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DETERMINATION OF MEAN ORBITAL ELEMENTS FROM INITIAL CONDITIONS FOR A VINTI BALLISTIC TRAJECTORY

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ABSTRACT

A method previously proposed for determining mean orbital elements for Vinti's spheroidal theory of drag-free satellite motion directly from either initial conditions or Keplerian osculating elements is shown feasible for application to ballistic trajectories. The method, originally intended for use with multi-revolution satellite orbits, is an iterative procedure involving a first-order Taylor's series expansion of position and velocity components at epoch time. The determination of mean orbital elements by this iterative method is shown to be a valid alternative to the factorization of two quartic polynomials arising in the inversion of the integrals of motion and solved by successive approximations carried through second order in the earth's oblateness parameter. Numerical results for a ballistic trajectory are presented, demonstrating that convergence of the iterative fitting to initial conditions is rapid and exact.

DETERMINATION OF MEAN ORBITAL ELEMENTS FROM INITIAL CONDITIONS FOR A VINTI BALLISTIC TRAJECTORY

INTRODUCTION

The determination of a set of constants of motion for an orbital satellite theory is a commonly encountered problem inasmuch as mathematical theories of satellite motion are generally given as functions of mean orbital elements rather than osculating elements. In practice, it is the set of initial conditions of position and velocity components at a given epoch time which is readily available, such as from nominal conditions for an orbit insertion maneuver or as output of a stepwise numerical integration technique for trajectory prediction. Such initial conditions are readily converted, by means of the Keplerian two-body transformations, into osculating orbital elements, but the problem of producing mean elements for use as constants of motion in an analytic development remains.

A method has been proposed (Reference 1) for determining mean orbital elements directly from initial conditions or Keplerian osculating elements for the spheroidal theory for satellite orbits developed by Vinti. The spheroidal theory (References 2 and 3) provides an algorithm for calculating an accurate reference orbit of any drag-free satellite moving in the gravitational field of an axially symmetrical oblate planet. As applied to the actual gravitational potential of the earth, the reference orbit accounts exactly for the effects of all zonal harmonic terms in the series expansion of the geopotential through the third, and it accounts for the major portion of the fourth zonal harmonic as well.

The spheroidal theory is applicable to all bounded orbits of arbitrary inclination and eccentricity. The method for determining mean orbital elements for Vinti's satellite theory was applied (Reference 1) to an actual trajectory corresponding to a near-earth satellite orbit of medium inclination and moderately high eccentricity which remained above the portion of the atmosphere inducing appreciable drag effects.

A recent paper (Reference 4) describes a method of utilizing differential coefficients to fit orbital parameters to assumed initial position and velocity vectors representing a ballistic trajectory. This latter method adopts an earlier spheroidal satellite theory by Vinti (Reference 5) which does not have the advantage of incorporating the effects of the third zonal harmonic term into the spheroidal potential. Also, the inversion of the integrals of motion in the form suggested by Izsak (Reference 6) for the solution of Vinti's dynamical problem is utilized for the co-ordinates, while the equations for the velocities are those given by Borchers (Reference 7). However, the method of determination of the Izsak-Borchers orbital elements from initial conditions is nearly identical to the method proposed earlier (Reference 1) in that both methods are iterative procedures involving a first-order Taylor's series expansion at epoch time dependent upon partial derivatives assuming the form of differential coefficients. The minor differences between the two iterative procedures shall be discussed further below, but it is to be emphasized that the methods are substantially the same.

The purpose of the present paper, appearing as a sequel to the earlier results (Reference 1), is to demonstrate the feasibility of applying the method as originally presented to ballistic trajectories. Such trajectories are elliptic (or circular) segments of orbital arcs which eventually intersect the earth's surface, i.e., their "perigee" heights are less than unity when measured in units of the earth's radius. These trajectories may correspond to the free-flight portion of an ascent to satellite orbit or to a spacecraft re-entry into the atmosphere.

