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Finite Difference Solutions of the Boundary Layer Equations

by

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NOMENCLATURE

a_x	x-component of acceleration of fluid particle
D	$(u_{i+1,n} - u_{i-1,n})/2u_{i,n}$
f	force per unit mass or some functional relationships
g_x	x-component of gravitational acceleration
k	thermal conductivity
L	characteristic length
m	defined by Eq. (3.16)
M	$\frac{(\Delta y)^2 u_\infty}{v \Delta x}$, a parameter arising in the finite difference solution
Nu	Nusselt number
p	thermodynamic pressure
Pr	Prandtl number
q	heat flow rate
q''	heat flux
Re	Reynolds number
s	slot height (height of secondary stream)
t	time
u	x-component of velocity
v	y-component of velocity
v'	$v(\Delta x/\Delta y)$
x, y	coordinates, defined in Fig. 2.1
δ_x, δ_y	central difference operators
Δ_x, Δ_y	forward difference operators
$\tilde{\Delta}_y$	difference operator defined by Eq. (2.45)
ϵ	defined by Eq. (2.39)

η $y[u_\infty/\nu x]^{1/4}$
 μ dynamic viscosity
 ν kinematic viscosity
 ξ a dimensionless x-coordinate defined by Eq. (4.19)
 ρ density
 τ shear stress
 ψ stream function

Subscripts

i y-index, $y_i = (i - 1)\Delta y$
 I maximum value of index i ; $u_{I,n}$ is assumed to be unknown and $u_{I+1,n}$ is assumed to be known from the inviscid solution
 n x-index; $x_n = n\Delta x$
 N value of n at location x_N , where the local free stream velocity is u_∞ . This is usually the value of n at the termination of the calculation.
 $\circ$ a specific location of interest--exact meaning depends upon problem
 p primary stream
 r a selectable reference quantity
 s inviscid solution, condition at edge of boundary layer at location n . Also secondary stream.
 w condition at the wall, $y = 0$
 ∞ free stream condition at location N

Superscripts

$*$ a dimensionless quantity
 k the k th approximation obtained from the iterative application of the corrector

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1. INTRODUCTION

An investigation of the feasibility of employing film cooling as an alternate thermal protection system for spacecraft during atmospheric re-entry was suggested by Mr. R. T. Swann of the Entry Structures Branch of the Langley Research Center. Film or slot cooling involves the mixing of the high enthalpy external stream with the coolant in the vicinity of a solid wall which is the surface being protected. A boundary layer forms along the wall, and eventually interacts strongly with the external mixing zone.

A consideration of this problem revealed that the boundary layer equations could be employed to approximate the most important flow configurations which occur in film cooling. However, the equations to be solved, given this simplification, are complex, non-linear, coupled partial differential equations. Due to the relative weakness of classical methods of boundary layer analysis [1,2]†, finite difference techniques appear to be the best approach to the formidable problem under consideration.

A literature search revealed that the use of finite difference techniques for the solution of the boundary layer equations has become popular only recently. As a consequence, discrepancies and inconsistencies are common among recent publications. For these reasons, it was deemed desirable to undertake a systematic analysis of the application of finite difference techniques to the boundary layer equations. A better understanding of the accuracy and stability

†Numbers in brackets refer to entries in REFERENCES.

of finite difference techniques and the relative advantages and disadvantages of the various difference analogs and solution planes is an easily justified prerequisite to the study of the difficult problem of interest. The preliminary results of this study which was begun during the summer of 1968 are the subject of this report.

Finite difference techniques were employed for many years in the study of heat conduction. Like the boundary layer equations, the transient heat conduction equation, often called the diffusion equation, is a *parabolic equation*. Hence, the two areas have much in common. An extensive examination of the application of finite difference techniques to the solution of the diffusion equation was recently performed by the author. This study was published in detail [3] and includes several powerful difference analogs recently proposed. This report draws heavily on the analysis and techniques given in [3].

The numerical solution of parabolic equations is, in some ways, more difficult than other types of equations due to the fact that the calculations may become unstable. Difference analogs of the diffusion equation were shown by the author to be unstable only if the technique tolerates a violation of the physical laws, i.e., the laws of thermodynamics [3]. The large meaningless fluctuations that occur in such cases generally do not stem from the growth of roundoff error but from the absurd results obtained from calculations in violation of the physical laws. Simple hand calculations resulted in

such fluctuations even if exact arithmetic were employed. The stability of possible difference analogs of the boundary layer equations will be analyzed again by considering the physical problem.

The finite difference analogs of the boundary layer equations reported in the literature were, in all cases, derived by replacing the derivatives in the differential equations by difference quotients obtained either from other references or by employing Taylor series expansions. Emphasis in this report will be placed on developing physical models to show the physical approximations being made and why these approximations could lead to an unstable scheme.

In the past, much emphasis in boundary layer analysis has been placed on obtaining similarity transformations and the so-called "similar" solutions [1,2]. The system of partial differential equations can be reduced to ordinary differential equations for such cases. Schlichting [1] states that this procedure "evidently constitutes a considerable mathematical simplification of the problem." The resulting ordinary differential equations are, however, nonlinear and for this reason are often solved numerically. Is this rather indirect historical approach being employed due to the belief that the simplicity of this class of solutions would be lost in all cases if finite difference techniques were applied directly? This question of similarity in finite difference solutions will be considered in Section 3. The objective of the analysis is threefold;

- (1) to clarify the physical characteristics of similar flows,
- (2) to show the advantages of the recognition of similarity in finite difference solutions in the physical plane, and
- (3) to provide detailed comparisons to enable the reader to judge for himself whether similarity transformations result in a mathematical simplification if numerical techniques are employed.

A number of classical similar boundary layers are solved and the results are presented in Section 4. Comparisons with other solutions to these classical problems are given. The finite difference algorithms are by no means limited to similar flows and several non-similar cases are considered.

Finite difference solutions of the boundary layer equations in the von Mises plane have been popular. A recent report by Kleinstein [4] reviews some of the past work in the von Mises plane and gives a detailed account of their investigations. Other recent investigations which employed the von Mises variables are given in [5,6]. The use of von Mises variables has been especially popular in mixing problems.

Since the von Mises transformation eliminates one equation (the continuity equation) and the dependent variable v , it might appear quite natural to employ this transformation. It results in an equation similar to the heat conduction equation, and considerable experience has already been gained in finite difference solutions of this equation. However, several severe problems arise if the von Mises variables are employed in finite difference solutions. The problems stem from the singularity occurring at locations where the velocity u becomes zero. The relative disadvantages of employing the von Mises plane in finite difference solutions are given in Section 2 in order to make use of the relationships derived there. Due to these disadvantages, this report is concerned only with obtaining finite difference solutions in the physical plane.

2. DISCRETE MODELS OF LAMINAR INCOMPRESSIBLE BOUNDARY LAYERS

Rather than beginning the analysis with a completely general case, i.e., a turbulent, compressible, nonisothermal, nonsteady flow of a real gas mixture, a very simple case will be considered. Namely, a steady two-dimensional, laminar boundary layer will be analyzed. All properties of the fluid forming this boundary layer will be assumed to be constants.

It is felt that the salient features of finite difference techniques can be best presented if the simplicity of this elementary case is preserved, thus allowing comparisons with solutions obtained by less powerful techniques to be made. At the same time, the features of a finite difference procedure are preserved as the complexity of the problem increases; hence, the algorithms which will be given can often easily be extended to more complex cases. The objective of this report is to present only the basic fundamentals of a new technique for formulating finite difference models in boundary layer analysis.

The numerical procedures for solving the energy equation do not differ greatly from those which will be employed for the similar momentum equation. In addition, the momentum equation is not coupled with the energy equations since constant properties have been assumed. Therefore, a detailed presentation will be given for the momentum equation only. Several different algorithms will be derived and analyzed.

If the usual boundary layer assumptions are employed, the boundary layer equations for a constant property fluid are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.1)$$

and

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (2.2)$$

Equations (2.1) and (2.2) represent a set of coupled, non-linear partial differential equations which must be solved. The continuity equation, Eq. (2.1), will be employed to eliminate the dependent variable v from the momentum equation, Eq. (2.2).

Consider a general boundary layer flow as depicted in Fig. 2.1.† With suction or blowing through the surface, the boundary condition

$$v(x,0) = v_0(x) \quad (2.3)$$

will be used in conjunction with Eq. (2.1). In the absence of suction or blowing, the function $v_0(x)$ would be zero which is a more common boundary condition.

Equation (2.2) governs the variation of the dependent variable u with respect to x and y . An examination of Eq. (2.2) shows that it is a parabolic equation and that the independent variable x is analogous to a more common independent variable in parabolic equations, time. An initial condition must be specified and it is assumed to be of the form

$$u(0,y) = u_0(y) \quad (2.4)$$

†Figure 2.1 is the first figure in Section 2. Figures are located at the end of each section.

Equation (2.2) is second order with respect to the spatial variable y ; hence, two boundary conditions are required.

$$u(x,0) = 0 \quad , \quad u(x,\infty) = u_s(x) \quad (2.5)$$

where $u_s(x)$ is the velocity at the edge of the boundary layer and is obtained from the inviscid flow solution.

The standard boundary layer assumption that $\frac{\partial p}{\partial y} = 0$ gives

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{1}{\rho} \frac{dp}{dx} = u_s \frac{du_s}{dx} = f(x) \quad (2.6)$$

Hence, the first term on the right-hand side of Eq. (2.2) is a specified function obtained from the inviscid flow solution and p is not an additional dependent variable.

Many investigators, e.g., Wu (7), Lane et al. [8], Flügge-Lotz and Blottner [9], and Grabow [10], employed finite difference algorithms which required an initial condition in v .

$$v(0,y) = v_1(y) \quad (2.7)$$

This requirement, however, is inconsistent with the boundary layer approximations. The problem is overspecified.

Consider, for example, a flow over a flat plate. Wu [4] employed the initial profile deduced from the physical problem, $v(0,y) = 0$. However, it is well known that this is incompatible with the Blasius solution. This initial condition caused unnecessarily large errors

in Wu's solution. Flügge-Lotz and Blottner [6] argued that the initial condition employed by Wu was indeed incompatible and that the similarity solution should be employed to determine the correct initial condition. Hence, it would not be possible to solve the Blasius problem with their algorithm since a prior knowledge of the solution is required to determine a compatible initial condition $v(0,y)$ in order to start the calculation.

It appears that difficulties of the nature just described stem from placing too little emphasis on the physical problem. No published finite difference solution of the boundary layer equations employed, to this author's knowledge, a physical model in order to obtain the governing equations. Instead, the difference representations were derived by a mechanical substitution of difference quotients for derivatives. The required difference quotients are often derived employing Taylor series expansions. It is felt that the power of these methods lies mainly in analyzing the truncation errors and the mathematical consistency of the difference analog. On the other hand, physical models hold many other advantages; some of these were discussed in [3].

The continuity, momentum, and energy equations in *differential* form are sometimes derived by balances on either finite elemental volumes or finite elements. The limit of the resulting expression, the finite difference equation, is then taken as the size of the spatial increments approach zero to give the governing partial differential equations. The desired difference relationships will again be derived from a consideration of the physical laws, rather than

starting from the differential equations and working backwards.

Let

$$\begin{aligned}x &= n(\Delta x) \quad , \quad 0 < n \leq N \\ y &= (i - 1)\Delta y \quad , \quad 0 < i \leq I^\dagger\end{aligned}\tag{2.8}$$

Hence,

$$u(y, x) = u_{i, n}.$$

The index n is used as the second index for the independent variable x , since it is commonly employed in other areas for the time index and occurs last in the sequence of indices. The discrete flow field is shown in Fig. 2.1. An explicit difference analog will be derived first. It will be called the standard explicit method. Two implicit finite difference analogs will then be derived. They will be referred to as the standard implicit and the Crank-Nicolson methods. These names were derived from those usually given to analogous finite difference representations of the diffusion equation [3].

2.1 THE STANDARD EXPLICIT ALGORITHM

Although the standard explicit method may not be the most useful algorithm, it shows most clearly the salient features of finite difference solutions of boundary layer problems in the physical plane. Hence, it will be considered first.

[†]Although it would be more convenient to define y as $i\Delta y$, this definition is employed since some computer languages, such as FORTRAN, do not permit the subscript zero.

Consider the momentum equation which is an expression of Newton's second law. Specifically:

$$\left\{ \begin{array}{l} \text{the acceleration of} \\ \text{a fluid element in} \\ \text{the x-direction} \end{array} \right\} = \left\{ \begin{array}{l} \text{pressure forces per} \\ \text{unit mass acting on} \\ \text{the element in the} \\ \text{x-direction} \end{array} \right\} + \left\{ \begin{array}{l} \text{viscous forces per} \\ \text{unit mass acting on} \\ \text{the element in the} \\ \text{x-direction} \end{array} \right\} \quad (2.9)$$

The Lagrangian viewpoint will be used to determine the particle acceleration, the convective acceleration. The verbal statement in the form of Eq. (2.9) is employed in contrast to equating these forces to the rate of efflux of x-momentum across the elements surfaces in order to obtain linear difference equations in a more direct manner.

The convective x-acceleration is determined by following the particle and is

$$a_x = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \quad (2.10)$$

Consider the meaning of the terms in Eq. (2.10) in order to obtain a better understanding of the approximations made in replacing Eq. (2.10) by a difference expression. Equation (2.10) can be written as

$$a_x \Big|_{i, n+\frac{1}{2}} = \frac{\Delta U}{\Delta t} = \frac{\Delta X}{\Delta t} \frac{\partial u}{\partial x} \Big|_{i, n+\frac{1}{2}} + \frac{\Delta Y}{\Delta t} \frac{\partial u}{\partial y} \Big|_{i, n+\frac{1}{2}} \quad (2.11)$$

where ΔX and ΔY are the distances the particle moves during the time interval Δt in the x- and y-directions, respectively, (see Fig. 2.2).

The first term represents the x-acceleration due to the particle's convection of a distance ΔX in the x-direction; the second term is the x-acceleration due to the particle's convection of a distance ΔY in the y-direction.

Consider Fig. 2.2 which represents a region of the grid around the point i, n . Δt is chosen such that *at this point* $\Delta x = \Delta X$. The objective is to determine the change in u , $(u_{i, n+1} - u_{i, n})$, as steps a distance $\Delta x = (x_{n+1} - x_n)$ downstream are made. Thus, the change in u with respect to x will be approximated by the central difference quotient

$$\left. \frac{\partial u}{\partial x} \right|_{i, n+\frac{1}{2}} = \frac{u_{i, n+1} - u_{i, n}}{\Delta x} \quad (2.12)$$

in all difference analogs. The term $\Delta X/\Delta t$, which is the velocity of the particle in the continuous system, should be approximated by the average velocity of the particle during the step

$$\frac{\Delta X}{\Delta t} = u_{i, n+\frac{1}{2}} = \frac{u_{i, n+1} + u_{i, n}}{2} \quad (2.13)$$

The continuity equation, which will be considered later, conveniently provides an average velocity in the y-direction for this interval.

It will presently be denoted as $v_{i, n+\frac{1}{2}}$.

Next, consider the term $\left. \frac{\partial u}{\partial y} \right|_{i, n+\frac{1}{2}}$. Δt varies with respect to both i and n . It was chosen such that $\Delta x = \Delta X$; thus, the y-increment Δy in the discrete system will, in general, not be equal to ΔY . The velocities at the locations $y \pm \Delta Y/2$ are undefined. Therefore, the central

differences which will be used to approximate $\frac{\partial u}{\partial y}$ must be based on the distance $2(\Delta y)$ which could be vastly different from the distance Δy . The approximations would, therefore, seem to be inconsistent with the physical problem. This appears to be a source of minor instabilities and of minor inconsistencies in the accuracy of the various techniques (see Section 4).

Linear difference equations will be maintained in the standard explicit method, and approximations must be chosen so that an explicit relationship for $u_{i,n+1}$ is obtained. Therefore, in this algorithm $u_{i,n+\frac{1}{2}}$ and $\frac{\partial u}{\partial y}\bigg|_{i,n+\frac{1}{2}}$ will be approximated by

$$u_{i,n+\frac{1}{2}} \approx u_{i,n}$$

and

$$\frac{\partial u}{\partial y}\bigg|_{i,n+\frac{1}{2}} \approx \frac{u_{i+1,n} - u_{i-1,n}}{2\Delta y}$$

Thus, the x-acceleration, Eq. (2.10), becomes

$$a_x\bigg|_{i,n+\frac{1}{2}} = u_{i,n} \frac{u_{i,n+1} - u_{i,n}}{\Delta x} + v_{i,n+\frac{1}{2}} \frac{u_{i+1,n} - u_{i-1,n}}{2\Delta y} \quad (2.14)$$

Next consider the right-hand side of Eq. (2.9), i.e., the forces acting on the particle. Since the change in the x-velocity of the particle as it moves from location n to $(n+1)$ is being determined, the right-hand side of Eq. (2.9) should be evaluated when the particle is at the location $(n + \frac{1}{2})$ (see Fig. 2.3).

The net pressure force is known from the external inviscid solution, and follows from Fig. 2.3 as $p_n(\Delta y)(\text{depth}) - p_{n+1}(\Delta y)(\text{depth})$.

The force per unit mass is determined by dividing by the mass of the element, $\rho \Delta y \Delta x (\text{depth})$. Hence,

$$f_p \Big|_{n+\frac{1}{2}} \approx - \frac{1}{\rho} \frac{(p_{n+1} - p_n)}{\Delta x} \quad (2.15)$$

This force is independent of i due to the boundary layer assumptions. If the functional relationship defined by Eq. (2.6) is known, f_p becomes

$$f_p \Big|_{n+\frac{1}{2}} = f[(n+\frac{1}{2})\Delta x] \quad (2.16)$$

The net viscous force in the x -direction per unit mass f_v for boundary layer flow is (see Fig. 2.3)

$$f_v \Big|_{n+\frac{1}{2}} = \frac{\tau_{i+\frac{1}{2}, n+\frac{1}{2}} - \tau_{i-\frac{1}{2}, n+\frac{1}{2}}}{\rho(\Delta y)} \quad (2.17)$$

The sign convention used is: a stress is positive if both the outward normal to the surface on which the stress acts and the stress itself have senses in either the positive or negative x -direction. For laminar boundary layer flow, Stoke's law of viscosity simplifies to Newton's law of viscosity.

$$\tau = \mu \frac{\partial u}{\partial y} \quad (2.18)$$

If a central difference quotient is substituted for $\frac{\partial u}{\partial y}$ but a continuous independent variable x is retained, Eq. (2.18) becomes

$$\tau_{i-\frac{1}{2}} = \mu \frac{(u_i - u_{i-1})}{\Delta y} \quad (2.19)^\dagger$$

It is important to note that the shear stresses are known only at locations between defined grid points, $(i + \frac{1}{2})\Delta y$ where $i = 1, 2, \dots, I$. The discretized field consists of points of known velocity (mass points) and points of known shear stress (stress points). The results of the finite difference solution give accurate values for these quantities only at these points in space.

If the approximation is to remain explicit the shear stresses must be evaluated from known velocities. Thus, in the standard explicit method, the shear stresses would be based on the velocities at location n and Eqs. (2.17) and (2.19) give

$$f_v \Big|_{i, n} = \frac{\nu}{(\Delta y)^2} (u_{i+1, n} - 2u_{i, n} + u_{i-1, n}) \quad (2.20)$$

or

$$f_v \Big|_{i, n} = \frac{\nu}{(\Delta y)^2} \delta_y^2 u_{i, n}$$

where δ is the central difference operator.

If Eqs. (2.14), (2.15), and (2.20) are substituted into Eq. (2.9) the following expression is obtained.

[†]Experimentally derived laws are, by necessity, determined in the form of Eq. (2.19). The form of Eq. (2.18) is obtained by deductive reasoning.

$$u_{i,n} \frac{(u_{i,n+1} - u_{i,n})}{\Delta x} + v_{i,n+\frac{1}{2}} \frac{(u_{i+1,n} - u_{i-1,n})}{2\Delta y} = - \frac{p_{i,n+1} - p_{i,n}}{\rho \Delta x} + \frac{v}{(\Delta y)^2} \delta_y^2 u_{i,n} \quad (2.21)$$

If Eq. (2.21) is divided by $(u_{i,n})/\Delta x$ and rearranged,

$$u_{i,n+1} = u_{i,n} - v'_{i,n+\frac{1}{2}} \left[\frac{u_{i+1,n} - u_{i-1,n}}{2u_{i,n}} \right] - \frac{p_{i,n+1} - p_{i,n}}{\rho u_{i,n}} + \frac{v \Delta x}{(\Delta y)^2 u_{i,n}} \delta_y^2 u_{i,n} \quad (2.22)$$

where $v'_{i,n+\frac{1}{2}} = v_{i,n+\frac{1}{2}} \left(\frac{\Delta x}{\Delta y} \right)$. Equation (2.22) still contains the unknown velocity $v'_{i,n+\frac{1}{2}}$. In order to eliminate this quantity, consider the conservation of mass with steady flow, stated as

$$\left\{ \begin{array}{l} \text{net rate of mass} \\ \text{flow into the} \\ \text{control volume} \\ \text{across the con-} \\ \text{trol surface.} \end{array} \right\} - \left\{ \begin{array}{l} \text{net rate of mass} \\ \text{flow out of the} \\ \text{control volume} \\ \text{across the con-} \\ \text{trol surface.} \end{array} \right\} = 0 \quad (2.23)$$

Since the velocity at the point $v_{i,n+\frac{1}{2}}$ is desired, Eq. (2.23) is applied to the control volume shown in Fig. 2.4 which gives

$$\left\{ \frac{\Delta y}{2} (u_{i-1,n} + u_{i,n}) + v_{i-1,n+\frac{1}{2}} \Delta x \right\} \rho (\text{depth}) - \left\{ \frac{\Delta y}{2} (u_{i-1,n+1} + u_{i,n+1}) + v_{i,n+\frac{1}{2}} \Delta x \right\} \rho (\text{depth}) = 0$$

or dividing by $(\rho)(\text{depth})(\Delta y)$

$$v'_{i, n+1/2} = v'_{i-1, n+1/2} + \frac{1}{2} (u_{i-1, n} + u_{i, n} - u_{i-1, n+1} - u_{i, n+1}) \quad (2.24)$$

Equation (2.24) now can be used to eliminate $v'_{i, n+1/2}$ from Eq. (2.22).

$$u_{i, n+1} = u_{i, n} - D \left[v'_{i-1, n+1/2} + \frac{1}{2} (u_{i-1, n} + u_{i, n} - u_{i-1, n+1} - u_{i, n+1}) \right] \\ - \frac{p_{n+1} - p_n}{\rho u_{i, n}} + \frac{u_\infty}{Mu_{i, n}} \delta_y^2 u_{i, n}$$

where $D = \frac{u_{i+1, n} - u_{i-1, n}}{2u_{i, n}}$, $M = \frac{u_\infty (\Delta y)^2}{v \Delta x}$, and u_∞ is the free stream velocity at location N. This equation rearranged in order to obtain an explicit expression for $u_{i, n+1}$ is

$$u_{i, n+1} = \frac{2}{2 - D} \left\{ u_{i, n} - D \left[v'_{i-1, n+1/2} + \frac{1}{2} (u_{i-1, n} + u_{i, n} - u_{i-1, n+1} - u_{i, n+1}) \right] \right. \\ \left. - \frac{p_{n+1} - p_n}{\rho u_{i, n}} + \frac{u_\infty}{Mu_{i, n}} \delta_y^2 u_{i, n} \right\} \quad (2.25)$$

The values of $u_{i, n}$ are known from the initial condition ($n = 0$) or from the results of the previous step ($n > 0$). The values of $u_{1, n+1}$ and $v'_{1, n+1}$ are known from the boundary conditions. Thus, $u_{i, n+1}$ can be calculated from Eq. (2.25) if the calculation starts at $i = 2$ and proceeds in the direction of increasing i .

The calculation procedure for the standard explicit algorithm is:

1. Calculate $u_{i, n+1}$ employing Eq. (2.25) for a single value of i .

2. Since $u_{i,n+1}$ is now known from the results of step 1, calculate $v'_{i,n+1/2}$ from Eq. (2.24).
3. If $i < I$, increment i and return to step 1. If $i = I$ and $n < N$, increment n , set the i index to 2; and return to step 1. If $i = I$ and $n = N$, the calculation has been completed.

The following points are to be noted.

- (i) The standard explicit method enables the solution of boundary layer type problems without solving nonlinear algebraic equations or simultaneous algebraic equations. It requires no iterative correction.
- (ii) No initial condition is required for v . This is compatible with the boundary layer approximations but in contrast with the literature.
- (iii) The only *direct* occurrence of Δx and Δy is in the coefficient M which is $[u_\infty(\Delta y)^2]/v\Delta x$. It will be shown later that this quantity is closely related to the similarity variable η .
- (iv) With the network given in Fig. 2.2, u is determined at locations (i,n) , v is determined at locations $(i,n + 1/2)$, and the shear stresses can be calculated at locations $(i + 1/2,n)$.
- (v) The change in u , $(u_{i,n+1} - u_{i,n})$, as the particle moves from location n to $(n + 1)$ is based on the viscous forces acting on the particle at location n .

