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A FUNDAMENTAL STUDY OF THE CURRENT DENSITY DISTRIBUTION IN A FLOWING NONEQUILIBRIUM PLASMA

by

Chong.S.Kim

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SUMMARY

In order to determine the current distributions in MHD channels in non-uniform plasma flow, the electrical conductivity and the Hall parameter are formulated in the form of the 'so-called' second approximation.

A successive iteration procedure is chosen, by which the electron temperature calculated during successive iterations provides the new spatial distribution of plasma parameters at this step of the iteration. The numerical solutions to the current and electron temperature distributions for uniform conductivity and Hall parameter less than one show diagonally symmetric patterns in a Faraday type channel. In fact, this result can be used as a first iteration for the solution of non-uniform plasma.

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NOTATION

ROMAN SYMBOLS

a	Speed of sound (meter/sec)
$A_{\alpha}i_{\beta}$	Integrals defined in Ref. (1)
$A_{o,1,2,3,4}$	Coefficients defined by and given by Bickley (Ref.51)
$\bar{B}$	Magnetic intensity vector (Weber/m^2)
$\bar{C}_s$	Random thermal velocity of s-species defined by $\bar{C}_s = \bar{W}_s - \bar{v}$ (meter/sec)
$\bar{E}$	Electric field with respect to laboratory (volts/meter)
$\bar{E}^1$	Effective electric field (see Eg. 1.3.13)(volts/meter)
$\bar{E}^*$	Electric field in co-ordinates moving with mass average gas velocity and given by $\bar{E}^* = \bar{E} + \bar{V} \times \bar{B}$ (volts/meter)
f_s	Distribution function of s-species in velocity-spatial space
$\bar{F}$	Force on a particle (Newton)
$\bar{g}_{\alpha\beta}$	Relative velocity between α - and β - species, defined by $\bar{g}_{\alpha\beta} = \bar{W}_{\alpha} - \bar{W}_{\beta}$ (meter/sec)
h	Plank's constant (Joules.sec), half of height of channel (meter), unit interval between adjacent datum points in the channel (meter)
hx, hy, hi, hj	Defined in Chapter 3 (meter)
$\bar{h}$	Relative heat flux vector, related to $\bar{q}$ with $\bar{h} = \bar{q} - \frac{5}{2}\bar{V}$ (Joules/m^2sec)
I	Total current (amps)
$\bar{j}$	Conduction current vector defined by $\bar{j} = \sum_s n_s q_s \bar{V}_s$ (amps/m^2)
$\bar{J}$	Electric current density vector defined by $\bar{J} = \sum_s n_s q_s \bar{v}_s$ (amps/m^2)
k	Boltzmann's constant (joules/^{o}K)
l	Half of insulator length (meter)
L	Half of electrode length (meter)
$M(x,y)$	See Equation (2.2.4a)

M	Mach number (dimensionless)
m_s	Mass of s-particle (Kg)
n_s	Number density of s-species ($1/m^3$)
$N(x,y)$	See Equation (2.2.46)
P	Hydrostatic pressure (Newton/m^2)
$\bar{\bar{P}}$, P_{ij}	Pressure tensor (Newton/m^2)
$P(x,y)$	See Equation (2.2.4C)
q_s	Electric charge of s-species (Coulomb)
$\bar{q}$	Heat Flux vector (Joules/m^2.sec)
q_{ij}	Value relating to Ohmic heating at the point (x_i,y_j) (see Sec.3.3) (Coulomb^2/m^2)
$Q_{\alpha\beta}$	Collisional cross section between α- and β- species (m^2)
$\bar{r}$	Spatial radius vector (meter)
$R_s(i)$	Collisional integral for s-species
$i = 0$	Mass transfer collisional integral (Kg/m^3.sec)
$i = 1$,	Momentum transfer collisional integral ($\bar{R}_s^{(1)}$) (Newton/m^3)
$i = 2$,	Energy transfer collisional integral (Joules/m^3.sec)
$R_{o,i}$	Resistance of loading circuit for "o"
	Resistance of internal plasma for "i" (ohm)
S_s	Seeding ratio of s-species (dimensionless)
t	Time (second)
T	Temperature (^oK)
$\bar{v}_s$	The average velocity of the s-species defined by $\bar{v}_s = < \bar{W}_s >$ (meter/sec)
$\bar{v}$	Mass average mean velocity in references to the laboratory given by $\bar{v} = \frac{1}{\rho} \sum_s \rho_s \bar{v}_s$ (meter/sec)
$\bar{V}_s$	Diffusion or drift velocity of s- species, given by $\bar{V}_s = \bar{v}_s - \bar{v}$ (equal to $< \bar{c} >_s$ in Ref.27) (meter/sec)

$\bar{W}_s$	Particle velocity of s-species in references to laboratory (meter/sec)
Z	Number of charges on a particle (dimensionless)
Z_s	The ground state degeneracies of s-species
	- partition function
	- Statistical weight
	- sum-over-states defined by $Z_s = (2L_s + 1)(2S_s + 1)$

GREEK SYMBOLS

α	See Appendix E
α_s	Degree of ionization of s-species (dimensionless)
β	Hall parameter (dimensionless)
γ	Ratio of specific heats
γ_s	Defined as $\gamma_s = m_s/kT_s$ (sec^2/m^2)
$\gamma_{\alpha,\beta}$	Defined as $\gamma_{\alpha\beta} = \gamma_\alpha \gamma_\beta/(\gamma_\alpha + \gamma_\beta)$ (dimensionless)
δ_s	Energy loss factor of s-species (dimensionless)
ϵ_i	Ionization energy (eV)
ϵ_{int_s}	Internal energy of s-species (Joules)
ϵ_s	Total energy of s-species (Joules)
ζ	See Appendix E
Λ	Ratio of Debye length to the average impact parameter (see Sec.1.6) (dimensionless)
λ_D	Debye length (meter)
$\mu_{\alpha\beta}$	The reduced mass defined by $\mu_{\alpha\beta} = m_\alpha m_\beta/(m_\alpha + m_\beta)$
$\nu_{\alpha\beta}$	Collision frequency between α and β species (sec^{-1})
ξ	See Appendix E
π	Stress tensor (Newton/m^2)
ρ^e	Excess charge density (Coulomb/m^3)
ρ, ρ_s	Mass density (Kg/m^3)

σ	Scalar electrical conductivity (mho/meter)
$\tau_{\alpha\beta}$	Collision time between α and β species (sec)
χ	Hall coefficient (m^3/mho. Weber)
ψ	Current stream function (Amps/m)
ω	Cyclotron frequency (1/sec)
$\Omega_{\alpha\beta}^{(\ell)}$	Weighted collision cross section integral between α and β species (Ref.1)

INTRODUCTION

A number of authors (Ref.1-9) have described the non-uniform conductivity problem. This non-uniformity is thought to be due to the multitemperature plasma model of a partially ionized steady state gas with the Hall effect and applied field effects. Thus these effects on MHD channels have been studied. For given initial conditions of specific geometry channel, a sequence of effects results, including reducing the output current expected, increasing the internal resistance, short-circuiting problem or likely a stability problem (Ref.11-23). This paper has been prepared with the purpose of bringing the non-uniform properties into the channel problem in connection with the practical choices of design parameters (electron cyclotron frequency, gas temperature, pressure or loading factor).

The present calculation permits heat flux in the momentum transfer equation to intensify the local heating due to joulean heating which may cause higher electrical conduction (Ref.24). An approximation is used under which the electron-atom or -ion mass ratio is negligible and the diffusion velocities of heavy particles are equal to zero. The results for the electrical conductivity and the Hall parameter are almost the same in the form as developed by Demetriades and Zhdanov (Ref.2,3) when the ion concentration was neglected. The plasma parameters (conductivity, the Hall parameter, electron density and the collision cross sections) are thus obtained in terms of electron temperature. They are essential to describe the non-uniform plasma due to non-equipartition ionization or nonequilibrium electron heating.

If no plasma parameter gradients in the duct are assumed as we will do for the calculation of current distribution, the interaction between current flow and plasma parameter variations involving the velocity variation cannot arise. Contrary to these assumptions, then, the gas must be compressible such that the current flow causes plasma parameter variations which in turn alter the current distribution. Apparently, an MHD problem such as this can be solved only by considering a set of consistent plasma parameters and current distribution equations. Herein, there are a couple of approximation methods to get a satisfactory answer based on the pre-mentioned detailed physical phenomena. A few authors have identified a reasonably simple approximation procedure with the aim of simplifying the problems without losing apparent significances. Iterative procedures to the solution of the current distribution in two-dimensional MHD channels with finite electrode segmentation are thus suggested (Ref.25). In order to solve more clearly the current distribution itself, the simpler case is solved wherein the uniform Hall parameter and conductivity are assumed. The Extrapolated Liebmann method for Laplace equation is used. However, the calculations are restricted to values of Hall parameter subjected to the condition under which the numerical solution for the current distribution converges over the channel, that is, Hall parameter less than or equal to one. This condition does not seem to be followed from the physical critical value of Hall parameter which brings on the stability problem in non-uniform conduction. But the remainder of the problem which involves formulating the iterative computation of successive states to account for the non-uniform Joulean heating is left to be analyzed. What we obtain for current and electron distribution in this paper is suggested to be used for a first iteration results.

Chapter 1.

Basic MHD Equations

Sec. 1.1 Species Equations

From the considerations of Maxwell conservation equation and applying it to each species to obtain energy and momentum equations, species equations desired to describe the plasma can be obtained. To begin with, there are the Boltzmann equation for each species in terms of particle velocity to get the transfer equation. Let us use the Boltzmann equation without any arguments of the force law whether the collision can be treated as binary or not,

$$\frac{\partial f_s}{\partial t} + \bar{W}_s \cdot \nabla_{\bar{r}} f_s + \frac{\bar{F}}{m_s} \cdot \nabla_{\bar{w}} f_s = \left(\frac{\partial f}{\partial t} \right)_c \qquad (1.1.1)$$

where $(\partial f/\partial t)_c$ represents the rate of change of the velocity distribution function of the particles in question due to collisions with all other particles. The force law per unit mass on a charged particle in the magnetic field will be given by

$$\frac{\bar{F}}{m_s} = \frac{q_s}{m_s} (\bar{E} + \bar{W}_s \times \bar{B})$$

when $\bar{E}$ is independent of the particle actual velocity. $\phi_s(\bar{W}_s)$ is assumed to be the each-species velocity dependent quantity. Multiplying (1.1.1) by $\phi_s(\bar{W}_s)$ and integrating it over the velocity space, we obtain so-called 'the equation of change',

$$\frac{\partial}{\partial t} (n_s <\phi_s>) + \nabla_{\bar{r}}(n_s < \bar{W}_s\phi_s>) - \frac{q_s}{m_s} n_s [\bar{E} \cdot < \nabla_{\bar{w}} \phi_s> +$$

$$< (\bar{W}_s \times \bar{B}) \cdot \nabla_{\bar{w}} \phi_s >] \int_{\bar{w}} \phi_s \left(\frac{\partial f}{\partial t} \right)_c d\bar{W} \qquad (1.1.2)$$

where $< >$ denotes the integration operator over the velocity space.

Using (1.1.2), the equations for species mass, momentum and energy conservation can now be derived by letting ϕ_s equal m_s, $m_s W_{si}$ (component of velocity in i-coordinate) ϵ_s in turn. Then as the results,

$$\frac{\partial \rho_s}{\partial t} + \nabla_{\bar{r}} \cdot (\rho_s \bar{V}_s) = R_s^{(o)} \qquad (1.1.3)$$

(see Appendix A)

$$\frac{\partial}{\partial t} (\rho_s \bar{V}_s) + \nabla_{\bar{r}} \cdot [\rho_s(\bar{v} \ddot{\bar{V}}_s + \bar{V}_s \bar{v})] + \nabla_{\bar{r}} \cdot (\rho_s \bar{v} \bar{v}) + \nabla_{\bar{r}} \bar{\bar{P}}_s$$

$$- q_s n_s [\bar{E}^* + \ddot{\bar{V}}_s \times \bar{B}] = \bar{R}_s^{(1)} \qquad (1.1.4)$$

(see Appendix B)

and

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_2 v^2 + \rho_2 \bar{V}_s\cdot\bar{v} + \frac{3}{2} kn_s T_s + n_s < \epsilon_{is} >\right] + \nabla_{\bar{r}}\left[n_s \bar{v}\left(\frac{1}{2} m_s v^2 + \frac{3}{2} k T_s + <\epsilon_{is}>\right)\right] + \nabla_{\bar{r}}\cdot\bar{q}_s + \nabla_{\bar{r}}\left(\frac{1}{2} m_s v^2 \bar{V}_s n_s\right) + \nabla_{\bar{r}}\left[n_s m_s \bar{v}(\bar{v}\cdot \bar{V}_s)\right] + \nabla_{\bar{r}}(\bar{v}\cdot\bar{\bar{P}}_s) - \bar{E}\,(q_s n_s \bar{v}_s) = R_s^{(2)} \qquad (1.1.5)$$

where $<\epsilon_{is}>$ represents the integration of the internal energy of s-species over the velocity space, (see Appendix C).

Sec.1.2 Over-All Species Equations

Since the total mass must be conserved, the sum of the mass transfer collision integral $R_s^{(o)}$ should be zero. Thus if we sum the mass equations of all components of species, we obtain

$$\frac{\partial\rho}{\partial t} + \nabla_{\bar{r}} \cdot (\rho\, \bar{v}) = 0 \qquad (1.2.1)$$

where ρ is the total mass density defined as $\rho = \sum_s \rho_s$.

To obtain the over-all momentum equation, the plasma is assumed to be quasineutral, that is, $q = \sum_s q_s = 0$. (This assumption is generally valid since in the most cases of practical interest in MHD problems, the characteristic length of plasma is generally greater than the Debye length. Thus, at any point of space, the force to bind the charged particles together is high enough to make the plasma quasineutral, Ref.26). The sum of the momentum transfer collision integral $\bar{R}_s^{(1)}$ is zero since the net change in momentum for all particles due to collisions is zero when only the elastic collisions are considered. That is, conservation of momentum during collisions. Here,the conduction current $\bar{j}$ is defined from a kinetic point of view as $\bar{j} = \sum_s q_s n_s \bar{V}_s$ in terms of the mean velocity or diffusion velocity. By using the definition of the total pressure $\bar{\bar{P}} = \sum_s \bar{\bar{P}}_s$, the overall momentum equation will be

$$\frac{\partial}{\partial t}(\rho\, \bar{v}) + \nabla_{\bar{r}}(\rho\, \bar{v}\, \bar{v} + \bar{\bar{P}}) = \rho^e\, \bar{E}^* + \bar{j} \times \bar{B}$$

where ρ^e is the excess net electron charge density due to deviation from the charge neutrality. This equation can be re-written by using over-all mass conservation equation as,

$$\rho\, \frac{D\bar{v}}{Dt} + \nabla_{\bar{r}}\, \bar{\bar{P}} = \rho^e\, \bar{E}^* + \bar{j} \times \bar{B} \qquad (1.2.2)$$

when we can neglect the body force $\rho^e\, \bar{E}$ and the current flow due to $\rho^e \bar{v}$, $\rho^e \bar{E}^*$ is neglected and it is obvious that the total electric current $\bar{J}$ then is equal to the conduction current, $\bar{J} = \bar{j}$. This is the case of most practical interest as we prementioned (Ref.27).

It should be noted that this over-all momentum equation is not sufficient to describe the motion of plasma until the total conduction current $\bar{j}$ and the pressure tensor $\bar{\bar{P}}$ are described elsewhere. Although it forces us to use the energy equation, this involves adding an energy term only to be given by the next higher moment equation. To make a closed set of equations, alternate procedures seem to be required along with some additional assumptions. Ohm's law will be introduced, so that naturally only both the species and over-all momentum equations would be used to obtain $\bar{j}$ related to any of the other variables. In fact, $\bar{j}$ frequently is used instead of $\bar{j}_e$ due to the small mass of electron during this procedure (see Chapter 1, Sec.1.3). However, unfortunately, Ohm's law turns out to involve the heat flux terms described only by considering higher order momentum terms. In the next section this argument is resolved by using a closed set of equations obtained by different methods but proved to be identical to each other for a weakly ionized gas. One of the procedures is inverting a large matrix for a gas mixture and the other is the successive approximation method.

Before finishing this section, it should be pointed out that in any of the cases to obtain Ohm's law, one more variable, electron temperature, should be given elsewhere to make the set of equations complete. Only for this purpose will the energy equation be introduced. One more factor to be mentioned is due to the inelastic collisions that occur between molecules with internal degrees of freedom. Because, clearly, mass and momentum should be conserved even in this case, although kinetic energy is no longer conserved, the equations (1.2.1) and (1.2.2) would not be changed.

Section 1.3 Ohm's Law for a Mixture Gas

It has been noted that in deriving Ohm's law, from the equation of motion is the basic starting point. The equation (1.1.4) is

$$\frac{\partial}{\partial t}\left(\rho_s \bar{v}_s\right) + \nabla_{\bar{r}} \cdot \left[\rho_s(\bar{v}\,\bar{\bar{V}}_s + \bar{\bar{V}}_s\,\bar{v})\right] + \nabla_{\bar{r}} \cdot$$

$$\left(\rho_s \bar{v}\,\bar{v}\right) + \nabla_{\bar{r}} \cdot \bar{\bar{P}}_s - q_s n_s\left[\bar{E}^* + \bar{V}_s \times \bar{B}\right] = \bar{R}_s^{(1)}$$

This will be rewritten as follows,

$$\rho_s \frac{D\bar{v}}{Dt} + \frac{\partial}{\partial t}\left(\rho_s \bar{\bar{V}}_s\right) + \nabla_{\bar{r}} \cdot \left[\rho_s(\bar{v}\,\bar{\bar{V}}_s + \bar{\bar{V}}_s\,\bar{v})\right]$$

$$+ \nabla_{\bar{r}} \cdot \bar{\bar{P}}_s - q_s n_s\left[\bar{E}^* + \bar{\bar{V}}_s \times \bar{B}\right] = \bar{R}_s^{(1)}$$

By separating the pressure tensor $\bar{\bar{P}}_s$ into a diagonal part $P_s I$ where I is the unit tensor, and an off-diagonal part, that is, a viscous part, $\bar{\bar{\tau}}_s$ the above equation becomes

$$\rho_s \frac{D\bar{v}}{Dt} + \nabla_{\bar{r}} P_s + \nabla_{\bar{r}} \bar{\bar{\tau}}_s + \frac{\partial}{\partial t} (\rho_s \bar{\bar{V}}_s) + \nabla_{\bar{r}} [\rho_s (\bar{v} \bar{V}_s + \bar{V}_s \bar{v})]$$

$$- q_s n_s [\bar{E}^* + \bar{\bar{V}}_s \times \bar{B}] = \bar{R}_s^{(1)} \tag{1.3.1}$$

Now it is generally valid in all but cases of extreme nonuniform that the gradient of the viscous part of the partial pressure tensors, $\nabla_{\bar{r}} \cdot \bar{\bar{\tau}}_s$ can be neglected comparing to the gradient of hydrostatic pressure $\nabla_{\bar{r}} P_s$. In fact, for a collision dominated plasma, it is generally the case that $\bar{\bar{V}}_e/\bar{v}_e \ll 1$ and similarly $|\bar{\bar{\tau}}_e|/p_e \ll 1$ (Ref.1), so that the gradient of τ_e can be neglected. As well, in all but cases of extreme nonuniformaties, we can assume that the macroscopic parameters of the plasma change only slightly within distances of the order of the mean free path and during times of the order of the collision times in the plasma. Then, we can neglect $(\rho_s \bar{V}_s)\ \partial/\partial t$ and the contributions of the diffusion of $\nabla_{\bar{r}} (\rho_s \bar{v} \bar{V}_s)$ and $\nabla_{\bar{r}} (\rho_s \bar{V}_s \bar{v}^s)$ to the momentum. Under this condition, the pressure defined as $\bar{\bar{P}}'_s \equiv \rho_s <\bar{c}'\bar{c}'>_s$ where $\bar{C}'_s = \bar{W}_s - \bar{v}_s$ (Ref.27), is identical to $\bar{\bar{P}}_s$. As a matter of fact, the conditions for the electron-fluid to be collision dominated (this is just the present case) are

$$\nu_{ee}^{-1} / \tau_o \ll 1 \ , \quad \bar{v}_e \nu_{ee}^{-1} / L \ll 1$$

where τ_o,L are the characteristic values for macroscopic change, (in Ref.28, $\bar{v}_e \nu_{ee}^{-1} / L \sim 0.01$ to 0.02). Some results (Ref.8) show that even the ionization relaxation time and length of the plasma are respectively orders of 100 μsec and 1 cm in most cases of practical interest compared with the collision time of the order of 10^{-5} sec. and mean free path of order of 10^{-5} cm.

