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STABILITY OF TANGENTIAL DISCONTINUITIES

T.G. NORTHROP
T.J. BIRMINGHAM

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Laboratory for Space Physics

ABSTRACT

The MHD stability of tangential discontinuities is first considered. We treat these discontinuities as structured forms rather than as sharp breaks in the magnetic field. An unfamiliar form of the MHD energy principle is applied, and stability is proved provided that there is no fluid flow tangent to the "discontinuity" plane. Perturbations which simply transform the system from one equilibrium to another are neutrally stable. Using comparison theorems we conclude that the observed stability of tangential forms in the solar wind implies near isotropy of the particle pressure in them.

STABILITY OF TANGENTIAL DISCONTINUITIES

The frequent appearance of tangential discontinuities in the solar wind flow (Burlaga, 1968; Siscoe et al., 1968; Burlaga and Ness, 1969) suggests that they are stable structures. Contrary to the implication of the name "discontinuities," these structures are often thick enough that their internal details can be observed within the time resolution of the measurements (Siscoe et al., 1968; Burlaga and Ness, 1969). They therefore are more appropriately named tangential forms. These forms are stationary in and convect with the solar wind. In the solar wind frame the magnetic field lines are straight and rotate in direction about an axis normal to the "discontinuity" plane. However, as stated above, the "discontinuity" really is thick and should be considered as a series of parallel planes (see Fig. 1). As the field rotates through the form, it may change magnitude also. [The special case where the magnitude is constant is a force-free field (Woltjer, 1958).] The rotation varies from monotonic, to non-uniform, to erratic (Siscoe et al., 1968; Burlaga and Ness, 1969). Because of this great variety of patterns, which have been seen to convect hundreds of earth radii (Fairfield, 1968; Burlaga and Ness, 1969), it seems that there should exist a universal proof of their stability.

There is in fact such a proof. This proof follows from an unfamiliar form of the MHD energy principle. This form was published recently by Grad and Rebhan (1969). They show that the second order variation in total energy (magnetic plus plasma) of an MHD system under a trial displacement $\underset{\sim}{\xi}(\underset{\sim}{r})$ can be written

$$\delta W = \frac{1}{2}\int\left\{\left(\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{\xi} - \underset{\sim}{B}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}\right)^2 + \underset{\sim}{\nabla}\left(P + \frac{B^2}{2}\right)\cdot\left(\underset{\sim}{\xi}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} - \underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{\xi}\right) + \frac{5}{3}P\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}\right)^2\right\} dV \quad (1)$$

where $\underset{\sim}{B}(\underset{\sim}{r})$ is the unperturbed (equilibrium) magnetic field at point $\underset{\sim}{r}$, $P(\underset{\sim}{r})$ is the unperturbed pressure, and the integration extends over the system. The necessary and sufficient condition for stability is that $\delta W \geq 0$ for all $\underset{\sim}{\xi}(\underset{\sim}{r})$ satisfying certain boundary conditions. The $\underset{\sim}{\xi}(\underset{\sim}{r})$ need not be normal modes. The energy variation linear in $\underset{\sim}{\xi}$ vanishes for equilibrium.

For an unbounded spatial system the energy principle exists only for the class of displacements $\underset{\sim}{\xi}$ which vanish as one goes infinitely far from the origin of an arbitrarily located coordinate system. Our stability analysis is therefore restricted to such perturbations. Practically speaking, we are thus limiting ourselves to perturbations of scale length short compared to the lateral (i.e., parallel to the discontinuity plane) extent of the observed tangential forms. Such is not a serious limitation, however, as tangential forms have a lateral extent of tens of R_E (Burlaga and Ness, 1969), and we expect any perturbing effect to be of much smaller dimension. Indeed our unbounded model is not even appropriate for treating perturbations of scale length comparable with the lateral size of the tangential forms.

Returning to Eq. (1), we see that the first and last terms in the integrand are positive definite. The middle term happens to vanish for all tangential forms. Therefore all tangential forms are hydromagnetically stable. The vanishing of the middle term is easily seen from the following relations. At equilibrium

$$\nabla P = \frac{4\pi}{c} \mathbf{j} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla\left(\frac{B^2}{2}\right) + \mathbf{B} \cdot \nabla \mathbf{B} \qquad (2)$$

$$\nabla\left(P + \frac{B^2}{2}\right) = \mathbf{B} \cdot \nabla \mathbf{B} \qquad (3)$$

Thus any situation, such as the tangential forms, where $\mathbf{B}$ does not vary as one moves along a field line, is stable. This stability is not recognizable from the usual (Bernstein et al., 1958) form of δ W.

