NASA TECHNICAL NOTE

NASA TN D-5718

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NASA TN D-5718

CONSERVATION LAWS OF LINEAR, HOMOGENEOUS SYSTEMS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1970

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TN D-5718		

4. Title and Subtitle
CONSERVATION LAWS OF LINEAR, HOMOGENEOUS SYSTEMS

5. Report Date
April 1970

6. Performing Organization Code

7. Author(s)
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8. Performing Organization Report No.
C-102

9. Performing Organization Name and Address
Electronics Research Center
Cambridge, Massachusetts 02139

10. Work Unit No.
129-02-21-02-25

11. Contract or Grant No.

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration

13. Type of Report and Period Covered
Technical Note

14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract A representation of the abstract operator for a linear, homogeneous system into a Lie group is given, which exhibits explicitly the metrics that define its conservation laws. The general nonuniform system with n degrees of freedom is shown to obey n^2 independent conservation laws, defined by metrics some or all of which depend on the space variable. While the constant metrics define groups, the remainder define groupoids under an appropriate binary operation; the intersections of groups and groupoids define groupoids. A Kronecker sum in the system operator is shown to belong to the corresponding Lie algebra.

The group representation and the corresponding Lie algebra provide an alternative means of describing the system, compared with the representation into the matricant group, with the system operator in a Lie algebra. They indicate clearly the manner in which a system is defined by its conservation laws. Uniform systems can be viewed as a limiting case, formally the same, but for which physical arguments have restricted attention to constant metrics.

17. Key Words
System Analysis
Linear Operators
Invariants
Conservation Laws
Lie Groups

18. Distribution Statement
Unclassified-Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	71	$3.00

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151

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SUMMARY

Conditions for a linear, uniform, homogeneous system to have nonsingular quadratic invariants, formed in the usual manner from constant metrics, are expressed in terms of properties of a Kronecker sum in the operator. Useful tests for the existence of such invariants are described. It is shown that system definition in terms of these invariants is not only restricted in its application but inconclusive. The interpretation of quadratic invariants, formed from metrics that depend on the space variable, as conservation laws is supported both formally and by physical arguments for the nonuniform system. A property of the Kronecker product and sum, that the Kronecker product of the matricants of two system operators is the matricant of the Kronecker sum of the operators, is then utilized to describe a useful procedure for obtaining the metrics, including the constant metrics. This property has application also in group-representation theory, and is used to obtain a representation of the abstract group onto a group of matrices that display the metrics defining the group.

The operator for a linear, lumped system is treated within the framework of the theory. Techniques developed for uniform distributed systems are found to be appropriate, and a group-theoretical interpretation is given.

I. INTRODUCTION

In this report we present a theory of the conservation laws, expressed as quadratic invariants, obeyed by a linear, homogeneous system. The theory applies to a distributed system that can be described by a matrix operator, over the field of complex numbers, relating a state vector and its derivative in a single space variable; and to a lumped-system operator relating a state vector at two generalized ports. The distributed-system operator is allowed to be a function of the space variable under a weak assumption of bounded differentiability, thereby admitting nonuniformity in the system due to any physical mechanism. The nonuniformity could be due, for example, to nonuniform boundaries in a wave-guiding system, material inhomogeneity, or nonuniform coupling in a single-frequency or parametric device. New insight into the quadratic

invariants of uniform systems is gained when these are considered in the context of the more general theory, and extended techniques for their study are developed. Operators for distributed and lumped systems are included in a unified theoretical treatment. We assume that the system has a finite number of degrees of freedom, but the fundamental results could be generalized.

Important results concerning the nonsingular quadratic invariants of linear, uniform, homogeneous systems, and of linear lumped systems, have been given by Pease (refs. 1, 2, and 3). A space is considered to be defined by a metric, which specifies how the length of a state vector is to be measured, and a conservation law to be determined by each metric for which the operator becomes rotational. Under certain conditions on the eigenvalues and structure of the operator, the system will obey a set of conservation laws determined by linearly independent nonsingular metrics, equal in number at least to the number of degrees of freedom of the system. The proof of this result provides a formal procedure for obtaining the metrics, a procedure that requires a determination of the eigenvalues, eigenvectors, and generalized eigenvectors of the operator. The metrics for which the operator becomes rotational can also be obtained by writing out and solving the governing equations. A set of linear, homogeneous equations is obtained, where the coefficient matrix is of squared order compared with the operator and the metrics. Either approach can be tedious if the operator matrix is not of small order.

Conversely, given two metrics of suitable form, a range of distributed-system operators that become rotational for either of these metrics can be determined. This range will in general be narrowed if more metrics are specified. The distributed system can be represented by a matricant — obtained by integrating the differential equations describing the system — which relates the state vector as a function of position to a boundary value. The specified metrics define a group for the abstract operator, the matricants determined by the metrics are a representation of the abstract group onto a Lie group, and the system operators are members of a corresponding Lie algebra. From this point of view, systems are defined by their conservation laws. We shall discover, however, that in general an operator is not determined uniquely by specifying all of the metrics for which it is rotational; in fact, the simplest class of operators for conservative systems will be determined only within a space of dimensionality equal to the number of degrees of freedom of the system. Most operators do not become rotational for any metric.

The coefficient matrix, obtained by writing out the equations governing the metrics, while of squared order, significantly takes the form of a Kronecker sum. This has some practical as well as important theoretical consequences. First of all, it will permit us to develop a formal parallel to the theory developed

for the operator, in terms of properties of the coefficient matrix. Existence criteria for nonsingular solutions, as well as other information about the operator and the metrics, are expressed in terms of more accessible properties of the coefficient matrix; and equivalent procedures for obtaining this information for an operator in the restricted class considered can be compared with those available for testing the operator directly.

With the theory reformulated in this way, we shall be able to complement the previous results by introducing metrics that vary with the space variable, and to confirm that these are required for a definitive theory of this and the larger class of operators. Quadratic invariants formed from such metrics, in general, can meaningfully be interpreted as conservation laws, and uniform systems can be considered a kind of "singularity" for which this interpretation breaks down. These and the constant metrics can be obtained in a way that indicates that they, as well as the matricants, have overt group-theoretical properties. Not only is a useful procedure for obtaining the metrics revealed, but a representation of the abstract group explicitly in terms of the metrics is indicated. The Kronecker sum, formed from the distributed-system operator, will then be recognized as the operator in the corresponding Lie algebra. Such a representation will be seen to be what is required for definitive system definition for the uniform system as well, which can be viewed as a limiting case when interpreting the invariants as conservation laws.

We shall also discuss briefly another class of metrics that depend on the space variable, and that determine ranges of nonuniform systems in a manner very similar to that in which the constant metrics determine ranges of uniform and nonuniform systems. While there is some basis for interpreting quadratic forms obtained from these as conservation laws, and while similar techniques for synthesis from these metrics can be employed, they have the same limitations for system definition as do the constant metrics.

Operators for lumped systems, giving as they do "black box" descriptions, are less definitive in circumscribing the kinds of systems they might describe. Their invariants have been obtained in a manner parallel to that for distributed systems, but this as well as the synthesis problem can in fact be reduced to the problem for uniform distributed systems. From the point of view of group-representation theory, this reduction in the range of systems that need be considered can be viewed as resulting from a homomorphic mapping of the group of abstract operators for the more general nonuniform system onto the full linear group.

The material presented in this paper, like the results obtained previously, has broad applicability. So that it may be most generally useful, the presentation is abstract. No attempt

is made to discuss sophisticated physical applications; but guidelines for the physical interpretation of results are indicated, and sufficient illustrative material is included to make the theoretical points clear.

II. BASIC CONCEPTS

In this section we summarize, for review and reference, some results concerning the quadratic invariants of linear, uniform, homogeneous systems. The analytical techniques by which these results have been obtained are outlined, for comparison with the approach in Section III, and several topics are pursued in more depth. Finally, we call attention to some tests for such invariants that are equivalent to applying directly the existence criteria that emerge in the formal development of the theory.

The converse problem to the central problem discussed in this section is perhaps of even more fundamental importance. We review current concepts in dealing with this problem in Sections IV and VII, at more logical points in the development of this paper.

The System

We assume first a linear, distributed system having a finite number of degrees of freedom, such as can be described by a first-order differential equation of the form

$$\frac{d\underset{\sim}{x}}{dz} = -j\underset{\sim}{R}\underset{\sim}{x} \tag{1}$$

Here $\underset{\sim}{x}$ is a column matrix, an n-dimensional state vector describing the state of the system as a function of position, and $\underset{\sim}{R}$ is an n×n matrix, the system operator. The components of $\underset{\sim}{x}$ are considered to be functions of the single space variable z, and the time dependence has been removed. Such a system is called homogeneous, for Eq. (1) is homogeneous in $\underset{\sim}{x}$.

As a simple example, on a voltage-current basis,

$$\underset{\sim}{x} = \begin{pmatrix} V \\ I \end{pmatrix} \tag{2}$$

the system operator for a uniform transmission line of characteristic impedance Z_0, having an associated propagation constant β, is

$$\underset{\sim}{R} = \begin{pmatrix} 0 & \beta Z_0 \\ \frac{\beta}{Z_0} & 0 \end{pmatrix} \tag{3}$$

The form of Eq. (1) is not as restrictive as may at first appear. It is not restricted altogether to the description of systems operating at a single frequency. Furthermore, order can be traded for dimensionality; and accordingly all systems of linear differential equations, of finite order and dimensionality, can be reduced to this form.

Metrics and Quadratic Forms

Before discussing the conservation laws governing such a system, we first introduce the concept of a quadratic form, or generalized inner product. This is a scalar quantity defined by

$$s(\underset{\sim}{K}) = \underset{\sim}{x}^{\dagger} \underset{\sim}{K} \underset{\sim}{x} \tag{4}$$

where the superscript † denotes the complex conjugate of the transposed vector, and where $\underset{\sim}{K}$ is a constant n×n matrix.

In order to suggest the appropriateness of such forms for describing physical properties of a system, we note that if the components of $\underset{\sim}{x}$ are voltage and current, as in Eq. (2), then with

$$\underset{\sim}{K} = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

the power may be written

$$P = \underset{\sim}{x}^{\dagger} \underset{\sim}{K} \underset{\sim}{x} = (V^* I^*) \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} = \frac{1}{4} (V^* I + V I^*) \tag{6}$$

We postpone further consideration of the physical significance of these forms — except to point out that $\underset{\sim}{K}$ is taken to be constant so that the interpretation of $s(\underset{\sim}{K})$ will not depend on position.

Equation (4) is a generalization of the unitary inner product $\underset{\sim}{x}^{\dagger}\underset{\sim}{x}$, for which $\underset{\sim}{K}$ is taken as the identity matrix $\underset{\sim}{I}$, and provides a more general definition of the square of the length of the vector $\underset{\sim}{x}$. The matrix $\underset{\sim}{K}$ is called the metric of the space in which $\underset{\sim}{x}$ is said to be embedded, and as such can be considered to define the space.

Appropriate Metrics and Invariant Forms

Of particular concern to us here will be those metrics $\underset{\sim}{K}$ for which $s(\underset{\sim}{K})$ is invariant under the operation $\underset{\sim}{R}$. In other words, for a given $\underset{\sim}{R}$, we are interested in those spaces, if any, in which $\underset{\sim}{x}$ can be embedded, such that its length will not vary with z. Associated with each such $\underset{\sim}{K}$ then is a conservation law, which predicates a quadratic form that remains constant as $\underset{\sim}{x}$ varies in accordance with Eq. (1). We shall refer to these $\underset{\sim}{K}$ as the metrics appropriate to $\underset{\sim}{R}$.

For $s(\underset{\sim}{K})$ to be invariant under the operation $\underset{\sim}{R}$, it is required, by Eqs. (1) and (4), that

$$\frac{ds}{dz} = \frac{d\underset{\sim}{x}^{\dagger}}{dz}\underset{\sim}{K}\underset{\sim}{x} + \underset{\sim}{x}^{\dagger}\underset{\sim}{K}\frac{d\underset{\sim}{x}}{dz} = j\underset{\sim}{x}^{\dagger}(\underset{\sim}{R}^{\dagger}\underset{\sim}{K}-\underset{\sim}{K}\underset{\sim}{R})\ \underset{\sim}{x} = 0 \qquad (7)$$

where $\underset{\sim}{R}^{\dagger}$ is the transpose conjugate of $\underset{\sim}{R}$. Since Eq. (7) must hold for any $\underset{\sim}{x}$, for $\underset{\sim}{K}$ to be appropriate to $\underset{\sim}{R}$ we must have

$$\underset{\sim}{R}^{\dagger}\underset{\sim}{K} = \underset{\sim}{K}\underset{\sim}{R} \qquad (8)$$

If $\underset{\sim}{K}$ is a nonsingular solution to Eq. (8), $\underset{\sim}{R}$ is said to be $\underset{\sim}{K}$-Hermitian.

Thus far a restriction has not in fact been imposed on the elements of $\underset{\sim}{R}$ that they may not be functions of z. On the assumption that the system is uniform, however, and $\underset{\sim}{R}$ is thus constant, results have been derived concerning the existence of appropriate, nonsingular $\underset{\sim}{K}$. Singular $\underset{\sim}{K}$ have not been considered important to the theory.

Necessary Conditions for Appropriate, Nonsingular $\underset{\sim}{K}$

For Eq. (8) to hold for some nonsingular $\underset{\sim}{K}$, $\underset{\sim}{R}^{\dagger}$ and $\underset{\sim}{R}$ must be related by a similarity transformation,

$$\underset{\sim}{R}^{\dagger} = \underset{\sim}{K}\underset{\sim}{R}\underset{\sim}{K}^{-1} \qquad (9)$$

This imposes conditions on the eigenvalues and structure of $\underset{\sim}{R}$.

The n eigenvalues of $\underset{\sim}{R}$ may be distinct, or some may be repeated. Associated with each distinct eigenvalue β_i is a linearly independent eigenvector $\underset{\sim}{x}^{(i,1)}$, satisfying

$$\underset{\sim}{R}\underset{\sim}{x}^{(i,1)} = \beta_i \underset{\sim}{x}^{(i,1)} \tag{10}$$

Associated with an eigenvalue repeated m_i times are m_i linearly independent eigenvectors and generalized eigenvectors, forming one or more chains. We shall use different subscripts to index repeated eigenvalues when these are associated with independent eigenvectors or separate chains, but we shall denote the number of eigenvalues equal to any β_i by m_i (i.e., if $\beta_i = \beta_j$, $m_i = m_j$). The chain of length ℓ_i, $1 \le \ell_i \le m_i$, associated with the eigenvalue β_i, consists of an eigenvector satisfying Eq. (10), and if $\ell_i > 1$, of generalized eigenvectors $\underset{\sim}{x}^{(i,k)}$, $k = 2,3,\ldots,\ell_i$, satisfying

$$\underset{\sim}{R}\underset{\sim}{x}^{(i,k)} = \beta_i \underset{\sim}{x}^{(i,k)} + \underset{\sim}{x}^{(i,k-1)} \tag{11}$$

where $\underset{\sim}{x}^{(i,k)}$ is said to be of rank k. A matrix is said to be semisimple if it has a complete set of n linearly independent eigenvectors — i.e., if the chains are all of length 1. It is necessarily semisimple if its eigenvalues are distinct.

Now it is readily shown by Eqs. (10) and (11) that if two matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are related by a similarity transformation, $\underset{\sim}{A} = \underset{\sim}{S}^{-1}\underset{\sim}{B}\underset{\sim}{S}$, and if $\underset{\sim}{A}$ has a chain $\underset{\sim}{x}^{(i,k)}$, $k = 1,2,\ldots,\ell_i$, with eigenvalue β_i, then $\underset{\sim}{B}$ has a chain $\underset{\sim}{S}\underset{\sim}{x}^{(i,k)}$, $k = 1,2,\ldots\ell_i$, with eigenvalue β_i.

Thus necessary conditions for the existence of a nonsingular solution $\underset{\sim}{K}$ to Eq. (9) are that $\underset{\sim}{R}$ and $\underset{\sim}{R}^\dagger$ have the same eigenvalues and structure. But the eigenvalues of $\underset{\sim}{R}^\dagger$ are the complex conjugates of the eigenvalues of $\underset{\sim}{R}$. Also, if $\underset{\sim}{R}$ has a chain $\underset{\sim}{x}^{(i,k)}$, $k = 1,2,\ldots,\ell_i$, with eigenvalue β_i, $\underset{\sim}{R}^\dagger$ will have a chain $\underset{\sim}{w}^{(i,k)}$, $k = 1,2,\ldots,\ell_i$, with eigenvalue β_i^* — where the $(\underset{\sim}{w}^{(i,k)})^\dagger$ are the reciprocal eigenvectors and generalized eigenvectors of $\underset{\sim}{R}$ with eigenvalue β_i. The conditions then become that the eigenvalues β_i of $\underset{\sim}{R}$ either be real,

$$\beta_i = \beta_i^* \,. \tag{12}$$

or occur in conjugate pairs,

$$\beta_i = \beta_j^* \tag{13}$$

such that chains of equal length are associated with β_i and β_j.

Sufficiency of Conditions, Degeneracy, and Formal Solution — $\underset{\sim}{R}$ Semisimple

That these conditions on the eigenvalues and structure of $\underset{\sim}{R}$ are also sufficient, and that there will then be at least n linearly independent, nonsingular $\underset{\sim}{K}$ appropriate to $\underset{\sim}{R}$, can be shown by exhibiting the appropriate $\underset{\sim}{K}$. In so doing, a formal procedure for their determination is described.

On the change of basis

$$\underset{\sim}{x} = \underset{\sim}{S}\underset{\sim}{x}' \tag{14}$$

where $\underset{\sim}{S}$ is a constant, nonsingular n×n matrix, Eq. (1) becomes

$$\frac{d\underset{\sim}{x}'}{dz} = -j\underset{\sim}{R}'\underset{\sim}{x}' \tag{15}$$

where

$$\underset{\sim}{R}' = \underset{\sim}{S}^{-1}\underset{\sim}{R}\underset{\sim}{S} \tag{16}$$

The quadratic form, Eq. (4), becomes

$$s'(\underset{\sim}{K}') = \underset{\sim}{x}'^{\dagger}\underset{\sim}{K}'\underset{\sim}{x}' \tag{17}$$

where the metric on the new basis is

$$\underset{\sim}{K}' = \underset{\sim}{S}^{\dagger}\underset{\sim}{K}\underset{\sim}{S} \tag{18}$$

Since $\underset{\sim}{S}$ is constant, $\underset{\sim}{K}'$ is constant. Accordingly, $s'(\underset{\sim}{K}')$ will be invariant under the operation $\underset{\sim}{R}'$ if

$$\underset{\sim}{R}'^{\dagger}\underset{\sim}{K}' = \underset{\sim}{K}'\underset{\sim}{R}' \tag{19}$$

If so, $s(\underset{\sim}{K})$ will be invariant under the operation $\underset{\sim}{R}$, since $s(\underset{\sim}{K})$ is independent of the basis.

In particular, $\underset{\sim}{S}$ can be chosen to be a modal matrix of $\underset{\sim}{R}$ — a nonsingular matrix, the columns of which consist of the n linearly independent eigenvectors and generalized eigenvectors of $\underset{\sim}{R}$

(the complete set of eigenvectors, if $\underset{\sim}{R}$ is semisimple), with each chain arranged in the order of increasing rank. Since $\underset{\sim}{R}$ is constant, $\underset{\sim}{S}$ can be chosen constant.

Assume first that $\underset{\sim}{R}$ is semisimple. Then under the transformation of Eq. (16), if $\underset{\sim}{S}$ is a modal matrix, $\underset{\sim}{R}$ will be diagonalized on the new basis. Thus

$$\underset{\sim}{R}' = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \beta_n \end{pmatrix} \tag{20}$$

where β_i, appearing in the ii position, is the eigenvalue that corresponds to the eigenvector $\underset{\sim}{x}^{(i,1)}$ appearing as the i'th column of $\underset{\sim}{S}$. If the β_i are real or occur in conjugate pairs, the condition on the structure is satisfied automatically for $\underset{\sim}{R}$ semisimple, and $\underset{\sim}{R}'^{\dagger}$ will be obtained from $\underset{\sim}{R}'$ by interchanging the positions of conjugate pairs of eigenvalues. This interchange of positions can be effected by a similarity transformation,

$$\underset{\sim}{R}'^{\dagger} = \underset{\sim}{P}^{-1}\underset{\sim}{R}'\underset{\sim}{P} \tag{21}$$

where $\underset{\sim}{P} = \underset{\sim}{P}^{-1}$ is a permutation matrix having a 1 in the ii position if $\beta_i = \beta_i^*$, or in the ij and ji positions if β_i is complex and $\beta_i = \beta_j^*$. Then Eq. (19) becomes

$$\underset{\sim}{R}'(\underset{\sim}{P}\underset{\sim}{K}') = (\underset{\sim}{P}\underset{\sim}{K}')\underset{\sim}{R}' \tag{22}$$

Any diagonal matrix will commute with $\underset{\sim}{R}'$. Accordingly, there will be at least n linearly independent metrics $\underset{\sim}{K}'$ appropriate to $\underset{\sim}{R}'$; namely,

$$\underset{\sim}{K}' = \underset{\sim}{P}^{-1}\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{pmatrix} \tag{23}$$

where $a_1, a_2, \ldots, a_n$ are arbitrary constants. Furthermore, if $\underset{\sim}{R}$ has

repeated eigenvalues, then for each $\beta_i = \beta_j$ two additional arbitrary constants are introduced, in the ij and ji positions of $\underset{\sim}{P}\underset{\sim}{K}'$. The p constants, $p \geq n$, can clearly be chosen to obtain p linearly independent, nonsingular $\underset{\sim}{K}'$.

Finally, a set of p linearly independent, nonsingular $\underset{\sim}{K}$ appropriate to $\underset{\sim}{R}$ is obtained by transforming back to the original basis. Inverting Eq. (18), we obtain for each $\underset{\sim}{K}'$,

$$\underset{\sim}{K} = (\underset{\sim}{S}^{\dagger})^{-1}\, \underset{\sim}{K}'\underset{\sim}{S}^{-1} \tag{24}$$

Since $\underset{\sim}{S}$ and $\underset{\sim}{K}'$ are nonsingular, $\underset{\sim}{K}$ is also; and the linear independence of the $\underset{\sim}{K}$ follows from that of the $\underset{\sim}{K}'$.

