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A neutral functional differential equation is a relationship in which the derivative of the state of a system at time t is specified in terms of the state at time t as well as the state and the derivative of the state for values of time preceding t. Many authors have considered such equations as may be seen by consulting [1], [2], [3]. Recently, Driver [4] considered a special class for which the derivative occurs linearly and proved the initial value problem is well-posed in the sense that a solution exists and depends continuously upon the initial data. To avoid discussing the differentiability properties of the solution, Hale and Meyer [5] introduced an integrated form of the equation which if differentiated would contain the derivative of the state with coefficients depending only on t. Hale and Cruz [6] gave a much more general version of [5] and proved again the problem was well-posed.

The present paper continues with the development in [6]. More specifically, we consider a class of equations which in some respects is more general than the ones considered in [6] and it has the advantage that it leads in a very natural manner to a discussion of the problem of the backward existence of solutions. After developing the basic theory of existence, uniqueness, continuous dependence and continuation of solutions, it is shown that solutions of most linear equations with bounded coefficients can not have a nonzero solution which approaches zero faster than an exponential.

1. Definition. Suppose $r \geq 0$ is a given real number, $R = (-\infty, \infty)$, E^n is a real or complex n-dimensional linear vector space with norm $|\cdot|$, $C([a,b], E^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into E^n with the topology of uniform convergence. If $[a,b] = [-r,0]$, we let $C = C([-r,0], E^n)$ and designate the norm of an element ϕ in C by $|\phi| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$. Single bars are generally used to denote norms in different spaces, but no confusion should arise. If $\sigma \in R$, $A \geq 0$ and $x \in C([\sigma-r, \sigma+A], E^n)$, then for any $t \in [\sigma, \sigma+A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. If Ω is an open subset of $R \times C$ and $f, D: \Omega \to E^n$ are given continuous functions, we say the relation

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) \tag{1.1}$$

is a functional differential equation. A function x is said to be a solution of (1.1) if there are $\sigma \in R$, $A > 0$ such that $x \in C([\sigma-r, \sigma+A), E^n)$, $(t,x_t) \in \Omega$, $t \in [\sigma, \sigma+A)$ and x satisfies (1.1) on $(\sigma, \sigma+A)$. Notice this definition implies that $D(t,x_t)$ and not $x(t)$ is continuously differentiable on $(\sigma, \sigma+A)$. For a given $\sigma \in R$, $\phi \in C$, $(\sigma,\phi) \in \Omega$, we say $x(\sigma,\phi)$ is a solution of (1.1) with initial value (σ,ϕ) or simply a solution of (1.1) through (σ,ϕ)

if there is an $A > 0$ such that $x(\sigma,\phi)$ is a solution of (1.1) on $[\sigma-r, \sigma+A)$ and $x_\sigma(\sigma,\phi) = \phi$.

Equation (1.1) is very general and includes ordinary differential equations $(r = 0)$ as well as the following:

(1.2) $$\frac{dx(t)}{dt} = f(t,x_t)$$

(1.3) $$\frac{d}{dt}[x(t) - bx(t-r)] = f(t,x_t), \quad b \neq 0,$$

(1.4) $$\frac{d}{dt}x(t-r) = f(t,x_t)$$

(1.5) $$\frac{d}{dt}x(t-\frac{r}{2}) = f(t,x_t).$$

In the classical terminology, for $r > 0$, equation (1.2) is called a retarded functional differential equation, equation (1.3) an equation of neutral type (because, if x is differentiable, the derivative occurs at t and $t - r$), equation (1.4) an equation of advanced type and equation (1.5) an equation of mixed type.

The initial value problem for equation (1.1) in general will not have a solution since it includes (1.4) and (1.5). Additional restrictions will be imposed on the function D so that the initial value problem is well defined. To formulate these restrictions, it is convenient to have

1.1 <u>Definition.</u> Suppose Ω is an open set in $R \times C$, $D: \Omega \to E^n$ is continuous, $D(t,\varphi)$ has a continuous Frechet derivative $D'_\varphi(t,\varphi)$ with respect to φ on Ω and

$$D'_\varphi(t,\varphi)\psi = \int_{-r}^{0} [d_\theta \mu(t,\varphi,\theta)]\psi(\theta)$$

for $(t,\varphi) \in \Omega$, $\psi \in C$, where $\mu(t,\varphi,\theta)$ is an $n \times n$ matrix with elements of bounded variation in $\theta \in [-r,0]$. For any β in $[-r,0]$ we say D is <u>atomic at β on Ω</u> if

$$\text{(1.6)} \qquad \begin{gathered} \mu(t,\varphi,\beta^+) - \mu(t,\varphi,\beta^-) = A(t,\varphi,\beta) \\ \det A(t,\varphi,\beta) \neq 0 \end{gathered}$$

where $A(t,\varphi,\beta)$ is continuous in (t,φ) and there is a scalar function $\gamma(t,\varphi,s,\beta)$ continuous for $(t,\varphi) \in \Omega$, $s \geq 0$, $\gamma(t,\varphi,0,\beta) = 0$ such that

$$\text{(1.7)} \qquad \left|\int_{\beta-s}^{\beta+s} [d_\theta \mu(t,\varphi,\theta)]\psi(\theta) - A(t,\varphi,\beta)\psi(\beta)\right| \leq \gamma(t,\varphi,s,\beta)|\psi|$$

for $(t,\varphi) \in \Omega$, $s \geq 0$, $\psi \in C$.