Consequently, the equation (1.3.1) becomes

$$\rho_s \frac{D\bar{v}}{Dt} + \nabla_{\bar{r}} P_s - q_s n_s [\bar{E}^* + \bar{\bar{V}}_s \times \bar{B}] = \bar{R}_s^{(1)} \tag{1.3.2}$$

and the equation (1.2.2) is rewritten as;

$$\rho \frac{D\bar{v}}{Dt} + \nabla P = \rho^e \bar{E}^* + \bar{j} \times \bar{B} \tag{1.3.3}$$

Eliminating $D\bar{v}/Dt$ from (1.3.2) and (1.3.3), we obtain the following algebraic system

$$(\nabla P_s - \frac{\rho_s}{\rho} \nabla P) + (\frac{\rho_s}{\rho} \rho^e - q_s n_s) \bar{E}^*$$

$$- q_s n_s \bar{\bar{V}}_s \times \bar{B} + \frac{\rho_s}{\rho} \bar{j} \times \bar{B} = \bar{R}_s^{(1)}$$

If we neglect the excess charge ρ^e (same as the plasma is assumed to be electrically neutral),

$$(\nabla P_s - \frac{\rho_s}{\rho} \nabla P) - q_s n_s \bar{\bar{V}}_s \times \bar{B} + \frac{\rho_s}{\rho} \bar{j} \times \bar{B} = \bar{R}_s^{(1)} \tag{1.3.4}$$

In the first approximation of collision integral $\bar{R}_s^{(1)}$, the magnitude of the impulse transmitted by collisions between particles of the s-kind with those of other components is taken to be proportional to the respective differences of the macroscopic component velocities. Then $\bar{R}_s^{(1)}$ is

$$\bar{R}_s^{(1)} = \sum_r \frac{n_s m_s m_r}{m_s + m_r} \nu_{sr} (\bar{\bar{V}}_r - \bar{\bar{V}}_s)$$

Inserting $R_s^{(1)}$ into (I-11), we get

$$\nabla P_s - \frac{\rho_s}{\rho} \nabla P - q_s n_s \bar{E}* - q_s n_s \bar{\bar{V}}_s \times \bar{B} + \frac{\rho_s}{\rho} \bar{j} \times \bar{B} = \sum_r \frac{n_s m_s m_r}{m_s + m_r} \nu_{sr} (\bar{V}_r - \bar{V}_s)$$

Thus, for electron particle

$$\nabla P_e - \frac{\rho_e}{\rho} \nabla P + en_e \bar{E}* + en_e \bar{\bar{V}}_e \times \bar{B} + \frac{\rho_e}{\rho} \bar{j} \times \bar{B} = \bar{R}_e^{(1)}$$

For ion particle

$$\nabla P_I - \frac{\rho_I}{\rho} \nabla P - en_I \bar{E}* - en_I \bar{\bar{V}}_I \times \bar{B} + \frac{\rho_I}{\rho} \bar{j} \times \bar{B} = \bar{R}_I^{(1)}$$

Using both equations to eliminate ∇ P, we finally get

$$\nabla P_e - \frac{\rho_e}{\rho} \nabla P_I + en_e (1 + \frac{\rho_e}{\rho}) \bar{E}* + en_e (\bar{\bar{V}}_e + \frac{\rho_e}{\rho_I} \bar{\bar{V}}_I) \times \bar{B} = \bar{R}_e^{(1)} - \frac{\rho_e}{\rho_I} \bar{R}_I^{(1)}$$

when the plasma is assumed to be electrically neutral, that is, ne = n_I since

$\frac{\rho_e}{\rho_I} = \frac{n_e m_e}{n_I m_I} = \frac{m_e}{m_I} \ll 1$, if we neglect the terms including

the factor of ρ_e / ρ_I, this equation is reduced to,

$$\nabla P_e + en_e \bar{E}^* + en_e \bar{\bar{V}}_e \times \bar{B} = \bar{R}_e^{(1)} \qquad (1.3.5)$$

Now that $m_e/m_I \ll 1$, $m_e/m_n \ll 1$ and if we assume $\bar{v}_n \doteq \bar{v}_I \doteq \bar{v}$ (that is, $\bar{V}_n, \bar{V}_I \doteq 0$),

$$\bar{j}_e = en_e \bar{\bar{V}}_e \doteq - \bar{j}$$

and the collision integral $\bar{R}_e^{(1)}$ becomes

$$\begin{aligned} \bar{R}_e^{(1)} &\doteq n_e m_e \sum_r \nu_{er} (\bar{\bar{V}}_r - \bar{\bar{V}}_e) \\ &\doteq - n_e m_e \sum_r \nu_{er} \bar{\bar{V}}_e \\ &\doteq \frac{m_e}{e} \bar{j} \sum_r \nu_{er} \end{aligned}$$

Thus, (1.3.5) becomes

$$\nabla P_e + en_e \bar{E}^* - \bar{j} \times \bar{B} = \frac{m_e}{e} \bar{j} \Sigma_r \nu_{er}$$

Re-arranging it to obtain,

$$\bar{E}^* + \frac{1}{en_e} \nabla P_e = \frac{m_e}{n_e e^2} \bar{j} \Sigma_r \nu_{er} + \frac{1}{n_e e} \bar{j} \times \bar{B}$$

We call it "first approximation" Ohm's law.

Let the effective electric field $\bar{E}' = \bar{E}^* + \frac{1}{en_e} \nabla P_e$ then

$$\bar{E}' = \frac{1}{\sigma} \bar{j} + \chi \bar{j} \times \bar{B} \qquad (1.3.6)$$

where

$$\frac{1}{\sigma} = (m_e/n_e e^2) \Sigma_r \nu_{er}$$

and

$$\chi = (n_e e)^{-1} \quad \text{respectively.}$$

The parameters σ and χ are called scalar electric conductivity and Hall coefficient.

Using the "13 moment" approximation (Ref.1,3,29) in diffusion equation which are taken to be proportional to the relative thermal flux of the particles, give $\bar{R}_s^{(1)}$ as

$$\bar{R}_s^{(1)} = \frac{4}{3} \Sigma_r n_s \mu_{sr} \nu_{sr} (\bar{\bar{V}}_r - \bar{\bar{V}}_s) + \frac{16}{3} \Sigma_r n_s n_r \mu_{sr} \Omega_{sr}^{(1)} A_{sr}^{(2)} \left(\frac{\bar{h}_r}{\gamma_r P_r} - \frac{\bar{h}_s}{\gamma_s P_s} \right)$$

for electron,

$$\bar{R}_e^{(1)} \doteq \frac{4}{3} \frac{m_e}{e} \bar{j} \Sigma_r \nu_{er} - n_e m_e \frac{\bar{h}_e}{P_e} \Sigma_{r\neq e} A_{er}^{(2)} \tau_{er}^{-1}$$

for ion,

$$\bar{R}_I^{(1)} \doteq \frac{4}{3} \frac{m_e}{e} \bar{j} \nu_{eI} + n_e m_e \frac{\bar{h}_e}{P_e} A_{eI}^{(2)} \tau_{eI}^{-1}$$

Here, we also neglected the terms multiplied by m_e/m_I and assumed that $n_e = n_I$, $\bar{j} = -\bar{j}_e$, since under the conditions here considered in the diffusion state the Debye radius is much smaller than the characteristic lengths of the system (electrical neutrality of plasma, Ref.26).

The electron relative heat flux vector $\bar{h}_e$ was found to be given by

$$\bar{h}_e = -\lambda_e \left[\bar{P}_e - \frac{e}{m_e} \frac{\tau_e^*}{1+\omega_e^2 \tau_e^{*2}} \bar{P}_e \times \bar{B} + \frac{e^2}{m_e^2} \frac{\tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}} (\bar{P}_e \times \bar{B}) \times \bar{B}\right]$$

where

$$\lambda_e = \frac{5}{2} \frac{k}{m_e} P_e \tau_e^*$$

$$\bar{P}_e \doteq \nabla T_e - \frac{m_e^2}{k\, n_e\, e} \bar{j}\, \nu_o$$

τ_e^* was defined by,

$$\tau_e^* \equiv 0.4\, \tau_{ee}^{-1} + 2.5\, \Sigma_{K \neq e}\, A_{eK}^{(5)}\, \tau_{eK}^{-1}$$

and ν_o by,

$$\nu_o \equiv \Sigma_{K \neq e}\, A_{eK}^{(2)}\, \tau_{eK}^{-1}$$

Thus

$$\frac{\bar{h}_e}{P_e} = -\frac{5}{2} \frac{k}{m_e} \tau_e^* \left[\nabla T_e - \frac{e}{m_e} \frac{\tau_e^*}{1+\omega_e^2 \tau_e^{*2}} (\nabla T_e) \times \bar{B} + \frac{e^2}{m_e^2} \frac{\tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}} (\nabla T_e \times \bar{B}) \times \bar{B} - \frac{m_e \nu_o}{k\, en_e} \bar{j} + \frac{\nu_o}{k\, n_e} \frac{\tau_e^*}{1+\omega_e^2 \tau_e^{*2}} - \frac{e\, \nu_o}{k\, n_e m_e} \frac{\tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}} (\bar{j} \times \bar{B}) \times \bar{B}\right]$$

Inserting this expression into $\bar{R}_e^{(1)}$, and replacing in (1.3.5) by this result, we obtain

$$\nabla P_e + en_e \bar{E}^* - \bar{j} \times \bar{B} = \frac{4}{3} \frac{m_e}{e} \bar{j}\, \Sigma_r\, \nu_{er} + \frac{5}{2} m_e k\, \tau_e^*\, \nu_o \Big[\nabla T_e - \frac{e}{m_e} \frac{\tau_e^*}{1+\omega_e^2 \tau_e^{*2}} (\nabla T_e) \times \bar{B} + \frac{e^2}{m_e^2} \frac{\tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}} (\nabla T_e \times \bar{B}) \times \bar{B} - \frac{m_e \nu_o}{k\, en_e} \bar{j} + \frac{\nu_o}{kn_e} \frac{\tau_e^*}{1+\omega_e^2 \tau_e^{*2}} (\bar{j} \times \bar{B}) - \frac{e\nu_o}{kn_e m_e} \frac{\tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}} (\bar{j} \times \bar{B}) \times \bar{B}\Big]$$

and rearrange it to obtain;

$$\bar{E}^* + \frac{1}{en_e} \nabla P_e - \frac{5}{2} \frac{k}{e} \nu_o \tau_e^* [\nabla T_e - \frac{e}{m_e} \frac{\tau_e^*}{1+\omega_e^2 \tau_o^{*2}} (\nabla T_e) \times \bar{B} + \frac{e^2}{m_e^2} \frac{\tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}}$$

$$(\nabla T_e \times \bar{B}) \times \bar{B}] = \frac{m_e}{n_e e^2} (\frac{4}{3} \nu_t - \frac{5}{2} \nu_o^2 \tau_e^*) \bar{j} + \frac{1}{n_e e} ($$

$$1 + \frac{5}{2} \frac{\nu_o^2 \tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}}) (\bar{j} \times \bar{B}) - \frac{5}{2} \frac{\nu_o^2}{m_e n_e} \frac{\tau_e^{*3}}{1+ \omega_e^2 \tau_e^{*2}} (\bar{j} \times \bar{B}) \times \bar{B}$$

where $\nu_t = \Sigma_r \nu_{er}$, we get 'second' approximation of Ohm's law.

Let the effective electric field $\bar{E}'$

$$\bar{E}' = \bar{E}^* + \frac{1}{en_e} \nabla P_e - \frac{5}{2} \frac{k}{e} \nu_o \tau_e^* [\nabla T_e - \frac{e}{m_e} \frac{\tau_e^*}{1+\omega_e^2 \tau_e^{*2}}$$

$$(\nabla T_e) \times \bar{B} + \frac{e^2}{m_e^2} \frac{\tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}} (\nabla T_e \times \bar{B}) \times \bar{B}]$$

The 'second approximation' Ohm's law to describe the nonuniform plasma in electric and magnetic fields is now including not only pressure gradients but also the temperature gradients and is given by

$$\bar{E}' = \frac{1}{\sigma} \bar{j} + \chi (\bar{j} \times \bar{B}) + \epsilon (\bar{j} \times \bar{B}) \times \bar{B} \qquad (1.3.7)$$

where the scalar electric conductivity is defined as

$$\frac{1}{\sigma} = \frac{4}{3} \frac{m_e \nu_t}{n_e e^2} - \frac{5}{2} \frac{m_e \nu_o^2}{n_e e^2} \tau_e^* \qquad (1.3.8)$$

the Hall coefficient as

$$\chi = \frac{1}{n_e e} + \frac{5}{2} \frac{\nu_o^2}{n_e e} \frac{\tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}} \qquad (1.3.9)$$

and the ion slip coefficient without considerations of ion-concentration as

$$\epsilon = - \frac{5}{2} \frac{\nu_o^2}{n_e m_e} \frac{\tau_e^{*3}}{1+\omega_e^2 \tau_e^{*2}} \qquad (1.3.10)$$

In the expression of the effective electrical field $\bar{E}'$, the gradient term of pressure should be taken into account in most cases of practical interests, since it involves the electron number density gradients, which were sometimes found to be of the same magnitude of the applied field at certain place of MHD channel, (Ref.30). Since the term proportional to the electron temperature gradient, however, accounts for only about a few per cent of the electron current flow, one may neglect this on purpose for simplification of the problem in a given MHD state.

Section 1.4 Energy Balance Equation

The energy balance equation is for describing the transfer energy into the plasma or the required energy to keep the plasma in the motion of the desirable state thermally and kinetically and in the balance between the interacting particles through collisions.

The calculation for energy transfer integrals generally has been taken to be the following,

$$R_s^{(2)} = - \Sigma_r \; 3 \, n_s \; \frac{m_s \, m_r}{(m_s + m_r)^2} \; \nu_{sr} \, k \, (T_s - T_r)$$

when we neglect the inelastic collisions between particles. As we might see from this expression, the ratio of energy transfer by elastic collisions between the atom and the ion species is much greater than that between the electron and these particles, because of the mass factor. Consequently, we expect that collisions between ions and atoms would tend to maintain their temperature at almost the same value, while the electron temperature may greatly exceed their temperature. Because of this, the plasma is expected to be described by the two temperatures, gas temperature and an electron temperature different than this. This is nonuniform process of plasma (Ref.31).

From Eq. (1.1.5) for electron, replacing $\bar{v}_e$ by $\bar{v} + \bar{\bar{V}}_e$, we obtain

$$\frac{\partial}{\partial t} \left(\tfrac{1}{2} \rho_e \bar{v}^2 + \rho_e \bar{\bar{V}}_e \bar{v} + \tfrac{3}{2} k n_e T_e + n_e < \epsilon_{ie} > \right) + \nabla_{\bar{r}} [(\tfrac{1}{2} \rho_e v^2 +$$

$$\tfrac{3}{2} k n_e T_e + n_e < \epsilon_{ie} > \bar{v}] + \nabla_{\bar{r}} \; \bar{q}_e + \nabla_{\bar{r}} \; [\tfrac{1}{2} \rho_e \, v^2 \, \bar{\bar{V}}_e] +$$

$$\nabla_{\bar{r}} \left(\rho_e \bar{v} \, (\bar{v} \cdot \bar{V}_e) \right) + \nabla_{\bar{r}} \; (\bar{v} \, \bar{\bar{P}}_e) - \bar{E} \circ \bar{j} + \bar{E} \; (e n_e \, \bar{v}) = R_e^{(2)}$$

The term representing the derivative with respect to time can be neglected in the case of steady state. By considering the very small mass of the electron and the small contribution of $e n_e \bar{v}$ and stress tensor $\bar{\bar{\tau}}_e$, we can eliminate several terms, so that finally, the electron energy equation becomes

$$\nabla_{\bar{r}} \; [(\tfrac{3}{2} k n_e T_e + n_e <\epsilon_{ie}>) \; \bar{v} \;] + \nabla_{\bar{r}} \; \bar{q}_e + \nabla_{\bar{r}} \; (P_e \bar{v}) = \bar{E}^* \, \bar{j} = R_e^{(2)} \tag{1.4.1}$$

In the lowest approximation

$$- \bar{E}\ \bar{j} = R_e^{(2)} \qquad (1.4.2)$$

This joule heating and collision energy loss equation will then determine the electron temperature of the plasma with the expression of $R_e^{(2)}$.

The point concerning inelastic collisions might be argued, since the species integrated contribution of inelastic collision frequencies of Maxwellian electrons to the electron energy loss is no longer identically zero and may cause important effects. However, in the mass conservation and momentum equations the inelastic effect is almost negligible insofar as the electron-electron collision frequency is much greater than the inelastic collision frequency of electrons (Ref. 31, 32).

Under the steady-state condition, the energy lost by the electrons per unit time in elastic collisions with heavy particles is

$$R_{el}^{(2)} = - 3\ kn_e \Sigma_r \frac{m_e}{m_r} (T_e - T_r)\tau_{er}^{-1}$$

$$= - \frac{3}{2}\ kn_e \Sigma_r\ \delta_{r,el}(T_e - T_r)\ \tau_{er}^{-1} \qquad (1.4.3)$$

where $\delta_{r,el}$ is defined as $\delta_{r,el} = 2\ m_e/m_r$ and for argon $\delta_{A,el} \doteq 2.7 \times 10^{-5}$.

No account has been taken of the average energy transfer through inelastic collisions as, namely, the collisional transitions between two discrete states and between the continuum and a discrete state and the radiative transitions between the continuum and a discrete state (Ref.32,34).

Sec. 1.5 Ionization of a Seeded Gas

Finally, we should derive the equation relating to T_e to the number density to make the set of equations complete.

When the diffusion of the electrons through the ionized gas is rapid enough, the average thermal energy of electron ($3kT_e/2$) will be considerably above that of heavy particles ($3kT/2$). (See Chapt. 1.4, under such conditions, the heavy particles can be considered immobile). Usually this causes the nonuniformity of plasma through the process as follows. This assumption can be proved valid so long as the energy loss factor is less than unity (Ref.31). Then the random velocity of the electron should be always much greater than its directed velocity even in a strong electric field. This provides the postulation for thermodynamical non-equilibrium for process which relax over longer times than that required for the velocity distribution of electron to come to equilibrium. Thus the process of ionization is completely dominated by the electron thermal energy, since in this case energy transferred between free electrons in a Maxwell distribution is well transferred very readily between free electrons and valence electrons if the electron number density is not too low (Ref.17,36). This is the case wherein the free electrons are strongly coupled thermally to the valence electrons of the atom rather than to the atom itself so as to be in Maxwell distribution between electrons themselves. This seems likely to be valid locally rather than in any point of space inside of a weakly ionized plasma. Thus, a much larger degree of ionization results locally and this non-equilibrium process relating T_e to the number density can be described along with pre-obtained equations (Refs. 13, 20 & 37).

Now then, the degree of ionization for a seeded plasma can be obtained on the assumption that volume ionization and recombination process are in equilibrium at the electron temperature so that the electrons can be distributed as a stationary Maxwell distribution of velocities. Then we may use the Saha equations which are, for seeding species (Refs. 14, 34)

$$\frac{n_{s^+}\, n_e}{n_s} = \frac{Z_{s^+}\, Z_e}{Z_s} \left(\frac{m_e k\, T_e}{2\pi \hbar^2} \right)^{3/2} \exp \left(- \frac{\epsilon_{i_s}}{k\, T_e} \right)$$

and for the inert carrier gas

$$\frac{n_{A^+}\, n_e}{n_A} = \frac{Z_{A^+}\, Z_e}{Z_A} \left(\frac{m_e k\, T_e}{2\pi \hbar^2} \right)^{3/2} \exp \left(- \frac{\epsilon_{i_A}}{k\, T_e} \right)$$

In these equations, ϵ_{i_s} and ϵ_{i_A} are the first ionization potentials of the seed and the carrier gas, respectively.

Here we neglect the increasing of the degree of ionization due to the magnetic field. The criteria is the ratio of the quanta of the electron gyration motion $\hbar\, \omega_e \equiv \hbar eB/m_e$ to the magnitude of the thermal energy kT_e (Ref.35). Since in the case of the magnetic intensity B of 1.7 Webers/m^2 and of the electron temperature of 2000°K the ratio $\hbar\, \omega_e/k\, T_e$ is around 1.1×10^{-3}, the elevation of ionization due to the magnetic field in the Saha equation can be negligible, so that under the above condition this form of the Saha equation always is applicable.

The total number density n will be

$$\begin{aligned} n &= n_{A^+} + n_A + n_{s^+} + n_s + n_e \\ &= n_A + n_s + 2\, n_e \end{aligned}$$

The last equation was given by assuming the electrical neutrality of the plasma Seeding ratio S_s is defined by

$$S_s = \frac{n_s + 2n_{s^+}}{n}$$

and the degrees of ionization by

$$\alpha_s = \frac{n_{s^+}}{n_s + n_{s^+}} \qquad \text{for seeding species}$$

and by

$$\alpha_A = \frac{n_{A^+}}{n_A + n_{A^+}} \quad \text{for carrier gas.}$$

Combining these definitions, we obtain

$$n_{s^+} = n\, \alpha_s\, S_s \tag{1.5.1a}$$

$$n_{A^+} = n\, (1-S_s)\, \alpha_A \tag{1.5.1b}$$

$$n_s = n\, (1-S_s)\, \alpha_s \tag{1.5.1c}$$

$$n_A = n\, (1-S_s)\, (1-\alpha_A) \tag{1.5.1d}$$

and

$$n_e = n\, [(1-S_s)\, \alpha_A + \alpha_s\, S_s\,] \tag{1.5.1e}$$

Inserting (1.5.1)'s into the Saha equations,

$$\frac{\alpha_s}{1-\alpha_s}\, [\alpha_s S_s + (1-S_s)\, \alpha_A\,] = \frac{a}{n}$$

$$\frac{\alpha_A}{1-\alpha_A}\, [\alpha_s S_s + (1-S_s)\, \alpha_A\,] = \frac{b}{n}$$

where a, b are given by

$$a,\, b = \frac{Z_{j^+}\, Z_e}{Z_j} \left(\frac{m_e k\, T_e}{2\, \pi \hbar^2} \right)^{3/2} \exp\left(- \frac{\epsilon_{ij}}{k\, T_e} \right), \quad j = s,\, A$$

These algebraic equations for α_A and α_s can be solved by iterating numerical method to the desirable accuracy. For the numerical scheme, if we rearrange them for α_A and α_s, then

$$\alpha_A = \frac{1}{1 + \frac{a}{b}\left(\frac{1}{\alpha_s} - 1\right)}$$

$$\alpha_s = \frac{-\{\frac{a}{n} + \alpha_A\, (1-S_s)\} + \sqrt{\{\, \frac{a}{n} + \alpha_A\, (1-S_s)\}^2 + 4\, S_s\, \frac{a}{n}}}{2\, S_s}$$

Because of several time's higher ionization potential of the carrier gas than that of the seed, we can carry out this numerical procedure as follows. For the first approximation, α_A may be set to zero and α_s than be calculated. This value of α_s is then used to calculate α_A. The process is repeated with the new value obtained for α_A. It turned out that this procedure converged quite rapidly over the considerably wide range of electron temperature to the accuracy of order of 10^{-8}. In the worst case, three iterations were required for argon plasma seeded with potassium.

Sec. 1.6 Collision Cross Sections

The collision cross section for momentum transfer between electrons and ions is

$$Q_{eI} = \frac{\pi}{2}\left[\frac{e^2}{4\pi\epsilon_o}\frac{\gamma_{eI}}{\mu_{eI}}\right]^2 \ln\sqrt{\lambda_{eI}^2+1}$$

$$\doteq \frac{\pi}{2}\left[\frac{e^2}{4\pi\epsilon_o}\frac{1}{kT_e}\right]^2 \ln\sqrt{\lambda_{eI}^2+1} \qquad (1.6.1)$$

where λ_{eI} is the ratio of the Debye length λ_D to the impact parameter for a 90^o collision of an electron of speed 3 kTe and defined by

$$\lambda_{eI} = 3\lambda_D\left[\frac{4\pi\epsilon_o}{e^2}\frac{\mu_{eI}}{\gamma_{eI}}\right] \doteq 3\lambda_D\left[\frac{4\pi\epsilon_o}{e^2}kT_e\right]$$

and the Debye screened length λ_D is introduced to prevent the collision cross section integral from diverging and defined by

$$\lambda_D = \frac{1}{\epsilon_o}\left(\frac{n_e e^2}{kT_e} + \frac{n_I e^2}{kT_I}\right) \doteq \frac{n_e e^2}{k\epsilon_o}\left(\frac{1}{Te}+\frac{1}{T_I}\right)$$

There has been some discussion (Ref. 8,31) concerning the introduction of λ_D into these calculations. However, when $\ln\Lambda \gg 1$ which may arise in many applications, the result based on the screened Coulomb potential and on the unscreened potential together with a Debye length cut-off have been commented on by Liboff, R. L.* . They yield the same value for the transport properties and in both cases the collision parameter is limited to distances of the order of λ_D.

