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VARIATION OF PARAMETERS AND THE LONG-TERM BEHAVIOR OF PLANETARY ORBITERS

PART 1
THEORY

BERNARD KAUFMAN

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VARIATION OF PARAMETERS AND THE LONG-TERM BEHAVIOR OF PLANETARY ORBITERS

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February 1970

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VARIATION OF PARAMETERS AND THE LONG-TERM BEHAVIOR OF PLANETARY ORBITERS

B. Kaufman

ABSTRACT

In this volume, the first of two, is contained the theory developed to determine the behavior of planetary orbiters under the perturbative influences of the oblateness of the central body, the presence of the Sun and atmospheric drag. The significant contribution of this study is the treatment of the third body which includes the medium-period as well as the long-period terms of the disturbing function in a system of singly-averaged equations. The variation of the elements is then expressed along with the effects of drag and oblateness in a set of equations to yield a time history of the orbit. These equations are easily adapted to an electronic computer. In test runs this program has been shown to be extremely fast and accurate to within short-period terms when compared with a complete numerical integration of the equations of motion.

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CONTENTS

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INTRODUCTION

This investigation is done in two parts. The theory of the orbital evolution appears in this document along with several sample cases. The detailed study of a parametric nature appears in Volume II.

The design of any mission to place a satellite in orbit about a planet must include a detailed investigation of the possible types of orbits that would be useful. These orbits will to a large extent be determined by the type of scientific investigations to be carried out in the vicinity of the planet. Of particular importance will be the distance of closest approach (periapsis) and the furthest distance reached (apoapsis). If periapsis is very low, then it is likely that the orbit will be affected by the atmosphere as well as the oblateness of the planet. When the apoapsis becomes large, then perturbations caused by other bodies can become significant. As one can easily see it is possible for the orbital motion to become highly complex when one considers the effects of oblateness, drag and other bodies and the possibilities of coupling between them.

It is precisely because of this complexity that a time history of the size, shape and orientation of the orbit should be obtained in order to gain the maximum scientific data from the satellite. Such a history should include not only lifetime predictions but also a reasonably accurate history of all of the orbital parameters for a variety of initial conditions. Because such a variety of conditions will be used, it also becomes essential that the model chosen to produce the time history be not only accurate but very fast. Any good n-body precision integration program of the Cowell or Encke type is capable of meeting the first of these criteria but certainly not the second. It was to meet both of the above criteria that the present investigation was undertaken.

THE SOLAR POTENTIAL: $R_{\odot}$

The first problem to be considered here is that of third-body perturbations. In the context used here this means the influence of the Sun on a satellite in orbit about a planet. Although this study does not include the Earth, the theory presented here can be used by including both the Sun and the Moon as disturbing bodies. To determine the exact effects of third-body perturbations, it is necessary to solve the three-body problem. This would normally require numerical integration of the equations of motion which, as pointed out earlier, becomes very time consuming when applied over periods of years for each orbit. Previous experience with such integrations has shown that in many cases the amplitudes of the short-term variations in the orbital elements are very small when compared with the values of the elements themselves. This means that one may "average" the disturbing function over the period of the satellite thereby eliminating all short-period terms. This averaging may be accomplished by several means such as appear in References 1 and 2. It is also possible to average this result once again over one revolution of the central body about the disturbing body (the Sun in our case). This process thus eliminates all "medium"-period terms leaving only the long-period perturbations for consideration. This doubly-averaged system was used in References 2, 3, and 4. As pointed out in Reference 4, however, this double-averaging technique has some severe limitations, especially for Mars where the interactions of oblateness and third-body effects become very complex. Therefore, the model used in this report is one of a set of singly-averaged equations which have been developed in a form which makes them readily adaptable to efficient programming.

The development of the solar disturbing function can be found in Reference 5, which was the preliminary investigation that motivated the present more detailed model. Because of the importance of this function in deriving the variational equations in a most useful form, the development will be repeated here.

With reference to Figure 1 we may write the solar disturbing function as follows:

$$R_\odot' = \mu_\odot \left[\frac{1}{\rho} - \frac{\vec{r}' \cdot \vec{r}}{r'^3}\right] \tag{1}$$

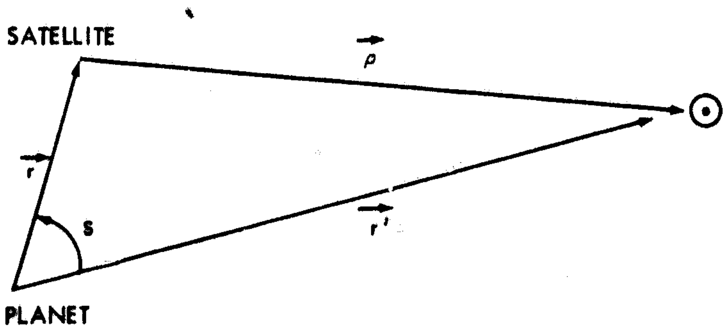

Figure 1—Sun-planet-satellite geometry.

where

$$\vec{\rho} = \vec{r}' - \vec{r}.$$

Taking note that

$$\rho^2 = |\vec{r}' - \vec{r}|^2 = (\vec{r}' - \vec{r}) \cdot (\vec{r}' - \vec{r})$$

$$= r'^2 + r^2 - 2rr' \cos S$$

$$= r'^2 \left(1 + \frac{r^2}{r'^2} - \frac{2r}{r'} \cos S\right)$$

we may rewrite Equation 1 as

$$R_\odot' = \frac{\mu_\odot}{r'}\left[\left(1 + \frac{r^2}{r'^2} - \frac{2r}{r'} \cos S\right)^{-1/2} - \frac{r \cos S}{r'}\right]. \tag{2}$$

For the cases that we will be considering, we note that $r/r' << 1$ and therefore we may expand the square root term in Equation 2 and neglect all terms of order $(r/r')^3$ and higher. This yields

$$R_{\odot}' = \frac{\mu_{\odot}}{r'}\left[1 - \frac{1}{2}\left(\frac{r^2}{r'^2} - \frac{2r}{r'}\cos S\right) + \frac{3}{8}\left(\frac{4r^2}{r'^2}\cos^2 S\right) - \frac{r}{r'}\cos S\right]$$

$$= \frac{\mu_{\odot}}{r'}\left[1 - \frac{1}{2}\frac{r^2}{r'^2} + \frac{3}{2}\frac{r^2}{r'^2}\cos^2 S\right]$$

$$= \frac{\mu_{\odot}}{r'}\left[1 + \frac{r^2}{r'^2}\left(\frac{3}{2}\cos^2 S - \frac{1}{2}\right)\right]$$

or

$$R_{\odot}' = \frac{\mu_{\odot}}{r'}\left\{1 + \frac{r^2}{r'^2}\left[\frac{3}{2}\left(\frac{\vec{r}\cdot\vec{r}'}{rr'}\right)^2 - \frac{1}{2}\right]\right\}. \qquad (3)$$

The motion of any satellite under the sole influence of a third body can be completely described by

$$\ddot{\vec{r}} = \vec{\nabla} R - \frac{\mu \vec{r}}{r^3} \qquad (4)$$

where $\vec{r}$ is the radius vector from the central body to the satellite, R is the disturbing function and $\vec{\nabla}$ is the gradient operator. This gradient is with respect to the state of the satellite but the first term of Equation 3 contains only elements of the Sun. It is obvious then that

$$\vec{\nabla}\frac{\mu_{\odot}}{r'} = 0$$

and we can drop this term and write

$$R_{\odot}' = \frac{\mu_{\odot} r^2}{2r'^3}\left[3\left(\frac{\vec{r}\cdot\vec{r}'}{rr'}\right)^2 - 1\right]$$

or alternatively

$$R_{\odot}' = \frac{\mu_{\odot} r^2}{2r'^3}\left[3\cos^2 S - 1\right]. \qquad (5)$$

Letting $\mu_{\odot} = n'^2 a'^3$, where n' is the mean motion of the Sun and a' is the semimajor axis, we may write Equation 5 as

$$R'_{\odot} = \frac{a^2 n'^2}{2}\left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \left[3\cos^2 S - 1\right]. \quad (6)$$

Now

$$\cos S = \frac{\vec{r}\cdot\vec{r}'}{rr'} = \vec{r}^{\,0}\cdot\vec{r}^{\,0\prime} \quad (7)$$

and

$$\vec{r}^{\,0} = \vec{P}\cos\theta + \vec{Q}\sin\theta \quad (8)$$

where θ is the true anomaly and

$$\vec{P} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\ \sin i\sin\omega \end{bmatrix} \quad (9)$$

$$\vec{Q} = \begin{bmatrix} -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\ -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \\ \sin i\cos\omega \end{bmatrix}. \quad (10)$$

$\vec{P}$ is a unit vector in the satellite's orbit plane from the center of the planet to pericenter, $\vec{Q}$ is a unit vector in the orbit plane perpendicular to $\vec{P}$ and in the direction of satellite motion, and i is the orbital inclination. For completeness and because we shall need it later, we define the third unit vector $\vec{R}$ as the cross product of $\vec{P}$ and $\vec{Q}$

$$\vec{R} = \vec{P}\times\vec{Q} = \begin{bmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{bmatrix}. \quad (11)$$

It is important to observe that $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are independent of θ. Also that

$$\vec{r}^{\,\prime 0} = \begin{bmatrix} \cos\Omega'\cos\phi' \\ \sin\Omega'\cos\phi' \\ \sin\phi' \end{bmatrix} \quad (12)$$

where Ω' is the right ascension of the Sun and ϕ' is the latitude of the Sun measured from the equator of the planet. Figure 2 shows the geometry involved in the above equations and the transformations to the equators of Venus and Mars appear in Appendix A.

