3), and thus, can be con-
sidered the graph theoretical analogue of that result. In

the next chapter, we will use Theorem 4.15 extensively in deriving some properties of FI matrices, and in particular, of ND matrices.

Theorem 4.16: An $n \times n$ (0,1) matrix A satisfies the conclusion of Theorem I.2.3 if and only if its bigraph G satisfies the conclusion of Theorem 4.15.

Proof: We will adopt the notation of the theorems cited.

Suppose that A satisfies the conclusion of Theorem I.2.3. The cases where $n = 1$ or $n = 2$ are trivial, so we may assume that $n \geqq 3$. We will proceed by induction on n. If $n = 3$, either $A_1 = A_2 = A_3 =$ the $1 \times 1$ matrix 1, or $A_1 = 1$ and $A_2$ is the $2 \times 2$ ND matrix. In either case G clearly satisfies the conclusion of Theorem 4.15. Assume that $n > 3$, and that the result holds for $s \times s$ (0,1) matrices, when $3 \leqq s < n$. Observe that the vertices of G corresponding to the rows of A numbered $1, \cdots, s-1$, and the columns of A numbered $1', \cdots, (s-1)'$, are the interior vertices of a simple path $P = v_a v_1, v_1, \cdots v_{(s-1)}, v_{(s-1)} v_b$, which avoids the bigraph of $A_s$, except for its end vertices $v_a$ and $v_b$ which correspond, respectively, to some row and column of $A_s$. By the conclusion of Theorem I.2.3, $A_s$ is ND. It follows that

$A_s$ satisfies the hypothesis, and therefore the conclusion of Theorem I.2.3. By our induction hypothesis, the bigraph $G_s$ of $A_s$ has a good path decomposition $G_s = P_0 \cup P_1 \cup \cdots \cup P_k$ satisfying the conclusion of Theorem 4.15. But then $P_0 \cup P_1 \cup \cdots \cup P_k \cup P$ is just such a decomposition for $G$.

Conversely, suppose $G$ satisfies the conclusion of Theorem 4.15. Again, we proceed by induction on $n$, and quickly settle the cases $n = 1, 2,$ and $3$. Now $G = P_0 \cup P_1 \cup \cdots \cup P_k$. Since the length (odd) of $P_k$ is at least $3$, $P_k$ has an even number (at least 2) of interior vertices $v_1, v_{1'}, \cdots, v_{(s-1)}, v_{(s-1)'}$. Now simply write a representative matrix $A$ by letting the $ith$ row and $jth$ column of $A$ correspond to $v_i$ and $v_j$, respectively, for $1 \leq 1, j \leq s - 1$. Let $A_s$ correspond to the bigraph of $B_{k-1} = P_0 \cup \cdots \cup P_{k-1}$, where $A_s$ occupies the n-s+1 remaining columns and the n-s+1 remaining rows of $A$, ordered in some arbitrary fashion. Then $A_s$ is at least $1 \times 1$ (since $P_k$ has odd length), and is FI by construction. But $B_{k-1}$ has no removable edges, for otherwise $G$ would. It follows that $A_s$ is ND. This completes the proof of the theorem.

## III.  APPLICATIONS OF THE MAIN RESULTS

In this chapter the results of Chapter II are applied
to the derivation of some basic properties of FI and ND
matrices.  In addition, some further results are developed
concerning these matrices.

### 1.  Examples of FI Matrices and Their Bigraphs

Some specific examples familiarize the reader with the
techniques of associating particular matrices with particular
bigraphs and illustrate the usefulness of bigraphs in identi-
fying FI matrices.  We also answer a few questions which arose
during the course of our research.

In [19], the author depends heavily on a result (stated
herein as Theorem II.3.1 of Chapter II) to determine whether
or not certain nonnegative matrices are irreducible.  In
general, for a given nonnegative matrix  A ,  it is probably
not as formidable a task to decide by inspection whether or
not  A  is FI as it is to decide whether or not  A  is irre-
ducible.  The former is not always an easy task, however, even
for matrices of fairly small order.  On the other hand, it is
usually quite easy to determine whether or not the associated
bigraph  G  has a good path decomposition.  The following
matrix and its bigraph illustrate this point.  It is difficult

to ascertain by inspection that the following matrix *does not* have an s×t zero submatrix with  s + t = n :

$$
\begin{array}{c c c c c c c c c}
 & 1' & 2' & 3' & 4' & 5' & 6' & 7' & 8' \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
4 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
7 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
$$

On the other hand, one may readily see that the corresponding bigraph has several good path decompositions:
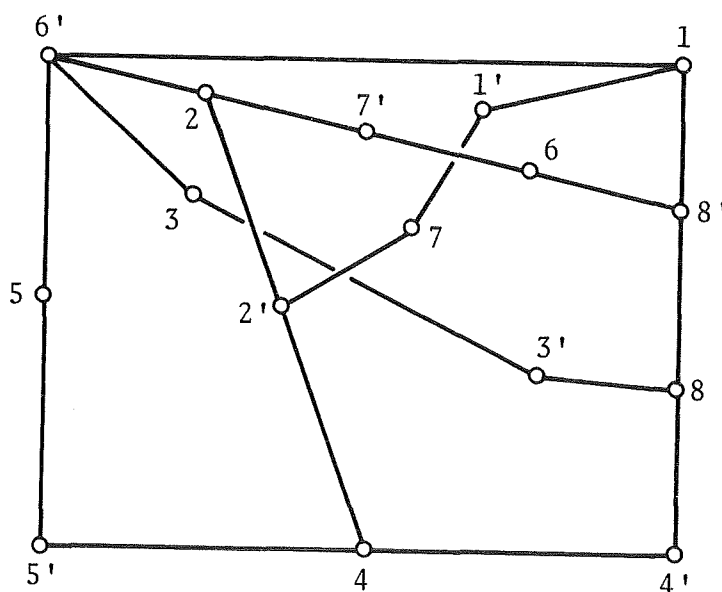
Two possible good path decompositions are:

| | | | | | |
|---|---|---|---|---|---|
| $P_0$ | = | 22' | $P_0$ | = | 18' |
| $P_1$ | = | 26'55'42' | $P_1$ | = | 16'27'68' |
| $P_2$ | = | 44'88'16' | $P_2$ | = | 22'44'88' |
| $P_3$ | = | 27'68' | $P_3$ | = | 11'72' |
| $P_4$ | = | 83'36' | $P_4$ | = | 83'36' |
| $P_5$ | = | 11'72' | $P_5$ | = | 45'56' |

Notice that both decompositions have the same number of component paths. We will show in the next section that for the bigraph of an FI matrix, this number is the same for any good path decomposition.

Notice also that for a given path decomposition, $P_0 \cup P_1$ is always a cycle and, in the example shown here, two different cycles were chosen for $P_0 \cup P_1$. We challenge the reader to discover in this example a cycle which *cannot* be taken for $P_0 \cup P_1$ for *some* good path decomposition. In the course of this research, the question arose as to whether *any* cycle in G could be taken for $P_0 \cup P_1$ in some good path decomposition. The following example shows that this is not the case, even when G corresponds to a matrix which is ND.

G: 
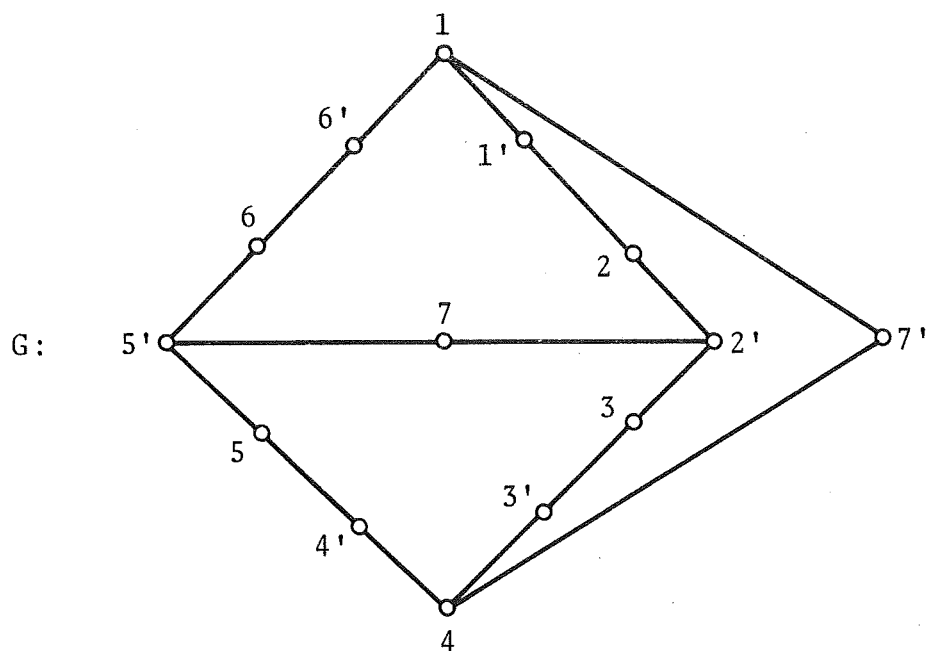
G is FI because it has the path decomposition

$$P_0 = 11'$$
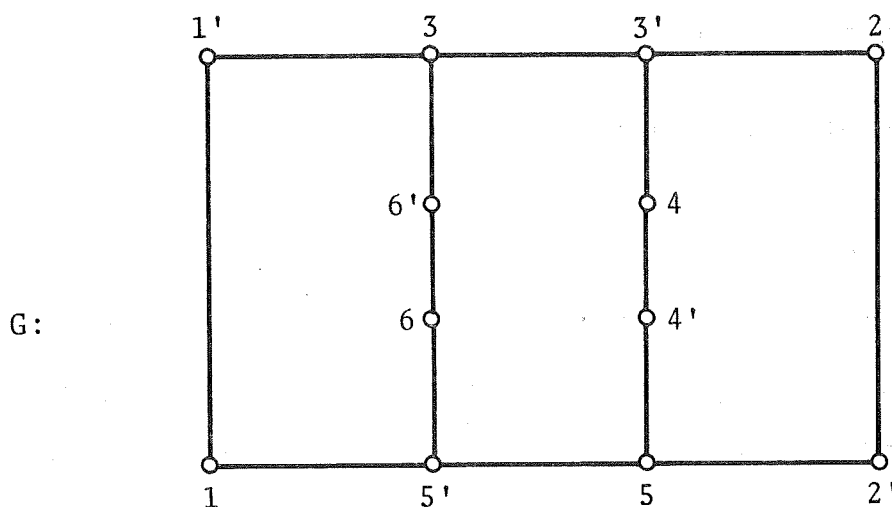
$$P_1 = 17'43'32'21'$$

$$P_2 = 16'65'72'$$

$$P_3 = 44'55'$$

However, the cycle 11'22'33'44'55'66'1 cannot correspond to $P_0 \cup P_1$ for *any* good path decomposition. Observe that the matrix corresponding to G is ND since each edge is incident to at least one divalent vertex and, hence, if e is any edge of G , any matrix representative of G-e would not satisfy Theorem II.2.5.