According to the calculations of weighted momentum transfer collision cross sections (Ref.3), $A_{eI}^{(2)}$ and $A_{eI}^{(5)}$ are given by

$$A_{eI}^{(2)} = \frac{6}{5}C_{eI}^{*} - 1 = -0.6$$

and

$$A_{eI}^{(5)} = \frac{5}{2} - \frac{6}{5}B_{eI}^{*} = 1.3$$

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* The Physics of Fluids, Vol.2, 40-46, 1959.

where C^*_{eI} and B^*_{eI} are defined by many authors including Zhdanov and Cowling (Ref.3, 38).

So long as the electron temperature increases not too much so that the condition under which $\ln \Lambda$ is not too high is satisfied, all the series of expressions for collisions between electrons and ions is quite satisfactory even when we neglect the higher approximation (Ref.6).

The differential momentum transfer cross section between the electron-argon atom is characterized by the Ramsauer-Townsend minimum at an electron energy near 0.3 eV (Ref.39). Because of the significance of this effect, the determination has been difficult. A few available theoretical computations and experimental data (Ref.40-43 including the works of Frost and Phelps) have been compared. To about 0.5 eV of electron energy the determination from the modified effective range formulae of O'Malley (Ref.39) was used to calculate the electron-argon momentum transfer collision cross section. That is,

$$Q_{ea} = \frac{4\pi}{k^2} \left(\sin^2\eta_o + \sin^2\eta_1 \right) a_o^2 \qquad (1.6.2)$$

where a_o is the Bohr radius

$$\frac{\sin\eta_o}{k} = 1.70 - 3.13\ EV^{\frac{1}{2}} + 0.92\ EV \ln EV + 1.23\ EV$$

and

$$\frac{\sin\eta_1}{k} = 0.6026\ EV^{\frac{1}{2}} - 0.589\ EV$$

In these expressions EV denotes the electron kinetic energy in electron volts. In more accurate approximation (Ref.39),

$$Q_{ea} = \frac{4\ \pi}{k^2} \ \Sigma^{\infty}_{L=0} \ (L + 1) \sin^2(\eta_L - \eta_{L+1})$$

where $k^2 = 2m\ E/\hbar^2$ is the wave number of incident electron particles, η_L represents the phase shifts of L-wave and L is the orbital angular quantum number. Kruger and Viegas (Ref.5) calculated the effective mean cross sections of argon and potassium to obtain the conductivity and Hall parameter.

Assuming solid elastic sphere models of argon and potassium for higher momentum transfer integrals, we used $B^*_{eA} = B^*_{eK} = 3/4$ $C^*_{eA} = C^*_{eK} = 5/6$ (Refs.2,3,9).

For the electron-potassium case, we used Nichol's results (Ref.14) and recalculated the cross section from the figure in terms of electron temperature around the region of interest.

$$Q_{eK} = (403\ EV + 150) \times 10^{-20}\ (m^2) \qquad (1.6.3)$$

COMMENTS ON RESULTS

In our calculations, we neglect the electron-electron

$$\left(\nu_{ee} = \sqrt{2}\ \nu_{ei},\ \frac{4\pi e^4 n_e}{m_e^3 v_e^3} \ln \frac{k^{\frac{3}{2}} T_e T^{\frac{1}{2}}}{e^3 n_e^{\frac{1}{2}}}\right)$$

ion-ion (Ref.44) and ion-atom collision integrals (Ref.26, $\nu_{in} \sim \left(\frac{m_e T_i}{m_i T_e}\right)^{\frac{1}{2}} \nu_{en}$).

Along with increasing the electron temperature, the differences of χ and σ from the first approximation remain greater than zero, the differences decreasing up to the electron temperature of around 2200°K. For electron temperatures above 2200°K, the deviations are so remarkable that at around 3000°K they differ by some 20% from the first approximation (Fig.2-4). To be sure, at electron temperatures above 2200°K, electron-electron collision effects which we neglected should be taken into account (Refs. 8, 17, 31, 36) while ion-ion and ion-neutral collision still remain at a negligible contribution. In spite of these corrections which may reduce the deviation closer to the first approximation, our computations ensure us that the errors in σ and χ seeming to be within a few per cent at the equilibrium state (Refs. 1,3) mislead us into underestimating the electrical conductivity and Hall parameter for non-equilibrium cases where electron temperature might be elevated to as much as twice the gas temperature. On the other hand, below a gas temperature of 2000°K (electron number density of around $3 \times 10^{18}/m^3$), violation of the electron distribution assumed to be Maxwellian should be naturally considered so that electron-electron energy exchange can be properly taken into account compared with electron-atom energy exchange. This consideration may give correct number density so as to determine more reasonable calculation of σ and β.

Number density at electron temperatures between 2200°K and 3000°K are near $5 \times 10^{18}\ m^3$ and $5 \times 10^{20}/m^3$ (Fig.1). This indicates that at the low electron density corresponding to the electron temperature below 2200°K, the significance of the correction to first approximation is not so important due to the invalidity of the Saha equation following from the violation of the electron distribution assumed to be Maxwellian due to the smallness of electron number density. Also our formulation can be used as a reasonably well explainable one so long as the electron-electron collisional effect is properly considered; well measured experimental data and theoretical results of various kinds of collision cross sections are available in terms of electron temperature. The relations of the conductivity to the current density are given in Fig.6. Since, in the present paper, we neglect all the inelastic collision including radiation energy loss, our results show the underestimating of current densities for the region where the radiation loss cannot be neglected. However, at the high current density for which radiation is no longer the dominating effect to energy equation, the results agree fairly well with those referred to. It enables us to expect that from the energy balance equation, the electron temperature was slightly overestimated at low current density, while underestimating at high values of current density (Fig.7) can be explained by the effective energy loss factor. With increases in electron temperature the role of inelastic collisions becomes greater and the energy loss factor which we assumed constant increases; this increases the degree of ionization sharply and causes the increasing of left hand terms in energy balance equation. Consequently the electron temperature can be higher than what we calculated (Ref.31). Within this error mainly due either to radiation energy or to inelastic collision, this figure represents the relation between input energy into plasma by joulean heating and thermal maintenance through collisions which cause the electron temperature to increase higher than the gas temperature. It should be mentioned, however, that the energy loss procedure with the invalidity of the Saha equation for low temperature (Refs. 17, 36, 45) is quite difficult to describe completely satisfactorily.

Chapter 2. Solution to Segmented Electrodes

Sec. 2.1 Isentropic Flow Equations

The static gas temperature T_∞ and static pressure P_∞ were referred to the stagnation temperature T_o and stagnation pressure P_o by the relation through isentropic expansion, which are good approximations for small degree of ionization (Refs. 27, 34).

$$\frac{P_o}{P_\infty} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_o}{T_\infty} = 1 + \frac{\gamma-1}{2} M^2$$

where M is Mach number of the flow near the entrance of the generator when the gas velocity is defined by

$$u_\infty = M \frac{\gamma\, k\, T_\infty}{\Sigma_s (\frac{n_s}{n})\, m_s}$$

and the isentropic sound speed a in the absence of electron heating as $a^2 = (\partial P/\partial \rho_s)$. By using the flow continuity equation,

$$n_o u_o = n_\infty u_\infty$$

n_∞ can be written in terms of Po, To together with the ideal gas law

$$P_s = n_s k\, T_s$$

$$n_\infty = n_o \left[1 + \frac{\gamma-1}{2} M^2\right]^{-\frac{1}{\gamma-1}}$$

$$= \frac{Po / k\, To}{\left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{1}{\gamma-1}}} \qquad (2.1.1)$$

so that the value of n in Chapter 1 is now replaced by n_{∞} , which is the number density at the entrance of channel.

Sec.2.2 Current Governing Equation

When we neglect the ion slip coefficient, the effective field $\bar{E}'$ becomes

$$\bar{E}' = \frac{1}{\sigma} \bar{J} + \chi \ (\bar{J} \times \bar{B})$$

If we take the rotation of the above equation, we arrive at

$$\nabla \times \bar{E}' = \nabla\left(\frac{1}{\sigma}\right) \times \bar{J} + \frac{1}{\sigma} \ (\nabla \times \bar{J}) + (\nabla\chi) \times (\bar{J} \times \bar{B}) +$$

$$\chi \ (\bar{B}.\nabla) \ \bar{J} - \chi \ (\bar{J}.\nabla) \ \bar{B} \qquad (2.2.1)$$

with $\nabla.\bar{B} = 0$ and the condition of continuity of electric current density $\nabla.\bar{J} = 0$ in the steady state problem (Appendix D). On the other hand, from the definition of effective electric field in Chapter 1, Sec. 1.3, rotation of effective electric field ensues

$$\nabla \times \bar{E}' = (\bar{B}.\nabla) \ \bar{u} - B(\nabla.\bar{u}) + \nabla\left(\frac{1}{en_e}\right) \times (\nabla P_e) + (\nabla\alpha) \times (\nabla T_e)$$

$$- (\nabla\xi) \times(\nabla T_e \times \bar{B}) + \xi \ \bar{B} \ (\nabla^2 T_e) - \xi \ (\bar{B}.\nabla)\nabla T_e +$$

$$(\nabla\zeta)\times [(\nabla T_e \times \bar{B}) \times \bar{B}] + \zeta \ (\bar{B}.\nabla) \ (\nabla T_e \times \bar{B}) \qquad (2.2.2)$$

(see Appendix E,F)

From $\nabla. \ \bar{J} = 0$, the current stream function ψ is introduced such that

$$\bar{J} = \left(- \frac{\partial\psi}{\partial y} \ , \ \frac{\partial\psi}{\partial x} \ , \ 0\right).$$

Here, the current lines will then be parallel to the tangential directions on the surface defined by the current stream function ψ.

For the simplified case of small magnetic Reynolds number, when the magnetic field in the fluid hardly differs from the external field, $\bar{B}$ is defined by (0,0,B) and B is constant through the channel (Ref.48). Furthermore, the velocity profile is assumed to be $\bar{u} = (u_x,u_y,0)$. Then it is advantageous to go over to two dimensional problems where

$$\psi = \psi \ (x, y)$$

$$\sigma = \sigma \ (x, y)$$

$$\chi = \chi \ (x, y)$$

(or $\beta = \beta \quad (x, y)$)

$$u_x = u_x \; (x, y)$$

and $u_y = u_y \; (x, y)$

Equating (2.2.1) to (2.2.2) and applying the above mentioned quantities to them,we arrive at

$$\nabla^2 \psi + M(x,y) \frac{\partial\psi}{\partial x} + N\,(x,y)\frac{\partial\psi}{\partial y} = P(x,y) \qquad (2.2.3)$$

where the functions of M(x,y), N(x,y) and P(x,y) are defined as

$$M(x,y) = \sigma \left[\frac{\partial}{\partial x}\left(\frac{1}{\sigma}\right) - \frac{\partial}{\partial y}\left(\frac{\beta}{\sigma}\right) \right] \qquad (2.2.4a)$$

$$N(x,y) = \sigma \left[\frac{\partial}{\partial y}\left(\frac{1}{\sigma}\right) + \frac{\partial}{\partial x}\left(\frac{\beta}{\sigma}\right) \right] \qquad (2.2.4b)$$

$$P(x,y) = \sigma \Bigg[- B\left(\frac{\partial u_x}{\partial x}\right) - B\left(\frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{1}{en_e}\right)\left(\frac{\partial P_e}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{en_e}\right)$$

$$\left(\frac{\partial P_e}{\partial x}\right) + \left(\frac{\partial \alpha}{\partial x}\right)\left(\frac{\partial T_e}{\partial x}\right) - \left(\frac{\partial \alpha}{\partial y}\right)\left(\frac{\partial T_e}{\partial y}\right) - B\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial T_e}{\partial x}\right) - B\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial T_e}{\partial y}\right)$$

$$+ B\xi\left(\frac{\partial^2 T_e}{\partial x^2}\right) + B\xi\left(\frac{\partial^2 T_e}{\partial y^2}\right) + B^2\left(\frac{\partial \zeta}{\partial y}\right)\left(\frac{\partial T_e}{\partial x}\right) + B^2\left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial T_e}{\partial y}\right) \Bigg]$$

(see Appendix G) (2.2.4c)

In these expressions β is the Hall parameter and defined as

$$\beta = \sigma \quad \chi \quad B.$$

The equations (2.2.3) is a non-homogeneous elliptic partial differential equation of a more general type with linear boundary conditions. Making some assumptions for conductivity and Hall parameter or velocity profiles leads (2.2.3) to the Poisson Equation or simply to the Laplace equation. The convergence of the scheme, which deals with the nonuniform plasma together with Saha's equation, energy balance equation and current governing equation (2.2.3), depends on whether the solution of (2.2.3) is stable or not with respect to electron temperature. D. A. Oliver et al (Ref.15) concluded that when the inequality $\beta^2 < 24/25$ is satisfied, the solution of the potential equation, conjugate equation to the current governing equation, would be stable even when nonequilibrium heating of the electrons occurs in a magnetic field. Others (Refs. 13, 17, 23, 49) argued the stability problem related to (2.2.3). In the present paper, however, we will not go into the problem of the stability limits in detail.

Sec. 2.3 Boundary Conditions of Faraday Type Generator

The Faraday type generator has the electrode segments insulated, long periodic structure (consequently, end effects will be neglected) and opposite segments connected by individual loads as shown by Fig.8. The current direction in the central part of the channel will then tend to be perpendicular to the magnetic field direction and also to the flow of the plasma. The effect of segmentation can also be explained by the Hall field that prevents the electron drift motion and also by a conducting electrode segment parallel to this field that will clearly have a short-circuiting action on the field. Thus a strong deviation from the uniform current distribution in the central portion of the channel can be expected near the electrode. Because of this, near the electrode reasonably strong effects due to electron temperature gradients and due to electron pressure gradients should occur. Furthermore, if boundary layer effects are introduced in order to describe the real channel flow in general, it is essential to consider those gradient terms together with a wall temperature different than the gas temperature. However, for the most simple case, neglecting these effects, the effective field $\bar{E}'$ involves only the real applied field $\bar{E}$, the additional field introduced from changing the stationary coordinate system into the system moving with plasma and the gradient terms of velocity. Thus equation (2.2.3) becomes

$$\nabla^2 \psi + M(x,y) \frac{\partial\psi}{\partial x} + N(x,y) \frac{\partial\psi}{\partial y} = P'(x,y) \qquad (2.3.1)$$

where

$$P'(x,y) = - B\sigma \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right].$$

Without ion-slip considerations, eq. (1.3.7) is

$$\bar{E}' = \frac{1}{\sigma} \bar{J} + \chi (\bar{J} \times \bar{B})$$

where the effective electric field $\bar{E}' = \bar{E}^*$ by neglecting velocity gradients.

The projections onto the x- and y- axis are respectively,

$$E'_x = \frac{1}{\sigma} J_x + \chi B J_y$$

$$E'_y = \frac{1}{\sigma} J_y - \chi B J_x$$

Defining β such that $\beta = \sigma \chi B$ as before, we have

$$\sigma E'_x = J_x + \beta J_y$$

$$\sigma E'_y = J_y - \beta J_x$$

, or

$$\sigma\ (E_x + u_y B) = J_x + \beta\ J_y \qquad (2.3.2a)$$

$$\sigma\ (E_y - u_x B) = J_y - \beta\ J_x \qquad (2.3.2b)$$

Solving these equations for Jx and Jy, we obtain

$$J_x = \frac{\sigma}{1 + \beta^2} (u_y\ B + E_x - \beta\ E_y + \beta\ u_x\ B) \qquad (2.3.3a)$$

$$J_y = \frac{\sigma}{1 + \beta^2} (\beta u_y B + \beta E_x + E_y - u_x B) \qquad (2.3.3b)$$

Since on the wall of the electrode uy = Ex = 0, from (2.3.2a), one of the boundary conditions is

$$J_x + \beta\ J_y = 0$$

In terms of stream function, it becomes

$$\frac{\partial\psi}{\partial y} - \beta\ \frac{\partial\psi}{\partial x} = 0 \qquad \text{on the electrode } (2.3.4a).$$

Another wall condition is on the insulator. Assuming a perfect insulator, we have $J_x = J_y = 0$ for the condition on the insulator. If we rewrite it in terms of stream function,

$$\psi = \text{Const.} \qquad \text{on the insulator} \quad (2.3.4b)$$

For such a configuration as shown in Fig.8, the periodicity of the conductors and insulators in a period of (2L + 2l) requires

$$\psi\ (x + 2L + 2l,\ y) = \psi(x,y) + I_y \qquad (2.3.5)$$

where Iy is the transverse current flowing through the electrode. It can be shown that the specification of Iy with Ix = 0, in the case of the Faraday type generator, determines a unique distribution of current within a gas (Ref.15). Moreover, the one-dimensional transverse current Iy, which gives the output current by multiplying it by the electrode width in the B direction will be given by the channel loading factor K defined as K = Ey/ux B. Thus, it is a known value, or a quantity which can be determined from outside of the channel. In fact, the internal current relation is from (2.3.3a) and (2.3.3b), in y direction

$$\beta\ J_x - J_y = \sigma\ (1 - K)\ u_x\ B \qquad (2.3.6)$$

The output current I or the total load current over one period of the channel will be constant and thus can be specified as

$$I = w \int_{L+1}^{2L+1} (\beta J_x - J_y)\, d x$$

Since there is no connection between the upstream end of the electrode and the downstream end, the integration of Jx over the electrode length should be zero when β is assumed to be constant or when we can neglect the correlations of the integration. Thus I becomes

$$I = - w \int_{L+1}^{2L+1} J_y \, dx$$

$$= - w \int_{L+1}^{2L+1} \frac{\partial \psi}{\partial x} \, d x$$

$$= - w \, I_y \qquad (2.3.7)$$

On the other hand, the R.H.S. of (2.3.6) is

$$w \int_{L+1}^{2L+1} \sigma \, (1-K) \, u_x B \, dx$$

$$= < \sigma > < u_x > (1-K) \, BLW \qquad (2.3.8)$$

when we neglect the correlations.

Thus,

$$I_y = <\sigma> < u_x > (K-1) \, BL \qquad (2.3.9)$$

where $< >$ represents the average value of the variations over the electrode.

The relation between loading current I and either the loading resistance R_o or internal resistance R_i of the plasma cannot be determined explicitly unless equipotential surface or current filament is solved over the channel. As a matter of fact, whether one chooses to solve the problem as described by equipotential surface or as described by current stream surface does not matter as their descriptions are identical to each other. However, the loading resistance, Ro, pertaining to the loading factor, K, is expressed most conveniently in terms of the potential, ϕ, as

$$R_o = \Delta \phi \, / \, I$$

$$\Delta \phi = -\int_{x'} \bar{E} \; d\bar{x}' = \int_{x'} (\nabla\phi) \; d\,\bar{x}'$$

In this expression the primed axis is along the eigenrays of the resistivity tensor (Ref.50), such that $\Delta \phi$ is the integration of $\bar{E}$ between two electrodes connected through an external resistance along the line normal to the equipotentials (Ref.54,55). Moreover, the internal resistance pertaining to joule losses is expressed in terms of potential difference between these electrodes measured in a coordinate system moving with the plasma as

$$R_i = \Delta \phi' \,/\, I$$

$$\Delta \phi' = \int_{\bar{h}} \bar{E}' \; d\,\bar{h} = \int_{\bar{h}} \frac{\bar{J}}{\sigma} \quad d\,\bar{h}$$

and the vector $\bar{h}$ is a distance vector lying along the current density vector, such that $\Delta \phi'$ is the integration of $\bar{J}/\sigma$ between these electrodes independent of the path (Refs. 11,12).

Although quite extensive work (Refs. 11,12,23,54) has been done for different geometries and for various Hall parameters, neither R_o nor R_i for a nonuniform plasma simplified to a one-dimensional problem is easy to determine uniquely, and thus further comparisons with experimental studies seem to be required.

Chapter 3. Numerical Solutions

Sec. 3.1 Current Stream Function

The current governing equation, namely, the equation for the current stream function ψ is as we developed, from eq. (2.3.1),

$$\nabla^2 \psi + M\,(x,y)\, \frac{\partial\psi}{\partial x} + N\,(x,y)\, \frac{\partial\psi}{\partial y} = P'\,(x,y)$$

where,

$$P'\,(x,y) = -\,B\,\sigma \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right]$$

For the first approximation, if we neglect velocity gradients,

$$\nabla^2 \psi + M(x,y)\, \frac{\partial\psi}{\partial x} + N\,(x,y)\, \frac{\partial\psi}{\partial y} = 0 \qquad (3.1.1)$$

Using the central difference method to discretize the differential expressions, we have

$$\frac{2}{h_i + h_{i-1}} \left(\frac{\psi_{i+1,j} - \psi_{i,j}}{h_i} - \frac{\psi_{i,j} - \psi_{i-1,j}}{h_{i-1}} \right) + \frac{2}{h_j + h_{j-1}} \left(\frac{\psi_{i,j+1} - \psi_{i,j}}{h_j} - \frac{\psi_{i,j} - \psi_{i,j-1}}{h_{j-1}} \right) + \frac{1}{2} \left(M_{i+\frac{1}{2},j} \frac{\psi_{i+1,j} - \psi_{i,j}}{h_i} + M_{i-\frac{1}{2},j} \frac{\psi_{i,j} - \psi_{i-1,j}}{h_{i-1}} \right) + \frac{1}{2} \left(N_{i,j+\frac{1}{2}} \frac{\psi_{i,j+1} - \psi_{i,j}}{h_j} + N_{i,j-\frac{1}{2}} \frac{\psi_{i,j} - \psi_{i,j-1}}{h_{j-1}} \right) = 0 \tag{3.1.2}$$

where

$$h_i = x_{i+1} - x_i$$

$$h_j = y_{j+1} - y_j$$

$$\psi_{i,j} = \psi(x_i, y_j)$$

and

$$M_{i+\frac{1}{2},j} = M(\tfrac{1}{2}(x_i + x_{i+1}), y_j)$$

$$N_{i,j+\frac{1}{2}} = N(x_i, \tfrac{1}{2}(y_j + y_{j+1}))$$

$$M_{i-\frac{1}{2},j} = M(\tfrac{1}{2}(x_{i-1} + x_i), y_j)$$

$$N_{i,j-\frac{1}{2}} = N(x_i, \tfrac{1}{2}(y_{j-1} + y_j))$$

(see Appendix H)

Rearranging (3.1.2) to obtain

$$\frac{1}{h_i}\left[\frac{2}{h_i + h_{i-1}} + \frac{M_{i+\frac{1}{2},j}}{2}\right]\psi_{i+1,j} + \frac{1}{h_j}\left[\frac{2}{h_j + h_{j-1}} + \frac{N_{i,j+\frac{1}{2}}}{2}\right]\psi_{i,j+1} - \left[\frac{2}{h_i\, h_{i-1}} + \frac{2}{h_j\, h_{j-1}} + \frac{M_{i+\frac{1}{2},j}}{2h_i} - \frac{M_{i-\frac{1}{2},j}}{2h_{i-1}} + \frac{N_{i,j+\frac{1}{2}}}{2h_j} - \frac{N_{i,j-\frac{1}{2}}}{2h_{j-1}}\right]\psi_{i,j} + \frac{1}{h_{i-1}}\left[\frac{2}{h_i + h_{i-1}} - \frac{M_{i-\frac{1}{2},j}}{2}\right] \cdot \psi_{i-1,j} + \frac{1}{h_{j-1}}\left[\frac{2}{h_j + h_{j-1}} - \frac{N_{i,j-\frac{1}{2}}}{2}\right]\psi_{i,j-1} = 0 \tag{3.1.3}$$

Equation (3.1.3) is a standard five points central-difference approximation for different rectangular mesh sizes. Since the calculation involves too many independent parameters to give general solutions, we limit ourselves to the case where the plasma is uniform and consequently uniform conductivity and Hall parameter are assumed. In this case, since $M(x,y) = N(x,y) \equiv 0$,

$$M_{i+\frac{1}{2},j} = N_{i,j+\frac{1}{2}} = M_{i-\frac{1}{2},j} = N_{i,j-\frac{1}{2}} \equiv 0$$

Then (3.1.3) reduces to Laplace's equation, and thus (3.1.3) becomes

$$\frac{1}{h_i + h_{i-1}} \frac{\psi_{i+1,j}}{h_i} + \frac{1}{h_j + h_{j-1}} \frac{\psi_{i,j+1}}{h_j} - \left(\frac{1}{h_i h_{i-1}} + \frac{1}{h_j h_{j-1}}\right) \psi_{i,j}$$

$$+ \frac{1}{h_i + h_{i-1}} \frac{\psi_{i-1,j}}{h_{i-1}} + \frac{1}{h_j + h_{j-1}} \frac{\psi_{i,j-1}}{h_{j-1}} = 0 \qquad (3.1.4)$$

For the same step sizes in which $h_x = h_i = h_{i-1}$, $h_y = h_j = h_{j-1}$,

$$\frac{1}{2h_x^2} \psi_{i+1,j} + \frac{1}{2h_y^2} \psi_{i,j+1} - \left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) \psi_{i,j}$$

$$+ \frac{1}{2h_x^2} \psi_{i-1,j} + \frac{1}{2h_y^2} \psi_{i,j-1} = 0$$

and in the simpler approach of the square net where $h_x = h_y = h$ it becomes,

$$\psi_{i+1,j} + \psi_{i,j+1} - 4\,\psi_{i,j} + \psi_{i-1,j} + \psi_{i,j-1} = 0 \qquad (3.1.5)$$

Sec. 3.2 Boundary Conditions

From (2.3.4a) and (2.3.4b) the boundary conditions on the surface of the electrodes and of the insulators are

$$\frac{\partial \psi}{\partial y} = \beta \frac{\partial \psi}{\partial x}$$

$$\psi = \text{CONST.}$$

Owing to linearity and to periodicity, we put $\psi = 0$ on the left insulator and $\psi = I$ on the right one.

