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I. Introduction

A few years ago, Synge (1966) reviewed the concept of the so-called "escape cone" of light emitted in the vicinity of gravitationally-intense objects (i.e. where $r/r_{Schwarzschild} \gtrsim 1$). In a region outside but close to the surface of such objects, not all of the light emitted escapes to a distant observer; rather, light only escapes if it is emitted within a cone which has the property that it monotonically shrinks in angle as its vertex approaches the surface of the object. Zel'dovich and Novikov (1965), and others have also discussed this interesting and unusual effect. These investigations have all dealt with the emission of light from a region exterior to a spherically symmetric mass distribution.

Because of the possible importance of general relativistic effects on the characteristics of stellar objects, we wish to discuss here the emission and escape of light from <u>inside</u> spherically symmetric mass distributions. In particular, we will explicitly analyze the interior Schwarzschild field. This model, although idealized, can be analyzed analytically; it should give useful insight into the relativistic effects and possible severe restrictions that can exist for the motion and escape of light from within a mass distribution.

We will show that there also exists a "cone effect" for light escaping from within the object. However, the "interior escape cone" does not monotonically shrink with decreasing radius of emission, as does the "exterior escape cone" discussed by Synge.

II. The Interior Field

For constant density, $\rho(r \leq r_b) = \rho_o$, $\rho(r > r_b) = 0$, we have the Schwarzschild interior field (Schwarzschild 1916), given by

$$ds^2 = [3/2\sqrt{(1 - \frac{r_b^2}{R^2})} - 1/2\,\sqrt{(1 - \frac{r^2}{R^2})}]^2\, c^2\, dt^2 \qquad (1)$$

$$- \frac{dr^2}{(1 - r^2/R^2)} - r^2(d\phi^2 + \sin^2\phi\, d\theta^2)$$

$0 \leq r \leq r_b$,

where

$$R^2 \equiv \frac{3c^2}{8\pi G\rho_o} \qquad \rho_o = \frac{3M}{4\pi r_b^3}$$

The field equations lead to an immediate restriction (Schwarzschild 1916): in order that the pressure never becomes infinite anywhere, r_b must be greater than $(9/4)m$. (G and c have been set equal to unity; the Schwarzschild radius of the sphere in these units is $r_s = 2m$.)

Solving the geodesic equations of motion for light ($ds^2 = 0$), we find $\dot{\phi} = 0$, or motion is restricted to a plane, chosen as $\phi = \pi/2$, and

$$r^2\dot{\theta} = \alpha = \text{constant} \qquad (2)$$

$$[3/2\sqrt{(1 - \frac{r_b^2}{R^2})} - 1/2\sqrt{(1 - \frac{r^2}{R^2})}]^2 = \dot{t} = \beta = \text{constant}\,, \qquad (3)$$

where a dot indicates differentiation with respect to an arbitrary parameter. Since the interior and exterior solutions match at r_b, we can show (Appendix A) that

$$\alpha/\beta = \ell \quad , \tag{4}$$

the classical impact parameter as measured at infinity.

Combining equations (2), (3), and (4) with $ds^2 = 0$, we find

$$\frac{dr}{d\theta} = r\sqrt{(1 - r^2/R^2)}\left\{ \frac{4r^2}{\ell^2\,[B - \sqrt{(1 - r^2/R^2)}]^2} - 1\right\}^{1/2} \quad , \tag{5}$$

where

$$B \equiv 3\,(1 - r_b{}^2/R^2).$$

Thus the motion of light is characterized by a single parameter, ℓ.

The quantity inside the braces in equation (5) must always be non-negative; therefore, at each $r \leq r_b$, the allowable range of ℓ is

$$0 \leq \ell \leq \ell_t = \frac{2r}{B - \sqrt{(1 - r^2/R^2)}} \quad . \tag{6}$$

ℓ_t $\left(\left.\frac{dr}{d\phi}\right|_{\ell = \ell_t} = 0\right)$ is always greater than r (classically, $\ell_t = r$). This is an illustration of the curvature of space-time, for ℓ cannot be given a real interpretation near the source, but only at

$r = \infty$. We must therefore be content to consider ℓ only as a mathematical constant of the motion.

