X-643-70-83

PREPRINT

NASA TM X- 63855

NEAR PARABOLIC SOLUTIONS OF THE TWO-BODY PROBLEM

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INTRODUCTION

The classical Kepler equations for the elliptic and hyperbolic cases of two-body motion are written

$$
\begin{aligned}
M &= E - e \sin E \\
M &= e \sinh H - H
\end{aligned}
\tag{1}
$$

respectively, where

$$M = \sqrt{\frac{\mu}{|a|^3}}\,(t - \tau) \quad . \tag{2}$$

Here a is the semi-major axis; τ is the time of pericenter passage; $\mu = G(m_1 + m_2)$, where G is the universal gravitational constant and m_1 and m_2 are the masses of the two bodies; and e is the eccentricity of the relative orbital motion. The quantities E and H are defined by the relations

$$
\begin{aligned}
\tan E/2 &= \sqrt{\frac{1-e}{1+e}}\,\tan \theta/2 \\
\tanh H/2 &= \sqrt{\frac{e-1}{1+e}}\,\tan \theta/2
\end{aligned}
\tag{3}
$$

where θ is the true anomaly. Note that we have used the sign convention for the semimajor axis, i.e., $a > 0$ for an ellipse and $a < 0$ for a hyperbola. In both cases a is obtained from the equation

$$a = \left(\frac{2}{r} - \frac{v^2}{\mu} \right)^{-1} \tag{4}$$

where r and v are the magnitudes of the position and velocity vectors respectively.

It is easily shown that Kepler's equation possesses a unique solution. When the eccentricity is not close to one, there are efficient iterative methods for obtaining this solution. However, when a becomes large and forces the eccentricity near unity, both the elliptic and hyperbolic equations suffer a critical loss of accuracy. These are designated "nearly-parabolic" orbits and their solution requires special treatment.

The parabolic motion is described by the cubic equation

$$2 \sqrt{\frac{\mu}{p^3}} \; (t - \tau) = \tan \theta/2 + \frac{1}{3} \tan^3 \theta/2 \qquad (5)$$

where p is the semilatus rectum and is such that

$$p = a\,(1-e^2)$$

for elliptic and hyperbolic motion and

$$p = 2q$$

for parabolic motion, where q is the distance between the focus and vertex of the parabola. It is easily shown that one and only one real root exists for this equation.

THE GAUSS METHOD

The classical method for solving Kepler's equation in the nearly parabolic region is an iteration approach due to Gauss, and a detailed description of the technique has been given by Herget[5]. The method requires the use of auxiliary functional quantities which are usually obtained from special tables. However, Benima et al[2] have derived series expansions for these quantities which makes the method suitable for high-speed computer solution. Following the notation of Benima, the equations for Gauss' solution are

$$a = \sqrt{\frac{1 + 9e}{10}} \qquad b = \frac{5(1-e)}{1 + 9e} \qquad c = \sqrt{\frac{5(1+e)}{1+9e}} \tag{6}$$

$$A = b \tan^2 w/2 \tag{7}$$

$$B\, a \sqrt{\frac{\mu}{2q^3}}\,(t - \tau) = \tan w/2 + \frac{1}{3} \tan^3 w/2 \tag{8}$$

$$\tan \theta/2 = c\, C \tan w/2 \tag{9}$$

where

$$A = \frac{15(E - \sin E)}{9E + \sin E}, \qquad B = \frac{20\sqrt{+A}}{9E + \sin E}$$
$$C = \frac{1}{\sqrt{+A}} \tan E/2 \tag{10}$$

for elliptic orbits and

$$A = \frac{15(H - \sinh H)}{9H + \sin H}, \qquad B = \frac{20\sqrt{-A}}{9H + \sinh H}$$
$$C = \frac{1}{\sqrt{-A}} \tanh H/2 \tag{11}$$

for hyperbolic orbits.

Since B and C are functions of A, they may be tabulated or, as developed by Benima et al, expanded as

$$B = \sum_{j=0}^{\infty} \beta_j A^j , \qquad C = \sum_{j=0}^{\infty} \gamma_j A^j . \tag{12}$$

This form is more efficient for computers, because now only the coefficients need be stored. The first eight coefficients for these expansions have been presented by the above authors.