DETERMINATION OF MEAN ORBITAL ELEMENTS

The constants of the motion q_i $(i = 1,2,\ldots, 6)$, which are the mean orbital elements for Vinti's spheroidal theory of satellite motion, include the following: the semi-major axis a, the eccentricity e, a parameter S corresponding to $\sin^2 i$ (where i is the inclination of the orbital plane to the Equator) in Keplerian two-body motion, and three parameters β_1, β_2, and β_3, which correspond to the negative of the time of passage through perigee τ, to the argument of perigee ω, and to the right ascension of the ascending node Ω, respectively, in the reduction to Keplerian motion. The given initial conditions for a satellite orbit are generally provided in the form of rectangular inertial position components x_0, y_0, z_0 and velocity components $\dot{x}_0, \dot{y}_0, \dot{z}_0$ specified for a particular epoch time t_0. For the case in which the given initial conditions are provided in the form of Keplerian osculating orbital elements, namely, a, e, i, τ, ω, and Ω, these may readily be transformed to inertial Cartesian position and velocity vector components by the usual two-body transformation equations.

The problem of determining the proper set of mean orbital elements from the given initial conditions has been approached by Vinti (Reference 8) through a method of factorization of two quartic polynomials arising in the inversion of the integrals of motion. This factorization is carried out iteratively, beginning with a zeroth-order solution corresponding to Keplerian two-body motion. A set of four non-linear equations that results is solved by a method of successive approximations carried through second order in the oblateness parameter J_2. The second-order transformation equations for the orbital elements used in the equations of the final solution for the satellite co-ordinates and velocities (References 9 and 10) are provided explicitly (Reference 8). To obtain the solution to the non-linear system to arbitrarily high order, a method applying the Newton-Raphson iteration scheme has been proposed by Borchers (Reference 7).

An alternative method for the determination of mean orbital elements for Vinti's spheroidal satellite theory, based upon differential corrections applied to position and velocity residuals at epoch time, has already been specified (Reference 1). A summary of this method appears in order here. This method is capable of determining mean elements directly from initial conditions, eliminating the need for numerical factorization through successive approximation, and has no connection to the traditional differential correction of satellite orbits utilizing observational data.

If the Keplerian osculating orbital elements, a, e, i, τ, ω, and Ω, are adopted as the constants of the motion in Vinti's spheroidal satellite theory, then the rectangular inertial position and velocity vectors predicted analytically by the theory for the epoch time t_0 may be denoted x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$. These components will differ from the initial conditions because the spheroidal theory incorporates an earth model considerably more sophisticated than the Keplerian elliptic model. Assuming that the required corrections to the orbital elements are sufficiently small so that their squares and higher powers may be neglected (as is traditionally the case with linear approximations), the residual differences in co-ordinates and velocities may be expressed by a truncated Taylor's series expansion as

$$x_0 - x = \sum_{i=1}^{6} \frac{\partial x}{\partial q_i} \Delta q_i ,$$

$$y_0 - y = \sum_{i=1}^{6} \frac{\partial y}{\partial q_i} \Delta q_i ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\dot{z}_0 - \dot{z} = \sum_{i=1}^{6} \frac{\partial \dot{z}}{\partial q_i} \Delta q_i .$$

Here the predicted values x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$ are considered functions of the six independent variables q_i (i = 1, 2, . . ., 6) which are to be improved by the respective additive increments Δq_i (i = 1, 2, . . ., 6). Note that the time t_0 has been omitted as an independent variable inasmuch as it remains a constant value throughout. The thirty-six differential coefficients appearing in the Taylor's expansions have been evaluated explicitly (References 1 and 11) and retain the accuracies of the original Vinti orbital theory. The above simultaneous system of six linear algebraic equations admits a solution for the six unknowns Δq_i, which are used to correct the orbital elements, as follows:

$$q_i' = q_i + \Delta q_i \quad , \ i = 1, 2, \ldots , 6 .$$

The corrected (primed) orbital elements are then used to predict analytically through the spheroidal theory a new set of position and velocity vectors at epoch time and to re-evaluate the differential coefficients. The process of improving the elements is then continued iteratively until the absolute values of the position and velocity residuals at epoch time, $|x_0 - x|$, $|y_0 - y|$, . . ., $|\dot{z}_0 - \dot{z}|$, reach some sufficiently small predetermined values.