Kuchowicz (1965) has very neatly solved equation (5) and found

$$r = R\,\frac{B\sqrt{[B^2 - 1 + (A+1-B^2)\sin^2\theta]} + \sqrt{[A+1-B^2]}\sin\theta}{B^2 + (A+1-B^2)\sin^2\theta}$$

where

$$A \equiv 4R^2/\ell^2.$$

Examining this equation shows that, for a sphere in the range

$$9/4\ m < r_b < 3m$$

(recall that r_b is proportional to ρ_o), light emitted at a radius r, with

$$\ell \geq \ell_c = r_b/\sqrt{(1-2m/r_b)}$$

does not escape the sphere (i.e., there exists a turning point $\leq r_b$). ℓ_c is less than or equal to ℓ_t in the region

$$r \geq r_b\left[\frac{r_b - 9/4\ m}{r_b - 3/2\ m}\right] \equiv r_c$$

In this region, some of the light emitted in the range $\ell = 0$ to ℓ_t never escapes past r_b. All of the light emitted from $r \leq r_c$ escapes past the boundary of the sphere. For $r_b > 3m$, all of the light emitted in the allowable range of ℓ at <u>any</u> r escapes.

These results are illustrated in Figure (1). In Figure (1a), for $r_b = 200m$, ℓ_t versus r is practically a straight line of slope one, for g_{00} and $g_{11} \to 1$ as ρ decreases. For $r_b \geq 3m$ (Figures [1a] and [1b]), ℓ_t is monatomically increasing with r. For $r_b < 3m$ (Figures [1c] - [1e]), ℓ_t reaches a maximum at

$$r_{max} = r_b/3 \ [(\frac{r_b}{m}) \ \frac{4r_b - 9m}{r_b - 2m}]^{1/2}$$

and then decreases to $\ell_t = \ell_c$ at $r = r_b$.

III. Cone of Escape

We consider the cone of emission of light escaping from the sphere at any $r \leq r_b$. From the metric, equation (1), the r and θ spatial displacements are

$$dl_\theta = r\, d\theta$$

$$dl_r = \sqrt{g_{11}}\, dr \quad .$$

The half-angle of the cone of emission as measured from the radial direction is then

$$\tan\theta = \frac{dl_\theta}{dl_r} = \frac{r}{\sqrt{g_{11}}} \left(\frac{d\theta}{dr}\right)_{\ell_c}$$

$$= \left\{\frac{4r^2}{\ell_c^2\,[B - \sqrt{(1-r^2/R^2)}]^2} - 1\right\}^{-1/2} \quad .$$

Figure (2) illustrates θ for various values of r_b. For $r_b \geq 3m$, $\theta = \pi/2$ for all r. (It should be clear that, while the spread from 0 to ℓ_t may be decreasing with decreasing r, if all the light emitted at r, in the allowable range 0 to ℓ_t, escapes the object, the escape cone is still $\pi/2$.) For $r_b < 3m$, we have the interesting result that, with decreasing r, the full cone angle of the envelope of escaping light initially decreases from π, reaches a minimum, and then increases to π at r_c. This is in contrast to the result for the exterior Schwarzschild field, where for $r < 3m$ (with an emitter <u>exterior</u> to the mass distribution), the corresponding cone envelope is monotonically decreasing.

The red shift of the light emitted of course increases as the point of emission moves closer to the center. For a photon emitted radially ($\ell = 0$) from a point at rest in the object, the red shift relative to an observer at infinity is simply

$$z = 1/\sqrt{g_{00}} - 1$$
$$= [3/2 \sqrt{(1 - \frac{r_b^2}{R^2})} - 1/2 \sqrt{(1 - \frac{r_b^2}{R^2})}]^{-1} - 1 \quad ,$$

which increases with decreasing r.