Theorem II.4.15 asserts that if $G = P_0 \cup \cdots \cup P_k$ is any good path decomposition for a bigraph $G$ of an ND matrix, no $P_i$ has length $1$ for $i = 1,2,\cdots,k$. In view of this result, it is natural to ask if there exists a bigraph $G$ corresponding to an ND matrix $A$ such that both end points of some edge of $G$ are nodes. The following example illustrates that this can happen.

G:

(figure: a ladder-like bigraph. Top row vertices left to right: 1', 3, 3', 2. Middle interior vertices: 6' and 4 (upper), 6 and 4' (lower). Bottom row vertices left to right: 1, 5', 5, 2'.)

It is obvious that $G$ is FI. To see that $G$ is ND, we observe that $33'$ and $55'$ are the only edges which could possibly be removable. If we remove $33'$, then $U = \{1,3,6\}$ and $V = \{2',3',4'\}$ are an independent set of vertices with $|U| + |V| = n$. Alternatively, observe that the removal of any edge of $G$ leaves a bigraph which is not 2-connected.

## 2. Basic Properties of ND Matrices

In this section we use the results of Chapter II to develop some basic properties of ND matrices. We first establish the useful concept of the "degree" of an FI matrix.

Theorem 2.1: Let $G$ be the bigraph of an $n \times n$ FI $(0,1)$ matrix $A$. Let $N = |E(G)|$. (Equivalently, $N$ = the number of positive entries in $A$.) Suppose $G = P_0 \cup P_1 \cup \cdots \cup P_k$ is any good path decomposition for $G$. Then $k = N - 2n + 1$.

Proof: We use induction on $n$. If $n = 1$, then $G = P_0$, $k = 0$, so $N - 2n + 1 = 1 - 2 + 1 = 0$, and the theorem holds. Let $n > 1$, and suppose the theorem holds for any $m \times m$ FI $(0,1)$ matrix $A'$ such that $m < n$. Suppose the length of $P_k$ is $L$, so that $P_k$ introduces $L$ new edges and $L-1$ new vertices to $G$, half green and half blue. Then $B_{k-1} = P_0 \cup P_1 \cup \cdots \cup P_{k-1}$ corresponds to an $\left(n - \frac{L-1}{2} \times n - \frac{L-1}{2}\right)$ matrix, which satisfies the induction hypothesis. Therefore, $(N - L) - 2\left(n - \frac{L-1}{2}\right) + 1 = k - 1$. It follows that $N - 2n + 1 = k$.

The above theorem shows that any good path decomposition of $G$ has the same length $k$, which depends only on the size $n$ and the number of positive entries $N$ of the FI matrix $A$. We call $k$ the *degree* of the FI matrix $A$ and also refer to $k$ as the *degree* of $G$.

The next theorem gives an upper bound estimate for the number of positive entries in an ND matrix.

Theorem 2.2: Let $G$ be the bigraph of an $n \times n$ ND $(0,1)$ matrix $A$, where $n \geq 3$. Let $N = |E(G)|$. Then $N \leq 3n - 3$.

Proof: We use induction on the degree $k$ of $G$. If $k = 1$, then $G$ is an $r$-cycle where $r$ is an even number $\geq 6$. In this case $N = 2n$ and $3 \leq n$. Hence $0 \leq n - 3$, so $N \leq 3n - 3$. Let $k > 1$, and suppose the theorem holds for ND matrices of degree less than $k$. Let $G = P_0 \cup P_1 \cup \cdots \cup P_k$ be a good path decomposition for $G$. Let $L$ be the length of $P_k$. Then $B_{k-1}$ satisfies the induction hypothesis, so $N - L \leq 3\left(n - \frac{L-1}{2}\right) - 3$. Therefore, $N \leq 3n - \frac{3}{2}(L-1) - 3 + L$. But by Theorem II.4.15 $3 \leq L$, so $-\frac{1}{2}L \leq -\frac{3}{2}$. It follows that $N \leq 3n - 3$.

We end this section by showing that for each $n$, there is only one $n \times n$ ND $(0,1)$ matrix $A$, up to p-equivalence, for which $N = 3n - 3$. This theorem illustrates how the bigraph concept can be used to illuminate what might otherwise be an exceedingly unattractive combinatorial argument.

Theorem 2.3: Let $G$ be the bigraph of an $n \times n$ ND $(0,1)$ matrix $A$, with $n \geq 3$. Let $N = |E(G)|$. Then $N = 3n - 3$ if and only if $A$ is p-equivalent to the matrix:

$$A' = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Proof: Suppose $A$ is p-equivalent to $A'$. It is then clear that $N = 3n - 3$ for the ND matrix $A'$, and therefore for $A$. Conversely, suppose for a given ND matrix $A$, $N = 3n - 3$. Observe that $A$ is p-equivalent to $A'$ if and only if $G$ has the form
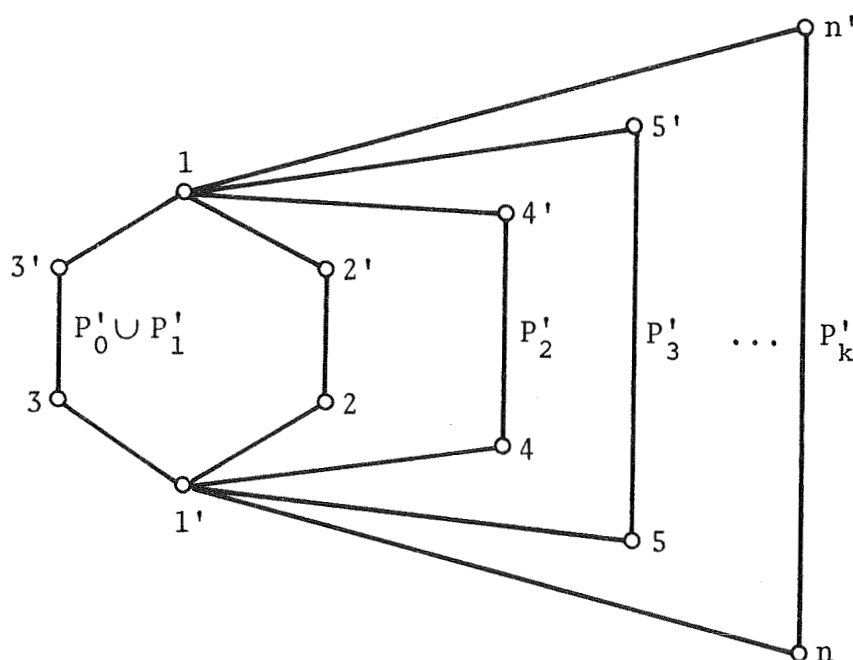
where $G = P_0' \cup P_1' \cup \cdots \cup P_k'$ . We will prove the theorem by showing that $G$ has this form. We proceed by induction on $n$ . If $n = 3$ , $G$ is a cycle of length $6$ , and the theorem is satisfied. We now suppose $n > 3$ , and that the theorem holds for $(n-1) \times (n-1)$ ND matrices with $N = 3n - 3$ . Since $n > 3$ , $k \geqq 1$ . Now suppose $P_k$ has length $L$ , where $L \geqq 3$ by Theorem II.4.15. Then $B_{k-1}$ is an $(m,m)$ bigraph with $|E(B_{k-1})| = M$ , which corresponds to an $m \times m$ ND $(0,1)$ matrix $A''$ . Observe that $M = N - L$ , and $m = n - \dfrac{L - 1}{2}$ . We now show that $L = 3$ . If $3 < L$ , we have

$$(1) \qquad\qquad -\frac{1}{2} L \;<\; -\frac{3}{2}$$

Now, by Theorem 2.2, $M \leqq 3m - 3$ , so we can write

$$N - L \;\leqq\; 3\left(n - \frac{L - 1}{2}\right) - 3 \;=\; 3n - \frac{3}{2} L + \frac{3}{2} - 3$$
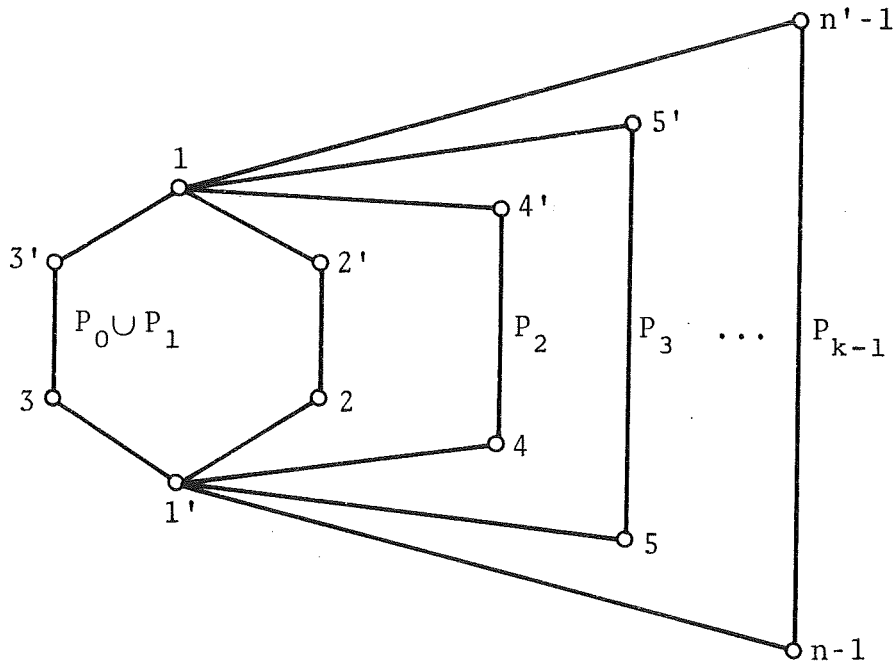
or by (1) above

$$N \;\leqq\; 3n - \frac{1}{2} L + \frac{3}{2} - 3 \;<\; 3n - \frac{3}{2} + \frac{3}{2} - 3 \;=\; 3n - 3$$

But this contradicts the assumption that $N = 3n - 3$ . It follows that $L = 3$ . This implies that $M = N - 3$ and $m = n - 1$ .

Now observe that if $m = 2$ , $B_{k-1}$ would be a cycle of length $4$ , and $G$ would not satisfy Lemma II.4.6. If $m = 1$ , then $n = 2$ , which contradicts our assumptions.

Therefore, $m \geqq 3$ , and $B_{k-1}$ satisfies the induction hypothesis and has a representation



with $B_{k-1} = P_0 \cup P_1 \cup \cdots \cup P_{k-1}$ . The vertices labeled with integers, we call green; those labeled with primed integers, we call blue. Since the length of $P_k$ is 3 , $P_k$ has interior vertices n and n' which are adjacent, respectively, to a blue vertex a' and a green vertex a . By definition a' and a coincide with vertices of $B_{k-1}$ . To complete the proof, it is sufficient to show that a = 1 and a' = 1' .