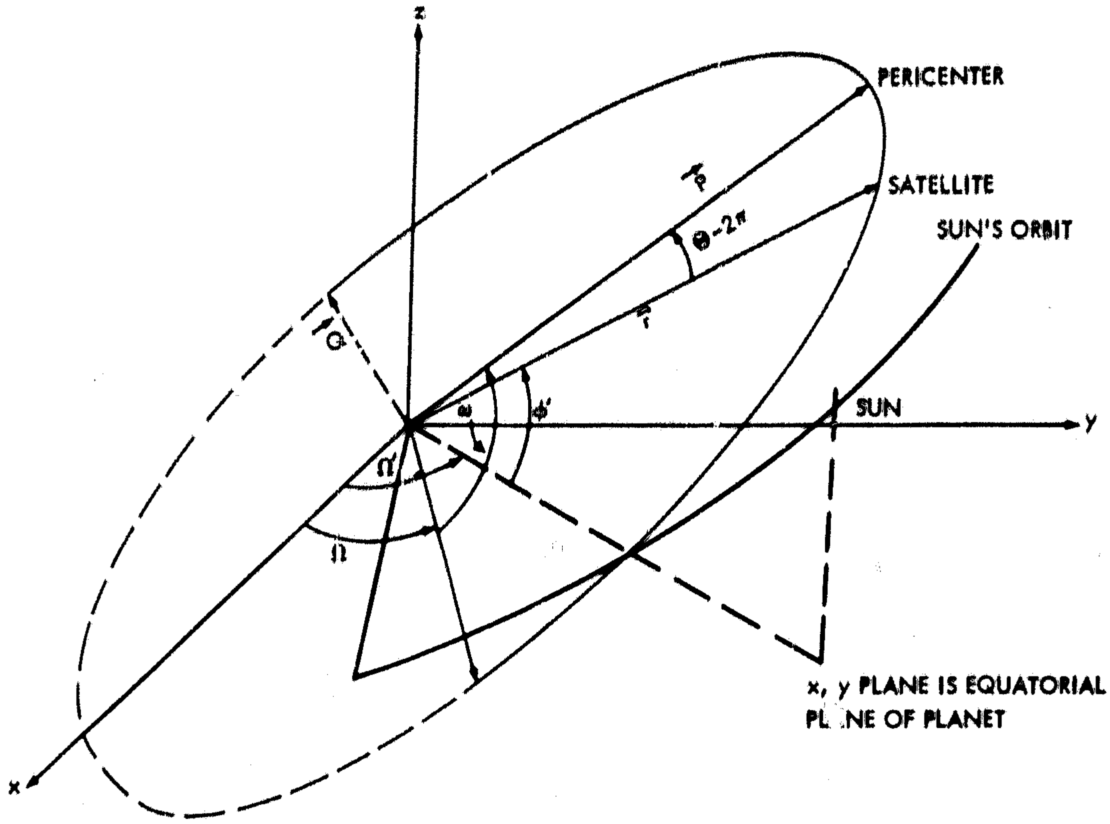

Figure 2—Planet-centered geometry.

Equation 7 may now be written as

$$\cos S = (\vec{P} \cdot \vec{r}'^{0}) \cos\theta + (\vec{Q} \cdot \vec{r}'^{0}) \sin\theta$$

and denoting

$$\left.\begin{aligned} \vec{P} \cdot \vec{r}'^{0} &= \alpha \\ \vec{Q} \cdot \vec{r}'^{0} &= \beta \end{aligned}\right\} \tag{13}$$

we have

$$\cos S = \alpha \cos\theta + \beta \sin\theta \tag{14}$$

where once again we point out that α and β are independent of θ. Substituting this into Equation 6 we have

$$R_{\odot}' = \frac{a^2 n'^2}{2} \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \left[3\left(\alpha^2 \cos^2\theta + 2\alpha\beta \sin\theta\cos\theta + \beta^2 \sin^2\theta\right) - 1\right].$$

Observing that

$$\frac{1 + \cos 2\theta}{2} = \cos^2 \theta$$

$$\frac{1 - \cos 2\theta}{2} = \sin^2 \theta$$

and

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

we may write the terms in the brackets as

$$\left[3\left(\alpha^2 \frac{1 + \cos 2\theta}{2} + \alpha\beta \sin 2\theta + \beta^2 \frac{1 - \cos 2\theta}{2}\right) - 1\right]$$

$$= \left[\left(\frac{3}{2}\alpha^2 + \frac{3}{2}\beta^2 - 1\right) + \frac{3}{2}\left(\alpha^2 - \beta^2\right)\cos 2\theta + 3\alpha\beta \sin 2\theta\right]$$

and after substituting this, we then have

$$R_{\odot}' = \frac{a^2 n'^2}{2}\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3\left[\left(\frac{3}{2}\alpha^2 + \frac{3}{2}\beta^2 - 1\right) + \frac{3}{2}\left(\alpha^2 - \beta^2\right)\cos 2\theta + 3\alpha\beta \sin 2\theta\right] . \qquad (15)$$

As explained earlier, Equation 15 will now be averaged over the period of the satellite, thereby eliminating any dependence of the disturbing function on the mean anomaly M. This not only simplifies the equations of motion but also eliminates all short-term variations over the orbital period. We consider therefore only those terms in Equation 15 dependent on θ and find that we need to integrate three terms over the period. The three integrals are as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 dM \qquad (16)$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \cos 2\theta \, dM \qquad (17)$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \sin 2\theta \, dM . \qquad (18)$$

In evaluating these integrals, we will make use of the following relationships

$$\frac{r}{a} = (1 - e\cos E) \tag{19}$$

$$M = E - e\sin E \tag{20}$$

$$dM = dE(1 - e\cos E) \tag{21}$$

where E is the eccentric anomaly.

Using these equations, we may write Equation 16 as

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 dM = \frac{1}{2\pi}\int_0^{2\pi}(1 - e\cos E)^3\, dE$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(1 - 3e\cos E - e^3\cos^3 E + 3e^2\cos^2 E)\, dE$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 3e\left(\frac{1 + \cos 2E}{2}\right)\right] dE\,.$$

Evaluating this integral, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 dM = 1 + \frac{3}{2}e^2 \tag{22}$$

a result valid for all e.

The integration of Equation 17 is much more complex

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2\cos 2\theta\, dM = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2\left[\cos^2\theta - \sin^2\theta\right] dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\left(\frac{r\cos\theta}{a}\right)^2 - \left(\frac{r\sin\theta}{a}\right)^2\right] dM$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \cos 2\theta \, dM = \frac{1}{2\pi}\int_0^{2\pi}\left[(\cos E - e)^2 - \left(1 - e^2\right)\sin^2 E\right] dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\cos^2 E - 2e\cos E + e^2 - \left(1 - e^2\right)\sin^2 E\right] dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1 + \cos 2E}{2} - 2e\cos E + e^2 - \frac{\left(1 - e^2\right)}{2}(1 - \cos 2E)\right] dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\left(\frac{1}{2} + e^2 - \frac{\left(1 - e^2\right)}{2}\right) - 2e\cos E + \left(\frac{1}{2} + \frac{\left(1 - e^2\right)}{2}\right)\cos 2E\right] dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{3e^2}{2} - 2e\cos E + \left(1 - \frac{e^2}{2}\right)\cos 2E\right] dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{3e^2}{2} - 2e\cos E + \left(1 - \frac{e^2}{2}\right)\cos 2E\right](1 - e\cos E)\, dE \, .$$

Dropping all periodic terms which will go to zero, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \cos 2\theta \, dM = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{3e^2}{2} + 2e^2\cos^2 E\right] dE$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{3e^2}{2} + e^2\right] dE$$

which yields

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \cos 2\theta \, dM = \frac{5e^2}{2} \tag{23}$$

again a result valid for all e.

For the last integral, Equation 18, we write the integrand in the following manner:

$$\left(\frac{r}{a}\right)^2 \sin 2\theta = \left(\frac{r}{a}\right)^2 [2 \sin\theta \cos\theta]$$

$$= 2\left[\left(\frac{r}{a} \sin\theta\right)\left(\frac{r}{a} \cos\theta\right)\right]$$

$$= 2\left(1 - e^2\right)^{1/2} \sin E \cos E - 2\left(1 - e^2\right)^{1/2} e \sin E .$$

We then have

$$\frac{\left(1 - e^2\right)^{1/2}}{\pi} \int_0^{2\pi} \sin E \cos E \, dM - \frac{e\left(1 - e^2\right)^{1/2}}{\pi} \int_0^{2\pi} \sin E \, dM$$

$$= \frac{\left(1 - e^2\right)^{1/2}}{\pi} \int_0^{2\pi} \left(\sin E \cos E - e \sin E \cos^2 E\right) dE$$

$$- \frac{e\left(1 - e^2\right)^{1/2}}{\pi} \int_0^{2\pi} (\sin E - e \sin E \cos E) \, dE$$

and conveniently all terms go to zero, leaving

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^2 \sin 2\theta \, dM = 0 . \tag{24}$$

We now define the solar disturbing function as

$$R_\odot = \frac{1}{2\pi} \int_0^{2\pi} R_\odot' \, dM$$

and substituting into this the results of evaluating the integrals, we finally obtain

$$R_\odot = \frac{a^2 n'^2}{2} \left(\frac{a'}{r'}\right)^3 \left[\left(\frac{3}{2} \alpha^2 + \frac{3}{2} \beta^2 - 1\right)\left(1 + \frac{3e^2}{2}\right) + \left(\frac{3}{2} \alpha^2 - \frac{3}{2} \beta^2\right) \frac{5e^2}{2}\right] \tag{25}$$

a result that is considerably simpler to use than was Equation 15.

VARIATION OF PARAMETERS

We shall here merely write down the equations for the variations in the Keplerian elements. The derivation of these equations may be found in any textbook on celestial mechanics (Reference 6)

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M} \tag{26}$$

$$\frac{de}{dt} = \frac{(1-e^2)}{ena^2}\frac{\partial R}{\partial M} - \frac{(1-e^2)^{1/2}}{ena^2}\frac{\partial R}{\partial \omega} \tag{27}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial R}{\partial i} \tag{28}$$

$$\frac{di}{dt} = -\frac{\csc i}{na^2(1-e^2)^{1/2}}\left[\frac{\partial R}{\partial \Omega} - \cos i\,\frac{\partial R}{\partial \omega}\right] \tag{29}$$

$$\frac{d\omega}{dt} = \frac{(1-e^2)^{1/2}}{ena^2}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial R}{\partial i} \tag{30}$$

$$\frac{dM}{dt} = n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{(1-e^2)}{nea^2}\frac{\partial R}{\partial e} \tag{31}$$

where in the present case $R = R_{\otimes}$.

Equation 26 yields immediately a very important result. Since we have averaged out all dependence of $R_{\otimes}$ on M, we have

$$\frac{da}{dt} = 0 \tag{32}$$

which means that there are no long-term or secular variations in the semimajor axis due to the presence of a third body.