IV. Transit Time of Light

It is interesting too to examine the transit time of light travelling within the sphere. From the geodesic equations, we find

$$\frac{dr}{dt} = \frac{\ell\, g_{00}}{r\sqrt{-g_{11}}} \sqrt{\left(\frac{r^2}{g_{00}\,\ell^2} - 1\right)} .$$

This equation can be integrated analytically, yielding

$$t - t_o = \frac{R}{\sqrt{(B^2-1)}} \sin^{-1} \frac{2[A(A-\sigma)(\sigma+1-B^2)(B^2-1)]^{1/2}}{\sigma(A+1-B^2)}$$

where

$$A \equiv 4R^2/\ell^2 \quad \text{and} \quad \sigma \equiv R^2 \left[\frac{B - \sqrt{(1-r^2/R^2)}}{r}\right]^2 .$$

In the limit of vanishing ρ, we have

$$t - t_o \approx \sqrt{(r^2-\ell^2)} + 4/3 \frac{(r^2-\ell^2)^{3/2}}{R^2} + 0\left(\frac{1}{R^{2n}}\right) ,$$

the classical result, plus terms of order $\frac{1}{R^{2n}}$. (See Figure [3].)

V. Conclusion

Gravity places unusual restrictions on the emission and escape of light from within a spherically symmetric mass distribution. There exist regions where the escape cone is less than π. While the interesting range of r_b here (or equivalently, ρ_0) is not terribly great, recent work by Kuchowicz (1968) indicates that other, perhaps more realistic configurations have a much greater latitude of dimensions within which these capture effects may exist.

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APPENDIX A

The exterior Schwarzschild solution is given as

$$ds^2 = (1-2m/r)\ dt^2 - \frac{dr^2}{(1-2m/r)} - r^2(d\phi^2 + r^2\sin^2\phi\ d\theta^2)$$

The relevant geodesic equations of motion for light ($ds^2 = 0$) are readily found to be

$$r^2\dot{\theta} = p = \text{constant} \tag{A1}$$

$$(1-2m/r)\dot{t} = c = \text{constant}\ . \tag{A2}$$

If we consider a test particle at infinity, of mass m and with velocity v as measured at infinity, equations (A1) and (A2) can be written as

$$r^2\dot{\theta} = p = \ell v/(1-v^2) \tag{A3}$$

$$(1-2m/r)\dot{t} + \dot{t} = c = 1/(1-v^2) \qquad , \tag{A4}$$

where ℓ is the classical impact parameter of the particle as measured at infinity. Therefore, for light, where $v = 1$, we have

$$p/c = \ell \,. \tag{A5}$$

Since the interior and exterior solutions match at r_b, we have from equations (2), (3), (A3), and (A4),

$$\alpha = p$$

$$\beta = c \,.$$

Therefore,

$$\alpha/\beta = p/c = \ell .$$

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FIGURE CAPTIONS

Figure 1. Range of light emitted ($\ell = 0$ to ℓ_t) as a function of r/r_b for various values of r_b. For $r_b \leq 3m$, not all the light emitted in the region $r_c \leq r < r_b$ escapes past r_b (shaded sections).

Figure 2. The half-angle of the cone of escaping light as a function of r/r_b, for various values of r_b. Curve (a), $\theta = \pi/2$, is valid for $r_b \geq 3m$. Curves (b), (c), and (d) correspond respectively to $r_b = 2.86$ m, 2.35 m, and 2.27 m.

Figure 3. The transit time, in units of m, from $r = 0$ to $r = r_b$ for a radial photon ($\ell = 0$), as a function of r_b. The transit time is longer than the classical case, especially in the limit as r_b approaches its minimum allowable value, (9/4) m.