$\underset{\sim}{R}$ Not Semisimple

More generally, if $\underset{\sim}{S}$ is a modal matrix of $\underset{\sim}{R}$, $\underset{\sim}{R}$ is transformed by Eq. (16) to a quasidiagonal matrix in the Jordan canonical form,

$$\underset{\sim}{R}' = \text{quasidiag}\,(\underset{\sim}{J}_1, \underset{\sim}{J}_2, \ldots) = \begin{pmatrix} \underset{\sim}{J}_1 & \underset{\sim}{0} & \cdots \\ \underset{\sim}{0} & \underset{\sim}{J}_2 & \ddots \\ \vdots & \ddots & \end{pmatrix} \tag{25}$$

a partitioned matrix in which, corresponding to each chain of length ℓ_i, associated with the eigenvalue β_i, there appears an $\ell_i \times \ell_i$ Jordan block,

$$\underset{\sim}{J}_i = \begin{pmatrix} \beta_i & 1 & 0 & \cdots & 0 \\ 0 & \beta_i & 1 & & \vdots \\ 0 & 0 & \beta_i & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & & 0 & \beta_i \end{pmatrix} \tag{26}$$

positioned in the same ℓ_i columns of $\underset{\sim}{R}'$ as the chain is positioned in $\underset{\sim}{S}$. By assumption, either β_i is real, in which case

$$\underset{\sim}{J}_i = \underset{\sim}{J}_i^* \tag{27}$$

or there exists an $\ell_i \times \ell_i$ Jordan block of $\underset{\sim}{R}'$ with complex eigenvalue $\beta_j = \beta_i^*$, such that

$$\underset{\sim}{J}_j = \underset{\sim}{J}_i^* \tag{28}$$

We define a permutation matrix $\underset{\sim}{P} = \underset{\sim}{P}^{-1}$ partitioned in the same manner as $\underset{\sim}{R}'$. Indexing the positions of the component submatrices by rows and columns of blocks, we put an $\ell_i \times \ell_i$ identity block $\underset{\sim}{I}_{\ell_i}$ in the ii position of $\underset{\sim}{P}$ (the position of $\underset{\sim}{J}_i$ in $\underset{\sim}{R}'$) if $\underset{\sim}{J}_i = \underset{\sim}{J}_i^*$, or in the ij and ji positions if β_i is complex and $\underset{\sim}{J}_i = \underset{\sim}{J}_j^*$. Then

$$\underset{\sim}{P}^{-1}\underset{\sim}{R}'\underset{\sim}{P} = \underset{\sim}{R}'^* = \text{quasidiag } (\underset{\sim}{J}_1^*, \underset{\sim}{J}_2^*, \ldots) \tag{29}$$

since the positions in $\underset{\sim}{R}'$ of each $\underset{\sim}{J}_i$ and $\underset{\sim}{J}_j$ satisfying Eq. (28) are interchanged by this permutation.

Let $\underset{\sim}{Q}_i$ be a modal matrix of $\underset{\sim}{J}_i^\dagger$. Then

$$\underset{\sim}{J}_i^\dagger \underset{\sim}{Q}_i = \underset{\sim}{Q}_i \underset{\sim}{J}_i^* \tag{30}$$

since $\underset{\sim}{J}_i^*$ is the Jordan canonical form of $\underset{\sim}{J}_i^\dagger$. Defining

$$\underset{\sim}{Q} = \text{quasidiag } (\underset{\sim}{Q}_1, \underset{\sim}{Q}_2, \ldots) \tag{31}$$

it follows that

$$\underset{\sim}{R}'^\dagger \underset{\sim}{Q} = \underset{\sim}{Q}\underset{\sim}{R}'^* = \underset{\sim}{Q}\underset{\sim}{P}^{-1}\underset{\sim}{R}'\underset{\sim}{P}$$

or

$$\underset{\sim}{R}'(\underset{\sim}{Q}\underset{\sim}{P}^{-1}) = (\underset{\sim}{Q}\underset{\sim}{P}^{-1})\underset{\sim}{R}' \tag{32}$$

Thus

$$\underset{\sim}{K}' = \underset{\sim}{Q}\underset{\sim}{P}^{-1} \tag{33}$$

is a solution to Eq. (19).

Each of the ℓ_i eigenvectors and generalized eigenvectors of $\underset{\sim}{J}_i^\dagger$, which make up the columns of $\underset{\sim}{Q}_i$, is specified only within an

arbitrary constant. The expressions for $\underset{\sim}{Q}$ and $\underset{\sim}{K}'$, then, each contain n arbitrary constants. Equation (33) reduces to Eq.(23) for $\underset{\sim}{R}$ semisimple, since postmultiplying instead of premultiplying by $\underset{\sim}{P}^{-1}$ simply permutes the arbitrary constants.

Again, if $\underset{\sim}{R}'$ has more than one Jordan block with the same eigenvalue, there will be further degeneracy in the $\underset{\sim}{K}'$. The blocks in $\underset{\sim}{Q}$ off the quasidiagonal are not in general square, however, and the manner in which additional arbitrary constants are introduced becomes more complicated.

As before, the p constants, $p \geq n$, can be chosen to obtain p linearly independent, nonsingular $\underset{\sim}{K}'$; and these can be transformed back to the original basis by Eq. (24).

Lumped Systems

Similar results are obtained for a lumped linear system, described by the equation

$$\underset{\sim}{x}_2 = \underset{\sim}{M}\underset{\sim}{x}_1 \tag{34}$$

where $\underset{\sim}{M}$ is an n×n nonsingular matrix relating a state vector at two generalized ports of the system. We seek metrics $\underset{\sim}{K}$ such that

$$\underset{\sim}{x}_2^{\dagger}\underset{\sim}{K}\underset{\sim}{x}_2 = \underset{\sim}{x}_1^{\dagger}\underset{\sim}{K}\underset{\sim}{x}_1 \tag{35}$$

which requires that

$$\underset{\sim}{M}^{\dagger}\underset{\sim}{K}\underset{\sim}{M} = \underset{\sim}{K} \tag{36}$$

If $\underset{\sim}{K}$ is a nonsingular solution to Eq. (36), $\underset{\sim}{M}$ is said to be $\underset{\sim}{K}$-Unitary.

Necessary conditions for $\underset{\sim}{M}$ to be $\underset{\sim}{K}$-Unitary are that the eigenvalues λ_i of $\underset{\sim}{M}$ either be of unit magnitude,

$$\lambda_i \lambda_i^* = 1 \tag{37}$$

or occur in pairs satisfying the cross-conjugacy relation

$$\lambda_i \lambda_j^* = 1 \tag{38}$$

such that chains of equal length are associated with λ_i and λ_j. Again these conditions are sufficient to ensure the existence of at least n linearly independent $\underset{\sim}{K}$ for which $\underset{\sim}{M}$ is $\underset{\sim}{K}$-Unitary.

The solutions to Eq. (36) are obtained as before by first transforming to $\underset{\sim}{M}'$, the Jordan canonical form of $\underset{\sim}{M}$. Then by a similar procedure the $\underset{\sim}{K}'$ appropriate to $\underset{\sim}{M}'$ are obtained in the form of Eq. (33). The positioning of the blocks in the permutation matrix depends in the same manner on the self-conjugacy or cross-conjugacy of the eigenvalues in the associated Jordan blocks. The $\underset{\sim}{Q}_i$ are, in this case, modal matrices of the $(\underset{\sim}{J}_i^\dagger)^{-1}$.

These results for a lumped system can be applied directly to the analysis of a uniform distributed system also, if Eq. (1) is first integrated. If $\underset{\sim}{R}$ is n×n and constant, Eq. (1) will have n linearly independent solutions for $\underset{\sim}{x}$. (The elements of $\underset{\sim}{R}$, in fact, need only be differentiable with bounded derivatives in the region of interest — this is known as the Lipschitz condition.) If any set of linearly independent solutions is arranged as the columns of an n×n matrix $\underset{\sim}{M}(z)$, then $\underset{\sim}{M}(z)$ satisfies

$$\frac{d\underset{\sim}{M}}{dz} = -j\underset{\sim}{R}\underset{\sim}{M} \tag{39}$$

The solutions to Eq. (39) are known as integral matrices of $\underset{\sim}{R}$. Since the columns of $\underset{\sim}{M}(z)$ are linearly independent, we can form its inverse at any value z_1 of z. Thus we can define

$$\underset{\sim}{M}(z,z_1) = \underset{\sim}{M}(z)\underset{\sim}{M}^{-1}(z_1) \tag{40}$$

As the postmultiplication of $\underset{\sim}{M}(z)$ is by a constant nonsingular matrix, $\underset{\sim}{M}(z,z_1)$ is still a nonsingular solution of Eq. (39). It is a particular integral matrix having the property

$$\underset{\sim}{M}(z_1,z_1) = \underset{\sim}{I} \tag{41}$$

$\underset{\sim}{M}(z,z_1)$ is known as the matricant of the distributed operator $\underset{\sim}{R}$. In view of Eqs. (39) and (41),

$$\underset{\sim}{x}(z) = \underset{\sim}{M}(z,z_1)\underset{\sim}{x}(z_1) \tag{42}$$

is a solution to Eq. (1), expressing $\underset{\sim}{x}$ in terms of a boundary value. Equation (42) provides a lumped representation of the section of the distributed system between z_1 and z.

By comparing the eigenvalues and structure of $\underset{\sim}{R}$ and $\underset{\sim}{M}(z,z_1)$, for $\underset{\sim}{R}$ constant, we could demonstrate the equivalence of the conditions for $\underset{\sim}{R}$ to be $\underset{\sim}{K}$-Hermetian and for $\underset{\sim}{M}(z,z_1)$ to be $\underset{\sim}{K}$-Unitary, with respect to the same number $p \geq n$ of linearly independent $\underset{\sim}{K}$ (except that the degeneracy of $\underset{\sim}{M}$ may be greater at discrete values of z). If $\underset{\sim}{R}$ is semisimple, $\underset{\sim}{M}(z,z_1)$ will be also, with the same eigenvectors; the eigenvalues β_i of $\underset{\sim}{R}$ are related to the eigenvalues λ_i of $\underset{\sim}{M}(z,z_1)$ by

$$\lambda_i = e^{-j\beta_i(z-z_1)} \tag{43}$$

The equivalence of Eqs. (12) and (13), the conditions on the β_i, and Eqs. (37) and (38), the conditions on the λ_i, is clear, and only at discrete values of z can $\lambda_i=\lambda_j$ if $\beta_i \neq \beta_j$. If $\underset{\sim}{R}$ is not semisimple, the generalized eigenvectors of $\underset{\sim}{R}$ and $\underset{\sim}{M}(z,z_1)$ will not be the same — in fact, the generalized eigenvectors of $\underset{\sim}{M}(z,z_1)$ will depend on z; but the chain of $\underset{\sim}{R}$ with eigenvalue β_i will be reflected into a chain of $\underset{\sim}{M}(z,z_1)$ with eigenvalue $e^{-j\beta_i(z-z_1)}$.

We do not need to compare details of the theory to see that we will obtain in fact the same appropriate metrics $\underset{\sim}{K}$ by either approach. For let $\underset{\sim}{K}$ be a metric appropriate to $\underset{\sim}{M}(z,z_1)$, satisfying Eq. (36) for all z. Differentiating, we obtain

$$\frac{d}{dz}\left(\underset{\sim}{M}^\dagger \underset{\sim}{K} \underset{\sim}{M}\right) = \frac{d\underset{\sim}{M}^\dagger}{dz}\underset{\sim}{K}\underset{\sim}{M} + \underset{\sim}{M}^\dagger \underset{\sim}{K}\frac{d\underset{\sim}{M}}{dz} = \underset{\sim}{0} \tag{44}$$

Substitution of Eq. (39) and its transpose conjugate gives

$$j\underset{\sim}{M}^\dagger\left(\underset{\sim}{R}^\dagger\underset{\sim}{K} - \underset{\sim}{K}\underset{\sim}{R}\right)\underset{\sim}{M} = \underset{\sim}{0} \tag{45}$$

Since $\underset{\sim}{M}$ is nonsingular, we can premultiply by $-j(\underset{\sim}{M}^\dagger)^{-1}$ and postmultiply by $\underset{\sim}{M}^{-1}$, yielding Eq. (8), which is the condition for $\underset{\sim}{K}$ to be appropriate to $\underset{\sim}{R}$. The steps are also reversible, except that Eq. (44) requires only that $\underset{\sim}{M}^\dagger\underset{\sim}{K}\underset{\sim}{M}$ be constant. But since we can then evaluate it at any value of z, by Eq. (41) this constant matrix must be $\underset{\sim}{K}$, which proves the converse.

We note for future reference that the preceding result applies as well if $\underset{\sim}{R}$ is not constant, for any constant solutions to Eq. (8), or to Eq. (36) if $\underset{\sim}{M}$ is the matricant.

Thus any linear, distributed system, described by a system operator $\underset{\sim}{R}$ satisfying the Lipschitz condition, has an equivalent lumped representation described by the matricant $\underset{\sim}{M}(z,z_1)$ of $\underset{\sim}{R}$,

the particular solution of Eq. (39) satisfying the boundary condition, Eq. (41). Only in special cases (including $\underset{\sim}{R}$ constant) are there exact procedures for obtaining this solution. But if the matricant representation is given, it is simple to obtain $\underset{\sim}{R}$. Inverting Eq. (39),

$$\underset{\sim}{R} = j \frac{d\underset{\sim}{M}}{dz} \underset{\sim}{M}^{-1} \tag{46}$$

The operators $\underset{\sim}{R}$ and $\underset{\sim}{M}(z,z_1)$ have the same appropriate metrics, constant $\underset{\sim}{K}$ satisfying both Eqs. (8) and (36).

We also point out that a lumped system — for constant, nonsingular $\underset{\sim}{M}$, not a matricant — can be represented equivalently by a uniform distributed system. We write

$$\underset{\sim}{M} = \underset{\sim}{S}\underset{\sim}{M}'\underset{\sim}{S}^{-1} \tag{47}$$

where $\underset{\sim}{S}$ is a modal matrix of $\underset{\sim}{M}$, and

$$\underset{\sim}{M}' = \text{quasidiag } (\underset{\sim}{J}_1, \underset{\sim}{J}_2, \ldots) \tag{48}$$

is its Jordan canonical form. The $\ell_i \times \ell_i$ Jordan blocks are given by Eq. (26), with λ_i replacing β_i. For each λ_i we define a β_i (which may be complex) as the solution to

$$e^{-j\beta_i L} = \lambda_i, \quad -\pi < \text{Re}\beta_i \leq \pi \tag{49}$$

where L is an arbitrarily chosen equivalent length. We define $\ell_i \times \ell_i$ blocks

$$\hat{\underset{\sim}{J}}_i(z) = e^{-j\beta_i z} \begin{pmatrix} 1 & -jz & \frac{(-jz)^2}{2!} & \frac{(-jz)^3}{3!} & \cdots & \frac{(-jz)^{\ell_i-1}}{(\ell_i-1)!} \\ 0 & 1 & -jz & \frac{(-jz)^2}{2!} & & \vdots \\ 0 & 0 & 1 & -jz & & \\ 0 & 0 & 0 & 1 & \ddots & \\ \vdots & & & & & -jz \\ 0 & \cdots & & & 0 & 1 \end{pmatrix} \tag{50}$$

These have the property

$$j \frac{d\hat{J}_i(z)}{dz} \hat{J}_i^{-1}(z) = J_i^{(\beta)} \tag{51}$$

where $J_i^{(\beta)}$ is the Jordan block of Eq. (26), with eigenvalue β_i. Also, J_i is the Jordan canonical form of $\hat{J}_i(L)$, and $\hat{J}_i(0) = I_{\ell_i}$. We then define

$$J_i(z) = \hat{T}_i^{-1} \hat{J}_i(z) \hat{T}_i \tag{52}$$

where $\hat{T}_i$ is a modal matrix of $\hat{J}_i(L)$, and $J_i(z)$ has the properties

$$\begin{aligned} J_i(L) &= J_i \\ J_i(0) &= I_{\ell_i} \end{aligned} \tag{53}$$

We form

$$\begin{aligned} T &= \text{quasidiag } (\hat{T}_1, \hat{T}_2, \ldots) \\ M''(z) &= \text{quasidiag } (\hat{J}_1(z), \hat{J}_2(z), \ldots) \\ M'(z) &= \text{quasidiag } (J_1(z), J_2(z), \ldots) \end{aligned} \tag{54}$$

It then follows that

$$M(z,0) = SM'(z)S^{-1} = ST^{-1}M''(z)TS^{-1} \tag{55}$$

is the matricant of a uniform distributed system, with operator

$$R = j \frac{dM}{dz} M^{-1} = (ST^{-1}) R' (ST^{-1})^{-1} \tag{56}$$

where

$$R' = \text{quasidiag } (J_1^{(\beta)}, J_2^{(\beta)}, \ldots) \tag{57}$$

and satisfies the boundary conditions

$$\underset{\sim}{M}(L,0) = \underset{\sim}{M}$$
$$\underset{\sim}{M}(0,0) = \underset{\sim}{I} \tag{58}$$

In general, $\underset{\sim}{S}$ is a modal matrix of $\underset{\sim}{M}(z,0)$ and $\underset{\sim}{M}'(z)$ is its Jordan canonical form only at $z = L$, but $\underset{\sim}{S}\underset{\sim}{T}^{-1}$ is a modal matrix of $\underset{\sim}{R}$. If $\underset{\sim}{M}$ is semisimple, $J_i(z) = e^{-j\beta_i z}$. Then $\underset{\sim}{M}'(z)$ is the Jordan canonical form of $\underset{\sim}{M}(z,0)$ and $\underset{\sim}{S}$ is a modal matrix of both $\underset{\sim}{M}(z,0)$ and $\underset{\sim}{R}$. In this case $\underset{\sim}{R}$ is determined quite simply as $\underset{\sim}{R} = \underset{\sim}{S}\underset{\sim}{R}'\underset{\sim}{S}^{-1}$, with the β_i obtained from Eq. (49).

The significance of this result is that we shall be able, without loss of generality, to develop the subsequent theory with a focus on distributed systems. Whatever the physical basis for the transformations they describe, lumped systems can be treated by techniques developed for uniform distributed systems. At this point we can observe that the metrics appropriate to $\underset{\sim}{M}$ will be the constant solutions to Eq. (36) for the matricant of Eq. (55), and may therefore be obtained as the solutions to Eq. (8) for $\underset{\sim}{R}$ in Eq. (56). That these will be all the appropriate $\underset{\sim}{K}$ is ensured by restricting $\mathrm{Re}\beta_i$ in each case to the interval indicated in Eq. (49) — in fact, we need only require that $\beta_i = \beta_j^*$ when $\lambda_i\lambda_j^* = 1$.

Hermitian Metrics

Interest has centered on real invariant quadratic forms as representing real physical quantities conserved by the system. For $s(\underset{\sim}{K})$ to be real, it is required that $\underset{\sim}{K}$ be Hermitian — i.e., $\underset{\sim}{K} = \underset{\sim}{K}^\dagger$. But the $\underset{\sim}{K}$ determined by the procedure described will not in general be Hermitian. (Hermitian $\underset{\sim}{K}'$ do not imply Hermitian $\underset{\sim}{K}$.)

Hermitian $\underset{\sim}{K}$ for which $\underset{\sim}{R}$ is $\underset{\sim}{K}$-Hermitian or $\underset{\sim}{M}$ is $\underset{\sim}{K}$-Unitary can, however, be formed from any set of appropriate, nonsingular $\underset{\sim}{K}$. If $\underset{\sim}{K}$ is appropriate to the operator, $\underset{\sim}{K}^\dagger$ is also appropriate, as can be seen by taking the transpose conjugate of both sides of Eq. (8) or Eq. (36). Moreover, the appropriate $\underset{\sim}{K}$ define a space — since Eqs. (8) and (36) are linear, linear combinations of appropriate metrics are appropriate also. Thus from the given set $\underset{\sim}{K}_i$ we can form appropriate metrics

$$\underset{\sim}{K}_{a_i} = e^{ja_i}\underset{\sim}{K}_i + e^{-ja_i}\underset{\sim}{K}_i^\dagger$$
$$\underset{\sim}{K}_{b_i} = j\left(e^{jb_i}\underset{\sim}{K}_i - e^{-jb_i}\underset{\sim}{K}_i^\dagger\right) \tag{59}$$

which will be Hermitian for a_i and b_i real. Furthermore, real a_i and b_i can always be chosen so that $\underset{\sim}{K}_{a_i}$ and $\underset{\sim}{K}_{b_i}$ are nonsingular.

It may be shown that, of the 2p Hermitian metrics obtained by Eq. (59) from a maximal set of p linearly independent, appropriate $\underset{\sim}{K}_i$, exactly p will be linearly independent. Given a set of p linearly independent, Hermitian metrics $\underset{\sim}{K}_{hi}$ appropriate to an operator describing a physical system, however, it does not follow that the physical significance of each of the associated quadratic forms will be apparent. But the $\underset{\sim}{K}_{hi}$ span a subspace of the space spanned by the $\underset{\sim}{K}_i$. Linear combinations $\sum_i c_i \underset{\sim}{K}_{hi}$ will be Hermitian for c_i real, will include all appropriate, Hermitian metrics, and may generate quadratic forms having clear physical interest. The physical interpretation of the invariant forms of a system presents a separate problem in every individual case, beyond the determination of a linearly independent set of mathematically appropriate forms.

By way of example, the system operator $\underset{\sim}{R}$ for a uniform transmission line, given by Eq. (3), has two distinct eigenvalues $\pm\beta$, which, assuming a lossless line, will be real. We therefore expect that there will be two linearly independent, nonsingular metrics $\underset{\sim}{K}$ appropriate to $\underset{\sim}{R}$; and we find as an Hermetian set,

$$\underset{\sim}{K}_1 = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underset{\sim}{K}_2 = \frac{1}{4}\begin{pmatrix} \frac{1}{Z_0} & 0 \\ 0 & Z_0 \end{pmatrix} \tag{60}$$

We have seen by Eq. (6) that the quadratic form constructed from $\underset{\sim}{K}_1$ represents the power. From $\underset{\sim}{K}_2$ we obtain

$$s(\underset{\sim}{K}_2) = \frac{1}{4}\left(\frac{|V|^2}{Z_0} + Z_0|I|^2\right) = \frac{1}{8Z_0}\left(|V+Z_0I|^2 + |V-Z_0I|^2\right) \tag{61}$$

which is the sum of the powers in the forward and backward waves. In this form the conservation law is somewhat unfamiliar. But the net power $s(\underset{\sim}{K}_1)$ can be written also as the difference between the powers in the two waves, and we can form

$$\underset{\sim}{K}_{+} = \frac{1}{2}\,(\underset{\sim}{K}_1+\underset{\sim}{K}_2) = \frac{1}{8}\begin{pmatrix} \frac{1}{Z_0} & 1 \\ 1 & Z_0 \end{pmatrix}$$

$$\underset{\sim}{K}_{-} = -\frac{1}{2}\,(\underset{\sim}{K}_1-\underset{\sim}{K}_2) = \frac{1}{8}\begin{pmatrix} \frac{1}{Z_0} & -1 \\ -1 & Z_0 \end{pmatrix} \qquad (62)$$

Then $s(\underset{\sim}{K}_+)$ and $s(\underset{\sim}{K}_-)$ are invariant also, and signify that power is conserved in each wave separately.