It is well known that some explicit finite difference schemes are unstable or may become unstable unless certain criteria are satisfied. Now consider the reason for such instabilities in the solution of the boundary layer equations.

Round-off errors or numerical errors will grow if an unstable solution procedure is being employed. It is often believed that the growth of these errors is the source of the meaningless fluctuations which occur if an unstable algorithm is employed. However, it would

appear that viscous effects would damp such errors if they were introduced. Why then, may an algorithm for the solution of a viscous flow problem become unstable?

It is the purpose of this treatment to show that the presence of instabilities stem from violations of physical laws. The arguments which will be given *are not intended as proofs*. They are given for the purpose of clarifying the source of instabilities. Although round-off errors would grow if an unstable scheme were employed, the meaningless fluctuations which often occur are due to the violations of physical laws which these schemes permit and would occur even if exact arithmetic were employed. A similar discussion with reference to the diffusion equation was given by the author [3].

Consider the change in the x-velocity component, u_i , proceeding from the location n to $n + 1$. Equation (2.22) gives

$$\begin{aligned} \left\{ \Delta u_i \right\}_{n \rightarrow n+1} &= -v'_{i, n+\frac{1}{2}} \left\{ \frac{u_{i+1} - u_{i-1}}{2u_i} \right\}_n + \frac{\Delta x}{u_{i, n}} \left\{ f_p \right\}_{n+\frac{1}{2}} \\ &\quad + \frac{\Delta x}{u_{i, n}} \left\{ f_v \right\}_n \end{aligned} \quad (2.26)$$

Consider what would happen if Eq. (2.26) were employed with large x-increments; i.e., let Δx approaches infinity. The coefficient of the last two terms of Eq. (2.26) is the time for the particle to travel the distance Δx , based on its velocity at location (i, n) . As Δx approaches infinity, this coefficient also approaches infinity; it is especially large if $u_{i, n}$ is near zero.

Consider first the pressure term.

$$\frac{\Delta x}{u_{i,n}} \left\{ \frac{f_p}{\rho u_{i,n}} \right\}_{n+1/2} = - \frac{p_{n+1} - p_n}{\rho u_{i,n}}$$

It is seen that this term depends on Δx due to the dependency of $(p_{n+1} - p_n)$ on Δx . The term will remain finite if $(p_{n+1} - p_n)$ remains finite. On the other hand, if the pressure gradient were a constant throughout the flow field, the effect of this force should be proportional to Δx and the force should become large in the discrete system as Δx becomes large. Of course, this force may be balanced by other forces which would limit the change in u if the other forces were represented correctly.

Next consider the change in $\{\Delta u_i\}_{n \rightarrow n+1}$ due to the viscous forces. It is seen that the change in the velocity of the fluid element as it moves from n to $(n + 1)$ is being based on the viscous forces evaluated at $n\Delta x$. Hence, this viscous force is constant and the size of this term is directly proportional to Δx . As Δx approaches infinity this term becomes infinite. This unbounded term is the basis for the instability of this scheme.

The change in the velocity of the element caused by the net viscous force is physically logical only if the net viscous force remains of the same sign if $u_{i,n}$ is replaced by $u_{i,n+1}$. If this step size is exceeded, the acceleration could be in a direction opposite to that of the resultant viscous force. Obviously, an approximation to Newton's second law which violates its sign could be catastrophic. This would be analogous to the violation of the second law of thermodynamics in

the solution of a heat conduction problem.

In order to determine the criterion which must be satisfied to avoid such potential sign violations, consider Eq. (2.26), but ignore the other term (e.g., consider a location where $v \approx 0$ and $\frac{\partial p}{\partial x} \approx 0$).

Then

$$(u_{i,n+1} - u_{i,n}) = \frac{\Delta x v}{u_{i,n} (\Delta y)^2} (u_{i+1,n} - 2u_{i,n} + u_{i-1,n}) \quad (2.27)$$

Thus, if

$$(u_{i+1,n} - 2u_{i,n} + u_{i-1,n}) > 0$$

then

$$(u_{i+1,n+1} - 2u_{i,n+1} + u_{i-1,n+1}) \geq 0$$

If the second inequality is substituted into Eq. (2.27), the following inequality results:

$$\frac{u_{i,n} (\Delta y)^2}{v \Delta x} \geq 2$$

or

$$M \left(\frac{u_{i,n}}{u_{\infty}} \right) \geq 2 \quad (2.28)$$

The criterion depends on the local velocity $u_{i,n}$; hence if

$u_{i,n}$ is small, the criterion can become restrictive. The criterion is the same as the one which arose in the solution of transient conduction problems with one spatial dimension, except that the time increment has been replaced by $\frac{\Delta x}{u_{i,n}}$ and the thermal diffusivity has been replaced by the kinematic viscosity ν .

The influence of the first term on the right-hand side of Eq. (2.26) must be studied further. It apparently gives rise to the criterion that

$$\left| \frac{u_{i+1,n} - u_{i-1,n}}{2u_{i,n}} \right| \leq 2 \quad (2.29)$$

This criterion causes little difficulty unless the flow field contains a step discontinuity across which the velocity ratio is greater than 5. It is interesting to note that this criterion contains neither Δx nor Δy . Physically, this criterion probably stems from the fact that $2\Delta y$ is not the distance the particle moves during the time interval $\Delta x/u_{i,n}$.

It might appear that some of the approximations used in the standard explicit method are inconsistent. For example, the pressure force at location $(n + \frac{1}{2})$ is being employed whereas the viscous forces are based on their values at location n . Obviously, the truncation error would be smaller if the viscous forces could also be evaluated at location $(n + \frac{1}{2})$; however, the results are not inconsistent due to the fact that a more accurate approximation is used for the pressure force than that employed for the viscous forces. On the other hand, if the shearing stress at $(i - \frac{1}{2})$ were always based on location $(n + 1)$ and the shearing stress at $(i + \frac{1}{2})$ were always based on location n , the approximation would be inconsistent with a

relatively large Δx and a consistency criterion would arise (see, e.g., Section 4.4 of [3]). An analysis of the truncation error showed that the approximation represented by Eq. (2.22) is consistent; i.e., the truncation error did vanish as Δx and Δy approached zero. A further verification of the consistency of the difference analog is the accurate results which were obtained with this method (see Section 4).

Next consider a comparison of the standard explicit algorithm with the solution of boundary layer problems in the von Mises plane. A boundary layer flow described by Eqs. (2.1) and (2.2) will be considered, but the pressure gradient will be assumed to be zero for further simplification. These equations in the von Mises plane reduce to (see [1])

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi} \left(v u \frac{\partial u}{\partial \psi} \right) \quad (2.30)$$

The argument usually given in favor of the use of the von Mises plane is that it eliminates the continuity equation and the dependent variable v . However, the dependent variable v in the procedure described in this section was also essentially eliminated from the momentum equation, and the alternate use of the continuity equation caused little additional work. Consider now what was accomplished by transforming Eqs. (2.1) and (2.2) to Eq. (2.30) if a finite difference technique were to be used.

If the standard explicit scheme described in Section 4.1 of [3] is employed in a finite difference solution of Eq. (2.30), the stability

criterion is

$$\frac{(\Delta\psi)^2}{v\Delta x u_{i,n}} \geq 2 \quad (2.31)$$

This would appear to have eliminated the troublesome areas where $u_{i,n}$ is small and made them the least troublesome. Why? The definition of $\Delta\psi$ is

$$\Delta\psi = u\Delta y$$

Hence, the criterion Eq. (2.31) is, in reality, identical to Eq. (2.28)!

In a boundary layer analysis, more detail is required near the surface in order to obtain accurate values for the shear stress on the surface and the heat flux through the surface. However, if constant ψ increments are employed, a coarse network results in the important area near the surface and a fine network occurs in the area of little interest-- the edge of the boundary layer. This, obviously, is a wasteful situation and the reason the stability criterion, Eq. (2.31), appears to be more easily satisfied. Lastly, since $\Delta y = \frac{\Delta\psi}{u}$, a singularity exists at the surface which makes the transformation back to the physical plane difficult.

Although the standard explicit algorithm is probably the least complex, the stability criterion can become burdensome. Implicit methods which remove this problem will be considered next. It will be seen that these methods require the same magnitude of effort as the explicit methods, because the equations can be reduced to a set of linear

simultaneous equations with a tridiagonal coefficient matrix.

2.2 THE STANDARD IMPLICIT AND THE CRANK-NICOLSON ALGORITHMS

The approximations which were employed in the standard explicit method were chosen such that

- (i) the resulting expression would be linear in $u_{i,n+1}$, and
- (ii) the resulting difference equation would contain a single unknown, $u_{i,n+1}$.

These two restrictions will now be removed and a set of nonlinear simultaneous algebraic equations will be obtained. The procedure to be employed in the solutions of these equations will then be considered.

Consider first the x-acceleration. Since linear equations will no longer be maintained, Eqs. (2.12) and (2.13) can be employed to determine the x-acceleration due to the particles convection of a distance Δx in the x-direction, or

$$\left[u \frac{\partial u}{\partial x} \right]_{i,n+\frac{1}{2}} \approx \frac{(u_{i,n} + u_{i,n+1})}{2} \left\{ \frac{u_{i,n+1} - u_{i,n}}{\Delta x} \right\} \quad (2.31)$$

The continuity of mass will be applied in an identical fashion which will result in the same expression for $v_{i,n+\frac{1}{2}}$. The term $\frac{\partial u}{\partial y} \Big|_{i,n+\frac{1}{2}}$ will be approximated by

$$\frac{\partial u}{\partial y} \Big|_{i,n+\frac{1}{2}} \approx \left\{ \frac{u_{i+1,n+1} + u_{i+1,n} - u_{i-1,n+1} - u_{i-1,n}}{4\Delta y} \right\} \quad (2.32a)$$

and

$$\frac{\partial u}{\partial y} \Big|_{i,n+\frac{1}{2}} \approx \left\{ \frac{u_{i+1,n+1} - u_{i-1,n+1}}{2\Delta y} \right\} \quad (2.32b)$$

in the Crank-Nicolson and the Standard Implicit methods, respectively.

The pressure force will be approximated in an identical fashion in all three schemes; that is, the acceleration of the particle is based on the pressure force when the particle is at the location $(n + \frac{1}{2})$. This is, of course, a known quantity. Equations (2.15) and (2.16) give this force in terms of the quantities in the discrete system.

Next consider the viscous forces. In the explicit method, the change in the u as the particle moved from locations n to $(n + 1)$ was based on the viscous forces acting on the particle at location n . This approximation was necessary in order to avoid the introduction of additional unknowns. However, this approximation was the source of the stringent stability criterion in the explicit method. The Crank-Nicolson and the Standard Implicit methods will employ, respectively

$$\{f_v\}_{n+\frac{1}{2}} = \frac{1}{2} \{f_v\}_n + \frac{1}{2} \{f_v\}_{n+1}$$

or

$$\{f_v\}_{n+\frac{1}{2}} = \frac{v}{2(\Delta y)^2} \left\{ \delta_y^2 u_{i,n} + \delta_y^2 u_{i,n+1} \right\} \quad (2.33a)$$

and

$$\{f_v\}_{n+1} = \frac{v}{(\Delta y)^2} \delta_y^2 u_{i,n+1} \quad (2.33b)$$

If these approximations are substituted into Eq. (2.9), the following expressions are obtained.

Crank-Nicolson

$$\begin{aligned}
 (u_{i,n+1} - u_{i,n}) = & \frac{-v'_{i,n+1/2}}{u_{i,n+1/2}} \left\{ \frac{u_{i+1,n+1} + u_{i+1,n} - u_{i-1,n+1} - u_{i-1,n}}{4} \right\} \\
 & - \frac{1}{\rho u_{i,n+1/2}} \{P_{n+1} - P_n\} + \frac{1}{2M} \left(\frac{u_{\infty}}{u_{i,n+1/2}} \right) \\
 & \cdot \left\{ \delta_y^2 u_{i,n} + \delta_y^2 u_{i,n+1} \right\} \quad (2.34)
 \end{aligned}$$

where

$$u_{i,n+1/2} \equiv \frac{(u_{i,n+1} + u_{i,n})}{2}$$

Standard Implicit

$$\begin{aligned}
 (u_{i,n+1} - u_{i,n}) = & - \frac{v'_{i,n+1/2}}{u_{i,n+1/2}} \left\{ \frac{u_{i+1,n+1} - u_{i-1,n+1}}{2} \right\} \\
 & - \frac{1}{\rho u_{i,n+1/2}} \{P_{n+1} - P_n\} \\
 & + \frac{1}{M} \left(\frac{u_{\infty}}{u_{i,n+1/2}} \right) \left\{ \delta_y^2 u_{i,n+1} \right\} \quad (2.35)
 \end{aligned}$$

Equations (2.34) and (2.35) still contain the unknown $v'_{i,n+1/2}$. An expression for this quantity was previously derived, Eq. (2.24).

$$v'_{i,n+1/2} = v'_{i-1,n+1/2} + \frac{1}{2} (u_{i-1,n} + u_{i,n} - u_{i-1,n+1} - u_{i,n+1}) \quad (2.24)$$

It must be remembered that $v'_{i-1, n+1/2}$ is a function of all $u_{k, n+1}$ for $k < i$. If Eq. (2.24) were substituted into Eqs. (2.34) and (2.35), the k th equations would contain, in general, k unknowns: $u_{k+1, n+1}$, $u_{k, n+1}$, $\dots$, $u_{2, n+1}$. Furthermore, the equations are nonlinear in several of these unknowns.

If the coefficients of those terms of Eqs. (2.34) and (2.35) which are enclosed in braces were known, the equation would be linear. Since these coefficients are unknown, consider how they might be approximated. One possibility is to base them on the results of the previous step, i.e., on the values at location $n\Delta x$. Equation (2.34) or (2.35) then could be solved iteratively. The results from the iteration would be employed to correct the coefficients after each iteration. Unfortunately, the iteration does not converge rapidly unless the initial approximation for these coefficients is accurate. The results of the previous iteration do not afford an accurate approximation especially when Δx is large. This problem is more severe for compressible flows since approximations for $T_{i, n+1}$ are required in order to calculate the density, viscosity, and thermal conductivity, and additional equations must be considered simultaneously in the iterations. Hence, a more accurate prediction of these coefficients is desirable.

A method which is similar to the predictor-corrector methods employed in the solution of ordinary differential equations will be used. A "predictor" will be derived to predict $u_{i, n+1}$. The continuity equation

will then be used to obtain $v'_{i,n+1/2}$ and the momentum equation employed iteratively as the "corrector." The continuity equation must be employed after each iteration to correct $v'_{i,n+1/2}$.† The predictor provides accurate estimates of the required dependent variables after the first few steps; hence, the corrector usually must be applied only once. The corrector, Eq. (2.34) or (2.35) adds the required stability to the overall process.

A predictor for $u_{i,n+1}$ is obtained by a Taylor series expansion around the point $n - 1$.

$$u_{i,n+1}^0 = u_{i,n-1} + (2\Delta x) \left. \frac{\partial u}{\partial x} \right|_{i,n-1} + \frac{(2\Delta x)^2}{2!} \left. \frac{\partial^2 u}{\partial x^2} \right|_{i,n-1} + \dots$$

If this series is truncated with three terms and if the $\left. \frac{\partial u}{\partial x} \right|_{i,n-1}$ and $\left. \frac{\partial^2 u}{\partial x^2} \right|_{i,n-1}$ are approximated by average and central difference quotients, respectively, the following expression is obtained.

$$u_{i,n+1}^0 = 3u_{i,n} - 3u_{i,n-1} + u_{i,n-2}, \quad n > 1 \quad (2.36a)$$

Equation (2.36a) is identical to that which is obtained if the differences $(u_{i,n} - u_{i,n-1})$ and $(u_{i,n-1} - u_{i,n-2})$ are extrapolated linearly to determine $(u_{i,n+1} - u_{i,n})$. Alternatively, it is the same as that obtained by passing a parabola through the points $u_{i,n-2}$,

†In a compressible flow problem, $u_{i,n+1}$ and $T_{i,n+1}$ must be predicted; v', μ, ρ , and k are then calculated. The momentum and energy equations serve as the corrector; however, v', μ, ρ , and k must be recalculated after each application of the corrector.

$u_{i,n-1}$, and $u_{i,n}$ and extrapolating this curve to the location $n + 1$.

The predictor given by Eq. (2.36a) is valid only for $n > 1$, and estimates for $u_{i,n+1}$ are also required for n equal to zero and one. A linear extrapolation of the velocities is used for $n = 1$; that is,

$$u_{i,2}^0 = 2u_{i,1} - u_{i,0} \quad (2.36b)$$

and the initial condition is used for $n = 0$, i.e.,

$$u_{i,1}^0 = u_{i,0} \quad (2.36c)$$

The approximation for $u_{i,1}^0$ is arbitrary so long as the iteration converges. The initial condition, Eq. (2.36c), was used in the examples presented in Section 4, and no difficulties arose.

Since the corrector must be applied iteratively, a means of terminating this iterative process is required. The superscript in $u_{i,n}^k$ denotes the k th approximation obtained from the corrector. The predicted value is $k = 0$. Rather than mechanically specifying, e.g., q iterations, it would be more desirable to have a meaningful test to determine "convergence" of the iteration. Additional iterations would then be performed only if necessary and the efficiency of the algorithm would be improved. What would be a suitable criterion?

Consider the application of a mass balance† to the control volume

†The use of a mass balance means that the tests described are also applicable to compressible flows.

shown in Fig. 2.5.

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{mass flow} \\ \text{into c.v.} \\ \text{across ad} \end{array} \right\} + \left\{ \begin{array}{l} \text{rate of} \\ \text{mass flow} \\ \text{into c.v.} \\ \text{across ab} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate of} \\ \text{mass flow} \\ \text{out of c.v.} \\ \text{across cd} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate of} \\ \text{mass flow} \\ \text{out of c.v.} \\ \text{across bc} \end{array} \right\} = 0 \quad (2.38)$$

The first two terms of Eq. (2.38) are constants throughout the iteration. The results of an iteration give the fourth term which is used in conjunction with Eq. (2.38) or Eq. (2.24) to calculate the third term, $\Delta x(\rho v)_{I, n+1/2}$. Thus, an examination of the change in this term between iterations provides a simple yet meaningful test of the degree of convergence of the iteration. Specifically, the change in $(\rho v')_{I, n+1/2}$ which is equal to $(\rho v)_{I, n+1/2}(\Delta x/\Delta y)$ is examined. The change in this quantity is identically equal to the change in

$$(\rho u)_{I, n+1/2} + \sum_{i=2}^{I-1} (\rho u)_{i, n+1}$$

which is $1/\Delta y$ times the rate of mass flow per unit depth through bc. The division by Δy removes the dependence of the test on the absolute size of the spatial increments.

The specific test employed was

$$(100 \text{ percent}) \left| \frac{[(\rho v')_{I, n+1/2}]^k - [(\rho v')_{I, n+1/2}]^{k-1}}{[(\rho v')_{I, n+1/2}]^k} \right| \leq \epsilon \quad (2.39a)$$

to reduce the probability of accidentally satisfying Criterion (2.39a), a second criterion

$$(100 \text{ percent}) \cdot \left| \frac{\rho u_{2, n+1}^k - \rho u_{2, n+1}^{k-1}}{\rho u_{2, n+1}^k} \right| \leq 0.1\epsilon \quad (2.39b)$$

was also employed. Criterion (2.39b) is generally less stringent than Criterion (2.39a).

A value of $\epsilon = 1$ was employed for the bulk of the results presented in Section 4 and proved to be sufficiently stringent for engineering calculations in all cases studied. Often, both tests were passed immediately and usually no iterations of the corrector were necessary after the calculations had proceeded a few steps. At times, the predicted and corrected values agreed to the six significant figures which were printed.

The solution of the linearized set of simultaneous equations represented by Eq. (2.34) or (2.35) will now be considered. In order to effect the solution with the algorithm of Section 3.1 of [3], it is convenient to rewrite the equations in the form

$$A_i x_{i-1} + B_i x_i + C_i x_{i+1} = b_i, \quad 1 < i \leq I$$

or the difference form of the momentum equation becomes

$$A_i u_{i-1, n+1}^k + B_i u_{i, n+1}^k + C_i u_{i+1, n+1}^k = b_i, \quad 1 < i \leq I \quad (2.40)$$

The respective constants for the *Crank-Nicolson Algorithm* are

$$\left. \begin{aligned} A_1 &= -(c_1 + c_2) , \quad B_1 = 2c_2 + 1 , \quad C_1 = c_1 - c_2 \\ b_1 &= u_{1,n} - c_1(u_{1+1,n} - u_{1-1,n}) + c_2(u_{1+1,n} + u_{1-1,n} - 2u_{1,n}) + c_3 \end{aligned} \right\} \quad (2.41)$$

where

$$c_1 = \frac{[v'_{i,n+\frac{1}{2}}]^{k-1}}{4u_{i,n+\frac{1}{2}}^{k-1}} ; \quad c_2 = \frac{u_\infty}{2Mu_{i,n+\frac{1}{2}}^{k-1}} ; \quad c_3 = - \frac{(p_{n+1} - p_n)}{\rho u_{i,n+\frac{1}{2}}^{k-1}}$$

These follow from the definitions of A_1 , B_1 , C_1 , and b_1 , Eq. (2.40), combined with considerable rearrangement of Eq. (2.34). The respective constants for the *Standard Implicit Algorithm* are

$$\left. \begin{aligned} A_1 &= -(c_1 + c_2) , \quad B_1 = 2c_2 + 1 , \quad C_1 = c_1 - c_2 \\ b_1 &= u_{1,n} + c_3 \end{aligned} \right\} \quad (2.42)$$

where

$$c_1 = \frac{[v'_{i,n+\frac{1}{2}}]^{k-1}}{2u_{i,n+\frac{1}{2}}^{k-1}} , \quad c_2 = \frac{u_\infty}{Mu_{i,n+\frac{1}{2}}^{k-1}} , \quad c_3 = - \frac{p_{n+1} - p_n}{\rho u_{i,n+\frac{1}{2}}^{k-1}}$$

These quantities are, of course, based on Eq. (2.35).

If the coefficients of Eq. (2.40) have been determined, the algorithm of Section (3.1) of [3] would require $8(I-1)$ operations to solve for the $(I-1)$ unknowns $u_{2,n+1}$, $u_{3,n+1}$, ..., $u_{I,n+1}$. The algorithm is essentially a Gauss Elimination. The applicability of this simple algorithm removes the biggest disadvantage of implicit methods.

The simultaneous equations can be solved with little or no additional work (per iteration) over that required with explicit difference equations.

The calculation procedure for the implicit techniques is:

1. Calculate the initial approximations for the coefficients as defined by Eqs. (2.41) or (2.42) based on the approximation of Eq. (2.36).
2. Solve the set of simultaneous equations represented by Eq. (2.40). Employ the algorithm represented by Eqs. (3.4) and (3.5) of [3] to effect this solution. This gives the k th approximation for the velocities $u_{i,n+1}^k$, $1 < i \leq I$.
3. Apply the convergence tests, Criteria (2.39a) and (2.39b). If these tests are satisfied, go to Step 4. If these tests are not satisfied, set $k = k + 1$ and return to Step 2.
4. If $n < N$, increment n and return to Step 1. If $n = N$, the calculation has been completed.

The following points concerning these two implicit techniques are to be noted.

1. Although the momentum and continuity equations are no longer "integrated" concurrently as they were in the explicit algorithm, the proposed implicit techniques do not require an initial condition for v .
2. The only direct occurrence of Δx and Δy is again in the coefficient M .
3. The Crank-Nicolson technique would generally be expected to be more accurate than the Standard Implicit technique; however, the difference equations are more complicated and it would require slightly more computational time for each x -increment.

4. Since one application of the corrector is often sufficient, the implicit methods require only slightly more computational effort than that required by the standard explicit method per step downstream.

The change in u as the particle moves from location n to $(n + 1)$ is based on the viscous forces acting on the particle at location n in the explicit algorithm. This gave rise to a rather stringent stability criterion to avoid potential violations of the sign of Newton's second law. In the Standard Implicit method, the viscous forces at the end of the step downstream are employed which eliminates completely such difficulties.

The Crank-Nicolson method bases the change in u on the average of the forces at location n and $(n + 1)$. If a sufficiently large Δx were employed, a sign violation might still occur. This is because the change in u due to the forces at n is still directly proportional to Δx ; however, now Δx can be twice as large before a potential violation could occur. In addition, the inclusion of the viscous forces at location $(n + 1)$ bound the maximum violation which can occur. This is in direct contrast to the explicit method. This then prohibits the growth of round-off error and damps oscillations which might occur.

A more detailed physical analysis of the stability of implicit schemes is given in [3], for the diffusion equation. Since this equation is so similar, further details will not be given here. Both methods appear to be unconditionally stable.