From Equation 27 we can derive the variation in eccentricity as follows: we first note that as above

$$\frac{\partial R}{\partial M} = 0$$

and therefore

$$\frac{de}{dt} = -\frac{(1-e^2)^{1/2}}{ena^2}\frac{\partial R_{\otimes}}{\partial \omega} .$$

Referring to Equation 25 we have

$$\frac{\partial R_\oplus}{\partial \omega} = \frac{\partial R_\oplus}{\partial \alpha}\frac{\partial \alpha}{\partial \omega} + \frac{\partial R_\oplus}{\partial \beta}\frac{\partial \beta}{\partial \omega}$$

$$\alpha = \vec{P} \cdot \vec{r}'^{\,0}$$

therefore

$$\frac{\partial \alpha}{\partial \omega} = \frac{\partial \vec{P}}{\partial \omega} \cdot \vec{r}'^{\,0}$$

and from Equation 9 it is easily seen that

$$\frac{\partial \vec{P}}{\partial \omega} = \vec{Q} .$$

Thus

$$\frac{\partial \alpha}{\partial \omega} = \vec{Q} \cdot \vec{r}'^{\,0} = \beta .$$

$$\beta = \vec{Q} \cdot \vec{r}'^{\,0}$$

$$\frac{\partial \beta}{\partial \omega} = \frac{\partial \vec{Q}}{\partial \omega} \cdot \vec{r}'^{\,0} = - \vec{P} \cdot \vec{r}'^{\,0} = -\alpha .$$

Therefore

$$\frac{\partial R_\oplus}{\partial \omega} = \frac{\partial R_\oplus}{\partial \alpha}\beta - \frac{\partial R}{\partial \beta}\alpha . \tag{33}$$

Now

$$\left.\begin{aligned} \frac{\partial R_\oplus}{\partial \alpha} &= \frac{3a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 \alpha\left(1 + 4e^2\right) \\ \frac{\partial R_\oplus}{\partial \beta} &= \frac{3a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 \beta\left(1 - e^2\right) \end{aligned}\right\} \tag{34}$$

and substituting these into Equation 33 and then into the equation for e, we have

$$\frac{de}{dt} = -\frac{\left(1-e^2\right)^{1/2}}{nea^2}\,\frac{3a^2n'^2}{2}\left(\frac{a'}{r'}\right)^3\left[\alpha\beta\left(1+4e^2\right)-\alpha\beta\left(1-e^2\right)\right]$$

$$= -\frac{\left(1-e^2\right)^{1/2}}{ne}\,\frac{3n'^2}{2}\left(\frac{a'}{r'}\right)^3\left(5\alpha\beta e^2\right)$$

or

$$\frac{de}{dt} = -\frac{15}{2}\left(1-e^2\right)^{1/2}\frac{n'^2}{n}\,e\alpha\beta\left(\frac{a'}{r'}\right)^3 . \tag{35}$$

For Equation 28 for the node, we have

$$\frac{d\Omega}{dt} = \frac{1}{na^2\left(1-e^2\right)^{1/2}\sin i}\left[\frac{\partial R_\odot}{\partial \alpha}\frac{\partial \alpha}{\partial i}+\frac{\partial R_\odot}{\partial \beta}\frac{\partial \beta}{\partial i}\right] \tag{36}$$

where all we need now is $\partial\alpha/\partial i$ and $\partial\beta/\partial i$. Now

$$\frac{\partial \alpha}{\partial i} = \frac{\partial \vec{P}}{\partial i}\cdot \vec{r}'^{\,0}$$

and

$$\frac{\partial \vec{P}}{\partial i} = \begin{bmatrix} \sin\Omega\sin\omega\sin i \\ -\cos\Omega\sin\omega\sin i \\ \cos i\sin\omega \end{bmatrix}$$

and using Equation 11 we have

$$\frac{\partial \alpha}{\partial i} = \vec{R}\cdot\vec{r}'^{\,0}\sin\omega . \tag{37}$$

In a similar manner we find

$$\frac{\partial \beta}{\partial i} = \vec{R}\cdot\vec{r}'^{\,0}\cos\omega . \tag{38}$$

Combining Equations 34, 37 and 38 and substituting into Equation 36, we have

$$\frac{\partial\Omega}{\partial t} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\left[\frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 3\alpha\left(1+4e^2\right)\left(\vec{R}\cdot\vec{r}'^0\right)\sin\omega + \frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 3\beta\left(1-e^2\right)\left(\vec{R}\cdot\vec{r}'^0\right)\cos\omega\right]$$

$$= \frac{3n'^2}{2}\left(\frac{a'}{r'}\right)^3 \frac{\left(\vec{R}\cdot\vec{r}'^0\right)}{n\left(1-e^2\right)^{1/2}\sin i}\left[\alpha\left(1+4e^2\right)\sin\omega+\beta\left(1-e^2\right)\cos\omega\right]$$

and letting

$$\gamma = \vec{R}\cdot\vec{r}'^0 \tag{39}$$

we have finally

$$\frac{d\Omega}{dt} = \frac{3}{2}\frac{n'^2}{n\left(1-e^2\right)^{1/2}\sin i}\left(\frac{a'}{r'}\right)^3\left[\alpha\gamma\left(1+4e^2\right)\sin\omega+\beta\gamma\left(1-e^2\right)\cos\omega\right]. \tag{40}$$

Next for di/dt, we note that the only term not derived is $\partial R_\oplus/\partial\Omega$.

$$\frac{\partial R_\oplus}{\partial\Omega} = \frac{\partial R_\oplus}{\partial\alpha}\frac{\partial\alpha}{\partial\Omega}+\frac{\partial R_\oplus}{\partial\beta}\frac{\partial\beta}{\partial\Omega} \tag{41}$$

where

$$\frac{\partial\alpha}{\partial\Omega} = \frac{\partial\vec{P}}{\partial\Omega}\cdot\vec{r}'^0$$

and

$$\frac{\partial\vec{P}}{\partial\Omega} = \begin{bmatrix}-\sin\Omega\cos\omega-\cos\Omega\sin\omega\cos i\\ \cos\Omega\cos\omega-\sin\Omega\sin\omega\cos i\\ 0\end{bmatrix}$$

$$= \begin{pmatrix}-P_y\\ P_x\\ 0\end{pmatrix}.$$

Therefore

$$\frac{\partial \alpha}{\partial \Omega} = \begin{pmatrix} -P_y \\ P_x \\ 0 \end{pmatrix} \cdot \vec{r}'^{\,0} \, . \tag{42}$$

Similarly

$$\frac{\partial \beta}{\partial \Omega} = \frac{\partial \vec{Q}}{\partial \Omega} \cdot \vec{r}'^{\,0}$$

and

$$\frac{\partial \vec{Q}}{\partial \Omega} = \begin{bmatrix} \sin\Omega \sin\omega - \cos\Omega \cos\omega \cos i \\ -\cos\Omega \sin\omega - \sin\Omega \cos\omega \cos i \\ 0 \end{bmatrix} .$$

Then

$$\frac{\partial \beta}{\partial \Omega} = \begin{pmatrix} -Q_y \\ Q_x \\ 0 \end{pmatrix} \cdot \vec{r}'^{\,0} \, . \tag{43}$$

Substituting Equations 34, 42 and 43 into Equation 41 and then along with Equation 33, the equation for inclination becomes

$$\frac{di}{dt} = -\frac{\csc i}{na^2\left(1-e^2\right)^{1/2}} \left\{ \frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 3\alpha\left(1+4e^2\right) \begin{pmatrix} -P_0 \\ P_x \\ 0 \end{pmatrix} \cdot \vec{r}'^{\,0} \right.$$

$$+ \frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 3\beta\left(1-e^2\right) \begin{pmatrix} -Q_y \\ Q_x \\ 0 \end{pmatrix} \cdot \vec{r}'^{\,0}$$

$$\left. - \cos i \left[\frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 3\alpha\left(1+4e^2\right)\beta + \frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 3\beta\left(1-e^2\right)(-\alpha) \right] \right\}$$

$$\frac{di}{dt} = -\frac{3n'^2 \csc i}{2n(1-e^2)^{1/2}}\left(\frac{a'}{r'}\right)^3 \left[\alpha(1+4e^2)\begin{pmatrix}-P_y\\ P_x\\ 0\end{pmatrix}\cdot \vec{r}'^0 + \beta(1-e^2)\begin{pmatrix}-Q_y\\ Q_x\\ 0\end{pmatrix}\cdot \vec{r}'^0 - \alpha\beta(1+4e^2)\cos i + \alpha\beta(1-e^2)\cos i\right]$$

or

$$\frac{di}{dt} = -\frac{3n'^2}{2n(1-e^2)^{1/2}}\left(\frac{a'}{r'}\right)^3 \frac{1}{\sin i}\left[\alpha(1+4e^2)\begin{pmatrix}-P_y\\ P_x\\ 0\end{pmatrix}\cdot \vec{r}'^0 + \beta(1-e^2)\begin{pmatrix}-Q_y\\ Q_x\\ 0\end{pmatrix}\cdot \vec{r}'^0 - 5\alpha\beta e^2\cos i\right] . \tag{44}$$

For $d\omega/dt$, we need only to find $\partial R_\oplus/\partial e$. Since α and β are independent of e, then from Equation 25 for $R_\oplus$ we have

$$\frac{\partial R_\oplus}{\partial e} = \frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 \left[3e\left(\frac{3}{2}\alpha^2 + \frac{3}{2}\beta^2 - 1\right) + 5e\left(\frac{3}{2}\alpha^2 - \frac{3}{2}\beta^2\right)\right]$$

$$= \frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 \left[12e\,\alpha^2 - 3e\,\beta^2 - 3e\right]$$

or

$$\frac{\partial R_\oplus}{\partial e} = \frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 \left[3e\left(4\alpha^2 - \beta^2 - 1\right)\right] \tag{45}$$

and since $\partial R_\oplus/\partial i$ has already been evaluated, we have

$$\frac{d\omega}{dt} = \frac{(1-e^2)^{1/2}}{ena^2}\frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 \left[3e\left(4\alpha^2-\beta^2-1\right)\right] - \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\left\{\frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 3\alpha(1+4e^2) \times \left(\vec{R}\cdot\vec{r}'^0\right)\sin\omega + \frac{a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3 3\beta(1-e^2)\left(\vec{R}\cdot\vec{r}'^0\right)\cos\omega\right\}$$

$$\frac{d\omega}{dt} = \frac{3}{2}\left(1-e^2\right)^{1/2}\frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3\left(4\alpha^2-\beta^2-1\right)$$

$$-\frac{3}{2}\frac{n'^2}{n}\gamma\frac{\cos i}{\left(1-e^2\right)^{1/2}\sin i}\left(\frac{a'}{r'}\right)^3\left[\alpha\left(1+4e^2\right)\sin\omega+\beta\left(1-e^2\right)\cos\omega\right]$$

Finally

$$\frac{d\omega}{dt} = \frac{3}{2}\frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3\left(1-e^2\right)^{1/2}\left\{\left(4\alpha^2-\beta^2-1\right)-\gamma\frac{\cos i}{\left(1-e^2\right)\sin i}\left[\alpha\left(1+4e^2\right)\sin\omega\right.\right.$$

$$\left.\left.+\beta\left(1-e^2\right)\cos\omega\right]\right\} . \qquad (46)$$

For the last equation dM/dt, we can readily calculate the only needed term as

$$\frac{\partial R_\oplus}{\partial a} = an'^2\left(\frac{a'}{r'}\right)^3\left[\left(\frac{3}{2}\alpha^2+\frac{3}{2}\beta^2-1\right)\left(1+\frac{3e^2}{2}\right)\right.$$

$$\left.+\left(\frac{3}{2}\alpha^2-\frac{3}{2}\beta^2\right)\frac{5e^2}{2}\right] = \frac{2}{a}R_\oplus . \qquad (47)$$