We observe further that the metric

$$\underset{\sim}{K}_3 = \frac{2\beta}{\omega}\,\underset{\sim}{K}_2 = \frac{1}{2\omega}\begin{pmatrix} \frac{\beta}{Z_0} & 0 \\ 0 & \beta Z_0 \end{pmatrix} = \frac{1}{2\omega}\begin{pmatrix} \frac{\omega\sqrt{LC}}{\sqrt{L/C}} & 0 \\ 0 & \omega\sqrt{LC}\sqrt{\frac{L}{C}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} \qquad (63)$$

where L and C are respectively the distributed inductance and capacitance per unit length, is also appropriate. Then

$$s(\underset{\sim}{K}_3) = \frac{1}{2}\,C|V|^2 + \frac{1}{2}\,L|I|^2 \qquad (64)$$

is invariant, and is recognized as the total electric and magnetic stored energy per unit length on the line.

There may then be any number of invariants of physical interest. Only p can be independent, but these may not give explicitly the information we seek. We chose the metrics Hermitian, after all, in order to obtain an explicit physical interpretation. An examination of the space of Hermitian metrics with a view to identifying invariants of physical interest can lead also to a recognition of relationships that may not otherwise be obvious. The same relation will obtain among the $s(\underset{\sim}{K})$ as among the $\underset{\sim}{K}$. If

$$\sum_i a_i \underset{\sim}{K}_i = \underset{\sim}{0} \tag{65}$$

then

$$\underset{\sim}{x}^\dagger \left(\sum_i a_i \underset{\sim}{K}_i \right) \underset{\sim}{x} = \sum_i a_i \underset{\sim}{x}^\dagger \underset{\sim}{K}_i \underset{\sim}{x} = \sum_i a_i s(\underset{\sim}{K}_i) = 0 \tag{66}$$

In this case,

$$\underset{\sim}{K}_3 = \frac{2\beta}{\omega} \underset{\sim}{K}_2 = \frac{2\beta}{\omega} (\underset{\sim}{K}_+ + \underset{\sim}{K}_-) \tag{67}$$

Then

$$s(\underset{\sim}{K}_3) = \frac{2\beta}{\omega} [s(\underset{\sim}{K}_+) + s(\underset{\sim}{K}_-)] \tag{68}$$

The stored energy per unit length is thus $\frac{2\beta}{\omega}$ times the sum of the powers in the forward and backward waves.

As another example, a lumped shunt susceptance jb is represented on a voltage-current basis by the operator

$$\underset{\sim}{M} = \begin{pmatrix} 1 & 0 \\ -jb & 1 \end{pmatrix} \tag{69}$$

This operator has appropriate metrics of the form

$$\underset{\sim}{K} = \begin{pmatrix} a_1 & a_2 \\ a_2 & 0 \end{pmatrix} \tag{70}$$

One choice of the arbitrary constants, for $\underset{\sim}{K}_1$, again gives the metric of Eq. (5), indicating that power is conserved. Another choice gives

$$\underset{\sim}{K}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{71}$$

which implies that

$$s(\underset{\sim}{K}_2) = VV^* \tag{72}$$

is conserved. The constancy of the voltage across the network appears here in a somewhat weakened form.

The $\underset{\sim}{K}$ of Eq. (70) can be chosen nonsingular, but $\underset{\sim}{K}_2$ is singular (as are $\underset{\sim}{K}_+$ and $\underset{\sim}{K}_-$ in Eq. (60)). The space of appropriate Hermetian metrics will include singular metrics, even if we have gone to the trouble to choose the metrics that span this space nonsingular, and these may be of interest. The nonsingularity of the $\underset{\sim}{K}_{hi}$ does not ensure the nonsingularity of linear combinations. Conversely, a linear combination of singular $\underset{\sim}{K}_{hi}$ can be nonsingular (as in Eq. (67)). In attempting to interpret our results physically, then, it is not important that we choose Hermitian metrics by Eq. (59) to be nonsingular, only that we choose a linearly independent set. Later, however, it will be significant to know that a nonsingular set exists.

The assumption that Hermitian metrics lead to quadratic forms that represent real physical invariants can also be misleading. For example, the operator for a lumped shunt resistance r,

$$\underset{\sim}{M} = \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} \tag{73}$$

has the same repeated eigenvalue $\lambda = 1$ as the operator of Eq.(69) for a lumped shunt susceptance jb. It will, therefore, also have appropriate Hermitian metrics, even though real power is not conserved. From one of these,

$$\underset{\sim}{K} = j\,\frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{74}$$

with $\underset{\sim}{x}$ as in Eq. (2), we obtain

$$s(\underset{\sim}{K}) = \frac{1}{2} \text{ Im } (VI^*) \tag{75}$$

The imaginary part of the complex power — a real quantity, to be sure — is thus conserved. This example (see also Example 2 of Section VI) suggests that the special significance attributed to Hermetian metrics should not be pushed too far.

While such simple systems are adequate vehicles for these several comments, examples of more interesting systems obeying conservation laws are given by Pease (refs. 1,2, and 4).

Testing the Operator

The conditions given for the existence of appropriate, nonsingular $\underset{\sim}{K}$ are conditions on the eigenvalues and structure of the system operator. It may be pointed out, however, that it is not necessary to obtain the eigenvalues, eigenvectors, and generalized eigenvectors directly, in order to determine whether these conditions are satisfied.

The eigenvalues β_i of $\underset{\sim}{R}$ are the roots of the characteristic equation,

$$|\underset{\sim}{R}-\beta\underset{\sim}{I}| = a_n\beta^n + a_{n-1}\beta^{n-1} +\ldots+ a_1\beta + a_0 = 0 \tag{76}$$

where $\underset{\sim}{I}$ is the identity matrix. It is well known that the condition for the roots of an algebraic equation to be real or occur in conjugate pairs is that the coefficients a_i be real, and so it is not necessary to solve Eq. (76) to obtain this information. In the case of the lumped-system operator, the eigenvalues λ_i of $\underset{\sim}{M}$ will satisfy Eq. (37) or Eq. (38) if and only if $\underset{\sim}{M}$ and $(\underset{\sim}{M}^\dagger)^{-1}$ have the same eigenvalues,and thus the same characteristic equations, and it is only necessary to write these out and compare. This provides a preliminary check on a necessary existence condition, relatively simple even for a matrix of large order.

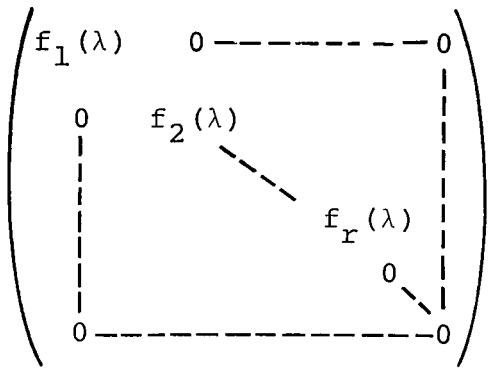
The suitability of the eigenvalues and structure of $\underset{\sim}{R}$ or $\underset{\sim}{M}$ for the existence of nonsingular solutions to Eq. (8) or Eq.(33) can be determined by putting the λ-matrices $\underset{\sim}{R}^\dagger - \lambda\underset{\sim}{I}$ and $\underset{\sim}{R} - \lambda\underset{\sim}{I}$, or $\underset{\sim}{M}^\dagger - \lambda\underset{\sim}{I}$ and $\underset{\sim}{M}^{-1} - \lambda\underset{\sim}{I}$, in their Smith normal forms. This is accomplished for a matrix $\underset{\sim}{A}$, defined over a field F, by a sequence of elementary λ-matrix transformations defined by the following operations on $\underset{\sim}{A} - \lambda\underset{\sim}{I}$:

1. The interchange of two rows (columns).

2. The multiplication of all elements in a row (column) by a nonzero scalar in F.

3. The addition, to all elements of a row (column), of the corresponding elements of another row (column) multiplied by a polynomial $p(\lambda)$ over F.

By a systematic application of these transformations, $\underset{\sim}{A} - \lambda\underset{\sim}{I}$ can be put in its Smith normal form,



where each $f_i(\lambda)$ is a polynomial in λ, with leading coefficient 1, which divides $f_{i+1}(\lambda)$; where $f_1 f_2 \cdots f_p$, $p = 1,2,\ldots,r$, is the greatest common divisor of all p-square minors of $\underset{\sim}{A} - \lambda\underset{\sim}{I}$; and where r is the rank of $\underset{\sim}{A}$. It can be shown that two matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ can be related by a similarity transformation if and only if their characteristic matrices $\underset{\sim}{A} - \lambda\underset{\sim}{I}$ and $\underset{\sim}{B} - \lambda\underset{\sim}{I}$ have the same Smith normal form (ref. 5).

A similar procedure, in which the complications of operating on polynomials are traded against the need to examine matrices of higher order, together with other procedures for obtaining information about the operator, will be described in Section III under "Testing the Kronecker Sum." The degree of $f_r(\lambda)$, which can be shown to be the minimum polynomial of $\underset{\sim}{A}$ (see Section IV), gives some measure of the degeneracy in $\underset{\sim}{A}$, but not as much information; therefore, we will not pursue this point.

III. DEVELOPMENT BY KRONECKER-SUM FORMALISM

In the previous section, we discussed conditions for an operator to have appropriate nonsingular metrics, a formal procedure for determining the metrics, and the degeneracy of solutions. The metrics could also be obtained by writing out Eq. (8) or Eq. (36) in detail, and solving the resulting set of linear, homogeneous

equations. Short of solving these equations, we could determine the rank of the coefficient matrix and thereby the number of linearly independent solutions we can expect. In general, this procedure, applied to a set of homogeneous equations, would not tell us whether matrices formed in a prescribed manner from the vector solutions could be chosen nonsingular. Moreover, this would seem to be only an ad-hoc procedure – not necessarily simpler; and not to provide a formal theory of the conservation laws based on properties of the system operator.

Writing out either of these equations, however, yields a coefficient matrix that can be identified as a Kronecker sum (which incidentally provides an efficient means for obtaining this matrix). The Kronecker sum has useful properties that will enable us to obtain expressions for the number of linearly independent metrics, in terms of the eigenvalues and structure of the system operator, by this approach also; and more significantly, to express the conditions on the operator for the existence of nonsingular solutions, and the degree and type of degeneracy in the operator, in terms of properties of a subspace of the space spanned by the eigenvectors and generalized eigenvectors of the coefficient matrix.

It is the number of repeated zero eigenvalues and the number of linearly independent eigenvectors with zero eigenvalue, together with certain upper bounds on these numbers, that must be determined in order to obtain the information we require. The upper bounds can be determined as the number of repeated zero eigenvalues and the number of linearly independent eigenvectors with zero eigenvalue of a related Kronecker sum in the operator. Matrices of the order $n^2 \times n^2$ must be examined, but the subspace spanned by the eigenvectors and generalized eigenvectors with zero eigenvalue is the simplest to investigate. Moreover, the Kronecker sums can be compared under a simpler transformation than that to the Smith normal forms. In any event, a formal equivalent to the theory developed in Section II is obtained by this approach, together with procedures for testing an operator that can be useful in particular problems and can, in fact, give somewhat more information than the tests described in Section II under "Testing the Operator." The formal equivalent to the procedure for obtaining the metrics may be considered to be provided by the procedure to be described in Section VI.

Kronecker Product

The theory of the Kronecker or direct matrix product, and of the Kronecker sum, is a standard topic in matrix theory (refs.3, 6, and 7). The direct product is defined and some of its properties are summarized below.

The direct product of two n×n matrices $\underset{\sim}{A} = (a_{ij})$ and $\underset{\sim}{B} = (b_{ij})$, denoted by $\underset{\sim}{A}\times\underset{\sim}{B}$, is defined as the $n^2\times n^2$ matrix

$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{pmatrix} a_{11}\underset{\sim}{B} & a_{12}\underset{\sim}{B} & \cdots & a_{1n}\underset{\sim}{B} \\ a_{21}\underset{\sim}{B} & a_{22}\underset{\sim}{B} & & \\ \vdots & & \ddots & \vdots \\ a_{n1}\underset{\sim}{B} & & \cdots & a_{nn}\underset{\sim}{B} \end{pmatrix} \tag{77}$$

Here $\underset{\sim}{A}\times\underset{\sim}{B}$ is written as a partitioned matrix, with n×n blocks each consisting of the matrix $\underset{\sim}{B}$ multiplied by an element a_{ij} of $\underset{\sim}{A}$. For our purposes we restrict attention to square matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ of the same order, although the theory applies more generally.

The elements of $\underset{\sim}{A}\times\underset{\sim}{B}$ can be written

$$(\underset{\sim}{A}\times\underset{\sim}{B})_{ij,kh} = a_{ik}b_{jh} \tag{78}$$

where double indices are used to indicate the ordering of the rows and columns of the direct product, each double index running through the values 11,12,...,1n,21,...,2n,...,n1,...,nn, in that order.

The direct product is distributive,

$$\begin{aligned} (\underset{\sim}{A}+\underset{\sim}{B}) \times \underset{\sim}{C} &= \underset{\sim}{A} \times \underset{\sim}{C} + \underset{\sim}{B} \times \underset{\sim}{C} \\ \underset{\sim}{A} \times (\underset{\sim}{B}+\underset{\sim}{C}) &= \underset{\sim}{A} \times \underset{\sim}{B} + \underset{\sim}{A} \times \underset{\sim}{C} \end{aligned} \tag{79}$$

and associative,

$$\underset{\sim}{A} \times (\underset{\sim}{B}\times\underset{\sim}{C}) = (\underset{\sim}{A}\times\underset{\sim}{B}) \times \underset{\sim}{C} \tag{80}$$

but not commutative,

$$\underset{\sim}{A} \times \underset{\sim}{B} \neq \underset{\sim}{B} \times \underset{\sim}{A} \tag{81}$$

It also satisfies a matrix-product relation,

$$(\underset{\sim}{A}\times\underset{\sim}{B})(\underset{\sim}{C}\times\underset{\sim}{D}) = (\underset{\sim}{A}\underset{\sim}{C}) \times (\underset{\sim}{B}\underset{\sim}{D}) \tag{82}$$

Let $\underset{\sim}{x}^{(r)} = (x_i^{(r)})$ be an eigenvector of $\underset{\sim}{A}$ with eigenvalue λ_r, and $\underset{\sim}{y}^{(s)} = (y_i^{(s)})$ be an eigenvector of $\underset{\sim}{B}$ with eigenvalue μ_s. Define $\underset{\sim}{u}^{(r,s)}$ as the n^2-dimensional vector having components

$$u_{ij}^{(r,s)} = x_i^{(r)} y_j^{(s)} \tag{83}$$

where the double index ij again represents a single index running through values 11 to nn in the same order as before. Then $\underset{\sim}{u}^{(r,s)}$ is an eigenvector of $\underset{\sim}{A}\times\underset{\sim}{B}$ with eigenvalue $\lambda_r\mu_s$.

If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are semisimple, Eq. (83) gives n^2 linearly independent eigenvectors, and $\underset{\sim}{A}\times\underset{\sim}{B}$ is semisimple also.

The eigenvalues of $\underset{\sim}{A}\times\underset{\sim}{B}$, whether or not $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are semisimple, are the n^2 multiplicative combinations $\lambda_r\mu_s$, with all multiplicities of the λ_r and the μ_s included. If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are nonsingular, their eigenvalues will be nonzero, and $\underset{\sim}{A}\times\underset{\sim}{B}$ will accordingly be nonsingular also.

The Kronecker Sum

The Kronecker sum of two n×n matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$, which we denote by $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$, is defined as the sum of two direct products, of the form

$$\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B}) = \underset{\sim}{A} \times \underset{\sim}{I} + \underset{\sim}{I} \times \underset{\sim}{B} \tag{84}$$

where $\underset{\sim}{I}$ is the n×n identity matrix. Like the direct product, this is not a commutative relation,

$$\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B}) \neq \underset{\sim}{\Lambda}(\underset{\sim}{B},\underset{\sim}{A}) \tag{85}$$

If $\underset{\sim}{x}^{(r)}$ is an eigenvector of $\underset{\sim}{A}$ with eigenvalue λ_r, and $\underset{\sim}{y}^{(s)}$ is an eigenvector of $\underset{\sim}{B}$ with eigenvalue μ_s, then $\underset{\sim}{u}^{(r,s)}$, with components defined by Eq. (83), is an eigenvector of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ with eigenvalue $\lambda_r+\mu_s$.

Again if $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are semisimple, $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ is semisimple also. In any case the eigenvalues of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ are the n^2 additive combinations $\lambda_r+\mu_s$, with all multiplicities of the λ_r and the μ_s included.

Degeneracy in $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$

From the eigenvalues λ_r of $\underset{\sim}{A}$ and μ_s of $\underset{\sim}{B}$, it is easy to determine the number of repeated eigenvalues of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ equal to $\lambda_r+\mu_s$,

for given values of r and s. There will be the same number of linearly independent eigenvectors and generalized eigenvectors of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ in chains with this eigenvalue, and these will span an eigensubspace. We shall have need to know the number of linearly independent eigenvectors in such an eigensubspace. In general, if $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are not semisimple, the derivation is involved, although the expressions are simple and will be given without proof.

First, however, let us determine the dimensionality of a given eigensubspace. The indices r and s are assumed to run over values indexing the linearly independent eigenvectors, or separate chains, of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ respectively. For each value of r there is a chain of $\underset{\sim}{A}$ of length ℓ_r, and for each value of s a chain of $\underset{\sim}{B}$ of length ℓ_s. Then, each λ_r is of multiplicity ℓ_r, and each μ_s is of multiplicity ℓ_s.

There can be more than one value of r for which the λ_r are equal, and more than one value of s for which the μ_s are equal. The number m_r of repeated eigenvalues of $\underset{\sim}{A}$ equal to λ_r, including multiplicities, is the same for each of these values of r. If we choose a representative value $\bar{r}$ from this set of r, we can denote the number of repeated eigenvalues of $\underset{\sim}{A}$ equal to λ_r by $m_{\bar{r}}$, for each r in this set. Similarly, we denote the number of repeated eigenvalues of $\underset{\sim}{B}$ equal to μ_s by $m_{\bar{s}}$. Then we can write

$$\sum_{r\sim\bar{r}} \ell_r = m_{\bar{r}}$$

$$\sum_{s\sim\bar{s}} \ell_s = m_{\bar{s}} \tag{86}$$

where by $r\sim\bar{r}$ we denote values of r such that $\lambda_r=\lambda_{\bar{r}}$, and by $s\sim\bar{s}$ values of s such that $\mu_s=\mu_{\bar{s}}$.

By the property indicated in Section III under "The Kronecker Sum," the eigenvalue $\lambda_r+\mu_s$ of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$, for given values of r and s, will be repeated $\ell_r\ell_s$ times. The number of repeated eigenvalues $\lambda_r+\mu_s$, for all $r\sim\bar{r}$ and all $s\sim\bar{s}$, is then

$$\sum_{\substack{r\sim\bar{r}\\ s\sim\bar{s}}} \ell_r\ell_s = \sum_{r\sim\bar{r}} \ell_r \sum_{s\sim\bar{s}} \ell_s = m_{\bar{r}}m_{\bar{s}} \tag{87}$$

But there is yet another way in which $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ can have repeated eigenvalues. For distinct values of $\bar{r}$, $\bar{r}_1\neq\bar{r}_2$, and of $\bar{s}$, $\bar{s}_1\neq\bar{s}_2$,

such that $\lambda_{\bar{r}_1} \neq \lambda_{\bar{r}_2}$ and $\mu_{\bar{s}_1} \neq \mu_{\bar{s}_2}$, it can be that $\lambda_{\bar{r}_1} + \mu_{\bar{s}_1} = \lambda_{\bar{r}_2} + \mu_{\bar{s}_2}$. Let us choose a representative pair $(\bar{\bar{r}},\bar{\bar{s}})$ from each set of ordered pairs $(\bar{r},\bar{s})$ for which the $\lambda_{\bar{r}} + \mu_{\bar{s}}$ are equal. We denote the pairs in such a set by $(\bar{r},\bar{s}) \sim (\bar{\bar{r}},\bar{\bar{s}})$. Then the number of repeated eigenvalues $\lambda_r+\mu_s=\lambda_{\bar{\bar{r}}}+\mu_{\bar{\bar{s}}}$, counting all $r\sim\bar{r}$ and $s\sim\bar{s}$, for all $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$, is

$$\sum_{(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})} \sum_{\substack{r\sim\bar{r} \\ s\sim\bar{s}}} \ell_r \ell_s = \sum_{(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})} m_{\bar{r}} m_{\bar{s}} \tag{88}$$

Equation (88) gives also the number of linearly independent eigenvectors and generalized eigenvectors with eigenvalue $\lambda_{\bar{\bar{r}}}+\mu_{\bar{\bar{s}}}$. Among the eigenvectors will be the $\underset{\sim}{u}^{(r,s)}$ defined by Eq. (83), for all $\bar{r}\sim\bar{r}$, $\bar{s}\sim\bar{s}$, $(\bar{r}\sim\bar{s})\sim(\bar{r},\bar{s})$. But unless either $\underset{\sim}{A}$ or $\underset{\sim}{B}$ is semi-simple, there will be more. The number of linearly independent eigenvectors with eigenvalue $\lambda_{\bar{\bar{r}}} + \mu_{\bar{\bar{s}}}$ will be given by

$$p_{\bar{\bar{r}}\bar{\bar{s}}} = \sum_{(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})} \sum_{\substack{r\sim\bar{r} \\ s\sim\bar{s}}} \min\ (\ell_r,\ell_s) \tag{89}$$

where by min (m,n) we denote the smaller of the integers m and n. Moreover, $p_{\bar{\bar{r}}\bar{\bar{s}}}$ will be within the limits

$$\sum_{(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})} \min\ (m_{\bar{r}},m_{\bar{s}}) \leq p_{\bar{\bar{r}}\bar{\bar{s}}} \leq \sum_{(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})} m_{\bar{r}} m_{\bar{s}} \tag{90}$$

and will attain the upper limit if and only if $\ell_r = \ell_s = 1$ for all $\bar{r}\sim\bar{r}$, $\bar{s}\sim\bar{s}$, $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$. A necessary, but not sufficient, condition for $p_{\bar{\bar{r}}\bar{\bar{s}}}$ to attain the lower limit is that there be only one $r\sim\bar{r}$ or only one $s\sim\bar{s}$ (i.e., $\ell_{\bar{r}} = m_{\bar{r}}$ or $\ell_{\bar{s}} = m_{\bar{s}}$) for each $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$

We shall be particularly interested in the special case where $m_{\bar{r}} = m_{\bar{s}}$ for all $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$. The limits then become

$$\sum_{\substack{\bar{r} \\ (\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})}} m_{\bar{r}} \leq p_{\bar{\bar{r}}\bar{\bar{s}}} \leq \sum_{\substack{\bar{r} \\ (\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})}} m_{\bar{r}}^2 \tag{91}$$

where the summations are over all $\bar{r}$ for which there exist $\bar{s}$ such that $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$. In this case the lower limit is attained if and only if $\ell_{\bar{r}} = m_{\bar{r}}$ or $\ell_{\bar{s}} = m_{\bar{s}}$ for each $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$.