Comparison of the method outlined above with that presented by Allen (Reference 4) shows striking similarities. A truncated Taylor's series expansion is given by Equations 54 and 55 (Reference 4), after elimination of the apparent typographical errors, as follows:

$$\beta_j = \beta_j^0 + \sum_{i=1}^{6} \frac{\partial \beta_j}{\partial \gamma_i} \Delta \gamma_i ,$$

where

$$\Delta \gamma_i = \gamma_i \, (\beta_j) - \gamma_i \, (\beta_j^0).$$

In this Taylor's expansion, $\beta_j = \beta_j \; (\gamma_i)$ is a component of a matrix consisting of the six Izsak - Borchers orbital elements. These are analogous to the mean orbital elements used in Vinti's spheroidal theory, without inclusion of the third zonal harmonic term in the reference geopotential (References 5 and 8). The parameters β_j^0 are the Kepler two-body orbital elements obtained by setting the principal oblateness parameter J_2 equal to zero in the Izsak - Borchers equations of motion and fitting to the initial vectors. Also, γ_i represents a component of a matrix consisting of the six oblate spheroidal co-ordinates and their time derivatives used in the solution of the Vinti dynamical problem. The differential coefficients $\partial\beta_j/\partial\gamma_i$ appearing in this form of the Taylor's series are in some sense "inverses" of the previous differential coefficients, inasmuch as they consist of partial derivatives of the orbital elements with respect to the co-ordinates and velocities rather than the reverse. However, the differential coefficients are evaluated by Allen for the simplified model of a spherical earth only and do not include any oblateness effects. In Allen's version of iterative improvement, the mean orbital elements are improved indirectly through the residuals $\Delta\gamma_i$ in the oblate spheroidal co-ordinates and velocities between the "exact" values $\gamma_i(\beta_j)$ based upon the latest corrected elements and the approximate values $\gamma_i(\beta_j^0)$ based upon the elements of the previous iteration. Note also that the six Taylor's series expansions may be solved successively in turn as independent equations rather than as a simultaneous system of six linear algebraic equations to be solved by Gaussian elimination.

NUMERICAL APPLICATIONS TO A BALLISTIC TRAJECTORY

The iterative method of determining a set of mean orbital elements from initial conditions for Vinti's spheroidal satellite theory has already been applied (Reference 1) to an actual satellite orbital trajectory corresponding to an inclination of 46 degrees, an eccentricity of 0.24, a period of 195 minutes, and apogee and perigee heights of 4600 and 1300 statute miles, respectively. The same method is now to be applied to a ballistic trajectory represented by initial conditions given in the first column of Table 1 as inertial rectangular co-ordinates and their time derivatives. These data are precisely those specified by Allen (Reference 4) with the units of the velocity components converted from units of earth radii per second, as presumably adopted by Allen, to earth radii per canonical time unit, the latter defined to be 806.823 seconds as indicated in the footnote. The second column of Table 1 provides the osculating Keplerian orbital elements corresponding to the initial conditions. These classical Keplerian elements were obtained through use of the two-body equations of motion, considering only the central term in the geopotential. The final column displays the converged values for the mean set of orbital elements for Vinti's satellite theory obtained after two iterations. A measure of the degree of improvement in the orbital elements is provided by Table 2 which displays the residuals in position and

TABLE 1

Determination of Mean Orbital Elements from Initial Conditions

Initial conditions*	Osculating Keplerian elements*	Mean Vinti elements*
$x = 0.4650$	$a = 0.78809935$	$a = 0.78675654$
$y = 0.7254$	$e = 0.59910027$	$e = 0.60188516$
$z = 0.6010$	$\sin^2 i = 0.37911289$	$S = 0.37863865$
$\dot{x} = 0.7915$	$-\tau = 1.1628914$	$\beta_1 = 1.1638504$
$\dot{y} = 0.03026$	$\omega = -0.66925097$	$\beta_2 = -0.69029205$
$\dot{z} = 0.08673$	$\Omega = 3.0392151$	$\beta_3 = 3.0260546$

*The position components x, y, z are in earth equatorial radii (e.r.), and the velocity components $\dot{x}$, $\dot{y}$, $\dot{z}$ are in e.r./c.u.t., where 1 canonical unit of time (c.u.t.) is equal to 806.823 seconds. The semi-major axis a is in e.r., the time of perigee passage τ and β_1 are in c.u.t., and the argument of perigee ω, the right ascension of the ascending node Ω, β_2, and β_3 are in radians.

velocity components at the epoch time, using the osculating Keplerian elements initially and then the mean Vinti elements obtained upon convergence after two iterations. As an indication of the convergence speed of the iterative method, Table 3 presents the residuals in position and velocity at each iteration, where iteration 0 corresponds to the use of Keplerian elements.