1.2 <u>Definition.</u> <u>A neutral functional differential equation (NFDE)</u> is a system (1.1) for which $D,f\colon \Omega \to E^n$ are continuous and D is atomic at zero on Ω.

A very special but important case of a NFDE is one in which $\Omega = (\tau,\infty) \times C$, $D(t,\varphi)$ is linear in φ

$$(1.8) \qquad D(t,\varphi) = \int_{-r}^{0} [d_\theta \mu(t,\theta)]\varphi(\theta)$$

$$B(t) = \mu(t,0) - \mu(t,0^-), \quad \det B(t) \neq 0,$$

$$\left|\int_{-s}^{0} [d_\theta \mu(t,\theta)]\varphi(\theta) - B(t)\varphi(0)\right| \leq \gamma(t,s)|\varphi|$$

for $(t,\varphi) \in \Omega$, where $B(t)$ is continuous and $\gamma(t,s)$ is continuous for $t \in (\tau,\infty)$, $s \geq 0$, $\gamma(t,0) = 0$. In particular, all retarded functional differential equations (RFDE)

$$(1.9) \qquad \frac{d}{dt} x(t) = f(t,x_t)$$

are included in the class of NFDE.

In [6], a NFDE was defined in a manner similar to the above for a class of operators $D(t,\varphi) = \varphi(0) - g(t,\varphi)$ even when $g(t,\varphi)$ is not differentiable in φ. The important difference here is not the smoothness of $g(t,\varphi)$ but the fact that $D(t,\varphi)$ need not be of this special form.

2. Fundamental Properties of NFDE

In this section, we give results on the existence, uniqueness, continuation and continuous dependence of solutions on initial data.

Theorem 2.1 (Existence). If Ω is an open set in $R \times C$ and (1.1) is a NFDE, then for any $(\sigma,\varphi) \in \Omega$, there is a solution of (1.1) passing through (σ,φ).

Proof: A function x is a solution of (1.1) through (σ,φ) if and only if there is an $\alpha > 0$ such that x satisfies the equation

$$(2.1)\qquad \begin{aligned} &D(t,x_t) = D(\sigma,\varphi) + \int_\sigma^t f(s,x_s)ds, \qquad t \in [\sigma,\sigma+\alpha], \\ &x_\sigma = \varphi. \end{aligned}$$

Let $\tilde{\varphi}: [-r,\infty) \to E^n$ be defined by $\tilde{\varphi}(t) = \varphi(t)$, $t \in [-r,0]$, $\varphi(t) = \tilde{\varphi}(0)$, $t \in [0,\infty)$. Then x is a solution of (2.1) on $[\sigma,\sigma+\alpha]$ if and only if $x(\sigma+t) = \varphi(t) + z(t)$, $-r \leq t \leq \alpha$, where $z(t)$ satisfies

$$(2.2)\qquad \begin{aligned} &D(\sigma+t,\tilde{\varphi}_t+z_t) = D(\sigma,\varphi) + \int_0^t f(\sigma+s,\tilde{\varphi}_s+z_s)ds, \qquad t \in [0,\alpha], \\ &z_0 = 0. \end{aligned}$$

Since $D(t,\varphi)$ is continuously differentiable in φ,

$$(2.3)\qquad D(t,\varphi+\psi) = D(t,\varphi) + D'_\varphi(t,\varphi)\psi + g(t,\varphi,\psi)$$

where

$$g(t,\varphi,0) = 0$$

$$|g(t,\varphi,\psi) - g(t,\varphi,\xi)| \leq \varepsilon(t,\varphi,\delta)|\psi-\xi|$$

for $(t,\varphi) \in \Omega$, $|\psi|, |\xi| \leq \delta$ and $\varepsilon(t,\varphi,\delta)$ is continuous in t,φ,δ for $(t,\varphi) \in \Omega$, $\delta \geq 0$ and $\varepsilon(t,\varphi,0) = 0$. Therefore, using (2.2) and (2.3), x is a solution of (2.1) if and only if $x_{\sigma+t} = \tilde{\varphi}_t + z_t$ and z satisfies

$$
\begin{aligned}
(2.4) \qquad D'_\varphi(\sigma+t,\tilde{\varphi}_t)z_t &= D(\sigma,\varphi) - D(t+\sigma,\tilde{\varphi}_t) - g(\sigma+t,\tilde{\varphi}_t,z_t) \\
&\quad + \int_0^t f(\sigma+s,\tilde{\varphi}_s+z_s)ds, \quad t \in [0,\alpha], \\
z_0 &= 0.
\end{aligned}
$$

Using the fact that D is atomic at 0 on Ω, we have (as long as $(\sigma+t,\tilde{\varphi}_t) \in \Omega$)

$$
\begin{aligned}
(2.5) \qquad z(t) = A^{-1}(\sigma+t,\tilde{\varphi}_t,0)\{&\int_{-r}^{0^-}[d_\theta\mu(\sigma+t,\tilde{\varphi}_t,\theta)]z_t(\theta) \\
&+ D(\sigma,\varphi) - D(t+\sigma,\tilde{\varphi}_t) - g(\sigma+t,\tilde{\varphi}_t,z_t) \\
&+ \int_0^t f(\sigma+s,\tilde{\varphi}_s+z_s)ds\}, \qquad t \in [0,\alpha], \\
z_0 = 0.&
\end{aligned}
$$