A significant feature of these results is that $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ can exhibit considerable degeneracy even when there is no degeneracy in $\underset{\sim}{A}$ or $\underset{\sim}{B}$. This will account for the fact that a nondegenerate operator can have a number of appropriate metrics.

The Equation $\underset{\sim}{A}\underset{\sim}{X} + \underset{\sim}{X}\underset{\sim}{B}^T = \underset{\sim}{C}$

One application of the Kronecker sum is in the solution of the equation

$$\underset{\sim}{A}\underset{\sim}{X} + \underset{\sim}{X}\underset{\sim}{B}^T = \underset{\sim}{C} \tag{92}$$

for $\underset{\sim}{X} = (x_{ij})$, $\underset{\sim}{X}$ an $n\times n$ matrix. Here $\underset{\sim}{A} = (a_{ij})$, $\underset{\sim}{B} = (b_{ij})$, and $\underset{\sim}{C} = (c_{ij})$ are given $n\times n$ matrices, and $\underset{\sim}{B}^T = (b_{ji})$ is the transpose of $\underset{\sim}{B}$. By Eq. (78), and with the use of the δ function,

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

Equation (92) can be written in detail as

$$\begin{aligned} \sum_k a_{ik}x_{kj} + \sum_h x_{ih}b_{jh} &= \sum_{kh} (a_{ik}\delta_{jh} + \delta_{ik}b_{jh})\, x_{kh} \\ &= \sum_{kh} (\underset{\sim}{A}\times\underset{\sim}{I} + \underset{\sim}{I}\times\underset{\sim}{B})_{ij,kh}\, x_{kh} = c_{ij} \end{aligned} \tag{93}$$

If we define n^2-dimensional column vectors x" and c" having components x_{ij} and c_{ij} respectively, where in each case the double index is treated as a single index running over n^2 values according to the ordering convention adopted at the beginning of this section, then Eqs. (93) are a rendering in detail of the single

matrix equation

$$\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})\ \underset{\sim}{x}'' = \underset{\sim}{c}'' \tag{94}$$

We shall be interested, in this section, in the nontrivial solutions of Eq. (94) when $\underset{\sim}{c}''$ is the null vector. A necessary and sufficient condition for

$$\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})\ \underset{\sim}{x}'' = \underset{\sim}{0} \tag{95}$$

to have a nontrivial solution is that the determinant of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ vanish, or equivalently, that $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ have a zero eigenvalue, which may be repeated. If $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ has a zero eigenvalue repeated p_m times, the rank r of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ can be anywhere in the range $n^2-p_m \leq r \leq n^2 - 1$. There will then be $n^2 - r$, or between 1 and p_m, linearly independent solutions to Eq. (95). The solutions $\underset{\sim}{x}''$ can also be thought of as vectors in the null space of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$, the space spanned by the linearly independent eigenvectors of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ with zero eigenvalue. This is a subspace of the zero-eigenvalue eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$, the space spanned by the p_m linearly independent eigenvectors and generalized eigenvectors with zero eigenvalue.

These remarks apply in general to linear homogeneous equations; but since $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ is a Kronecker sum, we can be more particular. The number of linearly independent engenvectors of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ with zero eigenvalue is given by Eq. (89) for $\lambda_r^{=} + \mu_s^{=} = 0$, and is in the range indicated by Eq. (90).

Furthermore, in our application $\underset{\sim}{A}$ and $\underset{\sim}{B}$ will be forms of the same operator. This will enable us to develop relations between Kronecker sums in different forms of the operator that are equivalent to relations between forms of the operator described in Section II.

Equation (92) has been studied by other approaches (refs.8 and 9), and this is not the only route by which we could arrive at the results to be obtained in this section. But we shall be interested more generally in Eq. (92) for $\underset{\sim}{C} = j(d\underset{\sim}{X}/dz)$, and we shall find that the operator in the form $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$ can be identified with the conservation laws in a quite fundamental sense. Expressing results in this section in terms of properties of the zero-eigenvalue eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$, then, exhibits them as one facet of the theory.

Structure of the Zero-Eigenvalue Eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$

By Eqs. (92) and (94), Eq. (8) can be written in the form

$$\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger}-\underset{\sim}{R}^{T})\underset{\sim}{k}'' = \underset{\sim}{0} \tag{96}$$

where $\underset{\sim}{k}''$ is an n^2-dimensional column vector formed from the elements of $\underset{\sim}{K}$ in accordance with the convention adopted. The coefficient matrix is of the form

$$\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T}) = \underset{\sim}{R}^{\dagger}\times\underset{\sim}{I}-\underset{\sim}{I}\times\underset{\sim}{R}^{T}= \begin{pmatrix} r^*_{11}\underset{\sim}{I}-\underset{\sim}{R}^{T} & r^*_{21}\underset{\sim}{I} & \cdots & r^*_{n1}\underset{\sim}{I} \\ r^*_{12}\underset{\sim}{I} & r^*_{22}\underset{\sim}{I}-\underset{\sim}{R}^{T} & & \\ \vdots & & \ddots & \vdots \\ & & & r^*_{n(n-1)}\underset{\sim}{I} \\ r^*_{1n}\underset{\sim}{I} & \cdots & r^*_{(n-1)n}\underset{\sim}{I} & r^*_{nn}\underset{\sim}{I}-\underset{\sim}{R}^{T} \end{pmatrix} \tag{97}$$

where the entries shown are n×n blocks. The nontrivial solutions to Eq. (96) are the eigenvectors of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$ with zero eigenvalue.

If we apply the results given in Section III (under "Degeneracy in $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$") in an analysis of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$ under various conditions on the eigenvalues and structure of $\underset{\sim}{R}$, we find that we can express the results of this analysis in terms of four significant parameters. By $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$, we refer to all ordered pairs $(\bar{r},\bar{s})$ such that $\beta^*_{\bar{r}} = \beta_{\bar{s}}$, where $\bar{r}$ and $\bar{s}$ index distinct eigenvalues of $\underset{\sim}{R}$. The expressions in Section III(under "Degeneracy in $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$") then apply with reference to the zero eigenvalue of $\Lambda(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$. Otherwise the notation is as explained there. We note that for the eigenvalues of $\underset{\sim}{R}$ to be all real or occur in conjugate pairs, it is required that the $\beta_{\bar{r}}$ be all real or occur in conjugate pairs, and in addition that $m_{\bar{r}} = m_{\bar{s}}$ for all $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$.

The first parameter is the number of repeated zero eigenvalues of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$, which from Eq. (88) is given by

$$p_m = \sum_{(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})} m_{\bar{r}} m_{\bar{s}} \tag{98}$$

where the summation is over all $(\bar{r},\bar{s})$ such that $\beta^*_{\bar{r}} = \beta_{\bar{s}}$, and $m_{\bar{r}}$ and $m_{\bar{s}}$ are the number of repeated eigenvalues β_r and β_s of $\underset{\sim}{R}$, including multiplicities, equal to $\beta_{\bar{r}}$ and $\beta_{\bar{s}}$ respectively.

The number of linearly independent eigenvectors of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger, -\underset{\sim}{R}^T)$ with zero eigenvalue (the number of linearly independent metrics appropriate to $\underset{\sim}{R}$) from Eq. (89) is given by

$$p = \sum_{(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})} \sum_{\substack{r\sim\bar{r} \\ s\sim\bar{s}}} \min\ (\ell_r, \ell_s) \tag{99}$$

where we drop the subscript on p, since we will be concerned only with the zero eigenvalue. In Eq. (99), ℓ_r and ℓ_s are the lengths of the chains of $\underset{\sim}{R}$ with eigenvalues β_r and β_s respectively.

That $m_{\bar{r}}$ and $m_{\bar{s}}$ in Eq. (98), and ℓ_r and ℓ_s in Eq. (99), can be interpreted as indicated, although strictly they refer to the eigenvalues and chains of $\underset{\sim}{R}^\dagger$ and $-\underset{\sim}{R}^T$, is easily shown.

The other two parameters are summations over all $\bar{r}$, independent of the character of the eigenvalues of R. The first is defined by

$$p'_m = \sum_{\bar{r}} m^2_{\bar{r}} \tag{100}$$

Let the values of ℓ_r for $r\sim\bar{r}$ be denoted by ℓ_{r_i}, $r\sim\bar{r}$, and be arranged such that

$$\ell_{r_i} \geq \ell_{r_{i+1}}, \quad i=1,2,\ldots$$

The fourth parameter is then defined by

$$p' = \sum_{\bar{r}} \sum_{\substack{i \\ r\sim\bar{r}}} (2i-1)\ \ell_{r_i} \tag{101}$$

The following relations can then be shown to hold:

$$\sum_{(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})} \min\ (m_{\bar{r}},m_{\bar{s}}) \leq p \leq p_m \leq p'_m$$

$$p \leq p' \tag{102}$$

$$n \leq p' \leq p'_m$$

where

A. $p>0$ if $p_m>0$, for then the set $(\bar{r},\bar{s})\sim(\bar{\bar{r}},\bar{\bar{s}})$ is not empty. From another point of view, if $p_m>0$, $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$ is singular, in which case Eq. (96) has at least one nontrivial solution.

B. $p \geq n$ if the eigenvalues of $\underset{\sim}{R}$ are real or occur in complex-conjugate pairs. In this case, $p=n$ if and only if a single chain is associated with each distinct real eigenvalue of $\underset{\sim}{R}$ and with at least one of each distinct conjugate pair. If chains of equal length are associated with distinct conjugate pairs of eigenvalues, this condition requires that a single chain be associated with each distinct eigenvalue of $\underset{\sim}{R}$.

C. $p_m=p'_m$ if and only if the eigenvalues of $\underset{\sim}{R}$ are real or occur in conjugate pairs.

D. $p=p'$ if and only if the eigenvalues of $\underset{\sim}{R}$ are real or occur in conjugate pairs, and chains of equal length are associated with conjugate pairs of eigenvalues.

E. $p'=p'_m$ if and only if $\underset{\sim}{R}$ is semisimple.

F. $p'=n$ if and only if a single chain is associated with each distinct eigenvalue of $\underset{\sim}{R}$.

Derivable from D and E, and from E and F, respectively,

G. $p=p'_m$ if and only if the eigenvalues of $\underset{\sim}{R}$ are real or occur in conjugate pairs, and $\underset{\sim}{R}$ is semisimple.

H. $p'_m=n$ if and only if the n eigenvalues of $\underset{\sim}{R}$ are distinct.

It follows from the above results that, if the eigenvalues of $\underset{\sim}{R}$ are real or occur in conjugate pairs,

$$n \leq p \leq p' \leq p_m = p'_m \tag{103}$$

The condition on the eigenvalues of $\underset{\sim}{R}$, therefore, is sufficient to ensure that there will be at least n linearly independent solutions to Eq. (96). There will also then be p_m' repeated zero eigenvalues $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$. But if $\underset{\sim}{R}$, and thus $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$, is not semisimple, it is interesting to observe that $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$ will have not only fewer than n^2 linearly independent eigenvectors, but in particular fewer than p_m' associated with the zero eigenvalue. It is characteristic of the class of results we have obtained that a property that depends on the eigenvalues and structure of a matrix depends only on the structure of the zero-eigenvalue eigensubspace of a properly formulated Kronecker sum. This will be illustrated more adequately as we proceed, after we introduce a slightly different Kronecker sum than the one we have been considering. We shall then find that the similarity requirement on $\underset{\sim}{R}$ and $\underset{\sim}{R}^\dagger$ for the existence of appropriate nonsingular metrics, which is equivalent to conditions on the eigenvalues and structure of $\underset{\sim}{R}$, can be in like manner relaxed.

Reformulation of the Theory

We have obtained explicit expressions for the number of linearly independent metrics appropriate to a system operator $\underset{\sim}{R}$, under various conditions on the operator, by an approach based on the Kronecker theory. We have expressed the results in terms of the parameters p, p_m, p', and p_m', which are well defined in terms of the eigenvalues and structure of $\underset{\sim}{R}$. We have also established relations among these parameters that are equivalent to the conditions on the eigenvalues and structure of $\underset{\sim}{R}$, including the conditions under which the metrics can be chosen nonsingular. These results indicate a dependence of such interior properties of $\underset{\sim}{R}$, and the character as well as the number of its appropriate metrics, on conditions on p, p_m, p', and p_m'. We have in effect reformulated our theoretical results in terms of conditions on these parameters, but the parameters are defined in terms of the interior properties of $\underset{\sim}{R}$. The unprimed parameters, however, are also well defined in terms of the structure of the zero-eigenvalue eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$. We would like a comparable interpretation of p' and p_m' in terms of simpler matrix properties.

In attempting to characterize $\underset{\sim}{R}$ and its appropriate metrics in terms of properties of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$, we encounter the difficulty that, while the parameters p and p_m are descriptive of simple properties of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$, p' and p_m' are values assumed by these parameters under conditions on $\underset{\sim}{R}$. We are able to bypass this difficulty, however, for we can show that p' and p_m' are also descriptive of simple properties of a different Kronecker sum formed from $\underset{\sim}{R}$. In fact, exactly as p and p_m are defined in terms of the structure of the zero-eigenvalue eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$, p' and p_m' are defined in terms of the structure of the zero-eigenvalue

eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$.

Thus $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$ will have

$$p_m' = \sum_{\bar{r}} m_{\bar{r}}^2 \tag{104}$$

repeated zero eigenvalues, where the summation is over all values of $\bar{r}$, regardless of the character of the eigenvalues of $\underset{\sim}{R}$. The number of independent eigenvectors of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$ with zero eigenvalue is

$$p' = \sum_{\bar{r}} \sum_{\substack{i \\ r\sim\bar{r}}} (2i-1)\ \ell_{r_i} \tag{105}$$

where again the summation is over all $\bar{r}$. The upper bounds on the dimensionalities of the zero-eigenvalue eigensubspace and the null space of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$, then, are respectively the dimensionalities of the zero-eigenvalue eigensubspace and the null space of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$.

We are thus able to restate the theory in the following terms:

> The number of linearly independent metrics appropriate to a system operator $\underset{\sim}{R}$ is given by the dimensionality of the null space of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$, which is not greater than the dimensionality $(\geq n)$ of the null space of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$. If the zero-eigenvalue eigensubspaces of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger-\underset{\sim}{R}^T)$ and $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$, which contain the null spaces, are of equal dimensionality, there will be at least n independent metrics. These can be chosen nonsingular if and only if the null spaces are also of equal dimensionality.

By C and D, under "Structure of the Zero-Eigenvalue Eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$" in Section III, only if the zero-eigenvalue eigensubspaces are of equal dimensionality can the null spaces be of equal dimensionality. The former condition is necessary and the latter is necessary and sufficient to ensure that the metrics can be chosen nonsingular.

We have also the following corollaries:

1. If the zero-eigenvalue eigensubspaces of $\Lambda(R^\dagger,-R^T)$ and $\Lambda(R,-R)$ are of equal dimensionality, and the null space of $\Lambda(R,-R)$ is of dimensionality n, then the null space of $\Lambda(R^\dagger,-R^T)$ will also be of dimensionality n, and the n linearly independent metrics can be chosen nonsingular.

2. If the zero-eigenvalue eigensubspaces of $\Lambda(R^\dagger,-R^T)$ and $\Lambda(R,-R)$ are of equal dimensionality n, then the null spaces will also be of dimensionality n, and the n linearly independent metrics can be chosen nonsingular.

Moreover, by C, E, and H:

> If the zero-eigenvalue eigensubspaces of $\Lambda(R^\dagger,-R^T)$ and $\Lambda(R,-R)$ are of equal dimensionality, and the null space of $\Lambda(R,-R)$ is of this same dimensionality, then the null space of $\Lambda(R^\dagger,-R^T)$ will also be of this dimensionality, and the metrics can be chosen nonsingular.

The condition given in Section II for the existence of nonsingular metrics appropriate to R was that R and $R^\dagger$ be similar matrices, which required that the eigenvalues and structure of R satisfy certain conditions. We have replaced these by the condition that the null spaces of $\Lambda(R^\dagger,-R^T)$ and $\Lambda(R,-R)$ be of the same dimensionality (p=p'). This condition can be expressed also in terms of the rank $r=n^2-p$ of $\Lambda(R^\dagger,-R^T)$ and the rank $r'=n^2-p'$ of $\Lambda(R,-R)$, and is that

$$r = r' \tag{106}$$

or that $\Lambda(R^\dagger,-R^T)$ and $\Lambda(R,-R)$ have the same rank.

It is well known that two matrics have the same rank if and only if they are equivalent. (By definition, A and B are equivalent if there exist nonsingular matrices P and Q such that $A=PBQ$.) Thus at the sacrifice of an increase in the order of the matrices, we have replaced the requirement that R and $R^\dagger$ be similar (which requires that $Q=P^{-1}$) with the weaker requirement that $\Lambda(R^\dagger,-R^T)$ and $\Lambda(R,-R)$ be equivalent.

We shall encounter, in the test for semisimplicity described below, another example of an interior, structural property of R that can be expressed as the equivalence of two Kronecker sums.

Testing the Kronecker Sum

These results provide a basis for testing an operator to obtain information about its appropriate metrics — and indeed about

the eigenvalues and structure of any matrix, apart from our interest in the conservation laws governing a system it might describe – by determining or comparing the dimensionalities of the null spaces and zero-eigenvalue eigensubspaces of appropriate Kronecker sums.

Test I: test for suitability of eigenvalues of $\underset{\sim}{R}$.- By C, in Section III, under "Structure of the Zero-Eigenvalue Eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$," the eigenvalues of $\underset{\sim}{R}$ are real or occur in complex-conjugate pairs if and only if $p_m=p'_m$. These parameters could be determined as in Test IV below, and compared. We indicate this approach only for consistency, as a formal alternative to the much simpler test described in Section II under "Testing the Operator." That preliminary test, which requires only writing out the characteristic polynomial of $\underset{\sim}{R}$, is clearly indicated in any test procedure.

Test II: determination of the number p of linearly independent metrics appropriate to $\underset{\sim}{R}$.- Since p is also the dimensionality of the null space of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$, it can be found from the rank r of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$, for $r=n^2-p$. The rank r can be determined, among other ways, by a sequence of elementary transformations defined by the following operations:

1. The interchange of two rows (columns).

2. The multiplication of all elements in a row (column) by a nonzero scalar.

3. The addition, to all elements of a row (column), of the corresponding elements of another row (column) multiplied by a scalar.

By a systematic application of these elementary transformations, any matrix can be put in its normal form,

$$\begin{pmatrix} \underset{\sim}{I}_r & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix}$$

where $\underset{\sim}{I}_r$ is the r×r identity matrix (ref. 5). We obtain p, then, from the order of $\underset{\sim}{I}_r$.

Test III: test for suitability of eigenvalues and structure of $\underset{\sim}{R}$.- By D, the eigenvalues of $\underset{\sim}{R}$ are real or occur in conjugate pairs, and chains of equal length are associated with conjugate pairs of eigenvalues, if and only if $p=p'$ – that is, $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$ and $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$ are equivalent matrices, having the same rank. The ranks of these matrices can be determined, and compared, by putting each

in its normal form. Unlike putting $\underset{\sim}{R}-\beta\underset{\sim}{I}$ and $\underset{\sim}{R}^{\dagger}-\beta\underset{\sim}{I}$ in their Smith normal forms, this does not involve the manipulation of polynomials and the determination of greatest common polynomial divisors. Furthermore, it is a property of equivalent matrices that one can be obtained from the other by elementary transformations, which permits a comparison without reducing each to its normal form.

Test IV: test for semisimplicity of $\underset{\sim}{R}$.- By E, $\underset{\sim}{R}$ is semisimple if and only if $p'=p'_m$. Since p'_m is the number of repeated zero-eigenvalues of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$, in principle p'_m could be determined by finding the term of lowest degree in the characteristic polynomial of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$ for which the coefficient does not vanish. Thus, a measure of the number of repeated eigenvalues of $\underset{\sim}{R}$, determined conventionally by finding the roots of the characteristic polynomial, can be obtained merely by inspection of the characteristic polynomial of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$. But while not requiring the solution of an algebraic equation, writing out the characteristic polynomial (or even determining a sufficient number of the polynomial coefficients to apply this test) involves considerable, if straightforward, calculation for a matrix of high order. Compared with the tests we are describing on $n^2\times n^2$ matrices, it is relatively simple to do for an $n\times n$ matrix, and is therefore indicated above in the first of these tests. But rather than write out the characteristic polynomial for an $n^2\times n^2$ matrix, we can avail ourselves of a more expeditious procedure to test $\underset{\sim}{R}$ for semisimplicity.

Denote the number p' of independent eigenvectors of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$ with zero eigenvalue by $p'(\underset{\sim}{\Lambda})$. In all there are p'_m linearly independent eigenvectors and generalized eigenvectors with zero eigenvalue, in chains $\underset{\sim}{x}^{(i,1)}, \underset{\sim}{x}^{(i,2)},\ldots\underset{\sim}{x}^{(i,\ell i)}$ such that

$$\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})\underset{\sim}{x}^{(i,1)} = \underset{\sim}{0}$$

$$\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})\underset{\sim}{x}^{(i,2)} = \underset{\sim}{x}^{(i,1)}$$

$$\vdots$$

$$\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})\underset{\sim}{x}^{(i,\ell_i)} = \underset{\sim}{x}^{(i,\ell_i-1)}$$

From the first two of these equations, we see that

$$[\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})]^2\,\underset{\sim}{x}^{(i,1)} = \underset{\sim}{0}$$

$$[\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})]^2\,\underset{\sim}{x}^{(i,2)} = \underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})\underset{\sim}{x}^{(i,1)} = \underset{\sim}{0}$$

Thus the $\underset{\sim}{x}^{(i,1)}$ and $\underset{\sim}{x}^{(i,2)}$ are eigenvectors of $[\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})]^2$ with zero eigenvalue.