The results of an application to improve the mean orbital elements for Vinti's satellite theory following the use of iterative factorization of the quartics through second order is summarized in Table 4. The initial conditions for the ballistic trajectory are identical to those included in Table 1. The second column now provides the mean orbital elements for Vinti's satellite theory determined by

TABLE 2

Magnitude of Residuals in Position and Velocity Components

Residual*	Initially (Keplerian elements)	Upon convergence (mean Vinti elements)
$\lvert \Delta x \rvert$	25032.81	0.33
$\lvert \Delta y \rvert$	9598.04	0.14
$\lvert \Delta z \rvert$	27478.12	0.24
$\lvert \Delta \dot{x} \rvert$	3282.7721	0.0236
$\lvert \Delta \dot{y} \rvert$	5.2298	0.0328
$\lvert \Delta \dot{z} \rvert$	6006.7360	0.0044

* The position residuals are in meters, and the velocity residuals in centimeters per second.

TABLE 3

Convergence of the Computed Position and Velocity

to the Initial Conditions

Iteration	Position residual*	Velocity residual*
0	0.006 019 025	0.008 659 108
1	0.000 085 298	0.000 064 523
2	0.000 000 068	0.000 000 051

* The position residual is defined by $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ and is given in e.r., and the velocity residual is defined by $\sqrt{(\Delta \dot{x})^2 + (\Delta \dot{y})^2 + (\Delta \dot{z})^2}$ and is given in e.r./c.u.t.

TABLE 4

Improvement of Mean Orbital Elements After Use of

Second-Order Factorization

Initial conditions*	Factored Vinti elements*	Mean Vinti elements*
x = 0.4650	a = 0.78675646	a = 0.78675653
y = 0.7254	e = 0.60188502	e = 0.60188516
z = 0.6010	S = 0.37864354	S = 0.37863868
$\dot{x}$ = 0.7915	β_1 = 1.1638503	β_1 = 1.1638504
$\dot{y}$ = 0.03026	β_2 = -0.69029421	β_2 = -0.69029207
$\dot{z}$ = 0.08673	β_3 = 3.0260707	β_3 = 3.0260546

* The units for all variables are as indicated in the footnote to Table 1.

iterative factorization of the quartic polynomials in the integrals of motion carried through second order in the earth's oblateness parameter. The final column gives the converged values for the improved mean set of orbital elements, again obtained after two iterations. Table 5 shows the residuals in position and velocity components with use of the factored elements initially and then the mean elements obtained upon convergence after two iterations. The convergence of the residuals in position and velocity is indicated in Table 6, where iteration 0 now corresponds to the use of the factored Vinti orbital elements. Note that near convergence is attained after only a single iteration.

The conclusions reached in the earlier study (Reference 1) are valid as well in the numerical application of the iterative method to a ballistic trajectory. The converged values of mean orbital elements for Vinti's spheroidal satellite theory

TABLE 5

Magnitude of Residuals in Position and Velocity Components

Residual*	Initially (factored elements)	Upon convergence (mean Vinti elements)
$\lvert \Delta x \rvert$	91.43	0.21
$\lvert \Delta y \rvert$	34.02	0.05
$\lvert \Delta z \rvert$	27.70	0.10
$\lvert \Delta \dot{x} \rvert$	0.4830	0.0059
$\lvert \Delta \dot{y} \rvert$	10.8230	0.0121
$\lvert \Delta \dot{z} \rvert$	1.3488	0.0103

* Units are as indicated in the footnote to Table 2.

