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THE SOLID OF MODEL UNIVERSES

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16. Abstract

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SUMMARY

A modified form of the red-shift—magnitude relation is used to develop a solid called the solid of model universes. The solid of model universes relates the nonlinear part of the apparent magnitude to the density parameter, acceleration parameter, the red shift, and the time since the beginning of expansion. The solid of model universes offers an improvement over the tabular method of presenting Robertson's models of the universe and permits the rapid comparison of simplified models with the exact model and the comparison of models determined from observation by various methods. The determination of the nonlinear part of the apparent magnitude will permit the use of the solid of model universes to estimate the density and acceleration parameter. The numerical data necessary for constructing the solid of model universes are presented herein.

INTRODUCTION

When general relativity is used to describe the universe, it is assumed that the universe is isotropic and homogeneous, only the ponderable matter field is present, and the universe is so constituted that the pressure is zero. Even under these restrictive conditions, general relativity gives 11 possible families of models of the universe, each family being composed of many members. It is usual to classify these models by the curvature constant and the cosmical constant. The usual way to present these models is in tabular form. (See refs. 1, 2, and 3.) The tabular method of presentation gives a list of models but does not give a direct indication of how changes in the model parameters would affect the model.

This report presents the solid of model universes. It is a three-dimensional display of model information, which is helpful in understanding the relationships between models, and which builds insight into the cosmological problem. The solid of model universes is for models that have zero pressure and for models in which the line element is given by the Robertson-Walker metric. In addition to presenting all the information usually presented in tables, the solid of model universes shows the relationship of the models of the universe to each other, and permits interpretation of the simplified models, such as the zero-density model, with respect to the total spectrum of possible models of the universe.

Lastly, if the nonlinear part of apparent magnitude can be obtained from observational data, the solid of model universes may prove helpful in interpreting the observational data.

SYMBOLS

C	absolute magnitude parameter, $5 \log_{10} \frac{c}{H_o} + M - 5$
c	speed of light in a vacuum, parsecs sec^{-1}
D_l	luminosity distance, cm
G	universal constant of gravitation, dyn cm^2 g^{-2}
H	Hubble parameter, $\frac{d}{dt} \log R$, sec^{-1}
K	red shift correction to apparent observed magnitude
k	curvature constant: +1, spherical space; 0, Euclidean space; -1, hyperbolic space
M	absolute magnitude of source
m	apparent magnitude
Δm	increment in apparent magnitude due to σ_o and q_o being different from 1.0, $m - m_{11}$
p	pressure
q	deceleration parameter, $-\frac{\ddot{R}}{RH^2}$
$\overline{q}$	modified deceleration parameter, $q + 4$
R	scale factor that describes geometrical history of universe, cm
$S(\omega)$	function of radial metric variable
T	time since the beginning of expansion, sec
t	time

Y	normalized scale factor, $\frac{R}{R_o}$	
z	red shift	
Λ	cosmical constant, $3H_o^2(\sigma_o - q_o)$, sec^{-2}	
ρ	density, g cm^{-3}	
σ	density parameter, $\frac{4\pi G\rho}{3H^2}$	
τ	time ratio, T/T_{11}	
ω	radial metric variable	

Subscripts:

c	indicates a critical value
E	expansion
m	maximum value
o	indicates present conditions
11	indicates linear model, the model where $\sigma_o = q_o = 1.0$

Dots denote differentiation with respect to time.

THE MATHEMATICAL BASIS OF THE SOLID OF MODEL UNIVERSES

The equation that defines the solid of model universes is obtained from the red-shift—magnitude relation for zero-pressure isotropic and homogeneous relativistic models of the universe. The red-shift—magnitude relation as used in this study was obtained from equation (14) of reference 1 by setting $K = 0$ and $M = M_o + \Delta M_o$. The relationship is

$$m = 5 \log_{10}\left[R_o(1 + z)S(\omega)\right] + M - 5 \tag{1}$$

where

$$R_o = c\left(H_o\sqrt{|3\sigma_o - q_o - 1|}\right)^{-1}$$

and if a new parameter C is defined as

$$C = 5 \log_{10} \frac{c}{H_o} + M - 5$$

equation (1) can be written

$$m = 5 \log_{10}\left(|3\sigma_o - q_o - 1|\right)^{-1/2} (1 + z)S(\omega) + C \tag{2}$$

which is the desired form of the red-shift—magnitude relation. The function $S(\omega)$ is defined as

$$\left.\begin{aligned} S(\omega) &= \sinh \omega \qquad & (k = -1) \\ S(\omega) &= \omega \qquad & (k = 0) \\ S(\omega) &= \sin \omega \qquad & (k = 1) \end{aligned}\right\} \tag{3}$$

and

$$\omega = \sqrt{|3\sigma_o - q_o - 1|} \int_0^z \left[2\sigma_o z^3 + \left(3\sigma_o + q_o + 1\right)z^2 + 2\left(q_o + 1\right)z + 1\right]^{-1/2} dz \tag{4}$$

$$k = \operatorname{sgn}\left(3\sigma_o - q_o - 1\right) \tag{5}$$

In the definition of C, the Hubble parameter is determined from observations of nearby galaxies where the velocity-distance relation is always linear, and the speed of light is measured in the laboratory. See appendix A for a further discussion of c and H_o. Because of the time and distance separation of the source and observer, the absolute magnitude cannot be measured; thus, the absolute magnitude parameter is an unknown in equation (2). The absolute magnitude is a function of the energy radiated by the source at the time of emission and is a function of the energetic process of the source, the state of this process, and the size of the source. However, because M can vary from source to source, the absolute magnitude parameter may be different for different sources. None of the terms that make up the absolute magnitude parameter, however, contain the model parameters σ_o and q_o; thus, for the present epoch, the absolute magnitude parameter of a source will not vary with the model. Therefore, because the absolute magnitude parameter is independent of the model universe, the density parameter σ_o and the deceleration parameter q_o are the terms that determine the model universe. Since these parameters appear only in the argument of the log term of equation (1), this term determines the model of the universe. If the argument of the log term in equation (1) is multiplied and divided by z, this equation becomes

$$m = 5 \log_{10}(1 + z)\left(z\sqrt{|3\sigma_o - q_o - 1|}\right)^{-1} S(\omega) + 5 \log_{10} z + C \tag{6}$$

The last two terms in equation (6) are equation (1) evaluated for $\sigma_o = q_o = 1.0$. (See ref. 1.) Because $5 \log_{10} z + C$ plots as a straight line on the plot of $\log_{10} z$ against m, it is often referred to as the linear form of the red-shift—magnitude relation and it is

$$m_{11} = 5 \log_{10} z + C \tag{7}$$

If equation (7) is subtracted from equation (6), then

$$\Delta m = m - m_{11} = 5 \log_{10}(1 + z)\left(z\sqrt{|3\sigma_o - q_o - 1|}\right)^{-1} S(\omega) \tag{8}$$

Equation (8) is called the incremental red-shift—magnitude relation and it contains all the model information of equation (1) and could be used to determine σ_o and q_o if a Δm can be extracted from the observational data.

The Solid of Model Universes

Equation (8) forms a solid in the $\Delta m, q_o, z$ coordinate system. This solid called the solid of model universes may be thought of as a set of warped surfaces stacked one on top of the other. (See fig. 1.) The density parameter σ_o has a constant value on each of the surfaces but changes from surface to surface. The solid of model universes is not a bounded solid as it can extend indefinitely in the z-direction. The solid is not really limited in the σ_o- and q_o-directions but is constrained by reasonable values of these parameters. For example, in the case of σ_o, a reasonable upper limit is between 5 and 10. The part of the solid of model universes shown in figure 1 occupies the following intervals: $0 \leqq z \leqq 2.5$; $-1.0 \leqq q_o \leqq 5.0$; and $0 \leqq \sigma_o \leqq 6.0$. This range of σ_o corresponds to a density variation from 0 to 1.2×10^{-28} g-cm^{-3} for $H_o = 2.43 \times 10^{-18}$ sec^{-1}. The traces of the σ_o = constant surfaces on any q_o = constant sections, that is, planes containing $\Delta m, z$ axes, are called model curves and carry the information necessary to describe models of the universe.

The top surface of the solid of model universes is the surface on which density is equal to zero. Because all physically significant models of the universe (those models in which $\sigma_0 > 0$) lie below the zero-density surface, the Δm associated with this surface is an upper boundary for Δm. The lower boundary for Δm is a function of the density parameter and is set by a maximum allowable value of σ_0 or a value of σ_0 that is selected so that undue restrictions are not placed on possible models of the universe. In the work reported herein, a σ_0 of 6.0 was adopted. A σ_0 of 6.0 means that the density of the universe is approaching the density of a galaxy. As observational evidence indicates that the universe is less dense than a galaxy, this limit for σ_0 should impose no undue limitations on models for the universe.

Table I contains the numerical data necessary for the construction of the solid of model universes.

The models of the universe contained in the solid of model universes can be described in terms of the curvature constant k and the cosmical constant Λ $\left(\Lambda = 3H_0^2(\sigma_0 - q_0)\right)$. As k can be ±1 or 0 and $\Lambda \lesseqqgtr 0$, there are nine possible combinations of the constants; each combination represents a family of model universes. Because k is fixed at the values given, the members of each family are differentiated by different values of Λ. By locating the surfaces on which $k = 0$ and $\Lambda = 0$ the location of the various families within the solid of model universes can be found. The two surfaces $k = 0$ and $\Lambda = 0$ intersect as shown in figure 2. One family of models, which is actually a single family, lies along the intersection. Four more families lie on these surfaces as they branch out from the intersection. The four remaining families lie in the mutually exclusive regions separated by the branches of the $k = 0$ and $\Lambda = 0$ surfaces. Table II (based on ref. 2) shows the types of model (that is expanding, oscillating, etc.) in each family.

As shown in table II the $k = +1$, $\Lambda > 0$ region of the solid of model universes is occupied by several different types of models. The A_1 models fall on the curve labeled A_1 in figure 2 and separate the 0 models from the A_1 models. The A_2 models shown in figure 2 as a point located at $\sigma_0 = 0$, $q_0 = -1.0$ instead of a line because all these models have q_0 values $\leqq -1.0$. The expression for the A_1 curve (it also gives the A_2 curve) is given by Stabell and Refsdal in reference 4.

Table II is a typical example of the tabular method of presenting the relativistic models of the universe. In this presentation there is no indication of the basic parameters σ_0 and q_0 which define k and Λ and consequently a model of the universe. In the solid of model universes, the z = constant sections contain the σ_0 and q_0 information. When the model information from table II is placed on a z = constant section, information relative to σ_0 and q_0 for each model can be obtained. The curves of constant σ_0 on the z-section were not shown in figure 2 in order to improve the clarity of the figure. Figure 3 shows the σ_o, $k = 0$, $\Lambda = 0$, and A_1 curves on a $z = 1.0$ section of the solid of model universes. Each point on the z = constant section shown in figure 3 can be described by either of two sets of coordinates, $\Delta m, q_o$ or σ_o, q_o. Because σ_o and q_o identify model curves each point on the section is the cut end of a model curve. Thus, if σ_o and q_o are known, figure 3 can be used to determine the k and Λ for the model and obtain a qualitative indication as to how variations in σ_o and q_o will change the model. Because the positions of the model curves relative to each other do not change as z changes, any z = constant section can be used to obtain qualitative information about the model.

Sections of the solid of model universes for constant q_o are also useful. These sections contain the model curves and give the variation of Δm with z and σ_o. Figure 4 shows the model curves for several values of q_o for $0 \leqq \sigma_o \leqq 6.0$ and $0 \leqq z \leqq 2.5$. The curves may be used in two ways (1) to find σ_o and q_o from a known distribution of Δm with z and (2) to determine how Δm varies with z when σ_o and q_o are known. This second use of figure 4 also permits the determination of the manner in which a model differs from the linear model.

The Solid of Model Universes and Time Since the Beginning of Expansion

Omer and Vanyo (ref. 5) give an expression for the time since the beginning of expansion which, in the notation of this paper, is

$$T = H_o^{-1} \int_a^1 \left[\left(\sigma_o - q_o \right) Y^2 - \left(3\sigma_o - q_o - 1 \right) + \frac{2\sigma_o}{Y} \right]^{-1/2} dY \tag{9}$$

where $Y = \frac{R}{R_o}$ and the lower limit a may be zero or have a finite value. As the formula for T is a definite integral, T can change only with σ_o and q_o. In the constant z sections of the solid of model universes, each point in the section carries a unique value of σ_o and q_o. Thus, curves of constant time since the beginning of expansion can be superimposed on the constant z sections of the solid of model universes. Time since the beginning of expansion is dependent on the value of H_o. As the exact value of H_o is not known, T was normalized by dividing equation (9) by T_{11}. This division gives the ratio T/T_{11} and removes the effect of the Hubble parameter. For values of $\sigma_o = q_o = 1.0$, equation (9) can be integrated in terms of simple functions and the expression for time is $T_{11} = 0.57079633 H_o^{-1}$. The time ratio for any model τ therefore is given by

$$\tau = \frac{T}{T_{11}} = (0.57079633)^{-1} \int_a^1 \left[\left(\sigma_o - q_o \right) Y^2 - \left(3\sigma_o - q_o - 1 \right) + \frac{2\sigma_o}{Y} \right]^{-1/2} dY \tag{10}$$

Table III contains the numerical data for adding curves of constant τ to the solid of model universes.

Equation (10) was used to compute curves of constant time ratios. These curves of constant time ratio are shown in figure 5 superimposed on a $z = 1.0$ section of the solid of model universes. A new abscissa scale was adopted in order to eliminate excessive crowding of the τ and σ_0 curves in the interval $-3.0 \leqq 0$. The scale adopted and used in figure 5 is $\log_{10} \bar{q}_0$ where $\bar{q}_0 = q_0 + 4$. Because q_0 is less than -1.0 in figure 5, the curve for the A_2 models is now visible. It starts at $q_0 = -1.0$ and loops across the top of the figure and enters the heavy boundary with the $\sigma_0 = 0.16$ curve. The heavy boundary curve in figure 5 is called the model boundary curve and combinations of σ_0 and q_0 that lie between this curve and Δm axis caused the radicand in equation (4) to become negative and no solutions for Δm exist in this region. Because the boundary curve is a function of σ_0, q_0, and z, its position varies as z varies. For $z < 1.0$, this curve moves toward the Δm axis. The model boundary curve also represents a region where the σ_0 and τ curves intermingle to such an extent that they are meaningless. Between the model boundary curve and the $q_0 = -1.0$ line, there are multiple intersections of the σ_0 and τ curves. The physically significant intersections have been marked by a small circle in figure 5; the other intersections have no physical meaning and should be disregarded. The meaningless intersections occur because the z constant sections of the solid of model universes are surfaces in the $\Delta m, q_0, \sigma_0$ coordinate system (fig. 6) and when this surface is compressed into a plane, the σ_0 and τ curves are superimposed in a manner that produces the false intersections.

DISCUSSION

A pseudo solid called the solid of model universes has been developed from a modified form of the red-shift—magnitude relation. It contains much information about uniform relativistic models of the universe that have zero pressure. In order to illustrate the information content, the z constant section model presentation is compared with the tabular form of model presentation. Table II is a typical tabular listing of models for the universe and includes no information on σ_0, q_0, and τ. This information is not included because of the complexity of the table. (See refs. 3 and 4.) If included, it is in the form of inequalities that give no quantitative information about σ_0, q_0, and τ. Because the z constant section of the solid of model universe gives information about σ_0, q_0, and τ for each family of model universes, it is considered to be superior to the tabular model presentation. The model curves of the solid of model universes show how the model curve for any model deviates from the linear model and the model curves may be useful in making preliminary estimates of a model of the observed universe from the distribution of Δm with z. The solid of model universes is also a useful tool for comparing simplified models of the universe, such as the zero-density model, with the exact model to learn how various simplifications restrict the spectrum of possible models.

The incremental red-shift—magnitude relation and the solid of model universes can be used to examine the adequacy of simplified models for the determination of a model of the universe from observational data. There are three simplified models of the universe: the zero-density (ref. 3), the $\Lambda = 0$ model (ref. 6), and the approximate model (ref. 7). The zero-density model and the $\Lambda = 0$ model are exact models because the simplifying assumptions $\sigma_o = 0$ for the zero-density case and $\sigma_o = q_o$ for $\Lambda = 0$ reduce the elliptic integral of equation (8) to one that can be integrated in terms of simple functions. The solution of the approximate model can also be obtained in terms of simple functions but it is not exact because the term $2\sigma_o z^3$ in the radicand of equation (4) for ω is assumed to be negligible with respect to the other terms in the expression for ω. All the simplified models and the exact model can be expanded in a Maclaurin series about $z = 0$. In all these cases, the series is the same out to terms of the order of z^2. This expansion model which is used in place of the other models is also compared with the exact model.