Table 2.1 gives a brief comparison of the implicit and explicit

methods. Relative to the Standard Implicit algorithm, the Crank-Nicolson algorithm has the added advantage of greater accuracy and the minor disadvantage of greater complexity. Why was the Standard Implicit algorithm presented? What relative advantage might it have?

It would appear to be advantageous if extremely large x -increments were employed (small values of M) in order to minimize the required computational time at the expense of accuracy. In this case, oscillations might occur with the Crank-Nicolson algorithms due to violations of physical laws. Such oscillations would not be expected if the Standard Implicit algorithm were employed. Even slight oscillations in the nodal velocities can have catastrophic influence on the shear stresses. On the other hand, if the Crank-Nicolson results were averaged to remove such oscillations occurring, the averaged results would again be more accurate than the Standard Implicit results. Further study is needed in this area. (See also Section 4.)

The difference equation for the Standard Implicit approximation is by no means unique. Several different ways of representing the convective x -acceleration could be employed. The name is based on the fact that the change in u as the particle moves from n to $(n + 1)$ is based on the viscous forces at the end of the step, location $(n + 1)$. This is the important difference.

TABLE 2.1 A COMPARISON OF IMPLICIT AND EXPLICIT METHODS

	Implicit Methods	Standard Explicit Method
Advantages	(i) Stability (ii) Greater accuracy in some cases (Crank-Nicolson only)	(i) A simple algorithm (ii) Minimum computational effort per step
Disadvantages	(i) The algorithms are relatively complex (ii) An iterative procedure must be employed (iii) More computational effort is required per step	(i) A stringent stability criterion must be satisfied

2.3 DETERMINATION OF THE SHEAR STRESS AT THE WALL

Knowledge of accurate velocity and temperature profiles in boundary layer analysis is of secondary importance. The determination of the heat flux and the shear stress at the surface usually is the primary objective. Since the determination of both the heat flux and the shear stress gives rise to the same problems, only the evaluation of the shear stress at the wall will be described.

The wall shear stress is given by

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \quad (2.43)$$

If the continuous variation of u with respect to y were known, the calculation would be trivial. In numerical solutions, u is determined only at discrete y locations and differentiation is no longer simple. The problem is further complicated in the present case since only forward differences can be employed.

Equation (2.16) of [3] relates the differential operator to the forward difference operators. One or more terms of this equation are commonly used to obtain approximations for a desired derivative. For example, one term gives

$$\mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \underset{(i=1)}{\approx} \frac{\mu}{\Delta y} (u_2 - u_1) = \tau_a \quad (2.44a)$$

Two terms give

$$\tau_b = \frac{\mu}{\Delta y} \left(-\frac{u_3}{2} + 2u_2 - \frac{3}{2}u_1 \right) \quad (2.44b)$$

and three terms give

$$\tau_c = \frac{\mu}{\Delta y} \left(\frac{u_4}{3} - \frac{3}{2}u_3 + 3u_2 - \frac{11}{6}u_1 \right) \quad (2.44c)$$

Unfortunately, the addition of more terms does not mean an automatic increase in accuracy. The mechanical use of such relationships is dangerous.

The shear stresses were written as functions of the discrete velocities u_i in the derivation of the governing difference equations. To be consistent, the same representations should be employed when the procedure is reversed. However, the stresses are only defined at discrete locations, $(i + \frac{1}{2})\Delta y$ and the surface is not a defined stress point (see Section 2.1).

The relationships between the defined shear stresses in the discrete system and Eqs. (2.44a) through (2.44c) will now be examined. Equation (2.44a) is $\tau_{3/2}$; i.e., if this approximation is employed, the shear stress at the location $\frac{\Delta y}{2}$ is used to approximate the shear stress at the wall. Equation (2.44b) represents an extrapolation to the surface of a line passing through $\tau_{3/2}$ and $\tau_{5/2}$. Equation (2.44c) is slightly different from the results obtained by extrapolating the second degree curve passing through $\tau_{3/2}$, $\tau_{5/2}$, and $\tau_{7/2}$ to the surface. If the result of this parabolic extrapolation is denoted as τ'_c , then

$$\tau_c = \frac{8}{9} \tau'_c + \frac{1}{9} \tau_b$$

An extrapolation more accurate than a linear extrapolation seems possible; however, a parabolic extrapolation tends to hook around and usually gives little or no improvement. A good compromise appears to be an average of these two extrapolations since they tend to bound the correct value.

An arithmetic average, $\frac{(\tau_b + \tau_c')}{2}$ gives

$$\tau_w = \frac{\mu}{\Delta y} \left(\frac{3}{16} u_4 - \frac{17}{16} u_3 + \frac{41}{16} u_2 - \frac{27}{16} u_1 \right) \quad (2.44d)$$

This expression was employed in the calculation of the shear stresses at the wall reported in Section 4.

In order to compare Eqs. (2.44a) through (2.44d) consider a stagnation flow. This flow provides a stringent test because, unlike flat plate flow, the shear stress decreases rapidly upon leaving the surface. Table 2.2 gives a comparison of results. A coarse network was employed.

Since Eq. (2.44d) or expressions similar to it will be employed frequently in Section 4, this first-order forward difference operation will be defined by the operator $\tilde{\Delta}_y$; that is

$$\tilde{\Delta}_y u_i = \frac{3}{16} u_{i+3} - \frac{17}{16} u_{i+2} + \frac{41}{16} u_{i+1} - \frac{27}{16} u_i \quad (2.45)$$

With this definition the shear stress at the wall $i = 1$, becomes

$$\tau_w = \mu \frac{\tilde{\Delta}_y u_1}{\Delta y}$$

TABLE 2.2 COMPARISON OF VARIOUS TECHNIQUES OF DETERMINING THE WALL SHEAR STRESS ($M_i = 4$, $m = 1$, $N_i = 25$)

i	$\frac{u_i}{u_\infty}$	τ_i^{*++}	$\tau_i^* [2]$
1	0	†	1.2326
(1.5)†	†	1.039	1.0340
2	0.415	†	
(2.5)†	†	0.679	0.6750
3	0.687	†	
(3.5)†	†	0.403	0.3980
4	0.848	†	

Equation	$\tau^*(\text{wall})$	% Error
2.44a	1.0389	-15.8
2.44b	1.2186	-1.10
2.44c	1.2462	+1.10
Parabolic Extrapolation	1.2496	+1.40
2.44d	1.2341	+0.10

†Undefined in discrete system

††The dimensionless shear stress is defined in Section 4.1 by Eq. (4.3).

$$u_{\infty} = u_s(x), \quad y \rightarrow \infty$$

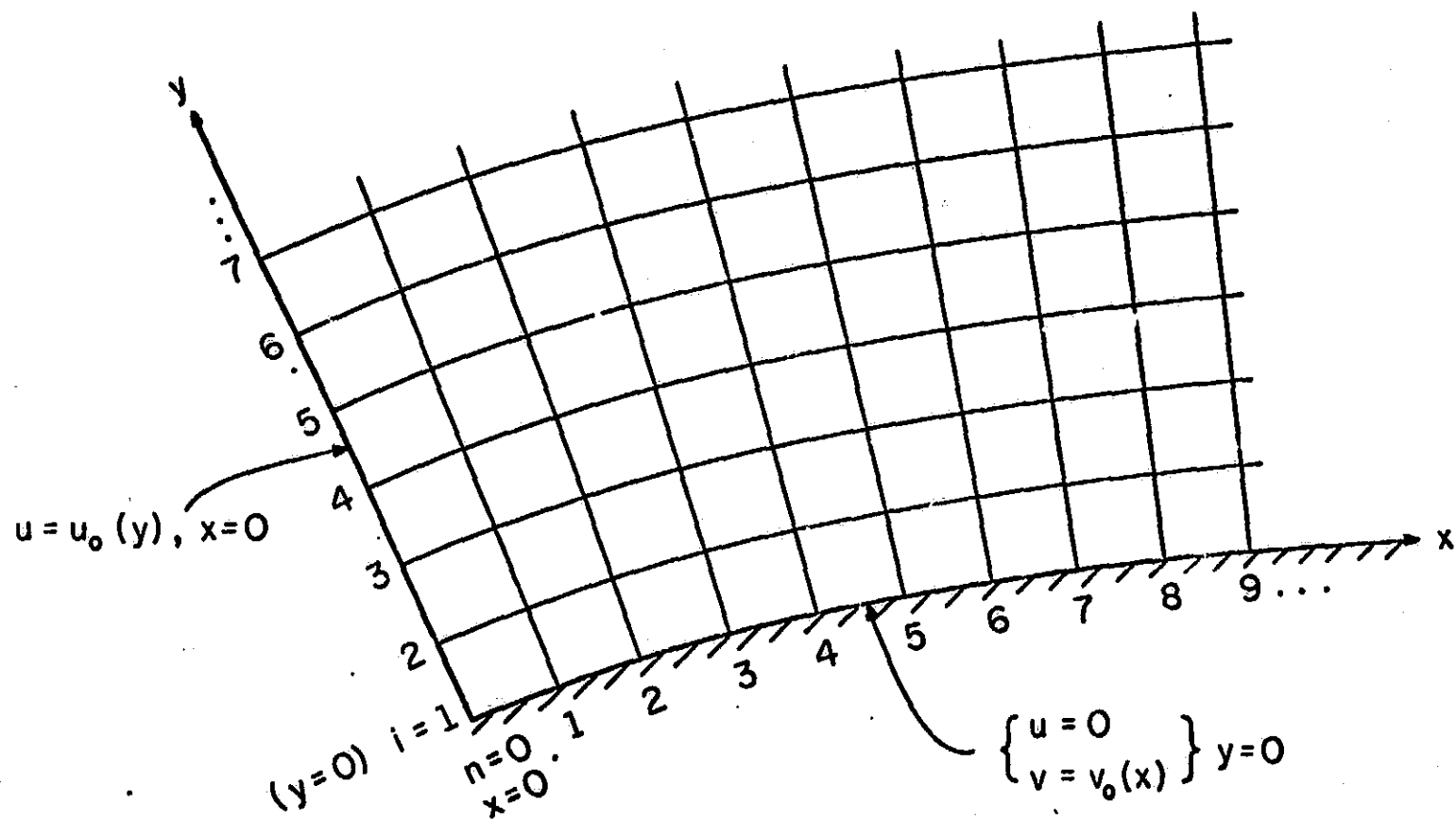


Figure 2.1 Discretized Flow Field

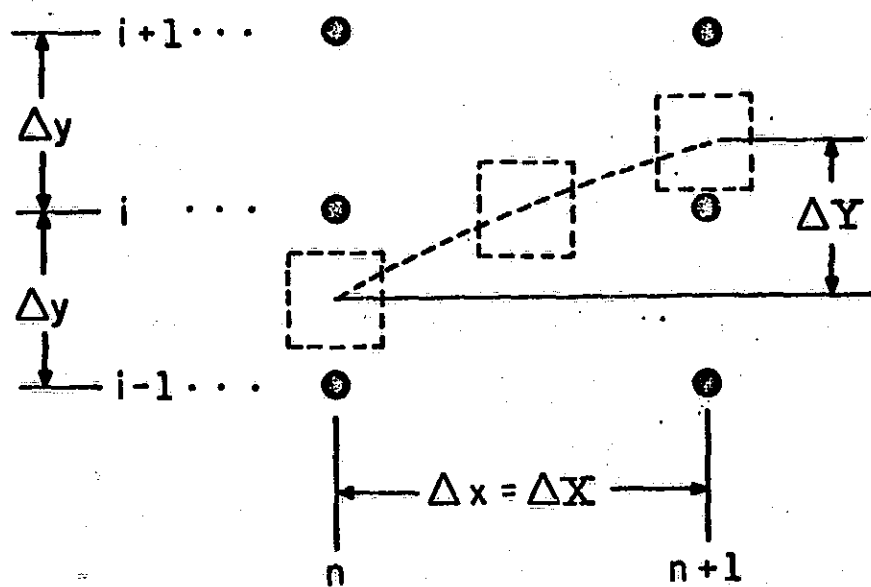


Figure 2.2 Pathline of Fluid Element

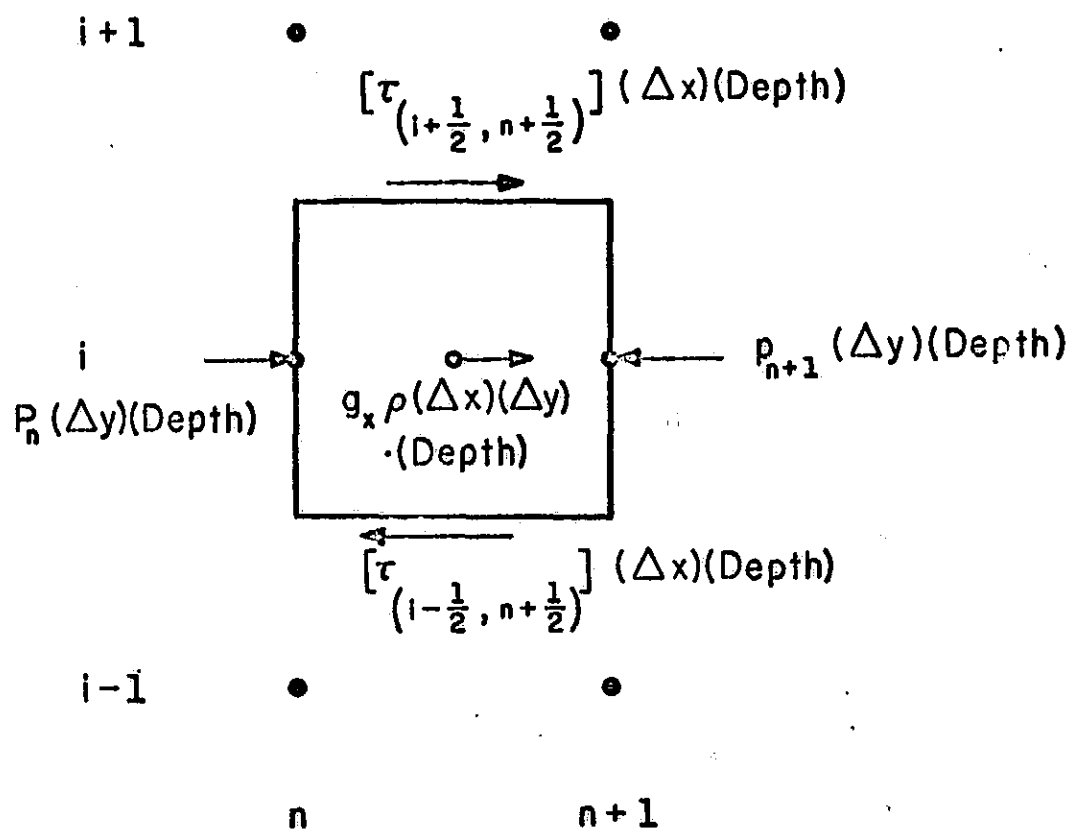


Figure 2.3 x-Forces Acting on Fluid Element

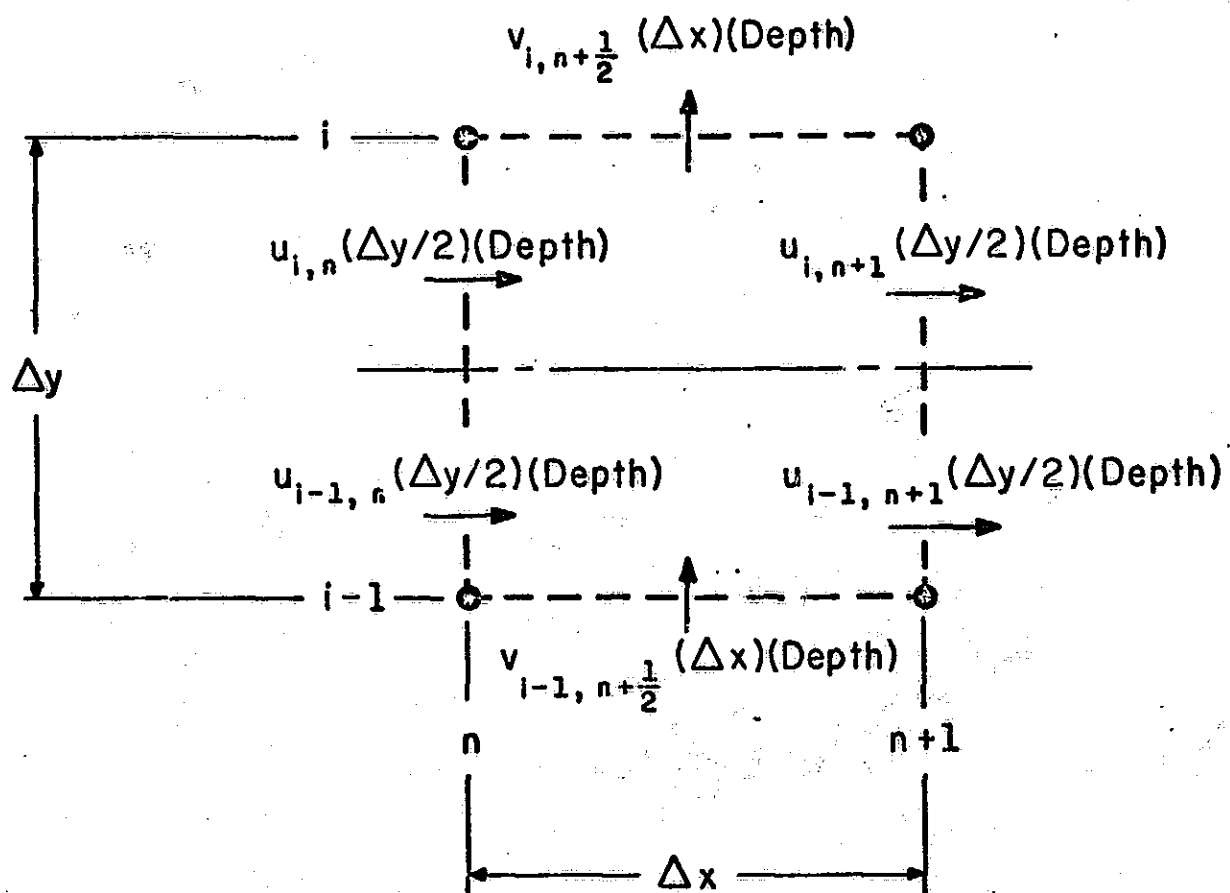


Figure 2.4 Control Volume for Application of Mass Balance

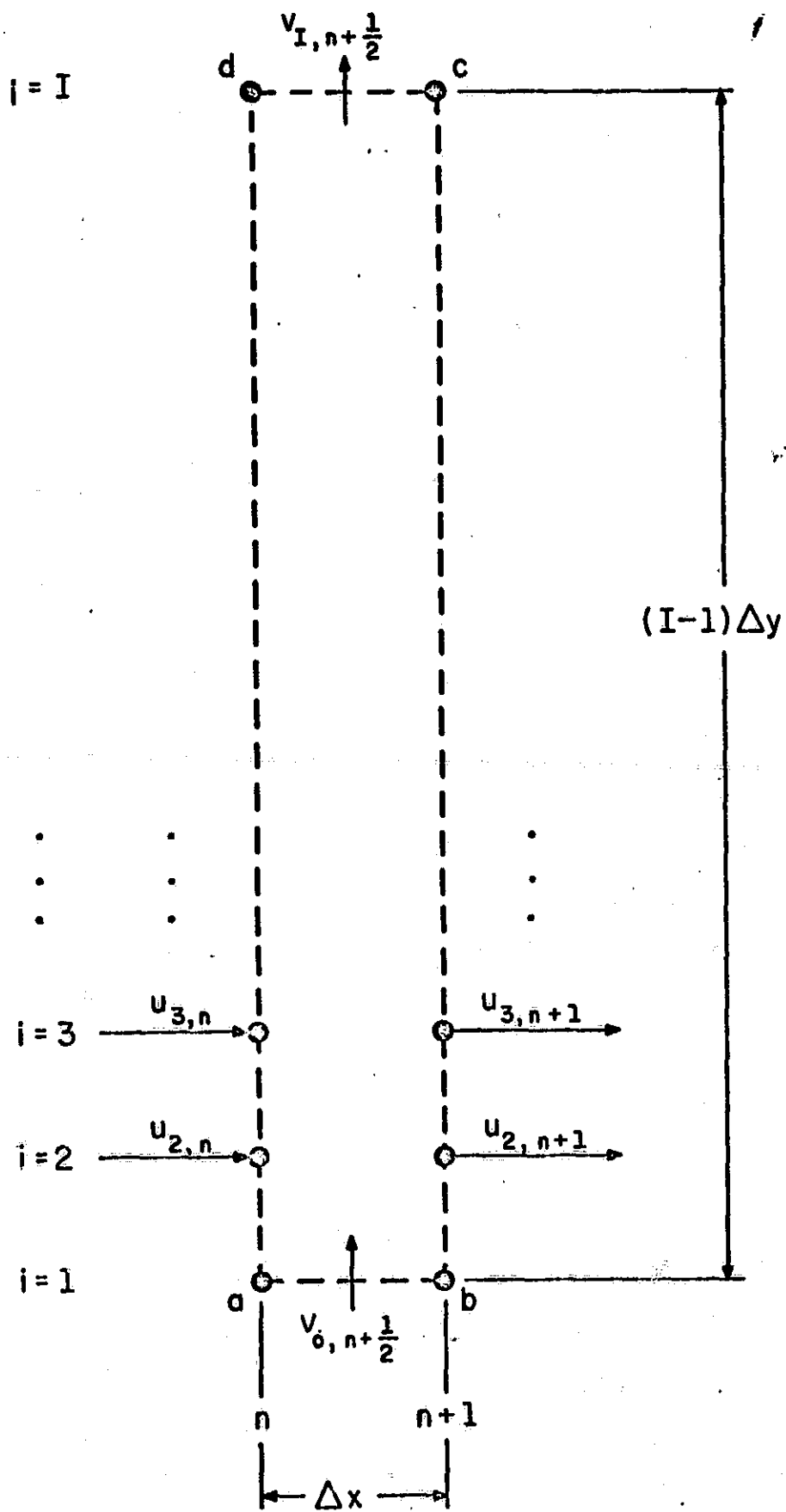


Figure 2.5 Control Volume for Application of Mass Balance in Iterative Implicit Methods

3. A SIMILARITY ANALYSIS OF THE BOUNDARY LAYER EQUATIONS IN FINITE DIFFERENCE FORM

It is well known that many boundary layers of interest are similar; that is, it is possible to define new independent and dependent variables which reduces the profiles (velocity, temperature, ...) to a single curve. The system of partial differential equations can be reduced to ordinary differential equations for such flows.

The ordinary differential equations which arise in similar boundary layer-type flows are nonlinear; hence, the integration of these equations is difficult. For this reason, these equations are often integrated numerically. Many similar solutions have been obtained by solving these transformed ordinary differential equations [1,2]. If a flow is similar, the recognition of this fact makes the results more widely applicable and their presentation is greatly simplified.

It will now be shown that similarity can be deduced in some cases by considering the governing finite difference equations in the *physical* plane. The analysis will perhaps add insight to the characteristics of similarity and discount the presumed advantage of classical similarity procedures over finite difference techniques. In addition, the analysis will help generalize the results obtained for non-similar flows. For example: If the presence of some characteristic length results in a non-similar flow, the answers often can still be generalized. Instead of a single curve, a family of profiles might be obtained (see Section 4).

3.1 THE BLASIUS PROBLEM

The Blasius Problem is that of an incompressible laminar flow over a flat plate of a constant property fluid. In this case, the pressure gradient is zero, i.e., $(p_{n+1} - p_n) = 0$, $0 \leq n \leq N$. The initial condition and the boundary conditions are

CONTINUOUS SYSTEM

$$u|_{x=0} = u_{\infty}$$

$$v|_{y=0} = 0$$

$$u|_{y=0} = 0$$

$$u|_{y \rightarrow \infty} = u_{\infty}$$

DISCRETE SYSTEM

$$u_{i,0} = u_{\infty}, i \geq 1 \quad (3.1a)$$

$$v'_{1,n+\frac{1}{2}} = 0, n \geq 0 \quad (3.1b)$$

$$u_{1,n} = 0, n \geq 0 \quad (3.1c)$$

$$u_{I+1,n} = u_{\infty}, I \text{ large} \quad (3.1d)$$

An examination of the initial condition shows that it is independent of i ; similarly, the boundary conditions are independent of n . Furthermore, if dimensionless velocities u/u_{∞} and v/u_{∞} were introduced, the boundary and initial conditions would introduce no parameters.

Next consider the difference form of the continuity equation.

Since $v'_{1,n+\frac{1}{2}}$ is zero, this equation contains no parameters and is independent of Δx and Δy . The finite difference form of the momentum equation for the three methods presented contains the single parameter

$$M = \frac{u_{\infty}(\Delta y)^2}{v\Delta x}$$

Thus

$$\frac{u_{i,n}}{u_{\infty}} = f_1(i,n,M) \quad (3.2)$$

However, M and N can be chosen arbitrarily, i.e., within the limits of the numerical algorithm being employed. Their values effect the error caused by replacing the continuous system by a discrete system, the truncation error. M and N also influence the stability of the explicit technique. Criterion (2.28) clearly shows the importance of M; the influence of N in the stability is implicit in $u_{i,n}$.