If we let $(Tz)(t) = 0$, $(Sz)(t) = 0$, $t \in [-r,0]$, and

$$
\begin{aligned}
(Tz)(t) &= A^{-1}(\sigma+t,\tilde{\varphi}_t,0)\{\int_{-r}^{0^-}[d_\theta\mu(\sigma+t,\tilde{\varphi}_t,\theta)]z_t(\theta) \\
&\quad + D(\sigma,\varphi) - D(t+\sigma,\tilde{\varphi}_t) - g(\sigma+t,\tilde{\varphi}_t,z_t)\} \\
(Sz)(t) &= A^{-1}(\sigma+t,\tilde{\varphi}_t,0)\int_0^t f(\sigma+s,\tilde{\varphi}_s+z_s)ds, \quad t \in [0,\alpha],
\end{aligned}
$$

then (2.5) is equivalent to the equation

$$z = Tz + Sz, \quad z \in C([-r,\alpha], E^n), \quad z_0 = 0.$$

One now proceeds as in [6] to show there are positive $\bar{\alpha},\bar{\beta}$ so that, if $\mathscr{A}(\bar{\alpha},\bar{\beta}) = \{\zeta\colon [-r,\bar{\alpha}] \to E^n$, continuous, $\zeta_0 = 0$, $|\zeta_t| \le \bar{\beta}$, $t \in [0,\bar{\alpha}]\}$, then $T\colon \mathscr{A}(\bar{\alpha},\bar{\beta}) \to C([-r,\bar{\alpha}], E^n)$ is a contraction, $S\colon \mathscr{A}(\bar{\alpha},\bar{\beta}) \to C([-r,\bar{\alpha}], E^n)$ is completely continuous and $T + S\colon \mathscr{A}(\bar{\alpha},\bar{\beta}) \to \mathscr{A}(\bar{\alpha},\bar{\beta})$. This implies the existence of a fixed point of $T + S$ in $\mathscr{A}(\bar{\alpha},\bar{\beta})$ and thus a solution of (1.1) through (σ,φ).

<u>Theorem 2.2 (Uniqueness)</u>. If Ω is an open set in $R \times C$ and (1.1) is a NFDE with $f(t,\varphi)$ locally lipschitzian in φ in each compact set of Ω, then for any $(\sigma,\varphi) \in \Omega$, there is a unique solution of (1.1) through (σ,φ).

<u>Proof</u>: The proof is essentially the same as the proof for ordinary differential equations if one uses the fact that a solution of (1.1) satisfies $x_{\sigma+t} = \tilde{\varphi}_t + z_t$ and z satisfies (2.5).

<u>Theorem 2.3 (Continuous Dependence)</u>. Suppose Ω is an open set in $R \times C$, $D_k\colon \Omega \to E^n$ is atomic at 0 on Ω and $A_k(t,\varphi,0)$ is the corresponding matrix of Definition 1.1, $k = 0,1,2,\ldots$. Suppose D_0, $A_k(t,\varphi,0)$ are uniformly continuous on closed bounded subsets of

Ω, D_k as well as the derivative $D'_{k,\varphi}$ with respect to φ converge to D_o, $D'_{o,\varphi}$ respectively as $k \to \infty$ uniformly on closed bounded subsets of Ω, $f_k: \Omega \to E^n$, $k = 0,1,2,\ldots$, are continuous, $f_k(t,\psi) \to f_o(s,\varphi)$ as $k \to \infty$, $(t,\psi) \to (s,\varphi)$ for all $(s,\varphi) \in \Omega$ and for any $(s,\varphi) \in \Omega$, there is a neighborhood $V(s,\varphi)$ of (s,φ) and a constant M such that

$$|f_k(t,\psi)| \leq M, \quad (t,\psi) \in V(s,\varphi), \quad k = 0,1,2,\ldots .$$

Finally, let $(\sigma^k,\varphi^k) \in \Omega$ be given, $(\sigma^k,\varphi^k) \to (\sigma^o,\varphi^o)$ as $k \to \infty$ and suppose $x^k = x^k(\sigma^k,\varphi^k)$ is a solution of

$$\frac{d}{dt} D_k(t,x_t) = f_k(t,x_t)$$

with initial value φ^k at σ^k. If x^o is defined on $[\sigma,-r,b]$, $b > \sigma$, and is the only solution through (σ^o,φ^o), then there is an integer k_o such that the x^k, $k \geq k_o$, can be defined on $[\sigma^k-r,b]$ and $x^k(t) \to x^o(t)$ as $k \to \infty$ uniformly on $[\sigma^o-r,b]$; that is, for any $0 < \varepsilon < b - \sigma^o + r$, there is a $k_1 = k_1(\varepsilon) \geq 0$ such that x^k, $k \geq k_1(\varepsilon)$, is defined on $[\sigma^o-r+\varepsilon,b]$ and $x^k(t) \to x^o(t)$ as $k \to \infty$ uniformly on $[\sigma^o-r+\varepsilon,b]$.

Proof: The proof is technically complicated but proceeds in a manner very similar to the one in [6] taking into account that a solution of

(1.1) satisfies $x_{\sigma+t} = \tilde{\varphi}_t + z_t$ where z satisfies (2.5).