Syrovatskii (1953) has solved the sharp boundary problem. [The solution is reproduced by Landau and Lifshitz (1960).] In his work, structure is ignored and the tangential discontinuity is represented as the sharp plane interface between two incompressible fluids. The magnetic field is tangent to and changes discontinuously (in magnitude and direction) across the interface. In the absence of tangential fluid flows at the interface, this configuration is hydromagnetically stable. Our result proves stability in the more general case of an arbitrarily structured transition regardless of the compressibility of the fluid.

One can not show from Eq. (1) the neutral stability of our structured system to perturbations which either (a) rotate the direction of $\mathbf{B}$ by an arbitrary angle uniformly in any discontinuity plane; (b) increase the magnitude of $\mathbf{B}$ by an arbitrary amount uniformly in any discontinuity plane; or (c) both rotate the direction of $\mathbf{B}$ and increase its magnitude. Any configuration obtained by processes a-c is another equilibrium, since for each such configuration $\nabla P = \mathbf{j} \times \mathbf{B}$. Further, these equilibria are dense: there is infinitesimally close to any equilibrium another equilibrium obtained by rotating the magnetic field of the initial equilibrium through an infinitesimal angle and/or by increasing its magnitude by an infinitesimal amount. The transition between such infinitesimally adjacent equilibria is continuous in the sense that the macroscopic variables (pressure, magnetic field, and density) change smoothly. One can pass by processes a-c between two finitely separated equilibria by a series of infinitesimal displacements $\underset{\sim}{\xi}$ through intermediate states, all of which are themselves equilibria. For each $\underset{\sim}{\xi}$, one expects δ W = 0.

The trouble is applying Eq. (1) to show that δ W = 0 for such $\underset{\sim}{\xi}$'s is that the $\underset{\sim}{\xi}$'s are not spatially local (in fact, $\underset{\sim}{\xi}$ becomes larger in magnitude as one gets further from the origin of coordinates) and surface terms which have been neglected in deriving Eq. (1) become important.

The neutral stability of such rotational perturbations is, however, evident from a more primitive form (Bernstein et al., 1958) of δ W

$$\delta W = -\frac{1}{2}\int \underset{\sim}{\xi} \cdot \underset{\sim}{F}\{\underset{\sim}{\xi}\}\, dV \tag{4}$$

where

$$\begin{aligned} \underset{\sim}{F}\{\underset{\sim}{\xi}\} - \rho\ddot{\underset{\sim}{\xi}} &= \underset{\sim}{\nabla}\left[\gamma P \underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} + \underset{\sim}{\xi}\cdot\underset{\sim}{\nabla} P\right] \\ &+ (\underset{\sim}{\nabla}\times\underset{\sim}{B}) \times \left[\underset{\sim}{\nabla}\times(\underset{\sim}{\xi}\times\underset{\sim}{B})\right] - \underset{\sim}{B}\times\left\{\underset{\sim}{\nabla}\times\left[\underset{\sim}{\nabla}\times(\underset{\sim}{\xi}\times\underset{\sim}{B})\right]\right\} \end{aligned} \tag{5}$$

is the force on the element of fluid (of mass density ρ) displaced from its equilibrium position by the amount $\underset{\sim}{\xi}$. (In Eq. (5) γ is the ratio of specific heats and all quantities take their equilibrium values).

In the Appendix we discuss the form of $\underset{\sim}{\xi}$ which produces (uniformly in each discontinuity plane) an infinitesimal rotation of $\underset{\sim}{B}$ and infinitesimal change in $|\underset{\sim}{B}|$. We conclude that without loss of generality $\underset{\sim}{\xi}$ can be represented as the 2-dimensional vector

$$\underset{\sim}{\xi} = x_1 G(z)\hat{e}_1(z) + \left\{ \epsilon(z) x_1 - \left[\gamma P(z) + B^2(z)\right]^{-1}\left[K' + \gamma P(z) G(z)\right] x_2 \right\} \hat{e}_2(z) \quad (6)$$