We accomplish this by showing that all other possible assignments for a and a' yield a graph with a removable edge. Without loss of generality we consider just six cases:

Case I: $n = 4$ . In this case, $B_{k-1}$ is a cycle of length 6 , and unless $d_{B_{k-1}}(a,a') = 3$ , we have $d_{B_{k-1}}(a,a') = 1$ . The theorem is satisfied in the former instance but in the latter, the edge aa' is removable.

We henceforth assume that $n \geqq 5$ .

Case II: $a = 1$ , $a' = 3'$ . In this case, if $G' = G - 13'$ , the cycle 1n'n3'31'22'1 can be taken as $P_0 \cup P_1$ for a good path decomposition, and therefore $G'$ is FI.

Case III: $a = 2$ , $a' = 3'$ . Let $G' = G - \{13',21'\}$ . Then the cycle 3'nn'22'14'41'33' can clearly be taken as $P_0 \cup P_1$ for some good path decomposition.

Case IV: $a = 1$ , $a' = 4'$ . Let $G' = G - 14'$ . Then let $P_2' = 1n'n4'41'$ replace $P_2$ in the decomposition for $B_{k-1}$ ; then $G' = P_0 \cup P_1 \cup P_2' \cup \cdots \cup P_{k-1}$ is FI.

Case V: $a = 3$ , $a' = 4'$ . Let $G' = G - \{14',31'\}$ . Then the cycle 3n'n4'41'22'13'3 can clearly be taken as $P_0 \cup P_1$ for a good path decomposition.

When  n > 5 ,  we must consider:

Case VI:  a = 4 ,  a' = 5' .  Let  G' = G - {15',41'} .
Then let  $P_3'$ = 14'4n'n5'51' ,  and we can write
G' = $P_0 \cup P_1 \cup P_3' \cup P_4 \cup \cdots \cup P_{k-1}$ .  This completes the
proof of the theorem.

3.   On Minc's Lower-Bound Estimate for the Permanent of
     an FI (0,1) Matrix

In 1969, Minc [11] announced that for all n×n FI (0,1)
matrices:

$$\text{per } A \geq N - 2n + 2$$

where  N  equals the number of positive entries in  A .
(Equivalently,  $N = \sum_{i,j} a_{ij}$ .)

In the following, we present a new, short proof of Minc's
inequality based upon preceding material in this paper.  Our
main result in this section is Theorem 3.2, which shows that,
in a sense, Minc's result is the best possible estimate in
terms of  n  and  N .

Observe that if  A  is an n×n (0,1) matrix, and if  B
is p-equivalent to  A ,  then  per B = per A = K .  It is
then clear that the number  K  is an invariant of the
p-equivalence class of  A ;  in fact, it corresponds to the

number of distinct 1-factors of the bigraph $G$ of $A$ (see Corollary II.3.4). Without danger of confusion, we will denote this unique number corresponding to an $(n,n)$ bigraph $G$ by "per $G$". Then, by definition, if $G$ is the bigraph of an $n \times n$ $(0,1)$ matrix $A$, per $G$ = per $A$.

Theorem 3.1: If $G$ is the bigraph of an $n \times n$ ND $(0,1)$ matrix $A$, and if $N = |E(G)|$, then per $G \geq N - 2n + 2$.

Proof: According to Theorem II.4.15, $G = P_0 \cup P_1 \cup \cdots \cup P_k$, where the length of $P_i$ is at least 3 $(i = 1, \cdots, k)$. We use induction on the degree $k$ of $G$ to show that per $G \geq k + 1$. For $k = 0$, per $G = 1$, and the theorem holds. Suppose $1 \leq r \leq k$, and per $B_{r-1} \geq (r - 1) + 1$. Now $P_r$ is a path of odd length $L \geq 3$, say, $P_r = v_0 v_1 \cdots v_L$. Let $X = \{v_0 v_1, v_2 v_3, \cdots, v_{L-1} v_L\}$ and $Y = \{v_1 v_2, v_3 v_4, \cdots, v_{L-2} v_{L-1}\}$. Then the 1-factors of $B_r$ will be the subgraphs $F \cup [Y]$ and $F' \cup [X]$, where $F$ is any 1-factor of $B_{r-1}$, and $F'$ is any 1-factor of $B_{r-1} - \{v_0, v_L\}$. Thus, $Y$ is contained in the edge set of exactly per $B_{r-1}$ 1-factors of $B_r$, and, by Lemma II.4.11, $X$ is contained in at least one other 1-factor of $B_r$. It follows that per $B_r \geq$ per $B_{r-1} + 1 \geq (r - 1) + 1 + 1 = r + 1$. This completes the induction argument. It follows that per $G \geq k + 1$. By Theorem 2.1, $k = N - 2n + 1$, and we have per $G \geq N - 2n + 2$.

Since per G = per A , this result yields Minc's estimate.

We now need the following graph-theoretical lemma:

Lemma 3.3: If the graph G is a forest, then the number of distinct 1-factors in G is at most one.