There are then $p'(\underset{\sim}{\Lambda}^2)$ independent eigenvectors of $[\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})]^2$ with zero eigenvalue, consisting of the eigenvectors and generalized eigenvectors of rank 2, with zero eigenvalue, of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$. If the chains are not all of length 1, $p'(\underset{\sim}{\Lambda})<p'(\underset{\sim}{\Lambda}^2)$. Similarly, there are $p'(\underset{\sim}{\Lambda}^3)$ independent eigenvectors of $[\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})]^3$ with zero eigenvalue, consisting of the eigenvectors and generalized eigenvectors up to rank 3, with zero eigenvalue, of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$; and if the chains are not all of length less than 3, $p'(\underset{\sim}{\Lambda}^2)<p'(\underset{\sim}{\Lambda}^3)$. Finally, if ℓ_m if the length of the longest chain associated with the zero eigenvalue of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$, then the null space of $[\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})]^{\ell_m}$, and of all higher powers of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$, consists of the entire p'_m-dimensional zero-eigenvalue eigensubspace of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$. We therefore have

$$n \leq p'(\underset{\sim}{\Lambda}) < p'(\underset{\sim}{\Lambda}^2) <\ldots< p'(\underset{\sim}{\Lambda}^{\ell_m}) = p'(\underset{\sim}{\Lambda}^{\ell_m+1}) = p'_m \quad (107)$$

or denoting the rank of $[\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})]^q$ by $r'(\underset{\sim}{\Lambda}^q)$,

$$n^2 - n \geq r'(\underset{\sim}{\Lambda}) > r'(\underset{\sim}{\Lambda}^2) >\ldots> r'(\underset{\sim}{\Lambda}^{\ell_m}) = r'(\underset{\sim}{\Lambda}^{\ell_m+1}) = n^2 - p'_m \quad (108)$$

since $r'(\underset{\sim}{\Lambda}^q) = n^2 - p'(\underset{\sim}{\Lambda}^q)$.

Since by Eq. (107) $p'(\underset{\sim}{\Lambda})=p'_m$ requires that $p'(\underset{\sim}{\Lambda})=p'(\underset{\sim}{\Lambda}^2)$, it follows that $\underset{\sim}{R}$ is semisimple if and only if $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$ and $[\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})]^2$ are equivalent, having the same rank. The ranks of these matrices can be compared as described in the previous test.

The value of p'_m could be found by determining the rank of the smallest power of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$ having the same rank as the next higher power. (Similarly, $p_m=n^2-r(\underset{\sim}{\Lambda}^{\ell_m})$ could be found by determining the rank $r(\underset{\sim}{\Lambda}^{\ell_m})$ of the smallest power of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^\dagger,-\underset{\sim}{R}^T)$ having the same rank as the next higher power; but, if $\ell_m=1$, it does not follow that $\underset{\sim}{R}$ is semisimple.)

Test V: test for multiplicity of chains associated with distinct eigenvalues of $\underset{\sim}{R}$.- By F, there will be a single chain associated with each distinct eigenvalue of $\underset{\sim}{R}$ if and only if $p'=n$. As described in Test II, $p'=n^2-r'$ can be determined from the rank r' of $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$.

Test VI: test for distinct eigenvalues of $\underset{\sim}{R}$.- By H, the eigenvalues of $\underset{\sim}{R}$ are distinct if and only if $p'_m=n$, and thus by

Eqs. (107) and (108), if and only if $[\Lambda(R,-R)]^2$ is of rank n^2-n.

These tests can be applied independently, each assuming no special conditions on R that must be determined by another test. But as we would expect from the theory on which these tests are based, results of some tests can in certain instances be combined to infer the results of others. If, for example, it is found by Test I that the eigenvalues of R are real or occur in conjugate pairs, then Eq. (103) is valid. If it is further determined, either by Test V that $p'=n$, or by Test VI that $p_m'=n$, it follows that $p=p'=n$. Then without having determined p directly, we know that there are n independent metrics appropriate to R, and without Test III that these can be chosen nonsingular. Or if by Tests I and IV it is determined that the eigenvalues of R are real or occur in conjugate pairs, and R is semisimple, then by E and G, $p=p'=p_m' \geq n$; and again it is not necessary to apply Test III in order to conclude that the metrics can be chosen nonsingular.

Lumped Systems

The theory concerning the metrics appropriate to a lumped system can be reformulated along the same lines that we have discussed in detail for the uniform distributed system. Using the technique described in Section III under "The Equation $AX+XB^T=C$," we can write Eq. (36), the condition for a metric K to be appropriate to a lumped-system operator M, in the form

$$\Lambda(M^\dagger,-(M^T)^{-1})\ k'' = 0 \tag{109}$$

The relations among the properties of $\Lambda(M^\dagger,-(M^T)^{-1})$ and $\Lambda(M,-M)$, the eigenvalues and structure of M, and the number and character of the nontrivial solutions to Eq. (109), are then in complete parallel with those among the properties of $\Lambda(R^\dagger,-R^T)$ and $\Lambda(R,-R)$, the eigenvalues and structure of R, and the number and character of the nontrivial solutions to Eq. (96) — where again p and p' are the dimensionalities of the null spaces, and p_m and p_m' of the zero-eigenvalue eigensubspaces, of $\Lambda(M^\dagger,-(M^T)^{-1})$ and $\Lambda(M,-M)$ respectively. It is only necessary to substitute everywhere, for the conditions given by Eqs. (12) and (13) on the eigenvalues of R, the conditions of Eqs. (37) and (38) on the eigenvalues of M.

Alternatively, we could first obtain an equivalent R for the given M, as discussed under "Lumped Systems" in Section II, and reduce the problem to the form we discussed in detail.

We have thus far been occupied with a parallel treatment of the operators R and M, or $M(z,z_1)$, for constant R, in terms of equivalent conditions on their eigenvalues and structure, translated

into identical conditions on certain Kronecker sums, and with reference to the same set of constant metrics. We shall extend this parallel through the following section. Later we shall exploit some dissimilar properties of $\underset{\sim}{R}$ and $\underset{\sim}{M}(z,z_1)$, and introduce other metrics whose dependence on the operators does not exhibit such simple parallels.

IV. THE CONVERSE PROBLEM

Distributed Systems

In the previous sections we addressed ourselves to the problem of what can be said about the metrics appropriate to a system operator, for distributed systems of a restricted class, and those appropriate to a lumped-system operator. Some attention has also been given to the converse problem, for uniform distributed systems, of how an operator is determined by its appropriate metrics (ref.1). A theoretical treatment of this problem is of interest in that it permits the synthesis of systems governed by specified conservation laws, and more basically in that it points to the significance of the conservation laws in defining a system. A critical look at the published results will show that they are only partly satisfactory in providing such an alternative approach to system definition. (See also Section VII.)

Given any two independent, nonsingular, Hermitian, constant metrics $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$, the range of operators $\underset{\sim}{R}$ for which $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ are appropriate metrics can be determined. For $\underset{\sim}{K}_1$ to be appropriate to $\underset{\sim}{R}$, by Eq. (8) it is necessary and sufficient that $\underset{\sim}{K}_1\underset{\sim}{R}$ be Hermetian – i.e., that

$$\underset{\sim}{K}_1\underset{\sim}{R} = \underset{\sim}{H}$$

where $\underset{\sim}{H}$ is any Hermitian matrix. Thus

$$\underset{\sim}{R} = \underset{\sim}{K}_1^{-1}\underset{\sim}{H} \tag{110}$$

gives the range of $\underset{\sim}{R}$ having $\underset{\sim}{K}_1$ as an appropriate metric. For $\underset{\sim}{K}_2$ to be appropriate also, it is necessary and sufficient that

$$\underset{\sim}{R}^{\dagger}\underset{\sim}{K}_2 = \underset{\sim}{K}_2\underset{\sim}{R}$$

where $\underset{\sim}{R}$ is in the range given by Eq. (110). On substitution of Eq. (110), this relation can be put in the form

$$J^{\dagger}H = HJ \tag{111}$$

where J is defined by

$$J = K_1^{-1}K_2 \tag{112}$$

Equation (111) places a further restriction on H. Thus any operator R of the form of Eq. (110), where H is an Hermitian solution to Eq. (111), will have both K_1 and K_2 as appropriate metrics.

Equation (110) defines a class of systems with operators having K_1 as an appropriate metric, such that $s(K_1)$ is an invariant. Similarly, $R = K_2^{-1}H$, where H is any Hermetian matrix, defines a class of systems for which $s(K_2)$ is invariant. The class of systems for which both $s(K_1)$ and $s(K_2)$ are invariant can be defined as the intersection of these two classes.

The extent of this intersection, of the range of R, will depend on the choice of K_1 and K_2, for these may imply other independent metrics that must be appropriate to any R in this range. It is easy to verify that if K_1 and K_2 are appropriate to R, then every metric of the set

$$K_m = K_1J^{m-1}, \quad m=1, 2,... \tag{113}$$

is also appropriate. For by Eqs. (110) to (113),

$$R^{\dagger}K_m = HK_1^{-1}K_1J^{m-1} = HJ^{m-1} = J^{\dagger}HJ^{m-2} = \ldots = (J^{\dagger})^{m-1}H$$

$$K_mR = K_1J^{m-1}K_1^{-1}H = K_1J^{m-2}K_1^{-1}K_2K_1^{-1}H = K_1J^{m-2}K_1^{-1}J^{\dagger}H$$

$$= \ldots = K_1K_1^{-1}(J^{\dagger})^{m-1}H = (J^{\dagger})^{m-1}H$$

Since every matrix satisfies its characteristic equation, the $n\times n$ matrices J^{m-1}, for $m>n$, must be linear combinations of the J^{m-1}, $m=1, 2,...,n$. Thus at most $K_1,K_2,...,K_n$ (and at least K_1,K_2) will be linearly independent. The more independent metrics implied by a specified K_1 and K_2, the more restricted will be the class of systems that can be designed that will be governed by the conservation laws $ds(K_1)/dz = 0$ and $ds(K_2)/dz = 0$.

Specifying additional metrics, other than those implied, will further restrict the class of systems — ultimately to $\underset{\sim}{R}=\underset{\sim}{I}$, for which all metrics are appropriate. It is in this sense that a system is said to be defined by its appropriate metrics.*

Let us, however, look at these results in a little more detail. Let us first ascertain the minimum range of $\underset{\sim}{R}$ for specified $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$. With $\underset{\sim}{J}$ defined by Eq. (112), and recalling the assumptions on $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$, it is easily checked that

$$\underset{\sim}{H} = \underset{\sim}{K}_1 \qquad \underset{\sim}{H} = \underset{\sim}{K}_2 \tag{114}$$

are independent, nonsingular solutions to Eq. (111). But Eq.(111) is of the form of Eq. (8), and we saw in Section II that if an equation of this form has one nonsingular solution, it has at least n independent solutions, which can be chosen Hermitian.Thus, by Eq. (110) there will be at least n independent operators $\underset{\sim}{R}$ having $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ as appropriate metrics — regardless of the number of additional independent metrics $\underset{\sim}{K}_1\underset{\sim}{J}^{m-1}$, m=3, 4,..., that are implied by $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$, and that restrict the range of $\underset{\sim}{R}$.

The number of independent metrics $\underset{\sim}{K}_1\underset{\sim}{J}^{m-1}$ is inversely related to the degeneracy in $\underset{\sim}{J}$. In the notation of Section III, if different subscripts are used to index repeated eigenvalues of $\underset{\sim}{J}$ when these are associated with independent eigenvectors or separate chains, and if ℓ_r is the length of the chain with eigenvalue η_r, the characteristic equation of $\underset{\sim}{J}$ can be written

$$\prod_r (\eta-\eta_r)^{\ell_r} = 0 \tag{115}$$

and is of degree η. However, this may not be the polynomial equation of lowest degree that $\underset{\sim}{J}$ satisfies. The latter will be of the form

$$\prod_{\bar{r}} (\eta-\eta_{\bar{r}})^{\max\, \ell_r,\ r\sim\bar{r}} = 0 \tag{116}$$

where the $\eta_{\bar{r}}$ are distinct, and max ℓ_r, $r\sim\bar{r}$, is the length of the longest chain with eigenvalue $\eta_r = \eta_{\bar{r}}$. The polynomial on the left

*The properties of an operator on a wave basis having an appropriate metric of canonical form for pairwise coupling have also been arrived at by an approach beginning with the specification of the metric (see ref. 10).

in Eq. (116) is known as the minimum polynomial of $\underset{\sim}{J}$, and is of degree not exceeding n. The less degeneracy in $\underset{\sim}{J}$, the larger will be the degree of the minimum polynomial, and thus the greater the number of linearly independent metrics $\underset{\sim}{K}_1\underset{\sim}{J}^{m-1}$.

But applying the results of Sections II and III, we see that the less degeneracy in $\underset{\sim}{J}$, the fewer linearly independent solutions $\underset{\sim}{H}$ there will be to Eq. (111), and the smaller will be the range of $\underset{\sim}{R}$. This explains the mechanism by which the range of $\underset{\sim}{R}$ is restricted by the specification of metrics $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ that imply additional metrics. The degeneracy in $\underset{\sim}{J}$ will be minimum when $\underset{\sim}{J}$ is semisimple with distinct eigenvalues, or when only one chain is associated with each distinct eigenvalue. The minimum polynomial will then be the same as the characteristic polynomial, and there will be n independent metrics in the set $\underset{\sim}{K}_1\underset{\sim}{J}^{m-1}$. But we have seen that under this condition, and only then, will there be only n independent solutions to Eq. (111), and thus a minimum n-dimensional range for $\underset{\sim}{R}$.

We also note that

$$\underset{\sim}{R}_k = \underset{\sim}{J}^{k-1}, \quad k = 1,2,... \tag{117}$$

are operators in the range of $\underset{\sim}{R}$. For

$$(\underset{\sim}{J}^{k-1})^\dagger \underset{\sim}{K}_1 = (\underset{\sim}{J}^\dagger)^{k-1} \underset{\sim}{K}_1 = (\underset{\sim}{J}^\dagger)^{k-2} \underset{\sim}{K}_2\underset{\sim}{K}_1^{-1}\underset{\sim}{K}_1 = (\underset{\sim}{J}^\dagger)^{k-2} \underset{\sim}{K}_1\underset{\sim}{K}_1^{-1}\underset{\sim}{K}_2$$

$$= (\underset{\sim}{J}^\dagger)^{k-2} \underset{\sim}{K}_1\underset{\sim}{J}$$

$$= \ldots = \underset{\sim}{K}_1\underset{\sim}{J}^{k-1}$$

$$(\underset{\sim}{J}^{k-1})^\dagger \underset{\sim}{K}_2 = (\underset{\sim}{J}^\dagger)^{k-1} \underset{\sim}{K}_2 = (\underset{\sim}{J}^\dagger)^{k-2} \underset{\sim}{K}_2\underset{\sim}{K}_1^{-1}\underset{\sim}{K}_2 = (\underset{\sim}{J}^\dagger)^{k-2}\underset{\sim}{K}_2\underset{\sim}{J}$$

$$= \ldots = \underset{\sim}{K}_2\underset{\sim}{J}^{k-1}$$

and so $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ are appropriate to each $\underset{\sim}{R}_k$.

If the degeneracy in $\underset{\sim}{J}$ is minimum, there will be n independent $\underset{\sim}{R}_k$. But the range of $\underset{\sim}{R}$ is then only n-dimensional, and so there are no more independent operators in this range. Furthermore, there will then be n independent metrics $\underset{\sim}{K}_m = \underset{\sim}{K}_1\underset{\sim}{J}^{m-1}$, m=1,2,...,n

which are all appropriate to each of the $\underset{\sim}{R}_k$, k=1, 2,...,n.

We cannot in general further restrict the range of $\underset{\sim}{R}$ selectively by the specification of additional metrics, such as to define each independent $\underset{\sim}{R}$ in this range uniquely in terms of its appropriate metrics. Any metric will be appropriate to $\underset{\sim}{R}_1=\underset{\sim}{I}$. Since the degeneracy in $\underset{\sim}{J}$ is minimum, $\underset{\sim}{R}_2=\underset{\sim}{J}$ will have exactly n independent, appropriate metrics; i.e., only the $\underset{\sim}{K}_m$ that we have already determined. Further degeneracy may or may not appear in successive powers of $\underset{\sim}{J}$. For example, if $\underset{\sim}{R}_2=\underset{\sim}{J}$ is semisimple with distinct eigenvalues η_r, then $\underset{\sim}{R}_k=\underset{\sim}{J}^{k-1}$, k=3, 4,..., will be semisimple with eigenvalues η_r^{k-1}. Clearly the η_r^{k-1}, for any k>2, may or may not be distinct (e.g., if $\eta_1=1$, $\eta_2=-1$, then $\eta_1^2=\eta_2^2=1$; but a situation of this kind is not representative). If successive powers of $\underset{\sim}{J}$ do not become more degenerate, the $\underset{\sim}{K}_m$, m=1,2,...,n, will be a maximal set of independent metrics appropriate to each of the independent operators $\underset{\sim}{R}_k$, k=2,3,...,n. (It is easily verified that another independent operator can be included, making a total of n, by adding $\underset{\sim}{I}$ to any one of the $\underset{\sim}{R}_k$, k=2,3,...,n.)

If the degeneracy in $\underset{\sim}{J}$ is not minimum, the range of $\underset{\sim}{R}$ will be greater than n-dimensional. There will still be q independent operators in this range of the form $\underset{\sim}{R}_k=\underset{\sim}{J}^{k-1}$, k=1,2,...,q $(2\leq q<n)$, where q is the degree of the minimum polynomial of $\underset{\sim}{J}$; and the q independent metrics $\underset{\sim}{K}_m=\underset{\sim}{K}_1\underset{\sim}{J}^{m-1}$, m=1,2,...,q, will be appropriate to each of these operators. Since the degeneracy in $\underset{\sim}{J}$ is not minimum, the total number of independent metrics appropriate to $\underset{\sim}{R}_2=\underset{\sim}{J}$ will, however, be greater than n. But let $\underset{\sim}{K}_i$ be a metric, appropriate to $\underset{\sim}{R}_2$, that is independent of the $\underset{\sim}{K}_m$, m=1,2,...,q. Then

$$\underset{\sim}{J}^{\dagger}\underset{\sim}{K}_i = \underset{\sim}{K}_i\underset{\sim}{J} \tag{118}$$

Postmultiplying both sides by $\underset{\sim}{J}$, and making use of Eq. (118) in the result, we obtain

$$(\underset{\sim}{J}^2)^{\dagger}\underset{\sim}{K}_i = \underset{\sim}{K}_i\underset{\sim}{J}^2 \tag{119}$$

and, accordingly, $\underset{\sim}{K}_i$ is appropriate also to $\underset{\sim}{R}_3=\underset{\sim}{J}^2$. By extension, $\underset{\sim}{K}_i$, and thus all of the metrics appropriate to $\underset{\sim}{R}_2$, are appropriate to each $\underset{\sim}{R}_k$, k=1,2,...,q. Again, further degeneracy may or may not appear in successive powers of $\underset{\sim}{J}$. (Consider, for example, a semisimple $\underset{\sim}{J}$ with repeated eigenvalues. The distinct eigenvalues raised to a given power will normally, but not necessarily, be distinct.) If successive powers of $\underset{\sim}{J}$ do not become more degenerate, the independent operators $\underset{\sim}{R}_k$, k=2,3,...,q, (and $\underset{\sim}{R}_2+\underset{\sim}{I}$) will have the same set of appropriate metrics.

We can easily construct examples also of nonsemisimple $\underset{\sim}{J}$, for which the degeneracy does not increase in powers of $\underset{\sim}{J}$. And if the degeneracy in $\underset{\sim}{J}$ is not minimum, we can often find a set of operators in the range of $\underset{\sim}{R}$ that are independent of the $\underset{\sim}{R}_k$, k=1,2,...,q, and which also have a common set of appropriate metrics. In fact, with $\underset{\sim}{R}$ substituted for $\underset{\sim}{J}$, Eq. (119) follows from Eq. (118) for any $\underset{\sim}{R}$ in the range; and while the degree of degeneracy in such an $\underset{\sim}{R}$ need not be the same as that in $\underset{\sim}{J}$, independent powers of $\underset{\sim}{R}$ having the same degree of degeneracy will have the same appropriate metrics.

We have indicated how the degeneracy in $\underset{\sim}{J}$ determines both the range of $\underset{\sim}{R}$ and the number of metrics implied by $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$, and gives some measure of the extent to which this range can be subdivided by the specification of additional metrics. We have found that specifying a set of nonsingular metrics, and stipulating that it be maximal, still in general determines $\underset{\sim}{R}$ only within a range.

Furthermore, this approach to system definition, at least in the form given, lacks generality. The procedure does not apply to a specified set of metrics that cannot be chosen nonsingular in linear combination, but which also define a range of operators. And there is a range of uniform systems that do not obey any conservation law. Specifying the constant $\underset{\sim}{K}$ appropriate to a non-uniform-system operator would in general be even less useful for discriminating among systems. (It does not follow from the existence of one constant, nonsingular solution to Eq. (8), or Eq.(36) for $\underset{\sim}{M}$ a matricant, that there need be more, if $\underset{\sim}{R}$ is not constant — but we shall see in Section V that the specification of z-dependent solutions to the latter equation will lead to a different interpretation of the conservation laws governing a range of systems determined by a variation of this procedure.) The results of this section suggest that the constant $\underset{\sim}{K}$ might play a role in a more general, exact scheme for system definition; but clearly something more must be incorporated in this scheme. What this must be, and its physical interpretation, will become clearer as we proceed, and the question will be resolved finally in Section VII.

Lumped Systems

We next ask how the range of lumped-system operators having specified appropriate metrics can be determined.

A set of constant $\underset{\sim}{K}$ defines a range of constant $\underset{\sim}{R}$, and thereby a range of matricants as solutions to Eqs. (39) and (41) for $\underset{\sim}{R}$ in this range. A range of lumped-system operators $\underset{\sim}{M}$ that are $\underset{\sim}{K}$-Unitary with respect to every $\underset{\sim}{K}$ in this set is given by the range of matricants evaluated at $z_1 = 0$ and any $z = L$.