We here adopt the orthogonal coördinate system x_1, x_2, z with $\hat{e}_1(z) = \underset{\sim}{B}(z)/|\underset{\sim}{B}(z)|$, $\hat{e}_z$ along the rotation axis and therefore normal to the discontinuity plane, and $\hat{e}_1 \times \hat{e}_2 = \hat{e}_z$. $P(z)$ and $B(z)$ in Eq. (6) are the pressure and magnetic field intensity of the initial equilibrium; $K' = (P' + \underset{\sim}{B}\cdot\underset{\sim}{B}')$ is the first order change in total pressure (magnetic plus thermal) produced by the perturbation and is independent of z (primes denote perturbations); $G(z) = \underset{\sim}{B}\cdot\underset{\sim}{B}' B^{-2} - P'(\gamma P)^{-1}$ is the first order change in ln $(|\underset{\sim}{B}|\rho^{-1})$ (The density ρ and P are related by $P\rho^{-\gamma}$ = const.); and $\epsilon(z) = (\underset{\sim}{B}'\cdot\hat{e}_2)|\underset{\sim}{B}|^{-1}$ is to first order the sine of the rotation angle. In the Appendix we also show that for $\underset{\sim}{\xi}$ as given by Eq. (6) $\underset{\sim}{F}\{\underset{\sim}{\xi}\}$ and hence δW vanish.

The importance of establishing MHD stability lies in comparison theorems (Kulsrud, 1964), which show that the hydromagnetic δW is less than any of the δW's obtained using one or more adiabatic invariants in a collisionless plasma. The following conditions must, however, be met for the comparison theorems to hold: (1) the γ of the gas in the MHD δW must be taken as 5/3, as it has been in deriving Eq. (1); (2) the pressure in the adiabatic δW's must be taken as isotropic. From the comparison theorems we thus conclude that <u>collisionless</u> tangential forms should be stable for isotropic pressures.

If the pressure anywhere in the tangential form were sufficiently anisotropic, the mirror and firehose instabilities (Rosenbluth, 1956; Chandrasekhar et al., 1958) would exist. These instabilities can be derived from the adiabatic δW's and are not prohibited by the comparison theorems which hold only for isotropic pressure. The long-term stability of the tangential forms is experimental evidence that the pressure in them is not excessively anisotropic.

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Figure Caption

The Structure of a Tangential Discontinuity. Five slices through the discontinuity are shown. $\underset{\sim}{B}$ changes arbitrarily in magnitude and direction from sheet to sheet but always remains in the plane of a sheet.

APPENDIX

We here outline the derivation of the 2-dimensional plasma displacement $\underset{\sim}{\xi}$, given by Eq. (6), which produces an infinitesimal rotation of $\underset{\sim}{B}$ and an infinitesimal change in $|\underset{\sim}{B}|$. We then show that for this $\underset{\sim}{\xi}$, $\underset{\sim}{F}\{\underset{\sim}{\xi}\}$ as given by Eq. (5) vanishes. We conclude by discussing an equilibrium-transforming 3-dimensional $\underset{\sim}{\xi}$ and prove that the changes in pressure and magnetic field produced by this 3-dimensional plasma displacement can always be effected by a 2-dimensional $\underset{\sim}{\xi}$.

Since the perturbed state is also a tangential discontinuity we demand that it satisfy three conditions:

$$(\underset{\sim}{B}+\underset{\sim}{B}')\cdot\nabla(\underset{\sim}{B}+\underset{\sim}{B}')=0 \tag{A-1}$$

$$(P+P')+\frac{|\underset{\sim}{B}+\underset{\sim}{B}'|^2}{2}=K+K' \tag{A-2}$$

$$P+P'=(P+P')(z);\quad \underset{\sim}{B}+\underset{\sim}{B}'=(\underset{\sim}{B}+\underset{\sim}{B}')(z) \tag{A-3}$$