Proof: We define a *pendant edge* of G to be an edge with precisely one monovalent vertex. Now, if the components of G consist only of edge graphs and vertex graphs, the theorem is evidently true. Suppose there is a component T of G which is not an edge graph or vertex graph. Now T is a tree, so by Theorem II.1.4, T has at least one monovalent vertex v . Since T is connected and not an edge graph, v must be adjacent to a vertex w with val w $\geq$ 2 . Hence, T has a pendant edge vw. Now suppose F is any 1-factor of G . Then E(F) covers both v and w , but this can only happen if vw $\epsilon$ E(F) . Therefore, the edges other than vw which are incident to w are not in E(F) and therefore not edges of any 1-factor of G , since F was chosen arbitrarily. It follows that the graph T-{v,w} has precisely as many distinct 1-factors as T . If T-{v,w} has a pendant edge, we repeat the process. Since E(G) is finite, we eventually arrive at a graph T' having no pendant edges, where T' has precisely as many 1-factors as T . But T'

must then be a graph whose components are edge graphs or vertex graphs. (Note that we cannot have both $E(T') = \emptyset$ and $V(T') = \emptyset$ because if $v$ is a vertex of a graph $G$, $G - v = \emptyset$ if and only if $G = \{v\}$.) Then if any of the components of $T'$ is a vertex graph, $T'$ has no 1-factors. If all the components of $T'$ are edge graphs, $T'$ has a single 1-factor. All the components of $G$ can be reduced in this manner, and since the number of distinct 1-factors of $G$ equals the product of the numbers of distinct 1-factors in each component, $G$ has at most one 1-factor.

The following theorem shows that Minc's lower bound can be achieved for every possible value of $n$ and $N$ for which there is an n×n ND (0,1) matrix with $N$ positive entries.

Theorem 3.2: Let $n$ be an integer, with $n \geq 3$. Suppose $2n \leq N \leq 3n - 3$. Then there is a bigraph $G$ corresponding to an n×n ND (0,1) matrix $A$ with $|E(G)| = N$, and $\text{per } G = \text{per } A = N - 2n + 2$.

Proof: Let $k = N - 2n + 1$. Then $k \geq 1$, by the inequality stated in the hypothesis. We construct the graph $G$ with the path decomposition $G = P_0 \cup P_1 \cup \cdots \cup P_k$ as follows: Let $L_i$ denote the length of the path $P_i$ for $i = 0, \cdots, k$. We assign the following lengths: $L_0 = 1$, $L_1 = N - 3k + 2$, and $L_2 = L_3 = \cdots = L_k = 3$. Then

$L_1 = N - 3(N - 2n + 1) + 2 = -2N + 6n - 1$

$\geq (-6n + 6) + 6n - 1 = 5$ . We construct $G$ according to its path decomposition as usual, with the end points of $P_0$ coincident with those of $P_1$ . We have that the cycle $B_1 = P_0 \cup P_1$ has length $\geq 6$ , so it is possible to pick vertices $u$ and $v$ of different color in $E(B_1)$ with $d_{B_1}(u,v) \geq 3$ . We now take $u$ and $v$ to be the end points of $P_i$ for $i = 2, \cdots, k$ (see fig. 3.1).
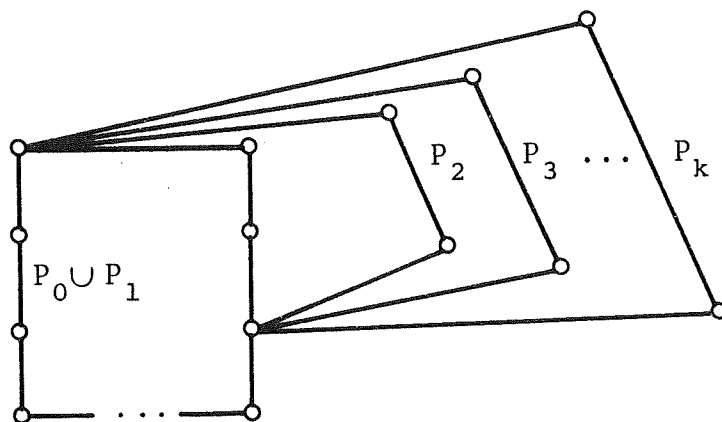


Figure 3.1

Clearly $G$ corresponds to an FI matrix $A$ by Theorem II.4.14; and, since every edge of $G$ is incident to at least one divalent vertex, no edge of $G$ is removable. It follows that $A$ is ND. It remains to show that per $A$ = per $G = k + 1 = N - 2n + 2$ .

We use induction on $k$ . The theorem is true for $k = 1$ . Let $2 \leq r \leq k$ and suppose per $B_{r-1} = (r - 1) + 1$ . As in the proof of Theorem 3.1, if $P_r = v_0 v_1 \cdots v_{L_r}$ , the the 1-factors of $B_r$ are partitioned into two classes, those containing the edges in $X$ and those containing the edges in $Y$ . The number in the former class equals per $(B_{r-1} - \{u,v\})$ . The number in the latter class equals per $B_{r-1}$ . Hence, per $B_r$ = per $B_{r-1}$ + per $(B_{r-1} - \{u,v\})$ . Again, per $(B_{r-1} - \{u,v\}) \geq 1$ by Lemma II.4.11. However, it is evident that $B_{r-1} - \{u,v\}$ is a forest, and by the Lemma 3.3 per $(B_{r-1} - \{u,v\}) \leq 1$ . Therefore, per $B_r$ = per $B_{r-1}$ + 1 = $r + 1$ . This completes the induction. Hence, per $G = k + 1 = N - 2n + 2$ .

### 4. Partial Results on Upper Bounds for the Permanent of an ND (0,1) Matrix

This author has conjectured that (following our usual notation) if $G$ is the bigraph of an $n \times n$ ND (0,1) matrix, with path decomposition $G = P_0 \cup P_1 \cup \cdots \cup P_k$ , and $|E(G)| = N$ , and if $u_j$ and $v_j$ are the end points of $P_j$ ($j = 2, \cdots, k$), then per $B_{j-1} >$ per $(B_{j-1} - \{u_j, v_j\})$ . It follows that, if this inequality is true, then per $B_j \leq 2$ per $B_{j-1} - 1$ . Since per $B_1$ always equals 2

for G corresponding to ND (0,1) matrices, our conjecture implies that

(1)        per G $\leq$ $2^{(k-1)} + 1$ = $2^{(N-2n)} + 1$

We can easily verify (1) in the case where n = 1,2,3 , or for k = 1,2,3 , or whenever N = 2n . We now define a special subclass of the bigraphs corresponding to ND (0,1) matrices:

Definition 4.1: Let G be the bigraph of an n×n ND (0,1) matrix A . G is said to be of class Z if for *every* divalent vertex v , the contraction G' of G by v does not correspond to an ND matrix A' .