Definition 2.1. If D is atomic at β on Ω and W is a subset of Ω, we say D is uniformly atomic at zero on W if there is an $N > 0$ such that $|A^{-1}(t,\varphi,\beta)| \leq N$, $|D'_\varphi(t,\varphi)| \leq N$ for all $(t,\varphi) \in W$ and $\gamma(t,\varphi,s,\beta) \to 0$ as $s \to 0$ uniformly for $(t,\varphi) \in W$.

If x is a solution of (1.1) on $[\sigma-r,a)$, $a > \sigma$, we say $\hat{x}$ is a continuation of x if there is a $b > a$ such that $\hat{x}$ is defined on $[\sigma-r,b)$, coincides with x on $[\sigma-r,a)$ and satisfies (1.1) on (σ,b). A solution x is noncontinuable if no such continuation exists; that is $[\sigma-r,a)$ is the maximal interval of existence of the solution x. If the conditions of the basic existence theorem are satisfied, then there is a solution of (1.1) on $[\sigma-r,a)$ for some $a > \sigma$. Zorn's lemma implies the existence of a noncontinuable solution of (1.1). It is also true that the maximal interval of existence is open.

The following theorem as well as the proof is based on the thesis of W. Melvin [7].

Theorem 2.4 (Continuation). Suppose Ω is an open set in $R \times C$, (1.1) is a NFDE and for any closed bounded set W in Ω with a δ-neighborhood also in Ω, f maps W into a bounded set in E^n, $D(t,\varphi)$, $D'_\varphi(t,\varphi)$ are uniformly continuous on W and D is uniformly atomic at zero on W. If x is a noncontinuable solution of

(1.1) on $[\sigma-r,b)$, then there is a t' in $[\sigma,b)$ such that $(t',x_{t'}) \notin W$.

<u>Proof</u>: The case $r = 0$ is known from ordinary differential equations. Therefore, suppose $r > 0$. Also, we may assume b finite. If there is a sequence $t_k \to b^-$ and a ψ in C such that $x_{t_k} \to \psi$, then the fact that $r > 0$ implies that $x(t)$ is uniformly continuous on $[\sigma-r,b)$ and $x(t) \to \psi(0)$ as $t \to b$. Therefore, if we define $x(b) = \psi(0)$, then (b,x_b) must belong to the boundary of Ω or x would be continuable beyond b. Also, the fact that x_t is continuous and the distance of (b,x_b) from any closed bounded set W is positive imply the existence of a t_W such that $(t,x_t) \notin W$ for $t_W \leq t < b$, a conclusion stronger than asserted.

If no such subsequence exists, there are two cases to consider: namely the cases where the set $V = \{(t,x_t)\}$ is bounded and unbounded. If this set is unbounded, then for any closed bounded set W in Ω, there is a constant k_W such that $|\phi| < k_W$ for $(t,\phi) \in W$. Let $k'_W = \max\{|x_\sigma|,k_W\}$. From hypothesis, there is a sequence $t_k \to b^-$ monotonically such that $|x_{t_k}| > k'_W$. From the property of the norm in C and the fact that $x_t(\theta) = x(t+\theta)$, this implies the existence of a t_W such that $(t,x_t) \notin W$ for $t_W \leq t < b$.

If the set $V = \{(t,x_t),\ t \in [\sigma,b)\}$ is bounded and has a δ-neighborhood in Ω, then this set is also closed since there are no subsequences $t_k \to b^-$ such that x_{t_k} converges. We wish to show there is an $\alpha > 0$ such that x is uniformly continuous on $[b-\alpha, b)$ and, therefore, $\{(t, x_t),\ t \in [\sigma, b)\}$ belongs to a compact set in Ω. This will obviously be a contradiction.

From the hypotheses on $\dot{V}$, D and Definitions 2.1 and (4.1), there are a $\beta_0 > 0$ and continuous functions $\gamma(s)$, $s \geq 0$, $\varepsilon(\beta)$, $0 \leq \beta \leq \beta_0$, $\gamma(0) = \varepsilon(0) = 0$, and a constant N such that

$$D(\sigma, \phi+\psi) = D(\sigma, \phi) + D_\phi'(\sigma, \phi)\psi + g(\sigma, \phi, \psi)$$

$$|A^{-1}(\sigma, \phi, 0)| \leq N, \quad |D_\phi'(\sigma, \phi)| \leq N,$$

$$|g(\sigma, \phi, \psi)| \leq \varepsilon(\beta)|\psi|$$

$$\left|\int_{-s}^{0} [d_\theta \mu(\sigma, \phi, \theta)]\psi(\theta)\right| \leq \gamma(s) \sup_{-s\leq\theta\leq 0}|\psi(\theta)|$$

Consequently,

$$(2.6) \qquad |D(\sigma, \phi+\psi) - D(\sigma, \psi)| \geq$$

$$|\psi(0)|/N - \gamma(s)|\psi| - N \sup_{-r\leq\theta\leq -s}|\psi(\theta)| - \varepsilon(\beta)|\psi|$$

for (σ, ϕ), $(\sigma, \phi+\psi) \in V$, $|\psi| \leq \beta$.