As far as the surfaces of the electrodes are concerned, the boundary conditions should be rewritten in the forward or backward difference approximation, such that

$$\psi_{i,j-1} = \frac{h_{j-1}}{h_j} (\psi_{i,j+1} - \psi_{i,j}) + \psi_{i,j} - \left[\left(\frac{h_{j-1}}{h_i} \right) \beta_{i+\frac{1}{2},j} \right.$$

$$\left. (\psi_{i+1,j} - \psi_{i,j}) + \left(\frac{h_{j-1}}{h_{i-1}} \right) \beta_{i-\frac{1}{2},j} (\psi_{i,j} - \psi_{i-1,j}) \right]$$

on the lower electrode (3.2.1a)

and

$$\psi_{i,j+1} = \frac{-h_j}{h_{j-1}} (\psi_{i,j} - \psi_{i,j-1}) + \psi_{i,j} + \left[\left(\frac{h_j}{h_i} \right) \beta_{i+\frac{1}{2},j} \right.$$

$$\left. (\psi_{i+1,j} - \psi_{i,j}) + \left(\frac{h_j}{h_{i-1}} \right) \beta_{i-\frac{1}{2},j} (\psi_{i,j} - \psi_{i-1,j}) \right]$$

on the upper electrode (3.2.1b)

The conditions of (3.2.1a) and (3.2.1b) correspond to (3.1.3). As we assumed when (3.1.4) was obtained, in the case of a uniform plasma, (3.2.1) is written as follows,

$$\psi_{i,j-1} = \frac{h_{j-1}}{h_j} (\psi_{i,j+1} - \psi_{i,j}) + \psi_{i,j} - \beta \left[\left(\frac{h_{j-1}}{h_i} \right) (\psi_{i+1,j} \right.$$

$$\left. - \psi_{i,j}) + \left(\frac{h_{j-1}}{h_{i-1}} \right) (\psi_{i,j} - \psi_{i-1,j}) \right]$$

on the lower electrode (3.2.2a)

$$\psi_{i,j+1} = \frac{-h_{j-1}}{h_j} (\psi_{i,j} - \psi_{i,j-1}) + \psi_{i,j} + \beta \left[\left(\frac{h_j}{h_i} \right) (\psi_{i+1,j} \right.$$

$$\left. - \psi_{i,j}) + \left(\frac{h_j}{h_{i-1}} \right) (\psi_{i,j} - \psi_{i-1,j}) \right]$$

on the upper electrode (3.2.2b)

and related to (3.1.5)

$$\psi_{i,j-1} = \psi_{i,j+1} - \beta (\psi_{i+1,j} - \psi_{i-1,j})$$

on the lower electrode (3.2.3a)

$$\psi_{i,j+1} = \psi_{i,j-1} + \beta\ (\psi_{i+1,j} - \psi_{i-1,j})$$

on the upper electrode (3.2.3b)

Sec. 3.3 OHMIC Heating

As in Chapter 2, Sec.2.2, the current density vector $\bar{J}(J_x, J_y, 0)$ was written in terms of current stream function ψ as

$$J_x = -\frac{\partial\psi}{\partial y}$$

$$J_y = \frac{\partial\psi}{\partial x}$$

In finite difference language, in the central difference method, these will be expressed by

$$J_x\Big|_{i,j} = -\frac{\partial\psi}{\partial y}\Big|_{i,j} = -\frac{1}{2h_j}\ (\psi_{i,j+1} - \psi_{i,j-1}) \qquad (3.3.1a)$$

$$J_y\Big|_{i,j} = \frac{\partial\psi}{\partial x}\Big|_{i,j} = \frac{1}{2h_i}\ (-\psi_{i-1,j} + \psi_{i+1,j}) \qquad (3.3.1b)$$

Thus, the ohmic heating defined in (1.4.2) as $\bar{J}^2/\sigma$ at the mesh point (i,j) is then, proportional to $q_{i,j}$ given by

$$q_{i,j} = [\ J_x\Big|_{i,j}]^2 + [J_y\Big|_{i,j}]^2 \qquad (3.3.2)$$

Corresponding to (3.1.5) and (3.2.3), $q_{i,j}$ becomes,

$$q_{i,j} = \frac{1}{4h^2}\ [(\psi_{i+1,j} - \psi_{i-1,j})^2 + (\psi_{i,j+1} - \psi_{i,j-1})^2] \qquad (3.3.3)$$

where h is the unit interval between adjacent datum points defined at Sec.3.1.

However, along the electrode surface this expression must be re-written from the boundary conditions on it as

$$q_{i,j} = \frac{(1+\beta^2)}{4h^2}\ (\psi_{i+1,j} - \psi_{i-1,j})^2 \qquad (3.3.4)$$

since

$$\bar{J}^2 = J_x^{\ 2} + J_y^{\ 2} = (1 + \beta^2)\ J_y^{\ 2} \quad \text{from (2.3.4a)}$$

Special care must be taken for calculations of $q_{i,j}$ along the insulator surface.

Since $\psi_x = 0$, using the five points approximation differentiated (Ref.51), we obtain

$$\psi_y \Big|_{i,j} = \frac{1}{24h} [A_o\psi_{i,j} + A_1\psi_{i,j+1} + A_2\psi_{i,j+2}$$
$$+ A_3 \psi_{i,j+3} + A_4 \psi_{i,j+4}]$$

thus

$$q_{i,j} = \frac{1}{24^2h^2} [A_o \psi_{i,j} + A_1 \psi_{i,j+1} + A_2 \psi_{i,j+2} + A_3\psi_{i,j+3}$$
$$+ A_4 \psi_{i,j+4}]^2 \qquad (3.3.5)$$

where the coefficients An (n = 0,------,4) are given in table of Bickley (Ref.51) as

	A_o	A_1	A_2	A_3	A_4
lower insulator	-50	96	-72	32	-6
upper insulator	6	-32	72	-96	50

At the points where the insulators and electrodes meet each other, it was empirically found to be difficult to solve either in case of Ohmic heating or of current density. In the present case, we ignored the boundary conditions on the electrode surface at those points and used

$$\psi_x \Big|_{i,j} = \frac{1}{2h} [\psi_{i+1,j} - \psi_{i-1,j}]$$

$$\psi_y \Big|_{i,j} = \frac{1}{24h} [A_o \psi_{i,j} + A_1\psi_{i,j+1} + A_2\psi_{i,j+2} + A_3\psi_{i,j+3} +$$
$$A_4 \psi_{i,j+4}] \qquad (3.3.6)$$

COMMENTS ON RESULTS

Using the IBM-1130, the solutions to Eq.(3.1.5) were obtained for various values of the Hall parameter not greater than one. The calculation was limited to one period of the channel, that is, a rectangular region (y = 0, y = 2H, x = 0, x = 2L + 21) (see Fig.8). This rectangular region is discretized into about 40 x 20 mesh points respectively in the x and y direction. As the approach developed, the standard five-point approximation (Ref.22,48) and successive over-relaxation (or extrapolated Liebmann Method) for Laplace's equation were used (Refs. 22, 48, 52, 53). The difficulty in finding the converging factor for the over-relaxation method was tentatively overcome by assuming this problem as a bounded boundary problem in which the converging factor is given and the accuracy of solution is expressed (Refs. 52,53). Thus, the converging factor was found to be between 1.6 and 2, depending on the number of mesh points of the problem. In this problem, the iterations were stopped after attaining a maximum relative difference of successive calculated values of the stream function at the same mesh point within the accuracy given by Refs. 52,53.

It is of note that we excluded the assumption of a diagonally symmetric current distribution hoping for our procedure to be extended later to the case of nonuniform plasma. In most cases of the computation the relative difference of stream function during successive iterations at the points, where insulators and electrodes met, showed an oscillating characteristic of this numerical procedure. The oscillation had a converging amplitude when the Hall parameter was not greater than one, and a diverging amplitude otherwise. This convinced us that the numerical instabilities for the Hall parameter greater than one when using the present successive over-relaxation procedure arise from the discontinuous boundary conditions. Many previous workers in this problem have pointed this out and thus called these points at the junctions of insulators and electrodes, particularly the upper point of the upstream junctions and the lower point of the downstream junctions as singular points.

In numerical calculation it was observed that there was also a kind of asymptotic instability when the calculation went further toward higher precisions; then, a slightly larger number of iterations was required for smaller values of the Hall parameter. The work of C. Lo Surdo (Ref.54) and Vataghin et al (Ref.28) also refers to this point.

The results (Fig.9-12) for current density distribution show that the current tends to flow diagonally between the pair electrodes. Consequently, the current is concentrated at one end on the upstream side of one electrode and at the other end on the downstream side of the other electrode. This tendency increases when the Hall parameter increases. The rapid variation of current density of distribution even at the electrode (which is short circuiting the electric field) is due to this concentration effect, (Fig.15). At a Hall parameter equal to 0.2, 35% of current leaves or arrives at 25% of the electrode surface. When the Hall parameter increases, this concentration at the tail of the lower electrode (and also the head of the upper electrode) also increases. (See Fig.15 in cases of the Hall parameter equal to 0.8 and 1.0).

In fact, the results from our calculations (from Fig.10 to Fig.12) are nearly identical to those of the prementioned authors and the references cited therein (Refs. 11, 12, 54) and reveal a diagonally symmetric distribution within the error of calculation. But still difficult problems remain in applying suitable boundary conditions to the 'singular points', which would, perhaps, keep the numerical procedures stable for Hall parameter greater than one. They would allow utilization of this method in cases of nonuniform plasma with a boundary layer, with a flowing current on the wall or with a leakage current flowing through the insulator surface.

To compute the Ohmic heating, we calculated $J^2 = J_x^2 + J_y^2$ at each mesh point which is proportional to the joulean dissipation energy. A reference value of J^2 was chosen, namely, the value at the center of the defined geometry, located midway between the pair of electrodes. Then all other calculated values were divided by the reference value such that each value indicated at each contour of Fig. (13,14) denotes the ratio of j^2 of that curve to j^2 at the center. Consequently, around singular points where the current is concentrated, a higher proportionality of joulean heating can be shown, still with a diagonally symmetric distribution.

With these results, we can calculate the electron temperature distribution in the channel with the aid of the energy balance equation so as to give the spatial distributions of the other plasma quantities for the next approximation. However, we assume uniform plasma flow in the first approximation.

CONCLUSIONS

We have assumed that plasma is in a steady state at any point of space and

1) me/ms $\doteq$ 1/2000 is negligible, where subscript s denotes heavy particles.

2) ρ^e, excess charge, contributions to momentum equation is negligible.

3) ne $\doteq$ n_1, electrical neutrality is assumed to be valid at any point in space. Thus the electrical force between charged particles is stronger than the diffusion force even in the presence of the magnetic field.

4) $\bar{v}_n = \bar{v}_I = \bar{v}$, that is, $\bar{V}_n$, $\bar{V}_I = 0$, so that $\bar{J} = \bar{j} = -\bar{j}_e$.

5) $|\bar{\bar{\tau}}_e| \,/\, P_e$ is assumed to be negligible.

6) The electron-electron momentum transfer cross section is not taken into account.

7) The ion-atom momentum transfer cross sections are not taken into account.

8) Ion-electron cross section is based on the first approximation consideration of coulomb force law. Thus higher order collisional cross sections are not considered.

9) Higher order momentum transfer collision integrals are assumed to be constants based on hard sphere models of heavy particles.

10) Energy loss factor is also assumed to be constant.

11) Radiation energy loss or inelastic collision effects are not taken into consideration.

12) For the energy balance equation, we completely neglect all the terms except joulean dissipation energy in the left hand side of this equation.

All these assumptions are thus acceptable when the plasma is a subsonic flow of the gas with temperature between 1500°K and 4000°K, with total pressure near 1 atm, and with seeding ratio between 0.001 and 0.1 Nevertheless, several studies have been carried out in the references cited in an effort to exclude a couple of the above assumptions. There are more accurate calculations of various kinds of collision cross sections, an emphasis on the ion-concentration, particularly in the presence of a strong magnetic or electric field, or on ambipolar diffusion consideration. Elsewhere, they have analyzed energy loss procedures in detail to reveal the effective energy loss factor (Refs. 33, 46, 27). So complext do the determination of plasma properties become in order to have a complete answer that the results originating from simplifications of the distribution function and Maxwell equation are different from each other in their forms. However, the values calculated from them are almost identical to each other under such conditions as we have used.

To make room for these plasma properties in the channel problem, further simplifications of expressions or approximate approaches to both of them have been inevitable. It is because either of mathematical difficulties or of physical phenomena. As a matter of fact, too many independent parameters have been introduced. An example of the former problem is the inhomogeneity of the current governing equation, particularly in the case of the 3-dimensional channel as it is in reality.

The coupling of terms of Maxwell's equation and the kinetic equations would be simple examples of this. A sharp increase of number density for electron temperature which might bring on the instability problem represents the latter example of physical phenomena quite well. To find proper boundary conditions when boundary layers are developed makes the problem more sophisticated. Thus, rearranging the whole problem in terms of fundamental plasma parameters (even for current governing equation) has been studied for the purpose of resolving these difficulties. Furthermore, relaxation effects require more proper experimental and theoretical research in real plasma flows.

But the problem confronted is not overwhelming. It is more or less soluble. Mapping method and modelling the conductivity have been used. (Ref. 27). A series expansion method has also been studied for partial analysis. But we can notice in most cases that they prevent us from easily introducing plasma nonuniformities to a realistic flow because of their restrictions. This is the case for the mapping method, in spite of neat analytical solutions of current distribution in uniform plasma. This is also true in the case of modelling the conductivity, which is restricted to a priori formulation. While having difficulties near singular points, a series expansion method seems to overcome these problems to some extent. Choosing fundamental expansion parameters (s) has been argued from the physical point of view in this case. Replacing the current or potential distribution by a circuit problem has been suggested (Ref.56). On the other hand, some advantages of iteration procedures by using fast digital computer have been supported by a series of authors (10,11,12,48,54). The choice of plasma parameter to make relations between the parameters is thus an important point to be argued. Discussion could arise as to whether or not this iteration procedure is stable so as to reveal a satisfactory result as its limit within the desirable precision. This point, however, is really beyond the scope of the present paper. Several subsidiary problems should be studied in more detail before arriving at a final overall solution. Nonetheless, in this paper a combination of Maxwell equations and hydrodynamic equations are identified as a current governing equation in order that one additional macroscopic variable, electron temperature profile may be solved. By connecting this with previous chapters, this would describe then the plasma properties of the channel with nonequilibrium ionization which is connected with non-uniformity of plasma.

As we mentioned earlier, it is, of course, an especially simple case that plasma flow is 'ideal' and end effects and heat flow between plasma and surroundings are neglected. In the case of an 'ideal' plasma the non-equilibrium effect manifests itself only in the electron temperature profile. The latter part of this paper thus has been prepared with the aim of formulating it. Subsidiary circuitry properties of the MHD channel can be obtained as we showed them in Chapter 2. Calculations on the uniform plasma without variational contributions due to non-equipartition electron heating reveal the initial first iteration as the electron temperature profiles. The current distribution of this case involves all the characteristics of uniform plasma which have been pointed out elsewhere. Despite uncertainties near singular points, these results are even good at forecasting what the nonuniform plasma will become in MHD channel. We have not even assumed what the first approximation of electron temperature distribution would be but calculated it directly from the equations. To proceed further toward more precise calculating of the nonuniform plasma, nevertheless, the parallel approach lies in determining the collisional properties of a gas such as a seeded mixture better and in overcoming the numerical instabilities confronted when we use extrapolated Liebmann Method for large Hall parameter.

In preparing this paper we have drawn freely on the published papers of many authors, and whenever possible, we have indicated the references used.

REFERENCES

1. Demetriades, S. T.; Argyropoulos, G. S. — Ohm's law in multicomponent nonisothermal plasma with temperature and pressure gradients. The Physics of Fluids, Vol.9, No.11, pp 2136 (1966)

2. DeVoto, R. S. — Transport properties of ionized monatomic gas. The Physics of Fluids, Vol.9, No.6, pp 1230 (1966)

3. Zhdanov, V. M. — Transport phenomena in a partly ionized gas. Journal of Applied Mathematics and Mechanics (Soviet, PMM), Vol. 26, No.2, pp 280 (1962)

4. Schweitzer, S.; Mitchner, M. — Electrical conductivity of partially ionized gases. 6th Symposium on Engineering Aspects of Magnetohydrodynamics, pp. 135 (1965) University of Pittsburgh, Pittsburgh, Pa.

5. Kruger, C. H.; Viegas, J. H. — Influence of the Ramsauer effect on nonequilibrium electron temperature. The Physics of Fluids, Vol.7, No.7, pp 1879 (1964)

6. DeVoto, R. S. — Transport coefficients of partially ionized argon. The Physics of Fluids, Vol.10, No.2, pp. 354 (1967)

7. Chmieleski, R. M.; Ferziger, J. H. — Transport properties of a nonequilibrium partially ionized gas. The Physics of Fluids, Vol.10, No,2, pp. 364 (1967)

8. Cool, Terrill A.; Zukoski, Edward E. — Recombination, Ionization, and Nonequilibrium Electrical Conductivity in Seeded Plasma. The Physics of Fluids, Vol.9, No.4, pp 780 (1966)

9. Pipkin, A. C. — Electrical conductivity of partially ionized gases. The Physics of Fluids, Vol. 4, No.1. pp. 154 (1961)

10. Argyropoulos, G. S.; Demetriades, S.T.; Kendig, A. P. — Influence of nonuniformaties, thermal diffusion, boundary layers and finite reaction rates in nonequilibrium JxB devices. AIAA Electrical Propulsion and Plasma-dynamics Conference, Colorado Springs, Sept.11-13 (1967) No.67-715.

11. Fischer, F. W. — Experimental Determination of the current distribution in a simulated MHD generator. International Symposium on MHD Electric Power Generation, Saltzburg, Austria (1966) SM-74/20.

12. Lengyel, L. L.
Salvat, M.
Internal Resistance and Potential Fall in MHD Generators. International Symposium on MHD Electric Power Generation, Saltzburg, Austria (1966) SM-74/16.

13. Kerrebrock, Jack L.
Nonequilibrium ionization due to electron heating; Theory and Experiments. AIAA J. Vol. No.6, pp. 1072 (1964).

14. Heighway, J. E.
Brayton cycle MHD power generator with non-equilibrium conductivity. Lewis Research Center 2659.

15. Oliver, David A.
Mitchner, M.
Nonuniform electrical conduction in MHD channels. AIAA J. Vol.5, No.8, pp. 1424, (1967)

16. Shair, F. H.
Theoretical performance comparison of working fluids in a nonequilibrium MHD generator. AIAA J., Vol.2, No.11, pp.1883 (1964)

17. Kerrebrock, Jack L.
Magnetohydrodynamic generators with nonequilibrium ionization. AIAA J., Vol.3, No.4, pp.591 (1965)

18. Sherman, Arthur
MHD channel flows with nonequilibrium ionization. The Physics of Fluids, Vol.9, No.9. pp.1782 (1966).

19. Sherman, Arthur
Nonequilibrium ionization and MHD phenomena. Technical Information Series, General Electric R64SD36, May (1964).

20. Lyman, Frederick A.
Goldstein, Arthur W.
Heighway, John E.
Effect of seeding and ion slip on electron heating in a MHD generator. Lewis Research Center, NASA TN D-2118.

21. Robben, F.
Nonequilibrium ionization in a magnetohydrodynamic generator. The Physics of Fluids, Vol.5 No.10, pp.1308 (RN) (1962).

22. Coney, M. W. E.
The electrodynamic effects of nonuniformity of magnetic field and fluid parameters in MHD generators. Plasma Physics, Vol.10, pp.499, (1968).

23. Kerrebrock, Jack L.
Segmented electrode losses in MHD generators with nonequilibrium ionization. AIAA J. Vol.4, No.11, pp. 1938 (1966).