Thus, for a given value of N and M, Eq. (3.2) becomes:

$$\frac{u_i}{u_{\infty}} = f_2(i) \quad (3.3)$$

Now consider the relationship between i and the independent variables.

By definition

$$y_i = (i - 1)\Delta y \quad (3.4)$$

or

$$(i - 1) = \frac{y_i}{\Delta y} \quad (3.5a)$$

where Δy must be eliminated. The definition of M gives

$$\Delta y = \left(\frac{Mv\Delta x}{u_{\infty}} \right)^{1/2} \quad (3.5b)$$

or, if x is employed to eliminate Δx which is equal to $\frac{x}{N}$

$$\Delta y = \left(\frac{M \nu x}{N u_{\infty}} \right)^{1/2} \quad (3.6)$$

The substitution of Eq. (3.6) into Eq. (3.5a) gives

$$(i - 1) = y_1 \sqrt{\frac{u_{\infty}}{\nu x}} \sqrt{\frac{N}{M}} \quad (3.7)$$

Let

$$\eta_1 = y_1 \sqrt{\frac{u_{\infty}}{\nu x}}$$

Then

$$i = \sqrt{\frac{N}{M}} \eta_1 + 1 \quad (3.8)$$

If Eq. (3.8) is substituted into Eq. (3.3)

$$\frac{u_1}{u_{\infty}} = f_3(\eta_1) \quad (3.9)$$

where

$$\eta_1 = (i - 1) \sqrt{\frac{M}{N}} \quad (3.10)$$

At this point the reader may feel that Eqs. (3.9) and (3.10) could have been deduced only with *a priori* knowledge that the flow field was similar. However, is it not obvious that the governing difference

equations and discrete boundary conditions permit the numerical calculation to proceed without choosing either Δy or Δx ? Only the quantity M is chosen. Thus, x has not been specified when the calculation is terminated with $n = N$. The x -location can be chosen arbitrarily.

It is without doubt easier to visualize the solution proceeding downstream as u increases; hence, the boundary layer is becoming thicker. This concept was employed throughout the presentation. On the other hand, it is equally correct to consider a *fixed value* of x ; hence, an increase in n means a decrease in both Δx and Δy . More resolution is obtained as n is increased which is clearly shown by Eq. (3.10). For example, $\eta = 4$ corresponds to a location in the boundary layer where the dimensionless u velocity is 0.96. If the ratio of $\frac{M}{N}$ were one, three grid points would lie between this location and the plate. If this ratio were decreased to $\frac{1}{9}$, eleven grid points would lie between these locations.

Next consider the determination of v . Only v' is calculated and this quantity is defined only between grid points. At termination, it is known at location $(N - \frac{1}{2})$. A completely analogous procedure gives

$$\frac{v'_i}{u_\infty} = f_4(\eta'_i) \quad (3.11)$$

where

$$\eta'_i = (i - 1) \sqrt{\frac{M}{(N - \frac{1}{2})}} \quad (3.12)$$

But

$$\frac{v'_1}{u_\infty} = \frac{v_1}{u_\infty} \frac{\Delta x}{\Delta y} \quad (3.13)$$

Again, Δx and Δy can be eliminated to give

$$\frac{v'_1}{u_\infty} = \frac{v_1}{u_\infty} \frac{x}{y} \frac{(i - 1)}{(N - \frac{1}{2})}$$

The substitution of Eq. (3.12) in order to eliminate $(i - 1)$ gives

$$v_1 \sqrt{\frac{x}{u_\infty v}} = v'_1 \sqrt{M(N - \frac{1}{2})}$$

or

$$v_1 \sqrt{\frac{x}{u_\infty v}} = f_5(\eta_1)$$

The numerical calculation gives u_1 and v'_1 . Equations (3.10) and (3.12) are employed to determine the corresponding values of η_1 and Eq. (3.14) relates v_1 to v'_1 . If N is large, the distinction between the location N of known u and the location $(N - \frac{1}{2})$ of known v need not be made.

N was defined as the maximum value of n , the value of n at which the calculation was terminated. However the results at more than one x -location, $N(\Delta x)$, may be desired. It is clear that N can take on any value; hence, the initial definition, although convenient for general problems, is too restrictive if the problem is similar or if

the solution can be effected without defining Δx . Many nonsimilar problems still retain this property. The difference between n and N is analagous to the difference between x and x' in the following integral

$$F(x) = \int_0^x f(x') dx' \quad (3.14)$$

or, in finite difference form

$$F(x) = \left\{ \frac{f_0}{2} + \left[\sum_{n=1}^{N-1} f_n \right] + \frac{f_N}{2} \right\} \Delta x \quad (3.15)$$

where x or $x_N = (N)(\Delta x)$ and $x' = x_n = (n)(\Delta x)$. Hence, unless Δx is specified, it should be assumed that N can have any positive integer value.

3.2 WEDGE FLOW

In the neighborhood of the stagnation point of a wedge whose included angle is equal to $\frac{2\pi m}{(m+1)}$, the velocity variation with respect to x is

$$u_s = cx^m, \quad m \geq 0 \quad (3.16)$$

This is the result of potential theory. The flow in this case is an accelerating flow.

There are two main differences between this case and the first case:

(1) The initial condition is now $u_{1,0} = 0$.

(2) The pressure gradient is no longer zero; hence, the free stream velocity is dependent on the x-location.

It is immediately seen that the new initial condition is also independent of i . Hence, this will not influence the similarity of the flow. If the profiles are to be made congruent, it is also clear that the velocity $u_{1,n}$ must be divided by the free stream velocity at location N , u_∞ .

The pressure term must now be examined to determine if it is independent of Δx . This term is given by Eqs. (2.6) and (2.16) and for the case of interest is

$$u_1 \frac{du_1}{dx} = f(x) = (cx^m)(cmx^{m-1}) \quad (3.17)$$

The terms of the momentum equation were divided by $u_\infty^2/\Delta x$ in order to reduce them to dimensionless quantities; hence, Eq. (3.17) in finite difference form, after dividing by $u_\infty^2/\Delta x$, becomes

$$\frac{\Delta x f[(n + \frac{1}{2})\Delta x]}{u_\infty^2} = \frac{m}{(n + \frac{1}{2})} \left(\frac{n + \frac{1}{2}}{N} \right)^{2m} \quad (3.18)$$

Equation (3.18) shows that the pressure term is independent of Δx and Δy . The parameters occurring in the momentum equation in this case are m and M .

For a given value of M , m , and N , u_1/u_∞ is again dependent only on η_1 ; that is

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$$u_i / u_\infty = f_6(\eta_i) \quad , \quad \eta_i = (i - 1)(M/N)^{1/2}$$

Since the flow is accelerating, the free stream velocity is dependent on the x-location; therefore, generality is lost if u_∞ is employed to reduce the terms of the finite difference equations to dimensionless quantities. Since N is to be free to take on any value, a different reference velocity must be introduced. Specifically, the pressure term, the dimensionless velocity, and M will be based on

$$u_r = c[N_r \Delta x]^m$$

Thus, in terms of the similarity variables, the results at location N are:

$$\frac{u_i}{u_\infty} = \frac{u_i}{u_r} \left(\frac{N_r}{N} \right)^m = \frac{u_i}{u_r} \frac{u_r}{u_{i+1,N}} \quad (3.19a)$$

where u_∞ is $u_{i+1,N}$ and u_∞/u_r is the dimensionless free stream velocity at location N which is employed in the finite difference calculation. Since M_r is also based on u_r , η_i becomes

$$\eta_i = (i - 1) \sqrt{\frac{M_r}{N} \left(\frac{N}{N_r} \right)^m} = (i - 1) \sqrt{\frac{M_r}{N} \frac{u_{i+1,N}}{u_r}} \quad (3.19b)$$

$u_{i+1,N}/u_r$ is simply M/M_r ; the definition of η_i has not been changed. The results are now meaningful for all values of N . The parameters

M_1 , N , and N_1 affect the desired solution only because they influence the various errors. The parameter m , however, is related to the wedge angle. A different solution is obtained for each value of m ; hence, a family of solutions is obtained. It is interesting to note that for $m = 1$, $\Delta\eta$ is independent of N . The influence of this on the accuracy will be seen in the results given in Section 4.1.

3.3 FLAT PLATE WITH MASS TRANSFER

Now consider the case of a flat plate with mass transfer through the plate. The usual assumption that the tangential velocity at the wall is zero, the no-slip condition, will be used. Thus the only change from the first case analyzed, the Blasius Problem, is that the boundary condition used in conjunction with the continuity equation is not $v_{1, n+\frac{1}{2}} = 0$, but

$$v_{1, n+\frac{1}{2}} = f[(n + \frac{1}{2})\Delta x]$$

The boundary condition in the continuous system is $v(y = 0) = f(x)$.

The form of the function $f(x)$ for similar velocity profiles is to be determined. Since the initial condition, the u boundary conditions, and the momentum equation remain the same, the solution would be of the form

$$\frac{u_1}{u_\infty} = f_7(\eta_1) \quad (3.20)$$

where

$$\eta = (i - 1) \sqrt{\frac{M}{N}} \quad (3.21)$$

if the flow is similar. It will be similar if the term

$$v'_{1, n+\frac{1}{2}} = v_{1, n+\frac{1}{2}} \frac{\Delta x}{\Delta y} = f[(n + \frac{1}{2})\Delta x] \frac{\Delta x}{\Delta y} \quad (3.22)$$

is independent of Δx and Δy .

In order to determine the form of $f[(n + \frac{1}{2})\Delta x] \frac{\Delta x}{\Delta y}$, Δy must first be eliminated. If Eq. (3.5b) is substituted into Eq. (3.22), the resulting equation is

$$v'_{1, n+\frac{1}{2}} = f[(n + \frac{1}{2})\Delta x] [\Delta x]^{\frac{1}{2}} \left(\frac{u_{\infty}}{\sqrt{M}} \right)^{\frac{1}{2}}$$

Thus, $f[(n + \frac{1}{2})\Delta x]$ must be proportional to $(\Delta x)^{-\frac{1}{2}}$ or

$$f[(n + \frac{1}{2})\Delta x] = \frac{C}{[(n + \frac{1}{2})\Delta x]^{\frac{1}{2}}} \quad (3.23)$$

The continuous function would be $f(x) = -\frac{C}{\sqrt{x}}$. Again, a different solution would be obtained for each different value of the constant C . The boundary layer equations apparently are not applicable for all values of the constant C .

Further study of similar flows may show that similarity analyses in the continuous system give more general results [11]. If this is the case, these analyses might complement finite difference solutions and make the results more general. The main point is that finite difference equations should also be examined in detail to avoid wasting computational effort by solving redundant cases. The advantages of such analysis is shown more clearly by the non-similar flows presented in Section 4.

4. RESULTS OF NUMERICAL CALCULATIONS

4.1 SIMILAR BOUNDARY LAYERS

Extensive numerical results and comparisons will be given for three similar boundary layer problems:

- (i) The Blasius problem.
- (ii) Flow over a wedge with an included angle of 90° .
- (iii) Stagnation flow.

The Blasius problem, flow over a flat plate at zero incidence, has probably been studied more completely than any other boundary layer problem. Although it is an elementary case, it provides a stringent test for finite difference schemes due to the singularity at the leading edge, $x = 0$. The convective term $v \frac{\partial u}{\partial y}$ is relatively large for this flow. The pressure term $-\frac{1}{\rho} \frac{dp}{dx}$, on the other hand, plays a major role in the other two flows. Hence, these three cases provide some breadth to the examination of the algorithms. The specifications of the three problems and the source of the published tabulated results which are used in all comparisons are given in Table 4.1. The Blasius problem and the stagnation flow are limiting cases of wedge flow; hence, the type of problem is simply specified by the value of m .

The percent errors which are given are based on the difference between the finite difference solutions and the published solution; hence, the published solutions are presumed to be correct to five significant figures. The error in the graphical comparisons may sometimes be exaggerated slightly due to interpolation error. Single precision arithmetic was used exclusively on the IBM 360/75 system. Roundoff errors were of the order of 0.005 percent.

TABLE 4.1. SPECIFICATIONS OF SIMILARITY SOLUTION USED IN EVALUATION OF FINITE DIFFERENCE ALGORITHMS

Problem	m	Included Wedge Angle	Source of Solution Used in Comparisons	Tabulated η Increments in Respective Reference
Blasius	0	0°	Reference 1 page 129	$\Delta\eta = 0.2$
Wedge Flow	1/3	90°	Reference 2 page 237	$\Delta\eta = 0.1225$
Separation Flow	1	180°	Reference 2 page 232	$\Delta\eta = 0.1$

Machine plots (CALCOMP) are used in most cases to remove human errors and give greater resolution. The computer drawn curves are usually *made continuous* by connecting the individual discrete points with straight lines.

Because the results are generalized by employing the similarity parameter η , x and y are unspecified; therefore, the dimensional shear stress is unknown. The dimensional shear stress at the wall follows from Eqs. (2.44d) and (2.45) as

$$\tau|_{y=0} = \mu \frac{\tilde{\Delta}_y u_1}{\Delta y} \quad (4.1)$$

The unknown Δy can be eliminated by introducing M and N . This gives

$$\tau|_{y=0} = \rho u_\infty^2 \sqrt{\frac{\mu}{\rho u_\infty x}} \sqrt{\frac{N}{M}} \tilde{\Delta}_y u_1^* \quad (4.2)$$

Hence, a logical dimensionless shear stress τ^* follows from Eq. (4.2) as

$$\tau_w^* = \frac{\tau|_{y=0}}{\rho u_\infty^2} [Re_x]^{1/2} \quad (4.2)$$

and

$$\tau_w^* = \sqrt{\frac{N}{M}} \tilde{\Delta}_y u_1^* = \frac{\tilde{\Delta}_y u_1^*}{\Delta \eta} \quad (4.4)$$

Since the solution to similar problems can be effected without specifying the x -location, the dimensionless wall shear stress defined by Eq. (4.4) must be independent of x , i.e., it must be a constant.

Hence, any variation in τ_w^* which is present in the numerical results is a direct consequence of numerical errors.

Consider first the results given in Fig. 4.1 which were obtained with the Crank-Nicolson algorithm. The results are for the problems outlined in Table 4.1. Relatively small values of M_x and the ratio M_x/N are used to exaggerate the error. Published results are also included in this figure for comparison. The shear stress, the computational time, and the errors in the shear stress and the discrete velocities are given in Table 4.2. The accuracy is seen to be excellent.

TABLE 4.2 EXECUTION TIMES, SHEAR STRESSES AND PERCENTAGE ERRORS--CRANK-NICOLSON ALGORITHM

m	Execution time (sec.)	τ_w^* [1 or 2]	τ_w^* (Finite-Diff.)	% Error in τ_w^*	Max. % Error in u_i^*	No. of i with $0 < u_i^* < 0.9$
0	0.1	0.33206	0.3315	-0.17	-0.35	8
1/3	0.1	0.75754	0.7578	0.05	Interpolation required	5
1	0.1	1.2326	1.2337	0.09	0.22	3

Other parameters employed in the solution are: $\epsilon = 1.0$, $M_x = 4$, $I = 15$, and $N_x = N = 25$ ($\Delta \eta = 0.4$ at $N = 25$).

The computation or execution times given are for the IBM 360/75 systems.† The reported times are approximate and are meant to serve only as a rough indication of the computational costs. The following points are to be noted:

- (i) A crude network is being used. Only three grid points were located at positions in the boundary layer where $0 < u^* < 0.9$ for the stagnation flow. However, the largest percentage error in the respective velocities was 0.22 percent, and the error in the wall shear stress, determined by extrapolation, was 0.09 percent.
- (ii) Although the computer program was written for generality and simplicity and not speed, the total computational time to solve all three problems was approximately 0.3 seconds! If M_∞ were reduced to 1.92 and N to 12, these solutions could be obtained with all errors still below 0.5 percent in approximately half this time.
- (iii) Although the shear stress at the wall is determined by extrapolation, the percent error in this quantity is of the same magnitude as the errors in the velocities. The maximum percentage error in the discrete velocities, incidentally, usually occurs in the smallest unknown velocity.
- (iv) The best accuracy was obtained for the stagnation flow even though the boundary layer is relatively thin for this case, and fewer grid points were located in the boundary layer.

Consider next the influence of M_∞ and N on the accuracy of the numerical calculation. It is well known that the expression for the truncation error caused by approximating a partial differential equation by a difference equation is composed of terms which are products of the increments of the independent variables and the higher order derivatives. Historically, the influence of these derivatives on the accuracy of finite difference solutions often has been disregarded. The emphasis has been placed on the magnitude

†The same programs were also run on a CDC 6600 system and the times were comparable.

of the increments in the independent variables. This is done because the derivatives of interest are not readily available. In addition, even if these quantities were estimated, the overall influence of these *localized* errors on the *global* accuracy is difficult to determine. Numerical experiments designed to estimate the truncation error by repeated calculations with different increments have often led to confusing results and incorrect conclusions. Young [12] in his survey of numerical analysis of elliptic and parabolic partial differential equations stated:

"None of the papers which have been written on error estimation (e.g., [51,52,47,48,49,35,65]) can be said to come close to satisfying the needs of the practical computer user."

- An attempt will now be made to at least partially satisfy that need.

The diffusion of heat, mass, and momentum gives rise to parabolic equations. In many diffusion problems, some rapid change occurs at some boundary or interface, and the diffusion of this input into the medium or media is to be described. A diffusion layer will be defined as the region where the influence of this change is strong. An example is the boundary layer in viscous flow. Initially, the diffusion layer is very thin, but its thickness approaches infinity as time, or the independent variable analogous to time, which is x in this case, approaches infinity. If the region were finite, the diffusion layer would, of course, be limited to the thickness of the region. If an accurate description of the diffusion process is to be given at some "time," several grid points must lie within the diffusion layer. If large "times" are of interest, large

increments will give completely satisfactory results. On the other hand, if small "times" are of interest, the spatial increments in the region of influence and the "time" increment might have to be several orders of magnitude smaller in order to effect a meaningful solution. Clearly, in the solution of such diffusion problems, the magnitude of the increments is a completely unsatisfactory criterion for estimating the influence of truncation errors. Somehow the influence of the magnitude of the derivatives must be included. How can this be done in practical problems?

Consider, as an example, the Blasius problem. It was found that the boundary layer thickness was dependent upon a single variable η where $\eta = y(u_\infty/\nu x)^{1/2}$. It is known that this problem can be accurately solved with the von Karman integral technique by approximating the boundary layer profile by a second degree curve. If several grid points at the x-location of interest were located inside the boundary layer which will be defined as $0 < u^+ < 0.9$, it would seem, by comparison, to be possible to accurately solve this problem with finite difference techniques also. Since, according to Eqs. (3.6) and (3.10),

$$\Delta\eta = (M/N)^{1/2} = \Delta y (u_\infty/\nu x)^{1/2}$$

and since $\eta = 3.4$ corresponds to the y-location where $u^+ = 0.9$, the criterion

$$\Delta\eta < 0.8$$

would guarantee four points within the boundary layer.

More generally, the edge of the viscous region for laminar boundary layers and shear layers including nonsimilar cases with arbitrary pressure gradients and variable properties will generally lie at a value of $\eta > 1.5$ when η is based on the local free stream velocity. Hence, the criterion

$$\Delta\eta = (M/N)^{1/2} < 0.3 \quad (4.5)$$

will usually guarantee four to five points within the viscous layer.

Criterion (4.5) essentially relates the size of the y-increment to the magnitude of the independent variable x . The criterion provides a means of taking into account the size of the derivatives with respect to the independent variable analogous to time in the solution of parabolic equations. The actual size of the y-increment becomes meaningful in the analysis of truncation error only if it is related to the x-location of interest.

Another way of specifying Criterion (4.5) is:

$$|u_{i+1,N} - u_{i,N}| < 0.25 (u_{\max,N} - u_{\min,N}), \quad 1 \leq i \leq I \quad (4.7)$$

Criterion (4.7) is only to be satisfied at the x-location of interest, $N\Delta x$; hence, the subscript N is used instead of n . Criterion (4.7) can be applied only after the solution has been effected whereas Criterion (4.5) provides a means of determining the required spatial

increment before the solution is effected and is, therefore, more practical.

As another example consider a nonsimilar heat conduction problem. Specifically, one-dimensional transient heat conduction in the y-direction in a solid of thickness L bounded by two parallel planes will be considered. The solid will be assumed to be initially at some steady-state temperature distribution and at time $t = 0$, some change occurs at one or both of the boundaries. Criterion (4.5) is still applicable except M is $(\Delta y)^2/(\alpha \Delta t)$ and η is $y/(\alpha t)^{1/2}$ where α is the thermal diffusivity. If the influence of the changes near the boundaries is to be shown at some small time $t = N\Delta t$, Criterion (4.5) will require $\Delta y \ll L$. If relatively large times were of interest, Criterion (4.5) might cease to be applicable because the deduced Δy could exceed L. For finite regions a second criterion is required to keep Δy from approaching or exceeding L. A suitable criterion is probably

$$\Delta y < (L/5) \quad (4.8)$$

However, the shape of the profile is important, and it is advisable to bear Criterion (4.7) in mind. If the region did not contain a characteristic length, e.g., a semi-infinite solid, there would be no need for Criterion (4.8).

It is perhaps only natural that the size of the increments in the similarity variable $\Delta \eta$ should strongly influence the accuracy of

the finite difference results. The importance of this ratio, however, is *not* confined to similar problems (see, e.g., Section 4.2).

If M/N were the only influence on the accuracy of the finite difference solutions, small values of M should obviously be chosen. For example, $M = 4$, and $N = 25$ would result in the same value of $\Delta\eta$ as $M = 40$ and $N = 250$, but the latter case would require an order of magnitude more computational effort. The role of M must be determined.

One might think that a second criterion which limits the magnitude of Δx would be sufficient; however, this is not the case. A limit also must be placed on the relative size of the increments, specifically on $M = (\Delta y)^2 u_{\infty} / \nu \Delta x$ and not on the absolute value of Δx . The only limitation that needs to be placed on Δx is that it cannot approach x_N . At least ten steps should be used in getting to the x -location of interest. Amazingly accurate results have been obtained in some instances, however, with N as small as one. The point is rather academic. The more important question is: Why are poor results sometimes obtained with N as large as 1000? Even more puzzling is the situation wherein a set of increments Δy_1 and Δx_1 give satisfactory results, whereas a second set with $\Delta x_2 = \Delta x_1$ and $\Delta y_2 = \Delta y_1 / 2$ give results which contain large errors. What is the source of the error? For example, consider the results given in Table 4.3 which gives the influence of Δy on the error in the shear stress for the Blasius problem with a fixed x -increment. The Crank-Nicolson algorithm was employed in the solutions. The

TABLE 4.3 INFLUENCE OF Δy ON THE TRUNCATION ERROR--
CRANK-NICOLSON ALGORITHM

	y-increment			
	Δy_1	$0.707 \Delta y_1$	$0.49 \Delta y_1$	$0.346 \Delta y_1$
Percent error in τ_w^*	7.0	0.5	0.14	129.0
$\Delta \eta$	1.15	0.82	0.56	0.4
M	8.0	4.0	1.92	0.96

Other parameters: $\epsilon = 1.0$, $N = 6$, $m = 0$, Δx --fixed.

truncation error at first decreases with decreasing Δy ; however, a catastrophic increase in the error then occurs with a further, modest decrease in Δy . The reason for this phenomenon lies in the value of M.

Consider again a simple problem--one-dimensional transient heat conduction in a solid medium. An analysis of the truncation error committed by replacing the partial differential equation governing this problem by a difference equation is of the form

$$E_T = C_1 (\Delta t) \left. \frac{\partial^2 T}{\partial t^2} \right|_{i,n} + C_2 (\Delta t)^2 \left. \frac{\partial^3 T}{\partial t^3} \right|_{i,n} + C_3 (\Delta y)^2 \left. \frac{\partial^4 T}{\partial y^4} \right|_{i,n} + \text{higher order terms}$$

where C_1 for the standard implicit algorithm would be equal in magnitude but opposite in sign to its value for the standard explicit algorithm. C_1 would be zero for the Crank-Nicolson method (see e.g., [3]). C_1 , C_2 and C_3 all are constants.

The expression for the truncation error indicates that the truncation error would decrease if either Δt or Δy were decreased. However, this was not the case for the results given in Table 4.3. The reason is that *the error committed in the discrete analog of the continuous system is dependent upon the variation of the discrete temperatures with respect to time.* That is, $\frac{\partial^2 T}{\partial t^2} \Big|_{i,n}$ should be replaced by $\frac{d^2 T_i}{dt^2} \Big|_n$ etc. These quantities must be approximately equal if the solution is acceptable. However, if *the time increment is too large relative to the time constants of the discrete elements*, these quantities will not be equal and the solution can become meaningless.