The use of the $\Lambda = 0$ restricts the possible models of the universe to a single surface in the solid of model universes. Because of the severe restriction on the choice of models to fit the observational data and because the $\Lambda = 0$ model is not either an extreme or limiting case, the use of this model to replace equations (1) and (5) for the analysis of data cannot be justified.

The zero-density model again restricts the possible models of the universe to a single surface of the solid of model universes; however, the zero-density model represents a limiting case for possible models of the universe and it will tend to overestimate q_o. (See figs. 3 and 4.) Suppose that a set of $\Delta m, z$ data fits the model curve for $\sigma_o = 0.75$, $q_o = 2.0$. (See fig. 4.) When the zero-density model is used to analyze the data, the computer must seek the model on the zero-density surface that best fits the data. As Δm and σ_o are fixed, the only parameter available for fitting is q_o. It can be seen from figures 3 and 4 that the Δm values on the zero-density surface that best fit the data will occur near $q_o = 4.0$ for the case under consideration. This value of q_o is twice the original value of that parameter. A study of figures 3 and 4 indicates that the zero density model should be satisfactory for the analysis of observational data if σ_o lies in the interval $0.16 \geqq \sigma_o > 0$ and q_o is greater than -1.0. In this region the overestimation of q_o will be small.

The region of use of the approximate model for $z \leqq 1.0$ is shown in figure 7. If the approximate model is used in the region between the curve labeled approximate model and the trace of the zero-density surface, an error equal to or less than 2 percent will be introduced into Δm. More details of the approximate model which appeared to offer the best simplified model substitute for the exact model are contained in reference 1.

The expansion model is obtained by the expansion of the argument of the logarithmic term in equation (1) in a Maclaurin series about $z = 0$. This form has a long history of development which is given by North. (See ref. 8.) To study the use of this form of the red-shift—magnitude relation, the result given by McVittie (ref. 9) is used. Replacing the argument of the log term in equation (1) with the expanded form gives

$$m_E = 5 \log_{10}\left\{z\left[1 + \left(\frac{1 - q_o}{2}\right)z\right]\right\} + C \tag{11}$$

and

$$\Delta m_E = 5 \log_{10}\left(1 + \frac{1 - q_o}{2} z\right) \tag{12}$$

where the subscript E denotes the expansion form. Because equations (11) and (12) derive from a Maclaurin expansion of ω about $z = 0$, a restriction that $z << 1$ is imposed on the use of these equations. In addition, the density parameter does not enter the coefficient of the series until terms on the order of z^3 and higher are retained. As the series is normally cut off after the z-squared term, the effects of density are neglected in the expansion model.

Equation (11) has been compared with the zero-density model (ref. 3) and with the $\Lambda = 0$ model (ref. 6), and both have shown that differences exist between these special exact solutions and equation (11). The maximum difference in the predicted apparent magnitudes of objects between the zero-density model and equation (11) is $1\overset{m}{.}9$ and between the $\Lambda = 0$ model and equation (11), the maximum difference is $0\overset{m}{.}4$. Solheim (ref. 10) without giving details or deviation diagrams states that the expansion form is of little use in the problem of determining a model of the universe.

The earlier work by Sherman and by Mattig that compared the expansion form of the red-shift—magnitude relation with special solutions of the general case is now extended to compare the expansion and general forms of the red-shift—magnitude relation. Deviation diagrams are obtained which show the danger of using series methods to approximate the general case. The comparison also illustrates the usefulness of the solid of model universes because the solid of model universes shows where the approximation lies and the models that are being approximated.

Equation (12) defines a surface in $\Delta m, q_o, z$ coordinates that is coincident with the q_o-axis at $z = 0$. (See fig. 8.) This surface is concave downward. As z increases from zero, the surface twists about the $q_o = 1.0$ line while the $z = 0$ line remains fixed. Because the incremental red-shift—magnitude relation and the incremental form of the expansion red-shift—magnitude relation are referred to the same standard, the constant z traces of the surface given by equation (12) can be superimposed on the constant z section of the solid of model universes. This superposition for $z = 0.2$ and 1.0 is shown in figures 9(a) and 9(b). A study of figures 9(a) and 9(b) shows that at three locations the surface defined by equation (12) intercepts the same σ_o and q_o in both sections of the solid of model universes. The σ_o and q_o for these points in the solid of model universes are 0, -1.0; 0,0; and 1.0, 1.0. These values of σ_o and q_o reduce equation (8) to equation (12); thus, at these points the expansion form of the red-shift—magnitude relation is an exact solution. Variations of σ_o and/or q_o from the noted values cause a rapid breakdown in the correspondence between equations (8) and (12). Figures 9(c) and 9(d) compare the general and expansion forms of the red-shift—magnitude relation between these points. Figure 9(e) was used to determine the models being approximated by the expansion form of the red-shift—magnitude relation out to $q_o = 1.85$. The results are given in table IV and figure 10. This approximation is poorest at $q_o = 1.85$ because of the difference in curvature of the Δm and Δm_E model curves at this value of q_o. (See fig. 9(e).) The data presented in figure 10 and table IV are interpreted in the following manner. If $q_o = 0.35$, then the expansion form is an approximation to the general case when $\sigma_o = 0.16$. The values of z given in table IV represent a conservative upper limit beyond which the approximation is not considered to be valid.

Because of the restricted range of q_o over which the expansion form of the red-shift—magnitude relation is a good approximation for the general form and then only for specific models of the universe, it is concluded that the expansion form should not be used for the analysis of observational data. Lastly, there is little hope of improving the expansion solution by retaining more terms in the series since convergence is very slow.

The constant z sections of the solid of model universes can be used to compare models of the universe obtained by different methods or models that are based on observational estimates. A comparison was made between a model obtained from the analysis of observational data with the red-shift—magnitude relation and models that are based on estimates of the time since the beginning of expansion and the present density of the universe. The model parameters obtained through the use of the red-shift—magnitude relation were $\sigma_o = 3.163 \pm 1.154$ and $q_o = 1.3010 \pm 0.519$. Abell (ref. 11) has estimated the present density of the universe as approximately 10^{-30} g-cm^{-3}. This density gives a σ_o of about 0.05. For the models of the universe considered in the solid of model universes, the time since the beginning of expansion should be equal to or greater than the age of the galaxy. This statement means that the age of the galaxy is a lower bound for the time since the beginning of expansion. Hoyle (ref. 12) has estimated the age of the galaxy to be about 15×10^9 years. For $H_o = 3.24 \times 10^{-18}$ sec^{-1}, this age gives a τ for the galaxy of approximately 2.0. The upper crosshatched area in figure 11 shows where the models based on the estimated density and time since the beginning of expansion would occur. The lower crosshatched area shows where the model predicted by the red-shift—magnitude relation from observed apparent magnitudes and red shifts of galaxies occurs. Since the same universe is being observed, the model regions should at least overlap. From figure 11, it is very easy to see that the τ of the model predicted by the red-shift—magnitude is too small and, in general, the lack of agreement is due to the large σ_o and q_o of this model.

If adequate methods can be found to extract Δm, the nonlinear part of the apparent magnitude, from the observed apparent magnitude of extra galactic objects, the solid of model universes will be useful for the estimation of the density and acceleration parameters. An approach to estimating Δm that depends on the noise in the measurement of apparent magnitudes and the boundary values of σ_o and q_o is discussed by Sherman (ref. 13).

CONCLUDING REMARKS

The incremental form of the red-shift—magnitude relation, a form that relates the model parameters to the nonlinear part of the apparent magnitude, was derived and interpreted in a special coordinate system as a solid. This solid called the solid of model universes provides presentations that show the relationship of the various families of models for the universe and relates these models to the time since the beginning of the expansion. The presentation of model information, which is an improved way to present Robertson's models, permits rapid evaluation of simplified models with respect to the exact model and it shows how the spectrum of models is limited by the use of a simplified model. It is also useful for comparing the various models obtained from observational data other than red shift and apparent magnitude and by methods of analysis other than the red-shift—magnitude relation. If the nonlinear part of the apparent magnitude (Δm) can be extracted from observational data, the solid of model universes will be useful for estimating the acceleration parameter, the density parameter, and the time since the beginning of expansion.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., October 31, 1969.

APPENDIX A

DISCUSSION OF THE TERM c/H_o

The luminosity distance for uniform zero pressure relativistic models of the universe as given in reference 3 is

$$D_l = R_o(1 + z)S(\omega)$$

where R_o, z, $S(\omega)$, and ω are as defined in the body of the report. The slope of the luminosity distance with respect to redshift dD_l/dz is the quantity of interest and is

$$\frac{dD_l}{dz} = \frac{c}{H_o\sqrt{|3\sigma_o - q_o - 1|}} S(\omega) + \frac{c(1 + z)}{H_o\sqrt{|3\sigma_o - q_o - 1|}} \frac{dS(\omega)}{dz}$$

where

$$\frac{dS(\omega)}{dz} = \cosh \omega \frac{d\omega}{dz} \qquad (k = -1)$$

$$= \frac{d\omega}{dz} \qquad (k = 0)$$

$$= \cos \omega \frac{d\omega}{dz} \qquad (k = +1)$$

and

$$\frac{d\omega}{dz} = \sqrt{|3\sigma_o - q_o - 1|} \left[2\sigma_o z^3 + \left(3\sigma_o + q_o + 1\right)z^2 + 2\left(q_o + 1\right)z + 1\right]^{-1/2}$$

The derivative of the luminosity distance at any point between the observer and the observed object is given by

APPENDIX A

$$\frac{dD_l}{dz} = \frac{c}{H_0\sqrt{|3\sigma_0 - q_0 - 1|}} \sinh\left\{\sqrt{|3\sigma_0 - q_0 - 1|}\int_0^z \left[2\sigma_0 z^3 + \left(3\sigma_0 + q_0 + 1\right)z^3 + 2\left(q_0 + 1\right)z + 1\right]^{-1/2} dz\right\}$$

$$+ \frac{c(1+z)}{H_0\sqrt{|3\sigma_0 - q_0 - 1|}} \left\{\cosh\sqrt{|3\sigma_0 - q_0 - 1|}\int_0^z \left[2\sigma_0 z^3 + \left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1\right]^{-1/2} dz\right\}$$

$$\times \sqrt{\frac{|3\sigma_0 - q_0 - 1|}{2\sigma_0 z^3 + \left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1}} \qquad (k = -1) \qquad \text{(A1)}$$

$$\frac{dD_l}{dz} = \frac{c}{H_0}\int_0^z \left[2\sigma_0 z^3 + \left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1\right]^{-1/2} dz$$

$$+ \frac{c(1+z)}{H_0} \frac{1}{\left[2\sigma_0 z^3 + \left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1\right]^{1/2}} \qquad (k = 0) \qquad \text{(A2)}$$

$$\frac{dD_l}{dz} = \frac{c}{H_0\sqrt{|3\sigma_0 - q_0 - 1|}} \sin\left\{\sqrt{|3\sigma_0 - q_0 - 1|}\int_0^z \left[2\sigma_0 z^3 + \left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1\right]^{-1/2} dz\right\}$$

$$+ \frac{c(1+z)}{H_0\sqrt{|3\sigma_0 - q_0 - 1|}} \left\{\cos\sqrt{|3\sigma_0 - q_0 - 1|}\int_0^z \left[2\sigma_0 z^3 + \left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1\right]^{-1/2} dz\right\}$$

$$\times \sqrt{\frac{|3\sigma_0 - q_0 - 1|}{2\sigma_0 z^3 + \left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1}} \qquad (k = +1) \qquad \text{(A3)}$$

In the three expressions for dD_l/dz, the derivative is dependent on the model parameters σ_0 and q_0, and these expressions are valid for all values of z. The primary interest is the value of dD_l/dz as z approaches zero. An inspection of equations (A1) to (A3) shows that this value is dependent on the behavior of the elliptic integral as z approaches zero. The argument of the circular and hyperbolic functions in equations (A1) and (A3) is the radial metric variable ω. (See eq. (4).) The variable ω gives the distance between an observer located at R_o and a source located at R. If successively closer sources are observed, R approaches R_o and ω approaches zero. At $R = R_o$, ω is equal to zero. In cosmology the red shift is given by

$$z = \frac{R_o}{R} - 1$$

so that as R approaches R_o, R_o/R approaches 1 and z approaches 0. Thus as z approaches 0, ω approaches 0. An inspection of equation (4) shows the only way that ω can be zero at $z = 0$ is for the elliptic integral to be zero. This behavior has been confirmed by a digital computer evaluation of the integral. Since the elliptic integral is zero, the first terms on the right-hand side of equations (A1) to (A3) are zero. In equations (A1) and (A3), the arguments of the cosine and hyperbolic cosine are zero; thus, these functions become unity and the second term on the right-hand sides of these equations reduces to cH_o^{-1}. The second term on the right of equation (A2) also reduces to cH_o^{-1} as z approaches 0. Thus, at $z = 0$, all three spaces have the same value of the derivative and

$$\left(\frac{dD_l}{dz}\right)_{z \to 0} = cH_o^{-1} \tag{A4}$$

for $k = 0$, $k = +1$ or $k = -1$. As c is the speed of light in a vacuum, equation (A4) shows that H_o^{-1} and thus H_o is dependent only on local conditions and is independent of the model universe.

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TABLE I.- Δm AS A FUNCTION OF σ_0, q_0, AND z

q_0	Values of Δm for $l = 0.001$ and σ_0 of –																	
	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004
-2.5	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004
-2.0	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003
-1.5	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003
-1.0	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002
-.5	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002
0.0	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001
.5	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001
1.0	.000	.000	.000	.000	.000	.000	.000	.000	.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000
1.5	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001
2.0	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001
2.5	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002
3.0	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002
3.5	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003
4.0	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003
4.5	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004
5.0	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004	-.004
5.5	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005
6.0	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

Values of Δm for $l = 0.005$ and σ_o of –

q_o	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0	.022	.022	.022	.022	.022	.022	.022	.022	.022	.022	.022	.022	.022	.022	.021	.021	.021	.021
-2.5	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019
-2.0	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016
-1.5	.014	.014	.014	.014	.014	.014	.014	.014	.013	.013	.013	.013	.013	.013	.013	.013	.013	.013
-1.0	.011	.011	.011	.011	.011	.011	.011	.011	.011	.011	.011	.011	.011	.011	.011	.011	.010	.010
-.5	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008
0.0	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005
.5	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.002	.002	.002	.002
1.0	.000	.000	.000	.000	.000	.000	.000	.000	.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000	-.000
1.5	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003	-.003
2.0	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.006	-.006	-.006	-.006
2.5	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008	-.008
3.0	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011	-.011
3.5	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.013	-.014	-.014	-.014
4.0	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016
4.5	-.018	-.018	-.018	-.018	-.018	-.018	-.018	-.018	-.018	-.019	-.019	-.019	-.019	-.019	-.019	-.019	-.019	-.019
5.0	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021
5.5	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024	-.024
6.0	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.027	-.027	-.027

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

q_o	Values of Δm for $l = 0.010$ and σ_o of –																	
	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0	.044	.044	.044	.044	.044	.044	.044	.043	.043	.043	.043	.043	.043	.043	.043	.042	.042	.042
-2.5	.038	.038	.038	.038	.038	.038	.038	.038	.038	.038	.038	.038	.037	.037	.037	.037	.037	.036
-2.0	.033	.033	.033	.033	.032	.032	.032	.032	.032	.032	.032	.032	.032	.032	.031	.031	.031	.031
-1.5	.027	.027	.027	.027	.027	.027	.027	.027	.027	.027	.027	.027	.026	.026	.026	.026	.026	.025
-1.0	.022	.022	.022	.022	.022	.022	.021	.021	.021	.021	.021	.021	.021	.021	.021	.020	.020	.020
-.5	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.015	.015	.015	.015	.014
0.0	.011	.011	.011	.011	.011	.011	.011	.011	.011	.011	.010	.010	.010	.010	.010	.010	.009	.009
.5	.006	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.004	.004	.004	.004
1.0	.000	.000	.000	.000	.000	.000	.000	.000	.000	-.000	-.000	-.000	-.000	-.001	-.001	-.001	-.001	-.001
1.5	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.005	-.006	-.006	-.006	-.006	-.006	-.007	-.007
2.0	-.010	-.010	-.010	-.010	-.010	-.010	-.010	-.010	-.010	-.011	-.011	-.011	-.011	-.011	-.011	-.012	-.012	-.012
2.5	-.015	-.015	-.015	-.015	-.015	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.016	-.017	-.017	-.017
3.0	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.021	-.022	-.022	-.022	-.022
3.5	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.026	-.027	-.027	-.027	-.027
4.0	-.031	-.031	-.031	-.031	-.031	-.031	-.031	-.031	-.031	-.031	-.031	-.031	-.031	-.032	-.032	-.032	-.032	-.032
4.5	-.036	-.036	-.036	-.036	-.036	-.036	-.036	-.036	-.036	-.036	-.036	-.036	-.036	-.037	-.037	-.037	-.037	-.037
5.0	-.041	-.041	-.041	-.041	-.041	-.041	-.041	-.041	-.041	-.041	-.041	-.041	-.041	-.042	-.042	-.042	-.042	-.042
5.5	-.046	-.046	-.046	-.046	-.046	-.046	-.046	-.046	-.046	-.046	-.046	-.046	-.046	-.047	-.047	-.047	-.047	-.047
6.0	-.051	-.051	-.051	-.051	-.051	-.051	-.051	-.051	-.051	-.051	-.051	-.051	-.051	-.051	-.052	-.052	-.052	-.052