24. Chen, Michael M.
Some elementary considerations concerning the uniformity of electrical conduction through gases. 7th Symposium on Engineering Aspects of MHD, pp.17 (1966). Princeton University, Princeton, N. J.

25. Oliver, D. A. — Nonuniform electrical conduction in MHD channels. 8th Symposium on Engineering Aspects of MHD, pp.104 (1967). Stanford University, Stanford, California.

26. Tanenbaum, B. S. — Plasma Physics. pp.119 (1967). McGraw-Hill Book Company.

27. Sutton, G. W.
Sherman, A. — Engineering Magnetohydrodynamics (1965). McGraw Hill Book Company.

28. Stachnov, I. P.
Stepanov, A. S. — Transport Equations for a Three-Component Plasma in a Magnetic Field. Soviet Physics-Technical Physics, Vol.9, No.3, pp.315 (1964).

29. Sherman, A.
Reshotko, E. — The nonequilibrium boundary layer along a channel wall. AIAA 6th Aerospace Sciences Meeting, New York, N.Y. Jan 22-24 (1968). Paper No.68-134.

30. Nighan, W. L. — Electron-transport phenomena in thermionic converter plasma. Journal of Applied Physics, Vol.39, No.1, pp.223 (1968).

31. Ginzberg, V. L.
Gurevich, A. V. — Nonlinear Phenomena in a plasma located in an alternating electromagnetic field. Soviet Physics-Uspekhi, Vol.70, pp.201, Feb (1960.)

32. Kruger, C. H.
Mitchner, M. — Kinetic theory of two temperature plasma. Institute for Plasma Research, Stanford University, SU-IPR Report No.94 (1966).

33. Demetriades, S. T. — Determination of energy loss factors for slow electrons in hot gases. The Physical Review Vol.158, No.2, pp.215 (1967)

34. Glass, I.I. — Theory and performance of simple shock tubes. UTIAS Review No.12, part 1, May (1958).

35. Dubner, V. M. — Thermal ionization in a magnetic field. High Temperature (Soviet Teplofizika Uysokikh Temperatur) Vol.4, No.2, pp.281 (1966)

36. Lyman, F. A.
Dugan, J. V. Jr., — Electron distribution function and number density in a nonequilibrium seeded gas. 7th Symposium on Engineering Aspects on MHD, pp.94 (1966). Princeton University, Princeton, N. J.

37. Potapov, A. V. — Chemical equilibrium of multitemperature systems. High Temperature (Soviet Teplofizika Uysokikh Temperatur) Vol.4, No.1, pp.48 (1966).

38. Chapman, S.
Cowling, T. G. — The Mathematical Theory of Nonuniform Gases (1964) Cambridge at the University Press.

39. O'Malley, T. F. — Extrapolation of electron-rare gas atom cross sections to zero energy. The Physical Review, Vol.130, No.3, pp.1020 (1963).

40. Lin, Shao-Chi. Resler, E. L. Kintrowitz, A. — Electrical conductivity of highly ionized argon produced by shock waves. Journal of Applied Physics. Vol.26, No.1, pp.95 (1955)

41. Iman-Rahajoe, S. Curtiss, C. F. Bernstein, R. B. — Numerical evaluations of quantum effects on transport cross sections. Journal of Chemical Physics, Vol.42, No.2, pp.530 (1965)

42. Smith, F. J. Mason, E. A. Monclick, L. Munn, R. J. — Transport collision Integrals for quantum gases obeying a 12-6 potential. Journal of Chemical Physics. Vol.42, No.2, pp.537 (1965)

43. Smith, F. J. Mason, E.A. Munn, R. J. — Classical collision integrals for the repulsive screened Coulomb potential. The Physics of Fluids. Vol.8, No.10, pp.1907 (RN) (1965)

44. Gurevich, A. V. — The influence of collisions between electrons on their velocity distribution in gases and semi-conductors in an electric field. Soviet Physics JETP, Vol.10, pp.215 (1960).

45. Viegas, J. R. Kruger, C. H. — Electron distribution function in a nonequilibrium plasma. 7th Symposium on Engineering Aspects on MHD, pp.96 (1966). Princeton University, Princeton, N. J.

46. McGregor, W. K. Brewer, L. E. — Equivalence of electron and excitation temperatures in an argon plasma. The Physics of Fluids, Vol.9, pp.826 (1966).

47. Lutz, M. A. — Radiant energy loss from a seeded plasma. AIAA J., Vol.3, No.4, pp.772 (TN) (1965).

48. Vatazhin, A. B. — Approximate calculation of the current distribution when a conducting fluid flows along a channel within a magnetic field. Journal of Applied Mathematics and Mechanics (Soviet PMM), Vol.26, No.3, pp.548 (1962).

49. McCune, J. E. — Nonlinear effects of fluctuations on MHD performance. 6th Symposium on Engineering Aspects of MHD, pp.89, (1965). University of Pittsburgh, Pittsburgh, Pa.

50. Dicks, J. R. Jr., Wu, Ying-chu Lin Shanklin, R. V. — An experimental and theoretical comparison of the performance of diagonal wall generators. Faraday Generators and Hall Current Generators. SM-74/169.

51. Southwell, R.V. Relaxation Methods in Theoretical Physics. Oxford (1946) at the Clarendon Press.

52. Sheldon, J.W. Iterative methods for the solution of elliptic partial differential equations. In Mathematical Methods for Digital Computers. Vol.1, edited by Ralston, Anthony and Wilf, Herbert S. (1967). John Wiley and Sons Inc.

53. Smith, G. D. Numerical Solution of Partial Differential Equations . Oxford University Press (1965).

54. Surdo, Lo C. Current and potential distribution in a Faraday type segmented electrode generator. International Symposium on MHD Electrical Power Generation, Salzburg, Austria (1966). SM-74/22.

55. Lengyel, Lajos L. Computation of the current distribution in MHD generators with allowance for the recombination effects. 8th Symposium on Engineering Aspects of MHD, pp.105 (1967). Stanford University, Stanford, California.

56. Ustimenko, Yu L. Calculating the potential distribution along the channel of a MHD generator by the replacement circuit method. High Temperature (Soviet Tep. Uys. Temperatur) Vol.5, No.4, pp. 676, (1967).

Physical Constants and Conversion Table

Physical Constants

$k = 1.38042 \times 10^{-23}$ Joules/^{o}K

$= 8.6164 \times 10^{-5}$ eV/^{o}K

$h = 6.6252 \times 10^{-34}$ Joules.sec

$\hbar = h/2\pi = 1.05443 \times 10^{-34}$ Joules.sec

$= 6.5817 \times 10^{-16}$ eV.sec

R(Gas Constant) $= 8.3149 \times 10^{3}$ Joules/Kg.mole^{o}K

1 ATM $= 1.013246 \times 10^{5}$ Newtons/m^{2}

$e = 1.60199 \times 10^{-19}$ Coulomb

$m_e = 9.1085 \times 10^{-31}$ Kg

$e/m_e = 1.75936$ Coulomb/Kg

a_o(Bohr Radius) $= 5.29172 \times 10^{-11}$ Meter

Conversion Table

1 Coulomb $= 3 \times 10^{9/2}$ (Newton m^{2})$^{1/2}$

1 Sec $= 9 \times 10^{9}$ Ohm.meter

1 Weber = 1 Volt.sec

1 Weber/meter^{2} $= 1/3 \times 10^{-9/2}$ (Kg/m^{3})$^{1/2}$

1 eV $= 1.60207 \times 10^{-19}$ Joules

Mass Calculations

Potassium (Z 19, A 39.100, N20)

$$m_k = 19 \times 1.67239 \times 10^{-27} + 20 \times 1.67470 \times 10^{-27} + \frac{22.31}{5.61} \times 10^{-29} (\text{Kg})$$

Argon (Z 18, A 39.944, N 22)

$$m_a = 18 \times 1.67239 \times 10^{-27} + 22 \times 1.67470 \times 10^{-27} + \frac{23.23}{5.61} \times 10^{-29} (\text{Kg})$$

Ion of Potassium

$$m_I = 19 \times 1.67239 \times 10^{-27} + 20 \times 1.67470 \times 10^{-27} + \frac{22.31}{5.61} \times 10^{-29}$$
$$-9.1083 \times 10^{-31} \quad (\mathrm{Kg})$$

Argon Ionization Potential (^1S_o)

$$\epsilon_{iA} = 15.755 \quad \mathrm{eV}$$

Potassium Ionization Potential ($^2S_{1/2}$)

$$\epsilon_{iK} = 4.339 \quad \mathrm{eV}$$

Partition Functions

for Argon $\quad Z_A = 1$

$Z_{A^+} = 6$

for Potassium

$Z_k = 2$

$Z_{k^+} = 1$

for electron

$Z_e = 2$

Ratio of specific heat for argon

$$\gamma = 1.67$$

APPENDIX A:

Derivation of Species Mass Conservation Equation (See Section 1.1)

$$\frac{\partial}{\partial t}\ (n_s m_s) + \nabla_{\bar{r}}\ (n_s m_s < \bar{w}_s >) = m_s \int_{w} \left(\frac{\partial f}{\partial t} \right)_c d\ \bar{W}$$

Since $\rho_s = n_s m_s < \bar{w}_s > = \bar{v}_s$ and $R_s^{(o)}$ is defined as

$$m_s \int_{\bar{w}} \left(\frac{\partial f}{\partial t} \right)_c d\ \bar{W}$$

this becomes

$$\frac{\partial \rho_s}{\partial t} + \nabla_{\bar{r}} \cdot (\rho_s\ \bar{v}_s) = R_s^{(o)}$$

APPENDIX B:

Derivation of Species Momentum Equation (see Section 1.1) in i-direction.

$$\frac{\partial}{\partial t}(n_s m_s <w_{s_i}>) + \nabla_{\bar{r}} \cdot (n_s m_s < \bar{w}_s w_{s_i} >) - \frac{q_s}{m_s} n_s m_s [\bar{E} \cdot (\nabla_{\bar{w}} w_{s_i})$$

$$+ < (\bar{w}_s \times \bar{B}) \cdot \nabla_{\bar{w}} w_{s_i} >] = m_s \int_{\bar{w}} w_{s_i} \left(\frac{\partial f}{\partial t} \right)_e d\bar{w}$$

Thus for the all directions of velocity space,

$$\frac{\partial}{\partial t}(n_s m_s <w_s>) + \nabla_{\bar{r}} \cdot (n_s m_s <\bar{w}_s \bar{w}_s> - q_s n_s [\bar{E} + <w_s> \times \bar{B}] = \bar{R}_s^{(1)}$$

That is,

$$\frac{\partial}{\partial t}(\rho_s V_s) + \nabla_{\bar{r}} \cdot (\rho_s < \bar{w}_s \bar{w}_s >) - q_s n_s [\bar{E} + \bar{v}_s \times \bar{B}] = \bar{R}_s^{(1)}$$

From the definition of a peculiar velocity $\bar{C}_s = \bar{W}_s - \bar{v}_s$ and the diffusion velocity of species $< \bar{C}_s > = \bar{V}_s$,

$$\frac{\partial}{\partial t}(\rho_s \bar{V}_s) + \nabla_{\bar{r}} \cdot [\rho_s(\bar{v}\,\bar{V}_s + \bar{V}_s \bar{v})] + \nabla_{\bar{r}} \cdot (\rho_s \bar{v}\,\bar{v}) + \nabla_{\bar{r}} (\rho_s < \bar{c}_s \bar{c}_s >)$$

$$- q_s n_s [\bar{E}^* + \bar{V}_s \times \bar{B}] = \bar{R}_s^{(1)}$$

Or, by defining the partial pressure tensor $\bar{\bar{P}}_s$ as $\rho_s < \bar{C}_s \bar{C}_s >$,

$$\frac{\partial}{\partial t}(\rho_s \bar{v}_s) + \nabla_{\bar{r}} \cdot [\rho_s(\bar{v}\,\bar{V}_s + \bar{V}_s \bar{v})] + \nabla_{\bar{r}} \cdot (\rho_s \bar{v}\,\bar{v}) + \nabla_{\bar{r}} \bar{\bar{P}}_s$$

$$- q_s n_s [\bar{E}^* + \bar{V}_s \times \bar{B}] = \bar{R}_s^{(1)}$$

where $\bar{E}^* = \bar{E} + \bar{v} \times \bar{B}$ is then the induced electric field in the moving frame of the gas

APPENDIX C:

Derivation of Species Energy Equation (see Section 1.1)

$$< \phi_s > = < \epsilon i_s > + \tfrac{1}{2} m_s < w_s^2 >$$

$$= < \epsilon i_s > + \tfrac{1}{2} m_s <(\bar{v} + \bar{c}_s).(\bar{v} + \bar{c}_s)>$$

$$= < \epsilon i_s > + \tfrac{1}{2} m_s < \bar{v}^2 > + \tfrac{1}{2} m_s < c_s^2 > + \tfrac{1}{2} m_s \bar{v}.< \bar{c}_2 >$$

$$+ \tfrac{1}{2} m_s < \bar{c}_s > \bar{v}$$

$$= < \epsilon i_s > + \tfrac{1}{2} m_s v^2 + \tfrac{1}{2} m_s < c_s^2 > + m_s \bar{v}. \bar{V}_s$$

Considering the random translational energy 1/2 ($m_s < C_s^2 >$)equal to $3kT_s/2$, this becomes

$$< \phi_s > = < \epsilon i_s > + \tfrac{1}{2} m_s v^2 + m_s \bar{v}. \bar{V}_s + \tfrac{3}{2} kT_s$$

$$< \bar{w}_s \phi_s > = < \bar{w}_s \epsilon i_s > + \tfrac{1}{2} m_s < \bar{w}_s w_s^2 >$$

where,

$$< \bar{w}_s \epsilon i_s > = < (\bar{v} + \bar{C}_s) \epsilon i_s >$$

$$= < \bar{v} \epsilon i_s > + < \bar{c}_s \epsilon i_s >$$

$$= \bar{v} < \epsilon i_s > + \bar{V}_s < \epsilon i_s >$$

$$< \bar{w}_s w_s^2 > = < \{ (\bar{v} + \bar{c}_s). (\bar{v} + \bar{c}_s)\} \bar{w}_s >$$

$$= < (v^2 + \bar{v}. \bar{c}_s + \bar{c}_s. \bar{v} + c_s^2) \bar{w}_s >$$

$$= < (\bar{v} + \bar{c}_s) (v^2 + \bar{v}.\bar{c}_s + \bar{c}_s. \bar{v} + c_s^2) >$$

$$= < (\bar{v} v^2 + \bar{v}(\bar{v}. \bar{c}_s) + \bar{v}(\bar{c}_s. \bar{v}) + \bar{v}.(c_s^2) + \bar{c}_s v^2$$

$$+ \bar{c}_s(\bar{v}. \bar{c}_s) + \bar{c}_s (\bar{c}_s.\bar{v}) + \bar{c}_s c_s^2 >$$

$$= \bar{v} v^2 + \bar{v} < \bar{c}_s^2 > + 2 \bar{v}(\bar{v}.< \bar{c}_s >) + v^2 < \bar{c}_s > + < \bar{c}_s c_s^2 >$$

$$+ 2 < \bar{c}_s (\bar{c}_s. \bar{v}) >$$

$$= \bar{v} v^2 + \bar{v} < \bar{c}_s^2 > + 2 \bar{v}(\bar{v}.\bar{V}_s) + v^2 \bar{V}_s + < \bar{c}_s c_s^2 >$$

$$+ 2 < \bar{c}_s (\bar{c}_s. \bar{v})>$$

Thus,

$$< \bar{w}_s \phi_s> = \bar{v} <\epsilon i_s> + \bar{\bar{V}}_s < \epsilon i_s > + \tfrac{1}{2} m_s [\bar{v} v^2 + \bar{v} <\bar{c}_s^2 > + v^2 \bar{\bar{V}}_s$$

$$+ < \bar{c}_s \, c_s^2 >] + m_s[\bar{v} (\bar{v} \cdot \bar{\bar{V}}_s) + < \bar{c}_s (\bar{c}_s \cdot \bar{v})>]$$

$$= \bar{v} < \epsilon i_s > + \bar{\bar{V}}_s < \epsilon i_s > + [\tfrac{1}{2} m_s v^2 + \tfrac{3}{2} kT_s] \bar{v} + \tfrac{1}{2} m_s v^2 \bar{\bar{V}}_s$$

$$+ \tfrac{1}{2} m_s < \bar{c}_s c_s^2 > + m_s[\bar{v} (\bar{\bar{V}}_s \cdot \bar{v}) + < \bar{c}_s(\bar{c}_s \cdot \bar{v})>]$$

$$= \bar{v} [\tfrac{1}{2} m_s v^2 + \tfrac{3}{2} kT_s + < \epsilon i_s >]$$

$$+ [\tfrac{1}{2} m_s < \bar{c}_s c_s^2 > + \bar{\bar{V}}_s < \epsilon i_s >]$$

$$+ \tfrac{1}{2} m_s v^2 \bar{\bar{V}}_s + m_s \bar{v} (\bar{\bar{V}}_s \cdot \bar{v}) + m_s \bar{v} < \bar{c}_s \bar{c}_s >$$

$$= \bar{v} [\tfrac{1}{2} m_s v^2 + \tfrac{3}{2} kT_s + < \epsilon i_s >]$$

$$+ \frac{1}{n_s} \bar{q}_s$$

$$+ \tfrac{1}{2} m_s v^2 \bar{\bar{V}}_s + m_s \bar{v} (\bar{\bar{V}}_s \cdot \bar{v}) + \frac{1}{n_s} \bar{v} \cdot \bar{\bar{P}}_s$$

where the flux of peculiar energy, that is, the heat flux energy $\bar{q}_s$ is defined by $\bar{q}_s = n_s [\tfrac{1}{2} m_s <\bar{C}_s \bar{C}_s^2 > + \bar{V}_s < \epsilon_{is}>]$, remembering the peculiar energy defined to be $\tfrac{1}{2} m_s <C_s> + <\epsilon_{is} >$.

$$< \nabla_{\bar{w}} \phi_s > = < \nabla_{\bar{w}} (\epsilon i_s + \tfrac{1}{2} m_s \bar{w}_s^2)>$$

$$= \frac{m_s}{2} < \nabla_{\bar{w}} \bar{w}_s^2 >$$

$$= m_s < \bar{w}_s >$$

$$= m_s \bar{v}_s$$

And

$$< (\bar{w}_s \times \bar{B}) \cdot \nabla_{\bar{w}} \phi_s> = m_s < (\bar{w}_s \times \bar{B}) \cdot \bar{w}_s> = 0$$

Then the desired species energy equation becomes

$$\frac{\partial}{\partial t} \; n_s \left[< \epsilon_{is} > + \tfrac{1}{2} m_s v^2 + \frac{3}{2} kT_s + m_s \bar{v} \bar{V}_s \right] + \nabla_{\bar{r}} \cdot \left[n_s \bar{v} \left(< \epsilon_{is} > \right.\right.$$

$$\left.\left. + \tfrac{1}{2} m_s v^2 + \frac{3}{2} kT_s \right)\right] + \nabla_{\bar{r}} \bar{q}_s + \nabla_{\bar{r}} \cdot \left[\tfrac{1}{2} n_s m_s v^2 \bar{V}_s \right] +$$

$$\nabla_{\bar{r}} \cdot \left[n_s m_s \bar{v} (\bar{V}_s . \bar{v}) \right] + \nabla_{\bar{r}} (\bar{v} . \bar{\bar{P}}_s) - \bar{E} (q_s n_s \bar{v}_s) = R_s^{(2)}$$

where $R_s^{(2)}$ is defined by

$$R_s^{(2)} = \frac{\partial}{\partial t} \left(\left(\frac{n_s m_s}{2} \right) < \bar{w}_s^2 > \right)_c$$

APPENDIX D:

Derivation of Equation (2.2.1)

From

$$\bar{E}' = \frac{1}{\sigma}\ \bar{J} + \chi\ (\bar{J}\ x\ \bar{B})$$

$$\nabla\ x\ \bar{E}' = \nabla\ x\ (\frac{\bar{J}}{\sigma}) + \nabla\ x\ \ \{\chi\ (\bar{J}\ x\ \bar{B})\ \}$$

$$= \nabla\ (\frac{1}{\sigma})\ x\ \bar{J} + \frac{1}{\sigma}\ \nabla x\ \bar{J} + (\nabla\chi)\ x\ (\bar{J}\ x\ \bar{B}) + \chi\ [$$

$$(\bar{B}\ .\ \nabla)\ \bar{J} - \bar{B}\ (\nabla\ .\bar{J}) - (\bar{J}\ .\nabla)\ \bar{B} + \bar{J}\ (\nabla.\bar{B})]$$

Since $\nabla.\bar{B} = 0$ and $\nabla.\bar{J} = 0$ in steady state

$$\nabla\ x\ \bar{E}' = \nabla\ (\ \frac{1}{\sigma}\ )\ x\ \bar{J} + \frac{1}{\sigma}\ (\nabla x\ \bar{J}) + (\nabla\chi)\ x\ (\bar{J}\ x\ \bar{B}) + \chi\ [$$

$$(\bar{B}.\nabla)\ \bar{J} - (\bar{J}.\nabla)\ \bar{B}\ ]$$

APPENDIX E:

Derivation of Eq. (2.2.2)

$$\nabla \times \bar{E}' = \nabla \times \bar{E}' + \nabla \times (\bar{u} \times \bar{B}) + \nabla \times \left(\frac{\nabla P_e}{en_e}\right) + \nabla \times (\alpha \nabla T_e)$$

$$- \nabla \times (\xi \nabla T_e \times \bar{B}) + \nabla \times [\zeta(\nabla T_e \times \bar{B}) \times \bar{B}]$$

$$\alpha = -\frac{5}{2} \frac{k}{e} \nu_o \tau_e^*$$

$$\xi = -\frac{k}{m_e} \frac{\nu_o \tau_e^{*2}}{1+\omega_e^2 \tau_e^{*2}}$$

$$\zeta = -\frac{ek}{m_e^2} \frac{\nu_o \tau_e^{*3}}{1+\omega_e^2 \tau_e^{*2}}$$

Since $\nabla . \bar{B} = 0$ and $\nabla \times \bar{E} = 0$, when we only consider the steady state case.

$$\nabla \times (\bar{u} \times \bar{B}) = (\bar{B}.\nabla)\, \bar{u} - \bar{B}\,(\nabla . \bar{u}) - (\bar{u}.\nabla)\, \bar{B}$$

$$\nabla \times \left(\frac{\nabla P_e}{en_e}\right) = \nabla \left(\frac{1}{en_e}\right) \times (\nabla P_e)$$

$$\nabla \times (\alpha \nabla T_e) = (\nabla \alpha) \times (\nabla T_e)$$

$$\nabla \times (\xi \nabla T_e \times \bar{B}) = (\nabla \xi) \times (\nabla T_e \times \bar{B}) + \xi[-\bar{B}\,(\nabla^2 T_e) + (\bar{B}.\nabla)\, \nabla T_e$$
$$- (\nabla T_e . \nabla)\, \bar{B}\,]$$

$$\nabla \times [\zeta\,(\nabla T_e \times \bar{B}) \times \bar{B}] = (\nabla \zeta) \times [(\nabla T_e \times \bar{B}) \times \bar{B}] + \zeta\,[\bar{B}\{\nabla T_e . (\nabla \times \bar{B})\}$$
$$+ (\bar{B}.\nabla)(\nabla T_e \times \bar{B}) - \{(\nabla T_e \times \bar{B}).\nabla\}\, \bar{B}\,]$$