Substituting Equations 45 and 47 into Equation 31, we have

$$\frac{\partial M}{\partial t} = n-\frac{2n'^2}{n}\left\{\left(\frac{a'}{r'}\right)^3\left[\left(\frac{3}{2}\alpha^2+\frac{3}{2}\beta^2-1\right)\left(1+\frac{3e^2}{2}\right)+\left(\frac{3}{2}\alpha^2-\frac{3}{2}\beta^2\right)\frac{5e^2}{2}\right]\right\}$$

$$-\frac{\left(1-e^2\right)}{nea^2}\left\{\frac{a^2n'^2}{2}\left(\frac{a'}{r'}\right)^3\left[3e\left(4\alpha^2-\beta^2-1\right)\right]\right\} . \qquad (48)$$

For convenience the six equations are written below:

$$
\left.\begin{aligned}
\left(\frac{da}{dt}\right)_3 &= 0 \\
\left(\frac{de}{dt}\right)_3 &= -\frac{15}{2}\left(1-e^2\right)^{1/2}\frac{n'^2}{n}\,e\alpha\beta\left(\frac{a'}{r'}\right)^3 \\
\left(\frac{d\Omega}{dt}\right)_3 &= \frac{3}{2}\,\frac{n'^2}{n\left(1-e^2\right)^{1/2}\sin i}\left(\frac{a'}{r'}\right)^3\left[\alpha\gamma\left(1+4e^2\right)\sin\omega+\beta\gamma\left(1-e^2\right)\cos\omega\right] \\
\left(\frac{di}{dt}\right)_3 &= -\frac{3}{2}\,\frac{n'^2}{n\left(1-e^2\right)^{1/2}\sin i}\left(\frac{a'}{r'}\right)^3\left[\alpha\left(1+4e^2\right)\begin{pmatrix}-P_y\\ P_x\\ 0\end{pmatrix}\cdot\vec{r}'^{\,0}+\beta\left(1-e^2\right)\begin{pmatrix}-Q_y\\ Q_x\\ 0\end{pmatrix}\cdot\vec{r}'^{\,0}-5\alpha\beta e^2\cos i\right] \\
\left(\frac{d\omega}{dt}\right)_3 &= \frac{3}{2}\frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3\left(1-e^2\right)^{1/2}\left\{\left(4\alpha^2-\beta^2-1\right)-\frac{\gamma\cos i}{\left(1-e^2\right)\sin i}\left[\alpha\left(1+4e^2\right)\sin\omega+\beta\left(1-e^2\right)\cos\omega\right]\right\} \\
\left(\frac{dM}{dt}\right)_3 &= n-\frac{2n'^2}{n}\left\{\left(\frac{a'}{r'}\right)^3\left[\left(\frac{3}{2}\alpha^2+\frac{3}{2}\beta^2-1\right)\left(1+\frac{3e^2}{2}\right)+\left(\frac{3}{2}\alpha^2-\frac{3}{2}\beta^2\right)\frac{5e^2}{2}\right]\right\} \\
&\qquad -\frac{\left(1-e^2\right)}{nea^2}\left\{\frac{a^2n'^2}{2}\left(\frac{a'}{r'}\right)^3\left[3e\left(4\alpha^2-\beta^2-1\right)\right]\right\}
\end{aligned}\right\} \tag{49}
$$

where the subscript means third-body variations only.

This completes the development of the variational equations for third-body perturbations only.

DRAG

For the variations due to the presence of an atmosphere, it will be convenient to transform Equations 26 through 31 to a slightly different form in which the perturbing force may be written in the following component form

$$\vec{F} = S\vec{U}_r + T\vec{U}_\theta + W\vec{U}_A \tag{50}$$

where S is the component along the unit radius vector $\vec{U}_r$; T is the component along the normal $\vec{U}_\theta$ to $\vec{r}$, in the orbital plane and such that it makes an angle less than 90° with the velocity vector; and W is the component along the perpendicular to the orbit plane where $\vec{U}_A = \vec{U}_r \times \vec{U}_\theta$. It can be shown from

Equations 9, 10 and 11 and the above definitions that $\vec{U}_r$, $\vec{U}_\theta$ and $\vec{U}_A$ have the following representations

$$\vec{U}_r = \begin{pmatrix} \cos\Omega\cos u - \sin\Omega\sin u\cos i \\ \sin\Omega\cos u + \cos\Omega\sin u\cos i \\ \sin u\sin i \end{pmatrix} \tag{51}$$

$$\vec{U}_\theta = \begin{pmatrix} -\cos\Omega\sin u - \sin\Omega\cos u\cos i \\ -\sin\Omega\sin u + \cos\Omega\cos u\cos i \\ \cos u\sin i \end{pmatrix} \tag{52}$$

$$\vec{U}_A = \begin{pmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{pmatrix} \tag{53}$$

where $u = \omega + \theta$. From Equation 4 we see that the partial derivatives $\partial R/\partial x$, $\partial R/\partial y$, $\partial R/\partial z$ are the components of the acceleration due to the disturbing function R. That is

$$\vec{\nabla} R = \frac{\partial R}{\partial x}\vec{i} + \frac{\partial R}{\partial y}\vec{j} + \frac{\partial R}{\partial z}\vec{k}.$$

In the development of the variation of parameters, it can be shown that if C_j represents any one of the six elements then

$$\frac{\partial R}{\partial C_j} = \vec{\nabla} R \cdot \frac{\partial \vec{r}}{\partial C_j} \qquad (j = 1, \cdots, 6) \tag{54}$$

where

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

It is obvious that

$$\vec{F} = \vec{\nabla} R.$$

and therefore we need only evaluate

$$\frac{\partial R}{\partial C_j} = \vec{F} \cdot \frac{\partial \vec{r}}{\partial C_j} \tag{55}$$

in terms of S, T and W and then use these partials of R in Equations 26 through 31. We will derive here only $\partial R/\partial a$ and the derivations of the other partials may be found in Reference 7. From Equation 8 we may write

$$\vec{r} = r\cos\theta\vec{P} + r\sin\theta\vec{Q} = \xi\vec{P} + \eta\vec{Q} . \tag{56}$$

Thus

$$\frac{\partial\vec{r}}{\partial a} = \vec{P}\frac{\partial\xi}{\partial a} + \vec{Q}\frac{\partial\eta}{\partial a} .$$

Then Equation 55 yields

$$\frac{\partial R}{\partial a} = \vec{F}\cdot\vec{P}\frac{\partial\xi}{\partial a} + \vec{F}\cdot\vec{Q}\frac{\partial\eta}{\partial a} . \tag{57}$$

Now we can write Equation 56 above as

$$\vec{r} = a(\cos E - e)\vec{P} + a\left(1-e^2\right)^{1/2}\sin E\vec{Q} \tag{58}$$

where

$$\left.\begin{aligned} \xi &= r\cos\theta = a(\cos E - e) \\ \eta &= r\sin\theta = a\left(1-e^2\right)^{1/2}\sin E . \end{aligned}\right\} \tag{59}$$

Then

$$\left.\begin{aligned} \frac{\partial\xi}{\partial a} &= \cos E - e = \frac{\xi}{a} \\ \frac{d\eta}{da} &= \left(1-e^2\right)^{1/2}\sin E = \frac{\eta}{a} \end{aligned}\right\} \tag{60}$$

also

$$\left.\begin{aligned} \vec{U}_r &= \vec{P}\cos\theta + \vec{Q}\sin\theta \\ \vec{U}_\theta &= -P\sin\theta + \vec{Q}\cos\theta . \end{aligned}\right\} \tag{61}$$

Then

$$\vec{F}\cdot\vec{P} = S\left(\vec{U}_r\cdot\vec{P}\right) + T\left(\vec{U}_\theta\cdot\vec{P}\right) + W\left(\vec{U}_A\cdot\vec{P}\right)$$

or

$$\vec{F} \cdot \vec{P} = S\cos\theta - T\sin\theta . \tag{62}$$

Similarly

$$\vec{F} \cdot \vec{Q} = S(\vec{U}_r \cdot \vec{Q}) + T(\vec{U}_\theta \cdot \vec{Q}) + W(\vec{U}_A \cdot \vec{Q})$$

or

$$\vec{F} \cdot \vec{Q} = S\sin\theta + T\cos\theta \tag{63}$$

and from Equations 59 we have

$$\left.\begin{aligned} \vec{F} \cdot \vec{P} &= \frac{1}{r}(S\xi - T\eta) \\ \text{and} \\ \vec{F} \cdot \vec{Q} &= \frac{1}{r}(S\eta + T\xi) . \end{aligned}\right\} \tag{64}$$

Substituting Equations 60 and 64 into Equation 57, we have

$$\begin{aligned} \frac{\partial R}{\partial a} &= \frac{1}{r}(S\xi - T\eta)\frac{\xi}{a} + \frac{1}{r}(S\eta + T\xi)\frac{\eta}{a} \\ &= \frac{1}{ra}(S\xi^2 - T\xi\eta) + \frac{1}{ra}(S\eta^2 + T\xi\eta) \\ &= \frac{S(\xi^2 + \eta^2)}{ra} \end{aligned}$$

or

$$\frac{\partial R}{\partial a} = S\frac{r}{a} . \tag{65}$$

In a similar manner we can derive the other partials as:

$$\frac{\partial R}{\partial e} = - Sa\cos\theta + Ta\sin\theta\left[1 + \frac{r}{a(1-e^2)}\right] \tag{66}$$

$$\frac{\partial R}{\partial M} = \frac{Sea \sin\theta}{\left(1-e^2\right)^{1/2}} + \frac{Ta^2\left(1-e^2\right)^{1/2}}{r} \tag{67}$$

$$\frac{\partial R}{\partial \Omega} = Tr \cos i - Wr \sin i \cos(\omega+\theta) \tag{68}$$

$$\frac{\partial R}{\partial \omega} = Tr \tag{69}$$

$$\frac{\partial R}{\partial i} = Wr \sin(\omega+\theta)\ . \tag{70}$$

Substituting Equations 66 through 70 into Equations 26 through 31, we have

$$\frac{da}{dt} = \frac{2}{n\left(1-e^2\right)^{1/2}}\left(Se \sin\theta + \frac{a\left(1-e^2\right)}{r}\,T\right) \tag{71}$$

$$\frac{de}{dt} = \frac{\left(1-e^2\right)^{1/2}}{na}\left[S \sin\theta + T(\cos E + \cos\theta)\right] \tag{72}$$

$$\frac{d\Omega}{dt} = \frac{r \sin(\omega+\theta)}{na^2\left(1-e^2\right)^{1/2}\sin i}\,W \tag{73}$$

$$\frac{di}{dt} = \frac{r \cos(\omega+\theta)}{na^2\left(1-e^2\right)^{1/2}}\,W \tag{74}$$

$$\frac{d\omega}{dt} = -\frac{\cos\theta\left(1-e^2\right)^{1/2}}{a\,ne}\,S + \frac{\sin\theta\left(1-e^2\right)^{1/2}}{a\,ne}\left[1 + \frac{r}{a\left(1-e^2\right)}\right]T - \frac{r \sin(\omega+\theta)\cot i}{a^2 n\left(1-e^2\right)^{1/2}} \tag{75}$$

$$\frac{dM}{dt} = n + \left[\frac{\left(1-e^2\right)\cos\theta}{a\,ne} - \frac{2r}{na^2}\right]S - \frac{\left(1-e^2\right)\sin\theta}{a\,ne}\left[1 + \frac{r}{a\left(1-e^2\right)}\right]T \tag{76}$$

where E is the eccentric anomaly.

