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NASA CR-1550

NASA CR-15

OPTIMAL CONTROL PROBLEMS AS EQUIVALENT LAGRANGE PROBLEMS

by Stephen K. Park

Prepared by
NORTH CAROLINA STATE UNIVERSITY
Raleigh, N. C.

for Langley Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1970

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Prepared under Contract No. NGL-34-002-032 by

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SUMMARY

Herein a method is described by which an optimal control problem with a compact, connected control region may be transformed into what is called a ψ-equivalent Lagrange problem in the calculus of variations. The notion of ψ-equivalence is explained and a condition sufficient to guarantee the ψ-equivalence of two problems is stated. In particular, it is shown that an optimal control problem with a closed parallepiped control region can, after an appropriate transformation, always be treated as a Lagrange problem.

The following discussion assumes the knowledge of some basic topics in the theory of functions of several variables - for example, portions of the material contained in reference 1, chapters I through IV, and reference 2, chapters I and II.

For any positive integer r let E^r denote the euclidean r-space and let n and m be fixed positive integers. Suppose also that fixed points $x^0 = (x_1^0, \ldots, x_n^0)$ and $x^1 = (x_1^1, \ldots, x_n^1)$ are given in E^n along with a compact (i.e., closed and bounded) and connected region U in E^m. We will first state a rather general optimal control problem and then discuss a method which enables us to formulate this problem as an equivalent classical Lagrange problem. This method provides a means of applying the standard tools of the calculus of variations to obtain solutions of the optimal control problem.

Suppose the real valued functions $g_0, g_1, \ldots, g_n$ are C^1 (reference 1, p. 41) on $E^n \times V$ where V is some open subset of E^m containing U. This is sufficient to guarantee that corresponding to each piecewise continuous control function $u = (u_1, \ldots, u_m)$ with $u(t) \in U$ there exists (locally) a unique piecewise smooth trajectory $x = (x_1, \ldots, x_n)$ (see, for example, reference 3) satisfying

$$\dot{x}(t) = g(x(t),u(t)) \tag{1}$$

where $g = (g_1, \ldots, g_n)$, $\dot{x} = \frac{dx}{dt}$, and $x(t_0) = x^0$. If in addition there exists a $t_1 > t_0$ (i.e. terminal time) such that the trajectory x satisfies (1) on $[t_0, t_1]$ and $x(t_1) = x^1$ then the control function u is said to be admissable. (The definition of an admissable control is not standardized - see reference 4, p. 10, reference 5, p. 278, reference 6, p. 225 - the difference being whether or not the associated trajectory is required to arrive at the terminal point x^1 for some terminal time t_1.) Let Ω_I denote the set of admissable control functions and with the function g_0 define a functional

$$J[u] = \int_{t_0}^{t_1} g_0(x(s), u(s))ds \tag{2}$$

on the set Ω_I. Notice that without loss of generality we may consider t_0 to be fixed (reference 4, p. 16) but that t_1, in general, depends on u. The optimal control problem considered, henceforth refered to as Problem I, is as follows: To find an (optimal) $u \in \Omega_I$ such that $J[u] \leq J[\overline{u}]$ for all $\overline{u} \in \Omega_I$. In all that follows, to avoid so-called vacuous arguments, assume that Problem I has at least one admissable control, i.e., that the set Ω_I is not empty.

To formulate Problem I as an equivalent Lagrange problem, let us suppose there exists a positive integer p and a C^1 function $\psi = (\psi_1, \ldots, \psi_m)$ which maps E^p onto U i.e., $\psi: E^p \to U$ with $\psi(E^p) = U$. The question of the existence of such a function will be considered later. In terms of ψ define new functions $f_0, f_1, \ldots, f_n$ by

$$f_i(x, z) = g_i(x, \psi(z)) \qquad i = 0, 1, 2, \ldots, n \tag{3}$$

where $x \in E^n$, $z \in E^p$. The functions f_0 and $f = (f_1, \ldots, f_n)$ then are C^1 on $E^n \times E^p$. Define Ω_{II} as the set of all piecewise continuous functions $z = (z_1, \ldots, z_p)$ (i.e. $z(t) \in E^p$) such that there exists a terminal time $t_1 > t_0$ and an associated (unique) trajectory $x^* = (x_1^*, \ldots, x_n^*)$ satisfying $x^*(t_0) = x^0$, $x^*(t_1) = x^1$ and

$$\dot{x}^*(t) = f\,(x^*(t), z(t)) \tag{4}$$

on $[t_0, t_1]$. This defines the functional

$$I[z] = \int_{t_0}^{t_1} f_0(x^*(s),\ z(s))ds \tag{5}$$

on Ω_{II} and what will be refered to as Problem II may now be stated as follows: To find a $z \varepsilon \Omega_{II}$ such that $I[z] \leq I[\bar{z}]$ for all $\bar{z} \varepsilon \Omega_{II}$. It should be mentioned that the definition of Ω_I (or Ω_{II}) contains the implicit assumption that t_1 is the first time ($> t_0$) for which $x(t_1) = x^1$ (or $x^*(t_1) = x^1$).