The time constant of the discrete elements, RC , is an indication of how rapidly the temperature of the element is capable of changing. The ratio of this time constant to the time increment Δt is essentially M for all one-, two-, or three-dimensional conduction problems with equal spatial increments h in all directions. That is

$$M = \frac{RC}{\Delta t} = \frac{\frac{h}{k\Delta A} (\Delta V \rho c)}{\Delta t} = \frac{h^2}{\alpha \Delta t}$$

since $\Delta V/\Delta A \approx h$. If the time increment is large relative to the time constants of the discrete elements, the discrete temperatures may change so much that one can no longer base the rate of change in internal energy partially or totally on the rate of heat flow to the element at the beginning of the time steps. If this is done, violations of physical laws are possible and catastrophic errors can

result. Analogously, in viscous flow, the change in the velocity u cannot be based totally or partially on the shear stresses at the beginning of the x -step unless this step is relatively small.

The absolute value of Δt or Δx is not the governing criterion. The limitation must be placed on the size of M . The finite-difference representation ceases to be physically meaningful if M is too small. How small is too small? Unfortunately, the criteria are dependent on the smoothness of the problem. Thus, a means of specifying the smoothness is required.

The change in a general dependent variable ϕ between locations n and $(n + 1)$ will be denoted as $\Delta_t \phi_{i,n}$ where Δ is the forward difference operator. If this change is of the same magnitude as $|\phi_{\max} - \phi_{i,n+1/2}|$ or $|\phi_{\min} - \phi_{i,n+1/2}|$ where $\phi_{\max}$ and $\phi_{\min}$ are the maximum and minimum boundary values of ϕ , then the criteria for the Crank-Nicolson algorithm become:

Heat Conduction (one-spatial dimension)

$$M = \frac{(\Delta y)^2}{\alpha \Delta t} > 1 \quad (4.9a)$$

if

$$|\Delta_t T_{i,n}| = O\{|T_{\max} - T_{i,n+1/2}| \text{ or } |T_{\min} - T_{i,n+1/2}|\}$$

Boundary Layers

In boundary layer problem, Δt is analogous to $\Delta x/u_{1,n+1/2}$, and ν is analogous to α . The criterion becomes:

$$M \frac{u_{1,n+1/2}}{u_{\infty}} = \frac{(\Delta y)^2}{\nu \Delta x} u_{1,n+1/2} > 1 \quad (4.9b)^\dagger$$

if

$$|\Delta_x u_{1,n}| = 0 \{ |u_{\max} - u_{1,n+1/2}| \text{ or } |u_{\min} - u_{1,n+1/2}| \}$$

If the fluid element is accelerating or decelerating, $|\Delta_x u_{1,n}|$ should be replaced by

$$|\Delta_x u_{1,n} - \Delta_x u_{1+1,n}|$$

A similar correction must be made in the consideration of the energy equation if heat is generated internally in the element.

On the other hand if $|\Delta_t \phi_{1,n}|$ is much less than $|\phi_{\max} - \phi_{1,n+1/2}|$ and $|\phi_{\min} - \phi_{1,n+1/2}|$, then considerably smaller values of M can be employed. The following limits are suggested for this case, although satisfactory results may be obtained in some cases for even smaller values of M .

[†]The simultaneous consideration of the thermal boundary layer gives rise to the additional criterion $Mu_{1,n+1/2}^* Pr > 1$.

Heat Conduction

$$M = \frac{(\Delta y)^2}{\alpha \Delta t} > 0.1 \quad (4.10a)$$

if

$$|\Delta_t T_{i,n}| \ll |T_{\max} - T_{i,n+\frac{1}{2}}|$$

and

$$|\Delta_t T_{i,n}| \ll |T_{\min} - T_{i,n+\frac{1}{2}}|$$

Boundary Layers

$$M \frac{u_{i,n+\frac{1}{2}}}{u_{\infty}} > 0.1 \quad (4.10b)$$

if

$$|\Delta_x u_{i,n}| \ll |u_{\max} - u_{i,n+\frac{1}{2}}|$$

and

$$|\Delta_x u_{i,n}| \ll |u_{\min} - u_{i,n+\frac{1}{2}}|$$

While a basis exists for Criteria (4.9a) and (4.9b), Criteria (4.10a)

and (4.10b) are only recommendations.

Values of M smaller than 0.1 may arise due to:

- (i) The use of unrealistically small "spatial" increments. The "time" truncation error is completely dominant in such cases and M should be made larger by increasing the size of the "spatial" increments.
- (ii) The desire to obtain "steady-state" solutions. As steady-state is approached, the time derivatives are also approaching zero. Hence, relatively large time increments (small M 's) are acceptable.

Values of $Mu_{i,n+\frac{1}{2}}^*$ smaller than 0.1 were seldom employed in this investigation. Although Criteria (4.9) and (4.10) are not very practical or absolute, more useful criteria are probably unnecessary because if physical laws are violated, meaningless oscillations will occur. A continuous examination of some quantity such as a heat flux or shear stress would vividly show areas where M is too small.

Stability considerations in the standard explicit method demand that the value of $Mu_{i,n}^*$ be greater than 2. In the standard implicit method, $\Delta_t T_{i,n}$ and $\Delta_x u_{i,n}$ are based on the heat flow rates and the shear stresses, respectively, at the end of the step. Hence, a criterion which limits the value of M apparently is unnecessary.

Table 4.4 shows the influence of Δy and M on the truncation error for the identical conditions employed in obtaining the results of Table 4.3 except the Standard Implicit algorithm is used. These results, unlike those of Table 4.3, show a monotonically decreasing error. This is an apparent advantage of this algorithm.

Tabulated velocity profiles are given in Tables 4.5 through 4.8 which show the influence of M_i and $\Delta \eta$. These tables also give:

TABLE 4.4 INFLUENCE OF Δy ON THE TRUNCATION ERROR--STANDARD IMPLICIT ALGORITHM

	y-increment			
	Δy_1	$0.707 \Delta y_1$	$0.49 \Delta y_1$	$0.346 \Delta y_1$
% error in τ_w^*	26.0	8.0	2.0	0.3
$\Delta \eta$	1.15	0.82	0.56	0.4
M	8	4.0	1.92	0.96

Other parameters: $\epsilon = 1.0$, $N = 6$, $m = 0$, Δx --fixed

- (1) The maximum percentage error in any discrete velocity at the location,
- (2) The shear stress and the percent error in this quantity,
- (3) The approximate computational time, and
- (4) The number of grid points in the boundary layer at the location.

All of the results for these four tables were obtained with the Crank-Nicolson algorithm. Results are given for both the Blasius problem ($m = 0$) and the stagnation flow ($m = 1$). Tables 4.9 and 4.10 give analogous quantities for flow over a 90° wedge and for the Blasius problem, and results from all three algorithms are given for comparison. Relatively large values of M were employed for these comparisons because of the requirements for stability of the Standard Explicit method. The Standard Explicit, The Standard Implicit, and the Crank-Nicolson algorithms are referred to in the tables by S-E, S-IM, and C-N, respectively.

Figure 4.2 shows the variation in percent error in the wall shear stress with N/M , for all three problems ($m = 0, 1/3, 1$) for two different values of ϵ , 1.0 and 0.1. These results were obtained

TABLE 4.5 THE INFLUENCE OF M_∞ -STAGNATION FLOW

		u_1^* (finite difference)				
		$M_\infty = 0.96$	1.92	4	8	16
η_1	$u_1^* [2]$	$N_\tau = 6$	12	25	50	100
0	0	0	0	0	0	0
0.4	0.4145	0.4154	0.4154	0.4154	0.4154	0.4154
0.8	0.6859	0.6870	0.6870	0.6870	0.6870	0.6870
1.2	0.8467	0.8478	0.8478	0.8478	0.8478	0.8478
1.6	0.9323	0.9336	0.9336	0.9336	0.9336	0.9336
2.0	0.9732	0.9743	0.9743	0.9743	0.9743	0.9743
2.4	0.9905	0.9913	0.9913	0.9913	0.9913	0.9913
Max. % error in u_1^*		0.22	0.22	0.22	0.22	0.22
τ_w^* (= 1.23259 [2])		1.2337	1.2337	1.2337	1.2337	1.2337
% error in τ_w^*		0.09	0.09	0.09	0.09	0.09
Execution time (sec.) ($N = N_\tau$)		0.0	0.0	0.1	0.2	0.3
Oscillations		none	none	none	none	none

Other parameters: $\epsilon = 1.0$, $I = 15$, $\Delta\eta = 0.4$, and $m = 1.0$, $1 < N \leq N_\tau$

Algorithm: Crank-Nicolson

TABLE 4.6 THE INFLUENCE OF M--BLASIUS PROBLEM

		u_i^* (finite difference)				
		M = 0.96	1.92	4	8	16
η_i	$u_i^* [1]$	N = 6	12	25	50	100
0	0	0	0	0	0	0
0.4	0.1328	0.2148	0.1325	0.1326	0.1324	0.1323
0.8	0.2647	0.3095	0.2642	0.2646	0.2641	0.2639
1.2	0.3938	0.4436	0.3932	0.3937	0.3931	0.3927
1.6	0.5168	0.5619	0.5161	0.5166	0.5159	0.5154
2.0	0.6298	0.6686	0.6288	0.6294	0.6286	0.6280
2.4	0.7290	0.7598	0.7275	0.7280	0.7273	0.7268
2.8	0.8115	0.8333	0.8092	0.8098	0.8091	0.8087
3.2	0.8761	0.8891	0.8729	0.8735	0.8730	0.8726
3.6	0.9233	0.9292	0.9196	0.9202	0.9199	0.9196
4.0	0.9555	0.9564	0.9517	0.9522	0.9520	0.9519
Max. % error in u_i^*		+ 61.7	-0.40	-0.35	-0.37	-0.40
τ_w^* (= 0.33206 [1])		0.7620	0.3311	0.3315	0.3310	0.3306
% error in τ_w^*		+129	-0.29	-0.17	-0.33	-0.44
Execution time (sec.)		0.0	0.0	0.1	0.2	0.3
Oscillations		strong	slight	none	none	none

Other parameters: $\epsilon = 1.0$, $I = 15$, $\Delta\eta = 0.4$, and $m = 0$

Algorithm: Crank-Nicolson

TABLE 4.7 INFLUENCE OF $\Delta\eta$ --BLASIUS PROBLEM

η	u_i^* [1]	u_i^* (finite difference)				
		$M = 4$	4	4	3.84	4
		$N = 400$	100	25	6	4
		$\Delta\eta = 0.1$	0.2	0.4	0.8	1.0
0	0	0	0	0	0	0
0.2	0.0664	0.0663	0.0663			
0.4	0.1328	0.1326	0.1325	0.1326		
0.6	0.1989	0.1988	0.1986			
0.8	0.2647	0.2645	0.2642	0.2646	0.2667	
1.0	0.3298	0.3295	0.3292			0.3355
1.2	0.3938	0.3934	0.3931	0.3937		
1.4	0.4563	0.4559	0.4555			
1.6	0.5168	0.5163	0.5159	0.5166	0.5216	
1.8	0.5748	0.5743	0.5738			
2.0	0.6298	0.6293	0.6288	0.6294		0.6420
2.4	0.7290	0.7284	0.7278	0.7280	0.7325	
2.8	0.8115	0.8110	0.8102	0.8098		
3.2	0.8761	0.8755	0.8747	0.8735	0.8725	
3.6	0.9233	0.9228	0.9220	0.9202		
4.0	0.9555	0.9551	0.9543	0.9522	0.9473	0.9457
4.4	0.9759	0.9755	0.9748	0.9728		
4.8	0.9878	0.9876	0.9870	0.9853	0.9807	
Max. % error in u_i^*		- 0.15	- 0.23	- 0.35	+ 0.93	+1.94
τ_w^* (= 0.33206 [1])		0.3317	0.3314	0.3315	0.3333	0.3363
% error in τ_w^*		- 0.10	- 0.21	- 0.17	+ 0.37	+1.28
Execution time (sec.)		4.0	0.6	0.1	0.0	0.0
No. of grid points, I		70	36	19	10	8
No. of points with $0 < u_i^* < 0.9$		33	16	8	4	3

Other parameters: $\varepsilon = 1.0$, $m = 0.0$

Algorithm: Crank-Nicolson

TABLE 4.8 INFLUENCE OF $\Delta\eta$ --STAGNATION FLOW

		u_i^* (finite difference)					
		$M_r = 4$	4	4	4	3.84	4
		$N_r = 400$	100	25	16	6	4
η_i	$u_i^* [2]$	$\Delta\eta = 0.1$	0.2	0.4	0.5	0.8	1.0
0	0	0	0	0	0	0	0
0.1	0.1183	0.1183					
0.2	0.2266	0.2267	0.2268				
0.3	0.3252	0.3253					
0.4	0.4145	0.4145	0.4147	0.4154			
0.5	0.4946	0.4947			0.4961		
0.6	0.5663	0.5664	0.5666				
0.7	0.6299	0.6299					
0.8	0.6859	0.6860	0.6862	0.6870		0.6884	
1.0	0.7779	0.7779	0.7782		0.7794		0.7809
1.2	0.8467	0.8468	0.8470	0.8478			
1.4	0.8968	0.8970	0.8971				
1.6	0.9323	0.9324	0.9327	0.9336		0.9368	
1.8	0.9568	0.9569	0.9571				
2.0	0.9732	0.9733	0.9735	0.9743	0.9749		0.9802
2.2	0.9839	0.9839	0.9841				
2.4	0.9905	0.9906	0.9907	0.9913		0.9938	
2.8	0.9970	0.9971	0.9972	0.9975			
Max. % error in u_i^*		+ 0.08	+ 0.09	+ 0.22	+0.30	+0.48	+0.72
$\tau_w^* (= 1.23259 [2])$		1.2330	1.2338	1.2337	1.2302	1.1936	1.1472
% error in τ_w^*		+ 0.03	+ 0.10	+ 0.09	-0.20	-3.16	-6.93
Execution time (sec.) ($N = N_r$)		3.0	0.4	0.0	0.0	0.0	0.0
No. of grid points, I		40	21	11	9	6	5
No. of points with $0 < u_i^* < 0.9$		14	7	3	2	1	1

Other parameters: $\varepsilon = 1.0$, $m = 1.0$, $1 < N < N_r$

Algorithm: Crank-Nicolson

TABLE 4.9 COMPARISONS OF THE RESULTS FROM THE THREE DIFFERENT ALGORITHMS--THE BLASIUS PROBLEM

		u_1^* (finite difference)					
		$M = 16; N = 100; \Delta\eta = 0.4$			$M = 16; N = 25; \Delta\eta = 0.8$		
η_1	$u_1^* [1]$	S-E	S-IM	C-N	S-E	S-IM	C-N
0	0	0	0	0	0	0	0
0.4	0.1328	0.1332	0.1320	0.1323			
0.8	0.2647	0.2657	0.2632	0.2639	0.2691	0.2694	0.2672
1.2	0.3938	0.3953	0.3917	0.3927			
1.6	0.5168	0.5189	0.5140	0.5154	0.5264	0.5251	0.5222
2.0	0.6298	0.6322	0.6262	0.6280			
2.4	0.7290	0.7313	0.7245	0.7268	0.7397	0.7320	0.7323
2.8	0.8115	0.8134	0.8060	0.8087			
3.2	0.8761	0.8772	0.8698	0.8726	0.8811	0.8682	0.8721
3.6	0.9233	0.9236	0.9168	0.9196			
4.0	0.9555	0.9552	0.9494	0.9519	0.9549	0.9427	0.9474
4.4	0.9759	0.9752	0.9707	0.9727			
4.8	0.9878	0.9870	0.9838	0.9853	0.9857	0.9777	0.9812
Max. % error in u_1^*		0.41	-0.72	-0.40	1.67	1.77	1.05
τ_w^* (= 0.33206 [1])		0.3329	0.3298	0.3306	0.3362	0.3371	0.3340
% error in τ_w^*		+0.24	-0.68	-0.44	+0.99	+1.51	+0.57
Execution time (sec.)		0.2	0.4	0.4	0.1	0.2	0.2

Other parameters: $\epsilon = 1.0$, $II = 30$, $m = 0$

TABLE 4.10 COMPARISONS OF THE RESULTS FROM THE THREE
DIFFERENT ALGORITHMS--FLOW OVER A 90° WEDGE

η_i	$u_i^* [2]$	u_i^* (finite difference)					
		$M_r = 18; N = N_r = 300; I = 22$			$M_r = 18; N = N_r = 75; I = 12$		
		S-E	S-IM	C-N	S-E	S-IM	C-N
0	0	0	0	0	0	0	0
0.2449	0.1756	0.1759	0.1757	0.1757			
0.4899	0.3311	0.3317	0.3314	0.3314	0.3334	0.3324	0.3322
0.7348	0.4669	0.4678	0.4673	0.4673			
0.9798	0.5833	0.5844	0.5838	0.5837	0.5876	0.5851	0.5849
1.2247	0.6811	0.6824	0.6816	0.6816			
1.4697	0.7614	0.7629	0.7619	0.7619	0.7668	0.7630	0.7629
1.7146	0.8258	0.8273	0.8262	0.8262			
1.9596	0.8761	0.8776	0.8763	0.8764	0.8816	0.8769	0.8769
2.2045	0.9142	0.9156	0.9144	0.9144			
2.4495	0.9422	0.9436	0.9423	0.9424	0.9473	0.9423	0.9424
2.6944	0.9623	0.9634	0.9623	0.9623			
2.9394	0.9761	0.9771	0.9760	0.9761	0.9803	0.9757	0.9759
Max. % error in u_i^*		0.19	+0.09	+0.09	0.74	+0.39	+0.33
τ_w^* (= 0.75754 [2])		0.7588	0.7583	0.7581	0.7633	0.7615	0.7609
% error in τ_w^*		0.17	+0.11	+0.09	0.77	+0.53	+0.45
Execution time (sec.)		0.9	1.0	1.1	0.1	0.2	0.2

Other parameters: $\epsilon = 1.0$, $m = 1/3$

with the Crank-Nicolson algorithm. Figure 4.3 gives results from the standard implicit algorithm for the same cases. Figure 4.4 shows the influence of both M and N/M on the accuracy for the Blasius problem.

The range of parameters investigated and the problems which were considered showed:

(1) The influence of ϵ †.

A value of $\epsilon = 1.0$ was sufficiently stringent to result in engineering accuracy in all cases. Some erratic and inconsistent error trends were obtained for the Blasius problem, however, unless ϵ was reduced to 0.1 (see Figs. 4.2 and 4.3). It is interesting to note that the less stringent value of ϵ actually resulted in slightly more accurate results for the Blasius problem. Negligible differences occurred for $m = 1/3$ and $m = 1$ when ϵ was changed from 1 to 0.1. The majority of the cases required a single application of the corrector if ϵ were equal to one after the calculation progressed beyond the first few steps. The stagnation flow usually required, with ϵ equal to one, four iterative applications of the corrector for the first step and a single application for all succeeding steps.

(2) The influence of $M \left[= \frac{(\Delta y)^2 u_\infty}{\nu \Delta x} \right]$ or $M_r \left[= \frac{(\Delta y)^2 u_r}{\nu \Delta x} \right]$.

The influence of M or M_r with $\Delta \eta$ fixed shows the importance of the x-truncation error. However, it must also be remembered that if M is made too small, i.e., if Δx is too large relative to $(\Delta y)^2$, physical laws will be violated, and the accuracy will be destroyed.

Consider first the results of the Crank-Nicolson algorithm; specifically, those results given in Tables 4.5, 4.6 and in Figs. 4.2 and 4.4. These results showed that for the Blasius problem, the accuracy was relatively independent of M as long as M was greater than 2. Slightly oscillating errors occurred at the leading edge for $M = 1.92$, and $M = 0.96$ resulted in errors which exceeded 100 percent

†This quantity is defined by Eq. (2.39).

(see Table 4.6 and Fig. 4.4). For flow over a 90° wedge, slight oscillations occurred at the leading edge with $M_t = 4$ and $N_t = 100$. Both of these problems are not very smooth at the leading edge. A step change occurs for the flat plate problem and the wedge flow accelerates very rapidly at the leading edge ($u_\infty(N = 1)/u_\infty(N = 100)$ is equal to 0.22). Criterion (4.9) required

$$M \frac{u_{1, n+1/2}}{u_\infty} > 1$$

The approximate values of this quantity at the grid point $i = 2, n = 1$ for these three cases were 1.0 ($m = 0, M = 1.92$), 0.5 ($m = 0, M = 0.96$), and 0.1 ($m = 1/3, M_t = 4$). Thus, the oscillations are understandable.

On the other hand, Table 4.5 shows that for the stagnation flow ($m = 1$), the influence of M_t was smaller than 0.01 percent for the range $0.96 < M_t < 16$. These remarkable results are a consequence of the smoothness of this problem. No singularity occurs at the leading edge, and the acceleration is gradual ($u_\infty(N = 1)/u_\infty(N = 100) = 0.01$).

It is also interesting to note that for stagnation flow

$$M \frac{u_{1+1, n}}{u_\infty} = \text{constant}$$

Hence, the probability of violating physical laws does not change by increasing M_t unless $\Delta\eta$ is changed. Values of $M(u_{1, n+1/2}/u_\infty)$ as small as 0.12 caused no difficulty. At the same time, if the energy equation is included in order to calculate the heat transfer rates from a surface at temperature T_w to the fluid whose temperature is $T_\infty (\neq T_w)$, then a singularity would exist at $x = 0$ in the thermal problem. Physical laws could then be violated very easily because near the stagnation point the fluid elements are moving slowly. Hence, their temperature, due to conduction from the surface will change so much during the large time interval $\Delta x/u_{1, n+1/2}$ that the second law of thermodynamics might be violated.

Since $M(u_{1+1, n}/u_\infty)$ is a constant for this stagnation flow, the occurrence of oscillations and their amplitudes are

dependent upon $\Delta\eta$ only. Employing large values of M_r and N gives better results only because the damping of these oscillations is dependent on N and not x_N . Table 4.11 gives the percent error in the heat flux for a Prandtl number of one. Since $\Delta\eta$ is constant

TABLE 4.11 ERROR IN HEAT FLUX FOR
STAGNATION FLOW ($Pr = 1$)

M_r	N_r	Percent Error in Heat Flux ($\Delta\eta = 0.4$)	
		$N_r = N$	$N = 6$
0.96	6	52.62	52.62
1.92	12	0.43	52.62
4.0	25	0.41	52.62
8.0	50	0.37	52.62
16.0	100	0.36	52.62

Note: The results for the hydrodynamic boundary layer for these cases are given in Table 4.5.

for these cases, changing M_r does not eliminate the oscillations near $x = 0$, and the error at $N = 6$ is independent of M_r .

Although $M(u_{2,n+1/2}/u_\infty)$ becomes small as x becomes large in the Blasius problem and in flow over a 90° wedge, the region far downstream caused no difficulty. The problems are very smooth in this region, and values of $M(u_{1,n+1/2}/u_\infty)$ as small as 0.1 were satisfactory.

Results obtained with the standard implicit algorithm exhibited two main differences:

- (i) Oscillations did not occur at small values of M .
- (ii) The truncation error generally was larger than corresponding errors obtained with the Crank-Nicolson algorithm as long as M was sufficiently large to avoid violations of physical laws.

The greater importance of the x -truncation error is vividly shown by the results for the stagnation flow given in Figs. 4.2 and 4.3.

(3) The influence of $\Delta\eta$.

$\Delta\eta$ is equal to $(M/N)^{1/2}$ or $\Delta y(u_\infty/vx)^{1/2}$. It indicates the importance of the y-truncation error. The size of the y-increment is, by itself, not meaningful; hence, it is related to u_∞ , v , and x . For the wedge flow (see Eq. (3.19)):

$$\Delta\eta = [(M_\infty/N)(N/N_\infty)^m]^{1/2}$$

The results obtained with the Crank-Nicolson algorithm showed that as long as M was sufficiently large to avoid violations of the physical laws, the error is mainly dependent on $\Delta\eta$ for a given problem. Since $\Delta\eta$ is independent of N or x_N for stagnation flow ($m = 1$), the error is also independent of N (see, e.g., Fig. 4.2). The tabulated results given in Tables 4.5 and 4.8 are the same to four significant figures for $1 < N < N_\infty$. (The values at $N = 1$ were slightly different for $\epsilon = 1$ but became identical for $\epsilon = 0.1$.) The significance of this result is that an accurate solution to this similar flow can be obtained by finite difference calculations in the physical plane with a single step in the x -direction. If some second order influence caused this flow to be non-similar, the accuracy of numerical results would be approximately independent of x_N . In contrast, the accuracy in Blasius problem was strongly dependent on x_N for small N because few grid points were located within the boundary layer. A small value of N always means a relatively large $\Delta\eta$ for the Blasius problem.