The Gauss procedure to determine θ from $(t - \tau)$ is to solve equation (8) for tan $w/2$ by successive approximation, beginning with $B = 1$. The value of tan $w/2$ obtained by solving the cubic equation with $B = 1$ permits the computation of A by equation (7). This in turn yields a new value for B by the series expansion (12) and permits a more accurate solution for tan $w/2$. This process is repeated until A reaches a desired accuracy, and then tan $\theta/2$ is computed from equation (9), having used the expansion (12) for the calculation of C. Rapid convergence of the method results from the condition that $\beta_1 = 0$ in equations (12). Numerical efficiency is increased if, after the first step through the algorithm, the cubic equation is solved by a single iteration using the solution of the cubic from the previous step as an initial guess. It should be noted that double precision accuracy requirements for other than short time intervals would force the calculation of more series coefficients than the eight given and hence the expense of considerable labor.

The calculation of $(t - \tau)$ when given θ, the reverse of the above problem, is not rapidly convergent using the Gauss method. For this calculation Benima et al introduce the series

$$C^{-2} = \sum_{j=0}^{\infty} \sigma_j A^j \tag{13}$$

which is somewhat more rapidly convergent than the series for C. The procedure is then to set $C^{-1} = 1$, obtain tan w/2 from equation (9), compute a value for A from equation (7), and then calculate a new C^{-2} from the series (13). This scheme is repeated until A reaches its final value, then B is computed from equation (12) and $(t - \tau)$ from equation (8).

The position and velocity may be written

$$\vec{r} = q\, D\, (1 - \tan^2 \theta/2)\, \hat{i}_\xi + 2q\, D \tan \theta/2\, \hat{i}_n$$

$$\vec{v} = \sqrt{\frac{\mu}{q(1+e)}} \left(\frac{1}{1 + \tan^2 \theta/2}\right) \left[-2 \tan \theta/2\, \hat{i}_\xi + (1+e - \tan^2 \theta/2)\hat{i}_n\right] \tag{14}$$

where $\hat{i}_\xi$ is a unit vector in the direction of pericenter and $\hat{i}_n$ is a unit vector in the direction of $\vec{\ell} \times \hat{i}_\xi$, where $\vec{\ell}$ is the angular momentum of the orbit. Here

$$D = \frac{1}{2} (1 + \cos E)$$

for an ellipse and

$$D = \frac{1}{2} (1 + \cosh H)$$

for a hyperbola. As before D may be tabulated as a function of A, expanded as

THE COEFFICIENTS β_j, γ_j, σ_j, AND δ_j*

j	β_j	γ_j	σ_j	δ_j
0	+ 1	+ 1	+ 1	+ 1
1	0	+ 2/5	- 4/5	- 1
2	- 3/175	+ 38/175	+ 8/175	+ 1/5
3	- 2/525	+ 328/2625	+ 8/525	+ 1/175
4	- 372/336875	+ 24726/336875	+ 1896/336875	+ 13/2625
5	- 8044/21896875	+ 114344/2627625	+ 28744/13138125	+ 1343/1010625
6	- 181646/1379503125	+ 59601184/2299171875	+ 26248/29859375	+ 423/875875
7	- 3229596/65143203125	+ 432281228/27918515625	+ 134584/372246875	+ 48703/275900625
0	+ 1	+ 1	+ 1	+ 1
1	0	+ 0.4	- 0.8	- 1
2	- 0.017142857	+ 0.21714286	+ 0.04571429	+ 0.2
3	- 0.003809524	+ 0.12495238	+ 0.01523810	+ 0.00571429
4	- 0.001104267	+ 0.07339814	+ 0.00562820	+ 0.00495238
5	- 0.000367358	+ 0.04351610	+ 0.00218783	+ 0.00132888
6	- 0.000131675	+ 0.02592289	+ 0.00087905	+ 0.00048295
7	- 0.000049577	+ 0.01548368	+ 0.00036155	+ 0.00017652

TABLE I

*From Benima, et al[2] page 125

$$D = \sum_{j=0}^{\infty} \delta_j A^j \tag{15}$$

or simply calculated from

$$D = (1 + AC^2)^{-1} . \tag{16}$$

The coefficients β_j, γ_j, σ_j and δ_j are presented in Table II of reference [2] and are reproduced in our Table I for completeness.