This definition, together with Theorems II.4.9 and II.4.10, implies that a bigraph G of class Z has the property that every divalent vertex v is adjacent to a divalent vertex w and that G-{v,w} corresponds to an ND matrix. Bigraphs of class Z appear to be rather difficult to construct, and we feel these graphs have many special properties. It is hoped that an intensive study of class Z will settle our conjecture, either by providing a counter example, or by succeeding in proving the conjecture true for bigraphs of class Z . Theorem 4.1 tells us that if (1) is true for bigraphs of class Z , then it is true for any bigraph G corresponding to an n×n ND (0,1) matrix.

The following example shows that class Z is not empty:
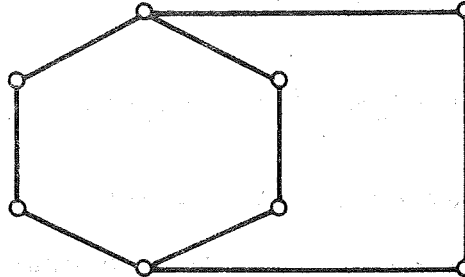


Theorem 4.1: Let G be the bigraph of an n×n ND (0,1) matrix A . Suppose that any bigraph of class Z satisfies inequality (1), above. Then G satisfies (1).

Proof: If n = 1 or 2 , then (1) is trivially satisfied. We may suppose, therefore, that n ≧ 3 . We use induction on n . If n = 3 , the theorem is clearly true. Suppose the theorem to be true for any bigraph corresponding to an (n-1)×(n-1) ND (0,1) matrix. If G is of class Z , our proof is complete. If G is not of class Z , then by Theorems II.4.9 and II.4.10, there is a divalent vertex v such that the contraction G' of G with respect to v corresponds to an (n-1)×(n-1) ND (0,1) matrix A' , with

per A = per A' . Notice that G' has two less edges than G . Now G' satisfies the induction hypothesis, so:

$$\text{per } A = \text{per } A' \leq 2^{((N-2)-2(n-1))} + 1 = 2^{(N-2n)} + 1 .$$

## 5. The Problem of Characterizing ND Matrices

The problem of finding a useful characterization of ND matrices (equivalently, of their bigraphs) in terms of some easily observed or easily calculated properties, appears to be very difficult. As far as this author is aware, no such result exists at present. The following example shows a bigraph G which satisfies Theorem II.4.15 but has a removable edge. This demonstrates that Theorem II.4.15 fails to characterize the bigraphs of ND matrices.

G:
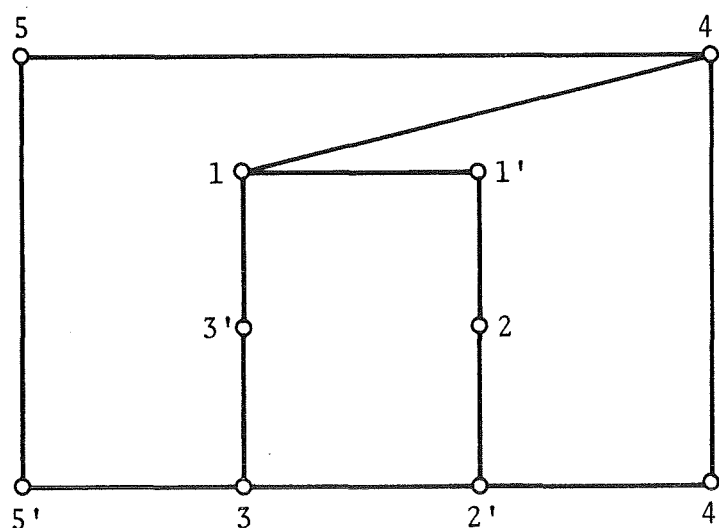
If we let

$$P_0 \cup P_1 \;=\; 11'22'33'1$$

$$P_2 \;=\; 14'42'$$

$$P_3 \;=\; 35'54'$$

then we can write $G = P_0 \cup P_1 \cup P_2 \cup P_3$ , which satisfies the necessary conditions stated in Theorem II.4.15; but the edges 14' and 32' are clearly removable.

During the study of this problem, the following question arose. Although 2-connectivity does not characterize the bigraphs of FI matrices, are the bigraphs of ND matrices characterized by being minimally 2-connected? By "minimally 2-connected", of course, we mean that the removal of any edge would destroy the property of being 2-connected. The example of $G$ on page 82 provides a negative answer to the question, and in this author's opinion, reflects the difficulty of the problem of characterizing bigraphs of ND matrices. $G$ corresponds to an FI matrix because we can write $G = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4$ if we define

$$P_0 \cup P_1 \;=\; 11'22'33'44'55'66'77'1$$

$$P_2 \;=\; 38'87'$$

$$P_3 \;=\; 79'94'$$

$$P_4 \;=\; 2(10)'(10)5'$$

G:



To see that G corresponds to an ND matrix, first observe that every edge except 77' is incident to a divalent vertex. Next we show that G-77' does not correspond to an FI matrix. If U = {5,6,7,9,10} and V = {1',2',3',7',8'} , then U ∪ V is an independent set of vertices of the (10,10) bigraph G-77', with |U| + |V| = 10 . We conclude that G is ND, but observe that G-77' is 2-connected.