The results given in Tables 4.7 and 4.8 show the strong dependence of the error on $\Delta\eta$ if $\Delta\eta > 0.8$ for $m = 0$ and, 1, respectively. The error was less than 0.5 percent in both problems, however, if $\Delta\eta$ was less than 0.5. In general, all results obtained substantiated the fact that the magnitude of $\Delta\eta$ provides an excellent indicator of the importance of truncation error regardless of the magnitude of v , u_∞ , or x .

4.2 NONSIMILAR BOUNDARY LAYERS

Three nonsimilar flows will be presented. The objectives of this section are:

- (i) To show the ease with which the solutions to nonsimilar problems can be obtained.

- (ii) To show that the generality of the results is not lost if the solution is effected with finite difference techniques. Additional parameters will be of importance since the flows are nonsimilar, and it will no longer be possible to present the solutions as single curves. Often a single additional parameter is sufficient to describe a nonsimilar flow and a family of curves results for such cases.
- (iii) To show that the parameters M and N/M still are of major importance and that the increments Δx and Δy still play minor roles.
- (iv) To show the effectiveness of film or slot cooling. The last of the three nonsimilar problems which will be presented is concerned with film cooling.

Excellent accuracy was obtained for the accelerating wedge flows because the boundary layer thickness generally increased more slowly with increasing x . Generally, the best accuracy was obtained for the stagnation flow; the boundary layer thickness is independent of x for this case. A rapidly decelerating flow would appear to provide the most stringent test for finite difference techniques since the boundary layer grows most rapidly for this case.

Specifically, consider the potential flow given by

$$u_x(x) = u_\infty - ax, \quad a > 0$$

or

$$u_x/u_\infty = 1 - x^*, \quad \text{where } x^* = ax/u_\infty \quad (4.11)$$

The flow can be interpreted as that which occurs in a channel with initially parallel walls ($u = u_\infty$, a constant) and which then enters a divergent section. The boundary layer growth is assumed to start at $x = 0$. The relative rates of growth of the boundary layers for the cases considered so far are given in Table 4.12. The solution

TABLE 4.12 VARIATION IN BOUNDARY LAYER GROWTH RATE

Type of Problem	Number of Grid Points in Boundary Layer		Rate of Growth
	Location: x_1	Location: $2x_1$	
180° wedge	p	p	$\delta = \text{constant}$
90° wedge	0.793p	p	δ proportional to $x^{1/3}$
Blasius	0.707p	p	δ proportional to $x^{1/2}$
$u = u_r - ax$	$0.53p(x^* = 0.6)^{\dagger}$	$p(x^* = 0.12)^{\dagger}$	---

$\dagger$ The number of grid points is based on $u/u_{\infty}|_{\delta} = 0.90$.

for the decelerating flow approaches the Blasius solution as x^* approaches zero. It deviates greatly from the Blasius solution as the point of separation is approached, $x^* = 0.12$.

The parameters of importance with the decelerating potential flow include those which arose in the Blasius problem. The pressure term also must be examined to determine what additional parameters are of importance. It is again convenient to divide the terms of the momentum equation by $u_r^2/\Delta x$ to make them dimensionless. The pressure term becomes

$$\frac{\Delta x}{u_r^2} \left[u_s \frac{du_s}{dx} \right]_{n+1/2} = (x^*|_{n+1/2} - 1) \Delta x^* \quad (4.12)$$

Even if the velocity u_{∞} at location $N \Delta x$ were employed to make the pressure term dimensionless, it would not be independent of Δx ; hence, the flow is nonsimilar. It is more convenient for this nonsimilar

case to base M and η on u_r . Thus

$$\frac{u_i}{u_r} = f_1(\eta_r|_i, x^*) \quad (4.13a)$$

or

$$\frac{u_i}{u_\infty} = f_1(\eta_r|_i, x^*)(1 - x^*)^{-1}$$

where:

$$\eta_r|_i = y_i(u_r/\nu x)^{1/2} = (i - 1)(M_r/N)^{1/2} \quad (4.14)$$

The dependency of the velocity profile on x^* is usually shown in the literature (see, e.g., [1], p. 164) by a plot of u/u_∞ (ordinate) v. η_r . Although this plot shows the changing shape of the profile with x^* , it does not represent a correct picture of the growth of the boundary layer with x^* . Since the results are to be applicable for any such flow, y and Δy must remain unspecified. Thus, a correct picture can only be given by a plot of y_i/y_1 or $y_i/\Delta y$ (ordinate) v. u/u_r with x^* as a parameter. This family of curves is given in Fig. 4.5. The discrete velocities and values for the shear stress at the wall, at $x^* = 0.0375, 0.075$, and 0.1125 are given in Table 4.13. The results obtained by Howarth [13] are given for comparison. Results given in Fig. 4.6 show the variation of the error

TABLE 4.13 COMPARISON OF VELOCITY PROFILES
AND WALL SHEAR STRESS--DECELERATING POTENTIAL
FLOW, $u/u_r = 1 - x^*$

	$x^* = 0.0375$		$x^* = 0.075$		$x^* = 0.1125$	
	u/u_∞		u/u_∞		u/u_∞	
η_r	Ref [13]	finite diff	Ref [13]	finite diff	Ref [13]	finite diff
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.4	0.108	0.108	0.078	0.078	0.034	0.033
0.8	0.222	0.222	0.168	0.168	0.085	0.084
1.2	0.338	0.338	0.267	0.267	0.152	0.151
1.6	0.455	0.454	0.372	0.372	0.234	0.232
2.0	0.567	0.566	0.480	0.479	0.325	0.324
2.4	0.670	0.669	0.585	0.584	0.426	0.423
2.8	0.760	0.759	0.682	0.681	0.527	0.524
3.2	0.834	0.833	0.769	0.767	0.625	0.622
3.6	0.891	0.891	0.839	0.838	0.716	0.712
4.0	0.934	0.933	0.894	0.893	0.794	0.791
4.4	0.962	0.961	0.934	0.933	0.858	0.855
4.8	0.979	0.979	0.961	0.961	0.908	0.905
5.2	0.990	0.989	0.979	0.978	0.943	0.941
5.6	0.995	0.995	0.989	0.989	0.967	0.965
6.0	0.998	0.998	0.995	0.994	0.982	0.981
$\tau_w^* [13]$	1.309	1.307	0.613	0.610	0.163	0.161
% error in τ_w^*	-0.15		-0.49		-1.23	
max. % error in u/u_∞	-0.22		-0.26		-3.03	
Parameters: M_r	2.4		2.4		2.43	
Δx^*	1.5625×10^{-4}		3.1250×10^{-4}		4.6296×10^{-4}	
I	70		80		90	
no. of grid points with $0 < u/u_\infty < 0.9$	36		40		47	

Other parameters: $\epsilon = 1.0$, $\Delta \eta_r = 0.1$

Algorithm: Crank-Nicolson

$$\tau_w^* = \nu^{1/2} (\partial u / \partial y)_{y=0} / u_r a^{1/2}$$

in the wall shear stress with Δx^* , M_1 , and N/M_1 .† Results for the Blasius problem are given for comparison. Numerical results obtained showed that:

- (i) The error is nearly independent of (Δx^*) and M but strongly dependent on $\Delta \eta$ and x^* . The importance of $\Delta \eta$ in these results is significant because it demonstrates the importance of this parameter for nonsimilar boundary layers.
- (ii) The variation of the error with N/M followed closely the results obtained for the Blasius problem if $x^* < 0.06$. The error in this region was dependent on $\Delta \eta$ but was relatively independent of x^* .
- (iii) As the point of separation was approached, the percent error in the shear stress at the wall grew rapidly. This is natural because this shear stress is approaching zero.
- (iv) A value of $\epsilon = 1$ was found to be sufficiently stringent for this case also. Like the Blasius problem, a smaller value of ϵ gave slightly *larger* errors but more consistent trends.

The prediction of the convective heat transfer from surfaces of varying temperature is an important problem which has received considerable attention. However, the techniques used in the past generally were restricted and did not permit a completely arbitrary temperature distribution. To treat such cases with finite difference techniques, on the other hand, would be straightforward.

If the relevant properties of the fluid media are constants and if viscous dissipation is neglected, the energy equation has the same form as the momentum equation. Due to this similarity, the addition of this equation with a finite difference technique is straightforward and will not be described. An additional parameter, the Prandtl

† N/M_1 is used as the ordinate instead of N/M for convenience. M and M_1 are identically equal at $x^* = 0$ and differ by only 12 percent at the point of separation.

number, arises when the thermal boundary layer is included. The energy equation is linear with these assumptions.

It is possible to obtain the solution for any arbitrary surface temperature distribution by an appropriate integration of the particular solution for the case when there is a step wall temperature change occurring at a finite distance from the leading edge of the plate [14]. This particular solution has been obtained by Eckert [15] using an integral technique. Cheema [14] is presently studying this problem by transforming the original partial differential equation and solving the resulting sequence of ordinary differential equations by Meksyn's procedure. If finite difference techniques were used, this indirect approach probably would not be employed. At the same time, the case of a step change in surface temperature provides the most stringent test of any arbitrary temperature distribution. For this reason, this particular solution will be presented.

Specifically, the laminar incompressible flow of a constant property fluid whose temperature is T_∞ is considered. The temperature of the plate is T_∞ from $x = 0$ to $x = x_0$, and T_w for $x > x_0$. The problem provides a stringent test of finite difference schemes due to the singularity at $x = x_0$. This is more severe than the singularity which occurs at the leading edge when $x_0 = 0$, because the boundary layer thickness in this case is finite. Hence, the time interval $\Delta x/u_{1,n}$ is large near the surface for x near x_0 , and violations of the second law of thermodynamics can occur easily.

The thermal boundary layer is not similar. It asymptotically approaches a similarity solution as x_0/x approaches zero. The finite difference equations clearly show that an additional parameter is of importance. This parameter is N_0/N or x_0/x where $N_0 \Delta x = x_0$. The choice of N_0 is arbitrary; however, its value influences the truncation error. Specifically,

$$\frac{T_i - T_w}{T_\infty - T_w} = f(\eta_i, Pr, x_0/x)$$

If the local Nusselt number, $Nu = hx/k$, is referenced to its value for $x_0 = 0$, it can be seen from the finite difference formulation that

$$\frac{Nu}{Nu}|_{x_0=0} = f(x_0/x, Pr)$$

Eckert's approximation solution is

$$\frac{Nu}{Nu}|_{x=0} = [1 - (x_0/x)^{3/4}]^{-1/3}, \quad Pr \geq 1$$

He assumed that the thermal boundary layer thickness δ_t was less than or equal to the hydrodynamic boundary layer thickness δ . Hence, his solution is restricted to $Pr \geq 1$, but the error would be expected to be small for $Pr = 0.72$, thus it also is used in the comparisons for $Pr = 0.72$.

The finite difference results along with Eckert's solution for comparison are given in Table 4.14 and Fig. 4.7. The finite difference

TABLE 4.14 COMPARISON OF RESULTS FOR FLOW OVER FLAT PLATE WITH STEP CHANGE IN TEMPERATURE

x_0/x	Nu/Nu $x_0=0$					
	Finite Difference Results					Ref. [15]
0.909	2.490	2.482	2.476		1.192†	2.44
0.806	1.922	1.913	1.920	1.920	1.992†	1.88
0.704	1.657	1.650	1.657*		1.658*	1.63
0.602	1.490	1.485	1.494*		1.489	1.47
0.495	1.363	1.360	1.363*	1.362*	1.363*	1.35
0.400	1.275	1.273	1.275*		1.275*	1.26
0.292	1.193	1.192	1.192*		1.193*	1.183
0.202	1.133	1.132	1.133	1.132*	1.133*	1.127
0.101	1.070	1.070	1.070		1.070	1.068
0.050	1.038	1.038	1.038	1.038	--	1.038
M	16	16	8	8	4	--
N_0	100	100	50	25	50	--
I	70	70	70	70	70	--
Pr	0.72	1.0	0.72	0.72	0.72	Pr $\geq$ 1

*interpolated between printed values. May contain an error of one in last digit.

†profile is oscillating.

solution indicates that for $0 \leq (x_0/x) < 0.9$, the error in Eckert's approximate solution is less than 2 percent for $Pr = 1$. The difference between the numerical results for $Pr = 0.72$ and $Pr = 1$ is generally less than 0.5 percent; hence, the dependency of this ratio on the Prandtl number for Prandtl numbers near 1 is weak. The

absolute error in the numerical results is estimated to be less than 0.5 percent for $0.1 < x_0/x < 0.9$.

The last nonsimilar problem which will be considered is a simplified film cooling problem. Film or slot cooling involves the mixing of two moving streams in the vicinity of a solid wall. A boundary layer forms along this wall and eventually interacts with the external mixing zone. Specifically, the coolant (the secondary stream) with a velocity u_s and a temperature T_s is ejected through a slot in a direction tangent to the surface. The objective is to protect the surface from the high temperature external stream (the primary stream) whose temperature and velocity are T_p and u_p . The flow field is depicted in Fig. 4.8. The mass flow rates and the slot spacing which are required in order to afford adequate thermal protection are desired.

It is assumed that:

- (1) The flow field is two-dimensional and steady.
- (2) The secondary and primary fluids are the same, and all relevant fluid properties are constants.
- (3) The usual boundary layer assumptions are satisfied in the mixing region and the boundary layer region.
- (4) The external pressure and velocity do not vary in a direction along the film cooled surface.
- (5) The initial velocity and temperature profiles of both streams are uniform and the velocity vectors are parallel.
- (6) The mixing region and the boundary layer are both laminar.
- (7) The splitter plate which initially separates the two streams has zero thickness at $x = 0$.

(8) Heat transfer by thermal radiation is negligible.

(9) Viscous dissipation is negligible.

With these assumptions, the incompressible boundary layer equations given in Section 2 can be applied throughout the region. The energy equation is again similar to the momentum equation.

It is assumed that the surface is being held at the temperature of the secondary stream T_s ; hence, the heat flux, q'' , is desired. The total rate of heat transfer q is also calculated. In order to show vividly the effectiveness of the coolant stream, the heat flux, the shear stress at the wall, and the total rate of heat flow is divided by the corresponding values without the secondary stream, $s = 0$.

Without a secondary stream, the hydrodynamic problem reduces to the Blasius problem; hence, the dimensionless shear stress is known to be a constant value, 0.33206 where

$$\tau_w^* = \frac{\tau|_{y=0}}{\rho u_\infty^2} [Re_x]^{1/2} = \tilde{\Delta}_y u_1^* (N/M)^{1/2} = \text{constant} \quad (4.15)$$

Another parameter, the Prandtl number Pr , arises from the energy equation. An analysis of the governing difference equations shows that the dimensionless temperature T^* and heat flux $[q'']^*$ are

$$T_i^* = \frac{T_i - T_s}{T_p - T_s} = f_1(Pr, \eta_1)$$

and

$$[q'']^* = \frac{Nu_x}{[Re_x]^{1/2}} = \tilde{\Delta}_y T_1^* (N/M)^{1/2} = f_2(Pr) \quad (4.16)$$

where

$$Nu_x = \frac{hx}{k} \quad \text{and} \quad q'' = h(T_p - T_w)$$

the thermal problem in this case is also similar.

The addition of the secondary stream adds two parameters, the velocity and height of the secondary stream. The mixing layer and the boundary layer both start out as similar flows; however, the problem ceases to be similar when the interaction between these regions becomes important. Far downstream the influence of the secondary stream vanishes and the problem again becomes similar.

Grid points 2 through IS lie in the secondary stream and the slot height becomes with this definition of the network: (See Fig. 4.8.)

$$s = (IS - 1/2) \Delta y \quad (4.17)$$

A dimensionless x location ξ logically follows from the definition $x = N \Delta x$ if M and s are introduced in order to eliminate Δx and Δy . This gives:

$$x = N/M (IS - 1/2)^{-2} [u_p s / \nu](s) \quad (4.18)$$

or

$$\xi = (x/s)(Re_s)^{-1} = N/M (IS - 1/2)^{-2} \quad (4.19)$$

A dimensionless y coordinate, $y^* = y/s$,† a dimensionless velocity, $u^* = u/u_p$, and a dimensionless temperature, $T^* = (T - T_s)/(T_p - T_s)$ will also be introduced. An examination of the difference equations and boundary conditions shows:

$$u_{i,N}^* = f_3(u_s^*, y_i^*, \xi_N) \quad (4.20)$$

and

$$T_{i,N}^* = f_4(Pr, u_s^*, y_i^*, \xi_N) \quad (4.21)$$

The ratio of the wall shear stress, the heat flux at the wall and the total rate of heat flow through the wall become, respectively:

$$\frac{\tau_w}{\tau_w|_{s=0}} = f_5(u_s^*, \xi_N) = \frac{\tilde{\Delta}_y u_1^* (N/M)^{1/2}}{0.33206} \quad (4.22)$$

$$\frac{q_w'}{q_w'|_{s=0}} = f_6(u_s^*, \xi_N, Pr) = \frac{\tilde{\Delta}_y T_1^* (N/M)^{1/2}}{f_2(Pr)} \quad (4.23)$$

†The similarity variable η could be used in place of the dimensionless y coordinate. Since the similar regions are of little interest, the use of the physically meaningful coordinate is more enlightening.

and

$$\frac{q}{q_{s=0}} = \frac{\int_0^x q_w''(\text{depth}) dx'}{\int_0^x q_w''|_{s=0}(\text{depth}) dx'} = f(u_s^*, \xi_N, \text{Pr}) \quad (4.24)$$

If the trapezoidal rule is employed in the integration of the finite difference results for the numerator of Eq. (4.24) and if Eq. (4.16) is introduced and the denominator is integrated, Eq. (4.24) becomes

$$\frac{q}{q_{s=0}} = \frac{\frac{\Delta x}{\Delta y} \left[\sum_{n=1}^{N-1} \tilde{\Delta}_y T_{1,n}^* + (\tilde{\Delta}_y T_{1,N}^*)/2 \right]}{2f_2(\text{Pr})[\text{Re}_x]^{1/2}} \quad (4.25)$$

If N and M are introduced in order to eliminate (Δx) and (Δy) , Eq. (4.25) can be further simplified to

$$\frac{q}{q_{s=0}} = \frac{\sum_{n=1}^{N-1} \tilde{\Delta}_y T_{1,n}^* + (\tilde{\Delta}_y T_{1,N}^*)/2}{2f_2(\text{Pr})[NM]^{1/2}} \quad (4.26)$$

The limiting values of Eqs. (4.22), (4.23) and (4.26) are given in Table 4.15.

Figure 4.9 shows a family of temperature profiles. The parameter is the dimensionless x -location, ξ . The change in the heating rate can be clearly seen by examining the slopes at y/s equal to zero. Figures 4.10 and 4.11 show the variation of the shear stress, the

TABLE 4.15 LIMITING VALUES OF SHEAR STRESS, HEAT FLUX, AND HEAT FLOW RATIOS

	Limit as $\xi_N \rightarrow 0$	Limit as $\xi_N \rightarrow \infty$
$\tau_w/\tau_w _{s=0}$	$(u_s/u_p)^{3/2}$	1
$q'_w/q'_w _{s=0}$	0	1
$q/q_{s=0}$	0	1

heat flux and the heat flow rate with ξ . Velocity ratios, u_s/u_p , of 0.25 and 0.5 were employed. The results vividly show the effectiveness of the coolant at small values of ξ . All results are for a Prandtl number of 0.72.

The heating rates will obviously be very large if initial boundary layers are not present in the primary stream. A uniform primary stream is unrealistic in most applications. It is only possible at a leading edge. If it is assumed that such a leading edge exists, the solution which has been presented could be used as an initial condition for the primary stream at the downstream slot. A uniform secondary stream is fairly realistic and this assumption will be retained.

Figures 4.12, 4.13 and 4.14 contain results for the case of a second identical slot being introduced at location ξ_0 . The velocity profiles including the initial conditions at $\xi = 0$ and $\xi = \xi_0$ are given in Fig. 4.12. Figures 4.13 and 4.14 give the heating rates etc., for u_s/u_p of 0.5 and 0.25, respectively. The reference values

of τ_w , q'_w and q employed for $\xi > \xi_0$ are those obtained by assuming the flat plate ($s = 0$) boundary layer starts at $\xi = 0$ and grows continuously. The total rate of heat transfer at $\xi = 0.45$ has been reduced to 15 percent and 35 percent of the flat plate value for u_s/u_p of 0.5 and 0.25, respectively. The location of the second slot was chosen arbitrarily, and these results do not represent an optimum case.

These results are far from being complete; they are only meant to give an indication of the drastic reduction in heating rates that can be affected by film cooling, to show the importance of the velocity ratio u_s/u_p and the initial boundary layer profiles, and to demonstrate the potentiality of finite difference methods. The generality of the results clearly demonstrate that this feature need not be lost if a finite difference approach is employed.

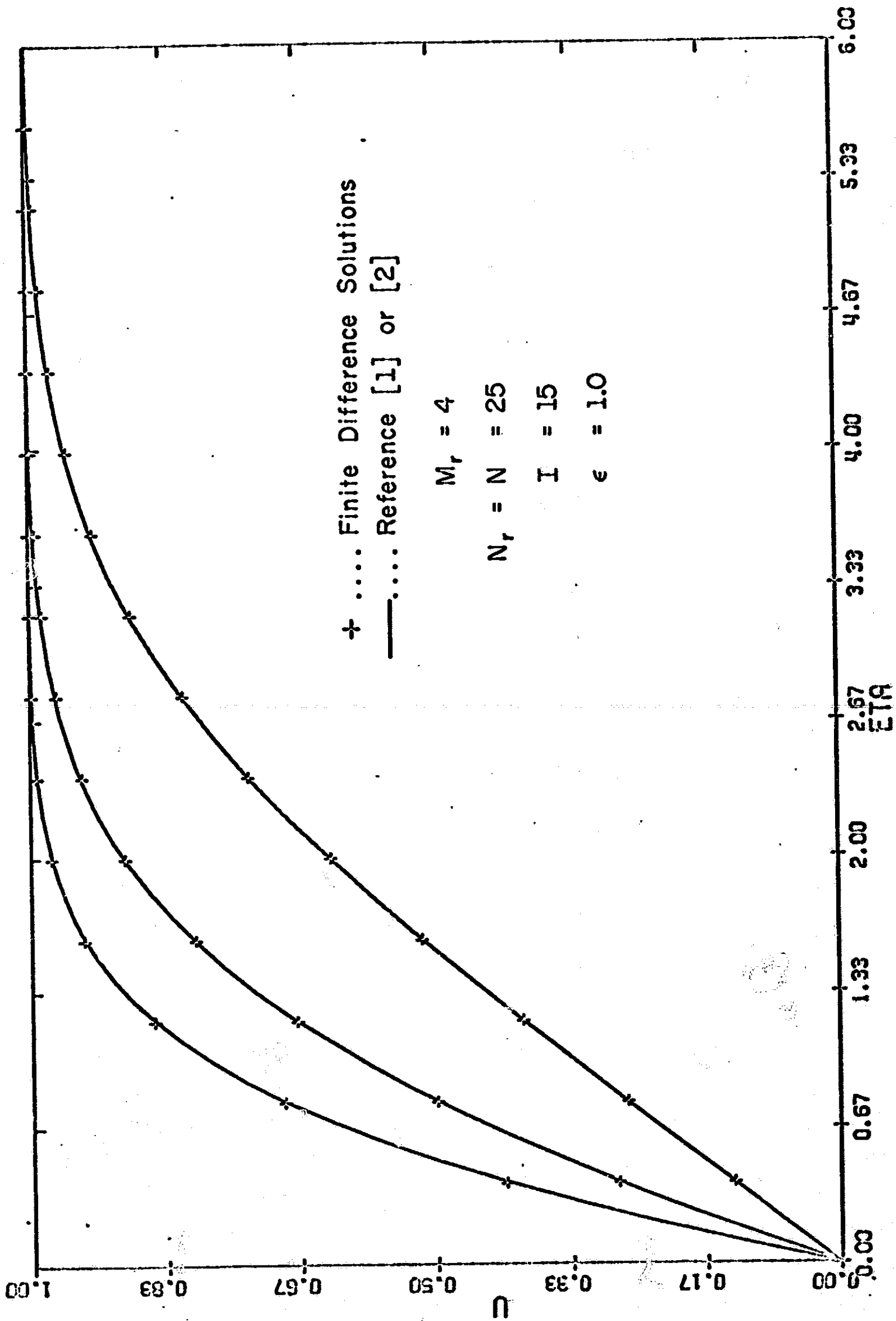


Figure 4.1 Comparisons Between Finite Difference Solutions and Published Solutions--Crank-Nicolson Algorithm