Thus

$$\nabla \times \bar{E}' = (\bar{B}\cdot\nabla)\,\bar{u} - \bar{B}\,(\nabla\cdot\bar{u}) - (\bar{u}\cdot\nabla)\,\bar{B} + \nabla\left(\frac{1}{en_e}\right) \times (\nabla P_e)$$

$$+ (\nabla\,\alpha)\times(\nabla\,T_e) - (\nabla\xi)\times(\nabla\,T_e \times \bar{B}) - \xi\,[\,-\,\bar{B}(\nabla^2 T_e)$$

$$+ (\bar{B}\cdot\nabla)\nabla\,T_e - (\nabla\,T_e\cdot\nabla)\,\bar{B}] + (\nabla\zeta)\times[(\nabla\,T_e \times \bar{B}) \times \bar{B}]$$

$$+ \zeta\,\bar{B}[(\nabla\,T_e)\cdot(\nabla \times \bar{B})] + \zeta\,(\bar{B}\cdot\nabla)\,(\nabla\,T_e \times \bar{B}) - \zeta\,\{(\nabla\,T_e \times \bar{B})\nabla\}\bar{B}$$

A variation of applied field $\bar{B}$ with co-ordinates is assumed not to be permitted. Then,

$$\nabla \times \bar{E}' = (\bar{B}\cdot\nabla)\,\bar{u} - \bar{B}(\nabla\cdot\bar{u}) + \nabla\left(\frac{1}{en_e}\right) \times (\nabla\,P_e) + (\nabla\,\alpha) \times (\nabla\,T_e)$$

$$- (\nabla\xi)\times(\nabla\,T_e \times \bar{B}) + \bar{B}\xi\,\nabla^2\,T_e - \xi\,(\bar{B}\cdot\nabla)\nabla\,T_e + (\nabla\,\zeta)\times[$$

$$(\nabla\,T_e \times \bar{B}) \times \bar{B}\,] + \zeta\,(\bar{B}\cdot\nabla)\,(\nabla\,T_e \times \bar{B})$$

APPENDIX F:

Derivation of Eq. (2.2.2)

$$\bar{u} = (u_x, u_y, 0) \quad , \quad \bar{B} = (0, 0, B)$$

$$(\bar{B}.\nabla)\ \bar{u} = B \frac{\partial}{\partial z} \bar{u} = 0 \text{ , since } \bar{u} = \bar{u}\ (x,y).$$

$$(\nabla \xi) \times (\nabla T_e \times \bar{B}) = (\nabla \xi) \times [\hat{i} \frac{\partial T_e}{\partial x} B - \hat{j} \frac{\partial T_e}{\partial y} B]$$

$$= - B\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial T_e}{\partial y}\right)\hat{i} - B\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial T_e}{\partial x}\right)\hat{j} + B\ [$$

$$\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial T_e}{\partial x}\right) + \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial T_e}{\partial y}\right)]\ \hat{k}$$

$$\bar{B}\ \xi\ (\nabla^2 T_e) = \hat{k}\ B\ \xi\ (\nabla^2\ T_e)$$

$$\xi\ (\bar{B}.\ \nabla)\ \nabla\ T_e = \hat{k}\ B\ \xi\ \frac{\partial}{\partial z}\ (\nabla\ T_e)$$

$$(\nabla\zeta) \times [(\nabla T_e \times \bar{B}) \times \bar{B}\ ] = - B^2\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial T_e}{\partial y}\right)\hat{i} - B^2\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial T_e}{\partial x}\right)\hat{j} + B^2\ [$$

$$\left(\frac{\partial \zeta}{\partial y}\right)\left(\frac{\partial T_e}{\partial x}\right) + \left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial T_e}{\partial y}\right)]\ \hat{k}$$

$$\zeta\ (\bar{B}.\ \nabla)(\nabla T_e \times \bar{B}) = \zeta\ B^2 \frac{\partial^2\ T_e}{\partial\ x\ \partial z}\ \hat{i} - \zeta\ B^2\ \frac{\partial^2\ T_e}{\partial y\ \partial z}\ \hat{j}$$

Thus,

$$\nabla \times \bar{E}' = - \bar{B}\ (\nabla.\ \bar{u}) + \nabla\ \left(\frac{1}{en_e}\right) \times (\nabla\ P_e) + (\nabla\ \alpha) \times (\nabla\ T_e) + B\left(\frac{\partial \xi}{\partial z}\right)$$

$$\left(\frac{\partial T_e}{\partial y}\right)\hat{i} + B\left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial T_e}{\partial x}\right)\hat{j} - B[\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial T_e}{\partial x}\right) + \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial T_e}{\partial y}\right)]\ \hat{k} + B\xi(\nabla^2 T_e)\hat{k}$$

$$- B\ \xi\ \frac{\partial}{\partial z}\ (\nabla\ T_e)\ \hat{k} - B^2\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial T_e}{\partial y}\right)\hat{i} - B^2\left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial T_e}{\partial x}\right)\hat{j} + B^2[\left(\frac{\partial \zeta}{\partial y}\right).$$

$$\left(\frac{\partial T_e}{\partial x}\right) + \left(\frac{\partial \zeta}{\partial x}\right)\left(\frac{\partial T_e}{\partial y}\right)]\hat{k} + \zeta B^2 \frac{\partial^2 T_e}{\partial x\ \ \partial z}\ \hat{i} - \zeta B^2\ \frac{\partial^2\ T_e}{\partial y\ \partial z}\ \hat{j}$$

and automatically, the following conditions should be satisfied when we neglect the ion slip effect and the variation of magnetic field,

$$\nabla\left(\frac{1}{en_e}\right) x\ (\nabla P_e)\Big|_{\hat{i}} + (\nabla\alpha)\ x\ (\nabla T_e)\Big|_{\hat{i}} + B\left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial T_e}{\partial y}\right) - B^2\left(\frac{\partial\zeta}{\partial z}\right)$$

$$\left(\frac{\partial T_e}{\partial y}\right) + B^2\zeta\ \frac{\partial^2 T_e}{\partial z\,\partial x} = 0$$

$$\nabla\left(\frac{1}{en_e}\right) x(\nabla P_e)\Big|_{\hat{j}} + (\nabla\alpha)x\ (\nabla T_e)\Big|_{\hat{j}} + \left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial T_e}{\partial x}\right) - B^2\left(\frac{\partial\zeta}{\partial z}\right)$$

$$\left(\frac{\partial T_e}{\partial y}\right) - B^2\zeta\ \frac{\partial^2 T_e}{\partial z\,\partial y} = 0$$

But if the variation of the plasma quantities in the z direction are negligible compared to the variation of the plasma quantities in the x-y plane, then we can ignore products of derivatives such as $(\partial/\partial z)\ (\partial/\partial y)$ or $(\partial/\partial z)\ (\partial/\partial x)$ in comparison with products such as $(\partial/\partial x)\ (\partial/\partial y)$, $(\partial^2/\partial x^2)$ or $(\partial^2/\partial y^2)$

Consequently, $\nabla\ x\ \bar{E}'$ is aligned with the z-direction and

$$\nabla\ x\ \bar{E}' = -B\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{1}{en_e}\right)\left(\frac{\partial P_e}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{en_e}\right)\left(\frac{\partial P_e}{\partial x}\right) +$$

$$\left(\frac{\partial\alpha}{\partial x}\right)\left(\frac{\partial T_e}{\partial x}\right) - \left(\frac{\partial\alpha}{\partial y}\right)\left(\frac{\partial T_e}{\partial x}\right) - B\left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial T_e}{\partial x}\right) - B\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial T_e}{\partial y}\right) +$$

$$B\ \xi\left(\frac{\partial^2 T_e}{\partial x^2}\right) + B\xi\left(\frac{\partial^2 T_e}{\partial y^2}\right) + B^2\left(\frac{\partial\zeta}{\partial y}\right)\left(\frac{\partial T_e}{\partial x}\right) + B^2\left(\frac{\partial\zeta}{\partial x}\right)\left(\frac{\partial T_e}{\partial y}\right)$$

APPENDIX G:

Derivation of Current Governing Equation (see Eq. 2.2.3)

From Eq. (2.2.1)

$$\nabla \times \bar{E}' = \nabla(\tfrac{1}{\sigma}) \times \bar{J} + \tfrac{1}{\sigma}(\nabla \times \bar{J}) + (\nabla\chi) \times (\bar{J} \times \bar{B}) + \chi\,(\bar{B}.\nabla)\,\bar{J} - \chi\,(\bar{J}.\nabla)\,\bar{B}$$

,where

$$\nabla\left(\frac{1}{\sigma}\right) \times \bar{J} = \hat{k}\left[\left(\frac{\partial\psi}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{1}{\sigma}\right) + \left(\frac{\partial\psi}{\partial y}\right)\frac{\partial}{\partial y}\left(\frac{1}{\sigma}\right)\right]$$

$$\frac{1}{\sigma}(\nabla \times \bar{J}) = \hat{k}\,(\tfrac{1}{\sigma})\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]$$

$$(\nabla\chi) \times (\bar{J} \times \bar{B}) = -\hat{k}\,B\left[-\frac{\partial\psi}{\partial y}\frac{\partial\chi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\chi}{\partial y}\right]$$

$$\chi\,(\bar{B}.\nabla)\,\bar{J} = 0$$

$$\chi\,(\bar{J}.\nabla)\,\bar{B} = 0$$

Thus,

$$\nabla \times \bar{E}' = \left(\frac{1}{\sigma}\right)\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] + \left[\left(\frac{\partial\psi}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{1}{\sigma}\right) + \left(\frac{\partial\psi}{\partial y}\right)\frac{\partial}{\partial y}\left(\frac{1}{\sigma}\right)\right]$$

$$- B\left[-\left(\frac{\partial\psi}{\partial y}\right)\left(\frac{\partial\chi}{\partial x}\right) + \left(\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\chi}{\partial y}\right)\right] \quad \text{only in z-direction}$$

$$= \left(\frac{1}{\sigma}\right)\left[\nabla^2\psi + \sigma\left\{\frac{\partial}{\partial x}\left(\frac{1}{\sigma}\right) - \frac{\partial\chi}{\partial y}\right\}\left(\frac{\partial\psi}{\partial x}\right) + \sigma\left\{\frac{\partial}{\partial y}\left(\frac{1}{\sigma}\right)\right.\right.$$

$$\left.\left. + B\frac{\partial\chi}{\partial x}\right\}\left(\frac{\partial\psi}{\partial y}\right)\right]$$

$$= \left(\frac{1}{\sigma}\right)\left[\nabla^2\psi + M(x,y)\frac{\partial\psi}{\partial x} + N(x,y)\frac{\partial\psi}{\partial y}\right]$$

APPENDIX H:

Discretization of the Current Governing Equation.

The derivations occuring in Sec.3.1 can be replaced by the finite central difference as follows,

$$[Q(x,y)\,\psi_x]_{i+\frac{1}{2},j} \doteq Q_{i+\frac{1}{2},j}\,\frac{\psi_{i+1,j}-\psi_{i,j}}{x_{i+1}-x_i}\,[Q\,(x,y)\,\psi_y]_{i,j+\frac{1}{2}}$$

Then,

$$\frac{\partial}{\partial x}\,[Q\,(x,y)\psi_x]\Big|_{i,j} \doteq \frac{2}{x_{i+1}-x_{i-1}}\Big[\,\{Q\,(x,y)\psi_x\}_{i+\frac{1}{2},j}-\{Q(x,y)\psi_x\}_{i-\frac{1}{2},j}$$

$$\Big] \doteq \frac{2}{x_{i+1}-x_{i-1}}\Big[\,Q_{i+\frac{1}{2},j}\,\frac{\psi_{i+1,j}-\psi_{i,j}}{x_{i+1}-x_i}-Q_{i-\frac{1}{2},j}\,\frac{\psi_{i,j}-\psi_{i-1,j}}{x_i-x_{i-1}}\Big]$$

Thus if we let $Q(x,y) = 1$, then

$$[\,\psi_{x,x}]_{i,j} \doteq \frac{2}{x_{i+1}-x_{i-1}}\Big[\,\frac{\psi_{i+1,j}-\psi_{i,j}}{x_{i+1}-x_i}-\frac{\psi_{i,j}-\psi_{i-1,j}}{x_i-x_{i-1}}\Big]$$

$$[\,\psi_{y,y}\,]_{i,j} \doteq \frac{2}{y_{j+1}-y_{j-1}}\Big[\,\frac{\psi_{i,j+1}-\psi_{i,j}}{y_{j+1}-y_j}-\frac{\psi_{i,j}-\psi_{i,j-1}}{y_j-y_{j-1}}$$

and letting $Q(x,y) = M(x,y)$ and then $Q(\,x,y) = N\,(x,y)$, in turn we obtain

$$[M(x,y)\psi_x\,]_{i,j} \doteq \tfrac{1}{2}\,\Big[M_{i+\frac{1}{2},j}\,\frac{\psi_{i+1,j}-\psi_{i,j}}{x_{i+1}-x_i}+M_{i-\frac{1}{2},j}\,\frac{\psi_{i,j}-\psi_{i-1,j}}{x_i-x_{i-1}}\Big]$$

$$[N(x,y)\psi_y]_{i,j} \doteq \tfrac{1}{2}\,\Big[N_{i,j+\frac{1}{2}}\,\frac{\psi_{i,j+1}-\psi_{i,j}}{y_{j+1}-y_j}+N_{i,j-\frac{1}{2}}\,\frac{\psi_{i,j}-\psi_{i,j-1}}{y_j-y_{j-1}}\Big]$$

Therefore the given equation for stream function can be rewritten as

$$\frac{2}{x_{i+1}-x_{i-1}}\left(\frac{\psi_{i+1,j}-\psi_{i,j}}{x_{i+1}-x_i}-\frac{\psi_{i,j}-\psi_{i-1,j}}{x_i-x_{i-1}}\right)+\frac{2}{y_{j+1}-y_{j-1}}\left(\frac{\psi_{i,j+1}-\psi_{i,j}}{y_{j+1}-y_j}\right.$$

$$\left.-\frac{\psi_{i,j}-\psi_{i,j-1}}{y_j-y_{j-1}}\right)+\frac{1}{2}\left[M_{i+\frac{1}{2},j}\frac{\psi_{i+1,j}-\psi_{i,j}}{x_{i+1}-x_i}+M_{i-\frac{1}{2},j}\frac{\psi_{i,j}-\psi_{i-1,j}}{x_i-x_{i-1}}+\right.$$

$$\frac{1}{2}\left[N_{i,j+\frac{1}{2}}\frac{\psi_{i,j+1}-\psi_{i,j}}{y_{j+1}-y_j}+N_{i,j-\frac{1}{2}}\frac{\psi_{i,j}-\psi_{i,j-1}}{y_j-y_{j-1}}\right.\doteq 0$$

Let $x_{i+1}-x_i=h_i$, $y_{j+1}-y_j=h_j$, then $x_{i+1}-x_{i-1}=h_i+h_{i-1}$ and $y_{j+1}-y_{j-1}=h_j+h_{j-1}$

Thus,

$$\frac{2}{h_i+h_{i-1}}\left(\frac{\psi_{i+1,j}-\psi_{i,j}}{h_i}-\frac{\psi_{i,j}-\psi_{i-1,j}}{h_{i-1}}\right)+\frac{2}{h_j+h_{j-1}}\left(\frac{\psi_{i,j+1}-\psi_{i,j}}{h_j}\right.$$

$$-\frac{\psi_{i,j}-\psi_{i,j-1}}{h_{j-1}}+\frac{1}{2}\left[M_{i+\frac{1}{2},j}\frac{\psi_{i+1,j}-\psi_{i,j}}{h_i}+M_{i-\frac{1}{2},j}\frac{\psi_{i,j}-\psi_{i-1,j}}{h_{i-1}}\right]+$$

$$\frac{1}{2}\left[N_{i,j+\frac{1}{2}}\frac{\psi_{i,j+1}-\psi_{i,j}}{h_j}+N_{i,j-\frac{1}{2}}\frac{\psi_{i,j}-\psi_{i,j-1}}{h_{j-1}}\right.\doteq 0$$

NUMBER	CONDITIONS				
	STAGNATION TEMPERATURE T_0, °K	STAGNATION PRESSURE P_0, ATM	MACH No. M, DIMENSION-LESS	STATIC GAS TEMPERATURE T_∞, °K	SEEDING RATIO S_k, DIMENSION-LESS
1	1, 800	1	0.8	1, 482.2	0.002
2	1, 800	1	0.8	1, 482.2	0.003
3	2, 002.5	1	1	1, 500	0.001
4	1, 821.6	1	0.8	1, 500	0.0065
5	2, 550.24	1	0.8	2, 100	0.004
6	2, 494.89	1	0.57	2, 250	0.001
7	2, 428.8	1	0.8	2, 000	0.004

	MAGNETIC INDUCTION B. WEBER/M^2	REFERENCES	
1	1.7		
2	1.7		
3	1.3	, TAB. 2 ;	, FIG. 1
4	1.7	, FIG. 4 ;	, FIG. 2, 3
5	1.7		
6	1.63	, TAB. 2	
7	1.7	, FIG. 1 ;	, FIG. 4, 5, 6

LEGENDS

n_e	ELECTRON NUMBER DENSITY (No/M^3)
T_e	ELECTRON TEMPERATURE (°K)
T_∞	GAS STATIC TEMPERATURE (°K)
S_s	SEEDING RATIO OF POTASSIUM (DIMENSIONLESS)
P_0	GAS STAGNATION PRESSURE (ATM)
σ	CONDUCTIVITY (MHO/M)
χ	HALL COEFFICIENT (M /MHO-WEBER)
β	HALL PARAMETER (DIMENSIONLESS)
j	CURRENT DENSITY (AMPS/M^2)
Ψ	CURRENT STREAM FUNCTION (AMPS/M)
ITN	NUMBER OF ITERATIONS REQUIRED TO COMPUTE TO GET THE DESIRABLE ϵ.
ϵ	MAXIMUM. DIFFERENCE OF THE CALCULATED VALUES AT THE SAME POINT BETWEEN ITERATIONS WHEN THE COMPU-