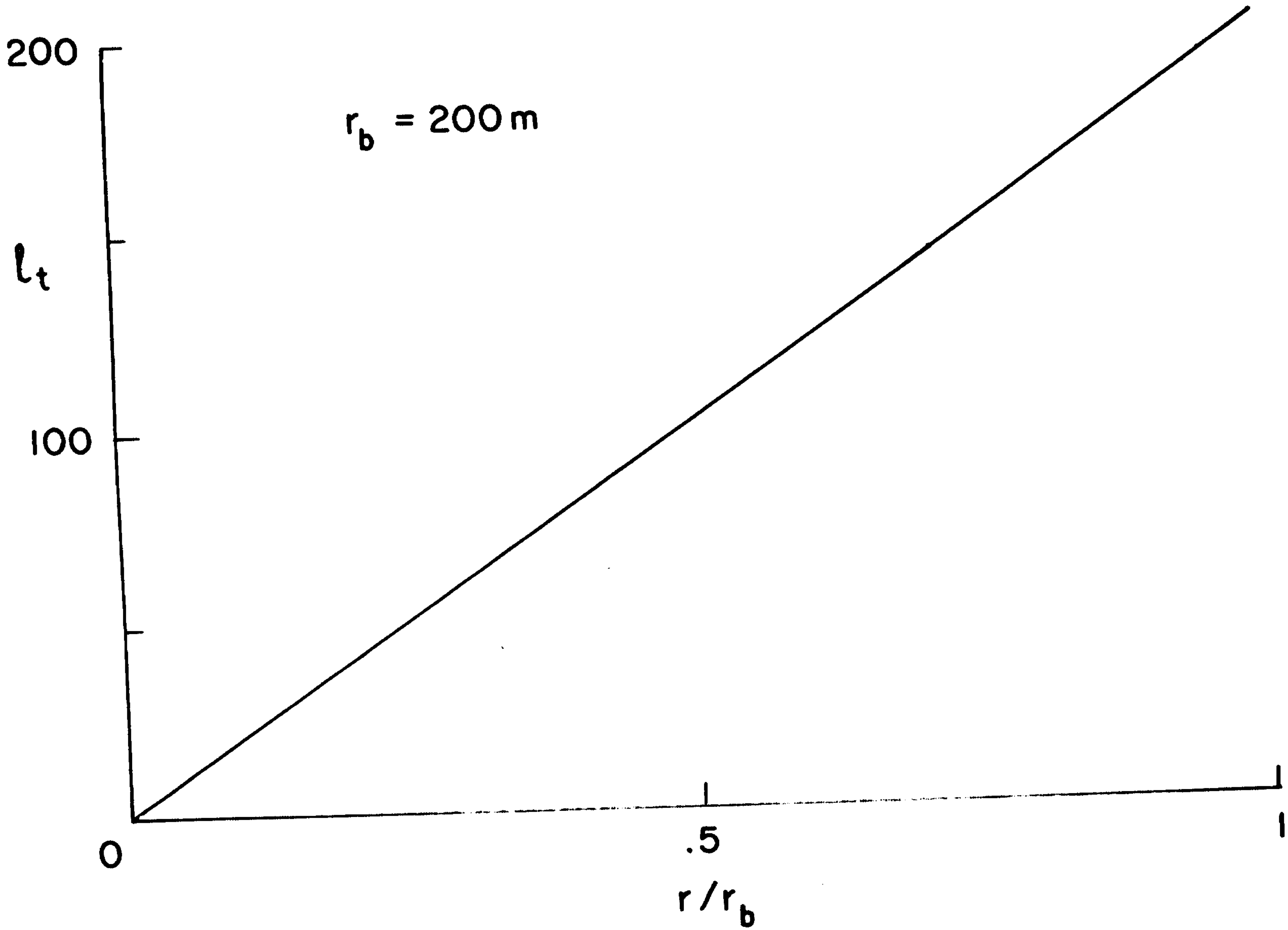

FIG. 1a