Now Problem II may be viewed as an optimal control problem with the (open) control region E^p and the set of admissable control functions Ω_{II}. However, with essentially no more than a change of notation, it may also be viewed as a classical Lagrange problem in the calculus of variations. Hereafter the latter viewpoint will be adopted. Before analyzing Problem II as a Lagrange problem, let us first define the previously mentioned concept of equivalence, then explain the significance of this concept, and finally state and prove a theorem which gives a condition that is sufficient to guarantee this equivalence.

Definition: Problem II will be said to be ψ-equivalent to Problem I if

(a) there exists a solution to Problem I, say $u \varepsilon \Omega_I$ with terminal time t_I, if and only if there exists a solution to Problem II, say $z \varepsilon \Omega_{II}$ with terminal time t_{II},

(b) in addition $t_I = t_{II}$ and,

(c) $\psi(z(t)) = u(t)$ for all $t \varepsilon [t_0, t_1]$ where $t_1 = t_I = t_{II}$.

From a practical standpoint the importance of this definition lies in the fact that when Problem II is ψ-equivalent to Problem I the following is true. From condition (a) if Problem I has a solution then by necessity there exists a solution to Problem II, and hence this solution (to Problem II) must satisfy all the necessary conditions of the calculus of variations and be related to the solution of Problem I by condition (c). Furthermore, if we can construct for Problem II a function $z \varepsilon \Omega_{II}$ which satisfies an appropriate sufficiency condition of the calculus of variations then we can conclude that the function u given by $u(t) = \psi(z(t))$ is an optimal control for Problem I. In other words, rather than directly attempting to solve Problem I, we may instead construct an appropriate ψ function and hence an equivalent Problem II. If Problem I has a solution we are assured Problem II does also and by solving Problem II we have at the same time (to within a transformation) solved Problem I. Loosely speaking, this procedure amounts to a "change of variables" in the control space and is analogous to the familiar change of variables technique in integration theory.

In order to determine a condition sufficient to guarantee that Problem II is ψ-equivalent to Problem I, let us recall the definitions of Ω_I and Ω_{II}. In particular note that corresponding to each admissable control function $u \varepsilon \Omega_I$ there exists the following:

(I-1) a terminal time $t_I > t_0$,

(I-2) a unique trajectory x (equation (1)),

(I-3) a real number $J[u]$ (equation (2)).

Similarly, to each $z \varepsilon \Omega_{II}$ there corresponds:

(II-1) a terminal time $t_{II} > t$,

(II-2) a unique trajectory x^* (equation (4)),

(II-3) a real number $I[z]$ (equation (5)).

The following lemma justifies the previous observation concerning a "change of variables" in the control space.

Lemma 1: If $u \varepsilon \Omega_I$, $z \varepsilon \Omega_{II}$ and $\psi(z(t)) = u(t)$ on $[t_0, t_{min}]$ where $t_{min} = \text{minimum } \{t_I, t_{II}\}$ then

(1) $t_{min} = t_I = t_{II}$

(2) $x(t) = x^*(t)$ on $[t_0, t_{min}]$

(3) $I[z] = J[u]$.

Proof: On $[t_0, t_{min}]$ $\dot{x}^*(t) = f(x^*(t), z(t))$ and $\dot{x}(t) = g(x(t), u(t))$. Therefore since $x^*(t_0) = x^0 = x(t_0)$ and since $f(x^*(t), z(t)) = g(x^*(t), u(t))$ it follows from the uniqueness of the trajectories that $x^*(t) = x(t)$ on $[t_0, t_{min}]$. In particular $x^*(t_{min}) = x(t_{min})$ hence $t_I < t_{II}$ implies $x^*(t_I) = x^1$ while $t_{II} < t_I$ implies $x(t_{II}) = x^1$. In either case this contradicts the assumption that t_I and t_{II} are the first terminal times for x and x^* respectively. Thus $t_I = t_{II} = t_{min}$. Finally

$$J[u] = \int_{t_0}^{t_I} g_0(x(s), u(s))\, ds$$

$$= \int_{t_0}^{t_I} f_0(x(s), z(s))\, ds$$

therefore since $t_I = t_{II}$ and $x(t) = x^*(t)$ we have

$$I[z] = \int_{t_0}^{t_{II}} f_0(x^*(s), z(s))\, ds$$

$$= J[u]$$

and this establishes the lemma.