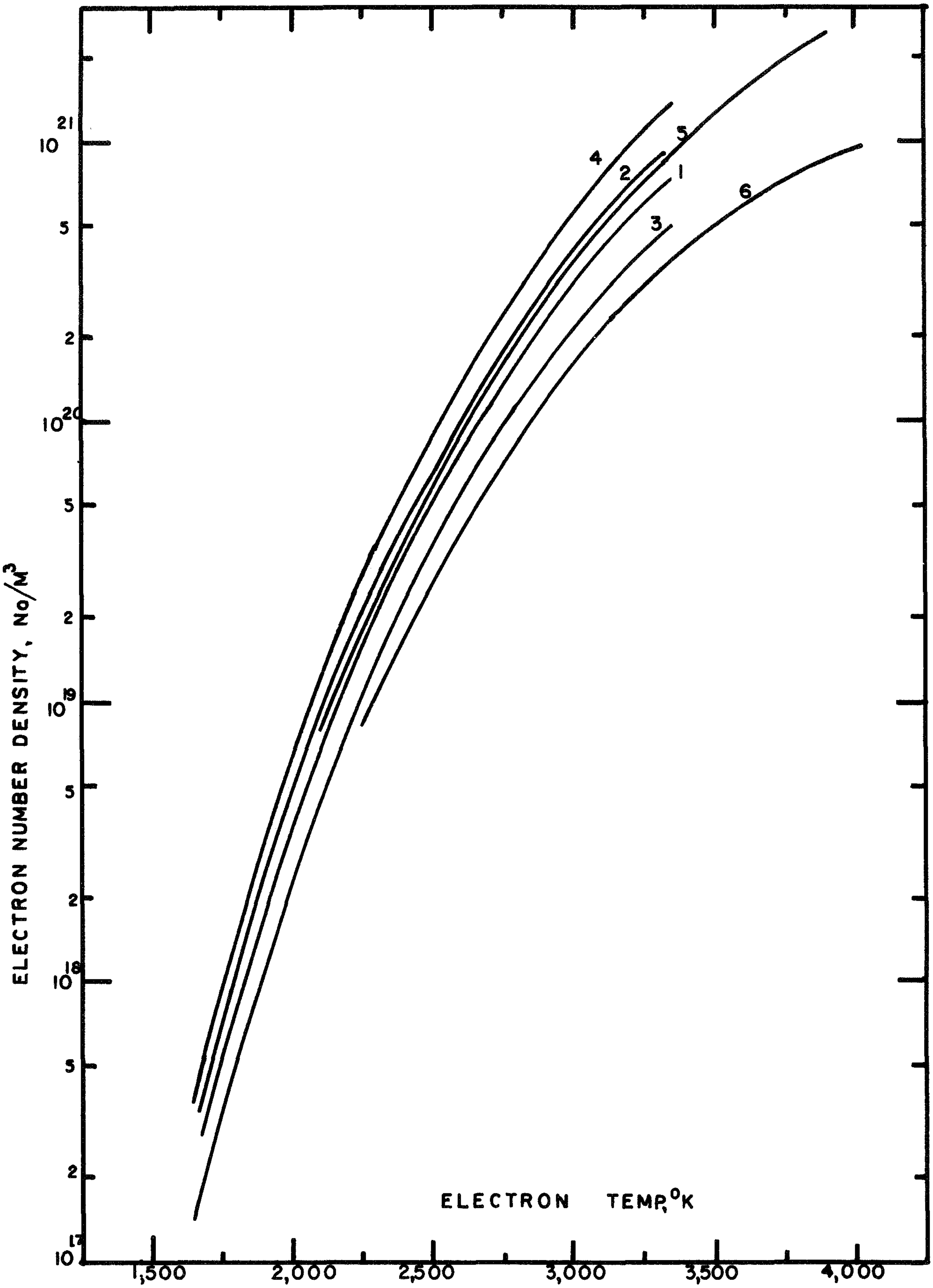

Fig. 1 Electron Number Density Against Electron Temperature

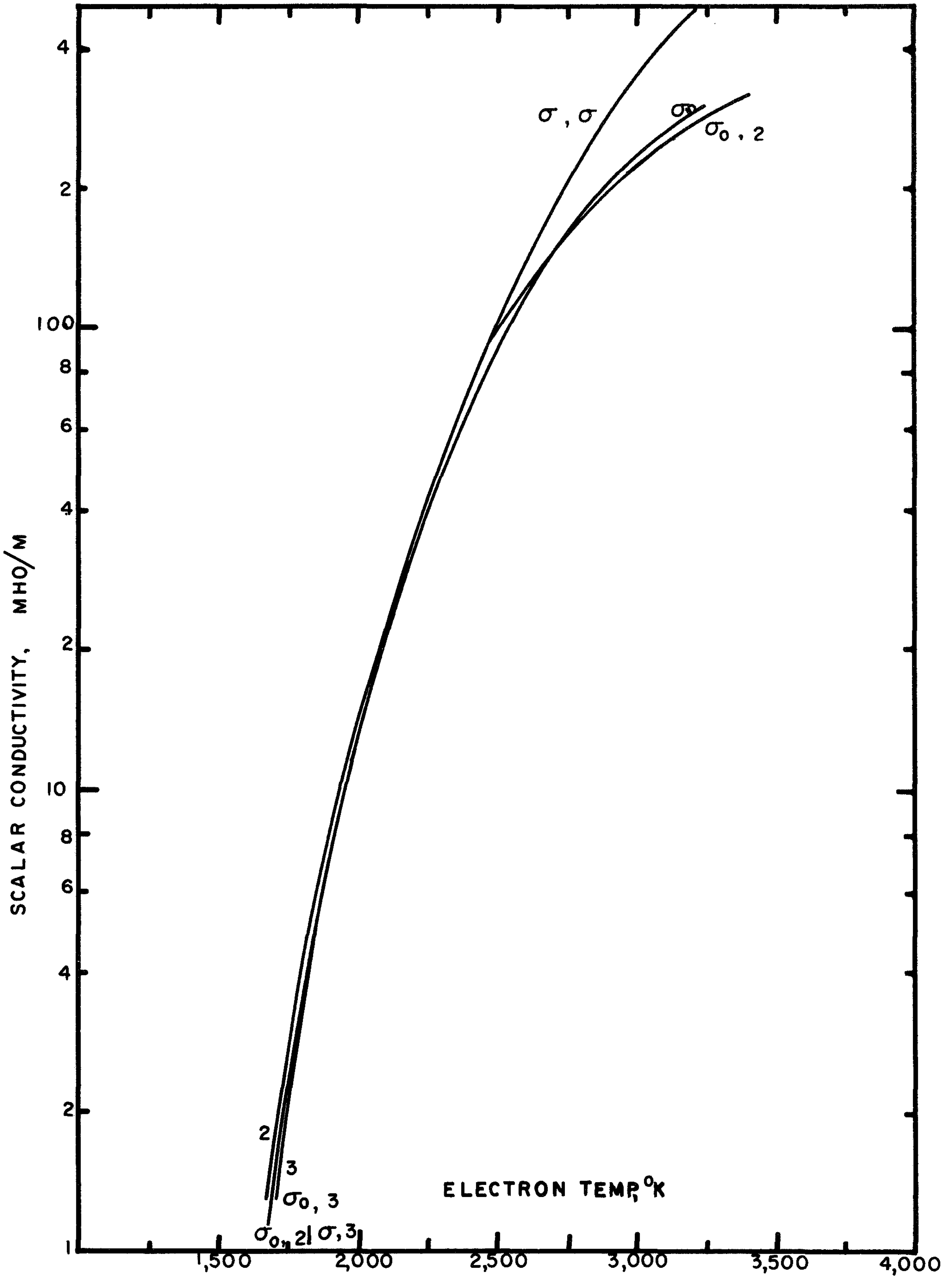

Fig. 2 Conductivity Calculations in terms of Electron Temperature
a) σ ; Calculations Based on the 2nd Approximation
b) σ_0 ; Calculations Based on the Conventional Definition

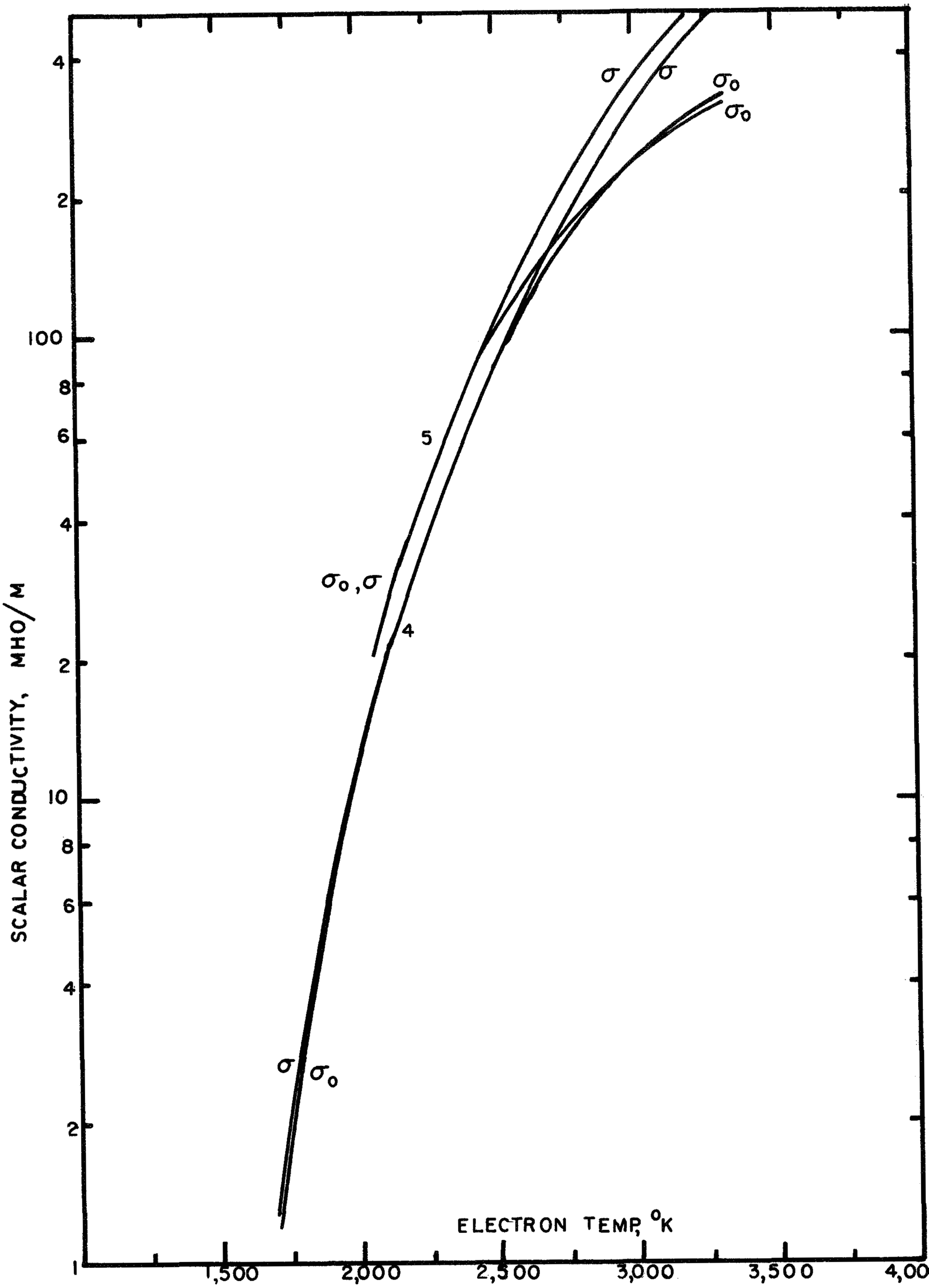

Fig. 3 Conductivity Calculations in terms of Electron Temperature

a) σ ; Calculations Based on the 2nd Approximation

b) σ_0 ; Calculations Based on the Conventional Definition

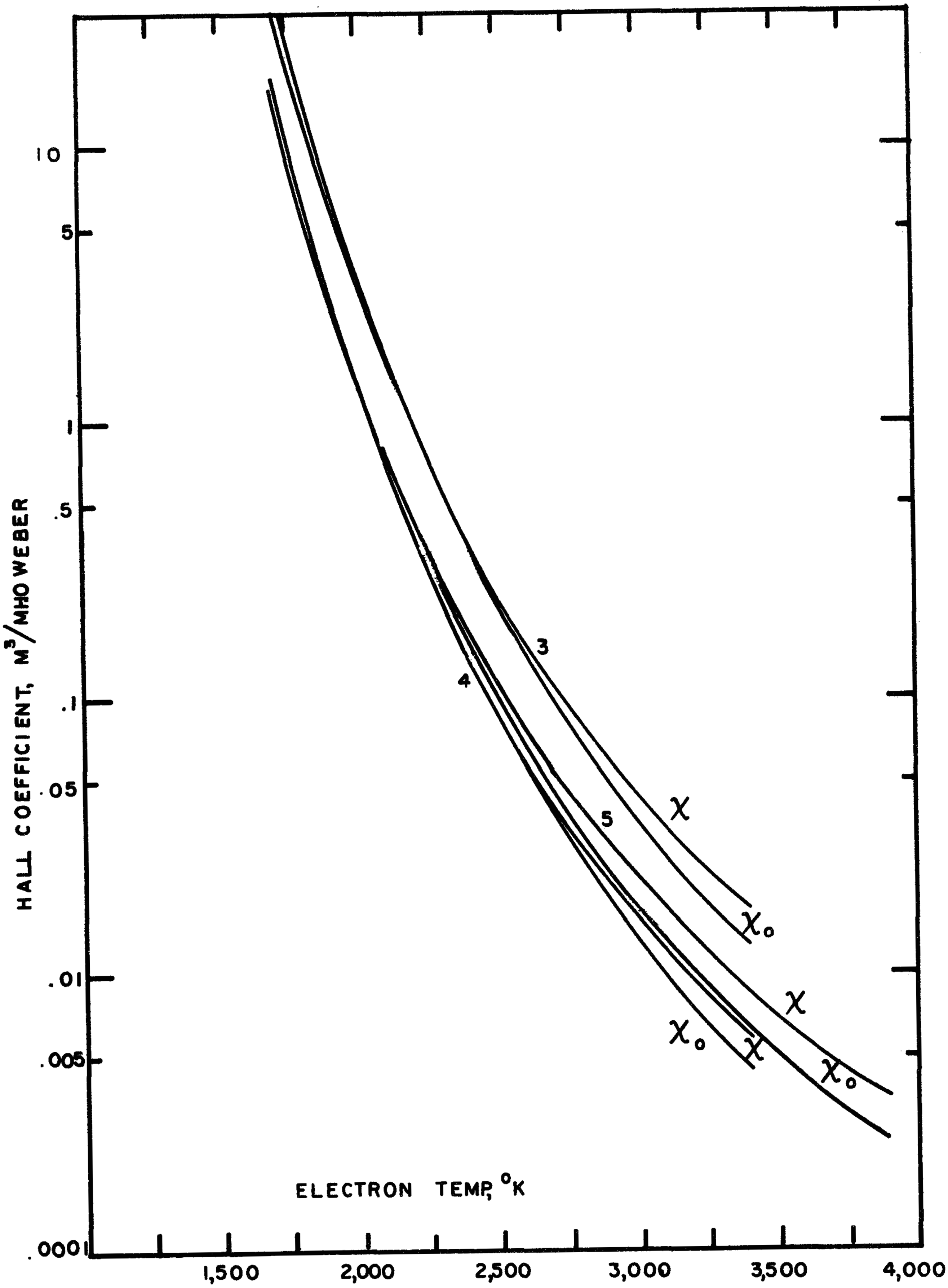

Fig. 4 Hall Coefficient Against Electron Temperature

a) χ ; Calculations Based on the 2nd Approximation

b) χ_0 ; Calculations Based on the Conventional Definition

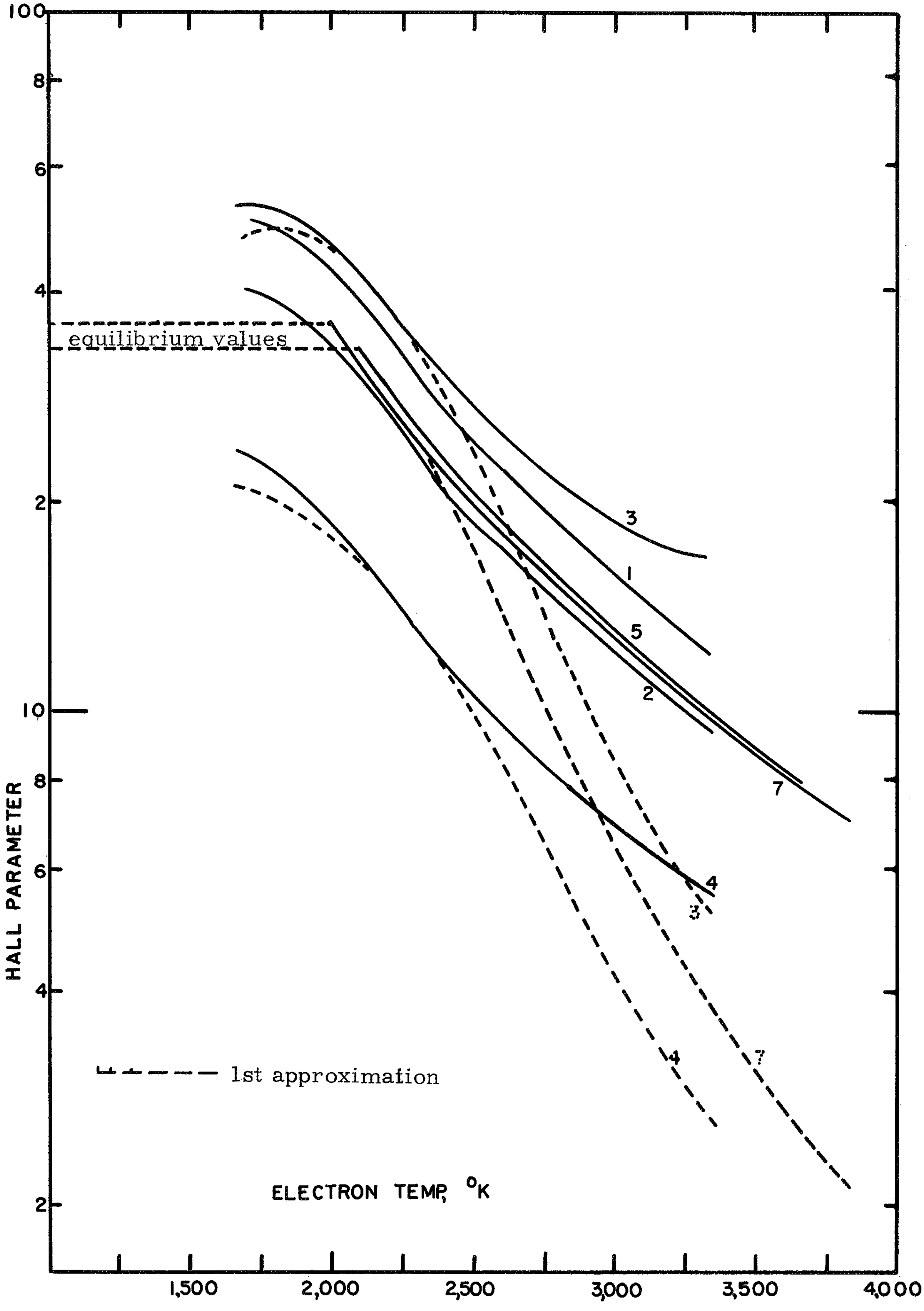

Fig. 5 Dependence of Hall Parameter on Electron Temperature By the Definition of $\beta = \sigma \chi$ B

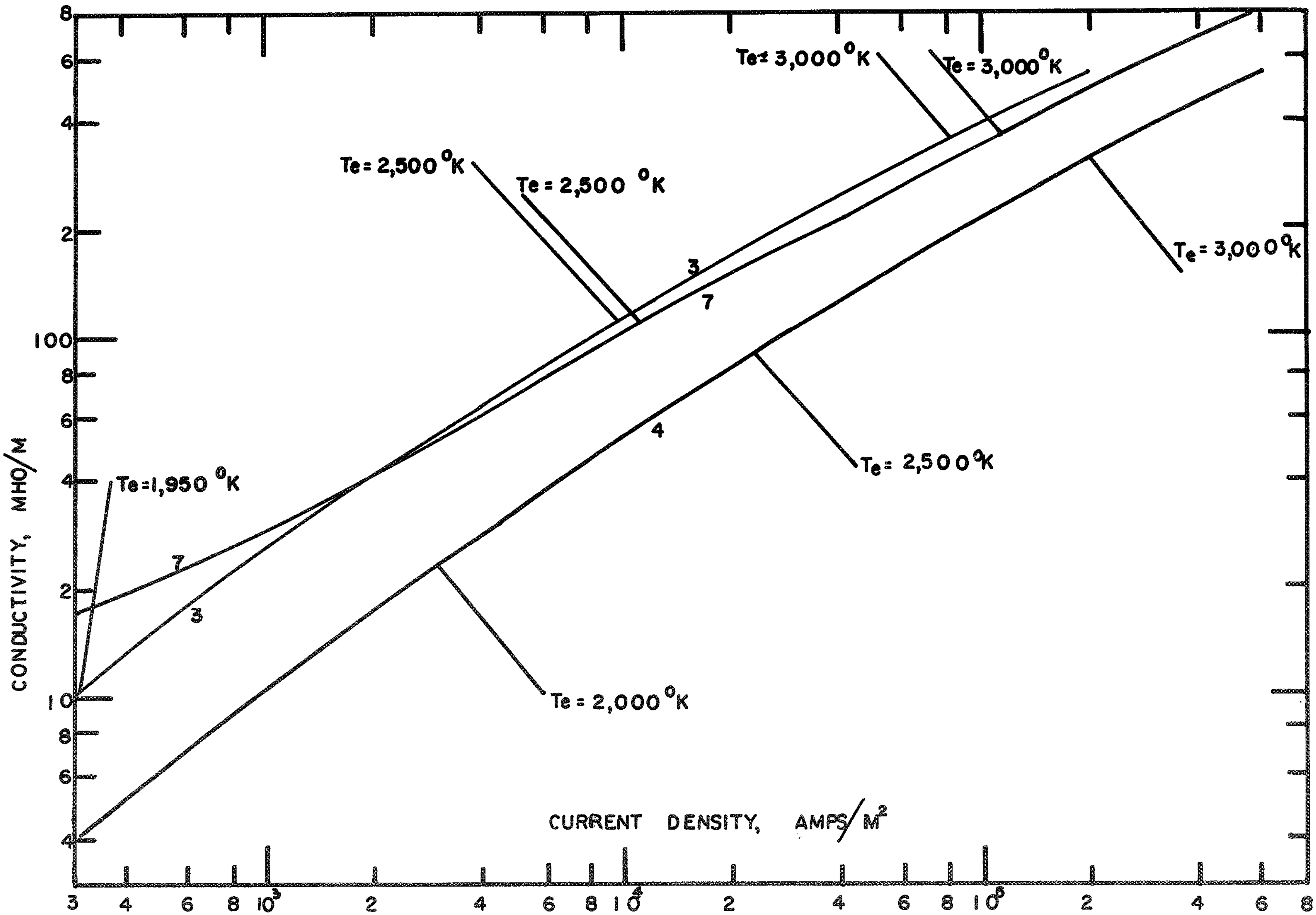

Fig. 6 Dependence of Conductivity on Current Density

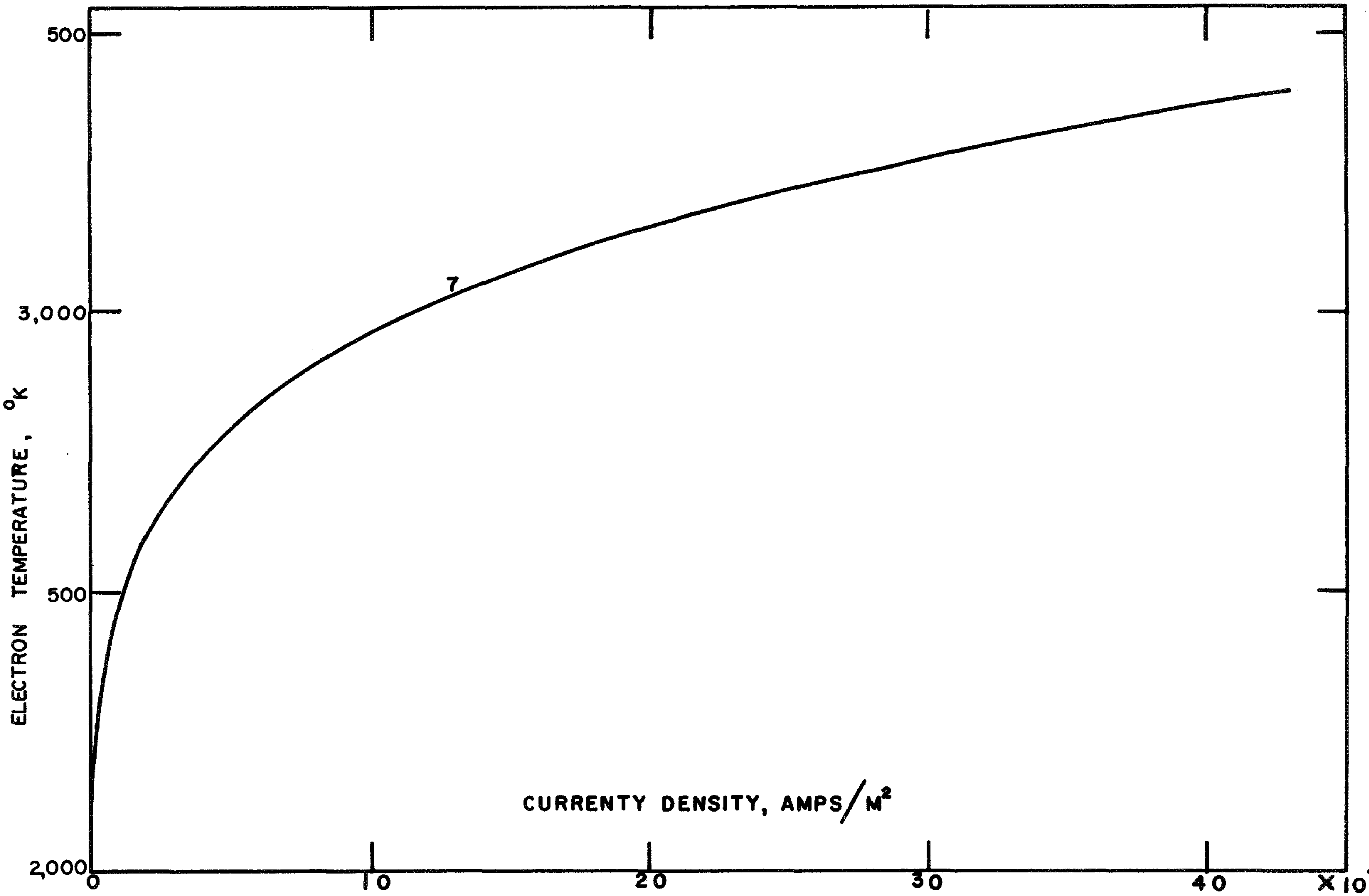

Fig. 7 Variation of Electron Temperature with respect to Current Density

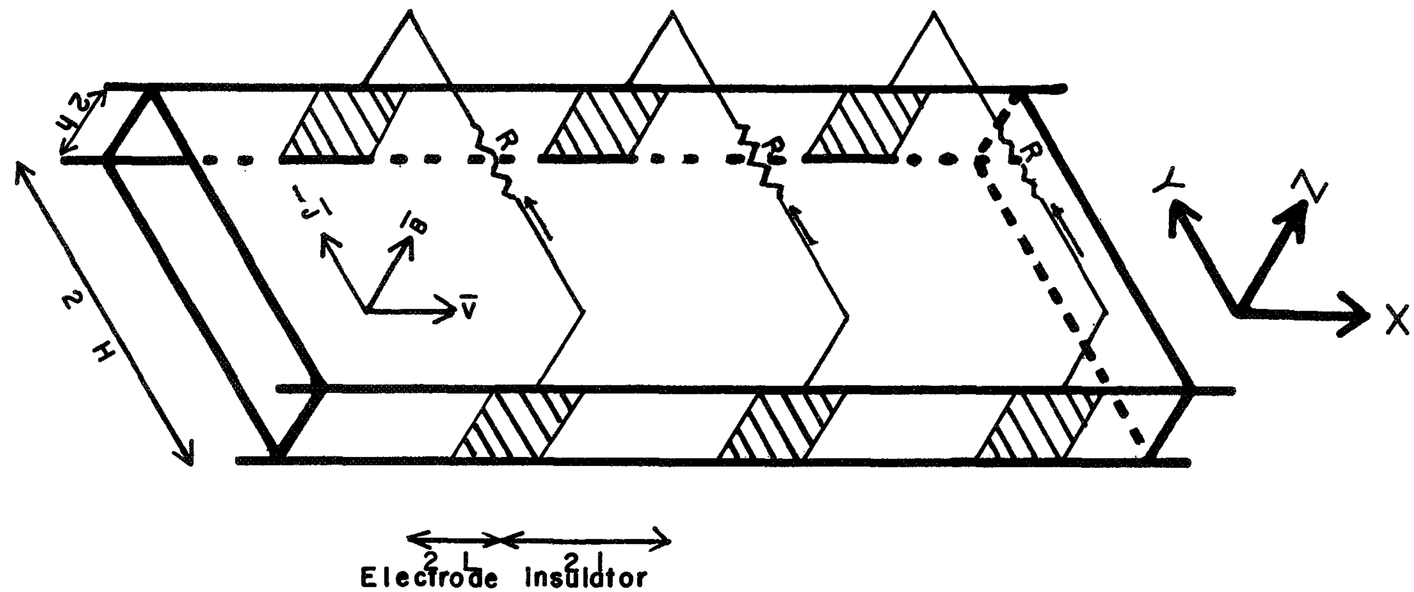

Fig. 8 Faraday Generator Channel Configuration Showing ;
a) Applied Field Direction
b) Current Flow Direction
c) Plasma Flow Direction
d) Channel Dimensionalized Geometry and Single Load Resistance Connections