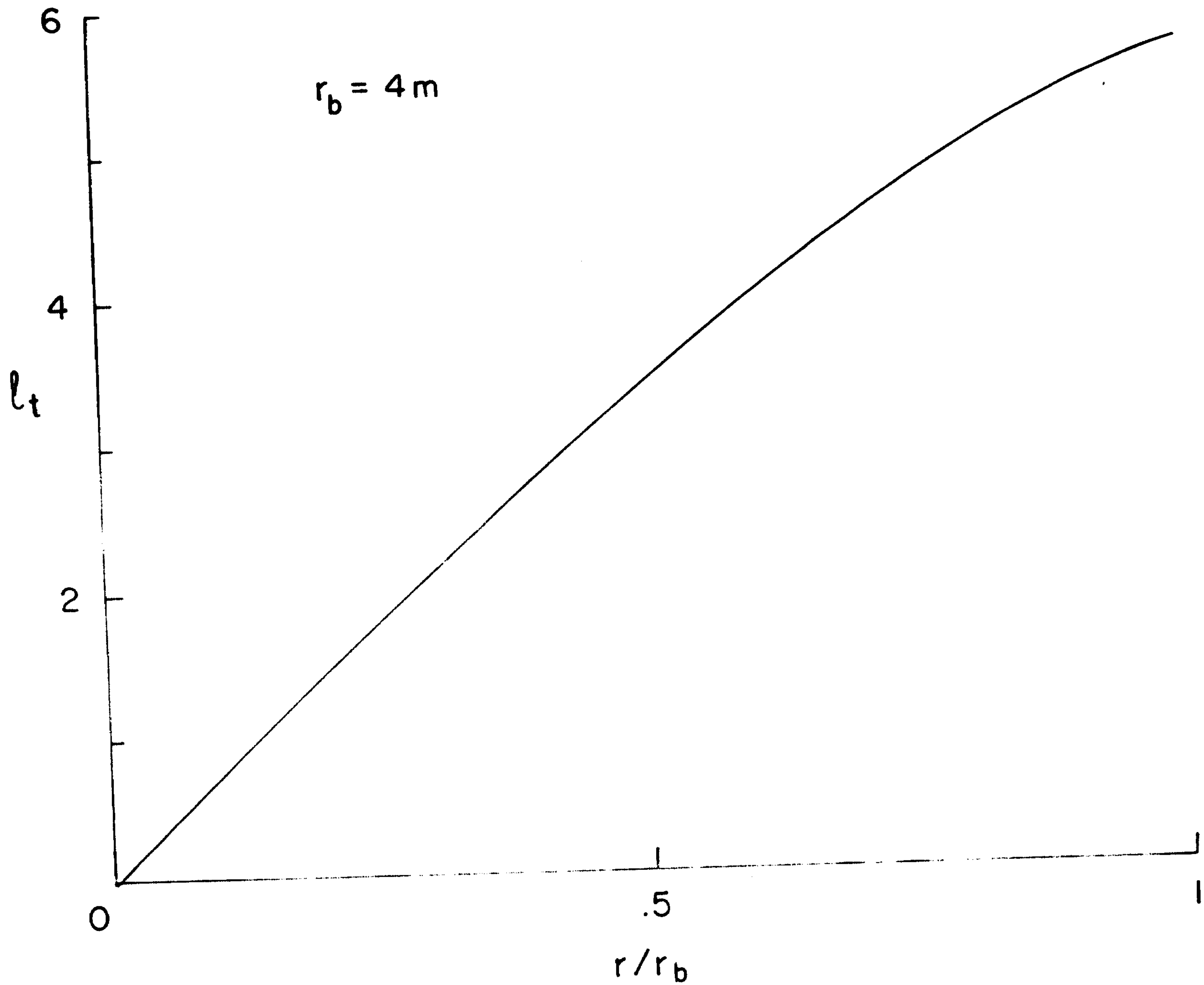

FIG. 15