If the functions u and z satisfy the hypothesis of lemma 1, it is valid to speak of a single common terminal time t_1, trajectory x, and functional value (i.e. $I[z] = J[u]$) associated with u and z. Furthermore, the notation $\psi(z(t)) = u(t)$ for $t \varepsilon [t_0, t_1]$ is unnecessary and we will simply write $\psi \circ z = u$ (reference 2, p. 9). Finally, it should be noted in passing that from lemma 1, conditions (a) and (c) of the definition imply (b).

Denote by $\psi[\Omega_{II}]$ the set of functions u such that $u = \psi \circ z$ for some $z \in \Omega_{II}$ i.e. if $u \in \psi[\Omega_{II}]$ then there exists a $z \in \Omega_{II}$ with terminal time t_{II} such that $u(t) = \psi(z(t))$ on $[t_0, t_{II}]$. In terms of $\psi[\Omega_{II}]$ one has the following theorem.

<u>Theorem</u> 1: If $\psi[\Omega_{II}] = \Omega_I$ then Problem II is ψ-equivalent to Problem I.

<u>Proof</u>: Let $u \in \Omega_I$ be a solution to Problem I. By hypothesis there exists a $z \in \Omega_{II}$ such that $\psi \circ z = u$. Consider any $z^* \in \Omega_{II}$ and there exists a $u^* \in \Omega_I$ with $\psi \circ z^* = u^*$. Since u is a solution $J[u] \leq J[u^*]$ and thus from lemma 1 $I[z] \leq I[z^*]$ i.e., z is a solution to Problem II.

Similarly if $z \in \Omega_{II}$ is a solution to Problem II then there exists a $u \in \Omega_I$ such that $\psi \circ z = u$ and as before $J[u] \leq J[u^*]$ for any $u^* \in \Omega_I$. From the remarks following lemma 1 condition (b) of the definition is satisfied in each case and this establishes the theorem.

Consider $z \in \Omega_{II}$ with terminal time t_1 and trajectory x. Since ψ is continuous the function $u = \psi \circ z$ is piecewise continuous and $u(t) \in U$. Furthermore since $x(t_0) = x^0$, $x(t_1) = x^1$ and $\dot{x}(t) = f(x(t), z(t)) = g(x(t), u(t))$ we have $u = \psi \circ z \in \Omega_I$. That is, independent of whether or not Problem II is ψ-equivalent to Problem I, the following is true:

<u>Lemma</u> 2: $\psi[\Omega_{II}] \subset \Omega_I$.

Therefore to show that Problem II is ψ-equivalent to Problem I it is only necessary to show that $\psi[\Omega_{II}] \supset \Omega_I$ i.e. that if $u \in \Omega_I$ there exists a $z \in \Omega_{II}$ such that $\psi \circ z = u$. In general the topological question of the existence of z is a very difficult one and depends strongly upon the nature of ψ and U. However for the particular case where it is possible to choose $p = m$ (i.e. to find a C^1 function ψ mapping E^m onto $U \subset E^m$) the existence of z is rather easily established. This case is treated in the following lemma.

Lemma 3: If there exists a compact connected set $A \subset E^m$ such that the restriction of ψ to A, denoted $\psi|A$, satisfies

(1) $\psi|A$ is one-to-one

(2) $\psi|A$ maps A onto U

then $\psi[\Omega_{II}] \supset \Omega_I$.

Proof: Purely for notational purposes let $\phi = \psi|A$. Since ϕ is continuous and one-to-one there exists a continuous inverse $\phi^{-1}: U \to A$ (reference 2, p. 10). Consider a function $u \in \Omega_I$ with trajectory x and terminal time t_1 and define a function $z = \phi^{-1} \circ u$. Now z is piecewise continuous with $z(t) \in A$ hence $\psi \circ z = \phi \circ z = \phi \circ \phi^{-1} \circ u = u$. Since $x(t_0) = x^0$, $x(t_1) = x^1$ and $\dot{x}(t) = g(x(t), u(t)) = f(x(t), z(t))$ it follows that $z \in \Omega_{II}$ and hence the lemma has been proved.