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

q_o	Values of Δm for $l = 0.015$ and σ_o of –																	
	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0	.066	.066	.066	.066	.066	.065	.065	.065	.065	.065	.065	.064	.064	.064	.063	.063	.062	.062
-2.5	.057	.057	.057	.057	.057	.057	.057	.057	.057	.056	.056	.056	.056	.055	.055	.054	.054	.053
-2.0	.049	.049	.049	.049	.049	.049	.049	.048	.048	.048	.048	.048	.047	.047	.046	.046	.045	.045
-1.5	.041	.040	.040	.040	.040	.040	.040	.040	.040	.040	.040	.039	.039	.039	.038	.038	.037	.037
-1.0	.032	.032	.032	.032	.032	.032	.032	.032	.032	.032	.031	.031	.031	.030	.030	.029	.029	.028
-.5	.024	.024	.024	.024	.024	.024	.024	.024	.024	.024	.023	.023	.023	.022	.022	.021	.021	.020
0.0	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.015	.015	.015	.014	.014	.013	.013	.012
.5	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.007	.007	.007	.006	.006	.005	.005	.005
1.0	.000	.000	.000	.000	.000	.000	.000	.000	.000	-.000	-.000	-.001	-.001	-.001	-.002	-.002	-.003	-.003
1.5	-.007	-.007	-.007	-.007	-.007	-.007	-.008	-.008	-.008	-.008	-.008	-.008	-.009	-.009	-.010	-.010	-.011	-.011
2.0	-.015	-.015	-.015	-.015	-.015	-.015	-.015	-.015	-.015	-.016	-.016	-.016	-.016	-.017	-.017	-.018	-.018	-.019
2.5	-.023	-.023	-.023	-.023	-.023	-.023	-.023	-.023	-.023	-.023	-.023	-.024	-.024	-.024	-.025	-.025	-.026	-.026
3.0	-.030	-.030	-.030	-.030	-.030	-.030	-.030	-.030	-.031	-.031	-.031	-.031	-.031	-.032	-.032	-.033	-.033	-.034
3.5	-.037	-.038	-.038	-.038	-.038	-.038	-.038	-.038	-.038	-.038	-.038	-.039	-.039	-.039	-.040	-.040	-.041	-.041
4.0	-.045	-.045	-.045	-.045	-.045	-.045	-.045	-.045	-.045	-.046	-.046	-.046	-.046	-.047	-.047	-.048	-.048	-.048
4.5	-.052	-.052	-.052	-.052	-.052	-.052	-.052	-.052	-.053	-.053	-.053	-.053	-.053	-.054	-.054	-.055	-.055	-.056
5.0	-.059	-.059	-.059	-.059	-.059	-.060	-.060	-.060	-.060	-.060	-.060	-.060	-.061	-.061	-.062	-.062	-.062	-.063
5.5	-.067	-.067	-.067	-.067	-.067	-.067	-.067	-.067	-.067	-.067	-.067	-.068	-.068	-.068	-.069	-.069	-.070	-.070
6.0	-.074	-.074	-.074	-.074	-.074	-.074	-.074	-.074	-.074	-.074	-.074	-.075	-.075	-.075	-.076	-.076	-.077	-.077

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

Values of Δm for $l = 0.100$ and σ_o of –

q_o	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0	.469	.467	.465	.462	.459	.456	.453	.445	.437	.422	.407	.392	.378	.349	.322	.295	.269	.243
-2.5	.393	.391	.390	.387	.384	.382	.379	.372	.365	.351	.337	.324	.311	.285	.259	.235	.211	.187
-2.0	.325	.324	.322	.320	.317	.315	.312	.306	.299	.287	.274	.262	.250	.226	.203	.180	.157	.135
-1.5	.263	.262	.261	.259	.256	.254	.252	.246	.240	.228	.217	.205	.194	.172	.150	.129	.108	.087
-1.0	.207	.206	.205	.203	.200	.198	.196	.190	.185	.174	.163	.153	.142	.121	.101	.081	.062	.042
-.5	.155	.154	.153	.151	.148	.146	.144	.139	.134	.124	.114	.104	.094	.075	.055	.037	.018	-.000
0.0	.106	.105	.104	.102	.100	.098	.096	.091	.087	.077	.068	.058	.049	.030	.012	-.005	-.023	-.040
.5	.060	.059	.058	.057	.055	.053	.051	.047	.042	.033	.024	.015	.006	-.011	-.028	-.045	-.062	-.078
1.0	.017	.016	.016	.014	.012	.010	.009	.004	.000	-.009	-.017	-.026	-.034	-.050	-.067	-.083	-.099	-.115
1.5	-.023	-.024	-.025	-.027	-.028	-.030	-.032	-.036	-.040	-.048	-.056	-.064	-.072	-.088	-.104	-.119	-.134	-.149
2.0	-.062	-.063	-.063	-.065	-.067	-.068	-.070	-.074	-.078	-.085	-.093	-.101	-.108	-.124	-.139	-.155	-.168	-.182
2.5	-.098	-.099	-.100	-.102	-.103	-.105	-.106	-.110	-.114	-.121	-.128	-.136	-.143	-.158	-.172	-.186	-.200	-.214
3.0	-.133	-.134	-.135	-.136	-.138	-.139	-.141	-.144	-.148	-.155	-.162	-.169	-.176	-.190	-.204	-.218	-.231	-.245
3.5	-.167	-.168	-.168	-.170	-.171	-.173	-.174	-.177	-.181	-.188	-.195	-.201	-.208	-.222	-.235	-.248	-.261	-.274
4.0	-.199	-.200	-.200	-.202	-.203	-.204	-.206	-.209	-.212	-.219	-.226	-.232	-.239	-.252	-.264	-.277	-.290	-.302
4.5	-.230	-.231	-.231	-.232	-.234	-.235	-.236	-.240	-.243	-.249	-.256	-.262	-.268	-.281	-.293	-.305	-.317	-.329
5.0	-.260	-.260	-.261	-.262	-.263	-.265	-.266	-.269	-.272	-.278	-.284	-.290	-.297	-.309	-.321	-.332	-.344	-.356
5.5	-.288	-.289	-.289	-.291	-.292	-.293	-.294	-.297	-.300	-.306	-.312	-.318	-.324	-.336	-.347	-.359	-.370	-.381
6.0	-.316	-.316	-.317	-.318	-.319	-.320	-.322	-.325	-.327	-.333	-.339	-.345	-.350	-.362	-.373	-.384	-.395	-.406

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

Values of Δm for $l = 0.150$ and σ_o of –

q_o	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0	.756	.751	.746	.736	.727	.717	.708	.685	.663	.620	.580	.541	.504	.435	.370	.310	.253	.199
-2.5	.608	.604	.600	.593	.585	.578	.570	.552	.533	.498	.465	.432	.400	.340	.284	.231	.180	.131
-2.0	.490	.487	.483	.477	.470	.464	.458	.442	.427	.396	.367	.339	.311	.258	.208	.160	.114	.069
-1.5	.390	.387	.385	.379	.373	.368	.362	.348	.335	.308	.282	.257	.232	.185	.139	.095	.053	.012
-1.0	.303	.301	.298	.293	.288	.283	.278	.266	.254	.230	.207	.184	.161	.118	.076	.036	-.003	-.041
-.5	.226	.224	.222	.217	.213	.208	.204	.193	.182	.160	.139	.118	.097	.057	.018	-.019	-.055	-.090
0.0	.157	.155	.153	.149	.145	.140	.136	.126	.116	.096	.076	.057	.038	.001	-.035	-.070	-.104	-.137
.5	.094	.092	.090	.086	.082	.078	.075	.065	.056	.037	.019	.001	-.017	-.051	-.085	-.118	-.150	-.181
1.0	.035	.034	.032	.028	.025	.021	.018	.009	.000	-.017	-.034	-.051	-.068	-.100	-.132	-.163	-.193	-.222
1.5	-.019	-.020	-.022	-.025	-.029	-.032	-.035	-.044	-.052	-.068	-.084	-.100	-.116	-.146	-.176	-.205	-.234	-.262
2.0	-.069	-.071	-.072	-.076	-.079	-.082	-.085	-.093	-.100	-.116	-.131	-.146	-.161	-.190	-.218	-.246	-.273	-.300
2.5	-.117	-.118	-.120	-.123	-.126	-.129	-.132	-.139	-.146	-.161	-.175	-.189	-.203	-.231	-.258	-.284	-.310	-.336
3.0	-.161	-.163	-.164	-.167	-.170	-.173	-.175	-.182	-.189	-.203	-.217	-.230	-.243	-.270	-.295	-.321	-.346	-.370
3.5	-.204	-.205	-.206	-.209	-.212	-.214	-.217	-.224	-.230	-.243	-.256	-.269	-.282	-.307	-.332	-.356	-.380	-.403
4.0	-.244	-.245	-.246	-.249	-.251	-.254	-.257	-.263	-.269	-.282	-.294	-.306	-.319	-.343	-.366	-.389	-.412	-.435
4.5	-.282	-.283	-.285	-.287	-.289	-.292	-.294	-.300	-.306	-.318	-.330	-.342	-.354	-.377	-.399	-.422	-.444	-.465
5.0	-.319	-.320	-.321	-.323	-.326	-.328	-.330	-.336	-.342	-.353	-.365	-.376	-.387	-.409	-.431	-.453	-.474	-.495
5.5	-.353	-.355	-.356	-.358	-.360	-.362	-.365	-.370	-.376	-.387	-.398	-.409	-.420	-.441	-.462	-.483	-.503	-.523
6.0	-.387	-.388	-.389	-.391	-.393	-.396	-.398	-.403	-.409	-.419	-.430	-.440	-.451	-.471	-.492	-.512	-.531	-.551

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

Values of Δm for $l = 0.200$ and σ_o of –

q_o	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0	1.161	1.145	1.129	1.099	1.070	1.043	1.017	.957	.901	.803	.717	.639	.568	.443	.333	.234	.144	.062
-2.5	.852	.843	.834	.817	.800	.783	.767	.727	.690	.619	.554	.494	.437	.334	.240	.154	.075	.001
-2.0	.660	.653	.646	.633	.621	.608	.596	.565	.536	.480	.427	.377	.329	.240	.158	.081	.010	-.057
-1.5	.514	.509	.504	.493	.483	.472	.462	.437	.412	.365	.320	.277	.235	.157	.084	.015	-.050	-.112
-1.0	.396	.391	.387	.378	.369	.360	.351	.330	.309	.267	.228	.190	.153	.082	.016	-.046	-.106	-.163
-.5	.295	.291	.287	.279	.271	.264	.256	.237	.218	.182	.146	.112	.079	.015	-.046	-.103	-.159	-.211
0.0	.207	.203	.200	.193	.186	.179	.172	.155	.138	.105	.073	.042	.012	-.047	-.103	-.157	-.208	-.257
.5	.128	.125	.122	.116	.109	.103	.097	.081	.066	.036	.006	-.022	-.050	-.104	-.156	-.206	-.254	-.301
1.0	.057	.054	.052	.046	.040	.034	.028	.014	.000	-.028	-.055	-.081	-.107	-.158	-.206	-.253	-.299	-.342
1.5	-.008	-.010	-.013	-.018	-.024	-.029	-.034	-.048	-.061	-.086	-.112	-.136	-.161	-.208	-.254	-.298	-.340	-.382
2.0	-.067	-.070	-.072	-.077	-.082	-.087	-.092	-.105	-.117	-.141	-.164	-.188	-.210	-.255	-.298	-.340	-.380	-.420
2.5	-.123	-.125	-.128	-.132	-.137	-.142	-.146	-.158	-.169	-.192	-.214	-.236	-.257	-.299	-.340	-.380	-.418	-.456
3.0	-.175	-.177	-.179	-.184	-.188	-.192	-.197	-.208	-.218	-.240	-.261	-.281	-.302	-.342	-.380	-.418	-.455	-.491
3.5	-.224	-.226	-.228	-.232	-.236	-.240	-.244	-.255	-.265	-.285	-.305	-.324	-.344	-.382	-.419	-.455	-.490	-.524
4.0	-.269	-.271	-.273	-.277	-.281	-.285	-.289	-.299	-.309	-.328	-.347	-.365	-.384	-.420	-.455	-.490	-.523	-.556
4.5	-.313	-.315	-.317	-.321	-.324	-.328	-.332	-.341	-.350	-.368	-.387	-.404	-.422	-.456	-.490	-.523	-.556	-.587
5.0	-.354	-.356	-.358	-.361	-.365	-.369	-.372	-.381	-.390	-.407	-.424	-.442	-.458	-.491	-.524	-.556	-.587	-.617
5.5	-.394	-.395	-.397	-.400	-.404	-.407	-.411	-.419	-.428	-.444	-.461	-.477	-.493	-.525	-.556	-.587	-.617	-.646
6.0	-.431	-.433	-.434	-.438	-.441	-.444	-.448	-.456	-.464	-.480	-.496	-.511	-.527	-.557	-.587	-.617	-.646	-.674

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

Values of Δm for $l = 0.250$ and σ_o of –

q_o	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0								1.344	1.175	.955	.796	.667	.557	.373	.220	.088	-.030	-.136
-2.5	1.164	1.142	1.121	1.081	1.044	1.008	.975	.897	.828	.706	.600	.506	.421	.271	.141	.025	-.080	-.176
-2.0	.839	.827	.815	.791	.768	.746	.725	.673	.625	.535	.453	.377	.307	.180	.066	-.037	-.132	-.220
-1.5	.637	.628	.619	.601	.584	.568	.551	.511	.473	.400	.333	.269	.209	.098	-.003	-.096	-.182	-.263
-1.0	.485	.477	.470	.456	.442	.428	.415	.382	.350	.288	.230	.175	.122	.024	-.067	-.152	-.231	-.306
-.5	.361	.355	.349	.337	.325	.313	.302	.273	.245	.192	.141	.092	.045	-.044	-.127	-.205	-.278	-.348
0.0	.256	.250	.245	.235	.224	.214	.204	.179	.154	.107	.061	.017	-.026	-.107	-.183	-.255	-.324	-.389
.5	.164	.159	.155	.145	.136	.127	.118	.095	.073	.030	-.011	-.052	-.091	-.165	-.236	-.303	-.367	-.428
1.0	.082	.078	.074	.066	.057	.049	.041	.020	.000	-.039	-.077	-.114	-.150	-.220	-.285	-.348	-.408	-.466
1.5	.009	.005	.001	-.007	-.014	-.022	-.030	-.048	-.067	-.103	-.138	-.173	-.206	-.271	-.332	-.391	-.448	-.503
2.0	-.059	-.062	-.066	-.073	-.080	-.087	-.094	-.111	-.128	-.162	-.195	-.227	-.258	-.319	-.377	-.432	-.486	-.538
2.5	-.121	-.124	-.127	-.134	-.140	-.147	-.153	-.170	-.186	-.217	-.248	-.278	-.307	-.364	-.419	-.472	-.523	-.572
3.0	-.178	-.181	-.184	-.190	-.197	-.203	-.209	-.224	-.239	-.268	-.297	-.325	-.353	-.407	-.459	-.509	-.558	-.605
3.5	-.232	-.235	-.238	-.243	-.249	-.255	-.261	-.275	-.289	-.317	-.344	-.371	-.397	-.448	-.498	-.545	-.592	-.637
4.0	-.282	-.285	-.288	-.293	-.298	-.304	-.309	-.323	-.336	-.362	-.388	-.414	-.438	-.487	-.534	-.580	-.625	-.668
4.5	-.329	-.332	-.335	-.340	-.345	-.350	-.355	-.368	-.381	-.406	-.430	-.454	-.478	-.525	-.570	-.613	-.656	-.698
5.0	-.374	-.377	-.379	-.384	-.389	-.394	-.399	-.411	-.423	-.447	-.470	-.493	-.516	-.560	-.604	-.646	-.687	-.727
5.5	-.417	-.419	-.422	-.426	-.431	-.436	-.440	-.452	-.463	-.486	-.508	-.530	-.552	-.595	-.636	-.677	-.716	-.755
6.0	-.457	-.460	-.462	-.466	-.471	-.475	-.480	-.491	-.502	-.524	-.545	-.566	-.587	-.628	-.668	-.707	-.745	-.782