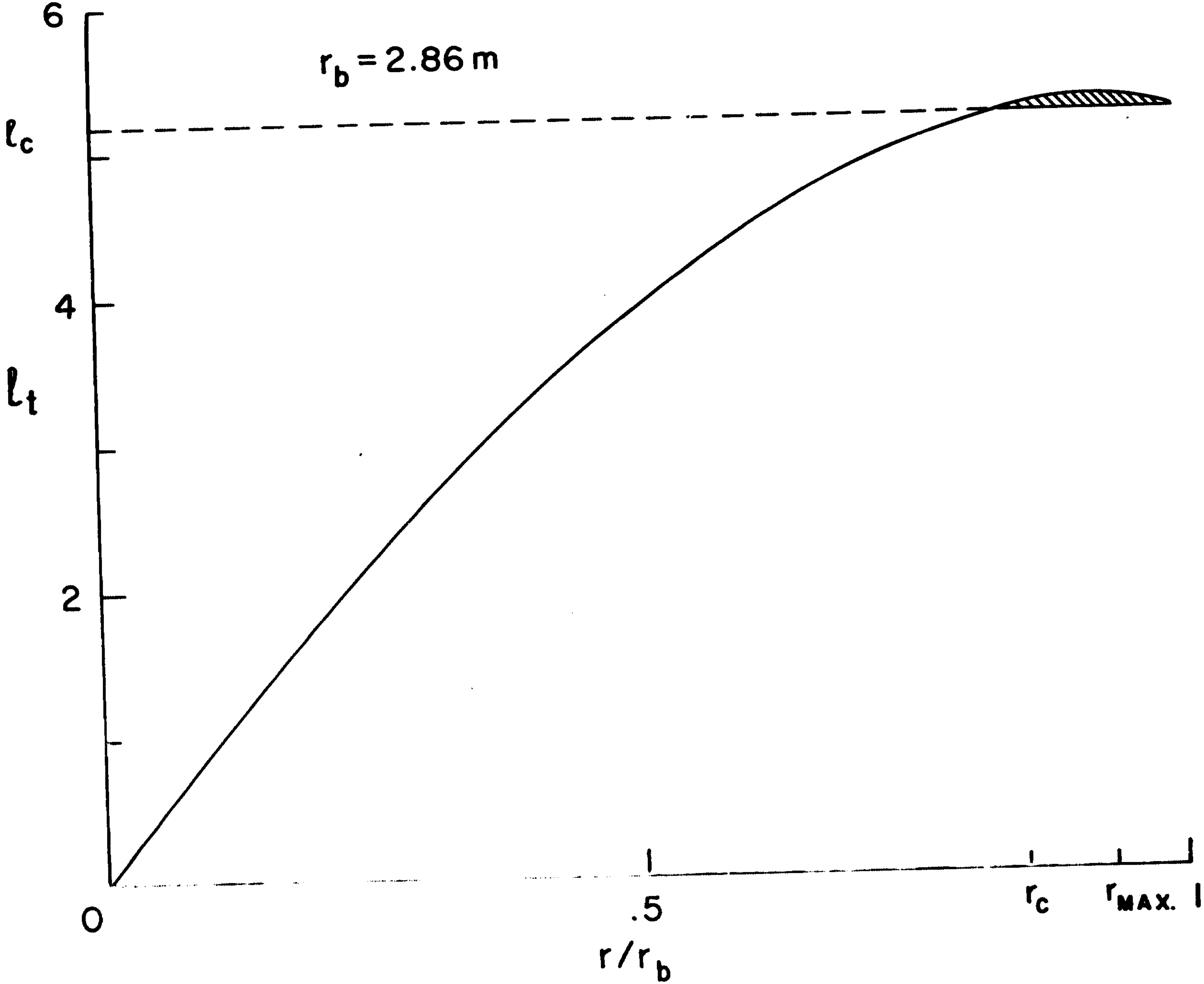

FIG. 1c