Notice that if G is the bigraph of an n×n FI (0,1) matrix A , and if G has a removable edge, then there exists a good path decomposition $G = P_0 \cup P_1 \cup \cdots \cup P_k$ ,

in which some $P_i$ $(i = 2, \cdots, k)$ has length one.  In view of this observation, perhaps the investigation of the various good path decompositions pertaining to a particular  G corresponding to an FI matrix would yield knowledge leading to a characterization of the bigraphs of ND matrices.

# IV. SUMMARY

In this chapter we briefly discuss those problems which have been settled only partially in the preceding pages, and we consider some future research projects which have arisen out of this dissertation.

## 1. Some Unsettled Issues

As far as we are aware no good upper-bound estimate, in terms of $n$ and $N$ alone, is known for the permanent of an $n \times n$ ND $(0,1)$ matrix having $N$ positive entries. The conjecture of Chapter III, Section 4, would provide such an estimate if it were proven true for ND matrices having bigraphs of class $Z$. On the basis of some recent preliminary investigations, it appears that problems involving permanent estimates for bigraphs of class $Z$ do not adapt very well to induction-type arguments. On the other hand, these studies indicate that there is hope that bigraphs of class $Z$ have a particularly simple structure. In this regard, we feel that the most promising approach to the problem might be a uniqueness type of argument, i.e., an attempt to construct a small collection of bigraphs of class $Z$ which, we would then hope to show, are the *only* members of class $Z$. In any event, this author intends to subject the bigraphs of class $Z$ to an intensive study.

Another unsettled problem, which appears to be a very difficult one, is the matter of finding a simple characterization for ND matrices and their bigraphs. Paralleling a remark in Chapter III, Section 5, one may observe that if G is the bigraph of an n×n (0,1) matrix A , then G has a removable edge if and only if there is some good path decomposition $G = P_0 \cup P_1 \cup \cdots \cup P_k$ , with some $P_i$ having length 1 , where $2 \leq i \leq k$ . This *is*, in fact, a characterization of the bigraph of an ND matrix, but not a very good one because little is known about the class D of good path decompositions for G . Perhaps an investigation of class D would be a fruitful new method for attacking the characterization problem. All that seems to be known at present about this class is the result of Theorem III.2.1, which concerns the degree of G . However, some interesting questions arise immediately: Do there exist any natural relations on the class D ? Can we find some sort of transformation on E(G) or V(G) which would map one good path decomposition into another? We hope to find some gratifying answers to these and other questions in the future.

2.  A Connection Between the ND Concept and a Problem
    of Erdös, Hajnal, and Moon

The notation of a bigraph corresponding to an n×n ND (0,1) matrix is closely related to a graph-theoretical problem of Erdös, Hajnal, and Moon [5]. We state here the weaker form

of their theory. The authors consider a graph  G  having  n
vertices.  G  is said to have property (n,p), where
$2 \leq p \leq n$ ,  if  G  does not contain the complete graph $K_p$ ,
but the graph  G'  formed by adding any new edge to  G ,
does.  One of the consequences of their results is that a
graph having property (n,p) must have at least $n(p-2)-\binom{p-1}{2}$
edges.  They offer an extension of their theory to bigraphs
as follows:

If  G  is an (n,m) bigraph, and if  $1 \leq k \leq n$  and
$1 \leq h \leq m$ ,  then  G  is said to have property (n,m,k,h) if
G  does not contain the complete bigraph $K_{k,h}$ ,  but the
bigraph  G'  formed by adding any new edge (with different
colored end points) to  G ,  does.  Erdös, et al, conjectured
that a bigraph having property (n,m,k,h) must contain at
least (k-1)m+(h-1)n+(k-1)(h-1) edges.  Bollobás [3] verified
that their conjecture was correct.

We now extend the original notion of Erdös, et al, to
bigraphs in a slightly different manner.

Definition 2.1:  Let  G  be an (n,m) bigraph, and
suppose  p  is an integer such that  $2 \leq p \leq m + n$ .  Then
G  is said to be of type (n,m,p) if for any integers  k  and
h ,  with  $1 \leq k \leq n$ ,  $1 \leq h \leq m$ ,  and  k + h = p ,

G does not contain the complete subgraph $K_{k,h}$ ; but the addition of any new edge (with different colored end points) to G destroys this property.

We now make the definitions and observations necessary to link the above notions with the ND concept.

Definition 2.2: Let G be an (n,m) bigraph. Then $\bar{G}$ , *the complement of* G is the bigraph defined as follows:

(1) $V(\bar{G}) = V(G) = S \cup T$ .

(2) If $v \in S$ and $w \in T$ , then $vw \in E(\bar{G})$ if and only if $vw \notin E(G)$ .

We observe that, if $U \subset S$ and $V \subset T$ , with $|U| = k$ and $|V| = h$ , then $[U \cup V] \subset \bar{G}$ is the complete bigraph $K_{k,h}$ if and only if $U \cup V$ is an independent set of $V(G)$ .

It is now clear that if G is the bigraph of an n×n ND (0,1) matrix A , then $\bar{G}$ has property (n,n,n). (The converse is also true, with a suitable adjustment of the terminology.) For this special case, Theorem III.2.2 tells us that G must have at least $n^2 - (3n-3)$ edges.

The determination of the minimal number of edges in a bigraph of type (n,m,p) might be an interesting future research problem. It also appears that the above observations indicate an interesting generalization of the FI concept.

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