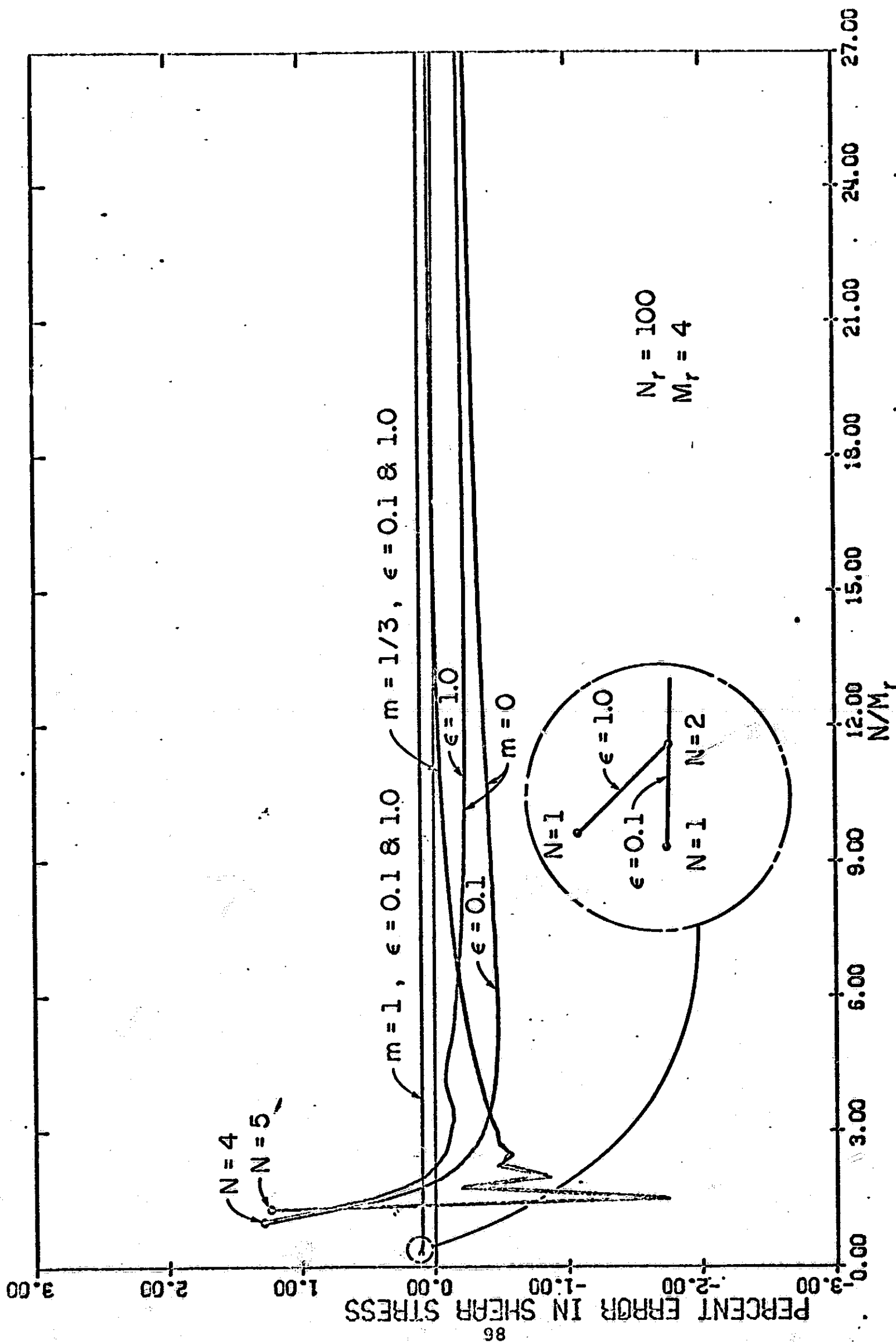


Figure 4.2 Influence of N/M_r and ϵ on Accuracy--Crank-Nicolson Algorithm

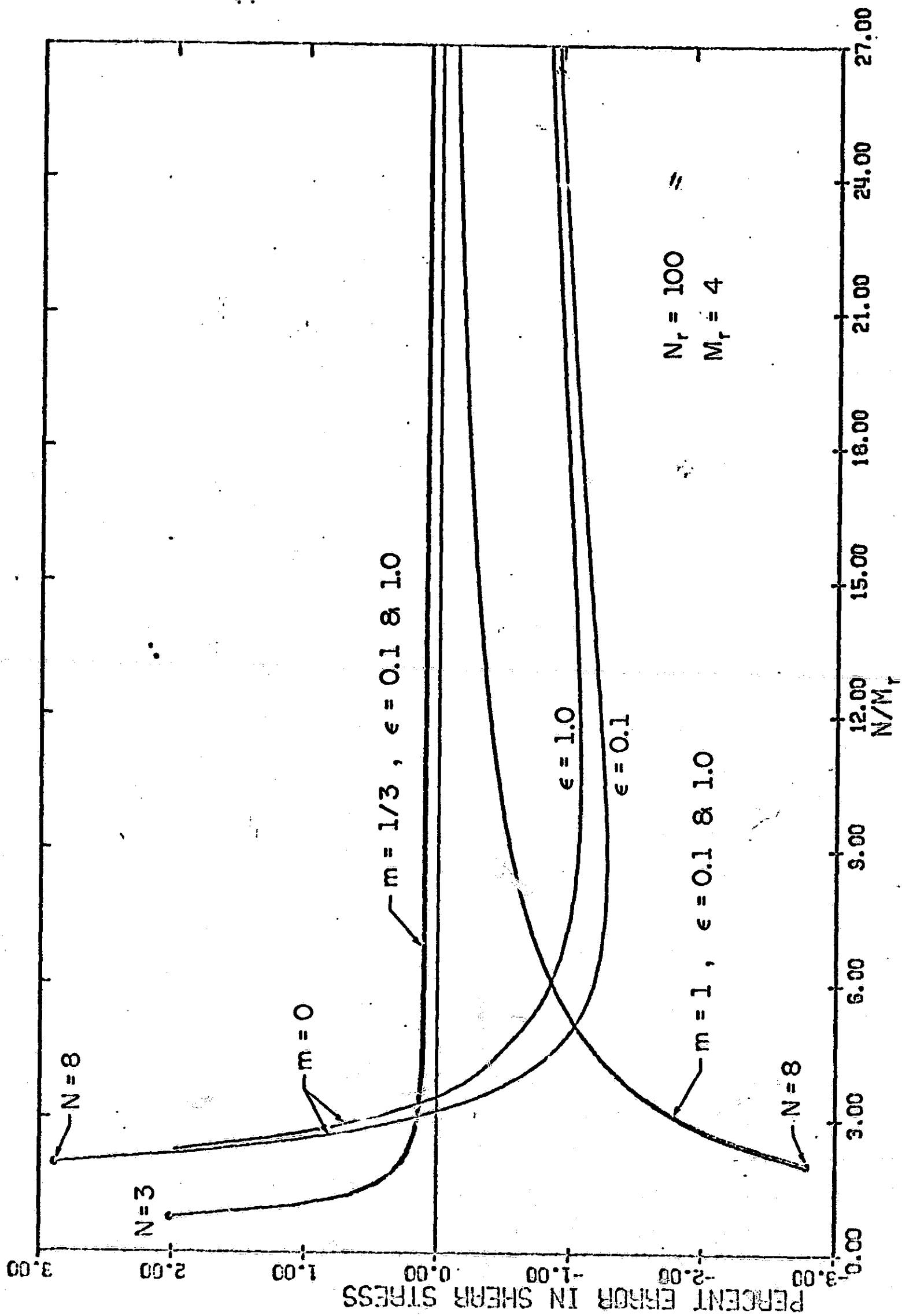


Figure 4.3 Influence of N/M_r and ϵ on Accuracy--Standard Implicit Algorithm

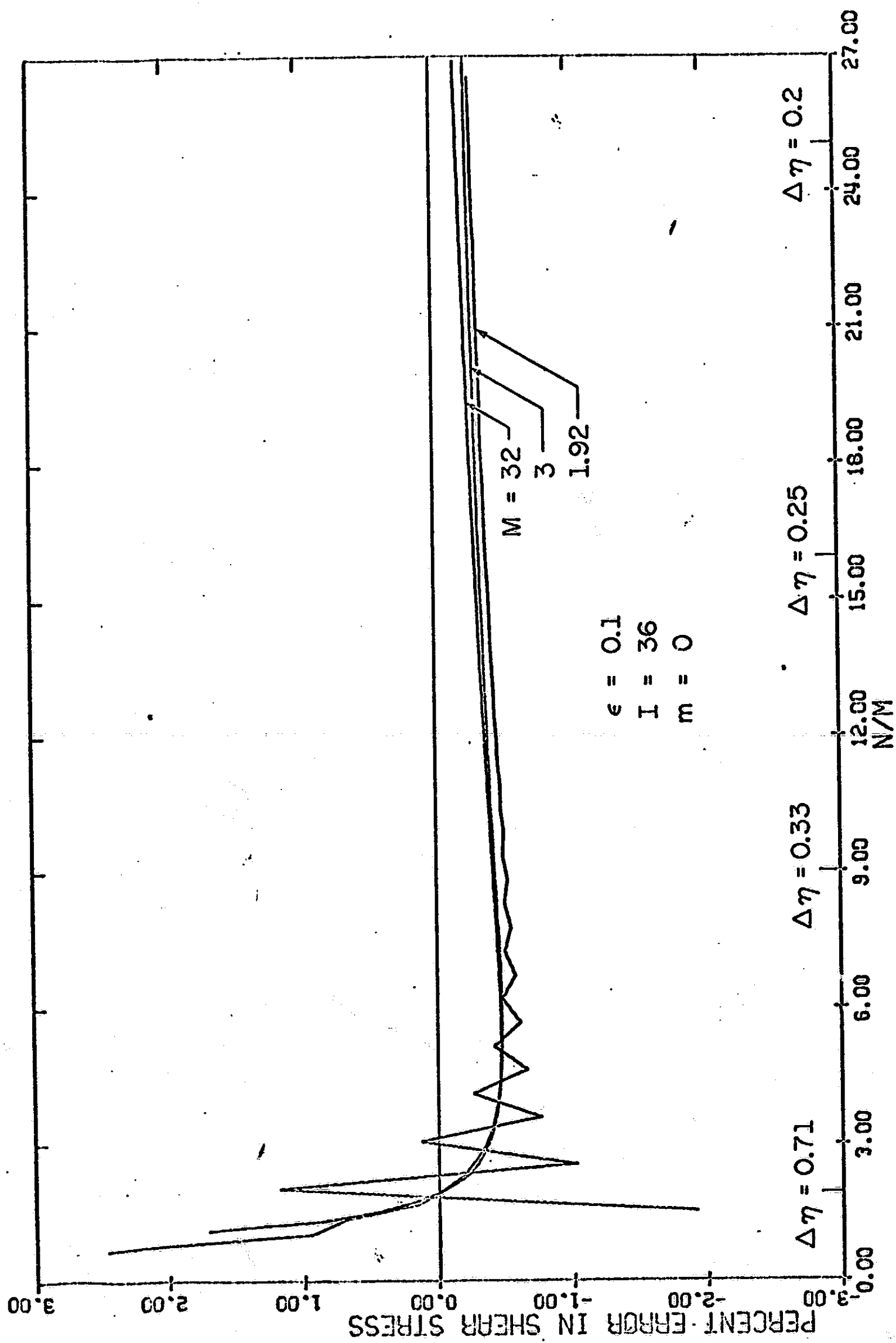


Figure 4.4 The Influence of M and $\Delta\eta$ on Accuracy--Crank-Nicolson Algorithm

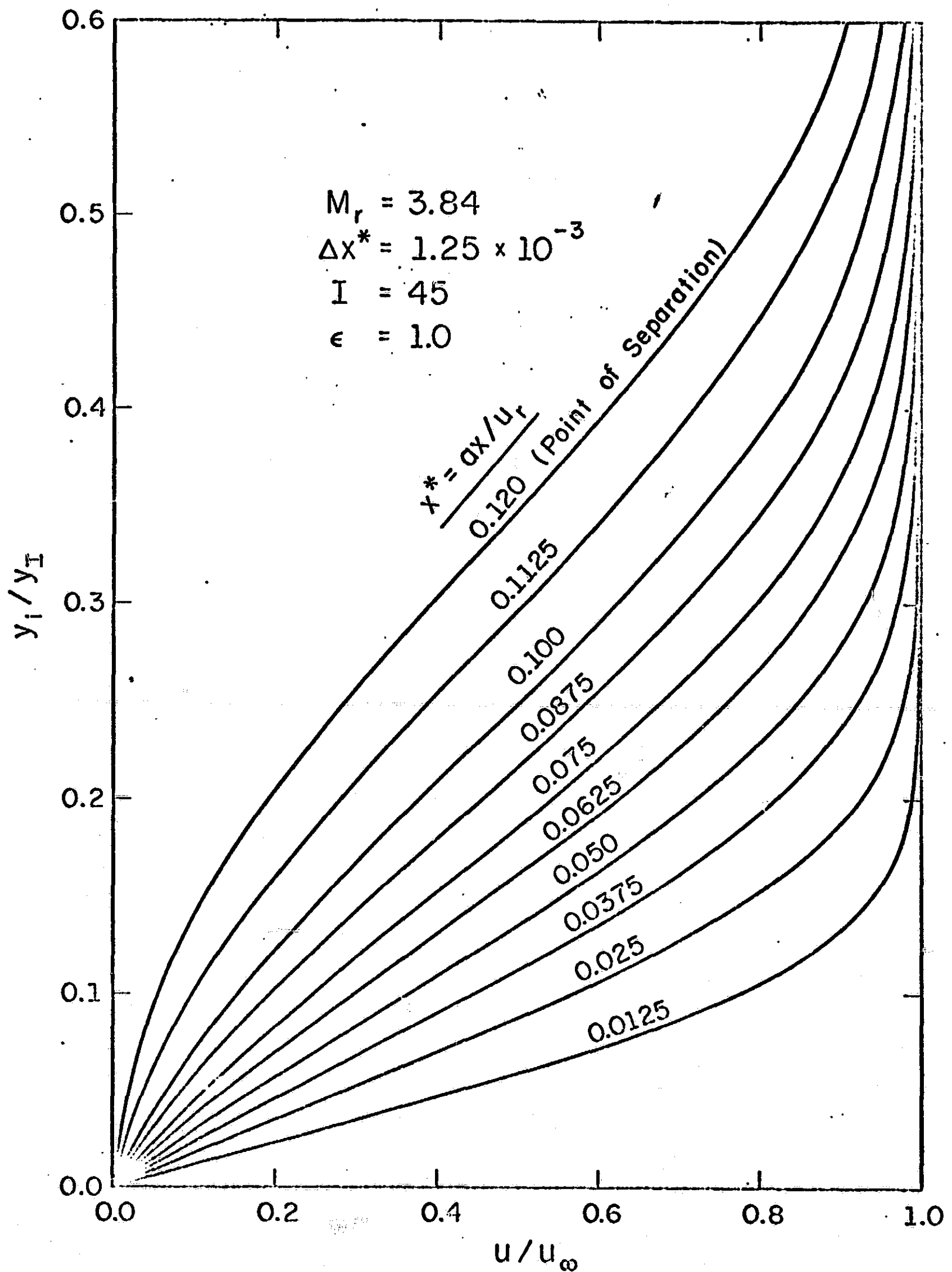


Figure 4.5 The Influence of x^* on the Velocity Distribution for the Potential Flow given by $u = u_r - ax$

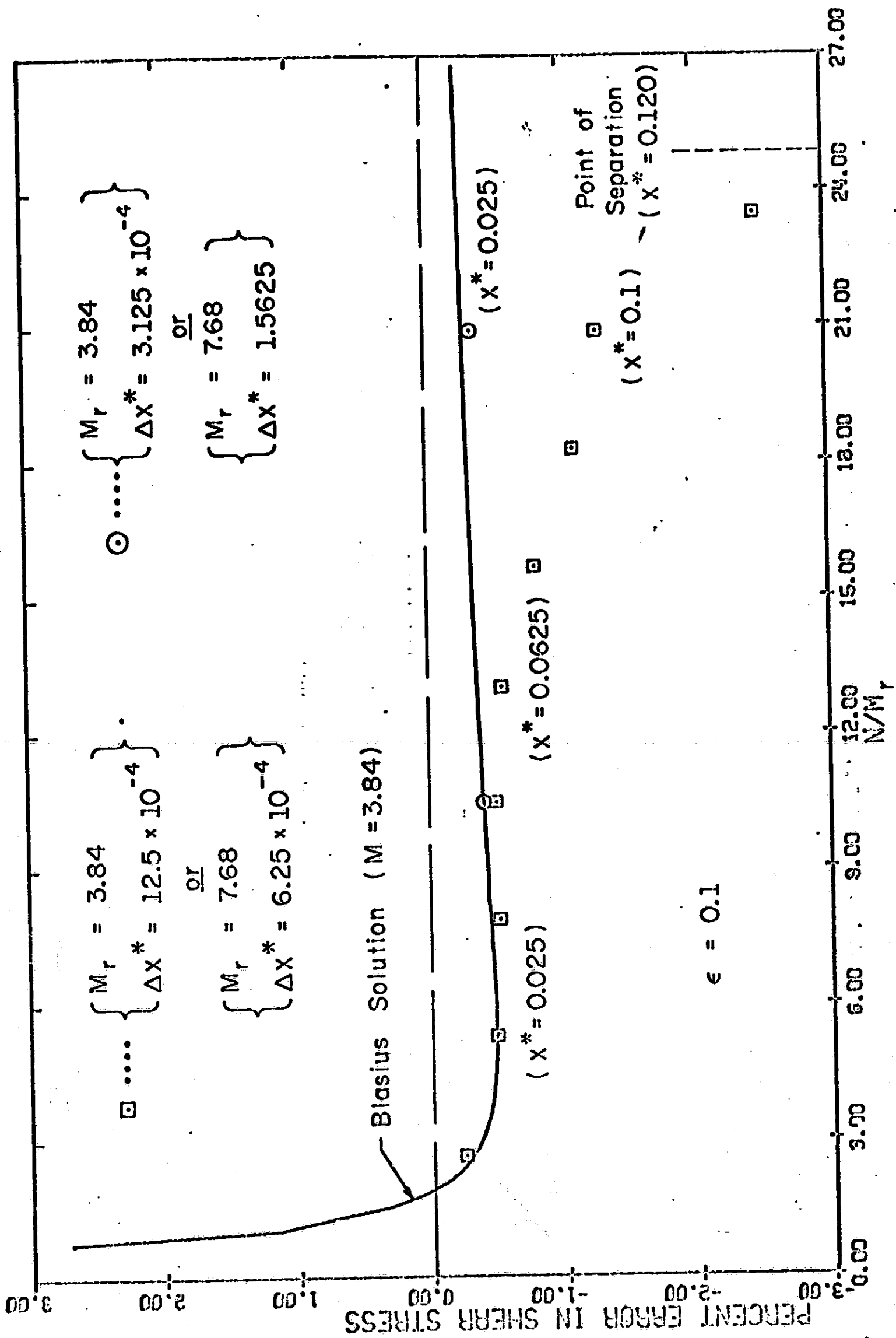


Figure 4.6 Influence of Various Parameters on Accuracy--Decelerating Potential Flow, $u = (u_r - ax)$ (Nonsimilar Boundary Layer)

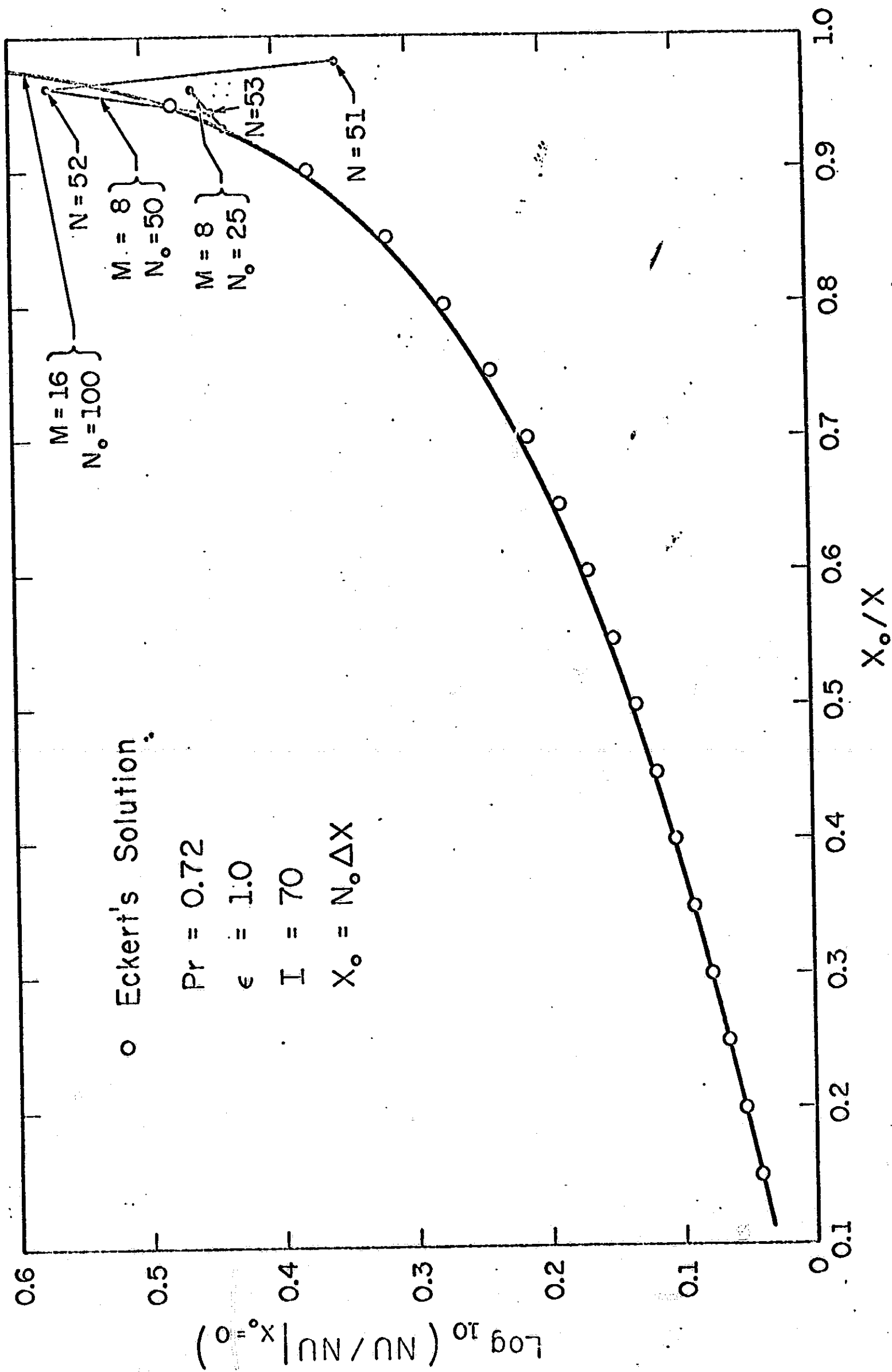


Figure 4.7 Comparison of Results for Flow over a Flat Plate with a Change in Temperature at $x = x_0$

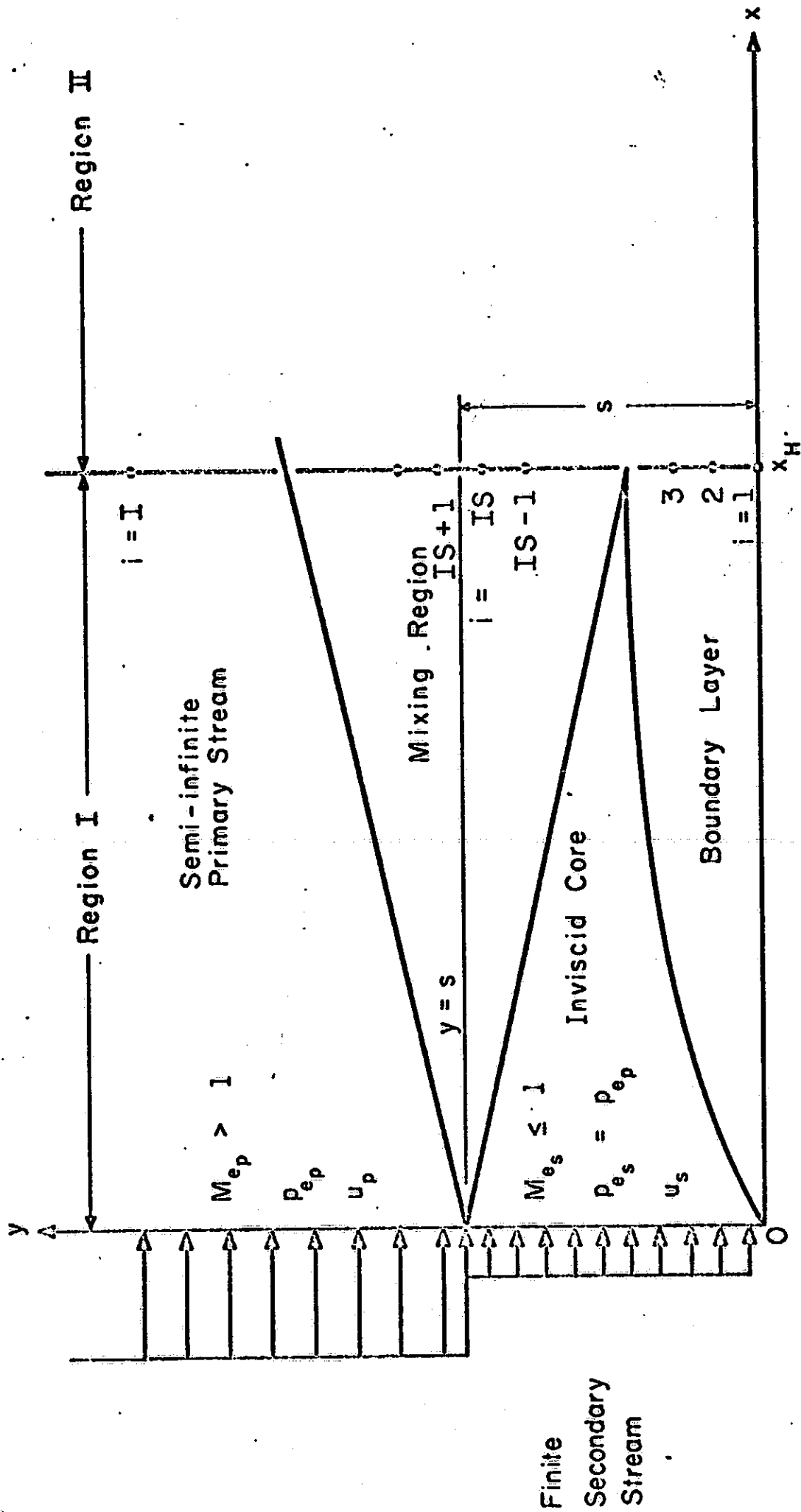


Figure 4.8 Nomenclature and Coordinate System for Flow Configuration being Investigated