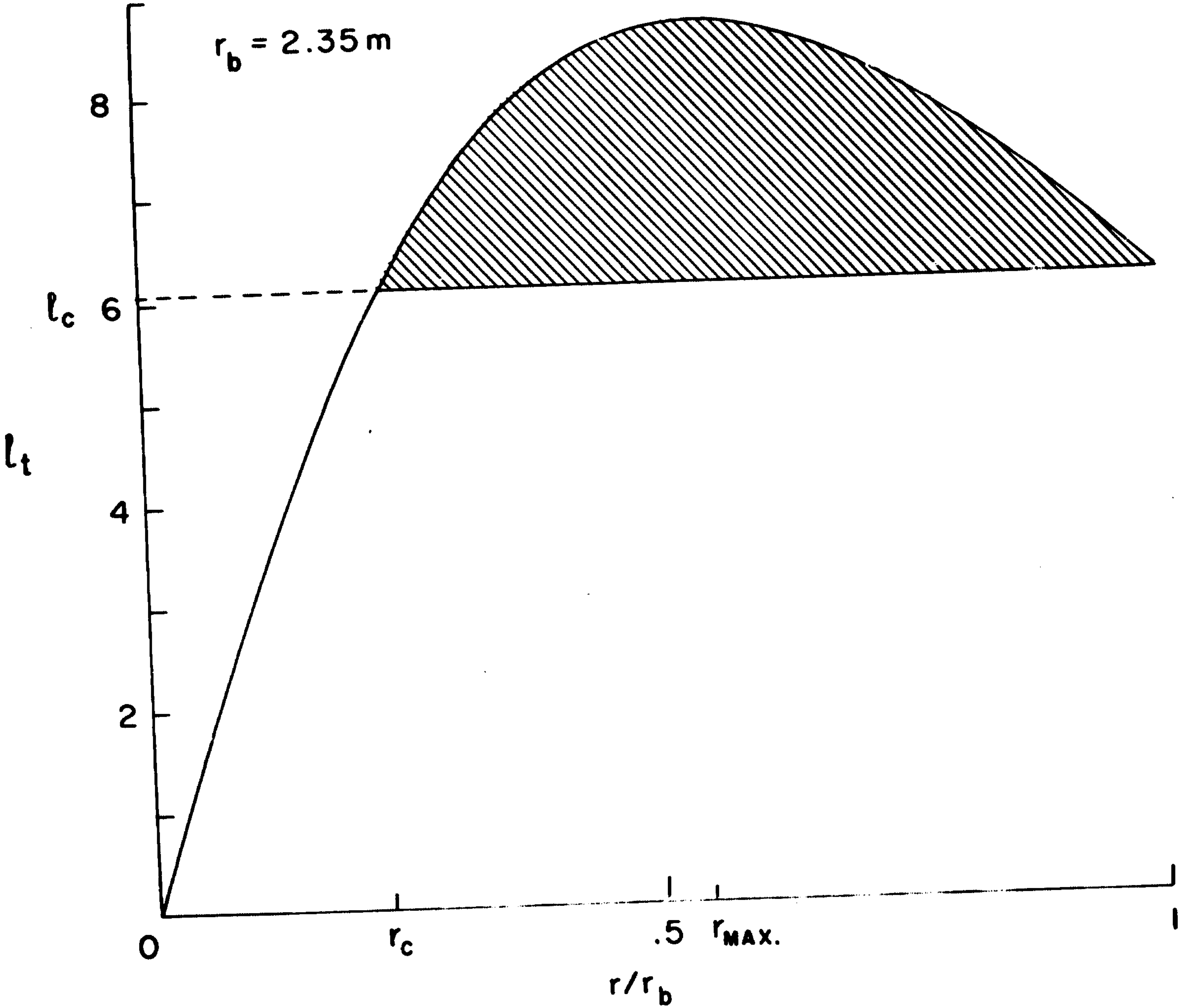

FIG. 1d