TABLE I.- Δm AS A FUNCTION OF σ_0, q_0, AND z – Continued

Values of Δm for $l = 0.300$ and σ_0 of –

q_0	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0										1.034	.789	.612	.469	.238	.053	-.104	-.242	-.365
-2.5				1.500	1.377	1.286	1.210	1.059	.940	.750	.598	.470	.358	.166	.004	-.137	-.263	-.377
-2.0	1.035	1.013	.992	.952	.914	.878	.844	.763	.691	.561	.447	.346	.254	.090	-.052	-.179	-.294	-.400
-1.5	.758	.744	.730	.704	.678	.653	.629	.571	.516	.415	.323	.239	.160	.018	-.109	-.224	-.330	-.428
-1.0	.570	.559	.549	.529	.509	.489	.470	.423	.379	.295	.217	.144	.076	-.050	-.165	-.270	-.368	-.459
-.5	.424	.415	.407	.390	.374	.357	.341	.302	.264	.192	.124	.060	-.001	-.114	-.219	-.316	-.406	-.491
0.0	.303	.296	.289	.275	.260	.246	.232	.199	.165	.102	.042	-.016	-.071	-.174	-.270	-.360	-.444	-.524
.5	.200	.194	.188	.175	.162	.150	.138	.108	.078	.021	-.033	-.085	-.135	-.230	-.319	-.403	-.482	-.557
1.0	.110	.104	.098	.087	.076	.065	.054	.027	.000	-.052	-.101	-.149	-.195	-.283	-.366	-.444	-.518	-.589
1.5	.029	.024	.018	.008	-.002	-.012	-.022	-.047	-.071	-.118	-.164	-.208	-.251	-.333	-.410	-.484	-.554	-.621
2.0	-.045	-.049	-.054	-.064	-.073	-.082	-.091	-.114	-.136	-.180	-.222	-.263	-.303	-.380	-.453	-.522	-.589	-.653
2.5	-.112	-.116	-.121	-.129	-.138	-.146	-.155	-.176	-.197	-.237	-.277	-.315	-.353	-.425	-.493	-.559	-.623	-.683
3.0	-.174	-.178	-.182	-.190	-.198	-.206	-.214	-.233	-.253	-.291	-.328	-.364	-.399	-.467	-.532	-.595	-.655	-.713
3.5	-.231	-.235	-.239	-.246	-.254	-.261	-.269	-.287	-.305	-.341	-.376	-.410	-.443	-.508	-.570	-.629	-.687	-.742
4.0	-.285	-.289	-.292	-.299	-.306	-.313	-.320	-.338	-.355	-.389	-.421	-.454	-.485	-.547	-.606	-.663	-.718	-.771
4.5	-.336	-.339	-.342	-.349	-.356	-.362	-.369	-.385	-.402	-.433	-.465	-.495	-.525	-.584	-.640	-.695	-.747	-.799
5.0	-.383	-.386	-.390	-.396	-.402	-.409	-.415	-.430	-.446	-.476	-.506	-.535	-.564	-.620	-.673	-.726	-.776	-.826
5.5	-.428	-.431	-.434	-.440	-.446	-.452	-.458	-.473	-.488	-.517	-.545	-.573	-.600	-.654	-.706	-.756	-.805	-.852
6.0	-.471	-.474	-.477	-.483	-.488	-.494	-.500	-.514	-.528	-.555	-.583	-.609	-.636	-.687	-.737	-.785	-.832	-.878

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

Values of Δm for $l = 0.400$ and σ_o of –

q_o	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0											.421	.265	.111	-.151	-.367	-.551	-.712	-.857
-2.5								1.279	1.012	.688	.462	.282	.130	-.122	-.328	-.505	-.661	-.802
-2.0	1.611	1.509	1.430	1.303	1.202	1.116	1.040	.880	.748	.534	.360	.212	.082	-.142	-.330	-.495	-.642	-.775
-1.5	.999	.970	.943	.891	.841	.795	.751	.649	.557	.395	.255	.131	.018	-.179	-.351	-.503	-.640	-.766
-1.0	.731	.712	.693	.657	.622	.589	.556	.479	.407	.276	.158	.051	-.047	-.224	-.380	-.521	-.650	-.768
-.5	.543	.528	.514	.486	.458	.432	.406	.343	.283	.172	.071	-.023	-.111	-.271	-.414	-.545	-.665	-.777
0.0	.396	.384	.372	.349	.326	.303	.281	.228	.177	.080	-.009	-.093	-.172	-.318	-.451	-.572	-.685	-.790
.5	.274	.263	.253	.233	.214	.194	.175	.128	.083	-.002	-.083	-.158	-.230	-.364	-.487	-.601	-.707	-.807
1.0	.169	.160	.151	.133	.116	.099	.082	.040	.000	-.077	-.150	-.219	-.285	-.410	-.524	-.631	-.731	-.826
1.5	.076	.068	.060	.045	.029	.014	-.002	-.039	-.076	-.146	-.212	-.276	-.338	-.453	-.561	-.661	-.756	-.846
2.0	-.006	-.014	-.021	-.035	-.049	-.063	-.077	-.111	-.145	-.209	-.271	-.330	-.387	-.495	-.596	-.691	-.782	-.867
2.5	-.081	-.088	-.094	-.108	-.121	-.133	-.146	-.178	-.208	-.268	-.325	-.381	-.434	-.536	-.631	-.721	-.807	-.889
3.0	-.150	-.156	-.162	-.174	-.186	-.198	-.210	-.239	-.268	-.323	-.377	-.428	-.479	-.575	-.665	-.751	-.833	-.911
3.5	-.213	-.219	-.224	-.236	-.247	-.258	-.269	-.296	-.323	-.375	-.425	-.474	-.521	-.612	-.698	-.780	-.858	-.933
4.0	-.272	-.277	-.283	-.293	-.304	-.314	-.324	-.350	-.375	-.424	-.471	-.517	-.562	-.648	-.730	-.809	-.883	-.955
4.5	-.327	-.332	-.337	-.347	-.357	-.366	-.376	-.400	-.424	-.470	-.515	-.559	-.601	-.683	-.761	-.836	-.908	-.978
5.0	-.379	-.383	-.388	-.397	-.407	-.416	-.425	-.448	-.470	-.514	-.556	-.598	-.639	-.717	-.792	-.864	-.933	-.999
5.5	-.427	-.432	-.436	-.445	-.454	-.462	-.471	-.493	-.514	-.556	-.596	-.636	-.675	-.749	-.821	-.890	-.957	-1.021
6.0	-.473	-.478	-.482	-.490	-.499	-.507	-.515	-.536	-.556	-.595	-.634	-.672	-.709	-.781	-.850	-.917	-.981	-1.043

TABLE I.- Δm AS A FUNCTION OF σ_0, q_0, AND z – Continued

Values of Δm for $l = 0.500$ and σ_0 of –

q_0	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0											-.377	-.288	-.380	-.611	-.827	-1.019	-1.191	-1.347
-2.5									.648	.437	.197	-.002	-.172	-.454	-.686	-.885	-1.062	-1.220
-2.0					1.494	1.292	1.151	.898	.709	.424	.204	.021	-.137	-.403	-.625	-.817	-.988	-1.142
-1.5	1.246	1.192	1.142	1.051	.970	.895	.827	.676	.546	.327	.144	-.014	-.154	-.396	-.602	-.784	-.946	-1.094
-1.0	.880	.850	.821	.766	.713	.663	.615	.505	.404	.227	.072	-.065	-.190	-.410	-.601	-.771	-.924	-1.065
-.5	.654	.632	.611	.569	.529	.491	.454	.365	.283	.133	-.000	-.122	-.234	-.434	-.611	-.770	-.915	-1.049
0.0	.485	.467	.450	.417	.384	.352	.321	.247	.177	.048	-.070	-.179	-.280	-.464	-.628	-.777	-.914	-1.042
.5	.347	.333	.318	.290	.263	.235	.209	.145	.084	-.031	-.136	-.235	-.327	-.497	-.650	-.790	-.920	-1.041
1.0	.231	.219	.206	.182	.158	.134	.111	.054	.000	-.103	-.199	-.289	-.374	-.531	-.674	-.807	-.930	-1.045
1.5	.131	.120	.109	.087	.065	.044	.023	-.027	-.076	-.169	-.257	-.340	-.419	-.566	-.700	-.825	-.942	-1.052
2.0	.041	.031	.021	.002	-.018	-.037	-.055	-.101	-.146	-.231	-.312	-.389	-.463	-.600	-.727	-.846	-.957	-1.062
2.5	-.039	-.048	-.057	-.075	-.093	-.110	-.127	-.170	-.211	-.289	-.365	-.436	-.505	-.634	-.755	-.867	-.973	-1.074
3.0	-.113	-.121	-.129	-.145	-.162	-.178	-.194	-.233	-.271	-.344	-.414	-.481	-.546	-.668	-.782	-.889	-.991	-1.087
3.5	-.180	-.188	-.195	-.210	-.225	-.240	-.255	-.291	-.327	-.395	-.461	-.524	-.585	-.701	-.809	-.912	-1.009	-1.101
4.0	-.242	-.249	-.256	-.271	-.285	-.298	-.312	-.346	-.379	-.444	-.506	-.565	-.623	-.733	-.837	-.935	-1.028	-1.117
4.5	-.300	-.307	-.314	-.327	-.340	-.353	-.366	-.398	-.429	-.490	-.548	-.605	-.660	-.764	-.863	-.957	-1.047	-1.132
5.0	-.355	-.361	-.367	-.380	-.392	-.404	-.416	-.446	-.476	-.533	-.589	-.643	-.695	-.795	-.890	-.980	-1.066	-1.149
5.5	-.406	-.412	-.418	-.429	-.441	-.453	-.464	-.493	-.520	-.575	-.628	-.679	-.729	-.825	-.916	-1.002	-1.085	-1.165
6.0	-.454	-.460	-.465	-.477	-.488	-.499	-.509	-.536	-.563	-.615	-.665	-.714	-.762	-.854	-.941	-1.025	-1.105	-1.182

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

q_o	Values of Δm for $l = 0.750$ and σ_o of –																	
	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0											-3.917	-2.035	-1.785	-1.812	-1.973	-2.153	-2.331	-2.502
-2.5									-5.257	-.624	-.680	-.835	-.997	-1.295	-1.556	-1.787	-1.995	-2.185
-2.0							.646	.538	.323	-.026	-.300	-.527	-.724	-1.054	-1.330	-1.570	-1.783	-1.977
-1.5		1.781	1.578	1.316	1.129	.978	.850	.591	.384	.058	-.201	-.418	-.607	-.928	-1.197	-1.432	-1.641	-1.831
-1.0	1.215	1.145	1.081	.963	.859	.764	.677	.485	.320	.044	-.185	-.382	-.557	-.858	-1.115	-1.340	-1.542	-1.727
-.5	.903	.859	.816	.736	.661	.591	.525	.373	.237	.001	-.202	-.380	-.541	-.821	-1.064	-1.278	-1.472	-1.650
0.0	.692	.658	.626	.564	.505	.449	.395	.269	.153	-.052	-.233	-.395	-.543	-.804	-1.032	-1.236	-1.422	-1.592
.5	.528	.502	.475	.424	.376	.328	.283	.175	.074	-.108	-.271	-.419	-.555	-.799	-1.015	-1.208	-1.386	-1.549
1.0	.394	.372	.349	.306	.264	.223	.184	.089	.000	-.164	-.312	-.448	-.574	-.803	-1.006	-1.191	-1.360	-1.517
1.5	.280	.260	.241	.203	.166	.130	.095	.011	-.069	-.218	-.354	-.480	-.598	-.812	-1.005	-1.180	-1.342	-1.493
2.0	.179	.162	.145	.111	.078	.046	.015	-.061	-.134	-.270	-.396	-.513	-.623	-.825	-1.008	-1.175	-1.330	-1.475
2.5	.090	.075	.059	.029	-.001	-.030	-.059	-.128	-.195	-.320	-.437	-.547	-.650	-.841	-1.015	-1.175	-1.323	-1.463
3.0	.010	-.005	-.019	-.046	-.073	-.100	-.126	-.190	-.252	-.368	-.477	-.580	-.678	-.859	-1.024	-1.177	-1.320	-1.454
3.5	-.064	-.077	-.090	-.115	-.140	-.165	-.189	-.248	-.305	-.414	-.517	-.614	-.706	-.878	-1.036	-1.183	-1.320	-1.449
4.0	-.132	-.144	-.156	-.179	-.202	-.225	-.248	-.303	-.356	-.458	-.555	-.646	-.734	-.898	-1.049	-1.190	-1.322	-1.447
4.5	-.194	-.206	-.217	-.239	-.260	-.282	-.303	-.354	-.404	-.500	-.592	-.679	-.762	-.918	-1.063	-1.199	-1.326	-1.447
5.0	-.253	-.263	-.274	-.294	-.315	-.335	-.354	-.403	-.450	-.541	-.628	-.710	-.789	-.939	-1.078	-1.209	-1.332	-1.449
5.5	-.308	-.318	-.327	-.347	-.366	-.385	-.403	-.449	-.494	-.580	-.662	-.741	-.817	-.960	-1.094	-1.220	-1.339	-1.452
6.0	-.360	-.369	-.378	-.396	-.414	-.432	-.450	-.493	-.535	-.617	-.696	-.771	-.844	-.981	-1.110	-1.232	-1.347	-1.456

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

Values of Δm for l = 1.000 and σ_o of –

q_o	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0											-4.282	-4.483	-3.393	-3.044	-3.105	-3.254	-3.428	-3.609
-2.5									-1.704	-1.928	-1.622	-1.679	-1.803	-2.085	-2.357	-2.610	-2.843	-3.061
-2.0							-1.388	-.129	-.227	-.539	-.820	-1.064	-1.279	-1.649	-1.962	-2.237	-2.485	-2.711
-1.5				1.348	1.084	.884	.719	.395	.143	-.247	-.553	-.808	-1.030	-1.405	-1.721	-1.996	-2.243	-2.468
-1.0	1.505	1.379	1.269	1.082	.924	.787	.665	.405	.189	-.160	-.444	-.684	-.895	-1.256	-1.562	-1.830	-2.071	-2.290
-.5	1.121	1.050	.983	.860	.749	.648	.554	.344	.163	-.144	-.400	-.622	-.819	-1.161	-1.453	-1.711	-1.943	-2.156
0.0	.880	.830	.781	.688	.602	.521	.444	.270	.114	-.157	-.389	-.593	-.777	-1.098	-1.376	-1.623	-1.847	-2.052
.5	.701	.661	.622	.547	.476	.408	.344	.194	.058	-.184	-.395	-.584	-.755	-1.058	-1.322	-1.558	-1.772	-1.970
1.0	.555	.522	.490	.427	.366	.308	.252	.121	.000	-.218	-.412	-.586	-.746	-1.032	-1.283	-1.508	-1.714	-1.905
1.5	.433	.404	.376	.322	.269	.218	.169	.052	-.057	-.256	-.434	-.597	-.746	-1.016	-1.255	-1.471	-1.669	-1.852
2.0	.327	.301	.277	.229	.182	.136	.092	-.013	-.112	-.295	-.460	-.612	-.753	-1.008	-1.236	-1.443	-1.633	-1.810
2.5	.232	.210	.188	.145	.102	.061	.021	-.075	-.165	-.334	-.488	-.631	-.764	-1.006	-1.223	-1.422	-1.605	-1.776
3.0	.148	.127	.107	.068	.030	-.008	-.044	-.132	-.216	-.373	-.518	-.652	-.777	-1.008	-1.216	-1.406	-1.583	-1.748
3.5	.071	.052	.034	-.002	-.037	-.072	-.105	-.187	-.265	-.412	-.547	-.674	-.793	-1.013	-1.212	-1.396	-1.566	-1.725
4.0	.000	-.017	-.034	-.067	-.100	-.131	-.163	-.239	-.312	-.449	-.577	-.697	-.811	-1.021	-1.212	-1.388	-1.553	-1.707
4.5	-.065	-.081	-.097	-.128	-.158	-.188	-.217	-.288	-.356	-.486	-.607	-.721	-.829	-1.030	-1.214	-1.384	-1.543	-1.692
5.0	-.126	-.141	-.155	-.184	-.212	-.240	-.268	-.335	-.399	-.522	-.637	-.745	-.849	-1.041	-1.218	-1.382	-1.536	-1.681
5.5	-.183	-.197	-.210	-.237	-.264	-.290	-.316	-.379	-.440	-.556	-.666	-.770	-.869	-1.053	-1.224	-1.383	-1.531	-1.672
6.0	-.236	-.249	-.262	-.288	-.313	-.338	-.362	-.422	-.479	-.590	-.695	-.794	-.889	-1.066	-1.231	-1.385	-1.529	-1.665