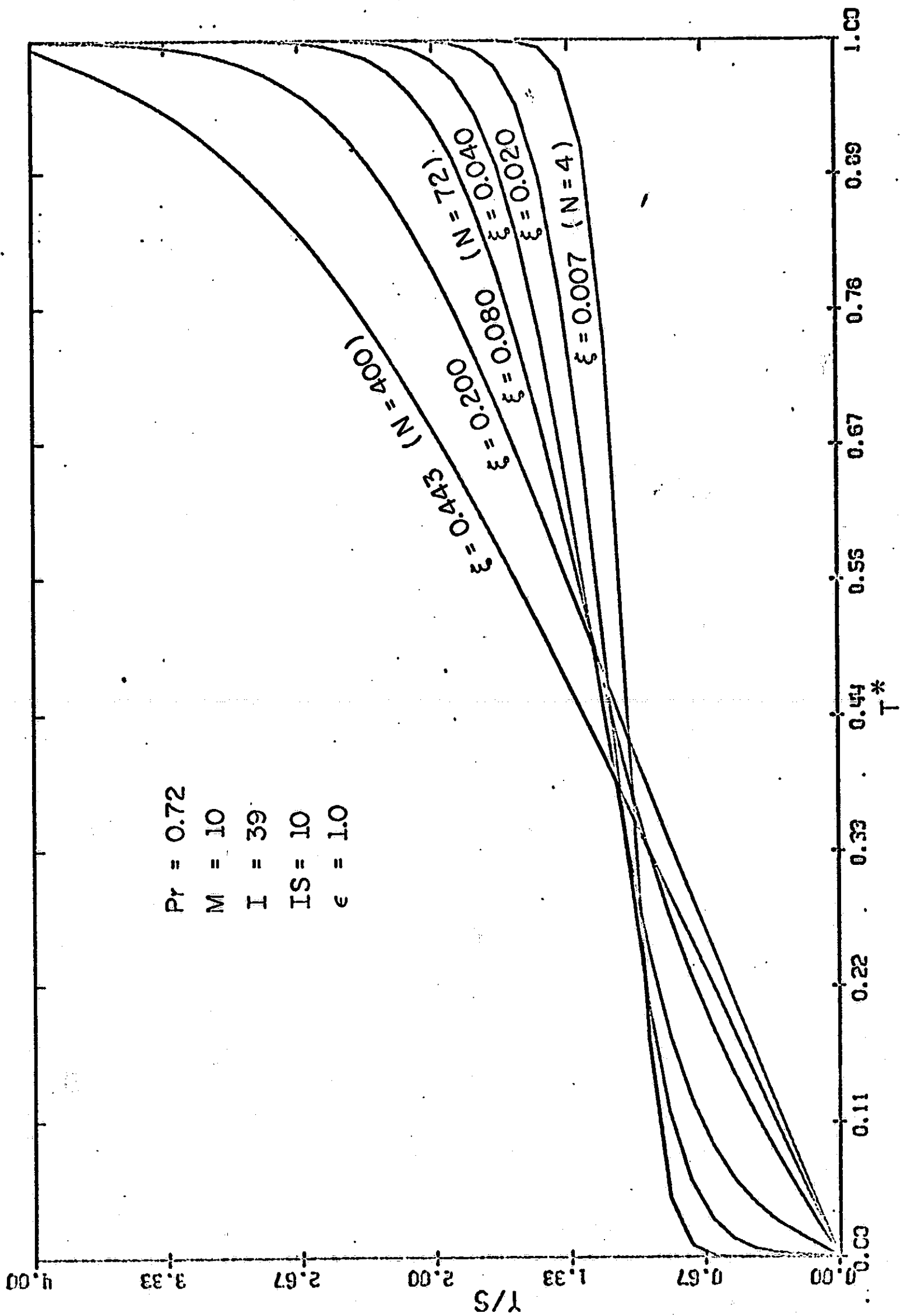


Figure 4.9 The Influence of ξ on the Temperature Distribution; $w_s/w_p = 0.5$

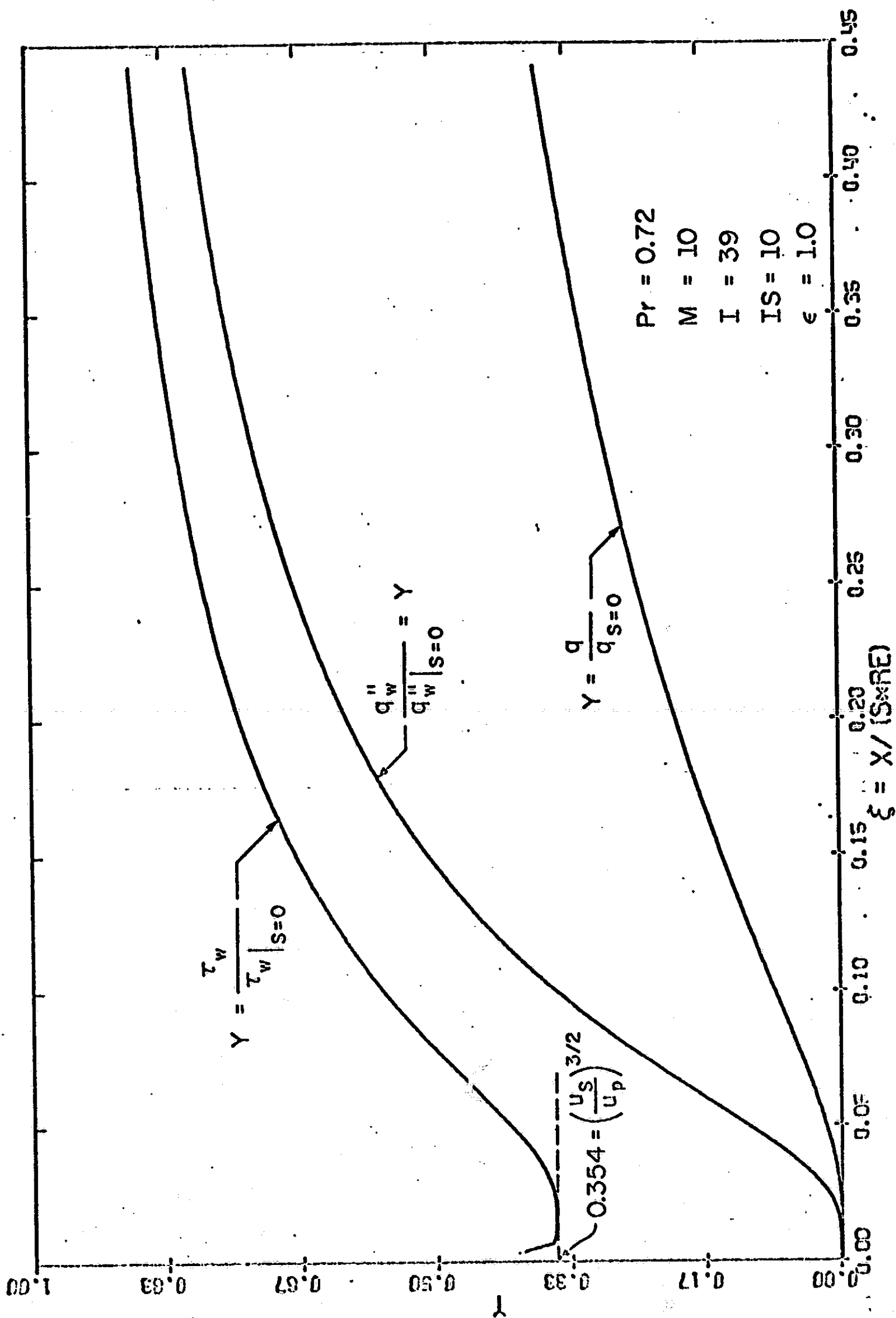


Figure 4.10 The Influence of ξ on the Shear Stress and Heating Rates; $u_s/u_p = 0.5$

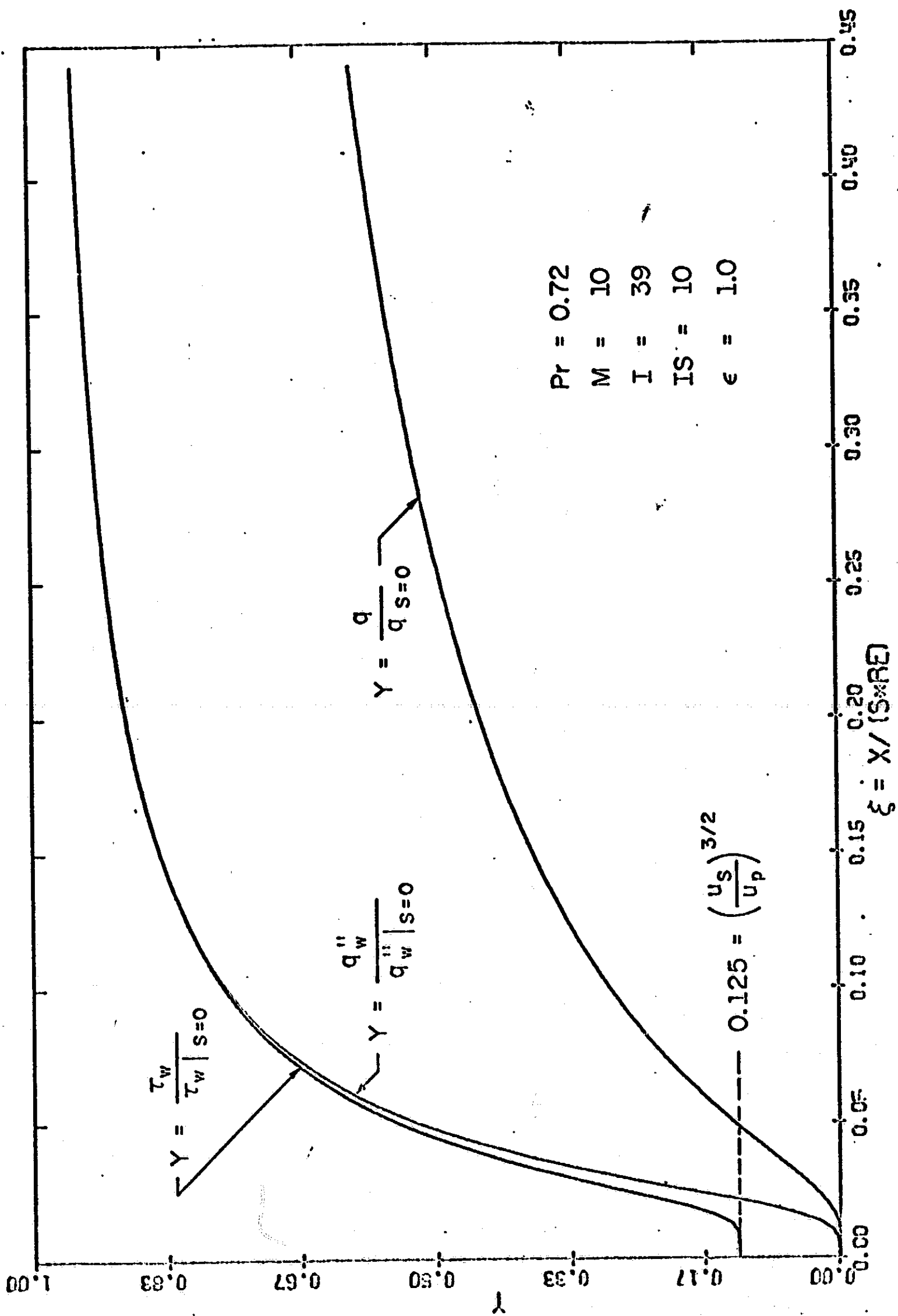


Figure 4.11 The Influence of ξ on the Shear Stress and Heating Rates; $u_s/u_p = 0.25$

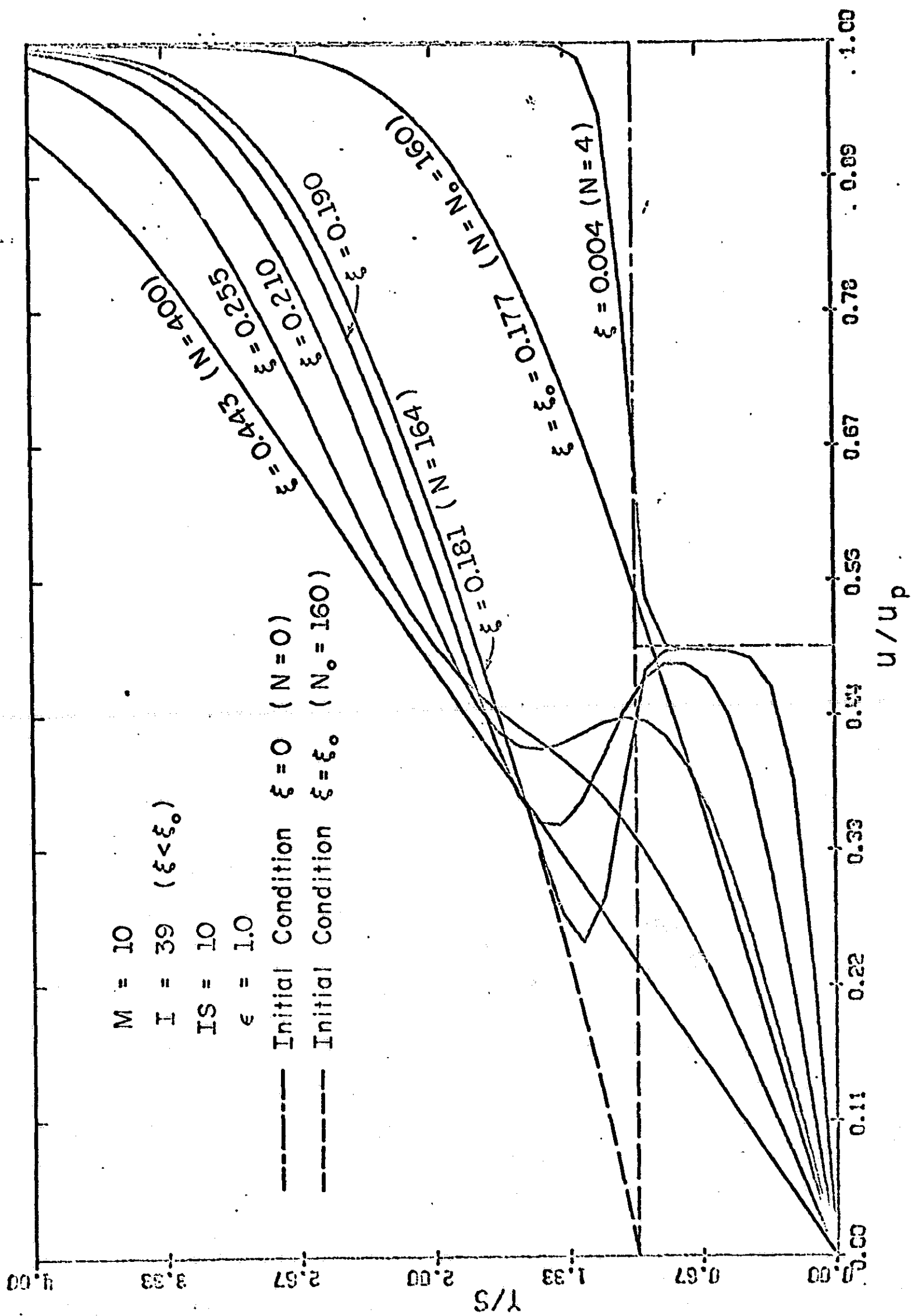


Figure 4.12 Velocity Profiles for $u_s/u_p = 0.5$; $\xi_0 = 0.177$ ($N_0 = 160$)

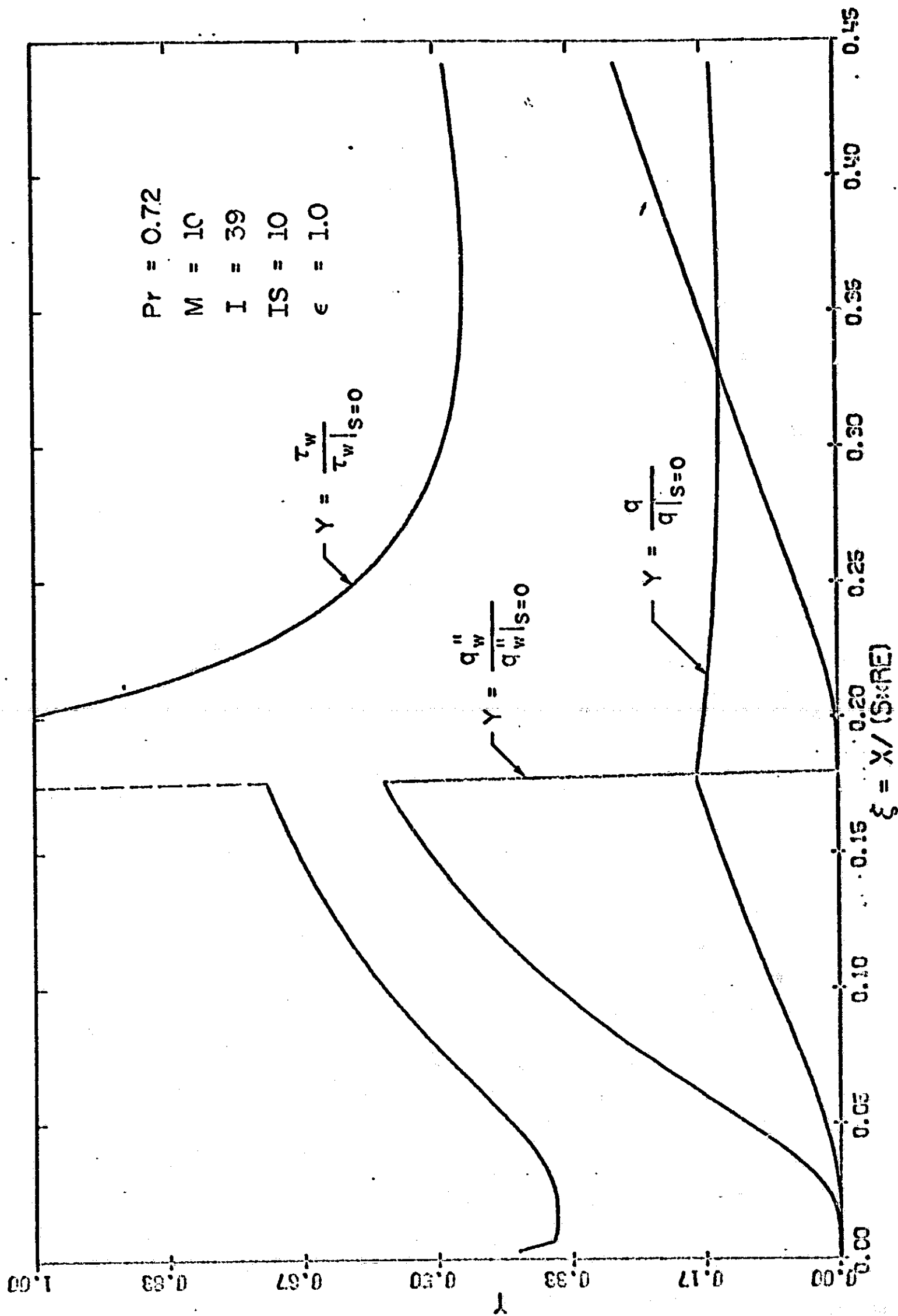


Figure 4.13 The Influence of ξ on the Shear Stress and Heating Rates; $u_s/u_p = 0.5$, $\xi_0 = 0.177$

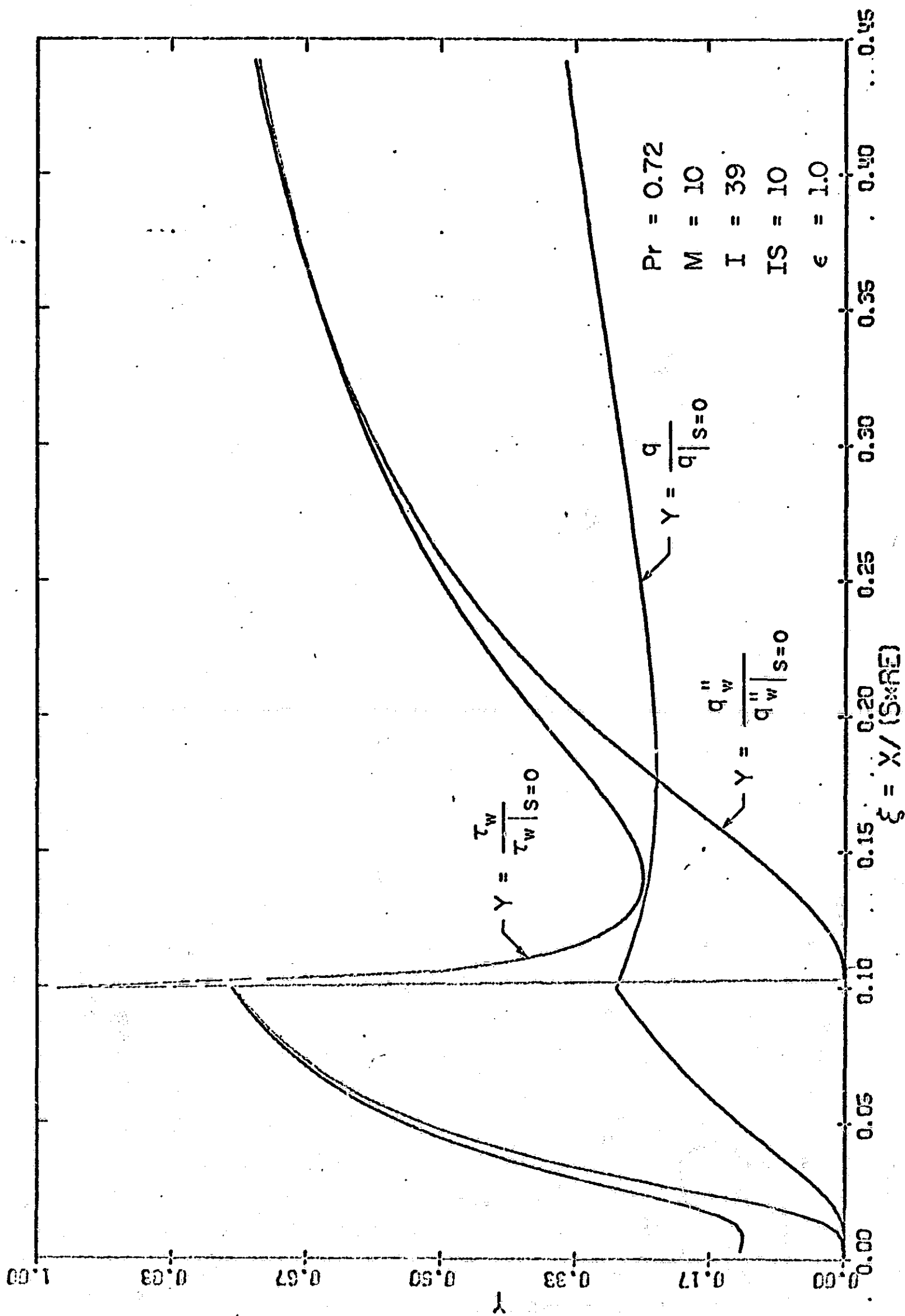


Figure 4.14 The Influence of ξ on the Shear Stress and Heating Rates; $u_s/u_p = 0.25$, $\xi_0 = 0.100$

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