The above form of the variational equations is known as Gauss's form. In the previous equations the variations of the elements were expressed in terms of the partial derivatives of the disturbing function with respect to the elements. In the Gaussian form we have the variations expressed in terms of the disturbing acceleration components. We can further resolve the acceleration in a different manner which is useful for the inclusion of atmospheric drag. We introduce components of the acceleration defined as follows

T′ component tangential to the orbit

N component normal to T′ and positive toward the central body and

W same as before.

As shown in Reference 8, we then have

$$\left.\begin{aligned}
T &= \frac{1 + e\cos\theta}{\left(1 + e^2 + 2e\cos\theta\right)^{1/2}}\, T' + \frac{e\sin\theta}{\left(1 + e^2 + 2e\cos\theta\right)^{1/2}}\, N \\
S &= \frac{e\sin\theta}{\left(1 + e^2 + 2e\cos\theta\right)^{1/2}}\, T' - \frac{1 + e\cos\theta}{\left(1 + e^2 + 2e\cos\theta\right)^{1/2}}\, N \\
W &= W
\end{aligned}\right\} \qquad (77)$$

In the present study we consider no lift forces to be present and the drag force acting as a negative tangential component (opposite to the velocity vector). We then have

$$\left.\begin{aligned}
T &= \frac{1 + e\cos\theta}{\left(1 + e^2 + 2e\cos\theta\right)^{1/2}}\, T' \\
S &= \frac{e\sin\theta}{\left(1 + e^2 + 2e\cos\theta\right)^{1/2}}\, T' \\
W &= N = 0\,.
\end{aligned}\right\} \qquad (78)$$

The drag force is the force per unit mass m of magnitude F and

$$F = \frac{C_D A \rho V^2}{2m} = T' \qquad (79)$$

where V is the speed,

C_D is the aerodynamic drag coefficient, A is the cross-sectional area of the satellite and ρ is the density. Utilizing Equations 73, 74 and 78, we see immediately:

$$\frac{d\Omega}{dt} = \frac{di}{dt} = 0\,. \qquad (80)$$

For the other equations we obtain after substituting Equations 78

$$\frac{da}{dt} = -\frac{C_D A}{m}\,\frac{\rho V^2}{n\left(1 - e^2\right)^{1/2}}\left(1 + e^2 + 2e\cos\theta\right)^{1/2} \qquad (81)$$

$$\frac{de}{dt} = -\frac{C_D A}{m}\,\frac{\rho V^2\left(1 - e^2\right)^{1/2}(\cos\theta + e)}{na\left(1 + e^2 + 2e\cos\theta\right)^{1/2}} \qquad (82)$$

$$\frac{d\omega}{dt} = -\frac{C_D A}{m} \frac{\rho V^2 (1-e^2)^{1/2} \sin\theta}{n\,ae(1+e^2+2e\cos\theta)^{1/2}} \tag{83}$$

$$\frac{dM}{dt} = n + \frac{C_D A}{m} \frac{\rho V^2 (1-e^2) \sin\theta}{n\,ae(1+e\cos\theta)} \left[\frac{1+e^2+e\cos\theta}{(1+e^2+2e\cos\theta)^{1/2}}\right] . \tag{84}$$

The presence of $\sin\theta$ in Equations 83 and 84 implies that $d\omega/dt$ and dM/dt are periodic with small amplitudes because of the coefficient $C_D A\rho/m$ and therefore will be neglected in this treatment. However a and e change secularly and must be included. We now make use of the fact that

$$r^2 \frac{d\theta}{dt} = |\vec{r} \times \vec{v}| = \left[\mu a(1-e^2)\right]^{1/2}$$

and obtain

$$\frac{dt}{d\theta} = \frac{(1-e^2)^{3/2}}{n(1+e\cos\theta)^2} \tag{85}$$

also using

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

and substituting into

$$V^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

we have

$$V^2 = \frac{n^2 a^2}{1-e^2}\left(1+e^2+2e\cos\theta\right) . \tag{86}$$

Making use of Equations 85 and 86 in Equations 81 and 82 we have

$$\frac{da}{d\theta} = -\frac{C_D A}{m} \frac{\rho a^2 V^3 (1-e^2)^{3/2}}{\mu(1+e\cos\theta)^2} \tag{87}$$

$$\frac{de}{d\theta} = -\frac{C_D A}{m} \frac{(1-e^2)^{3/2} \rho V(e+\cos\theta)}{(1+e\cos\theta)^2} \tag{88}$$

where

$$\mu = n^2 a^3 .$$

We now average the above equations over the orbit and letting $\bar{\dot{a}}$ and $\bar{\dot{e}}$ represent the perturbations in a and e over the orbit, we have

$$\bar{\dot{a}} = -\frac{C_D A a^2 (1-e^2)^{3/2}}{2\pi m \mu} \int_{-\pi}^{\pi} \frac{\rho V^3}{(1 + e\cos\theta)^2} \, d\theta \tag{89}$$

$$\bar{\dot{e}} = -\frac{C_D A (1-e^2)^{3/2}}{2\pi m} \int_{-\pi}^{\pi} \frac{\rho V (e + \cos\theta)}{(1 + e\cos\theta)^2} \, d\theta . \tag{90}$$

The integration is done using Gaussian quadratures, which is very fast and reasonably accurate. The use of quadratures was suggested by Uphoff in Reference 4. The density ρ is taken from several models and is calculated as a function of altitude. These models can be found in References 9 and 10.

It is to be stressed that we here have considered a nonrotating atmosphere; otherwise $d\Omega/dt$ and di/dt would be nonzero. We have also bypassed the complexities of assuming an exponential density profile by averaging the effects over one revolution of the satellite.

OBLATENESS

In this section we will merely list the variational equations due to the oblateness of the central body. The equations in J_2 and J_2^2 are derived in detail in Reference 11 and those in J_3 and J_4 were taken from Reference 12.

Equations in J_2, J_2^2

$$\left(\frac{da}{dt}\right)_{o,J_2} = 0 \tag{91}$$

$$\left(\frac{de}{dt}\right)_{o,J_2} = -\frac{45n}{32}\frac{J_2^2 R_e^4}{P^4} e\left(1-e^2\right)\sin^2 i \sin 2\omega \left(\frac{14}{15} - \sin^2 i\right) \tag{92}$$

$$\left(\frac{d\Omega}{dt}\right)_{o,J_2} = -\frac{3n J_2 R_e^2}{2P^2}\cos i - \frac{9nJ_2^2 R_e^4}{4P^4}\cos i \left[\frac{3}{2} - \frac{5}{6}\sin^2 i + \left(\frac{1}{6} + \frac{5}{24}\sin^2 i\right) e^2 + \left(\frac{7}{12} - \frac{5}{4}\sin^2 i\right) e^2 \cos 2\omega\right] \tag{93}$$

$$\left(\frac{di}{dt}\right)_{O,J_2} = \frac{135}{104}\frac{J_2^2 R_e^4}{P^4} ne^2 \sin 2i \sin 2\omega \left(\frac{14}{15} - \sin^2 i\right) \quad (94)$$

$$\left(\frac{d\omega}{dt}\right)_{O,J_2} = \frac{3}{2}\frac{nJ_2 R_e^2}{P^2}\left(2 - \frac{5}{2}\sin^2 i\right) - \frac{9}{4}\frac{nJ_2^2 R_e^4}{P^4}\left\{2 - \frac{23}{4}\sin^2 i + \frac{55}{16}\sin^4 i + \frac{e^2}{4}\left(7 - \frac{9}{2}\sin^2 i + \frac{75}{4}\sin^4 i\right) + \frac{\cos 2\omega}{4}\left[\left(7 - \frac{15}{2}\sin^2 i\right)\sin^2 i + e^2\left(7 + 5\sin^2 i - \frac{45}{4}\sin^4 i\right)\right]\right\} \quad (95)$$

where

R_e is the equatorial radius of the planet

$P = a(1 - e^2)$ is the semilatus rectum and the subscripts O, J_2 mean oblateness and J_2 terms only.