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

Values of Δm for $l = 1.500$ and σ_o of –

q_o	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0											-2.220	-4.640	-8.610	-6.505	-5.893	-5.864	-6.011	-6.236
-2.5									-.779	-6.369	-3.704	-3.396	-3.389	-3.599	-3.885	-4.185	-4.484	-4.780
-2.0							-2.443	-1.716	-1.384	-1.519	-1.774	-2.028	-2.265	-2.694	-3.073	-3.418	-3.738	-4.039
-1.5				.820	.674	.480	.303	-.062	-.355	-.815	-1.180	-1.487	-1.755	-2.215	-2.607	-2.956	-3.273	-3.568
-1.0	1.990	1.710	1.499	1.179	.933	.730	.556	.199	-.086	-.538	-.898	-1.200	-1.465	-1.918	-2.303	-2.643	-2.952	-3.237
-.5	1.491	1.358	1.239	1.032	.854	.698	.557	.256	.004	-.408	-.744	-1.030	-1.283	-1.717	-2.089	-2.417	-2.715	-2.989
0.0	1.215	1.125	1.041	.887	.748	.522	.506	.249	.028	-.345	-.655	-.923	-1.162	-1.576	-1.931	-2.247	-2.533	-2.798
.5	1.017	.947	.882	.759	.645	.540	.441	.218	.021	-.316	-.603	-.854	-1.079	-1.472	-1.812	-2.115	-2.391	-2.645
1.0	.859	.803	.749	.646	.549	.459	.373	.176	.000	-.308	-.574	-.809	-1.021	-1.395	-1.720	-2.011	-2.276	-2.521
1.5	.728	.680	.634	.546	.462	.382	.306	.130	-.029	-.313	-.560	-.780	-.981	-1.336	-1.648	-1.927	-2.182	-2.419
2.0	.615	.573	.533	.455	.381	.310	.242	.083	-.063	-.325	-.555	-.763	-.953	-1.292	-1.590	-1.859	-2.105	-2.333
2.5	.515	.479	.443	.374	.307	.243	.181	.036	-.098	-.342	-.558	-.755	-.935	-1.258	-1.544	-1.803	-2.040	-2.261
3.0	.426	.393	.361	.299	.238	.180	.123	-.010	-.135	-.362	-.566	-.752	-.923	-1.232	-1.507	-1.757	-1.986	-2.200
3.5	.346	.316	.287	.229	.174	.120	.068	-.055	-.171	-.385	-.577	-.754	-.917	-1.213	-1.478	-1.718	-1.940	-2.147
4.0	.272	.245	.218	.165	.114	.064	.016	-.099	-.208	-.408	-.591	-.759	-.915	-1.199	-1.454	-1.686	-1.901	-2.102
4.5	.204	.179	.154	.105	.058	.012	-.033	-.141	-.244	-.433	-.606	-.767	-.916	-1.189	-1.435	-1.660	-1.868	-2.063
5.0	.141	.117	.094	.049	.005	-.039	-.081	-.182	-.279	-.458	-.623	-.776	-.920	-1.183	-1.420	-1.638	-1.840	-2.029
5.5	.082	.060	.038	-.004	-.046	-.086	-.126	-.222	-.313	-.484	-.641	-.788	-.926	-1.179	-1.408	-1.619	-1.816	-2.000
6.0	.027	.006	-.014	-.054	-.093	-.132	-.169	-.260	-.346	-.509	-.660	-.800	-.933	-1.177	-1.399	-1.604	-1.795	-1.974

TABLE I.- Δm AS A FUNCTION OF σ_o, q_o, AND z – Continued

q_o	Values of Δm for $l = 2.000$ and σ_o of –																	
	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0											-1.859	-3.266	-4.509	-6.636	-8.375	-9.539	-9.925	-9.743
-2.5									-.778	-4.659	-7.637	-5.643	-5.280	-5.319	-5.630	-6.035	-6.494	-7.006
-2.0							-1.075	-3.759	-2.573	-2.439	-2.646	-2.900	-3.157	-3.648	-4.108	-4.546	-4.970	-5.391
-1.5				-.041	.150	.024	-.131	-.494	-.802	-1.304	-1.710	-2.057	-2.365	-2.902	-3.371	-3.798	-4.197	-4.578
-1.0	.386	1.914	1.610	1.186	.878	.630	.421	-.001	-.334	-.858	-1.272	-1.622	-1.929	-2.459	-2.915	-3.324	-3.702	-4.057
-.5	1.801	1.601	1.430	1.146	.912	.711	.534	.163	-.140	-.630	-1.024	-1.359	-1.654	-2.163	-2.601	-2.992	-3.351	-3.686
0.0	1.505	1.373	1.254	1.041	.857	.692	.543	.221	-.051	-.500	-.869	-1.185	-1.466	-1.952	-2.371	-2.745	-3.087	-3.406
.5	1.296	1.197	1.104	.935	.782	.644	.516	.232	-.012	-.424	-.768	-1.066	-1.332	-1.795	-2.195	-2.553	-2.881	-3.186
1.0	1.133	1.053	.977	.835	.706	.586	.473	.221	.000	-.379	-.700	-.981	-1.233	-1.675	-2.058	-2.401	-2.715	-3.007
1.5	.997	.930	.865	.744	.631	.525	.426	.199	-.003	-.354	-.655	-.920	-1.159	-1.581	-1.948	-2.278	-2.580	-2.860
2.0	.880	.822	.766	.660	.560	.466	.376	.170	-.016	-.342	-.625	-.876	-1.103	-1.506	-1.859	-2.176	-2.467	-2.737
2.5	.778	.727	.678	.583	.493	.408	.326	.137	-.034	-.339	-.606	-.844	-1.061	-1.446	-1.786	-2.091	-2.372	-2.633
3.0	.687	.642	.597	.511	.430	.352	.277	.103	-.057	-.342	-.594	-.821	-1.028	-1.398	-1.725	-2.020	-2.291	-2.543
3.5	.605	.563	.523	.445	.370	.299	.230	.068	-.081	-.350	-.589	-.805	-1.003	-1.359	-1.674	-1.959	-2.221	-2.466
4.0	.530	.492	.455	.383	.314	.248	.184	.033	-.107	-.361	-.589	-.795	-.985	-1.327	-1.631	-1.907	-2.161	-2.398
4.5	.460	.425	.391	.325	.261	.199	.140	-.002	-.134	-.375	-.591	-.789	-.971	-1.301	-1.595	-1.862	-2.109	-2.339
5.0	.396	.364	.332	.270	.211	.153	.097	-.037	-.161	-.390	-.597	-.786	-.962	-1.280	-1.564	-1.823	-2.063	-2.287
5.5	.336	.306	.276	.219	.163	.108	.056	-.070	-.189	-.407	-.604	-.786	-.955	-1.262	-1.538	-1.790	-2.023	-2.241
6.0	.280	.252	.224	.170	.117	.066	.016	-.104	-.216	-.424	-.614	-.789	-.952	-1.249	-1.516	-1.761	-1.988	-2.200

TABLE I.- Δm AS A FUNCTION OF σ_0, q_0, AND z – Concluded

Values of Δm for $l = 2.500$ and σ_0 of –

q_0	0.00	0.50	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
-3.0											-1.759	-2.800	-3.674	-4.923	-5.685	-6.116	-6.334	-6.425
-2.5									-.907	-3.373	-6.065	-10.393	-9.556	-8.354	-8.813	-9.960	-12.787	-11.899
-2.0							-.720	-10.358	-3.946	-3.364	-3.501	-3.753	-4.034	-4.612	-5.194	-5.788	-6.416	-7.107
-1.5				-1.029	-.376	-.412	-.536	-.882	-1.198	-1.731	-2.174	-2.559	-2.906	-3.523	-4.080	-4.603	-5.113	-5.624
-1.0	2.720	2.041	1.661	1.160	.806	.525	.290	-.179	-.549	-1.129	-1.589	-1.980	-2.325	-2.926	-3.453	-3.936	-4.392	-4.835
-.5	2.069	1.800	1.580	1.227	.946	.710	.504	.079	-.265	-.815	-1.257	-1.632	-1.963	-2.538	-3.037	-3.490	-3.912	-4.315
0.0	1.761	1.587	1.433	1.167	.941	.744	.568	.192	-.120	-.630	-1.045	-1.401	-1.716	-2.263	-2.738	-3.166	-3.563	-3.937
.5	1.546	1.416	1.297	1.084	.897	.728	.575	.242	-.042	-.513	-.902	-1.239	-1.538	-2.059	-2.512	-2.919	-3.294	-3.647
1.0	1.379	1.275	1.177	.999	.839	.693	.558	.259	.000	-.437	-.802	-1.120	-1.405	-1.902	-2.334	-2.723	-3.081	-3.416
1.5	1.240	1.153	1.071	.918	.778	.649	.528	.257	.020	-.387	-.730	-1.032	-1.302	-1.777	-2.192	-2.564	-2.907	-3.227
2.0	1.122	1.047	.976	.842	.717	.601	.492	.245	.026	-.354	-.678	-.965	-1.223	-1.677	-2.075	-2.433	-2.762	-3.070
2.5	1.018	.953	.890	.770	.658	.553	.454	.226	.023	-.333	-.641	-.913	-1.160	-1.596	-1.978	-2.323	-2.640	-2.936
3.0	.926	.867	.811	.703	.601	.505	.414	.203	.013	-.322	-.613	-.873	-1.109	-1.528	-1.897	-2.230	-2.536	-2.821
3.5	.843	.790	.738	.640	.547	.458	.374	.178	-.000	-.317	-.594	-.843	-1.069	-1.472	-1.828	-2.150	-2.446	-2.722
4.0	.766	.718	.671	.581	.495	.413	.334	.151	-.017	-.317	-.581	-.819	-1.036	-1.425	-1.769	-2.081	-2.368	-2.635
4.5	.696	.652	.609	.525	.445	.369	.295	.123	-.035	-.320	-.573	-.801	-1.010	-1.386	-1.719	-2.021	-2.299	-2.559
5.0	.631	.590	.550	.472	.398	.326	.257	.095	-.055	-.327	-.568	-.788	-.990	-1.353	-1.675	-1.968	-2.239	-2.491
5.5	.571	.533	.495	.422	.353	.285	.220	.067	-.076	-.335	-.567	-.778	-.973	-1.324	-1.637	-1.922	-2.185	-2.431
6.0	.514	.478	.443	.375	.309	.246	.184	.038	-.097	-.345	-.568	-.772	-.960	-1.300	-1.604	-1.882	-2.138	-2.377

TABLE II. - MODELS OF THE UNIVERSE IN WHICH $\rho \geqq 0$ and $p = 0$

[Based on ref. 2]

Λ \ k	-1	0	+1	
<0	0	0	0	
$=0$	M_1	M_1	0	
>0	M_1	M_1	$<\Lambda_c$ $=\Lambda_c$ $>\Lambda_c$	0, M_2 A_1, E, A_2 M_1

0 oscillating models that expand from zero radius to some maximum value of R, R_m, and then contract to zero radius at which point the expansion cycle begins again.

M_1 a monotonically expanding model that expands from zero radius. Some of these models pass through an inflection point before expanding monotonically.

M_2 a model that starts by contracting to a minimum radius and then expands monotonically to a de Sitter state.

A_1 a model that expands from a singular state and asymptotes to the constant-radius Einstein model. This model is a limiting case and separates 0 models from M_1 models.

E constant-radius Einstein model.

A_2 Eddington LeMaitre models that expand asymptotically from the Einstein model to de Sitter model and separate the M_1 and M_2 models.

Λ_c that value of the cosmical constant that gives the Einstein constant-radius model of the universe.

TABLE III.- VALUES OF q_o FOR SPECIFIED VALUES OF τ AND σ_o

τ	q_o for σ_o of –																	
	0.00	0.05	0.10	0.20	0.30	0.40	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
0.6															5.790	4.785	3.965	3.145
.7										5.968	5.210	4.535	3.900	2.815	1.915	1.098	.371	-.292
.8				6.038	5.690	5.405	5.160	4.588	4.098	3.249	2.554	1.941	1.400	.440	-.366	-1.090	-1.722	-2.292
.9	5.010	4.679	4.445	4.069	3.753	3.468	3.223	2.713	2.254	1.499	.851	.285	-.194	-1.060	-1.788	-2.434	-3.004	-3.511
1.0	3.635	3.273	3.054	2.694	2.409	2.155	1.910	1.431	1.004	.296	-.290	-.809	-1.256	-2.044	-2.710	-3.293	-3.808	-4.277
1.1	2.635	2.288	2.070	1.725	1.440	1.202	.988	.509	.113	-.540	-1.087	-1.559	-1.975	-2.701	-3.319	-3.856	-4.332	-4.769
1.2	1.893	1.554	1.343	1.006	.737	.514	.301	-.139	-.512	-1.134	-1.649	-2.090	-2.483	-3.162	-3.733	-4.231	-4.683	-5.089
1.3	1.338	1.007	.797	.475	.213	-.009	-.207	-.631	-.981	-1.572	-2.056	-2.473	-2.842	-3.478	-4.019	-4.492	-4.914	-5.300
1.4	.908	.577	.375	.065	-.185	-.400	-.590	-.991	-1.332	-1.892	-2.352	-2.750	-3.100	-3.705	-4.218	-4.668	-5.074	-5.441
1.5	.573	.249	.047	-.252	-.497	-.700	-.887	-1.272	-1.598	-2.134	-2.575	-2.953	-3.289	-3.867	-4.358	-4.793	-5.181	-5.537
1.6	.307	-.017	-.211	-.505	-.740	-.939	-1.113	-1.491	-1.805	-2.318	-2.743	-3.106	-3.428	-3.984	-4.460	-4.879	-5.257	-5.602
1.7	.092	-.224	-.414	-.705	-.931	-1.126	-1.297	-1.659	-1.961	-2.460	-2.868	-3.221	-3.532	-4.070	-4.532	-4.940	-5.308	-5.647
1.8	-.080	-.395	-.582	-.865	-1.087	-1.275	-1.442	-1.793	-2.086	-2.570	-2.966	-3.307	-3.609	-4.133	-4.584	-4.983	-5.345	-5.677
1.9	-.220	-.534	-.719	-.994	-1.212	-1.396	-1.557	-1.901	-2.186	-2.654	-3.040	-3.372	-3.667	-4.180	-4.621	-5.013	-5.370	-5.698
2.0	-.336	-.649	-.830	-1.101	-1.313	-1.492	-1.651	-1.985	-2.264	-2.721	-3.097	-3.442	-3.711	-4.214	-4.648	-5.035	-5.388	-5.713
2.1	-.431	-.743	-.922	-1.189	-1.396	-1.572	-1.727	-2.053	-2.325	-2.773	-3.142	-3.461	-3.745	-4.240	-4.668	-5.051	-5.400	-5.723
2.4	-.636	-.946	-1.118	-1.372	-1.569	-1.736	-1.882	-2.191	-2.449	-2.874	-3.226	-3.531	-3.804	-4.283	-4.701	-5.076	-5.419	-5.738
2.5	-.685	-.993	-1.163	-1.414	-1.608	-1.773	-1.917	-2.222	-2.475	-2.895	-3.242	-3.545	-3.815	-4.291	-4.707	-5.080	-5.422	-5.740
2.6	-.726	-1.034	-1.202	-1.450	-1.642	-1.803	-1.946	-2.246	-2.497	-2.911	-3.256	-3.555	-3.824	-4.297	-4.711	-5.083	-5.424	-5.742
2.7	-.762	-1.068	-1.236	-1.480	-1.669	-1.829	-1.970	-2.267	-2.515	-2.925	-3.266	-3.564	-3.831	-4.301	-4.714	-5.085	-5.426	-5.743
2.8	-.792	-1.098	-1.263	-1.505	-1.693	-1.851	-1.990	-2.284	-2.529	-2.936	-3.274	-3.570	-3.836	-4.305	-4.716	-5.087	-5.427	-5.743
2.9	-.819	-1.123	-1.288	-1.527	-1.713	-1.869	-2.007	-2.298	-2.541	-2.944	-3.281	-3.575	-3.840	-4.307	-4.718	-5.088	-5.428	-5.744
3.0	-.841	-1.146	-1.309	-1.546	-1.730	-1.885	-2.021	-2.309	-2.551	-2.951	-3.286	-3.579	-3.843	-4.309	-4.719	-5.089	-5.428	-5.744

TABLE IV.- VALUES OF σ_o THAT GIVE GOOD CORRESPONDENCE BETWEEN EQUATIONS (8) AND (12) FOR VARIOUS VALUES OF q_o AND z

q_o	σ_o	z
-1.0	0	∞
-.8	0	0.2
-.5	0	.2
-.2	0	.2
0	0	∞
.35	.16	1.0
.60	.40	1.0
.74	.58	1.0
.86	.75	1.0
1.0	1.00	∞
1.85	3.00	.3