If $x(t)$ is not uniformly continuous for t in $[\sigma-r, b)$, There are an $\varepsilon > 0$, a monotone decreasing sequence of positive numbers Δ_k, $\Delta_k \to 0$ as $k \to \infty$ and a sequence of real numbers t_k with t_k, $t_k - \Delta_k$ in $[\sigma, b)$ such that $|x(t_k) - x(t_k-\Delta_k| \geq \varepsilon$

for all k. For any $s > 0$, the fact that x is uniformly continuous on $[\sigma-r,b-s]$ implies for any $\varepsilon' > 0$ the existence of a $\Delta > 0$ such that $|x(t) - x(t')| \leq \varepsilon'$ for $|t-t'| < \Delta$, t,t' in $[\sigma-r,b-s]$. Also, since $D(\sigma,\phi)$ is uniformly continuous on V, we can choose Δ so that $|D(t,\phi) - D(t',\phi)| \leq \varepsilon'$ for $(t-t') < \Delta$, $(t,\phi) \in V$, $(t',\phi) \in V$.

Suppose $0 < \beta \leq \beta_0$ is given, choose $\varepsilon' < \min(\beta,\varepsilon)$ and K sufficiently large that $|\Delta_k| < \Delta$, $k \geq K$. For each $k \geq K$, let

$$s_k = \inf\{t \in [\sigma,b) : |x(t) - x(t-\Delta_k)| \geq \min(\beta,\varepsilon)\}.$$

This sequence of numbers is well defined since $|x(t_k) - x(t_k-\Delta_k)| > \varepsilon$. From (4.2),

$$\begin{aligned}
|D(s_k,x_{s_k}) &- D(s_k-\Delta_k,x_{s_k-\Delta_k}) \\
&\geq |D(s_k,x_{s_k}) - D(s_k,x_{s_k-\Delta_k})| \\
&\qquad - |D(s_k,x_{s_k-\Delta_k}) - D(s_k-\Delta_k,x_{s_k-\Delta_k})| \\
&\geq |D(s_k,x_{s_k}) - D(s_k,x_{s_k-\Delta_k})| - \varepsilon' \\
&\geq \min(\beta,\varepsilon)/N - \gamma(s)\beta - N\varepsilon' - \varepsilon(\beta)\min(\beta,\varepsilon) \overset{\text{def}}{=} \bar{\varepsilon}.
\end{aligned}$$

Now one can obviously choose β_0,s,ε' so that $\bar{\varepsilon} > 0$. Consequently, the hypothesis that $x(t)$ is not uniformly continuous on $[\sigma-r,b)$ implies that $D(t,x_t)$ is not uniformly continuous on $[\sigma,b)$.

On the other hand,

$$D(t+\tau, x_{t+\tau}) - D(t, x_t) = \int_t^{t+\tau} f(s, x_s)ds$$

for all t, $t + \tau$ in $[\sigma, b)$. Since $|f(s, x_s)| \leq M$ for $(s, x_s) \in W$ and some constant M, the function $D(t, x_t)$ is uniformly continuous on $[\sigma, b)$. This contradiction completes the proof of the theorem.

To improve on Theorem 2.4 we suppose $D(t, \phi)$ is continuous in t, ϕ, linear in ϕ, and in fact satisfies

$$(2.7) \qquad D(t, \phi) = A(t)\phi(0) + \int_{-r}^{0} [d_\theta \nu(t, \theta)]\phi(\theta), \ \det A(t) \neq 0$$

$$\left|\int_{-s}^{0} [d_\theta \nu(t, \theta)]\phi(\theta)\right| \leq \gamma(t, s) \sup_{\theta \in [-s, 0]} |\phi(\theta)|$$

for a continuous matrix $A(t)$ and scalar function $\gamma(t, s)$, $\gamma(t, 0) = 0$, $t \in R$, $s \geq 0$.

In the proof of Theorem 2.4, the assumption that a δ-neighborhood of W belonged to Ω was used only to show that relation (2.6) was valid for some β. When $D(t, \phi)$ is linear in ϕ and satisfies (2.7), $\varepsilon(\beta)$ in (2.6) can be taken identically zero. The proof above of Theorem 2.4 for this case yields

<u>Theorem 2.5</u>. Suppose Ω is an open set in $R \times C$, $f\colon \Omega \to E^n$ is continuous and maps closed bounded subsets of Ω into bounded sets and $D\colon R \times C \to E^n$ satisfies (2.7). If $(\sigma, \phi) \in \Omega$ and x is a

noncontinuable solution of (1.1) on $[\sigma-r,b)$ through (σ,ϕ), then for any closed bounded set W in Ω there is a $t_W \in [\sigma,b)$ such that $(t,x_t) \notin W$ for $t \in [t_W,b)$.

3. Backward Continuation.

We say a function $x \in C([\sigma-r-\alpha,\sigma],E^n)$, $\alpha > 0$, is a solution of (1.1) on $[\sigma-r-\alpha,\sigma]$ through (σ,ϕ) if $x_\sigma = \phi$ and for any $\tau \in [\sigma-\alpha,\sigma]$, x is a solution of (1.1) on $[\tau-r,\sigma]$ through (τ,x_τ). We sometimes refer to x as a backward continuation of ϕ by (1.1).

Theorem 3.1. Suppose Ω is an open set in $R \times C$ and D in (1.1) is atomic at $-r$ on Ω. If $(\sigma,\phi) \in \Omega$, then there is an $\alpha > 0$ and a solution of (1.1) through (σ,ϕ) on $[\sigma-r-\alpha,\sigma]$. If, in addition $f(t,\phi)$ is locally Lipschitzian in ϕ, then the solution is unique.