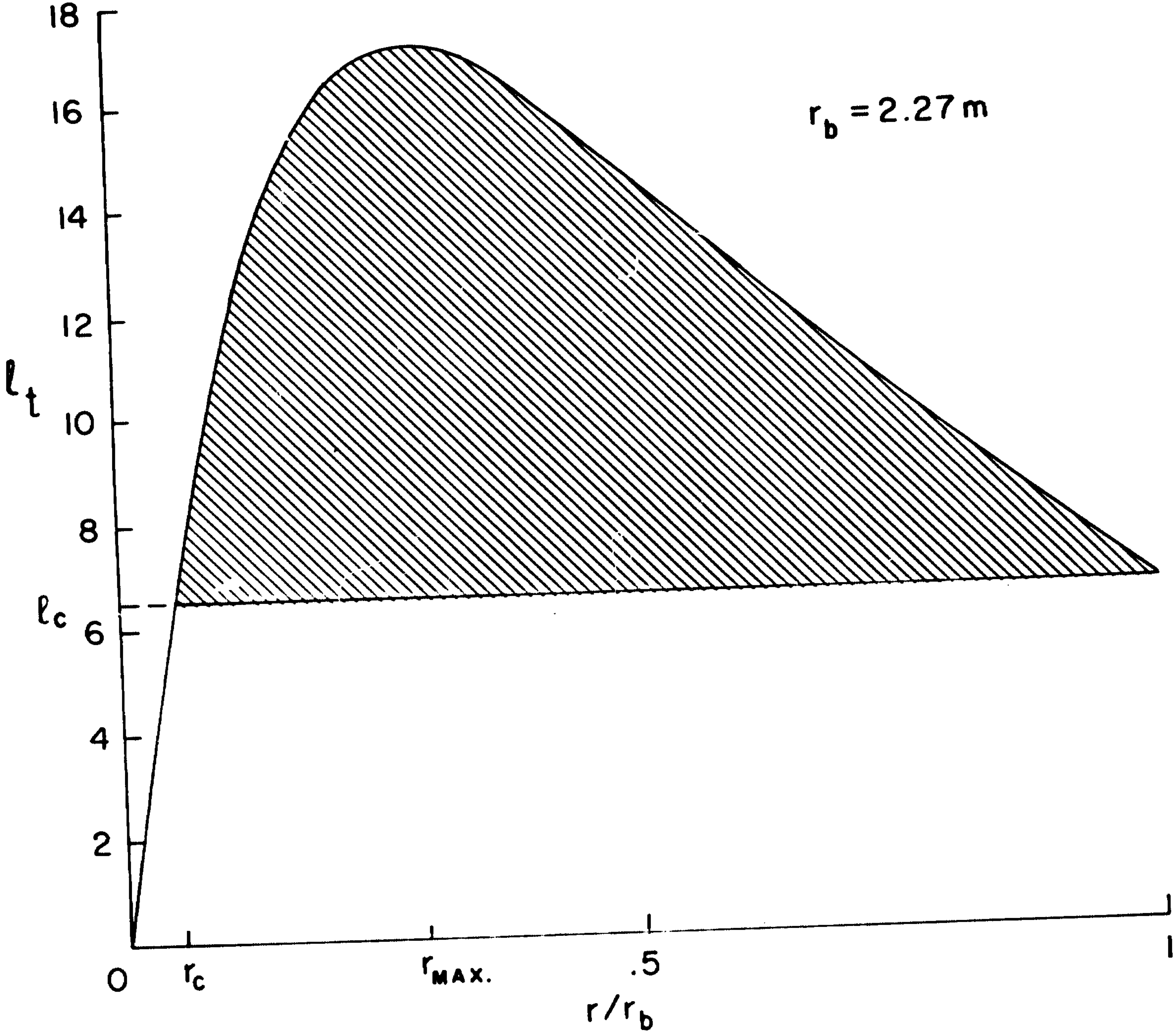

FIG. 1e