THE UNIVERSAL METHOD

In place of the separate equations given above to describe elliptic, parabolic and hyperbolic motion, a universally valid Kepler equation may be written[1]

$$\begin{gathered} \sqrt{\mu}\,(t-t_o) = A\,\beta^3\,S(\alpha\beta^2) + B\,\beta^2\,C(\alpha\beta^2) + r_o\,\beta \\ A = 1 - \alpha r_o, \qquad B = \frac{\bar{r}_o \cdot \bar{v}_o}{\sqrt{\mu}}, \qquad \alpha \equiv \frac{1}{a} = \frac{2}{r_o} - \frac{{v_o}^2}{\mu} \end{gathered} \tag{17}$$

where $\vec{r}_o$ and $\vec{v}_o$ are the position and velocity vectors at time $t = t_o$ and the C and S functions are defined by the series

$$S(x) = \sum_{i=0}^{\infty} \frac{(-x)^i}{(2i + 3)!} , \qquad C(x) = \sum_{i=0}^{\infty} \frac{(-x)^i}{(2i + 2)!} . \tag{18}$$

The symbols A, B, C and β defined above are independent of these of the previous section. Given the time t and the position and velocity at t_o, β is found from equation (17). The position and velocity at time t are

$$\vec{r} = \left[1 - \frac{\beta^2}{r_o} C(\alpha\beta^2)\right] \bar{r}_o + \left[(t - t_o) - \frac{\beta^3}{\sqrt{\mu}} S(\alpha\beta^2)\right] \bar{v}_o$$

$$\vec{v} = \frac{\sqrt{\mu}}{r r_o} \left[\alpha\beta^3 S(\alpha\beta^2) - \beta\right] \bar{r}_o + \left[1 - \frac{\beta^2}{r} C(\alpha\beta^2)\right] \bar{v}_o \qquad (19)$$

$$r = A \beta^2 C(\alpha\beta^2) + B\left[\beta - \alpha \beta^3 S(\alpha\beta^2)\right] + r_o.$$

The universal Kepler equation, which is independent of the pericenter location has several important features. For $\alpha = 0$, the parabolic case, equation (17) reduces as it should to the cubic equation

$$f(\beta) = \sqrt{\mu} (t - t_o) \qquad (20$$

where

$$f(\beta) = \frac{1}{6} A \beta^3 + \frac{1}{2} B \beta^2 + r_o \beta. \qquad (21)$$

For the parabolic case β may then be identified as $\sqrt{p} (\tan \theta/2 - \tan \theta_o/2)$. Moreover if we define

$$y(\beta) = A \beta^3 S(\alpha\beta^2) + B \beta^2 C(\alpha\beta^2) + r_o \beta \qquad (22)$$

we find that

$$\frac{dy}{d\beta} = r_o + A \beta^2 C(\alpha\beta^2) + B\left[\beta - \alpha \beta^3 S(\alpha\beta^2)\right] = r \qquad (23)$$

when $\frac{dy}{d\beta}$ is evaluated at the root of $y(\beta) = \sqrt{\mu} (t-t_o)$. We also find that $y(\beta)$ has inflection points given by

$$\beta_{inf} = \frac{1}{\sqrt{\alpha}} \tan^{-1} \left(- \frac{B\sqrt{\alpha}}{A} \right) \tag{24}$$

when $\alpha > 0$ and

$$\beta_{inf} = \frac{1}{\sqrt{-\alpha}} \tanh^{-1} \left(\frac{-B\sqrt{-\alpha}}{A} \right) \tag{25}$$

when $\alpha < 0$. Thus there is only one inflection point for hyperbolic motion and an infinite number given by

$$\beta_{inf} = \beta_{inf_o} + \frac{n\pi}{\sqrt{\alpha}}, \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{26}$$

for elliptic motion, where β_{inf_o} denotes the principal value of the arctan in equation (24). The value of y at an inflection point is given by $y(\beta_{inf}) = \dfrac{\beta_{inf} + B}{\alpha}$ for all α.

Various techniques exist in the literature for the solution of equation (17) for β when $(t-t_o)$ is given. In general an iterative method will be found most efficient, but in certain instances other methods of solution may be advantageous[4].

For the case under consideration, that of nearly parabolic motion, α is a small quantity. This fact has been utilized in expanding the solution of equation (17) as a power series in α[3]. However, under this situation the universal Kepler equation may be solved very efficiently for β, given $(t-t_o)$ by calculating an initial value for β from the cubic equation (20) and then applying a Newton-Raphson iteration technique. Optimal computing forms for the C and S functions have been obtained[6] which greatly increase the numberial efficiency when solving the universal formulation. This method of solution then has advantage over the Gauss method in that it is not tied to periapsis location, and moreover it is a special solution for nearly parabolic motion only in the selection of an appropriate starting value for the iteration procedure. In direct numerical comparison with the Gauss procedure of Benima et al, the methods displayed equality for short time intervals (approximately three iterations for double precision accuracy), but otherwise the universal technique displayed marked superiority. Figures 1 and 2 are graphs of $y(\beta)$ and $f(\beta)$ for elliptic and hyperbolic orbits respectively showing the approximating features of the cubics $f(\beta)$ expanded about inflection points.