Condition A-1 requires that the total magnetic field not vary as one moves along a field line. Condition A-2 asserts that the total pressure is everywhere constant, while from condition A-3 the partial pressures and $\underset{\sim}{B}$ itself may change but only in the direction perpendicular to the discontinuity planes. In linearized form these conditions are

$$\underset{\sim}{B}\cdot\nabla\underset{\sim}{B}'+\underset{\sim}{B}'\cdot\nabla\underset{\sim}{B}=0 \tag{A-1a}$$

$$P'+\underset{\sim}{B}\cdot\underset{\sim}{B}'=K' \tag{A-2a}$$

$$P'=P'(z);\quad \underset{\sim}{B}'=\underset{\sim}{B}'(z) \tag{A-3a}$$

In terms of the plasma displacement $\underset{\sim}{\xi}$, the pressure perturbation P' and the magnetic field perturbation $\underset{\sim}{B}'$ are (Bernstein et al., 1958)

$$P' = -\underset{\sim}{\xi} \cdot \nabla P - \gamma P \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} \tag{A-4}$$

$$\underset{\sim}{B}' = \underset{\sim}{B} \cdot \underset{\sim}{\nabla} \underset{\sim}{\xi} - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} \underset{\sim}{B} - \underset{\sim}{B} \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} \tag{A-5}$$

We next assume that $\underset{\sim}{\xi}$ has only x_1 and x_2 components, use A-4 and A-5 for P' and $\underset{\sim}{B}'$ ($\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P = \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} \underset{\sim}{B} = 0$ for this 2-dimensional $\underset{\sim}{\xi}$), and obtain expressions for the $\underset{\sim}{\xi}$ components from the conditions A-1a - A-3a. We find that the most general 2-dimensional $\underset{\sim}{\xi}$ is

$$\underset{\sim}{\xi} = \left[x_1 G(z) + H(x_2, z)\right] \hat{e}_1(z) + \left\{\epsilon(z) x_1 - (\gamma P + B^2)^{-1}\left[K' + \gamma P G(z)\right] x_2 + L(z)\right\} \hat{e}_2(z) \tag{A-6}$$

In (A-6) L represents a translation of plasma in the x_2 direction, uniform in each discontinuity plane, but varying arbitrarily from one discontinuity plane to another. Since a tangential discontinuity is at each z infinite and <u>uniform</u> in the x_2 direction, the translation L does not effect either a pressure or magnetic field change. Hence we take L = 0. Similar arguments also justify dropping H.

When we now reintroduce A-6 into Eqs. A-4 and A-5, we find

$$\underset{\sim}{B}' = |\underset{\sim}{B}| \left[\hat{e}_2 \epsilon + \hat{e}_1 \frac{K' + \gamma P G}{\gamma P + B^2}\right] \tag{A-7}$$

$$P' = -\gamma P \left[G - \frac{K' + \gamma P G}{\gamma P + B^2}\right] \tag{A-8}$$

Solving for ϵ and G (using A-2a for K'), we obtain

$$\epsilon = \frac{\underset{\sim}{B}' \cdot \hat{e}_2}{|\underset{\sim}{B}|} \tag{A-9}$$

and

$$G = \frac{\underset{\sim}{B} \cdot \underset{\sim}{B}'}{B^2} - \frac{P'}{\gamma P} \tag{A-10}$$

ϵ is to first order the sine of the rotation angle. From the relation $P\rho^{-\gamma}$ = const., it is easily shown that G is the first order change in $\ln(|\underset{\sim}{B}|\rho^{-1})$.

Proving that $\underset{\sim}{F}\{\underset{\sim}{\xi}\}$ as given by Eq. (5) vanishes for $\underset{\sim}{\xi}$ as given by A-6 is a straightforward exercise in differentiation. The individual terms appearing in $\underset{\sim}{F}$ are

$$\underset{\sim}{\nabla}\left[\gamma P(\nabla \cdot \underset{\sim}{\xi})\right] = \hat{e}_z \frac{\partial}{\partial z}\left[\gamma P(\gamma P + B^2)^{-1}(B^2 G - K')\right]$$

$$\underset{\sim}{\nabla}(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P) = 0$$

$$(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \left[\underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B})\right] = -\hat{e}_z\left[\epsilon B^2 \hat{e}_2 \cdot \frac{\partial \hat{e}_1}{\partial z} + \frac{\partial |B|}{\partial z}|B|(\gamma P + B^2)^{-1}(K' + \gamma P G)\right]$$

$$-\underset{\sim}{B} \times \left\{\underset{\sim}{\nabla} \times \left[\underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B})\right]\right\} = -\hat{e}_z\left\{\epsilon B^2 \hat{e}_1 \cdot \frac{\partial \hat{e}_2}{\partial z} + |B|\frac{\partial}{\partial z}\left[|\underset{\sim}{B}|(\gamma P + B^2)^{-1}(K' + \gamma P G)\right]\right\}$$

The sum of these four terms is zero. To show this we use the facts that $\partial/\partial z(\hat{e}_1 \cdot \hat{e}_2) = 0$ and $\partial/\partial z\,(P + B^2/2) = 0$.