Rather than continuing to pursue the problem in full generality let us consider the (mathematically) simpler case of Problem I with a control region U consisting of all vectors $u = (u_1, \ldots, u_m)$ such that $a_i \leq u_i \leq b_i$ where a_i, b_i $(b_i > a_i)$ are arbitrary real numbers for $i = 1, 2, \ldots, m$. The parallelepiped U corresponds physically to a control system in which the m controllers are free to move independently (of each other) within a range determined by the upper and lower bounds b_i, a_i - a situation which is of extreme importance in applications. By virtue of lemma 3 the problem of constructing the ψ-function (when U is a parallelepiped) becomes almost trivial. Temporarily, for purposes of discussion, suppose U is the one-dimensional interval $[a, b]$ then A must also be an interval, say $[c, d]$. On $[c, d]$ ψ must be one-to-one, onto, and C^1 and hence it must be strictly monotone. The derivative of ψ must vanish at c and d and $\psi(c) = a$, $\psi(d) = b$ (or $\psi(c) = b$, $\psi(d) = a$). An obvious choice of a function which has exactly these properties is $\psi(x) = \frac{b-a}{2} \sin x + \frac{b+a}{2}$ with $A = [-\frac{\pi}{2}, \frac{\pi}{2}]$. We can immediately generalize this idea to the previous case where

$$U = \{(u_1, \ldots, u_m): \quad a_i \leq u_i \leq b_i \quad i = 1, 2, \ldots, m\} \tag{6}$$

and consider $\psi = (\psi_1, \ldots, \psi_m)$ defined by

$$\psi_i(z) = \frac{b_i - a_i}{2} \sin z_i + \frac{b_i + a_i}{2} \qquad i = 1, 2, \ldots, m \tag{7}$$

where $z \in E^m$. In addition define $A \subset E^m$ as

$$A = \{(z_1, \ldots, z_m): \; -\frac{\pi}{2} \leq z_i \leq \frac{\pi}{2} \quad i = 1, 2, \ldots, m\}$$

Clearly ψ and A satisfy the hypothesis of lemma 3. Summarizing the results of theorem 1 and lemmas 2 and 3 we have the following.

Theorem 2: If Problem I has the parallelepiped control region U given by (6) then Problem II as defined by (3) and (5) is ψ-equivalent to Problem I where ψ is given by (7).

In order to illustrate the concepts discussed to this point - particularly theorem 2 - let us consider a standard example of a control problem. In this example we will construct a ψ-equivalent problem (Problem II) and later, after discussing, in general, problem II as a Lagrange problem, apply the results of this discussion to the example.

Example: The linear time optimal problem (see reference 4, chapter III). Consider problem I with x^0 and x^1 fixed and

$$g_0(x,u) = 1$$

$$g_i(x,u) = \sum_{j=1}^{n} \alpha_{ij} x_j + \sum_{j=1}^{m} \beta_{ij} u_j \qquad i = 1, 2, \ldots, n$$

and for simplicity take $a_i = -1$, $b_i = 1$ (i.e. $-1 \leq u_i \leq 1$) for $i = 1, 2, \ldots, m$. Thus $\psi = (\psi_1, \ldots, \psi_m)$ becomes $\psi_i(z) = \sin z_i$ for $i = 1, 2, \ldots, m$ and the ψ-equivalent problem II becomes: To find a piecewise continuous function $z = (z_1, \ldots, z_m)$ and a corresponding piecewise smooth function $x = (x_1, \ldots, x_n)$ which together satisfy

$$\dot{x}_i = \sum_{j=1}^{n} \alpha_{ij} x_j + \sum_{j=1}^{m} \beta_{ij} \sin z_j \qquad i = 1, 2, \ldots, n$$

the end conditions

$$x(t_0) = x^0, \; x(t_1) = x^1$$

and minimize the integral

$$\int_{t_0}^{t_1} dt = t_1 - t_0$$

for some $t_1 > t_0$. We will return to this example shortly.

Following the original formulation of Problem II (see equations (3) and (5)) it was stated that Problem II is, with a change of notation, a Lagrange problem. In particular it is the following Lagrange Problem: To find a piecewise smooth vector function $y = (y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+p})$ which satisfies the constraint equations

$$\dot{y}_i - f_i\,(y_1, \ldots, y_n, \dot{y}_{n+1}, \ldots, \dot{y}_{n+p}) = 0 \qquad i = 1, 2, \ldots, n$$

the boundary conditions

$$\begin{array}{lll} y_i(t_0) = x_i^0 & y_i(t_1) = x_i^1 & i = 1, 2, \ldots, n \\ y_{n+j}(t_0) = 0 & y_{n+j}(t_1) - \text{free} & j = 1, 2, \ldots, p \end{array}$$