In contrast to the Gauss method, the reverse calculation using the universal Kepler equation is as efficient as the procedure outlined above. If β is a given quantity, $(t-t_o)$ may be simply evaluated from equation (17) with no numerical difficulty. The calculation with $(\theta-\theta_o)$ as a given quantity is a little more involved. The dependence of the eccentric anomaly and its hyperbolic counterpart on the true anomaly may be written

$$\tan\frac{E-E_o}{2} = \sqrt{\alpha}\, d$$

$$\tanh\frac{H-H_o}{2} = \sqrt{-\alpha}\, d \tag{27}$$

$$d = \frac{r_o \sin\frac{\theta-\theta_o}{2}}{\sqrt{p}\cos\frac{\theta-\theta_o}{2} - B\sin\frac{\theta-\theta_o}{2}}$$

$$= \frac{r_o}{\cot\left(\frac{\theta-\theta_o}{2}\right)\left[\sqrt{r_o\,(2-\alpha r_o) - B^2}\right] - B}$$

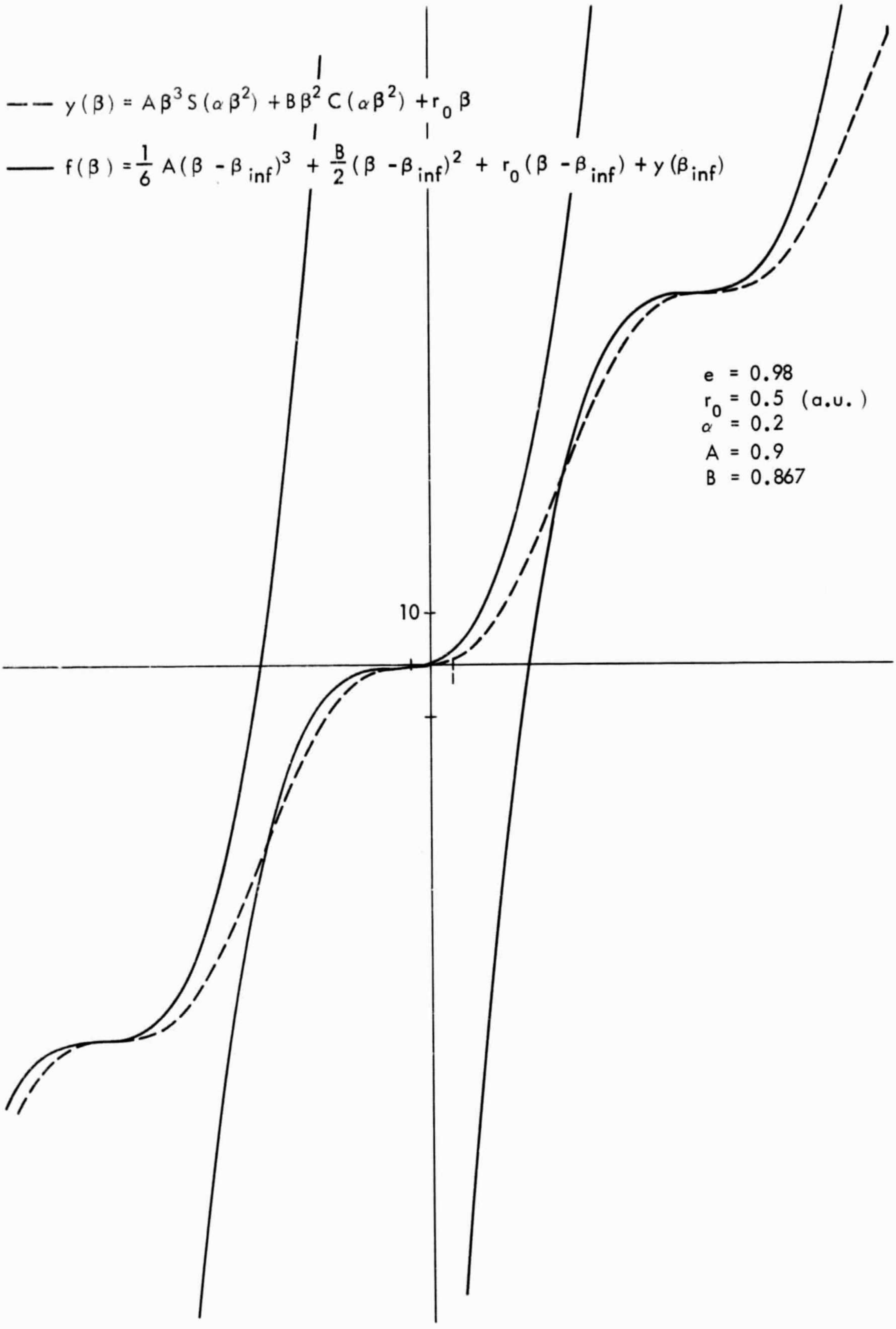
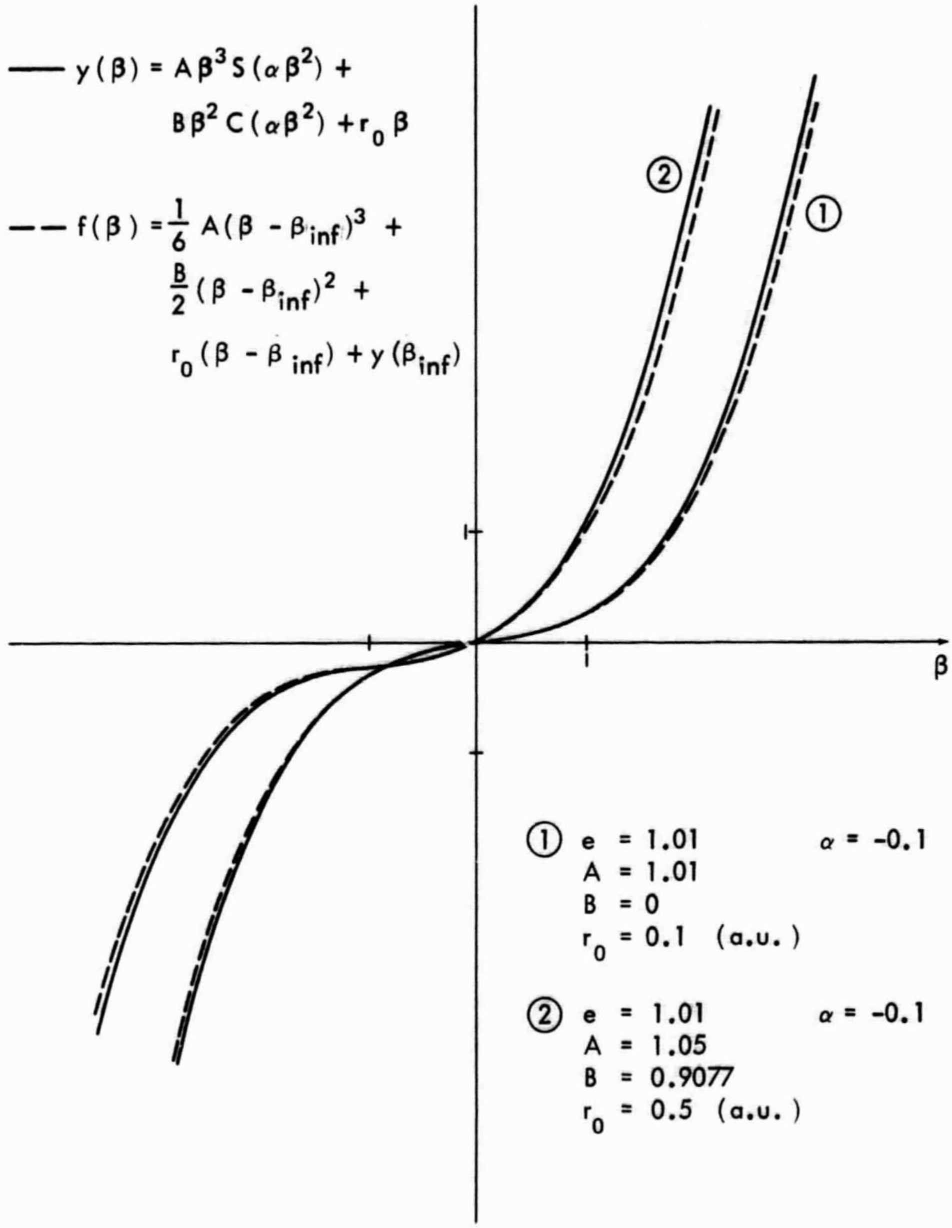