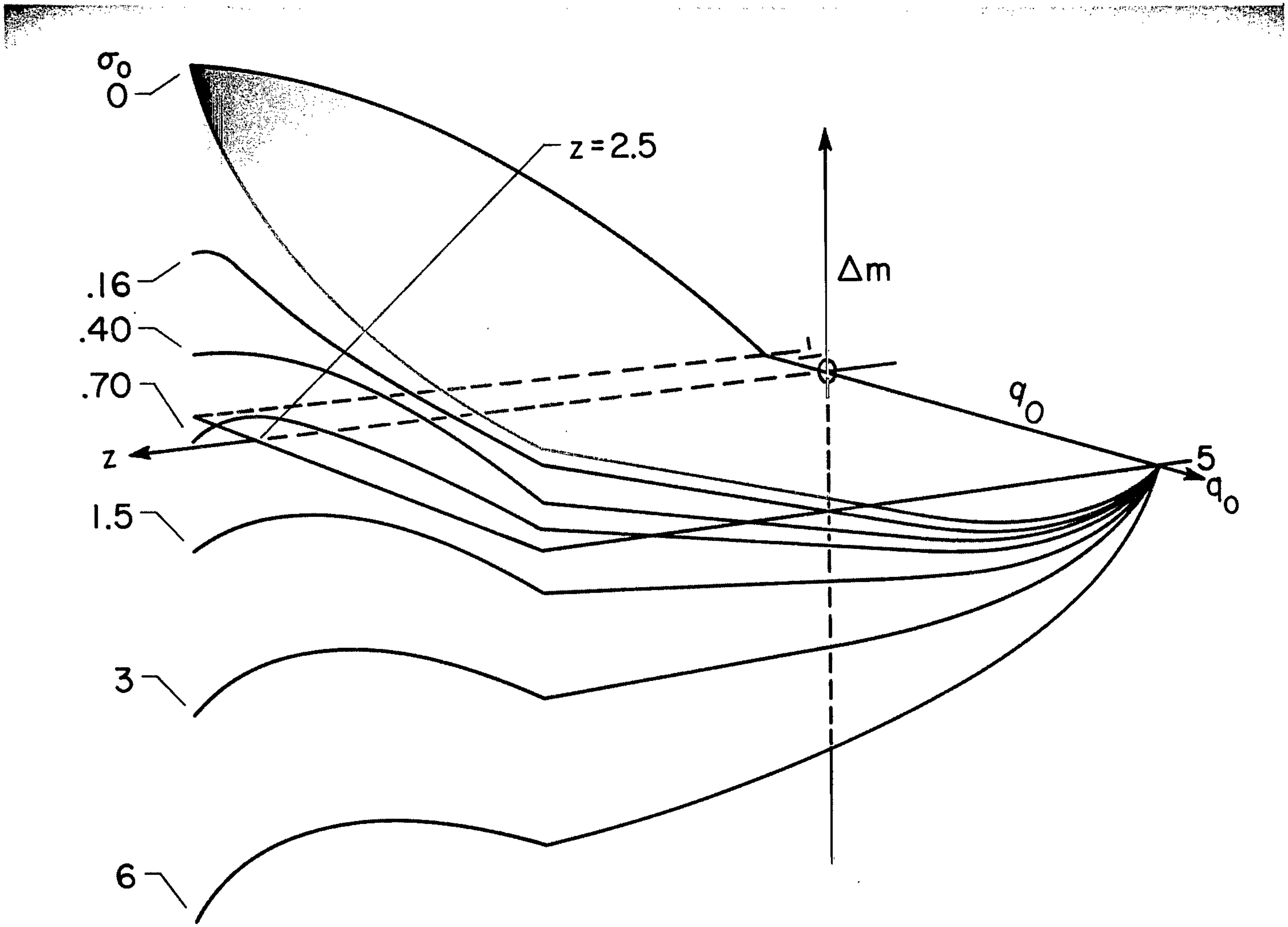


Figure 1.- The solid of model universes.

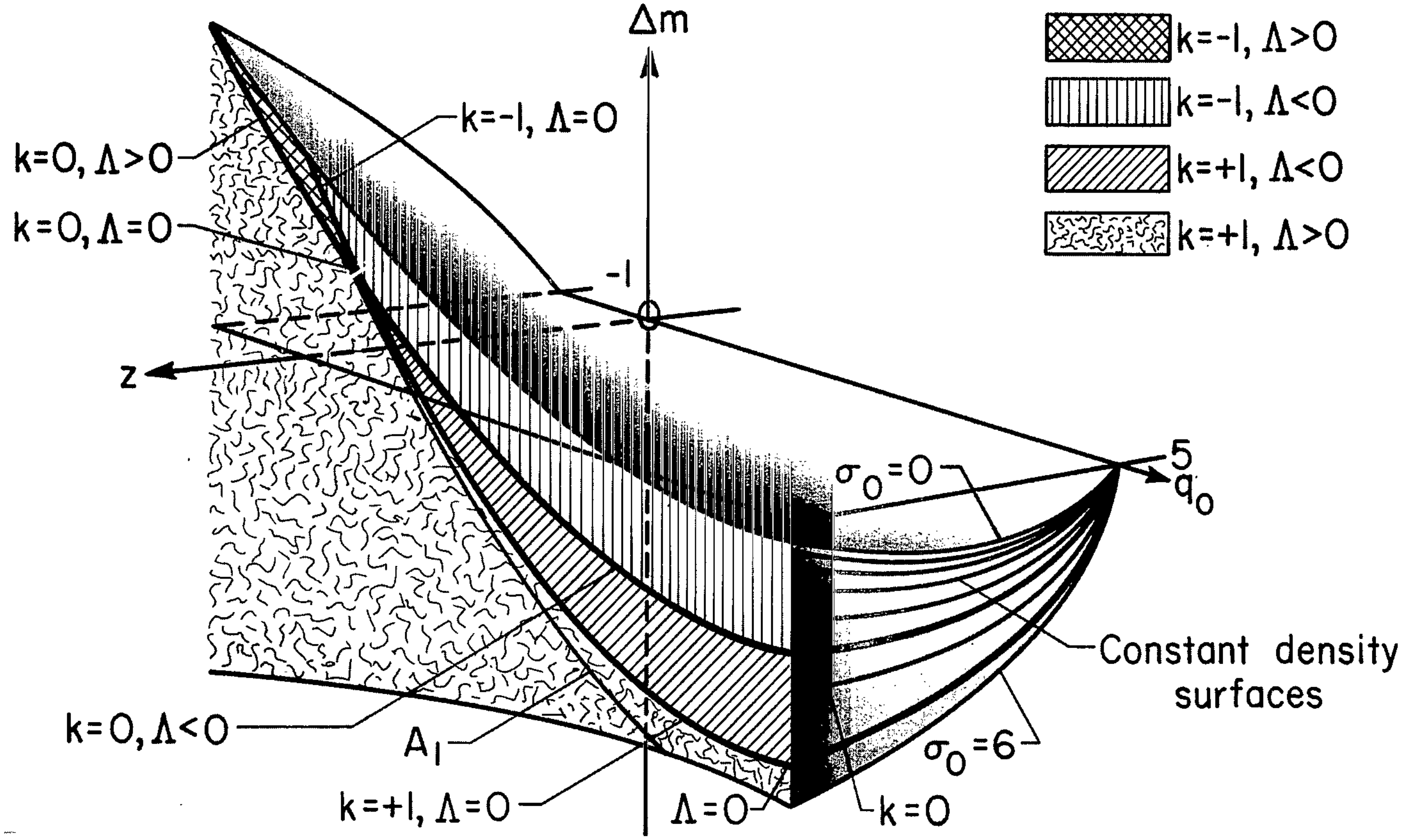


Figure 2.- The regions and surfaces for the various models of the universe.

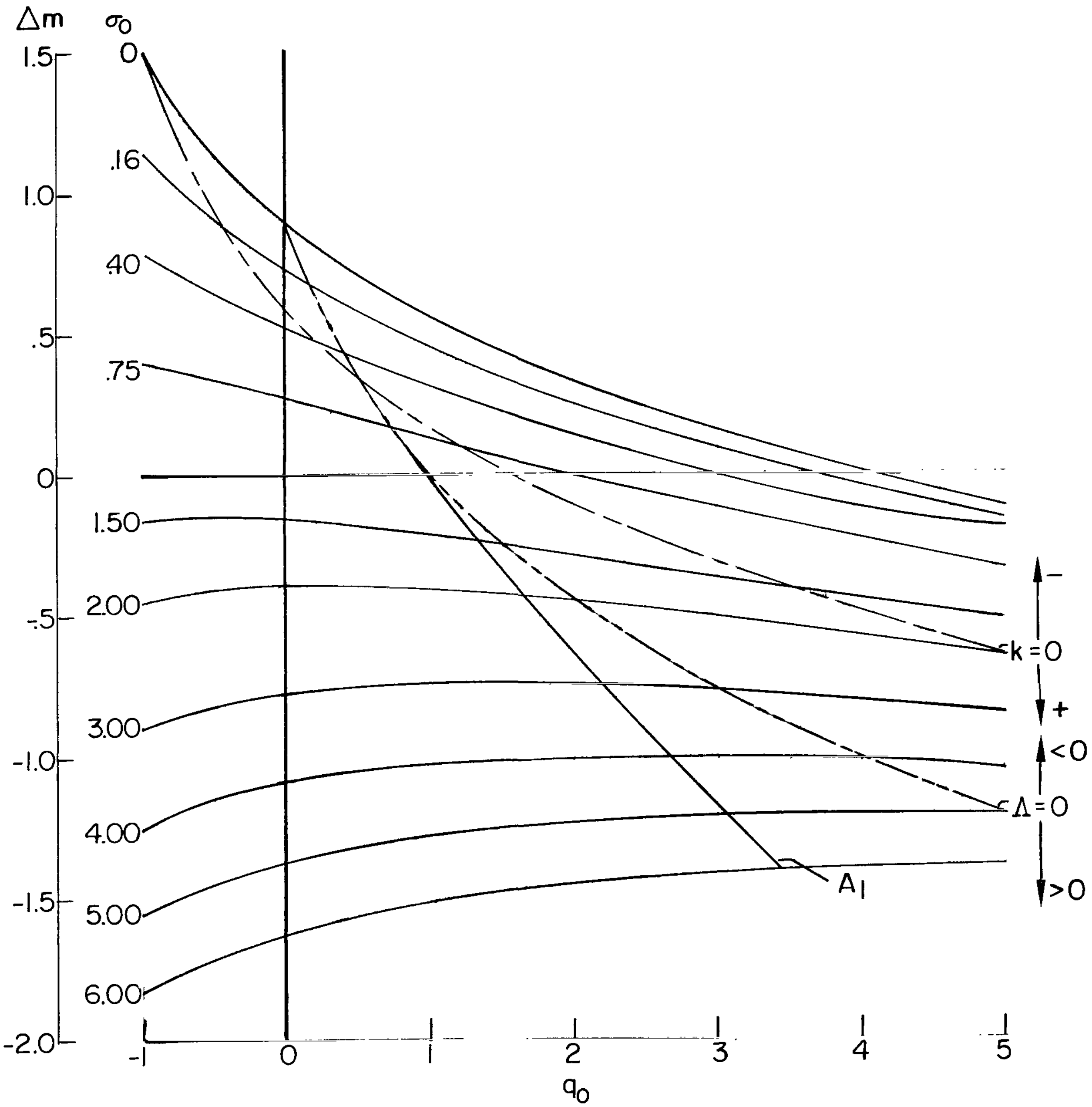


Figure 3.- Section of the solid of model universes at $z = 1.0$ with curves of constant σ_0 shown. The curve A_1 separates the M_1 expanding models from the oscillating models.

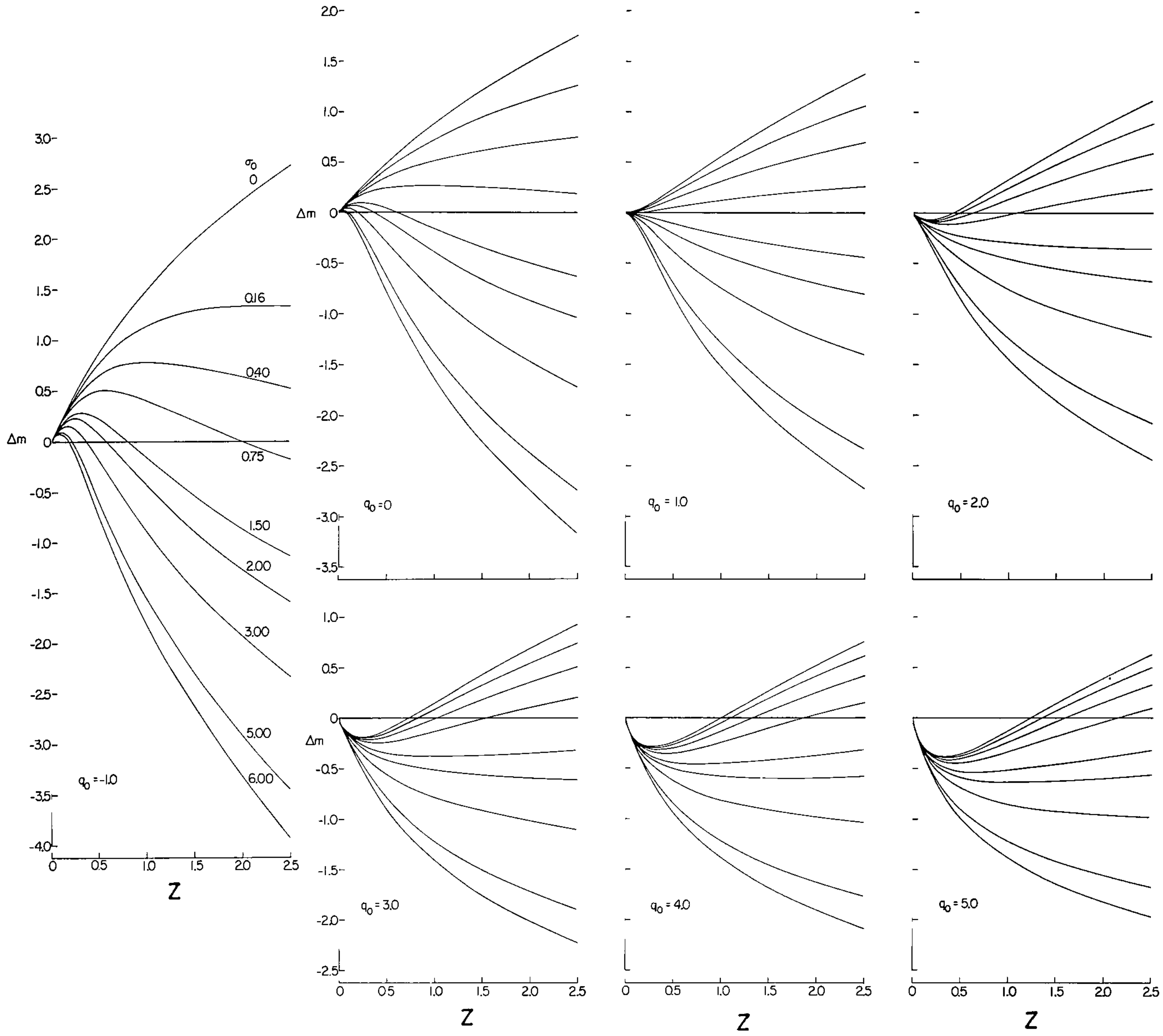


Figure 4.- Sections of the solid of model universes for constant values of q_0.