The metrics of Eqs. (5) and (71), for example, define constant $\underset{\sim}{R}$ in the range

$$\underset{\sim}{R} = \begin{pmatrix} a_1 & 0 \\ a_2 & a_1 \end{pmatrix} \tag{120}$$

where a_1 and a_2 are real constants. The range of matricants for $\underset{\sim}{R}$ in this range is given by

$$\underset{\sim}{M}(z,0) = e^{-ja_1 z} \begin{pmatrix} 1 & 0 \\ -ja_2 z & 1 \end{pmatrix} \tag{121}$$

Evaluated at $z = L \neq 0$, with $a_1 = 2\pi n/L$ and $a_2 = b/L$ (n=any integer), this is the operator $\underset{\sim}{M}$ of Eq. (69) for a lumped shunt susceptance jb, and for which we found the specified metrics were appropriate. More generally $\underset{\sim}{M}$ describes a lossless system at the edge of a pass band - reciprocal and symmetrical if $e^{-ja_1 L} = \pm 1$.

It is interesting to note, as in this example, that certain metrics may define distributed systems that do not appear to be of physical interest, but which in turn give rise to lumped systems of interest.

For a given $\underset{\sim}{R}$, $\underset{\sim}{M}(L,0)$ depends on the choice of L. Furthermore, $\underset{\sim}{K}$-Unitary $\underset{\sim}{M}$ can be obtained as the matricant, evaluated at $z = L$, of $\underset{\sim}{R}$ outside this range, including $\underset{\sim}{R}(z)$, and can describe a variety of physical systems.

The significant point, however, is that for any choice of $z = L \neq 0$, the technique described gives the complete range of nonsingular $\underset{\sim}{M}$ for which the specified $\underset{\sim}{K}$ are appropriate. For given any $\underset{\sim}{M}$ in this range -- which could describe a section of a nonuniform distributed system, a network of lumped elements, or an arbitrary transformation -- $\underset{\sim}{M}$ has an equivalent representation as a section, of length L, of a uniform system described by an operator $\underset{\sim}{R}$ for which these metrics are appropriate. This operator will therefore be in the range of constant $\underset{\sim}{R}$ defined by the specified metrics, and $\underset{\sim}{M}$ will be its matricant evaluated at $z = L$.

Even for a given choice of $z = L$, there will not be a one-to-one correspondence between $\underset{\sim}{M}$ and $\underset{\sim}{R}$ in their respective ranges.

As indicated in Section II under "Lumped Systems," and as may be seen in the above example, there are still an infinite number of choices for $\underset{\sim}{R}$ in its range from which we can obtain a given $\underset{\sim}{M}$.

But the range of $\underset{\sim}{M}$, which is independent of the choice of L, is determined by the range of constant $\underset{\sim}{R}$ defined by the specified metrics. Lumped systems can therefore be treated by the techniques developed for uniform distributed systems. The relation between operators for lumped and uniform distributed systems will be indicated more formally in Section VII, and it is particularly appropriate that they can be so related. For if we further delimit the range of $\underset{\sim}{R}$, in a manner to be discussed, we will in general further restrict the range of $\underset{\sim}{M}$; but unless this is done by specifying additional constant $\underset{\sim}{K}$, we shall not be distinguishing between operators in this range on the basis of what can be interpreted strictly as conservation laws. We shall find that this is a feature of uniform systems in particular.

V. NONUNIFORM SYSTEMS

In this section we generalize some of the previous results to nonuniform systems, and we introduce some metrics that we allow to vary with z in a manner appropriate to the conservation laws governing such systems. We consider two ways in which this can be done, each proceeding from a different interpretation of the conservation laws and providing different insights into the more general problem, but either providing a basis for analysis or synthesis. By ascertaining what is required of a metric for it to determine a conservation law, we justify the assumption that these must be constant for uniform systems as a special case. But the z-dependent metrics introduced in this section will later have application in characterizing uniform systems as well.

The condition for a quadratic form $s(\underset{\sim}{K})$, $\underset{\sim}{K}$ constant, to be invariant under the operation $\underset{\sim}{R}$, was given in Section II to be that $\underset{\sim}{K}$ satisfy

$$\underset{\sim}{R}^{\dagger}\underset{\sim}{K} = \underset{\sim}{K}\underset{\sim}{R} \tag{8}$$

This condition was derived without assuming $\underset{\sim}{R}$ constant, and accordingly it applies to both uniform and nonuniform systems. But the subsequent conditions on $\underset{\sim}{R}$, or on $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger}, -\underset{\sim}{R}^{T})$ and $\underset{\sim}{\Lambda}(\underset{\sim}{R},-\underset{\sim}{R})$, for the existence of constant (nonsingular or singular) solutions to Eq. (8), the expressions for the number of these that will be linearly independent, and the procedures for obtaining constant solutions were arrived at on the assumption that $\underset{\sim}{R}$ is constant.

If $\underset{\sim}{R}$ is not constant, the same procedures can be used formally to obtain solutions to Eq. (8). The number of independent solutions will depend as before on the conjugacy properties of the eigenvalues of $\underset{\sim}{R}$, the dimensionality of the eigensubspaces, and their organization into chains of generalized eigenvectors -- any or all of which, however, may depend on position. But if they do not, then on a Jordan basis they will determine a number p of independent, constant solutions to Eq. (19),

$$\underset{\sim}{R}'^{\dagger}\underset{\sim}{K}' = \underset{\sim}{K}'\underset{\sim}{R}' \tag{19}$$

where p has the same dependency as before on the eigenvalues and structure of $\underset{\sim}{R}$ -- although the eigenvalues, eigenvectors, and generalized eigenvectors are allowed to vary with z subject to the above constraints. If the eigenvectors and generalized eigenvectors, and thus the modal matrix $\underset{\sim}{S}$, are constant, the $\underset{\sim}{K}'$ can be transformed back to the same number p of independent, constant metrics $\underset{\sim}{K}$ on the original basis by Eq. (24). In this case, all of the theoretical results given for constant $\underset{\sim}{R}$ apply (although, as we shall see, we must be somewhat more careful in interpreting the conservation laws). The theory developed thus far, then, applies to a somewhat broader class of operators than assumed. If the modal matrix is not constant, however, there is no guarantee that any or all of the $\underset{\sim}{K}'$, in any set of independent situations to Eq. (19), will transform to constant $\underset{\sim}{K}$.

There will still be the same number p of independent solutions to Eq. (8). But if $\underset{\sim}{K} = \underset{\sim}{K}(z)$, then apart from any question of $s(\underset{\sim}{K})$ having the same interpretation for all z in terms of properties of the system, Eq. (8) no longer gives the condition on $\underset{\sim}{K}$ for $s(\underset{\sim}{K})$ to be invariant. Equation (7) then becomes

$$\frac{ds}{dz} = \frac{d\underset{\sim}{x}^{\dagger}}{dz}\underset{\sim}{K}\underset{\sim}{x} + \underset{\sim}{x}^{\dagger}\frac{d\underset{\sim}{K}}{dz}\underset{\sim}{x} + \underset{\sim}{x}^{\dagger}\underset{\sim}{K}\frac{d\underset{\sim}{x}}{dz}$$

$$= j\underset{\sim}{x}^{\dagger}(\underset{\sim}{R}^{\dagger}\underset{\sim}{K} - j\frac{d\underset{\sim}{K}}{dz} - \underset{\sim}{K}\underset{\sim}{R})\underset{\sim}{x} = 0$$

and the condition for $s(\underset{\sim}{K})$ to be invariant is more generally

$$\underset{\sim}{R}^{\dagger}\underset{\sim}{K} - \underset{\sim}{K}\underset{\sim}{R} = j\frac{d\underset{\sim}{K}}{dz} \tag{122}$$

For constant $\underset{\sim}{K}$, Eq. (122) reduces to Eq. (8).

If $s(\underset{\sim}{K})$ is invariant, and $\underset{\sim}{K} = \underset{\sim}{K}(z)$, then

$$\underset{\sim}{x}^{\dagger}(z)\underset{\sim}{K}(z)\underset{\sim}{x}(z) = \underset{\sim}{x}^{\dagger}(z_1)\underset{\sim}{K}(z_1)\underset{\sim}{x}(z_1) \tag{123}$$

where z_1 is any fixed value of z. By Eq. (42), the matricant $\underset{\sim}{M}(z,z_1)$ must then satisfy

$$\underset{\sim}{M}^{\dagger}(z,z_1)\underset{\sim}{K}(z)\underset{\sim}{M}(z,z_1) = \underset{\sim}{K}(z_1) \tag{124}$$

Equations (122) and (124) are equivalent, and one can be obtained from the other with the use of Eqs. (39) and (41).

Equation (122) is of the same form as Eq. (92), with $\underset{\sim}{C} = j(d\underset{\sim}{K}/dz)$ chosen as the derivative of the unknown $\underset{\sim}{K}$ instead of as a constant matrix. The derivation of Eq. (94) from Eq. (92) is not affected if $\underset{\sim}{C}$ is of this form, and accordingly Eq. (122) can be written

$$\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})\underset{\sim}{k}'' = j\,\frac{d\underset{\sim}{k}''}{dz} \tag{125}$$

This is a differential equation in the form of Eq. (1), where in this case the operator $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$ is a Kronecker sum, and $\underset{\sim}{k}''$ is an n^2-dimensional state vector formed from the elements of $\underset{\sim}{K}$ in the manner described in Section III. We shall find in Section VII that $\underset{\sim}{k}''$ is a state vector of the system in not only a formal, but a very real sense. If $\underset{\sim}{R}$ satisfies the Lipschitz condition, $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$ will also; for its elements are simply elements of $-\underset{\sim}{R}$, conjugates of elements of $\underset{\sim}{R}$, or sums of these. The same condition that ensures the existence of a matricant, then, ensures the existence of n^2 independent solutions to Eq. (125). The results of Section III give, for $\underset{\sim}{R}$ constant, the number p of independent solutions that can be chosen constant.

Thus it is only necessary that $\underset{\sim}{R}$ satisfy the Lipschitz condition for there to exist n^2 independent, invariant forms $s(\underset{\sim}{K})$. Let us look more carefully, for the general case where $\underset{\sim}{R}$ is not restricted to be a constant operator, at the conditions for such forms to represent invariant properties of the system. A system is defined by the elements r_{ij} of its operator $\underset{\sim}{R}$, and the variables chosen as components of the state vector $\underset{\sim}{x}$. A consistent definition of what we mean by any property of the system requires that it depend in the same way on the r_{ij} and the components of $\underset{\sim}{x}$ everywhere in the system — that is, that it does not depend explicitly on z. Thus it is reasonable to interpret an invariant form $s(\underset{\sim}{K})$ as a conservation law, although $\underset{\sim}{K}$ is a function of z, if $\underset{\sim}{K}$ can be written as an explicit function of the $r_{ij}(z)$ only:

$$\underset{\sim}{K} = \underset{\sim}{K}(r_{ij}(z)) \tag{126}$$

Constant $\underset{\sim}{K}$ are a special case of Eq. (126), for both uniform and nonuniform systems. If the r_{ij} are constant, then conversely $\underset{\sim}{K}$ must be constant to be of this form. Thus we are justified in having restricted our attention to constant $\underset{\sim}{K}$ in considering the conservation laws governing a uniform system. But for a nonuniform system, any solution $\underset{\sim}{K}(z)$ to Eq. (122) will meet this requirement if, on substitution of z expressed in terms of any of the r_{ij} by inverting $r_{ij} = r_{ij}(z)$, $\underset{\sim}{K}(r_{ij})$ is single valued.

All $s(\underset{\sim}{K}) = f(x_i, r_{ij})$ derived in this manner from z-dependent solutions to Eq. (122) are not likely to be identifiable as familiar invariants such as the power or the energy. But there is no guarantee either that the p independent invariants we have been considering for uniform systems will have simple physical interpretations, and Eq. (126) is the logical extension of constant $\underset{\sim}{K}$ to nonuniform systems. Furthermore, if the system is nonuniform, it is only by an examination of this larger class of metrics that we can be sure we have identified all the invariants of clear physical interest. (See example in the following section.)

Although, for constant $\underset{\sim}{K}$, an invariant form $s(\underset{\sim}{K})$ always represents an invariant property of the system, by the same token we must be careful in interpreting this property if the system is nonuniform. Suppose, for example, that the system includes a capacitance that varies in some manner $C = C_0 g(z)$, $g(0) = 1$, and it is determined that $(1/2)C_0|V|^2$ is invariant, where V is the voltage. If the system were uniform, g(z) constant, we could conclude that the stored electric energy is conserved; but in general we can only say that $|V|^2$ is invariant. (The conservation of electric energy — the invariance of $s(\underset{\sim}{K}) = (1/2)C|V|^2$, where then $\underset{\sim}{K} = \underset{\sim}{K}(z)$ — would have to be indicated by a z-dependent solution to Eq. (122).) On the other hand, an invariant form $(1/2)\mathrm{Re}\ VI^*$ implies power conservation in any case, for the expression for the power does not depend on z-dependent parameters of the system.

By introducing a still different set of $\underset{\sim}{K}(z)$, we can parallel for nonuniform systems the technique discussed in Section IV for determining the range of operators with specified appropriate metrics. But first let us point out that the results given there are not all restricted to uniform systems. If constant $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ are specified, it is not in fact necessary to choose solutions $\underset{\sim}{H}$ to Eq. (111) to be constant. The arbitrary constants in the general Hermitian solution can as well be arbitrary functions of z. This leads by Eq. (110) to the complete range of systems, uniform and nonuniform, with operator $\underset{\sim}{R}$ having constant $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ as appropriate metrics. It is not necessary that there be any more

independent, constant metrics appropriate to an $\underset{\sim}{R}$ in this range, however, if $\underset{\sim}{R}$ is not constant — unless there are more in the set $\underset{\sim}{K}_1\underset{\sim}{J}^{m-1}$ implied by $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$.

More generally, if the conjugacy properties of $\underset{\sim}{M}(z,z_1)$, the dimensionality of the eigensubspaces, and their organization into chains of generalized eigenvectors do not depend on z (except at discrete values) there will be p linearly independent solutions to the equation

$$\underset{\sim}{M}^{\dagger}(z,z_1)\underset{\sim}{K}(z)\underset{\sim}{M}(z,z_1) = \underset{\sim}{K}(z) \tag{127}$$

where p depends on the degeneracy in $\underset{\sim}{M}(z,z_1)$ exactly as was determined for uniform systems. Note that for any lossless, nonuniform system, with at least the one invariant $s(\underset{\sim}{K})$, $\underset{\sim}{K}$ constant, representing the power, there must by our earlier results be at least n independent, nonsingular solutions to Eq. (127) — for there exists an appropriate nonsingular $\underset{\sim}{K}$ at every value of z.

Equation (127) implies that

$$\underset{\sim}{x}^{\dagger}(z)\underset{\sim}{K}(z)\underset{\sim}{x}(z) = \underset{\sim}{x}^{\dagger}(z_1)\underset{\sim}{K}(z)\underset{\sim}{x}(z_1) \tag{128}$$

For a given z, $\underset{\sim}{K}(z)$ can be considered a constant metric. The $s(\underset{\sim}{K})$ formed from solutions to Eq. (127) will then represent properties of the system that are invariant when transformed from z_1 to z, but the invariant properties will in general depend on z. But this suggests that we can define a nonuniform system in terms of its z-dependent invariant properties in a manner parallel to the way in which we defined a uniform system in terms of its z-independent invariant properties.

Given two independent, nonsingular, Hermetian metrics $\underset{\sim}{K}_1(z)$ and $\underset{\sim}{K}_2(z)$, we can consider an $\underset{\sim}{M}(z,z_1)$ for which these are appropriate as represented by an equivalent uniform line from z_1 to z. The system operator for this uniform line can be denoted by $\underset{\sim}{R}_0(z)$, with z as a parameter. Defining

$$\underset{\sim}{J}(z) = \underset{\sim}{K}_1^{-1}(z)\ \underset{\sim}{K}_2(z) \tag{129}$$

we obtain the Hermitian solutions to

$$\underset{\sim}{J}^{\dagger}(z)\underset{\sim}{H}(z) = \underset{\sim}{H}(z)\underset{\sim}{J}(z) \tag{130}$$

Since $\underline{\underline{H}} = \underline{\underline{K}}_1, \underline{\underline{K}}_2$ are nonsingular solutions to Eq. (130), there must be at least n independent solutions. There may not be the same number at every value of z, and the $\underline{\underline{H}}(z)$ may be otherwise not well behaved, if the conjugacy properties of the eigenvalues or the degeneracy in $\underline{\underline{J}}$ changes with z. But we can always choose at least n independent solutions. We thus determine a range of $\underline{\underline{R}}_0(z)$ by

$$\underline{\underline{R}}_0(z) = \underline{\underline{K}}_1^{-1}(z)\ \underline{\underline{H}}(z) \tag{131}$$

Since $\underline{\underline{R}}_0(z)$ is the operator for a uniform line, it is straightforward to determine its matricant as the solution to the equation

$$j\frac{d\underline{\underline{M}}_0(z',z_1)}{dz'} = \underline{\underline{R}}_0(z)\underline{\underline{M}}_0(z',z_1),\ \underline{\underline{M}}_0(z_1,z_1) = \underline{\underline{I}} \tag{132}$$

where $\underline{\underline{M}}_0(z',z_1)$ contains z as a parameter. Corresponding to the range of $\underline{\underline{R}}_0(z)$ is a range of $\underline{\underline{M}}_0(z',z_1)$ having $\underline{\underline{K}}_1(z)$ and $\underline{\underline{K}}_2(z)$ as appropriate metrics, and in particular this is true at $z' = z$. Thus $\underline{\underline{K}}_1(z)$ and $\underline{\underline{K}}_2(z)$ are solutions to Eq. (127), with

$$\underline{\underline{M}}(z,z_1) = \underline{\underline{M}}_0(z',z_1)\Big|_{z' = z} \tag{133}$$

and the $\underline{\underline{M}}(z,z_1)$ are lumped representations for the range of systems defined by these conservation laws. If the $\underline{\underline{H}}(z)$ are sufficiently well behaved that the $\underline{\underline{M}}(z,z_1)$ are differentiable, then the latter are matricants from which a range of system operators can be determined by

$$\underline{\underline{R}}(z) = j\frac{d\underline{\underline{M}}}{dz}\,\underline{\underline{M}}^{-1} \tag{134}$$

This approach to system definition, in terms of metrics defined by Eq. (127), has the same limitations as the parallel approach for uniform systems. For this purpose, we shall find that the metrics defined by Eq. (122) are more significant. These are quite different definitions of $\underline{\underline{K}}(z)$, for with $\underline{\underline{R}}$ given by Eq. (134), we find that solutions to Eq. (127) must satisfy

$$\underline{\underline{R}}^{\dagger}\underline{\underline{K}} - \underline{\underline{K}}\underline{\underline{R}} = j\left[\frac{d\underline{\underline{K}}}{dz} - (\underline{\underline{M}}^{\dagger})^{-1}\,\frac{d\underline{\underline{K}}}{dz}\,\underline{\underline{M}}^{-1}\right] \tag{135}$$

and so do not in general satisfy Eq. (122). Incidentally, Eq. (135) indicates that we cannot obtain or characterize the solutions

to Eq. (127) by a parallel treatment of $\underset{\sim}{R}$, as we could for uniform systems. The solution of Eq. (135) for $\underset{\sim}{K}$ requires the determination of $\underset{\sim}{M}$, and Eq. (135) thus offers no real alternative to Eq. (127) in the study of nonuniform systems with this interpretation of the conservation laws.

VI. USE OF DIRECT PRODUCT TO OBTAIN THE CONSERVATION LAWS

The conservation laws governing a distributed, uniform or nonuniform system can be determined from another property of the Kronecker product and sum, if the matricant is known. Like the procedure described in Section III, and unlike that of Section II, this procedure does not require determining the eigenvalues, eigenvectors, and generalized eigenvectors of $\underset{\sim}{M}(z,z_1)$, or of $\underset{\sim}{R}$ obtained by Eq. (46). If only $\underset{\sim}{R}$ is known, it is still sometimes easier to determine the matricant (e.g., by the method of projectors (ref.3), for $\underset{\sim}{R}$ constant) than the eigenvalues and modal matrix. If $\underset{\sim}{R}$ is not constant, but satisfies the Lipschitz condition, the matricant can be determined at least approximately (refs. 3 and 11); and in either case this procedure offers a method of solution for Eq. (122), providing additional information beyond that given by the constant $\underset{\sim}{K}$.

It is well known that the direct product of the matricants $\underset{\sim}{M}_A(z,z_1)$ and $\underset{\sim}{M}_B(z,z_1)$ of two n×n operators $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is the matricant of the Kronecker sum $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$. If, that is,

$$\frac{d\underset{\sim}{M}_A}{dz} = -j\underset{\sim}{A}\underset{\sim}{M}_A,\ \underset{\sim}{M}_A(z_1,z_1) = \underset{\sim}{I}$$

$$\frac{d\underset{\sim}{M}_B}{dz} = -j\underset{\sim}{B}\underset{\sim}{M}_B,\ \underset{\sim}{M}_B(z_1,z_1) = \underset{\sim}{I} \qquad (136)$$

then, writing $\underset{\sim}{M}_A(z,z_1) \times \underset{\sim}{M}_B(z,z_1)$ as $\underset{\sim}{M}_A \times \underset{\sim}{M}_B(z,z_1)$,

$$\frac{d(\underset{\sim}{M}_A \times \underset{\sim}{M}_B)}{dz} = -j\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})[\underset{\sim}{M}_A \times \underset{\sim}{M}_B],\ \underset{\sim}{M}_A \times \underset{\sim}{M}_B(z_1,z_1) = \underset{\sim}{I} \qquad (137)$$

It is easily verified that

$$\frac{d(\underset{\sim}{M}_A \times \underset{\sim}{M}_B)}{dz} = \frac{d\underset{\sim}{M}_A}{dz} \times \underset{\sim}{M}_B + \underset{\sim}{M}_A \times \frac{d\underset{\sim}{M}_B}{dz} = -j[(\underset{\sim}{A}\underset{\sim}{M}_A) \times \underset{\sim}{M}_B + \underset{\sim}{M}_A \times \underset{\sim}{B}\underset{\sim}{M}_B)]$$

Writing $\underset{\sim}{M}_B = \underset{\sim}{I}\underset{\sim}{M}_B$ in the first term and $\underset{\sim}{M}_A = \underset{\sim}{I}\underset{\sim}{M}_A$ in the second term, and with the use of Eq. (82) and the definition, Eq. (84), of $\underset{\sim}{\Lambda}(\underset{\sim}{A},\underset{\sim}{B})$, Eq. (137) follows directly. Since the eigenvalues of $\underset{\sim}{M}_A \times \underset{\sim}{M}_B$ are products of the eigenvalues of $\underset{\sim}{M}_A$ and the eigenvalues of $\underset{\sim}{M}_B$, and $\underset{\sim}{M}_A$ and $\underset{\sim}{M}_B$ are nonsingular, $\underset{\sim}{M}_A \times \underset{\sim}{M}_B$ is also nonsingular — as required of a matricant. The boundary condition follows from those in Eq. (136) and the definition of the direct product.