Proof: The proof of this theorem follows the same lines as the proof of Theorem 1.1 except all extensions are made to the left of $\sigma - r$ rather than to the right of σ. The assertion of uniqueness is proved in a manner similar to the proof of Theorem 1.2.

If D in (1.1) is atomic at zero and $-r$ on Ω, then Theorems 1.1 and 3.1 imply for any $(\sigma,\phi) \in \Omega$, there are $\alpha > 0$, $\beta > 0$ and a continuous n-vector function x on $[\sigma-r-\alpha,\sigma+\beta]$ such that $x_\sigma = \phi$, $D(t,x_t)$ is continuously differentiable and satisfies (1.1) on $(\sigma-\alpha,\sigma+\beta)$. This is the same type of result that is known for ordinary differential equations $(r = 0)$.

Let U be the values of $(t,\sigma,\phi) \in R \times R \times C$ for which $x_t(\sigma,\phi)$ is defined and for each $(t,\sigma) \in R \times R$, let $U(t,\sigma) = \{\phi \in C: (t,\sigma,\phi) \in U\}$. Also, define $T(t,\sigma)\phi = x_t(\sigma,\phi)$. If D is atomic at 0 and $-r$, then $T(t,\sigma): U(t,\sigma) \to T(t,\sigma)U(t,\sigma)$ is one-to-one. With further conditions on D, f, we can prove

Theorem 3.2. Suppose Ω is an open set in $R \times C$, D in (1.1) is atomic at 0 and $-r$ on Ω, and U, $U(t,\sigma)$ are defined as above. If the functions D, $A(t,\phi,0)$, $A(t,\phi,-r)$ of Definition 1.1 are uniformly continuous on closed bounded subsets of Ω and $f(t,\phi)$ is locally Lipschitzian in ϕ, then the mapping $T(t,\sigma): U(t,\sigma) \to T(t,\sigma)U(t,\sigma)$ is a homeomorphism.

Proof: This is a consequence of Theorem 3.1, Theorem 1.4 and the extension of Theorem 1.4 to the backward continuation of functions ϕ by (1.1).

For RFDE, it is generally impossible to find a solution through (σ,ϕ) defined to the left of σ. In fact, if such a solution exists on $[\sigma-r-\alpha,\sigma]$, $\alpha > 0$, then x must be continuously differentiable on $(\sigma-\alpha,\sigma)$. On the other hand, $x(\sigma+\theta) = \phi(\theta)$ for $\theta \in (-\alpha,0)$ and ϕ may only be continuous. Even if ϕ is continuously differentiable, there may not be a solution through (σ,ϕ) to the left of σ for a RFDE. If the differential equation is

$$\dot{x}(t) = f(t,x_t) \tag{3.1}$$

it is certainly necessary for $\dot{\phi}(0) = f(\sigma,\phi)$ if a solution of (3.1) exists on $[\sigma-r-\alpha,\sigma]$, $\alpha > 0$, through (σ,ϕ).

We prove

Theorem 3.3. If Ω is an open set in $R \times C$, $f: D \to R^n$ is atomic at $-r$ on Ω, $(\sigma,\phi) \in \Omega$ and there is an α, $0 < \alpha < r$ such that $\dot{\phi}(\theta)$ is continuous for $\theta \in [-\alpha,0]$, $\dot{\phi}(0) = f(\sigma,\phi)$, then there are an $\bar{\alpha} > 0$ and a unique solution x of (3.1) on $[\sigma-r-\bar{\alpha},\sigma]$ through (σ,ϕ).

Proof: A function x is a solution of (3.1) on $[\sigma-r-\alpha,\sigma]$ through (σ,ϕ) if and only if $x_\sigma = \phi$, $(t,x_t) \in \Omega$, $t \in [\sigma-\alpha,\sigma]$ and

$$f(t,x_t) = \dot{x}(t), \qquad t \in [\sigma-\alpha,\sigma]. \tag{3.2}$$

For any $\alpha > 0$, let $\hat{\phi}: [-r-\alpha,0] \to E^n$, $\hat{\phi}(t)$, $t[-r,0]$, $\hat{\phi}(t) = \phi(-r)$, $t \in [-r-\alpha,-r]$. Then x is a solution of (3.2) if and only if $x(\sigma+t) = \hat{\varphi}(t) + z(t)$ and z satisfies

$$f(\sigma+t,\hat{\phi}_t+z_t) = \dot{\phi}(t), \qquad t \in [-\alpha,0]. \tag{3.3}$$

If $f(t,\phi+\psi) = f(t,\phi) + f'_\phi(t,\phi)\psi + g(t,\phi,\psi)$, then the definition of the derivative implies that $g(t,\phi,\psi)$ is continuous in (t,φ,ψ), $g(t,\phi,0) = 0$ and

$$|g(t,\phi,\psi) - g(t,\varphi,\xi)| < \varepsilon(t,\phi,\beta)|\psi-\xi|, \qquad |\psi|, |\xi| \le \beta$$

where $\varepsilon(t,\phi,\beta)$ is continuous in (t,ϕ,β) for $(t,\phi) \in \Omega$, $\beta \geq 0$, and $\varepsilon(t,\phi,0) = 0$. If we make use of this in (3.3), then x is a solution of (3.2) if and only if $x(\sigma+t) = \hat{\phi}(t) + z(t)$ and $z(t)$ satisfies

$$(3.4) \qquad f'_\phi(\sigma+t,\hat{\phi}_t)z_t = -f(\sigma+t,\hat{\phi}_t) - g(\sigma+t,\hat{\phi}_t,z_t) + \dot{\phi}(t), \quad t \in [-\alpha,0],$$
$$z_0 = 0.$$