$y(\beta) = A\beta^3 S(\alpha\beta^2) + B\beta^2 C(\alpha\beta^2) + r_0\beta$
$f(\beta) = \frac{1}{6} A(\beta - \beta_{inf})^3 + \frac{B}{2}(\beta - \beta_{inf})^2 + r_0(\beta - \beta_{inf}) + y(\beta_{inf})$
$e = 0.98$
$r_0 = 0.5$ (a.u.)
$\alpha = 0.2$
$A = 0.9$
$B = 0.867$
10
1

Figure 1



— $y(\beta) = A\beta^3 S(\alpha\beta^2) + B\beta^2 C(\alpha\beta^2) + r_0\beta$
-- $f(\beta) = \frac{1}{6} A(\beta - \beta_{inf})^3 + \frac{B}{2}(\beta - \beta_{inf})^2 + r_0(\beta - \beta_{inf}) + y(\beta_{inf})$
2
1
β
1 e = 1.01 α = -0.1
A = 1.01
B = 0
r_0 = 0.1 (a.u.)
2 e = 1.01 α = -0.1
A = 1.05
B = 0.9077
r_0 = 0.5 (a.u.)

Figure 2

where we express

$$p = a(1-e^2) = 2r_o - \alpha r_o^2 - B^2$$

to retain significant digits which would otherwise be lost when e is close to unity.

Define

$$x = \frac{E-E_o}{2\sqrt{\alpha}} , \qquad x = \frac{H-H_o}{2\sqrt{-\alpha}} \tag{28}$$

for elliptic and hyperbolic motion respectively. There then results the universal equation

$$x - \alpha\, x^3\, S(\alpha x^2) = d\left[1 - \alpha x^2\, C(\alpha x^2)\right] \tag{29}$$

where x is related to the universal variable β of equation (17) by

$$x = \frac{1}{2}\,\beta . \tag{30}$$

The solution of equation (29) may be easily obtained by a Newton-Raphson method, yielding

$$x_{i+1} = x_i - \frac{\left[x_i - \alpha x_i^{\,3}\, S(\alpha x_i^{\,2})\right] - d\left[1-\alpha x_i^{\,2}\, C(\alpha x_i^{\,2})\right]}{\left[1-\alpha x_i^{\,2}\, C(\alpha x_i^{\,2})\right] + \alpha d\left[x_i - \alpha x_i^{\,3}\, S(\alpha x_i^{\,2})\right]} . \tag{31}$$

The authors have found that full double precision accuracy is usually attained in only two iterations with the use of

$$x_0 = \frac{1}{\sqrt{\alpha}} \tan^{-1} (\sqrt{\alpha}\, d)$$

$$x_0 = \frac{1}{\sqrt{-\alpha}} \tanh^{-1} (\sqrt{-\alpha}\, d) \tag{32}$$

as initial values for the elliptic and hyperbolic cases respectively. Then by equations (30), (17) and reduction formulas for the C and S functions[1] we may calculate the time from

$$\frac{\sqrt{\mu}}{2} (t-t_0) = 4Ax^3 S(4\alpha x^2) + 2 Bx^2 C(4\alpha x^2) + r_0 x$$

or (33)

$$\frac{\sqrt{\mu}}{2} (t-t_0) = Ax^3 S(\alpha x^2) + r_0 x + \left[1-\alpha x^2 S(\alpha x^2)\right] \left\{Ax^3 C(\alpha x^2) + Bx^2\left(1-\alpha x^2 S(\alpha x^2)\right)\right\} .$$

It should be noted that another technique for the solution of the reverse calculation which is more efficient than the Gauss method lies in the use of the unified form of Lambert's theorem, due to Lancaster and Blanchard[7], where a series expansion is given for the normalized time of flight in the near parabolic region.

References

1. Battin, R. H., Astronautical Guidance, McGraw-Hill Inc., 1964.

2. Benima, B., J. R. Cherniack, and B. A. Marsden, "The Gauss Method for Solving Kepler's Equation in Nearly Parabolic Orbits", Publications of the Astronomical Society of the Pacific, Vol. 81,