<u>Equations in J_3</u>

$$\left(\frac{dP}{dt}\right)_{O,J_3} = 2P \tan i \left(\frac{di}{dt}\right)_{O,J_3} \quad (96)$$

$$\left(\frac{de}{dt}\right)_{O,J_3} = \frac{3}{8}\frac{nR_e^3 J_3}{P^3}(1 - e^2)\cos\omega \sin i (5\cos^2 i - 1) \quad (97)$$

$$\left(\frac{d\Omega}{dt}\right)_{O,J_3} = \frac{3nR_e^3 J_3}{8P^3} e \sin\omega \cot i (15\cos^2 i - 11) \quad (98)$$

$$\left(\frac{di}{dt}\right)_{O,J_3} = \frac{3nR_e^3 J_3}{8P^3} e \cos\omega \cos i (5\cos^2 i - 1) \quad (99)$$

$$\left(\frac{d\omega}{dt}\right)_{O,J_3} = \frac{3nR_e^3 J_3}{8P^3}\frac{(1 + 4e^2)}{e}\sin\omega \sin i (5\cos^2 i - 1) - \left(\frac{d\Omega}{dt}\right)_{O,J_3}\cos i \quad (100)$$

where

$$\left(\frac{da}{dt}\right)_{O,J_4} = \frac{\left(\frac{dP}{dt}\right)_{O,J_3} + 2ae\left(\frac{de}{dt}\right)_{O,J_3}}{(1 - e^2)} \quad (101)$$

Equations in J_4

$$\left(\frac{dP}{dt}\right)_{0,J_4} = 2P \tan i \left(\frac{di}{dt}\right)_{0,J_4} \tag{102}$$

$$\left(\frac{de}{dt}\right)_{0,J_4} = -\frac{15 n R_e^4 J_4}{32P^4} e\left(1-e^2\right) \sin 2\omega \sin^2 i \left(7\cos^2 i - 1\right) \tag{103}$$

$$\left(\frac{d\Omega}{dt}\right)_{0,J_4} = \frac{15 n R_e^4 J_4}{32P^4} \cos i \left\{2\left(7\cos^2 i - 3\right) + e^2 \left[7\cos^2 i - 1 + 4\sin^2\omega\left(7\cos^2 i - 4\right)\right]\right\} \tag{104}$$

$$\left(\frac{di}{dt}\right)_{0,J_4} = \frac{15 n R_e^4 J_4}{64P^4} e^2 \sin 2\omega \sin 2i \left(7\cos^2 i - 1\right) \tag{105}$$

$$\left(\frac{d\omega}{dt}\right)_{0,J_4} = -\frac{15 n R_e^4 J_4}{16P^4} \left\{8 - 28\sin^2 i + 21\sin^4 i - \sin^2\omega \sin^2 i \left(7\cos^2 i - 1\right) + e^2\left[6 - 14\sin^2 i + \frac{63}{8}\sin^4 i + \sin^6\omega\left(6 - 35\sin^2 i + \frac{63}{2}\sin^4 i\right)\right]\right\} \tag{106}$$

where a definition identical to Equation 101 holds with the proper change in the subscripts.

THE TOTAL VARIATIONAL EQUATIONS

As shown in Reference 3, if a doubly-averaged solar disturbing function is used, then the resulting set of equations separates into two uncoupled systems of 3 equations each. This set of uncoupled equations was used in Reference 13 along with the variation in eccentricity obtained with a singly-averaged disturbing function. However that study included only third-body effects. Using these uncoupled equations, one can obtain initial conditions for various types of behavior due to third-body effects. Williams and Lorell in Reference 3 give a mathematical description of the motion and Uphoff in Reference 4 gives a very good description of the interactions of the various perturbations. Using the doubly-averaged third-body disturbing function and superimposing medium period terms, Uphoff defines the regions in which the doubly-averaged equations yield very accurate results and then defines those regions where one must resort to the singly-averaged equations. A simple analysis of long-term third-body effects may also be obtained by merely looking at these doubly-averaged terms. A value of 135° or 315° in ω will result in long-period third-body perturbations that will raise periapsis; ω = 45° or 225° will lower periapsis and no change is obtained for ω = 0°, 90° or 180°. As Uphoff points out, the long periodic variations in eccentricity are more prominent at high inclinations, while at low inclinations the medium-period terms are dominant. Also the medium-period variations increase as the eccentricity grows under the influence of the long-period perturbations. It is unfortunate that when one goes to the singly-averaged equations the ability to obtain this type of information analytically is lost.

For the purposes of determining the total variations, the equations may now be written in the following manner.

$$\frac{da}{dt} = \bar{\dot{a}} + \left(\frac{da}{dt}\right)_{O,J_3} + \left(\frac{da}{dt}\right)_{O,J_4} \tag{107}$$

$$\frac{de}{dt} = \left(\frac{de}{dt}\right)_3 + \bar{\dot{e}} + \left(\frac{de}{dt}\right)_{O,J_2} + \left(\frac{de}{dt}\right)_{O,J_3} + \left(\frac{de}{dt}\right)_{O,J_4} \tag{108}$$

$$\frac{d\Omega}{dt} = \left(\frac{d\Omega}{dt}\right)_3 + \left(\frac{d\Omega}{dt}\right)_{O,J_2} + \left(\frac{d\Omega}{dt}\right)_{O,J_3} + \left(\frac{d\Omega}{dt}\right)_{O,J_4} \tag{109}$$

$$\frac{di}{dt} = \left(\frac{di}{dt}\right)_3 + \left(\frac{di}{dt}\right)_{O,J_2} + \left(\frac{di}{dt}\right)_{O,J_3} + \left(\frac{di}{dt}\right)_{O,J_4} \tag{110}$$

$$\frac{d\omega}{dt} = \left(\frac{d\omega}{dt}\right)_3 + \left(\frac{d\omega}{dt}\right)_{O,J_2} + \left(\frac{d\omega}{dt}\right)_{O,J_3} + \left(\frac{d\omega}{dt}\right)_{O,J_4} \tag{111}$$

$$\frac{dM}{dt} = \left(\frac{dM}{dt}\right)_3 . \tag{112}$$

In these equations one may include or not include any of the indicated terms in order to determine the effects of each. However with the exception of Equation 112 (which is included only for completeness), the above system yields an accurate description of the average motion of the orbit. As discussed earlier one of the principal advantages of this system is that the interaction of the medium and long-period terms is treated in a more realistic manner. This becomes particularly important in the case of Mars, where oblateness and third-body effects can combine to cause very complex behavior.

SAMPLE CASES

Figures 3 and 4 show a comparison between the variational method described here and an Encke n-body numerical integration program. The slight differences noticed here are due to short-period terms which are present in the n-body program and to differences in constants. The curve in Figure 3 produced by the n-body program took more than one hour of 360/95 time, while the variational method took 30 seconds on the same machine and only half of this was spent in the computation step.

Figure 5 from Reference 4 shows a comparison with a Cowell n-body program. Again the agreement between the two is excellent.

Figures 6 and 7 show the importance of including the atmosphere of a Martian orbiter, particularly when a long lifetime is required.

These few sample cases illustrate the usefulness of the program outlined above. The extreme speed and reasonable accuracy should prove the program to be a valuable aid in determining the orbital behavior for a multitude of initial conditions.

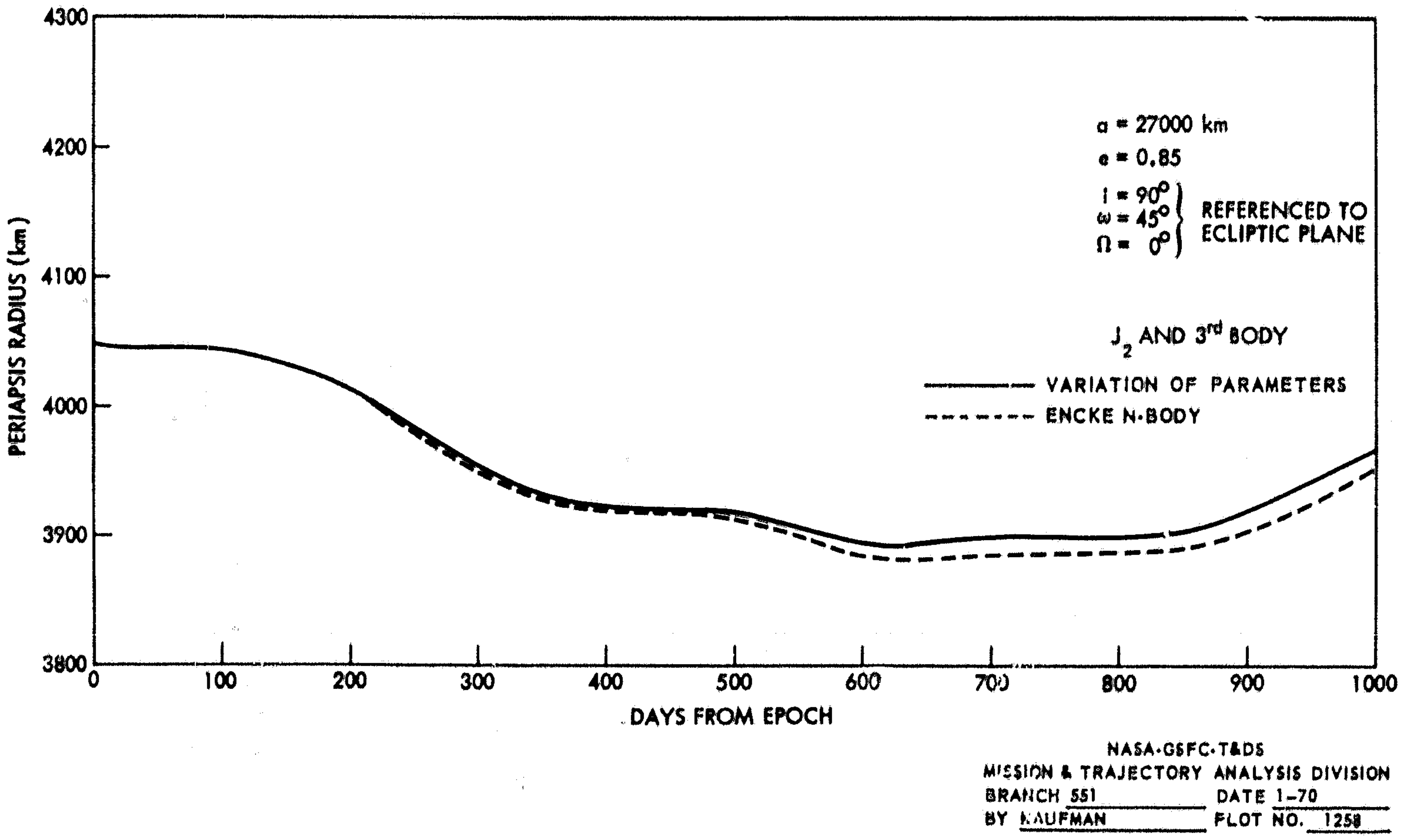

Figure 3—Mars: Periapsis vs time. Encke and variational comparison.