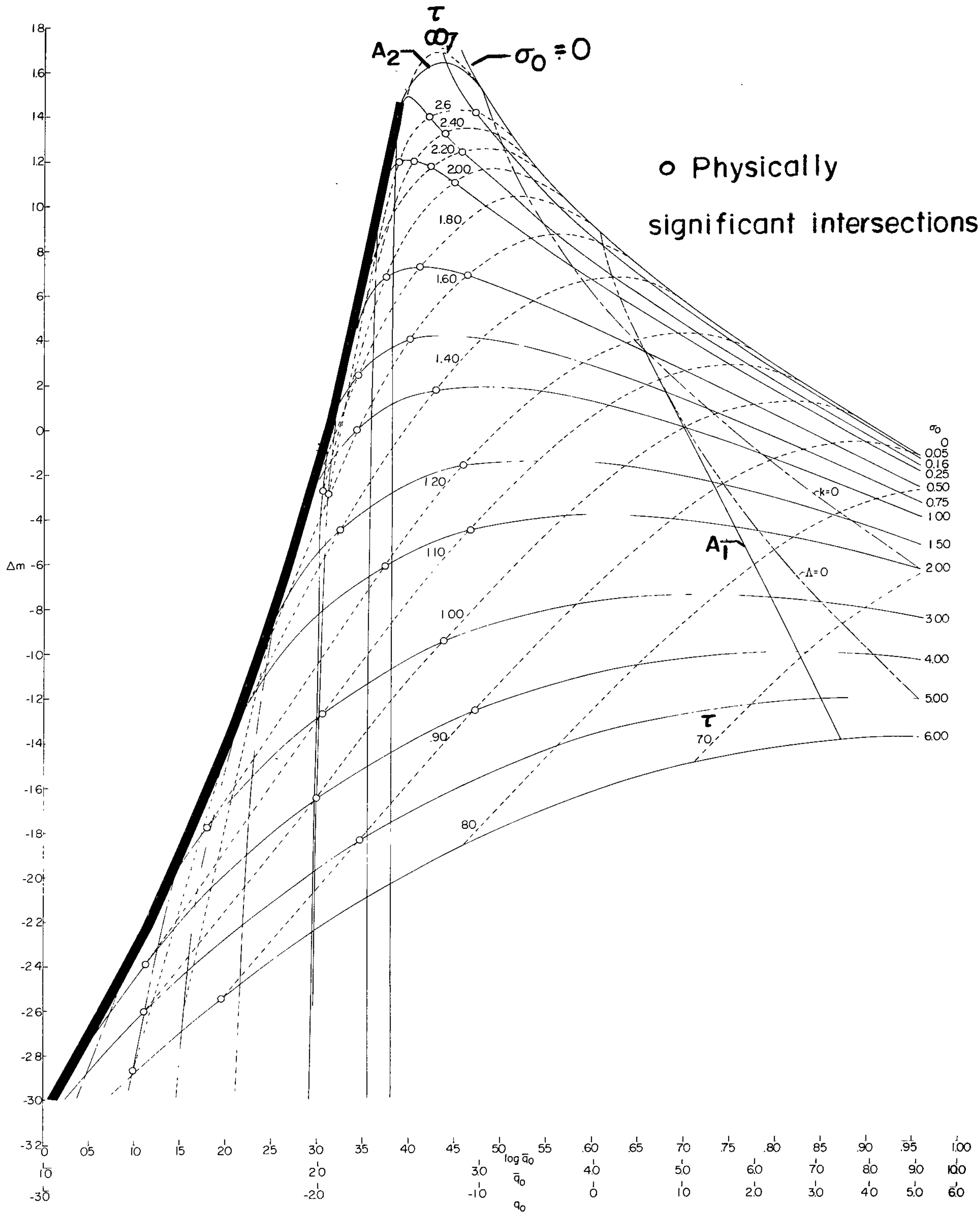


Figure 5.- The section for $z = 1.0$ of the solid of model universes with constant time curves added. This figure is independent of H_0.

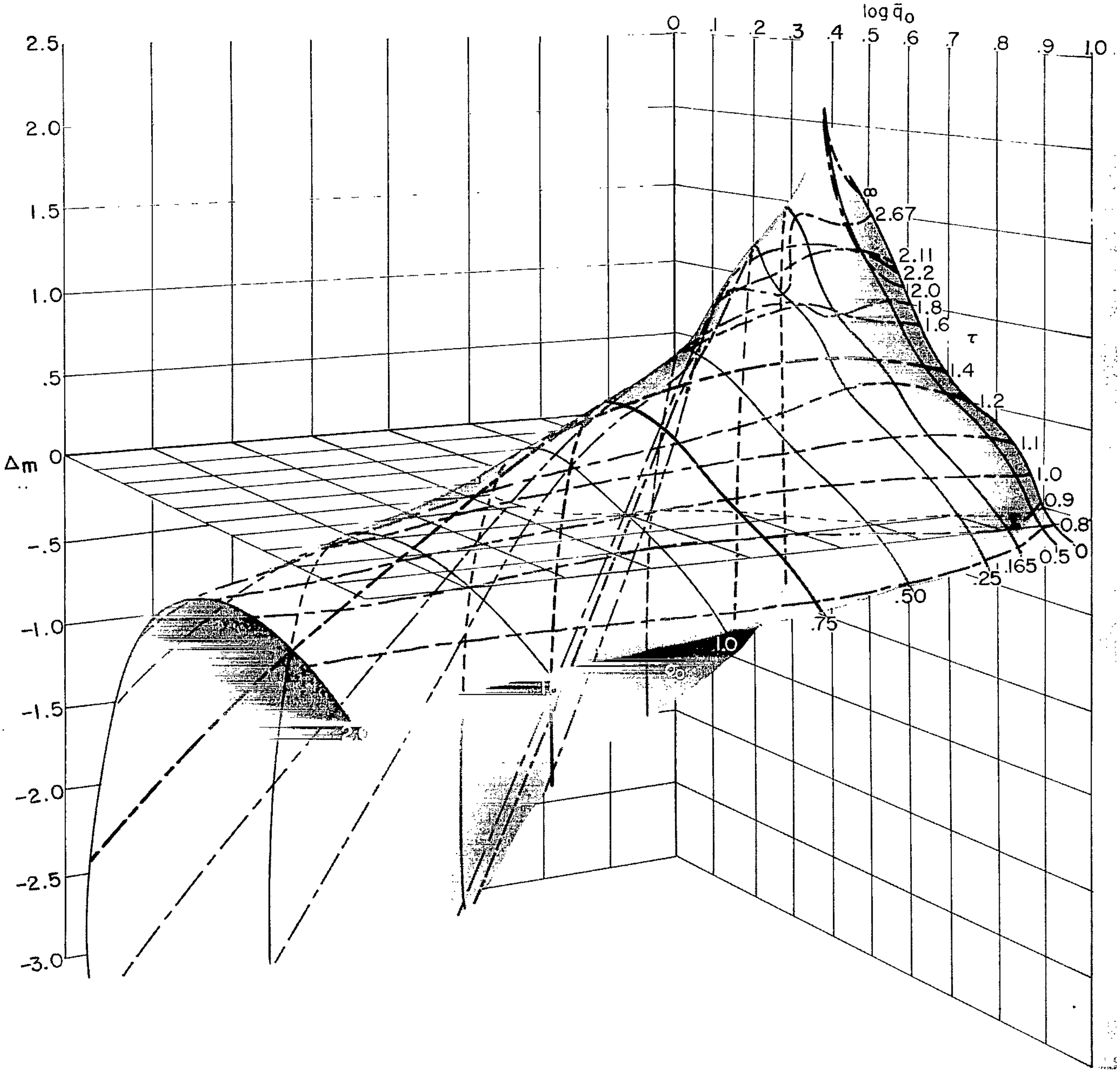


Figure 6.- The surface represented by the section z = 1.0 of the solid of model universes.

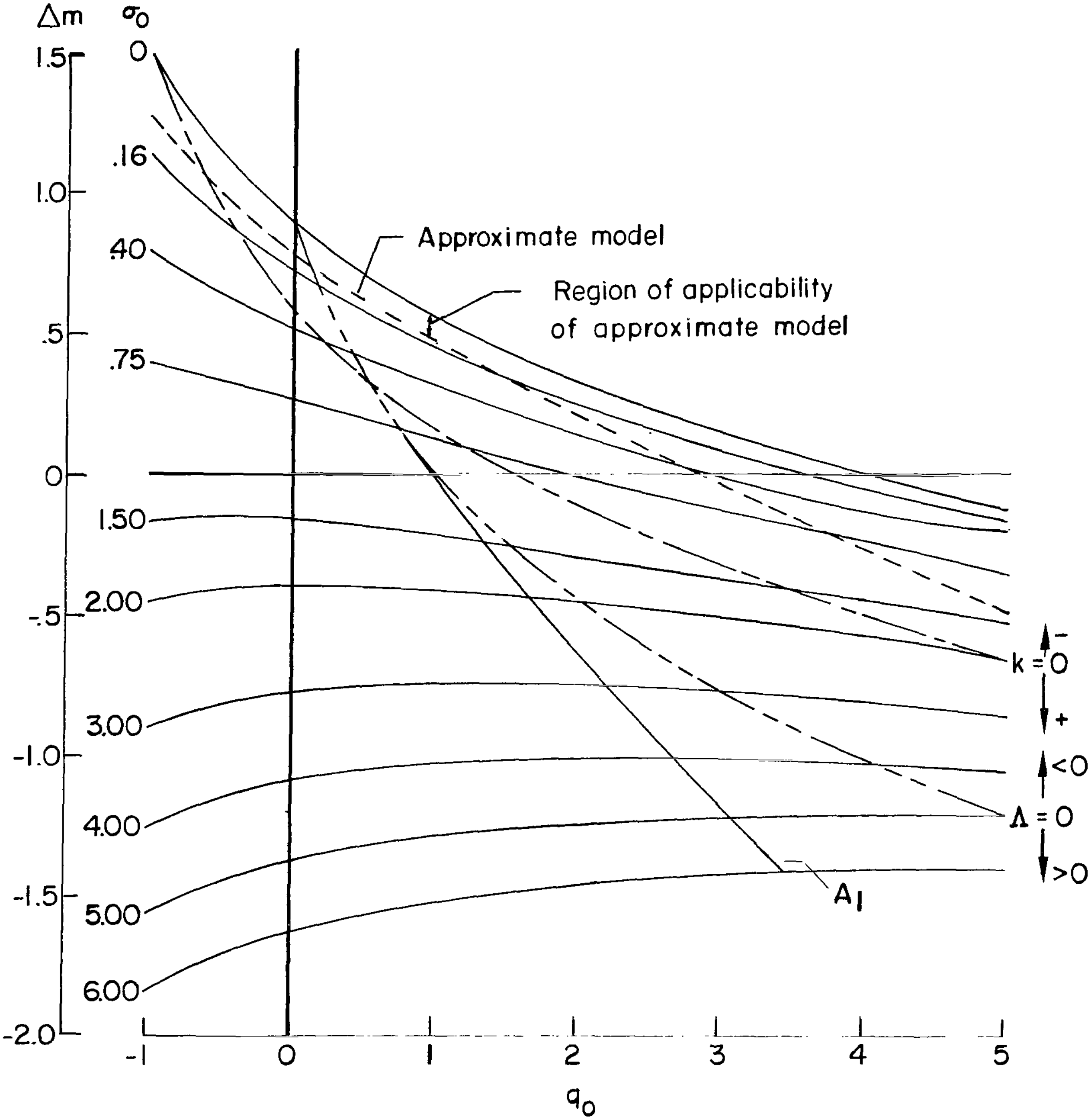


Figure 7.- Region where approximate model is good approximation for the exact model.

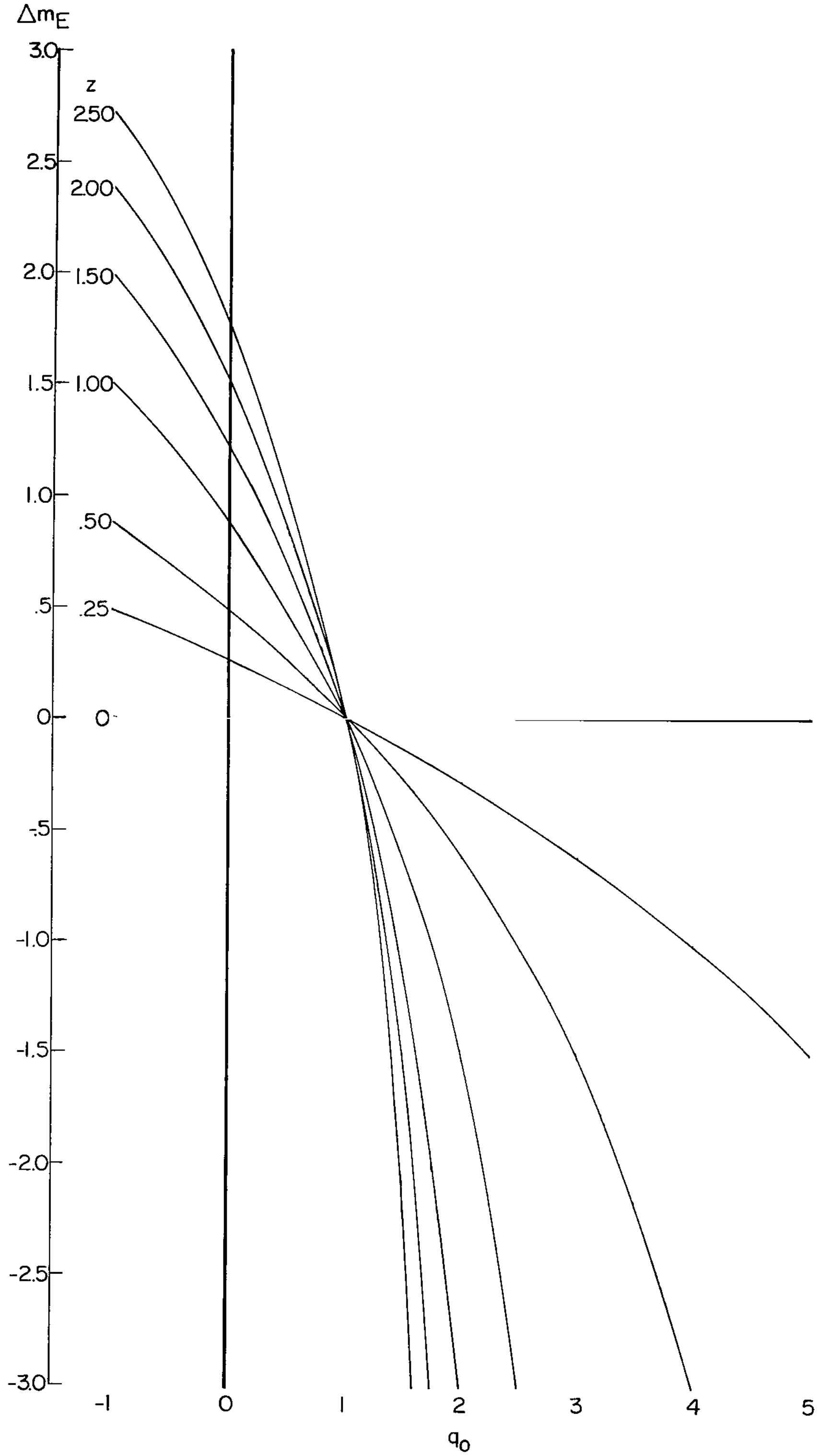
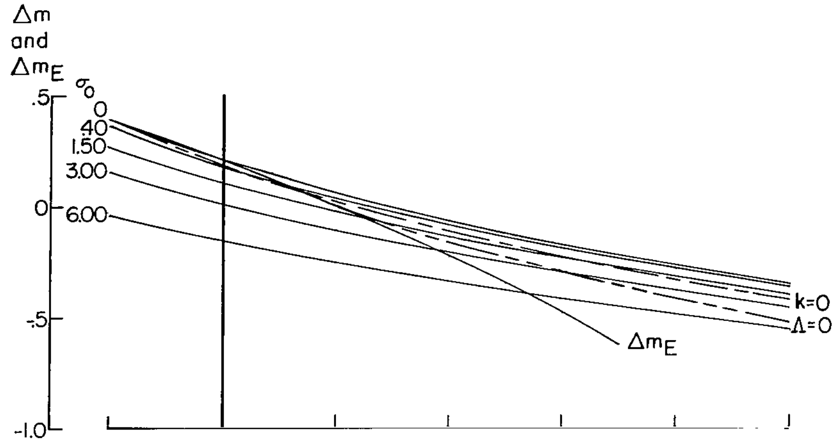
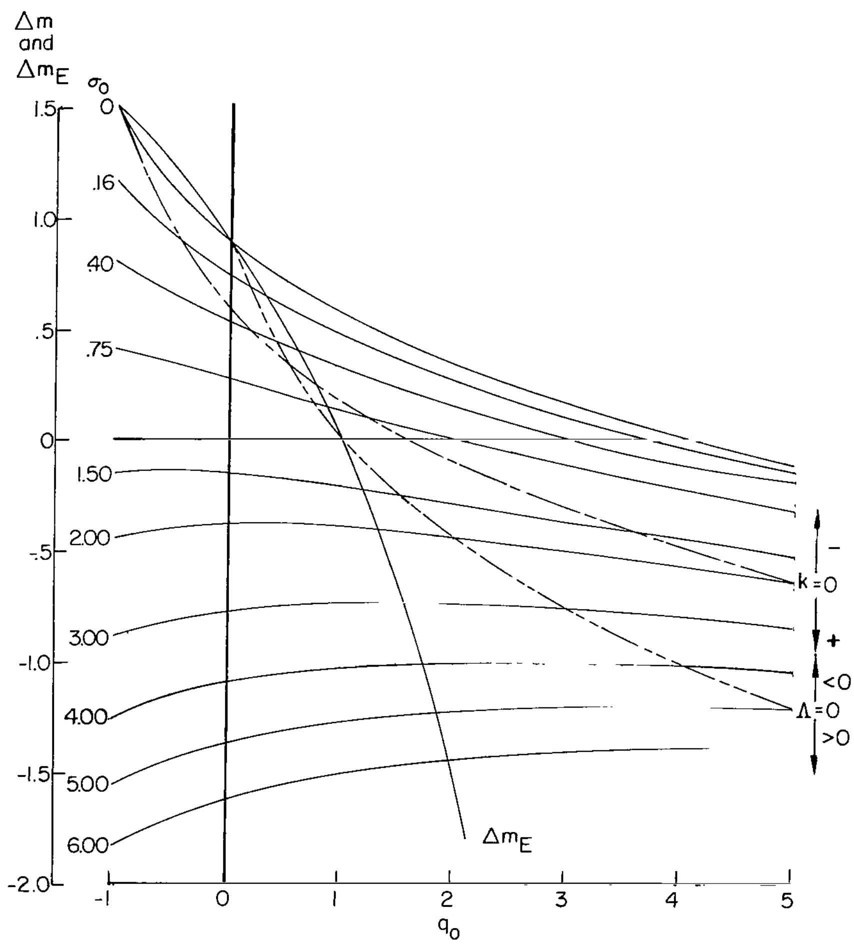


Figure 8.- Traces of the surface described by the approximate incremental red-shift—magnitude relation for constant values of the red shift z.