If we let $A(t,\phi,-r) \overset{\text{def}}{=} B(t,\phi)$ be the function defined in Definition 1.1, then $x(\sigma+t) = \hat{\phi}(t) + z(t)$ is a solution of (3.2) if and only if $z(t)$ satisfies

$$(3.5) \qquad z(t) = B^{-1}(\sigma+t,\hat{\phi}_t)\{-\int_{-r+}^{0}[d\mu_\theta(\sigma+t,\hat{\phi}_t,\theta)]z_t(\theta) - f(\sigma+t,\hat{\phi}_t)$$
$$-g(\sigma+t,\hat{\phi}_t,z_t) + \dot{\phi}(t)\}, \qquad t \in [-\alpha,0],$$
$$z_0 = 0.$$

For any $\beta > 0$, let $B_\beta = \{\psi \in C: |\psi| \leq \beta\}$. For any ν, $0 < \nu < 1/4$, there are $\alpha > 0$, $\beta > 0$, such that $(\sigma+t,\phi+\psi) \in \Omega$,

$$|B^{-1}(\sigma+t,\phi+\psi)|\varepsilon(\sigma+t,\phi+\psi,\beta) < \nu$$
$$|B^{-1}(\sigma+t,\phi+\psi)|\gamma(\sigma+t,\phi+\psi,\alpha,-r) < \nu$$

for $(t,\psi) \in I_\alpha x B_\beta$, where $\gamma(t,\phi,\alpha,-r)$ is defined in Definition 1.1.

Choose α,β so that these relations are satisfied. For any nonnegative real $\bar{\alpha},\bar{\beta}$, let $\mathscr{A}(\bar{\alpha},\bar{\beta})$ be the set defined by

$$\mathscr{A}(\bar{\alpha},\bar{\beta}) = \{\xi \in C([-r-\bar{\alpha},0],E^n): \xi_0 = 0, \quad \xi_t \in B_{\bar{\beta}}, \quad t \in [-\bar{\alpha},0]\}.$$

For any $0 < \bar{\beta} < \beta$, there is an $\bar{\alpha}$, $0 < \bar{\alpha} < \alpha$, so that $|\hat{\varphi}_t - \phi| < \beta - \bar{\beta}$. Thus $|\xi_t + \hat{\phi}_t - \phi| \leq \bar{\beta} + \beta - \bar{\beta} = \beta$ and $(\sigma+t, \hat{\phi}_t + \xi_t) \in \Omega$ for $t \in [-\bar{\alpha},0]$, $\xi \in \mathscr{A}(\bar{\alpha},\bar{\beta})$. Further restrict $\bar{\alpha}$ so that

$$|B^{-1}(\sigma+t,\hat{\phi}_t)| \cdot |f(\sigma+t,\hat{\phi}_t) - f(\sigma,\phi)| \leq \nu\bar{\beta}$$

$$|B^{-1}(\sigma+t,\hat{\phi}_t)| \cdot |\dot{\phi}(0) - \dot{\phi}(t)| \leq \nu\bar{\beta}$$

for $t \in [-\bar{\alpha},0]$.

For any $\xi \in \mathscr{A}(\bar{\alpha},\bar{\beta})$, define the transformation $T: \mathscr{A}(\bar{\alpha},\bar{\beta}) \to C([-r-\bar{\alpha},0], E^n)$ by the relation

$$(3.6) \qquad (T\xi)(t) = 0, \qquad t \in [-r,0]$$

$$(T\xi)(t) = B^{-1}(\sigma+t,\hat{\phi}_t)\{-\int_{-r+}^{0}[d_\theta \mu(\sigma+t,\hat{\phi}_t,\theta)]\dot{z}_t(\theta) - f(\sigma+t,\hat{\phi}_t)$$
$$+ f(\sigma,\phi) - g(\sigma+t,\hat{\phi}_t,z_t) + \dot{\phi}(t) - \dot{\phi}(0)\}, \quad t \in [-\bar{\alpha},0].$$

By hypothesis $\dot{\phi}(0) = f(\sigma,\phi)$ and therefore the fixed points of T in $\mathscr{A}(\bar{\alpha},\bar{\beta})$ coincide with the solutions x of (3.3) on $[\sigma-r-\bar{\alpha},\sigma]$ with $x(\sigma+t) = \hat{\phi}(t) + z(t)$ where $z_t \in \mathscr{A}(\bar{\alpha},\bar{\beta})$, $t \in [-\bar{\alpha},0]$.

We now show that T is a contraction on $\mathscr{A}(\bar{\alpha},\bar{\beta})$. It is clear from (3.6) and the above restriction on $\bar{\alpha},\bar{\beta}$ that

$$|(T\xi)(t)| \leq \nu\bar{\beta} + \nu\bar{\beta} + \nu\bar{\beta} + \nu\bar{\beta} = 4\nu\bar{\beta} \leq \bar{\beta}$$

$$|(T\zeta)(t) - (T\xi)(t)| \leq \nu|\zeta_t-\xi_t| + \nu|\zeta_t-\xi_t| \leq \frac{1}{2}|\zeta_t-\xi_t|$$

for all $t \in [-\bar{\alpha},0]$, ζ, $\xi \in \mathscr{A}(\bar{\alpha},\bar{\beta})$. Therefore, $T: \mathscr{A}(\bar{\alpha},\bar{\beta}) \to \mathscr{A}(\bar{\alpha},\bar{\beta})$ and T is a contraction. Thus, there is a unique fixed point in $\mathscr{A}(\bar{\alpha},\bar{\beta})$ and this proves the theorem.