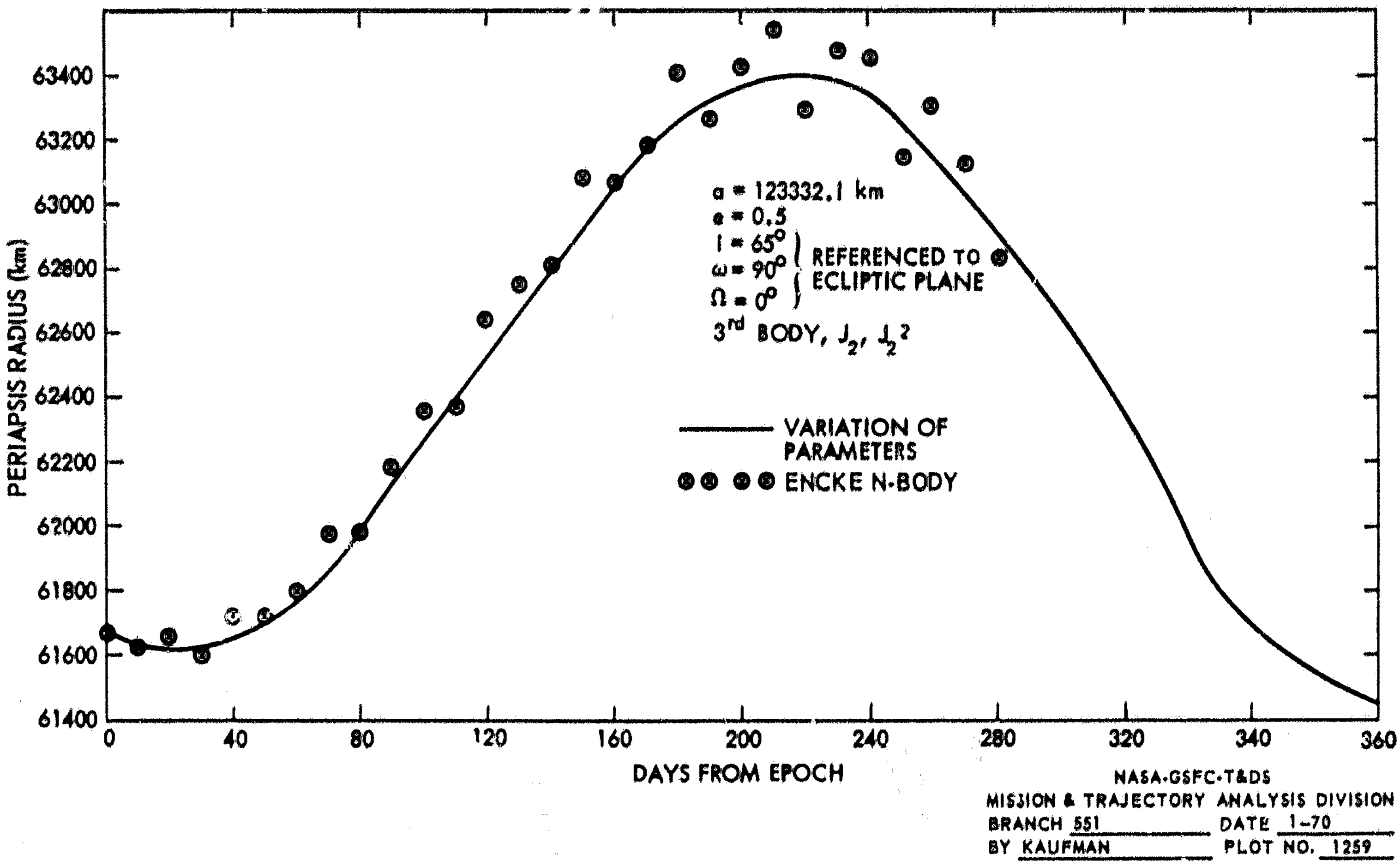

Figure 4—Mars: Periapsis radius vs time. Encke and variational comparison.

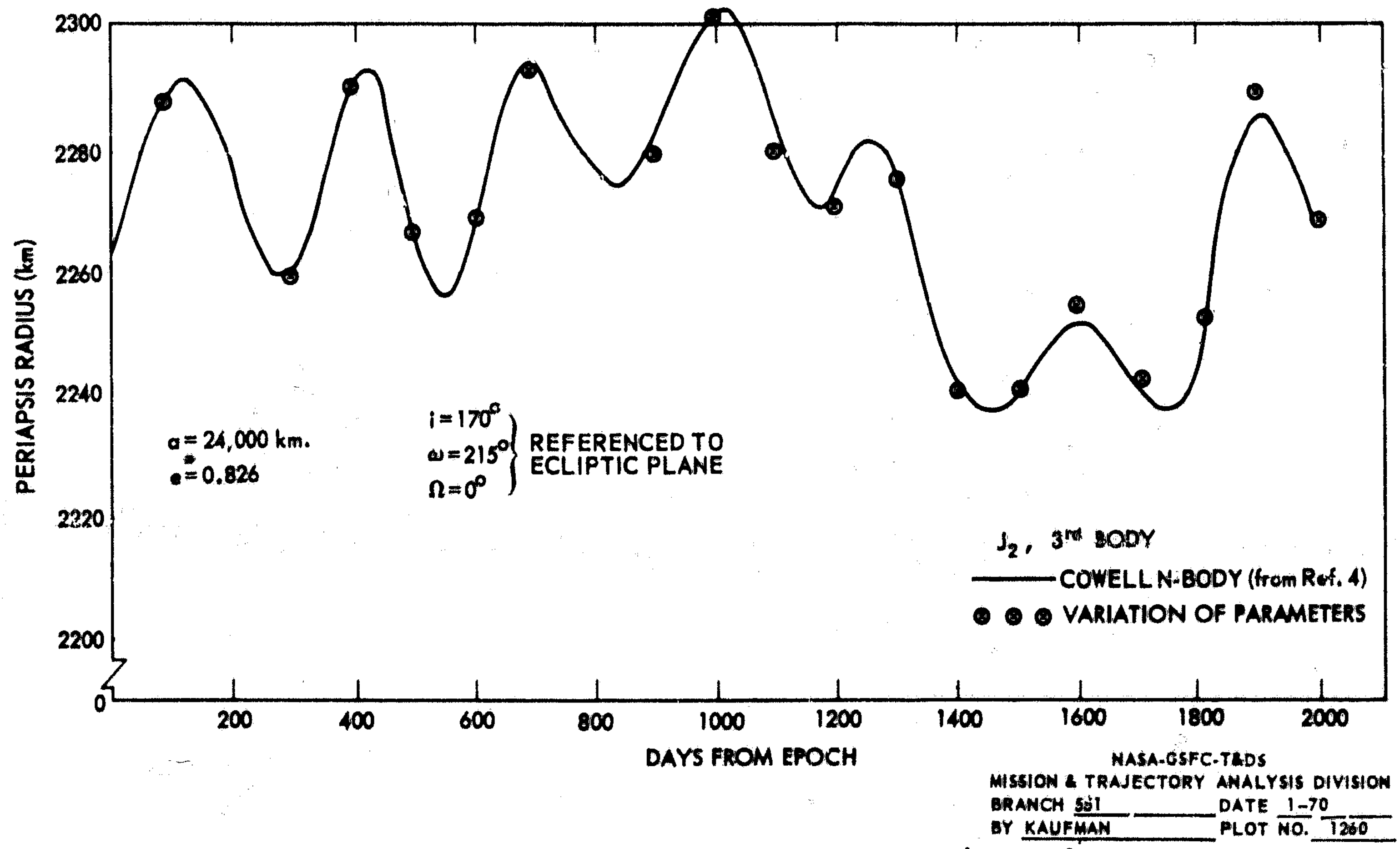

Figure 5—Mars: Periapsis vs time. Cowell and variational comparison.

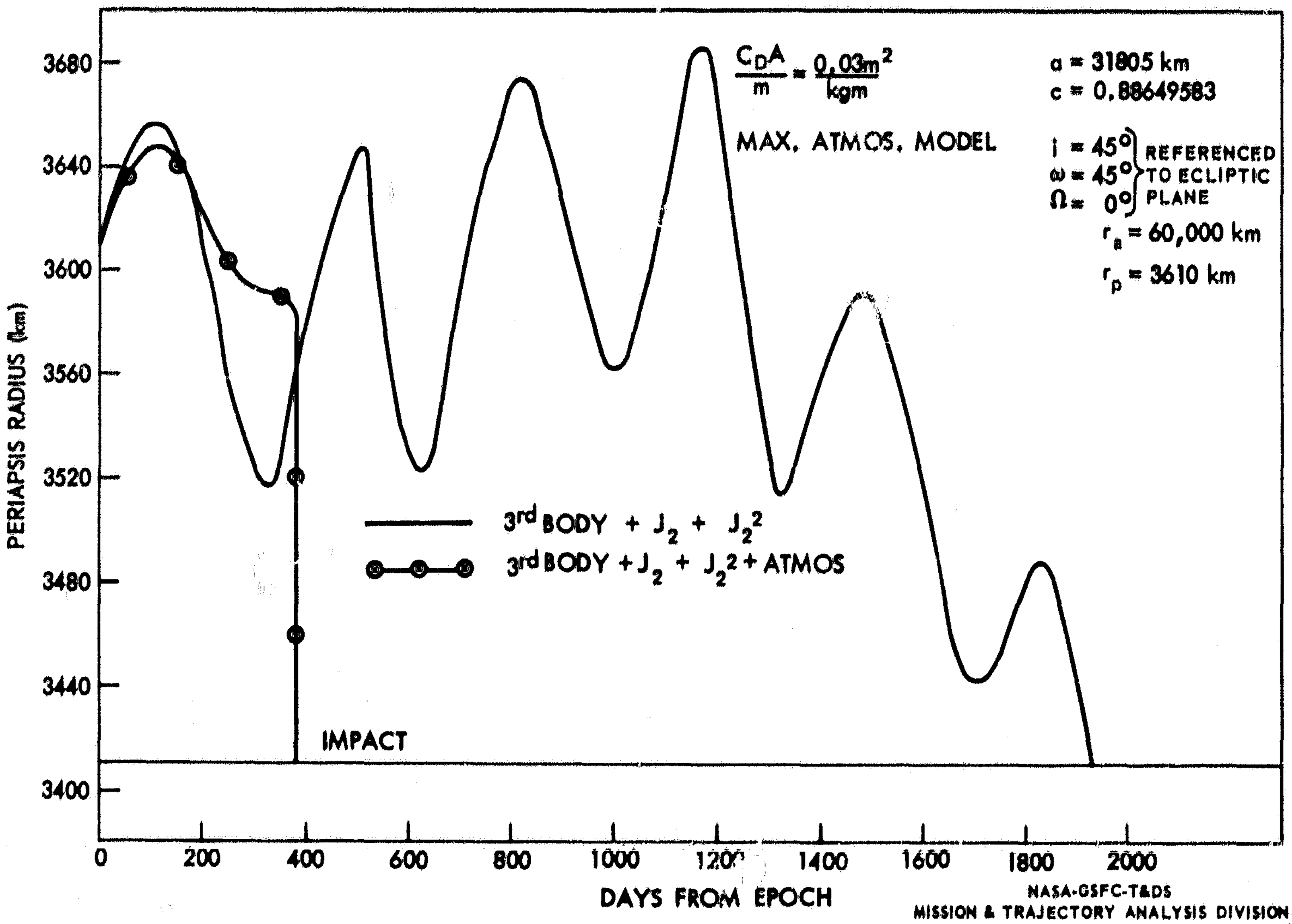

Figure 6—Mars: Periapsis vs time from epoch.