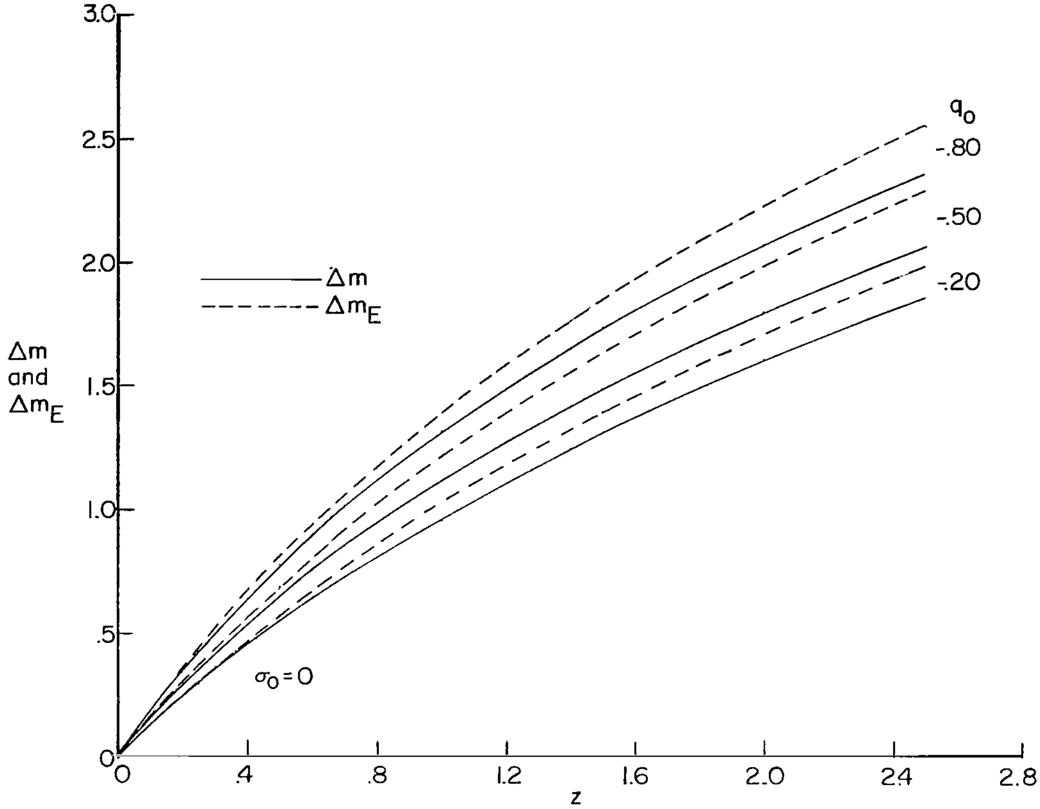


(a) z = 0.2.

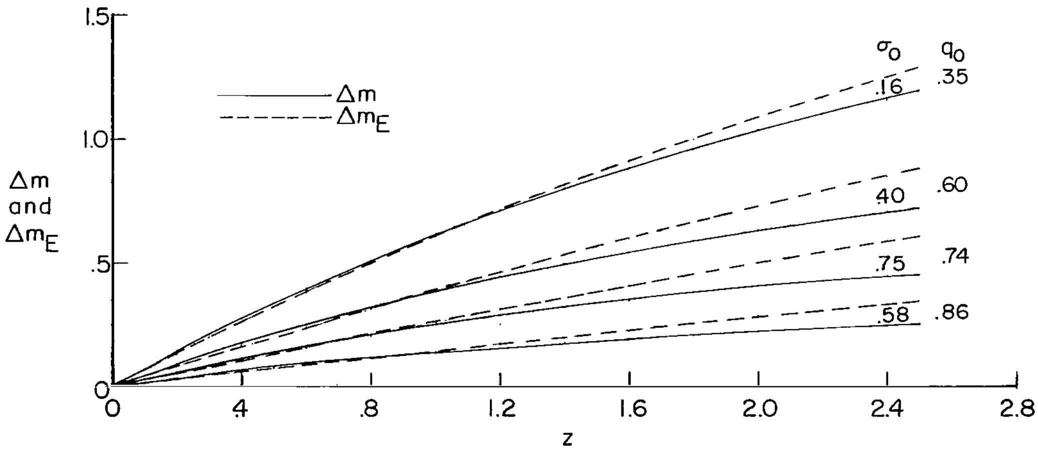


(b) z = 1.0.

Figure 9.- Comparison of the expansion form of the red-shift—magnitude relation with the solid of model universes.

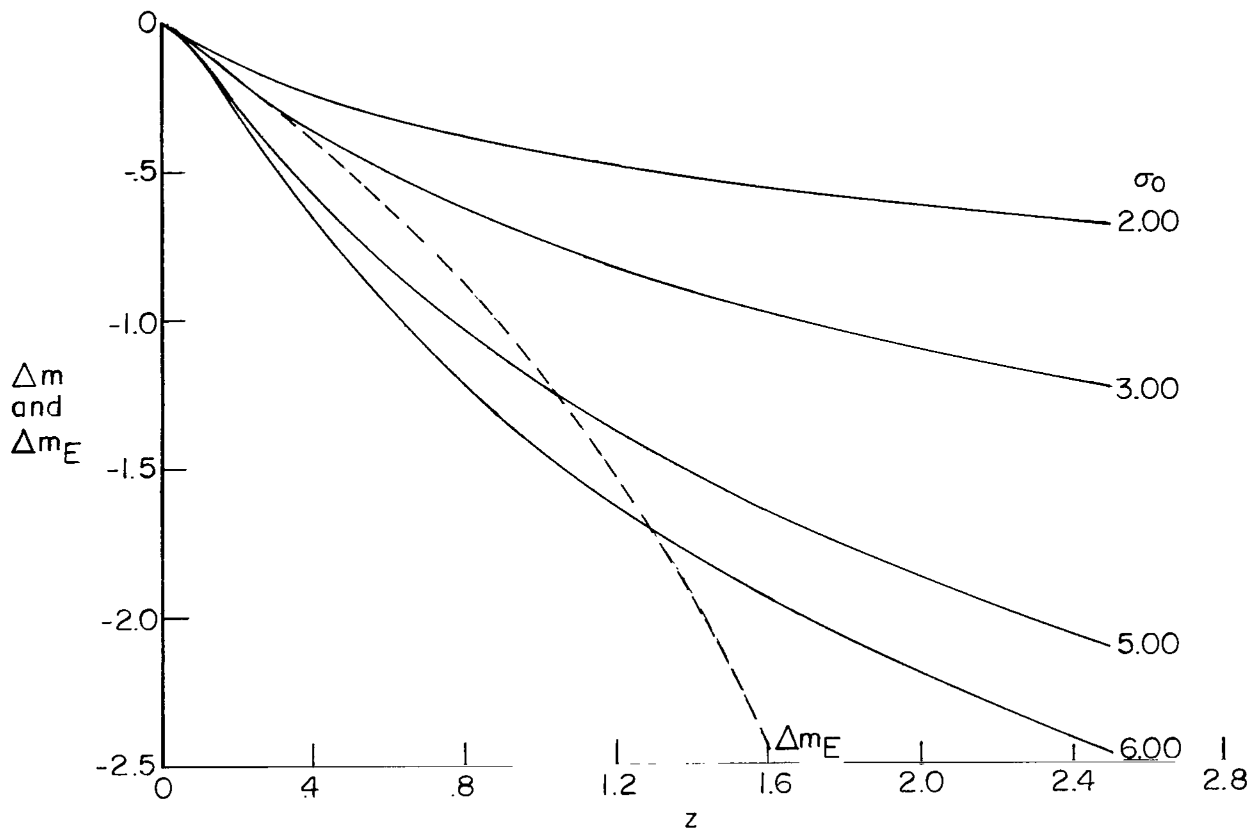


(c) $-1.0 < q_0 < 0$.



(d) $0 < q_0 < 1.0$.

Figure 9.- Continued.



(e) $q_0 = 1.85$.

Figure 9.- Concluded.

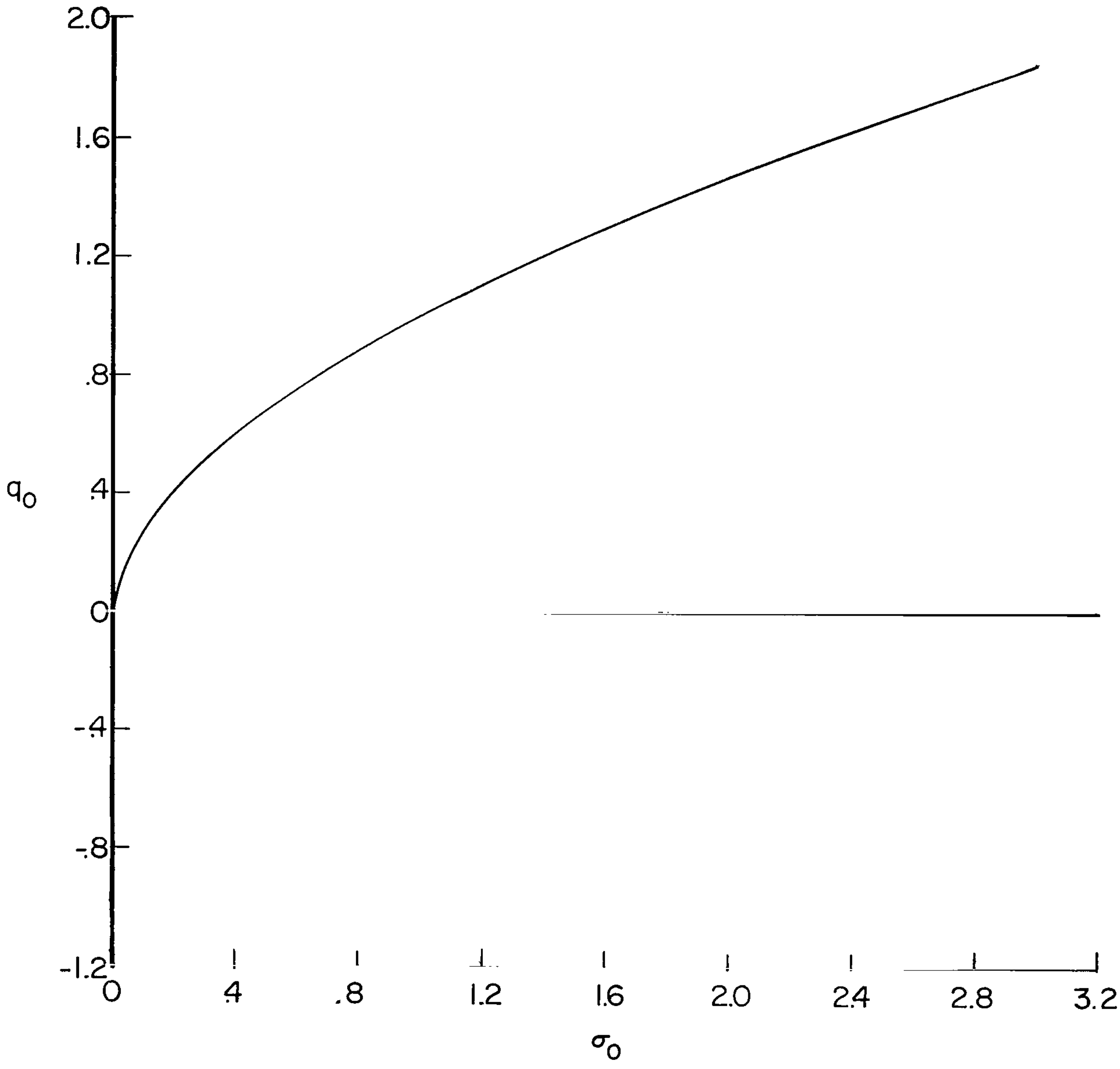


Figure 10.- The σ_0 and q_0 that give the greatest correspondence between Δm and Δm_E for $q_0 \geqq 0$. The curve was computed for $q_0 = 1.85$.

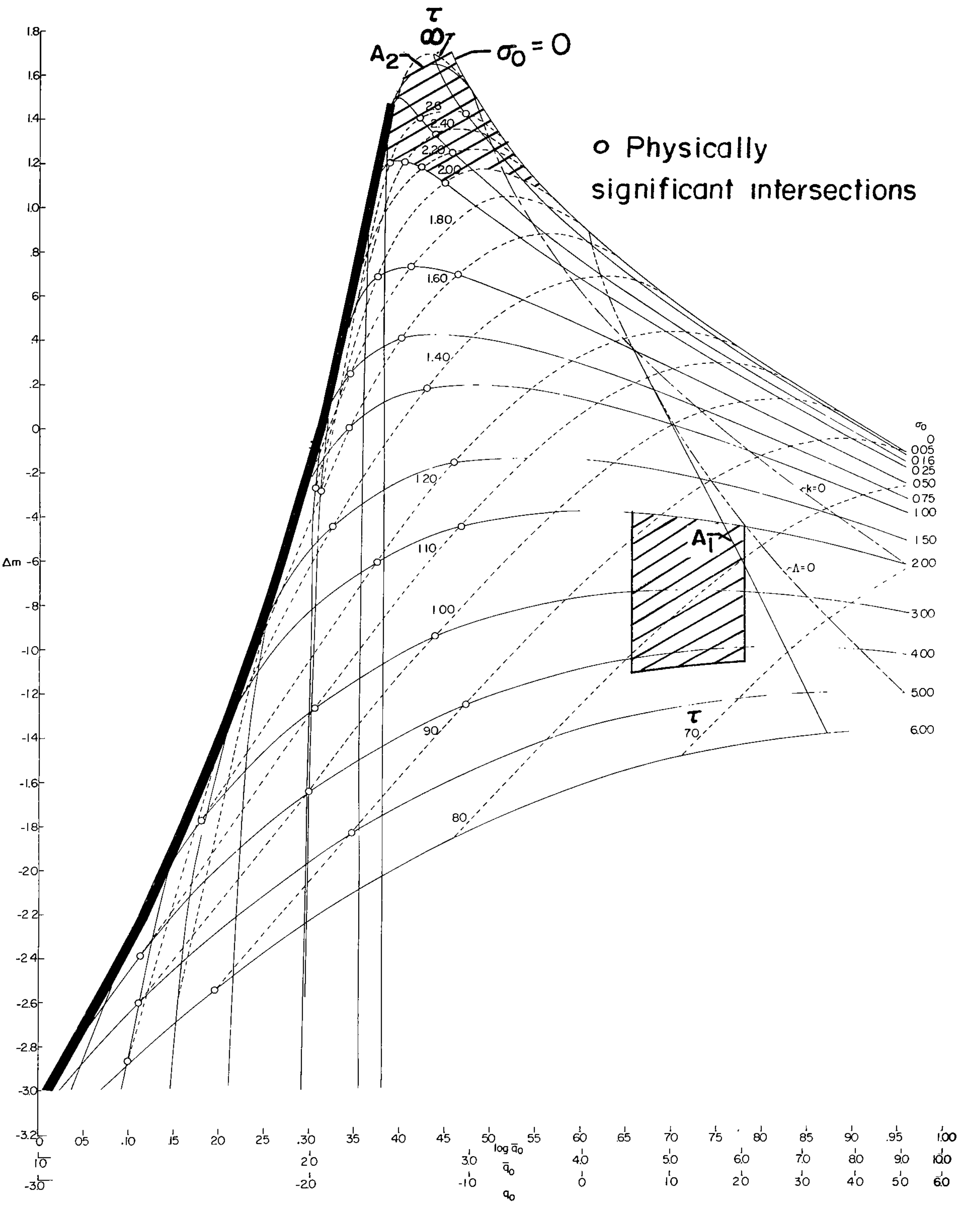


Figure 11.- Comparison of models of the universe obtained from the red-shift—magnitude relation with the models based on estimates of the time since the beginning of expansion and density. See text for explanation of shaded area.

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