We found in Section V that Eq. (122) can be put in the form of Eq. (125), and that the same condition that ensures the existence of a matricant for $\underset{\sim}{R}$ ensures the existence of n^2 linearly independent solutions $\underset{\sim}{k}"$ to Eq. (125). If any set of linearly independent solutions is arranged as the columns of an $n^2 \times n^2$ matrix $\underset{\sim}{K}"(z)$, then $\underset{\sim}{K}"(z)$ satisfies

$$\frac{d\underset{\sim}{K}"}{dz} = -j\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger}, -\underset{\sim}{R}^{T})\ \underset{\sim}{K}" \tag{138}$$

By Eq. (137), Eq. (138) is satisfied by

$$\underset{\sim}{K}"(z,z_1) = \underset{\sim}{M}_{\underset{\sim}{R}^{\dagger}} \times \underset{\sim}{M}_{-\underset{\sim}{R}^{T}},\quad \underset{\sim}{K}"(z_1,z_1) = \underset{\sim}{I} \tag{139}$$

where $\underset{\sim}{M}_{\underset{\sim}{R}^{\dagger}}(z,z_1)$ is the matricant of $\underset{\sim}{R}^{\dagger}$ and $\underset{\sim}{M}_{-\underset{\sim}{R}^{T}}(z,z_1)$ is the matricant of $-\underset{\sim}{R}^{T}$. To relate $\underset{\sim}{M}_{\underset{\sim}{R}^{\dagger}}$ to the matricant $\underset{\sim}{M}(z,z_1)$ of $\underset{\sim}{R}$, we take the transpose conjugate of both sides of Eq. (39) and postmultiply by $(\underset{\sim}{M}^{\dagger})^{-1}$, to obtain

$$\frac{d\underset{\sim}{M}^{\dagger}}{dz}\ (\underset{\sim}{M}^{\dagger})^{-1} = j\underset{\sim}{M}^{\dagger}\underset{\sim}{R}^{\dagger}(\underset{\sim}{M}^{\dagger})^{-1} \tag{140}$$

But

$$\frac{d\underset{\sim}{M}^{\dagger}}{dz}\ (\underset{\sim}{M}^{\dagger})^{-1} = -\underset{\sim}{M}^{\dagger}\ \frac{d(\underset{\sim}{M}^{\dagger})^{-1}}{dz} \tag{141}$$

since $(d/dz)[\underset{\sim}{M}^{\dagger}(\underset{\sim}{M}^{\dagger})^{-1}] = d\underset{\sim}{I}/dz = \underset{\sim}{0}$. Substituting Eq. (141) in Eq. (140) and premultiplying by $-(\underset{\sim}{M}^{\dagger})^{-1}$, we obtain

$$\frac{d(\underset{\sim}{M}^{\dagger})^{-1}}{dz} = -j\underset{\sim}{R}^{\dagger}(\underset{\sim}{M}^{\dagger})^{-1} \tag{142}$$

Clearly, also, $[\underset{\sim}{M}^{\dagger}(z_1,z_1)]^{-1} = \underset{\sim}{I}$. Thus $(\underset{\sim}{M}^{\dagger})^{-1}$ is the matricant of $\underset{\sim}{R}^{\dagger}$,

$$\underset{\sim}{M}_{\underset{\sim}{R}^{\dagger}} = (\underset{\sim}{M}^{\dagger})^{-1} \tag{143}$$

Similarly, we find that

$$\underset{\sim}{M}_{-\underset{\sim}{R}^{T}} = (\underset{\sim}{M}^{T})^{-1} \tag{144}$$

Thus

$$\underset{\sim}{K}'' = (\underset{\sim}{M}^{\dagger})^{-1} \times (\underset{\sim}{M}^{T})^{-1} \tag{145}$$

A nonsingular solution to Eq. (138) is obtained from the matricant of $\underset{\sim}{R}$ by Eq. (145). The columns of $\underset{\sim}{K}''$ thus form a set of n^2 linearly independent solutions $\underset{\sim}{k}_i''$ to Eq. (125).

Any linear combination $\sum a_i \underset{\sim}{k}_i''$ of these solutions, where the a_i are independent of z, is also a solution. An elementary transformation, as defined in Section III under "Testing the Kronecker Sum," on the columns of $\underset{\sim}{K}''$, will result in a new matrix with columns that are linear combinations of the columns of $\underset{\sim}{K}''$. Furthermore, it is a property of elementary transformations that they do not change the rank of a matrix. The columns of the new matrix, therefore, will also form a set of linearly independent solutions to Eq. (125).

All constant solutions to Eq. (125) will be linear combinations of the columns of $\underset{\sim}{K}''$, obtained by taking the direct product of $(\underset{\sim}{M}^{\dagger})^{-1}$ and $(\underset{\sim}{M}^{T})^{-1}$. The constant solutions can be obtained by successive elementary transformations on the columns of $\underset{\sim}{K}''$, with the assurance that after each step we will be left with a set of linearly independent solutions. A scalar multiplier employed in an elementary transformation must be chosen independent of z, and the functional forms of the elements of $\underset{\sim}{K}''$ will serve as a guide in making elementary transformations that will display the constant $\underset{\sim}{k}''$. If the required transformations are not evident, we can systematically obtain as many constant elements as possible in the first row. The constant $\underset{\sim}{k}''$ must then be linear combinations of the columns headed by the constant elements, and by elementary transformations on these we can obtain as many constant elements as possible in the second row. Continuing in this way, we will obtain the largest set of linearly independent $\underset{\sim}{k}''$ that can be chosen constant.

The constant metrics $\underset{\sim}{K}$ appropriate to $\underset{\sim}{R}$ are obtained by rearranging the elements of the constant $\underset{\sim}{k}''$ as n×n matrices in accordance with the convention we have adopted. As the constant solutions to Eq. (122), they will be the constant solutions to Eq. (8). The remaining independent solutions $\underset{\sim}{K}(z)$ to Eq. (122) are obtained similarly from the remaining $\underset{\sim}{k}''(z)$, and for nonuniform systems these may imply additional conservation laws. Note that if $\underset{\sim}{K}$ is a solution to Eq. (122), $\underset{\sim}{K}^{\dagger}$ is also; and again the solutions can be chosen Hermetian.

Example 1:

The operator $\underset{\sim}{R}$ for a uniform transmission line, given by Eq. (3), has the matricant

$$\underset{\sim}{M}(z,0) = \begin{pmatrix} \cos \beta z & -jZ_0 \sin \beta z \\ -j \frac{1}{Z_0} \sin \beta z & \cos \beta z \end{pmatrix} \tag{146}$$

By Eq. (145), we obtain

$$\underset{\sim}{K}'' = \begin{pmatrix} \cos^2 \beta z & j \frac{1}{Z_0} \sin \beta z \cos \beta z & -j \frac{1}{Z_0} \sin \beta z \cos \beta z & \frac{1}{Z_0^2} \sin^2 \beta z \\ jZ_0 \sin \beta z \cos \beta z & \cos^2 \beta z & \sin^2 \beta z & -j \frac{1}{Z_0} \sin \beta z \cos \beta z \\ -jZ_0 \sin \beta z \cos \beta z & \sin^2 \beta z & \cos^2 \beta z & j \frac{1}{Z_0} \sin \beta z \cos \beta z \\ Z_0^2 \sin^2 \beta z & -jZ_0 \sin \beta z \cos \beta z & jZ_0 \sin \beta z \cos \beta z & \cos^2 \beta z \end{pmatrix} \tag{147}$$

Writing the i'th column

$$\underset{\sim}{k}_i'' = \begin{pmatrix} k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \end{pmatrix}_i \tag{148}$$

we see that $(1/4)(\underset{\sim}{k}_2''+\underset{\sim}{k}_3'')$ gives the elements of $\underset{\sim}{K}_1$, and $(1/4Z_0)(\underset{\sim}{k}_1''+Z_0^2\underset{\sim}{k}_4'')$ the elements of $\underset{\sim}{K}_2$, in Eq. (60). If $\underset{\sim}{k}_1''$ and $\underset{\sim}{k}_2''$ are replaced respectively by $(1/4)(\underset{\sim}{k}_2''+\underset{\sim}{k}_3'')$ and $(1/4Z_0)(\underset{\sim}{k}_1''+Z_0^2\underset{\sim}{k}_4'')$, the result of elementary transformations, it is clear that we cannot then obtain another column with constant elements by replacing $\underset{\sim}{k}_3''$ or $\underset{\sim}{k}_4''$ by a linear combination $a_3\underset{\sim}{k}_3''+a_4\underset{\sim}{k}_4''$, a_3 and a_4 constant.

Example 2:

The operator $\underset{\sim}{R}$ for an exponentially tapered transmission line is of the form

$$\underset{\sim}{R} = \begin{pmatrix} 0 & r_{12}(z) \\ r_{21}(z) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega L \\ \omega C & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta Z_0 e^{\sigma z} \\ \frac{\beta}{Z_0} e^{-\sigma z} & 0 \end{pmatrix} \tag{149}$$

where $L=L_0e^{\sigma z}$ is the inductance per unit length, $C=C_0e^{-\sigma z}$ is the capacitance per unit length, $Z_0 = \sqrt{L_0/C_0}$, and $\beta = \omega\sqrt{LC} = \omega\sqrt{L_0C_0}$. The matricant is

$$\underset{\sim}{M}(z,0) = \begin{pmatrix} e^{(1/2)\sigma z}\left(\cos\gamma z - \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) & -j\frac{\beta Z_0}{\gamma} e^{(1/2)\sigma z}\sin\gamma z \\ -j\frac{\beta}{\gamma Z_0} e^{(-1/2)\sigma z}\sin\gamma z & e^{-(1/2)\sigma z}\left(\cos\gamma z + \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \end{pmatrix} \tag{150}$$

where $\gamma^2=\beta^2-\frac{1}{4}\sigma^2$. By Eq. (145) we obtain $\underset{\sim}{K}''$, with the column vectors:

$$\underset{\sim}{k}_1''= \begin{pmatrix} e^{-\sigma z}\left(\cos\gamma z + \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right)^2 \\ j\,\frac{\beta Z_0}{\gamma}\sin\gamma z\left(\cos\gamma z + \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \\ -j\,\frac{\beta Z_0}{\gamma}\sin\gamma z\left(\cos\gamma z + \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \\ \left(\frac{3Z_0}{\gamma}\right)^2 e^{\sigma z}\sin^2\gamma z \end{pmatrix} \quad \underset{\sim}{k}_2''= \begin{pmatrix} j\,\frac{\beta}{\gamma Z_0}e^{-\sigma z}\sin\gamma z\left(\cos\gamma z + \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \\ 1 - \frac{\beta^2}{\gamma^2}\sin^2\gamma z \\ \frac{\beta^2}{\gamma^2}\sin^2\gamma z \\ -j\,\frac{\beta Z_0}{\gamma}e^{\sigma z}\sin\gamma z\left(\cos\gamma z - \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \end{pmatrix}$$

$$\underset{\sim}{k}_3''= \begin{pmatrix} -j\,\frac{\beta}{\gamma Z_0}e^{-\sigma z}\sin\gamma z\left(\cos\gamma z + \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \\ \frac{\beta^2}{\gamma^2}\sin^2\gamma z \\ 1 - \frac{\beta^2}{\gamma^2}\sin^2\gamma z \\ j\,\frac{\beta Z_0}{\gamma}e^{\sigma z}\sin\gamma z\left(\cos\gamma z - \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \end{pmatrix} \quad \underset{\sim}{k}_4''= \begin{pmatrix} \left(\frac{\beta}{\gamma Z_0}\right)^2 e^{-\sigma z}\sin^2\gamma z \\ -j\,\frac{\beta}{\gamma Z_0}\sin\gamma z\left(\cos\gamma z - \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \\ j\,\frac{\beta}{\gamma Z_0}\sin\gamma z\left(\cos\gamma z - \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \\ e^{\sigma z}\left(\cos\gamma z - \frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right)^2 \end{pmatrix} \tag{151}$$

Clearly $(1/4)(\underset{\sim}{k}_2''+\underset{\sim}{k}_3'')$ gives the elements of

$$\underset{\sim}{K}_1 = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{152}$$

and so the power is conserved (see Eq. (6)). Successive elementary transformations, using the systematic procedure we have described, reveal that there are no constant metrics independent of $\underset{\sim}{K}_1$. But if we substitute $e^{\sigma z} = (1/\beta Z_0) r_{12}$ and $e^{-\sigma z} = (Z_0/\beta) r_{21}$ where these appear explicitly, and in the same manner seek linear combinations of the $\underset{\sim}{k}_i''$ that are not explicit functions of z, we find that $(\beta/Z_0)\underset{\sim}{k}_1'' + j\sigma\underset{\sim}{k}_2'' + \beta Z_0 \underset{\sim}{k}_4''$ gives the elements of

$$\underset{\sim}{K} = \begin{pmatrix} r_{21} & j\sigma \\ 0 & r_{12} \end{pmatrix} \tag{153}$$

from which we can form the Hermetian solution to Eq. (122),

$$\underset{\sim}{K}_2 = \frac{1}{4\omega}(\underset{\sim}{K}+\underset{\sim}{K}^{\dagger}) = \frac{1}{2}\begin{pmatrix} C & j\frac{1}{2}\frac{\sigma}{\omega} \\ -j\frac{1}{2}\frac{\sigma}{\omega} & L \end{pmatrix} \tag{154}$$

Equation (154) implies the invariance of

$$s(\underset{\sim}{K}_2) = \underset{\sim}{x}^{\dagger}\underset{\sim}{K}_2\underset{\sim}{x} = \frac{1}{2}C|V|^2 + \frac{1}{2}L|I|^2 + j\frac{1}{4}\frac{\sigma}{\omega}(V^*I - VI^*) \tag{155}$$

Thus the total stored energy plus σ/ω times the imaginary part of the complex power is conserved.

If $\underset{\sim}{k}_1''$ and $\underset{\sim}{k}_2''$ are replaced respectively by $(1/4)(\underset{\sim}{k}_2''+\underset{\sim}{k}_3'')$ and $(\beta/Z_0)\underset{\sim}{k}_1'' + j\sigma\underset{\sim}{k}_2'' + \beta Z_0\underset{\sim}{k}_4''$, we are left with $\underset{\sim}{k}_3''$ and $\underset{\sim}{k}_4''$. Two additional independent conservation laws can be derived from these on substituting $z = (1/\sigma)\ln(r_{12}/\beta Z_0)$ or $z = (1/\sigma)\ln(\beta/Z_0 r_{21})$ everywhere; but they do not have an obvious physical interpretation.

VII. GROUP REPRESENTATION WITH THE SYSTEM INVARIANTS

We observed in Section IV that the constant, nonsingular metrics appropriate to a system operator define this operator only within a range. It is also known that the matricant has group properties, and the group representation it provides is not unique (refs. 3 and 7). Rather than considering, in the usual manner, group properties of the matricant as defined by appropriate, nonsingular, constant $\underset{\sim}{K}$, we shall find that the constant $\underset{\sim}{K}$, singular and nonsingular, are embodied explicitly in another group representation. That they are neither necessary nor sufficient for such a representation, however, will indicate the extent of their contribution in defining a system. We shall see what is needed to complete the representation, and shall find that it has both physical interest and some further group properties.

A nonempty set S is said to be a ring under a Lie product if it is closed under defined binary operations of addition (+) and multiplication (·), and provided, for arbitrary elements s_1, $s_2, s_3 \varepsilon S$:

1. $(s_1+s_2) + s_3 = s_1 + (s_2+s_3)$.
2. $s_1 + s_2 = s_2 + s_1$.
3. There exists an additive zero element $s_0 \varepsilon S$ such that, for each $s_i \varepsilon S$, $s_i + s_0 = s_i$.
4. For each $s_i \varepsilon S$ there exists $-s_i \varepsilon S$ such that $s_i + (-s_i) = s_0$.
5. $s_1 \cdot (s_2 \cdot s_3) + s_2 \cdot (s_3 \cdot s_1) + s_3 \cdot (s_1 \cdot s_2) = s_0$ (Jacobi identity).
6. $s_1 \cdot (s_2+s_3) = s_1 \cdot s_2 + s_1 \cdot s_3$.
7. $s_1 \cdot s_2 = -(s_2 \cdot s_1)$.

A multiplicative operation having properties 5, 6, and 7 defines a Lie product; and an example is given by the commutator of matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$, defined by

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = [\underset{\sim}{A}, \underset{\sim}{B}] = \underset{\sim}{A}\underset{\sim}{B} - \underset{\sim}{B}\underset{\sim}{A} \tag{156}$$

where $\underset{\sim}{A}\underset{\sim}{B}$ is interpreted as ordinary matrix multiplication.

Let S be a subspace of a linear vector space over a field F such that S is closed under addition over V, where V is a subfield of F. (If $s_1, s_2 \varepsilon S$ and $v_1, v_2 \varepsilon V$, then $v_1 s_1 + v_2 s_2 \varepsilon S$.) Then if S is a ring under a Lie product, S is said to be a Lie algebra over V.

Let $\underset{\sim}{K}$ be a given constant metric, and let $\underset{\sim}{R}_i$ be an operator defined over the field F of complex numbers. Let (+) denote ordinary matrix addition, and the commutator be taken as the Lie product. Then it is easily verified that $\underset{\sim}{S} = -j\underset{\sim}{R}$, where $\underset{\sim}{R}$ is the range of operators for which

$$\underset{\sim}{R}_i^{\dagger} \underset{\sim}{K} = \underset{\sim}{K} \underset{\sim}{R}_i, \quad \underset{\sim}{R}_i \varepsilon \underset{\sim}{R} \tag{8}$$

is a Lie algebra over the field V of real numbers. The metric $\underset{\sim}{K}$ thus defines a Lie algebra for $\underset{\sim}{S}$. Furthermore, the specification of additional independent, constant metrics will define a Lie subalgebra.

A nonempty set G, together with a binary operation ($\circ$), is called a group provided:

1. If $g_1, g_2 \varepsilon G$, then $g_1 \circ g_2 \varepsilon G$ (the group property).
2. $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$, for all $g_1, g_2, g_3 \varepsilon G$ (associativity).
3. There exists an identity element $1 \varepsilon G$ such that, for all $g_i \varepsilon G$,

$$g_i \circ 1 = 1 \circ g_i = g_i$$

4. For every $g_i \varepsilon G$, there exists an inverse $g_i^{-1} \varepsilon G$, such that

$$g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = 1$$

It can be shown that the identity and the inverse must be unique. If G has only the group property, G is called a groupoid.

Again let $\underset{\sim}{K}$ be a given constant metric. Then it is easily verified that $\underset{\sim}{M}(z, z_1)$, where $\underset{\sim}{M}$ is the range of matricants for which

$$\underset{\sim}{M}_i^{\dagger} \underset{\sim}{K} \underset{\sim}{M}_i = \underset{\sim}{K}, \quad \underset{\sim}{M}_i \varepsilon \underset{\sim}{M} \tag{36}$$

is a group under ordinary matrix multiplication. It is a continuous, finite group, in that it can be parameterized in terms of continuous values of a finite number of parameters. We have seen that if $\underset{\sim}{M}_i$ is a matricant of $\underset{\sim}{R}_i$, then Eqs. (8) and (36) have the same constant solutions $\underset{\sim}{K}$. The metric $\underset{\sim}{K}$ thus defines a group $\underset{\sim}{M}$, and an associated Lie algebra for $\underset{\sim}{S}$. The group $\underset{\sim}{M}$ is known as a Lie group. The specification of additional independent, constant metrics will define a Lie subgroup and an associated Lie subalgebra. (A rigorous treatment of what is meant by continuous, finite, and Lie groups is beyond the scope of this discussion. There are many excellent references; e.g., refs. 12 and 13.)

The group $\underset{\sim}{M}$ can also be thought of as a matrix representation of an abstract group G of transformations characterizing a range of systems. A homomorphism of a group G into a group or groupoid H is defined as a mapping $g \to g\alpha$ of each $g\varepsilon G$ to $h = g\alpha$, $h\varepsilon H$, satisfying the condition

$$(g_1 \circ g_2)\ \alpha = (g_1\alpha) \circ (g_2\alpha),\ g_1, g_2 \varepsilon G \tag{157}$$

The binary operations for G and H need not be the same. We denote by $G\alpha$ the set of $g\alpha$ for all $g\varepsilon G$. Then $G\alpha$ is a subset of H, and the mapping is onto $G\alpha$ (i.e., for each $h\varepsilon G\alpha$, there exists a $g\varepsilon G$ such that $g\alpha = h$). It can be shown (ref. 14) that if α is a homomorphism of a group G into a group or groupoid H, then $G\alpha$ is a group. The group $G\alpha$ (or the mapping) is called a representation of the group G. If the mapping is one-to-one (i.e., $g_1\alpha = g_2\alpha$ implies $g_1 = g_2$), the mapping is called an isomorphism, G and $G\alpha$ are said to be isomorphic, and as the groups are essentially the same, we write $G \cong G\alpha$. The representation is then said to be faithful. Thus $G \cong \underset{\sim}{M}$ can be interpreted as a statement that $\underset{\sim}{M}$ is a faithful matrix representation of an abstract group G.

Let $\underset{\sim}{M}'$ be the set of all nonsingular, differentiable, n×n matrices $\underset{\sim}{M}_i(z, z_1)$, where $\underset{\sim}{M}_i(z_1, z_1) = \underset{\sim}{I}$. Again it is easily verified that these form a group under matrix multiplication. Then $G' \cong \underset{\sim}{M}'$ is the abstract group encompassing all systems having n degrees of freedom that can be described by a matricant, $\underset{\sim}{M}' = G'\alpha$ is a faithful representation of this group, and the set with elements $-j\underset{\sim}{R}_i = (d\underset{\sim}{M}_i/dz)\underset{\sim}{M}_i^{-1}$, $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}'$, is the Lie algebra associated with $\underset{\sim}{M}'$.