Theorem 3.3 is a generalization of a result of Hastings [8].

Corollary 3.1. Suppose Ω is an open set in $R \times C$, $f: D \to E^n$ is continuous, atomic at $-r$, and the solution $x(\sigma,\phi)$ of (3.1) through any $(\sigma,\phi) \in \Omega$ is unique. If $T(t,\sigma): C \to C$, $t \geq \sigma$, is defined by $T(t,\sigma)\phi = x_t(\sigma,\phi)$, then $T(t,\sigma)$ is one-to-one.

Proof: If the assertion is not true, then there are $\psi \neq \phi$ in C and a $t_1 > \sigma$ such that $x_{t_1}(\sigma,\phi) = x_{t_1}(\sigma,\psi)$, $x_t(\sigma,\phi) \neq x_t(\sigma,\psi)$, $0 \leq t < t_1$. If $x(t) = x(\sigma,\phi)(t)$, $y(t) = x(\sigma,\psi)(t)$, then $\dot{x}(t) = f(t,x_t)$, $\dot{y}(t) = f(t,y_t)$ for all $t > 0$ in the domain of definition of x. Since f is assumed to be atomic at $-r$, Theorem 3.3 implies there are an $\alpha = \alpha(t_1) > 0$ and a unique solution of (3.1) on $[t_1-r-\alpha,t_1-r]$ through (t_1,x_{t_1}), (t_1,y_{t_1}). Since $(t_1,x_{t_1}) = (t_1,y_{t_1})$ by hypotheses, it follows that $(t,x_t) = (t,y_t)$ for $t_1 - \alpha \leq t \leq t_1$.

This is a contradiction and proves the corollary.

4. Rate of Approach to Zero of Solutions of Linear Equations.

In this section, we prove

Theorem 4.1. Suppose $\Omega = (\tau,\infty) \times C$, $D(t,\varphi)$, $f(t,\varphi)$ in (1.1) are linear in φ, there is a positive constant k such that $|D(t,\varphi)| \leq k|\varphi|$, $|f(t,\varphi| \leq k|\varphi|$, $(t,\varphi) \in \Omega$ and D is uniformly atomic at 0 and $-r$ on Ω. For any $(\sigma,\varphi) \in \Omega$, there is a unique solution $x(\sigma,\varphi)$ of (1.1) through (σ,φ) which exists on (τ,∞) and, if a solution $x(t)$ approaches zero faster than any exponential as $t \to \infty$, then $x(t) \equiv 0$ for all $t \in (\tau,\infty)$.

Proof. The existence and uniqueness of the solution $x(\sigma,\varphi)$ on (τ,∞) follows from the results in Sections 2 and 3. Furthermore, following the same arguments as in Lemma II.1 in [5], one can show there are positive constants a, $b > 0$ such that for any $\sigma \in (\tau,\infty)$,

$$|x_t| \leq ae^{b|t-\sigma|}|x_\sigma|, \qquad t \in (\tau,\infty).$$

Suppose there is a $t \in (\tau,\infty)$ such that $|x_t| > 0$. Since $x(\sigma)$ approaches zero faster than any exponential as $\sigma \to \infty$, for any $\alpha > 0$, there is a constant $K(\alpha,t)$ such that $|x_\sigma| \leq K(\alpha,t)e^{-\alpha\sigma}$, $\sigma \in [t,\infty)$. Therefore, for $\sigma \geq t$,

$$0 < |x_t| \leq ae^{-bt}K(\alpha,t)e^{-(\alpha-b)\sigma}.$$

If α is chosen such that $\alpha > b$, then for σ sufficiently large, this gives a contradiction and proves the theorem.

The above theorem generalizes a result of Wright [9] for differential-difference equations.

For autonomous linear RFDE, one can prove that no nonzero solution can approach zero faster than any exponential as $t \to \infty$ provided that f is atomic at $-r$. The basic idea of the proof proceeds in the same manner but requires an estimate of the solution at time t in terms of the solution and the derivatives of the solution at time $\sigma > t$. For this case, D. Henry [10] using properties of entire functions has proved a much stronger result; namely, any solution approaching zero faster than any exponential as $t \to \infty$ must be identically zero after a fixed time (depending only on the equation and not the solution) even when f is not atomic at $-r$.

For nonautonomous linear periodic RFDE, examples are known (see [11]) for which nonzero solutions can approach zero faster than any exponential as $t \to \infty$. However, these examples have an f which is not atomic at $-r$.

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13. ABSTRACT

This paper is concerned with the development of neutral functional differential equations. Specifically, a class equations are considered which in some respect are more general than the ones considered by Hale and Cruz. This class has the advantage of leading in a very natural manner to the problem of the backward existence of solutions. The basic theory of existence, uniqueness, continuous dependence and continuation of solutions is developed and it is then shown that solutions of most linear equations with bound coefficients cannot have a nonzero solution which approaches zero faster than exponential.

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