and minimizes the integral

$$\int_{t_0}^{t_1} f_0(y_1, \ldots, y_n, \dot{y}_{n+1}, \ldots, \dot{y}_{n+p})\, dt$$

for some $t_1 > t_0$. The change of notation involved is nothing more than $x_i = y_i$ for $i = 1, \ldots, n$ and $z_j = \dot{y}_{n+j}$ for $j = 1, 2, \ldots, p$. Whether one chooses to consider Problem II in terms of the (x, z) formulation or the y formulation is purely a matter of taste. In either case by applying the results of, for example, references 5, 6, and 7 the following fundamental set of necessary conditions (in (x,z) notation) may be obtained.

Theorem 3: If $z \varepsilon \Omega_{II}$ is a solution to Problem II with corresponding trajectory x and terminal time t_1 it is necessary that there exist a sectionally smooth vector function $\lambda = (\lambda_1, \ldots, \lambda_n)$ and a scalar $\lambda_0 \leq 0$ such that for each $t \varepsilon [t_0, t_1]$

(1) $(\lambda_0, \lambda(t)) \neq (0,0)$

(2) $\dot{x}_i(t) = \frac{\partial H}{\partial \lambda_i} (x(t), z(t), \lambda(t)) \qquad i = 1, 2, \ldots, n$

$\dot{\lambda}_i(t) = -\frac{\partial H}{\partial x_i} (x(t), z(t), \lambda(t)) \qquad i = 1, 2, \ldots, n$

(3) $H(x(t), \xi, \lambda(t)) \leq H(x(t), z(t), \lambda(t))$ for all $\xi \varepsilon E^p$

(4) $H(x(t), z(t), \lambda(t)) = 0$

(5) $\frac{\partial H}{\partial z_i} (x(t), z(t), \lambda(t)) = 0 \qquad i = 1, 2, \ldots, p$

where $H(x, z, \lambda) = \sum_{i=0}^{n} \lambda_i f_i (x, z)$

Theorem 3 is an immediate consequence of the multiplier rule, the corner conditions, the transversality conditions, and the Weierstrass condition. The first four conditions are the usual maximum principle and condition (5) are the Mayer equations pertaining to $(\dot{y}_{n+1}, \ldots, \dot{y}_{n+p})$ and the corresponding transversality conditions. The advantage of condition (5) is that the determination of $z(t)$ (and hence $u(t)$) has been reduced to the problem of algebraically solving the p equations $\frac{\partial H}{\partial z_i} = 0$. This is illustrated nicely by considering the previous example for which (recall m = p)

$$H(x,z,\lambda) = \lambda_0 + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_j \lambda_i + \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} \lambda_i \sin z_j$$

$$\frac{\partial H}{\partial z_j}(x,z,\lambda) = \sum_{i=1}^{n} \beta_{ij} \lambda_i \cos z_j \qquad j = 1, 2, \ldots, m$$

and thus for all t

$$\left[\sum_{i=1}^{n} \beta_{ij} \lambda_i (t) \right] \cos z_j (t) = 0 \qquad j = 1, 2, \ldots, m$$

The fact that $\sum_{i=1}^{n} \beta_{ij} \lambda_i(t)$ cannot be zero at more than a finite number of t's in $[t_0, t_1]$ follows (if the so-called general position condition holds (reference 4, p. 116)) from the analyticity of λ (see reference 4, p. 118). Hence, $\cos z_j (t) = 0$ $(j = 1, 2, \ldots, m)$ except perhaps for a finite number of t's and thus $z_j(t) = \pm \frac{\pi}{2}$ i.e. $u_j(t) = \psi(z_j(t)) = \pm 1$ which is the familiar bang-bang principle.

There has been a vast amount of research in the calculus of variations and there is an obvious need to determine what portions of this research has application in the theory of optimal control, (reference 6 contains much of the work done along this line). It is generally recognized that optimal control problems (of the type considered herein) with open control regions can be viewed as Lagrange problems (see reference 4, Chap. V). However, if one is faced with a closed control region, as is almost always the case, it is not at all obvious what, if any, results from the calculus of variation may be directly applied to the problem. The significance of the ideas discussed in this paper is that, at least when U is compact and connected, a very straight-forward method is available for transforming an optimal control problem into a Lagrange problem that is ψ-equivalent to it. In addition any information obtained about a solution of the Lagrange problem may be immediately related to information about an optimal control for the control problem.

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NASA-Langley, 1970 —— 19 CR-1550