The entire procedure outlined so far in this Appendix can also be applied to the case where $\underset{\sim}{\xi}$ is assumed to have a z-component in addition to its x_1 and x_2 components. Neglecting trivial arbitrarinesses (terms equivalent to H and L in A-6), we find

$$\underset{\sim}{\xi} = x_1 G(z)\hat{e}_1(z) + \Big\{ C(z)\, x_1 - (\gamma P + B^2)^{-1}\left[K' + \gamma P G(z)\right] x_2 - \frac{\partial C}{\partial z} x_2 \Big\}\, \hat{e}_2(z) + C(z)\, \hat{e}_z \tag{A-11}$$

For this $\underset{\sim}{\xi}$ it is also straightforwardly shown that $\underset{\sim}{F}\{\underset{\sim}{\xi}\} = 0$.

We note from (A-11) that if C is independent of z so that the z-translation of plasma is uniform among discontinuity planes, no other component of $\underset{\sim}{\xi}$ is modified. In this case C is a trivial rigid translation of the entire system. However, when C depends on z the plasma density and hence the plasma pressure are changed in a non-trivial manner by the plasma displacement C. The magnetic pressure is also changed since the moving, conducting plasma carries field lines along with it. The dependence of plasma pressure and magnetic pressure on density is not the same however, so that in order to maintain constant total pressure in the z-direction, the plasma expands (or contracts, depending on the sign of $\partial C/\partial z$) in the x_2 direction.

The pressure and magnetic field perturbations produced by the 3-dimensional $\underset{\sim}{\xi}$ are

$$\underset{\sim}{B}' = |\underset{\sim}{B}| \left\{ \hat{e}_2\left(\epsilon - C\, \hat{e}_2 \cdot \frac{\partial \hat{e}_1}{\partial z}\right) + \hat{e}_1 \left[\frac{K' + \gamma P G}{\gamma P + B^2} - \frac{C}{|\underset{\sim}{B}|}\frac{\partial |\underset{\sim}{B}|}{\partial z}\right]\right\} \tag{A-12}$$

$$P' = -\gamma P\left[G - \frac{K' + \gamma P G}{\gamma P + B^2}\right] - C\frac{\partial P}{\partial z} \tag{A-13}$$

It is possible to relate ϵ, G, and C to changes in pressure and magnetic field, just as was done in Eqs. A-9 and A-10 for the two dimensional $\underset{\sim}{\xi}$.

Suppose now we define $\tilde{\epsilon}$ and $\tilde{G}$ as

$$\tilde{\epsilon}(z) = \epsilon - C\,\hat{e}_2 \cdot \frac{\partial \hat{e}_1}{\partial z} \tag{A-14}$$

$$|\underset{\sim}{B}|^2 \frac{\gamma P \tilde{G}}{\gamma P + B^2} = |\underset{\sim}{B}|^2 \frac{\gamma P G}{\gamma P + B^2} + C\frac{\partial P}{\partial z} \tag{A-15}$$

It may then be readily verified that A-12 and A-13 become

$$\underset{\sim}{B}' = |\underset{\sim}{B}|\left[\hat{e}_2\,\tilde{\epsilon} + \hat{e}_1\,\frac{k' + \gamma P \tilde{G}}{\gamma P + B^2}\right] \tag{A-16}$$

$$P' = -\gamma P\left[\tilde{G} - \frac{k' + \gamma P \tilde{G}}{\gamma P + B^2}\right] \tag{A-17}$$

Comparison with A-7 and A-8 indicates that these same changes $\underset{\sim}{B}'$ and P' can be effected by the 2 dimensional $\underset{\sim}{\xi}$

$$\underset{\sim}{\xi} = x_1 \tilde{G}\,\hat{e}_1 + \left[\tilde{\epsilon}\,x_1 - (\gamma P + B^2)^{-1}(k' + \gamma P \tilde{G})x_2\right]\hat{e}_2 \tag{A-18}$$

Eqs. A-14 - A-18 illustrate the fact that the same changes in magnetic field and pressure can result from several different plasma motions. In particular changes due to an infinitesimal three dimensional plasma displacement which transforms one tangential discontinuity into another can <u>always</u> be duplicated by a $\underset{\sim}{\xi}$ having only x_1 and x_2 components.