TABLE 6

Convergence of the Computed Position and Velocity

to the Initial Conditions

Iteration	Position residual*	Velocity residual*
0	0.000 015 900	0.000 013 810
1	0.000 000 107	0.000 000 083
2	0.000 000 037	0.000 000 021

* The remarks in the footnote to Table 3 apply here as well.

as determined from initial conditions is virtually independent of whether the process of second-order factorization of the quartics is utilized prior to the iterative Taylor's expansion. This is seen from the nearly identical values for the mean Vinti elements presented in Tables 1 and 4. Hence the iterative method of determining mean orbital elements for a Vinti ballistic trajectory may be used as a valid alternative to the factorization procedure. However, if second-order factorization is applied, then the mean orbital elements are corrected, through subsequent application of the iterative improvement method, only by increments of the third order in the oblateness parameter. Convergence of the iterative fitting to initial conditions in position and velocity is extremely rapid, and decreases in the residual components to very low tolerances are achieved, both with and without use of factorization to determine initial elements.

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APPENDIX

In the interest of thoroughness, efforts were undertaken to duplicate the numerical calculations of Allen (Reference 4). The following sequence of tabular displays shows that this effort was, at best, only partially successful. In Table 7 appear the adopted ballistic initial conditions and, in the second column, the Keplerian orbital elements associated by Allen with the initial conditions. The latter are seen to differ substantially from the Keplerian orbital elements given in Table 1. The final column displays the converged values for the mean Vinti elements, obtained after three iterations. Table 8 shows the residuals in position and velocity components with use of Allen's given osculating Keplerian elements initially and then the mean Vinti elements obtained upon convergence after three iterations. The convergence of the residuals in position and velocity is indicated in Table 9, where iteration 0 now corresponds to the use of Allen's given Keplerian elements. Finally, the converged values for the mean Vinti elements, obtained after three iterations, shown in the right column of Table 7 may be contrasted with Allen's converged mean orbital elements, obtained after four iterations. After the required manipulations described in the footnote to Table 7 are performed, the latter appear as follows: a = 0.78681 e.r., e = 0.56072, S = 0.36196, β_1 = 1.22172 c.u.t., β_2 = -0.71421 rad., and β_3 = 2.94749 rad.

TABLE 7

Determination of Mean Orbital Elements from Initial Conditions Using Given Osculating Keplerian Elements

Initial conditions*	Given osculating Keplerian elements*	Mean Vinti elements*
$x = 0.4650$	$a = 0.78771$	$a = 0.78675653$
$y = 0.7254$	$e = 0.55812$	$e = 0.60188518$
$z = 0.6010$	$\sin^2 i = 0.36201$	$S = 0.37863866$
$\dot{x} = 0.7915$	$-\tau = 1.23247$	$\beta_1 = 1.1638504$
$\dot{y} = 0.03026$	$\omega = -0.73416$	$\beta_2 = -0.69029211$
$\dot{z} = 0.08673$	$\Omega = 2.95757$	$\beta_3 = 3.0260546$

*The units for all variables are as indicated in the footnote to Table 1. The given osculating Keplerian elements were obtained from Allen's data (Reference 4) by adding one anomalistic period to the perigee time and by subtracting 2π and π from the argument of perigee and the longitude of node, respectively, at perigee time.

TABLE 8

Magnitude of Residuals in Position and Velocity Components

Residual*	Initially (given Keplerian elements)	Upon convergence (mean Vinti elements)
$\lvert \Delta x \rvert$	20660.69	0.17
$\lvert \Delta y \rvert$	14224.53	0.14
$\lvert \Delta z \rvert$	20418.46	0.10
$\lvert \Delta \dot{x} \rvert$	2994.5567	0.0059
$\lvert \Delta \dot{y} \rvert$	48612.7070	0.0180
$\lvert \Delta \dot{z} \rvert$	5396.8830	0.0015

*Units are as indicated in the footnote to Table 2.

TABLE 9

Convergence of the Computed Position and Velocity

to the Initial Conditions

Iteration	Position residual*	Velocity residual*
0	0.005 071 016	0.061 987 744
1	0.003 066 079	0.000 820 141
2	0.000 010 031	0.000 005 074
3	0.000 000 037	0.000 000 024

* The remarks in the footnote to Table 3 apply here as well.