Assume $\underset{\sim}{P}' = G'\beta$ and $\underset{\sim}{Q}' = G'\gamma$ are two faithful matrix representations of G', where for all $g_i \varepsilon G'$,

$$\begin{aligned} g_i\beta &= \underset{\sim}{P}_i,\ \underset{\sim}{P}_i \varepsilon \underset{\sim}{P}' \\ g_i\gamma &= \underset{\sim}{Q}_i,\ \underset{\sim}{Q}_i \varepsilon \underset{\sim}{Q}' \end{aligned} \tag{158}$$

and the $\underset{\sim}{P}_i \varepsilon \underset{\sim}{P}'$ and $\underset{\sim}{Q}_i \varepsilon \underset{\sim}{Q}'$ are of order n×n. Let $\underset{\sim}{N}' = G'\eta$ be defined by the mapping, for all $g_i \varepsilon G'$,

$$g_i\eta = (g_i\beta) \times (g_i\gamma) = \underset{\sim}{P}_i \times \underset{\sim}{Q}_i = \underset{\sim}{N}_i, \ \underset{\sim}{N}_i \varepsilon \underset{\sim}{N}' \qquad (159)$$

where $\underset{\sim}{P}_i \times \underset{\sim}{Q}_i$ is the matrix direct product defined by Eq. (77). By Eq. (82), and since β and γ are homomorphisms, we can write

$$(g_1\eta) \circ (g_2\eta) = (\underset{\sim}{P}_1 \times \underset{\sim}{Q}_1)(\underset{\sim}{P}_2 \times \underset{\sim}{Q}_2) = (\underset{\sim}{P}_1\underset{\sim}{P}_2) \times (\underset{\sim}{Q}_1\underset{\sim}{Q}_2)$$

$$= [(g_1\beta) \circ (g_2\beta)] \times [(g_1\gamma) \circ (g_2\gamma)] = [(g_1 \circ g_2)\beta]$$

$$\times [(g_1 \circ g_2)\gamma] = (g_1 \circ g_2)\eta$$

for arbitrary $g_1, g_2 \varepsilon G'$. Thus $\underset{\sim}{N}'$ is a groupoid under matrix multiplication, and η is a homomorphism of the group G' onto $\underset{\sim}{N}'$. It follows that $\underset{\sim}{N}'$ is a group and a representation of G' — but not that the representation is faithful.

Let $G \cong \underset{\sim}{M}$ be the group defined by some set of constant metrics $\underset{\sim}{K}$. G is a subgroup of G', and $\underset{\sim}{M} = G\alpha$ is a subgroup of $\underset{\sim}{M}'$. Then $\underset{\sim}{P} = G\beta$ and $\underset{\sim}{Q} = G\gamma$, defined by Eq. (158) for $g_i \varepsilon G$, are subgroups of $\underset{\sim}{P}'$ and $\underset{\sim}{Q}'$ respectively, and faithful representations of G. This follows since β and γ are homomorphisms of the group G into the groups $\underset{\sim}{P}'$ and $\underset{\sim}{Q}'$, and since the representations $\underset{\sim}{P}'$ and $\underset{\sim}{Q}'$ are faithful. Similarly, η is a homomorphism of G into $\underset{\sim}{N}'$; and $\underset{\sim}{N} = G\eta$, defined by Eq. (159) for $g_i \varepsilon G$, is a subgroup of $\underset{\sim}{N}'$ and a representation (not necessarily faithful) of G.

Now we note that taking the inverse of the transpose conjugate of all members of the group $\underset{\sim}{M}'$, or the inverse of the transpose, constitutes a one-to-one mapping of $\underset{\sim}{M}'$ onto itself. Furthermore, each of these mappings is a homomorphism, since for arbitrary $\underset{\sim}{M}_1, \underset{\sim}{M}_2 \varepsilon \underset{\sim}{M}'$,

$$\left(\underset{\sim}{M}_1^{\dagger}\right)^{-1}\left(\underset{\sim}{M}_2^{\dagger}\right)^{-1} = \left[\left(\underset{\sim}{M}_1\underset{\sim}{M}_2\right)^{\dagger}\right]^{-1}$$

$$\left(\underset{\sim}{M}_1^{T}\right)^{-1}\left(\underset{\sim}{M}_2^{T}\right)^{-1} = \left[\left(\underset{\sim}{M}_1\underset{\sim}{M}_2\right)^{T}\right]^{-1}$$

If, that is, we define $\underset{\sim}{P}' = \underset{\sim}{M}'\rho$ and $\underset{\sim}{Q}' = \underset{\sim}{M}'\tau$, where for all $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}'$,

$$\underset{\sim}{M}_i\rho = \left(\underset{\sim}{M}_i^{\dagger}\right)^{-1} = \underset{\sim}{P}_i,\ \underset{\sim}{P}_i \varepsilon \underset{\sim}{P}'$$
$$\underset{\sim}{M}_i\tau = \left(\underset{\sim}{M}_i^{T}\right)^{-1} = \underset{\sim}{Q}_i,\ \underset{\sim}{Q}_i \varepsilon \underset{\sim}{Q}' \qquad (160)$$

then ρ and τ are isomorphisms. Accordingly, $\underset{\sim}{M}' \cong \underset{\sim}{P}'$ and $\underset{\sim}{M}' \cong \underset{\sim}{Q}'$. We can also write

$$\underset{\sim}{P}' = G'\alpha\rho = G'\beta$$
$$\underset{\sim}{Q}' = G'\alpha\tau = G'\gamma \qquad (161)$$

and since α is an isomorphism as well, $\beta = \alpha\rho$ and $\gamma = \alpha\tau$ are isomorphisms. Thus $G' \cong \underset{\sim}{P}'$ and $G' \cong \underset{\sim}{Q}'$, and $\underset{\sim}{P}'$ and $\underset{\sim}{Q}'$ are faithful representations of G'. Also, then, $\underset{\sim}{P} = \underset{\sim}{M}\rho$ and $\underset{\sim}{Q} = \underset{\sim}{M}\tau$, defined by Eq. (160) for $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}$, are faithful representations of G. It follows that $\underset{\sim}{N}' = G'\eta$, where for all $g_i \varepsilon G'$

$$g_i\eta = \underset{\sim}{P}_i \times \underset{\sim}{Q}_i = \left(\underset{\sim}{M}_i^{\dagger}\right)^{-1} \times \left(\underset{\sim}{M}_i^{T}\right)^{-1} = \underset{\sim}{N}_i,\ \underset{\sim}{N}_i \varepsilon \underset{\sim}{N}' \qquad (162)$$

is a representation of G', and the subgroup $\underset{\sim}{N} = G\eta$ is a representation of G.

Thus the $\underset{\sim}{K}''$ of Eq. (145), deriving from all $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}'$, are seen to form a group $\underset{\sim}{N}'$, and the $\underset{\sim}{K}''$ deriving from $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}$ a subgroup $\underset{\sim}{N}$, that are moreover representations respectively of G' and G. But are these representations faithful? Inversion of Eq. (138),

$$j\,\frac{d\underset{\sim}{K}''}{dz}\,(\underset{\sim}{K}'')^{-1} = \underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger}, -\underset{\sim}{R}^{T}) \qquad (163)$$

shows that $\underset{\sim}{K}''$ determines the Kronecker sum $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger}, -\underset{\sim}{R}^{T})$. Inspection of Eq. (97) reveals further that if $\underset{\sim}{\Lambda}(\underset{\sim}{R}_1^{\dagger}, -\underset{\sim}{R}_1^{\dagger}) = \underset{\sim}{\Lambda}(\underset{\sim}{R}_2^{\dagger}, -\underset{\sim}{R}_2^{T})$, then $\underset{\sim}{R}_2 = \underset{\sim}{R}_1 + a\underset{\sim}{I}$, a real. Thus, while strictly the representations are not faithful, $\underset{\sim}{K}''$ determines $\underset{\sim}{R}$ within the identity.

The columns $\underset{\sim}{k}''$ of a given $\underset{\sim}{K}'' \varepsilon \underset{\sim}{N}'$ or $\underset{\sim}{K}'' \varepsilon \underset{\sim}{N}$ are generators of n^2 independent invariants $s(\underset{\sim}{K})$ that define the system operator within the identity. Not only do a set of constant $\underset{\sim}{K}$ define a group, and a range $\underset{\sim}{M}$ of $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}'$ that constitutes a matrix representation of the group, but written as $\underset{\sim}{k}''$ they are explicitly displayed as linear combinations of the columns in another matrix representation of the group. The system is not defined by the constant $\underset{\sim}{K}$ alone,

and we see that the $\underset{\sim}{K}(z)$ are essential to complete the group-theoretical representation. Moreover, this extension of the set of constant $\underset{\sim}{K}$ is not just formalistic; for we have seen that, for nonuniform systems in general, invariants obtained from the $\underset{\sim}{K}(z)$ reasonably admit of interpretation as conservation laws. From this point of view, the class of uniform systems can be considered as an atypical case, represented by a "singular" surface of $\underset{\sim}{N}'$ — or if in the range defined by the constant $\underset{\sim}{K}$, by a singular surface of $\underset{\sim}{N}$. This singularity represents an inability to interpret the invariants as conservation laws, but not an inability to obtain pre-images from the representation.

If $\underset{\sim}{K}"\varepsilon\underset{\sim}{N}$ (or $\underset{\sim}{K}"\varepsilon\underset{\sim}{N}'$) is the matricant of $\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$, then $-j\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$ is a member of the Lie algebra associated with $\underset{\sim}{N}$ (or $\underset{\sim}{N}'$). This can be verified directly. To verify closure under commutation, let $-j\underset{\sim}{R}_3$ be the commutator of two operators $-j\underset{\sim}{R}_1$ and $-j\underset{\sim}{R}_2$ in a Lie algebra defined by a set of constant $\underset{\sim}{K}$,

$$-j\underset{\sim}{R}_3 = [-j\underset{\sim}{R}_1,-j\underset{\sim}{R}_2] \tag{164}$$

Then $-j\underset{\sim}{R}_3$ is also in the algebra. It is easily checked that

$$[-j\underset{\sim}{\Lambda}(\underset{\sim}{R}_1^{\dagger},-\underset{\sim}{R}_1^{T}),\ -j\underset{\sim}{\Lambda}(\underset{\sim}{R}_2^{\dagger},-\underset{\sim}{R}_2^{T})] = -j\underset{\sim}{\Lambda}(\underset{\sim}{R}_3^{\dagger},-\underset{\sim}{R}_3^{T}) \tag{165}$$

The columns of the matricant of the operator $\underset{\sim}{\Lambda}(\underset{\sim}{R}_3^{\dagger},-\underset{\sim}{R}_3^{T})$, therefore, with $-j\underset{\sim}{\Lambda}(\underset{\sim}{R}_3^{\dagger},-\underset{\sim}{R}_3^{T})$ obtained by commutation of $-j\underset{\sim}{\Lambda}(\underset{\sim}{R}_1^{\dagger},-\underset{\sim}{R}_1^{T})$ and $-j\underset{\sim}{\Lambda}(\underset{\sim}{R}_2^{\dagger},-\underset{\sim}{R}_2^{T})$, will also yield in linear combination the $\underset{\sim}{k}"$ formed from the specified $\underset{\sim}{K}$. The other conditions for the $\underset{\sim}{K}$ to define a Lie algebra for $-j\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger},-\underset{\sim}{R}^{T})$, obtained by Eq. (163) for $\underset{\sim}{K}"\varepsilon\underset{\sim}{N}$, can likewise be verified.

Kronecker sums in this particular form, then, might be described as providing a representation of conservation laws onto their algebra.

If the constant $\underset{\sim}{K}$ define a group, what can be said of the range of systems having an invariant $s(\underset{\sim}{K})$, where $\underset{\sim}{K} = \underset{\sim}{K}(z)$? The range of matricants will be the solutions to Eq. (124),

$$\underset{\sim}{M}^{\dagger}\underset{\sim}{K}(z)\underset{\sim}{M} = \underset{\sim}{K}(z_1) \tag{124}$$

with the boundary condition $\underset{\sim}{M}(z_1,z_1) = \underset{\sim}{I}$. If we define a binary operation for arbitrary $\underset{\sim}{M}_1$, $\underset{\sim}{M}_2$ in this range by

$$\underset{\sim}{M}_3 = \underset{\sim}{M}_1 \circ \underset{\sim}{M}_2 = \underset{\sim}{M}_2 \underset{\sim}{M}_1^{-1} \underset{\sim}{M}_2 \tag{166}$$

it is readily verified that $\underset{\sim}{M}_3$ also satisfies Eq. (124). Thus the matricants in this range form a groupoid under the defined binary operation. The binary operation is not associative, however, and the groupoid does not contain an identity element. Thus it is not a group. The mapping $\underset{\sim}{M}_i \rightarrow (\underset{\sim}{M}_i^{\dagger})^{-1} \times (\underset{\sim}{M}_i^{T})^{-1}$ is to a groupoid under this same binary operation. Again it is a homomorphism, and the $\underset{\sim}{K}$" form a representation of the groupoid defined by the invariant $s(\underset{\sim}{K})$.

By way of example, consider the exponential line of Example 2 in Section VI. The metrics $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2(z)$ of Eqs. (152) and (154) are solutions respectively to Eqs. (36) and (124), for the matricant of Eq. (150). Let us denote this matricant by $\underset{\sim}{M}_1$. But $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ are also solutions for

$$\underset{\sim}{M}_2 = \begin{pmatrix} e^{(1/2)\sigma z} & 0 \\ 0 & e^{-(1/2)\sigma z} \end{pmatrix} \tag{167}$$

Thus $\underset{\sim}{M}_1$ and $\underset{\sim}{M}_2$ are in the group defined by $\underset{\sim}{K}_1$, and in the groupoid defined by $\underset{\sim}{K}_2$. We can obtain another member of the group,

$$\underset{\sim}{M}_3 = \underset{\sim}{M}_1 \underset{\sim}{M}_2 = \begin{pmatrix} e^{\sigma z}\left(\cos \gamma z - \frac{1}{2}\frac{\sigma}{\gamma} \sin \gamma z\right) & -j \frac{\beta z_0}{\gamma} \sin \gamma z \\ -j \frac{\beta}{\gamma z_0} \sin \gamma z & e^{-\sigma z}\left(\cos \gamma z + \frac{1}{2}\frac{\sigma}{\gamma} \sin \gamma z\right) \end{pmatrix} \tag{168}$$

which satisfies $\underset{\sim}{M}_3^{\dagger} \underset{\sim}{K}_1 \underset{\sim}{M}_3 = \underset{\sim}{K}_1$, but which is not a member of the groupoid $(\underset{\sim}{M}_3^{\dagger} \underset{\sim}{K}_2(z) \underset{\sim}{M}_3 \neq \underset{\sim}{K}_2(0))$. We can obtain another member of the groupoid,

$$\underset{\sim}{M}_4=\underset{\sim}{M}_2\underset{\sim}{M}_1^{-1}\underset{\sim}{M}_2=\begin{pmatrix} e^{(1/2)\sigma z}\left(\cos\gamma z+\frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) & j\frac{\beta z_0}{\gamma}e^{(1/2)\sigma z}\sin\gamma z \\ j\frac{\beta}{\gamma z_0}e^{-(1/2)\sigma z}\sin\gamma z & e^{-(1/2)\sigma z}\left(\cos\gamma z-\frac{1}{2}\frac{\sigma}{\gamma}\sin\gamma z\right) \end{pmatrix} \tag{169}$$

which satisfies $\underset{\sim}{M}_4^{\dagger}\underset{\sim}{K}_2(z)\underset{\sim}{M}_4=\underset{\sim}{K}_2(0)$, and it is seen that $\underset{\sim}{M}_4$ is also a member of the group.

Members of a groupoid, defined by some $\underset{\sim}{K}_2(z)$, which are generated by other members $\underset{\sim}{M}_1$ and $\underset{\sim}{M}_2$ that are also members of a group defined by constant $\underset{\sim}{K}_1$, will always be members of the group. For if $\underset{\sim}{M}_1^{\dagger}\underset{\sim}{K}_1\underset{\sim}{M}_1=\underset{\sim}{K}_1$ and $\underset{\sim}{M}_2^{\dagger}\underset{\sim}{K}_1\underset{\sim}{M}_2=\underset{\sim}{K}_1$, then

$$\begin{aligned}(\underset{\sim}{M}_1\circ\underset{\sim}{M}_2)^{\dagger}\underset{\sim}{K}_1(\underset{\sim}{M}_1\circ\underset{\sim}{M}_2) &= \underset{\sim}{M}_2^{\dagger}\left(\underset{\sim}{M}_1^{-1}\right)^{\dagger}\underset{\sim}{M}_2^{\dagger}\underset{\sim}{K}_1\underset{\sim}{M}_2\underset{\sim}{M}_1^{-1}\underset{\sim}{M}_2 \\ &= \underset{\sim}{M}_2^{\dagger}\left(\underset{\sim}{M}_1^{-1}\right)^{\dagger}\underset{\sim}{K}_1\underset{\sim}{M}_1^{-1}\underset{\sim}{M}_2=\underset{\sim}{M}_2^{\dagger}\underset{\sim}{K}_1\underset{\sim}{M}_2=\underset{\sim}{K}_1\end{aligned} \tag{170}$$

The role of the conservation laws in defining a linear system is thus clear. A system having n degrees of freedom is defined by its n^2 independent invariants $s(\underset{\sim}{K})$. The constant $\underset{\sim}{K}$, if any, including singular $\underset{\sim}{K}$ obtained from linear combinations of the columns of the nonsingular matrix $\underset{\sim}{K}$", define a group — the intersection of the groups defined by the constant $\underset{\sim}{K}$ separately. The $\underset{\sim}{K}(z)$, which we have seen in general lead to valid conservation laws, define a groupoid — the intersection of the groupoids defined by the $\underset{\sim}{K}(z)$ separately. By Eq. (170), intersections of groups and groupoids define subgroupoids. The intersection of all groups and groupoids is a subgroupoid that defines the system operator within the identity.

The n^2 conservation laws cannot, however, be specified arbitrarily. A specified $\underset{\sim}{K}$" must be chosen in the group $\underset{\sim}{N}'$, which is a subgroup of the group formed by taking all possible matrix direct products $\underset{\sim}{P}_i\times\underset{\sim}{Q}_j$ of the $\underset{\sim}{P}_i\varepsilon\underset{\sim}{P}'$ and $\underset{\sim}{Q}_i\varepsilon\underset{\sim}{Q}'$. (The latter is a representation of a group-theoretical direct product, a slightly different concept that we have not discussed.) This in turn is a subgroup of the group of $n^2\times n^2$ matricants. That is to say, the groups and groupoids defined by some sets of $\underset{\sim}{K}$ are disjoint, and others are included in one another.

The $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}$, the group defined by the specified constant $\underset{\sim}{K}$, evaluated at $z_1 = 0$ and any $z = L$, is also a group. The mapping is a homomorphism of the group with elements $\underset{\sim}{M}_i(z,0)$, $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}$, onto the group with elements $\underset{\sim}{M}_i(L,0)$, for all $\underset{\sim}{M}_i(z,0) \varepsilon \underset{\sim}{M}$. It is not an isomorphism, since for any $\underset{\sim}{M}_i(L,0)$, $\underset{\sim}{M}_i(z,0) \varepsilon \underset{\sim}{M}$, there will exist $\underset{\sim}{M}_j(z,0) \varepsilon \underset{\sim}{M}$, $\underset{\sim}{M}_j(z,0) \neq \underset{\sim}{M}_i(z,0)$, such that $\underset{\sim}{M}_j(L,0) = \underset{\sim}{M}_i(L,0)$ — the equivalent distributed system for a lumped system is not unique. Also the mapping depends on the choice of $z = L$. But for each $z = L \neq 0$, the $\underset{\sim}{M}_i(L,0)$, $\underset{\sim}{M}_i(z,0) \varepsilon \underset{\sim}{M}$, will include all nonsingular matrices $\underset{\sim}{K}$-Unitary with respect to the specified $\underset{\sim}{K}$, and will provide an unfaithful representation of G. It is thus not necessary to determine $\underset{\sim}{M}_i(z,0) \not\varepsilon \underset{\sim}{M}$ that map to the group of nonsingular $\underset{\sim}{K}$-Unitary matrices — such pre-images as satisfy $\underset{\sim}{M}_i^{\dagger}(z,0) \underset{\sim}{K}(z) \underset{\sim}{M}_i(z,0) = \underset{\sim}{K}(0)$, with $\underset{\sim}{K}(L) = \underset{\sim}{K}(0)$, and where the specified $\underset{\sim}{K}$ are among the $\underset{\sim}{K}(0)$. Furthermore, for each $\underset{\sim}{M}_i(L,0)$, $\underset{\sim}{M}_i(z,0) \varepsilon \underset{\sim}{M}$, among the $\underset{\sim}{M}_j(z,0) \varepsilon \underset{\sim}{M}$ that map to $\underset{\sim}{M}_i(L,0)$ there will exist matricants of constant operators $\underset{\sim}{R}$. It is thus not necessary either to determine all $\underset{\sim}{M}_i \varepsilon \underset{\sim}{M}$ — which requires obtaining the matricants of all $\underset{\sim}{R}$, including $\underset{\sim}{R}(z)$, with $-j\underset{\sim}{R}$ in the associated Lie algebra — in order to obtain the group of lumped-system operators $\underset{\sim}{M}_i(L,0)$, $\underset{\sim}{M}_i(z,0) \varepsilon \underset{\sim}{M}$, defined by the specified constant $\underset{\sim}{K}$.

VIII. CONCLUSIONS

A distributed system described by a matricant $\underset{\sim}{M}(z,z_1)$ that is a member of various groups and groupoids, with an operator $-j\underset{\sim}{R}$ in corresponding algebras, can be described also by a matricant $\underset{\sim}{K}''(z,z_1)$ with the same group properties, and an operator $-j\underset{\sim}{\Lambda}(\underset{\sim}{R}^{\dagger}, -\underset{\sim}{R}^{T})$ in corresponding algebras. The latter representation exhibits explicitly the metrics that define the conservation laws governing the system. It establishes and elucidates the significance of all the system invariants in characterizing a system. Their interpretation as conservation laws can be supported also by physical arguments.

This representation can be viewed as the end result of a theory developed along parallel lines to existing theory for linear, uniform, homogeneous systems. It provides new insights and techniques for the analysis and synthesis of uniform systems in terms of their conservation laws, and extends to a larger class of systems.

The representation provides a model that can be studied for synthesis from any number of specified metrics. Synthesis may in principle be carried out directly by filling in the columns of $\underset{\sim}{K}''(z,z_1)$, since this is an equivalent matricant for the system. The problem is that this must be done under the constraint $\underset{\sim}{K}'' \varepsilon \underset{\sim}{N}'$. But an intriguing feature of this approach is that one is specifying the system and its conservation laws simultaneously and

explicitly in forming the matricant $\underset{\sim}{K}''(z,z_1)$. The constraint under which this must be done is in fact a fundamental property of the group that is revealed by this representation. There are inherent restrictions on the conservation laws that can be obeyed together by a system.

Mode coupling in systems is a particularly appropriate area of application for this theoretical model. Considerable physical insight can be obtained by focusing attention on conservative exchanges other than the power exchange between modes. As should be expected, the physical information to be specified or obtained is increased if the coupling or the coupled systems are nonuniform, although the number of linearly independent invariants from constant metrics will in general be fewer. The representation provides a basis for characterizing, classifying, and synthesizing such systems.

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