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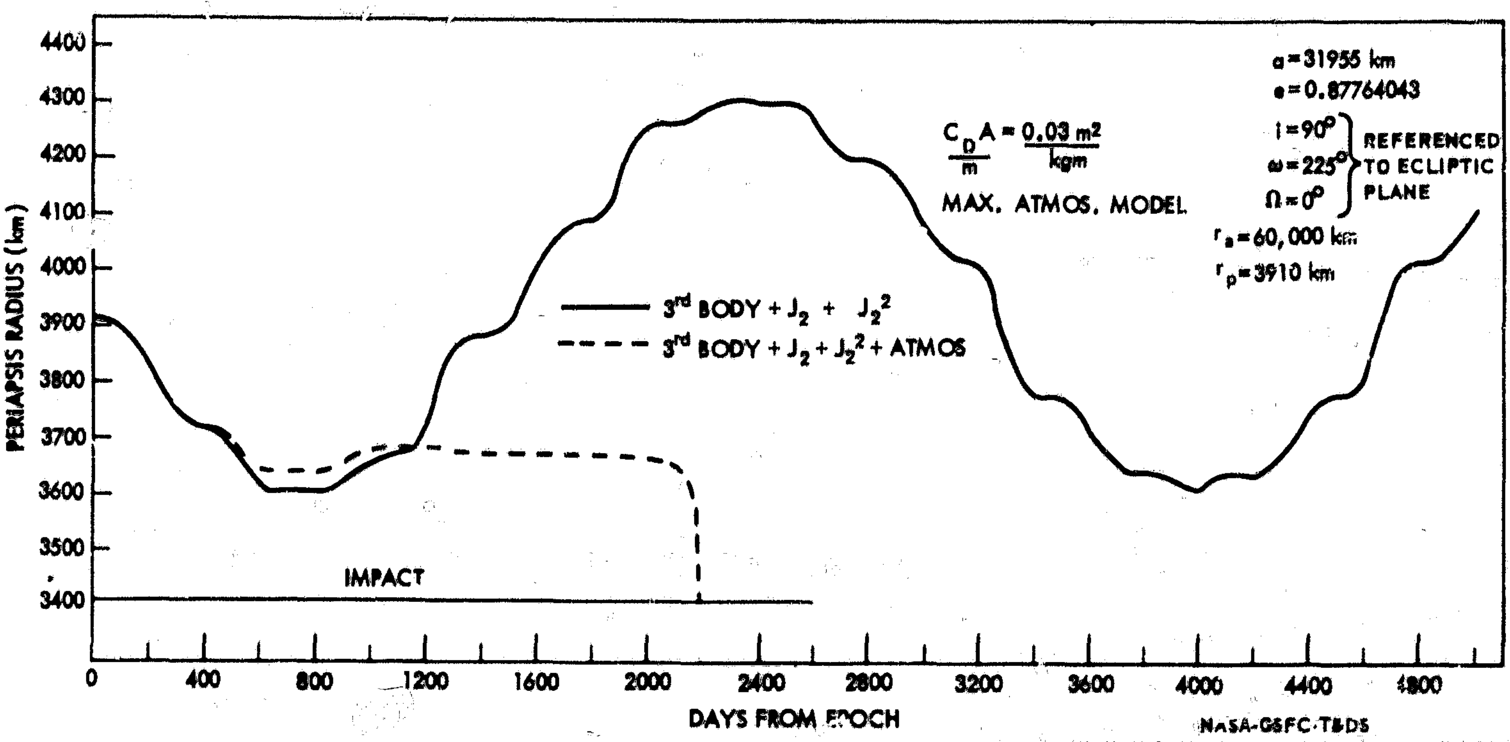

Figure 7—Mars: Periapsis vs time.

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APPENDIX A

Mars and Venus Transformations

Mars Transformation

To calculate Mars-centered position of the Sun at time t in mean Mars equator of date (t). Given the Earth-centered position in mean equinox and Earth equator of date.

1. Position of Mars' Pole at beginning of year t'.

$$\alpha_0' = 317.^\circ 793416667 + 0.^\circ 6520833 \times 10^{-2}\,(t' - 1950.0)$$

$$\delta_0' = 54.^\circ 6575 + 0.^\circ 35 \times 10^{-2}\,(t' - 1950.0)$$

Secular variation

$$T = \frac{(\text{Julian date})_t - (\text{Julian date})_{t'}}{365.25}$$

then

$$\alpha_0 = \alpha_0' - 0.^\circ 001013\ T$$

$$\delta_0 = \delta_0' - 0.^\circ 000631\ T$$

2. Determine Ω: The longitude of the ascending node of the Mars orbit along the ecliptic from the vernal equinox of the Earth.

$$T_e = \frac{(\text{Julian date})_t - 2415020.0}{36525.0}$$

$$\Omega = 48.^\circ 78644167 + 0.^\circ 77099167\ T_e - 0.^\circ 13888889 \times 10^{-5}\ T_e^2$$

3. Calculate i: The inclination of the Martian orbit plane to the ecliptic plane.

$$i = 1.^\circ 850333333 - 0.^\circ 675 \times 10^{-3}\ T_e + 0.12611111 \times 10^{-4}\ T_e^2$$

4. Calculate ϵ: The obliquity of the ecliptic.

$$\epsilon = 23.^\circ 45229444 - 0.^\circ 130125 \times 10^{-1}\ T_e - 0.^\circ 16388889 \times 10^{-5}\ T_e^2 + 0.^\circ 50277778 \times 10^{-6}\ T_e^3$$

5. Calculate Unit Vector $\vec{e}$ along pole of Mars

$$\vec{e} = (\cos\alpha_0 \cos\delta_0, \sin\alpha_0 \cos\delta_0, \sin\delta_0)$$

6. Calculate Unit Vector $\vec{O}$ perpendicular to orbit of Mars

$$\vec{O} = (\sin i \sin\Omega, -\sin i \cos\Omega \cos\epsilon - \cos i \sin\epsilon, -\sin i \cos\Omega \sin\epsilon + \cos i \cos\epsilon)$$

7. Calculate $\vec{O} \times \vec{e}$ unit vector to ascending node of equator of Mars on the orbit of Mars (autumnal equinox).

8. Calculate $\vec{EQ}$ unit vector to the ascending node of the equator of Mars on the equator of Earth.

$$\vec{EQ} = (-\sin\alpha_0, \cos\alpha_0, 0)$$

9. Calculate $\omega_{♂}$: The arc of the equator of Mars from its ascending node on the equator of Earth to its ascending node on the orbit of Mars.

$$\cos\omega_{♂} = \vec{EQ} \cdot (\vec{O} \times \vec{e})$$

$$\sin\omega_{♂} = |\vec{EQ} \times (\vec{O} \times \vec{e})|$$

10. Calculate $\Omega_{♂}$: Longitude of the ascending node of the equator of Mars on the equator of the Earth measured from Earth's vernal equinox.

$$\Omega_{♂} = \alpha_0 + \pi/2$$

11. Calculate $i_{♂}$: The inclination of the Martian equator with respect to the Earth's equator.

$$i_{♂} = \pi/2 - \delta_0$$

12. Calculate Rotation Matrix (P)

$$P_{11} = \cos\omega_{♂} \cos\Omega_{♂} - \sin\omega_{♂} \sin\Omega_{♂} \cos i_{♂}$$

$$P_{12} = \cos\omega_{♂} \sin\Omega_{♂} + \sin\omega_{♂} \cos\Omega_{♂} \cos i_{♂}$$

$$P_{13} = \sin\omega_{♂} \sin i_{♂}$$

$$P_{21} = -\sin\omega_{♂} \cos\Omega_{♂} - \cos\omega_{♂} \sin\Omega_{♂} \cos i_{♂}$$

$$P_{22} = -\sin\omega_♂ \sin\Omega_♂ + \cos\omega_♂ \cos\Omega_♂ \cos i_♂$$

$$P_{23} = \cos\omega_♂ \sin i_♂$$

$$P_{31} = \sin\Omega_♂ \sin i_♂$$

$$P_{32} = -\cos\Omega_♂ \sin i_♂$$

$$P_{33} = \cos i_♂ .$$

Let $\vec{r}_{\oplus ♂}$ be a vector from the Earth to Mars in mean Earth equator and equinox of date (t'). Let $\vec{r}_{\oplus p}$ be a vector from the Earth to the Sun in mean Earth equator and equinox of date. Then

$$\vec{r} = \vec{r}_{\oplus p} - \vec{r}_{\oplus ♂} .$$

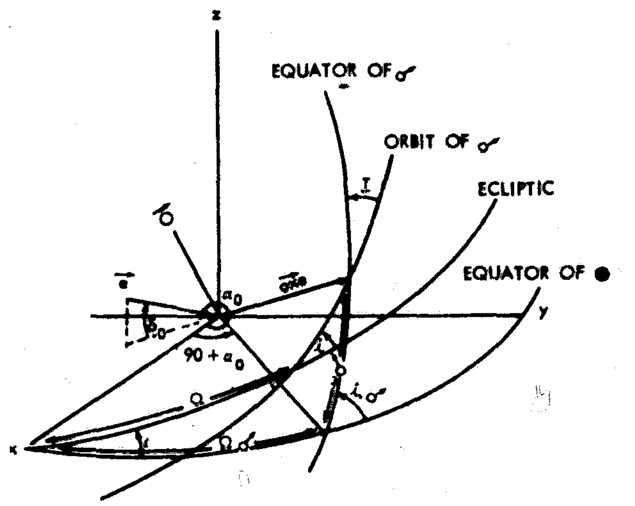
Let $\vec{r}_o$ be the position of the Sun in Mars mean equator and equinox of date. Then

$$\vec{r}_♂ = (P)\,\vec{r}$$

also

$$\vec{v}_♂ = (P)\,\vec{v} .$$

X axis is now towards ascending node of equator of Mars on orbit of Mars.



X♂ axis is toward ascending node of equator of Mars on orbit of Mars

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Figure A-1.

Venus Transformation (Taken from Reference 14).

Position of Pole

α = 98 - 0.0015551 (t - 1964.5) deg.

δ = - 69 - 0.0007748 (t - 1964.5) deg.

t - 1964.5 in tropical years

Δ = 180°075 (1964.5)

Let A be rotation matrix from mean Earth equator and equinox of 1952 to mean Earth equator and equinox of date. Then

$$\vec{X} = S_3 S_2 S_1 A \vec{X}_{1950.0}$$

where $\vec{X}$ is Venus equator and equinox of date

$$S_1 = \begin{pmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\delta_0 & \cos\delta_0 \\ 0 & -\cos\delta_0 & \sin\delta_0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} -\cos\Delta & -\sin\Delta & 0 \\ \sin\Delta & -\cos\Delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$