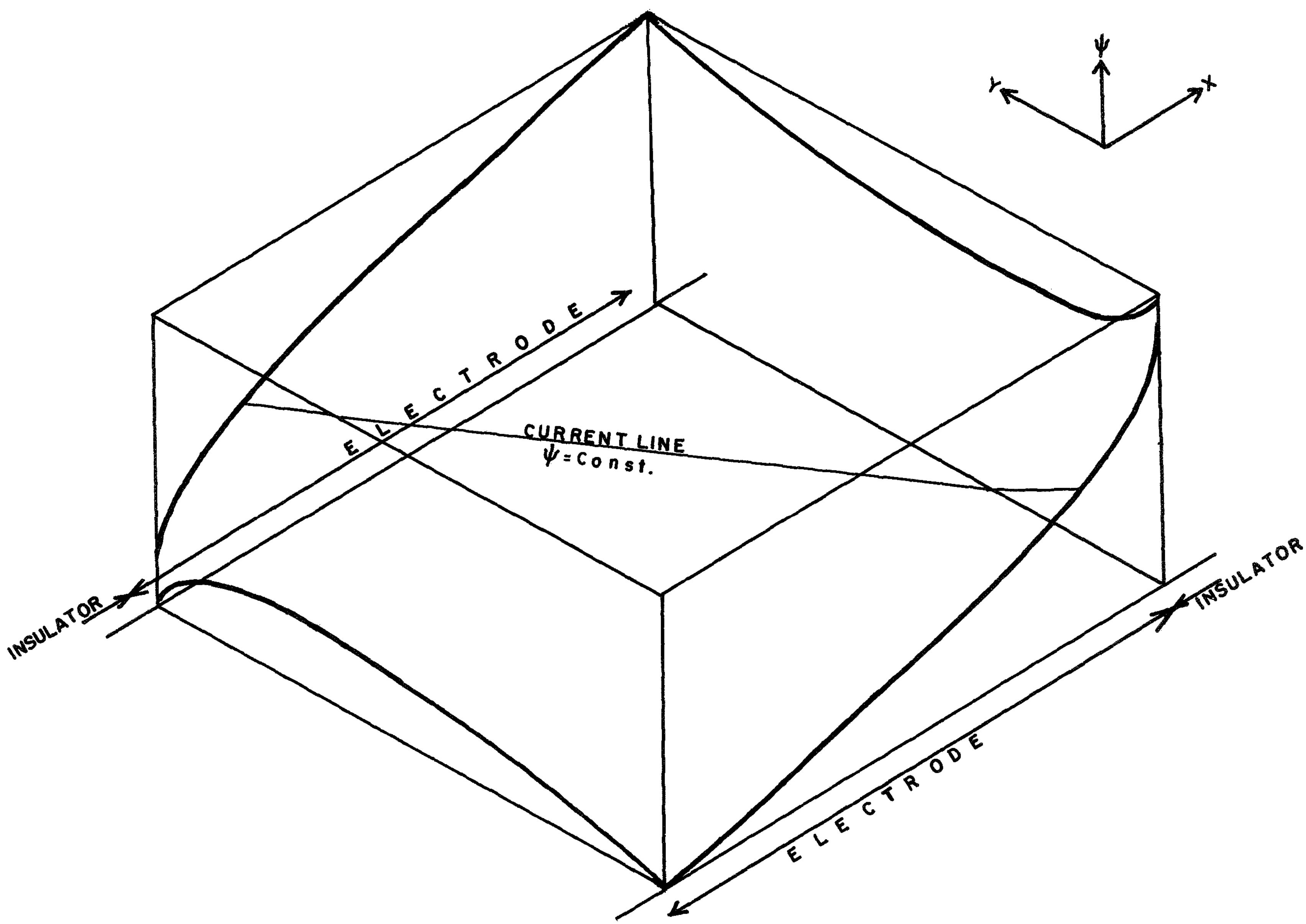

Fig. 9 One of Typical Solutions to the Current Governing Equation in the Faraday Channel in Case of Uniform Plasma at β = 1.0, Showing; a) Surface of Current Stream Function, b) One of Current Lines

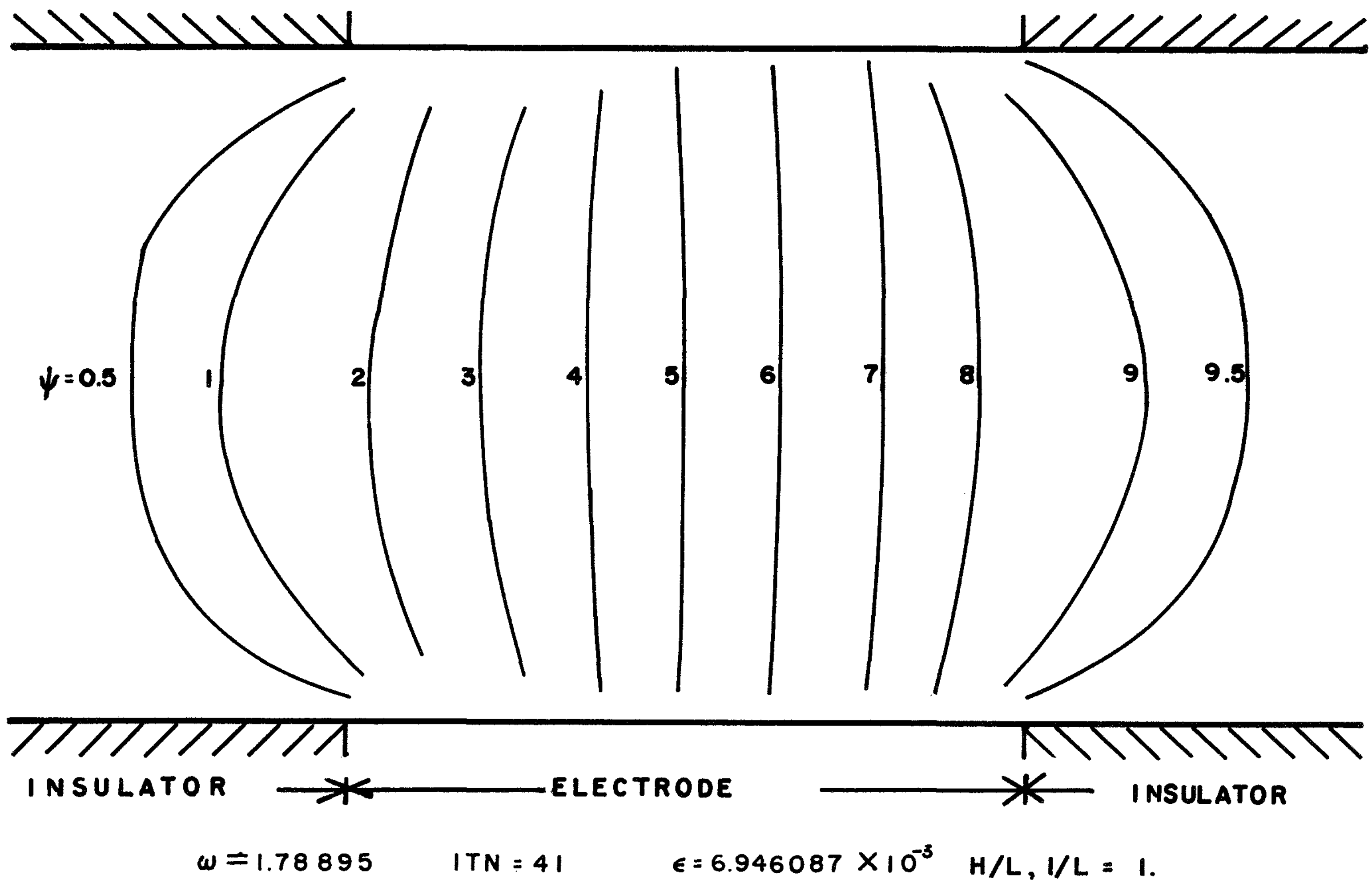

Fig. 10 Current Distribution in the Uniform Plasma Flow in the Faraday Channel at $\beta = 0.0$

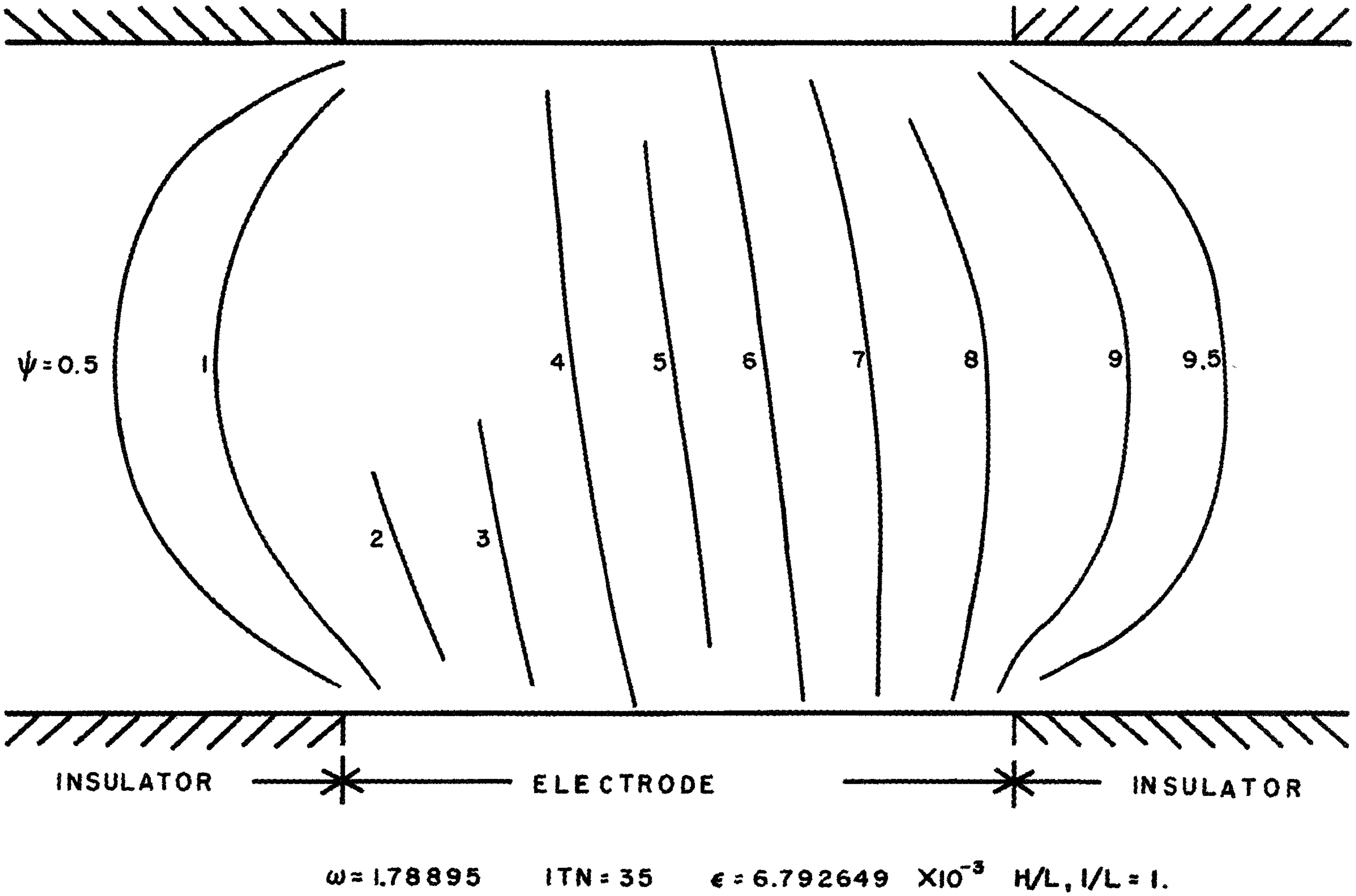

Fig. 11 Current Distribution in the Uniform Plasma Flow in the Faraday Channel at $\beta = 0.2$

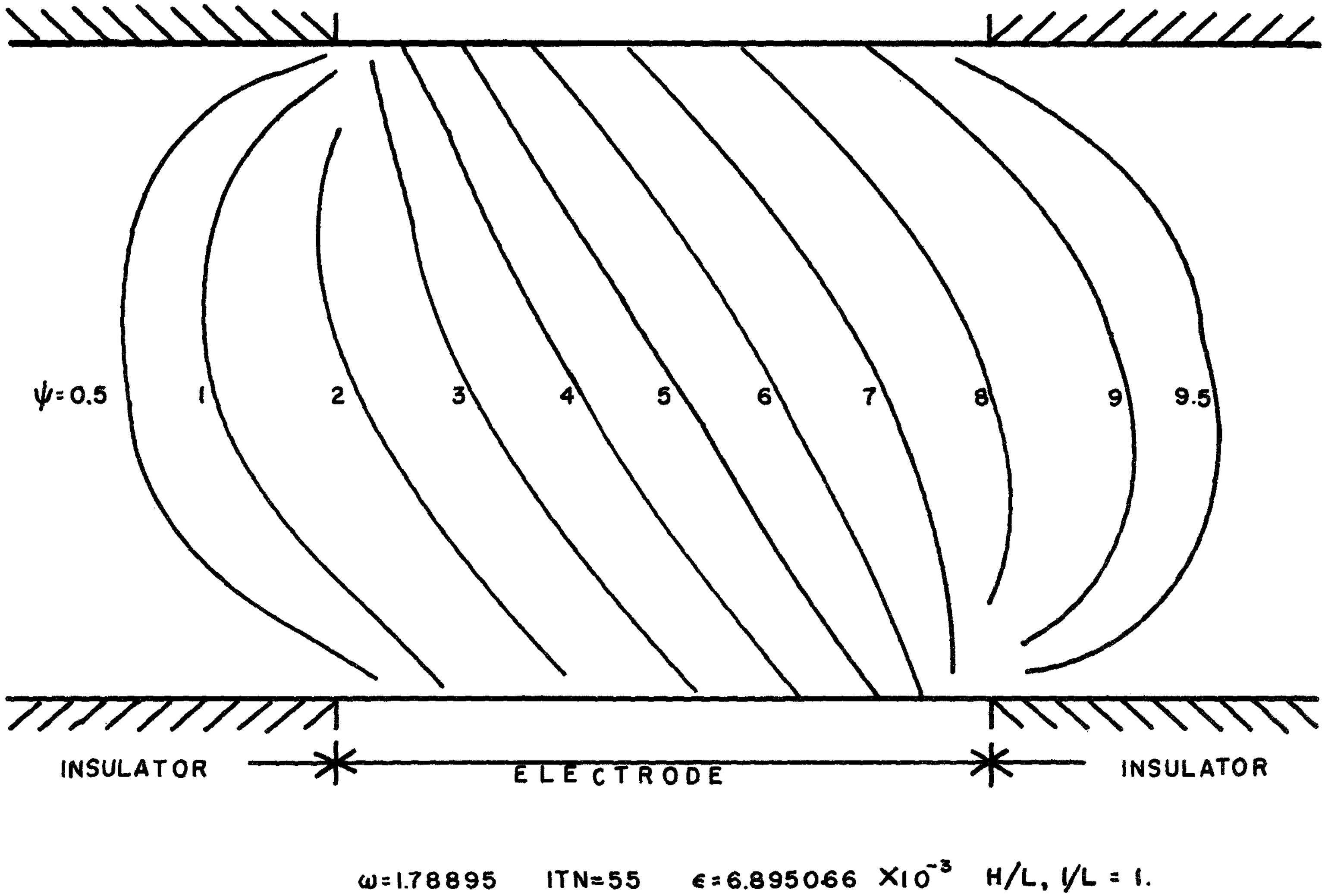

Fig. 12 Current Distribution in the Uniform Plasma Flow in the Faraday Channel at $\beta = 1.0$

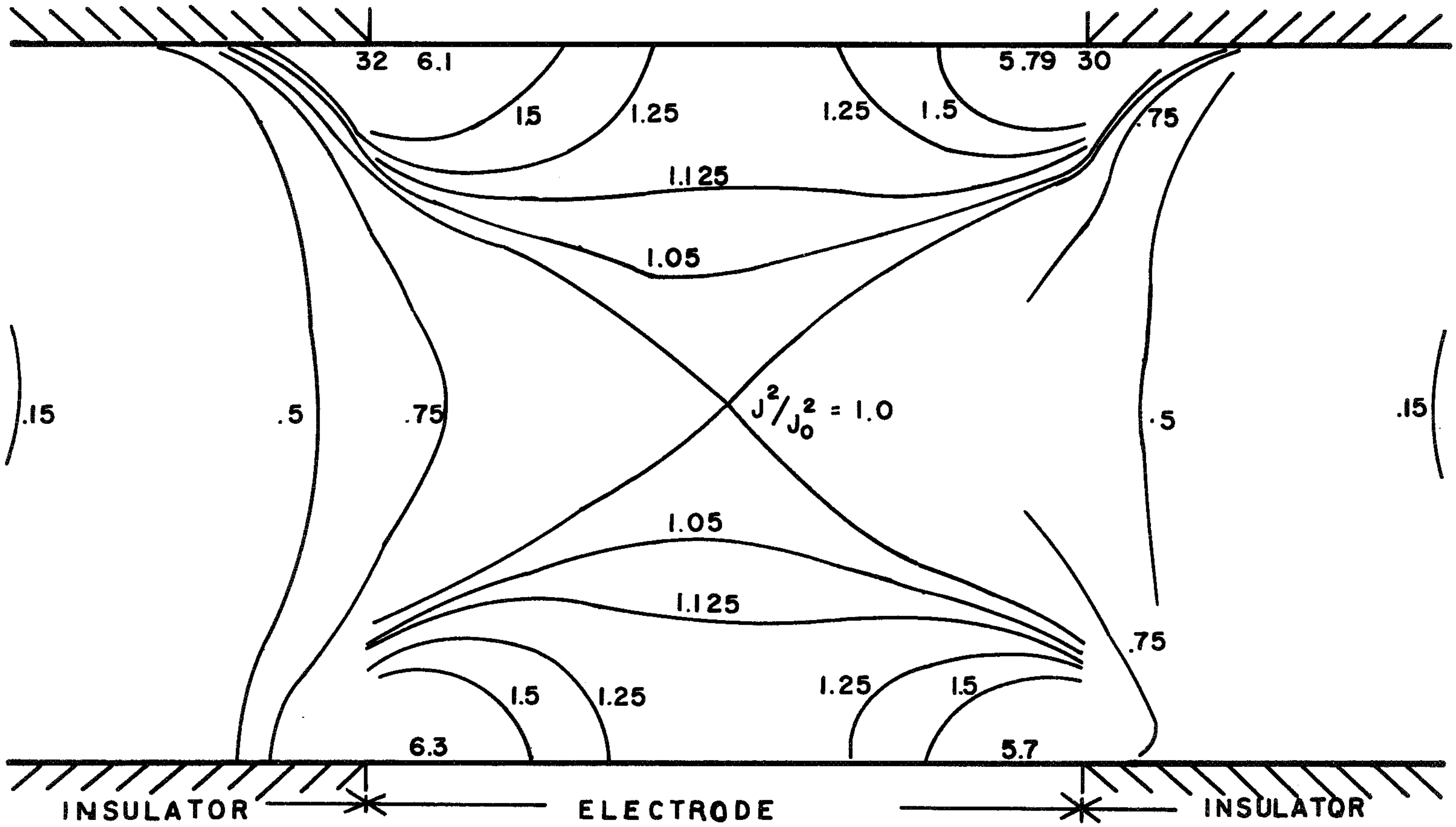

Fig. 13 Calculation of Electron Temperature Distribution by Using Simple Energy Balance Equation and the Obtained Current Distribution at $\beta = 0.0$, Showing Normalized Values of j^2 in Reference to the Value at the Center of the Channel.

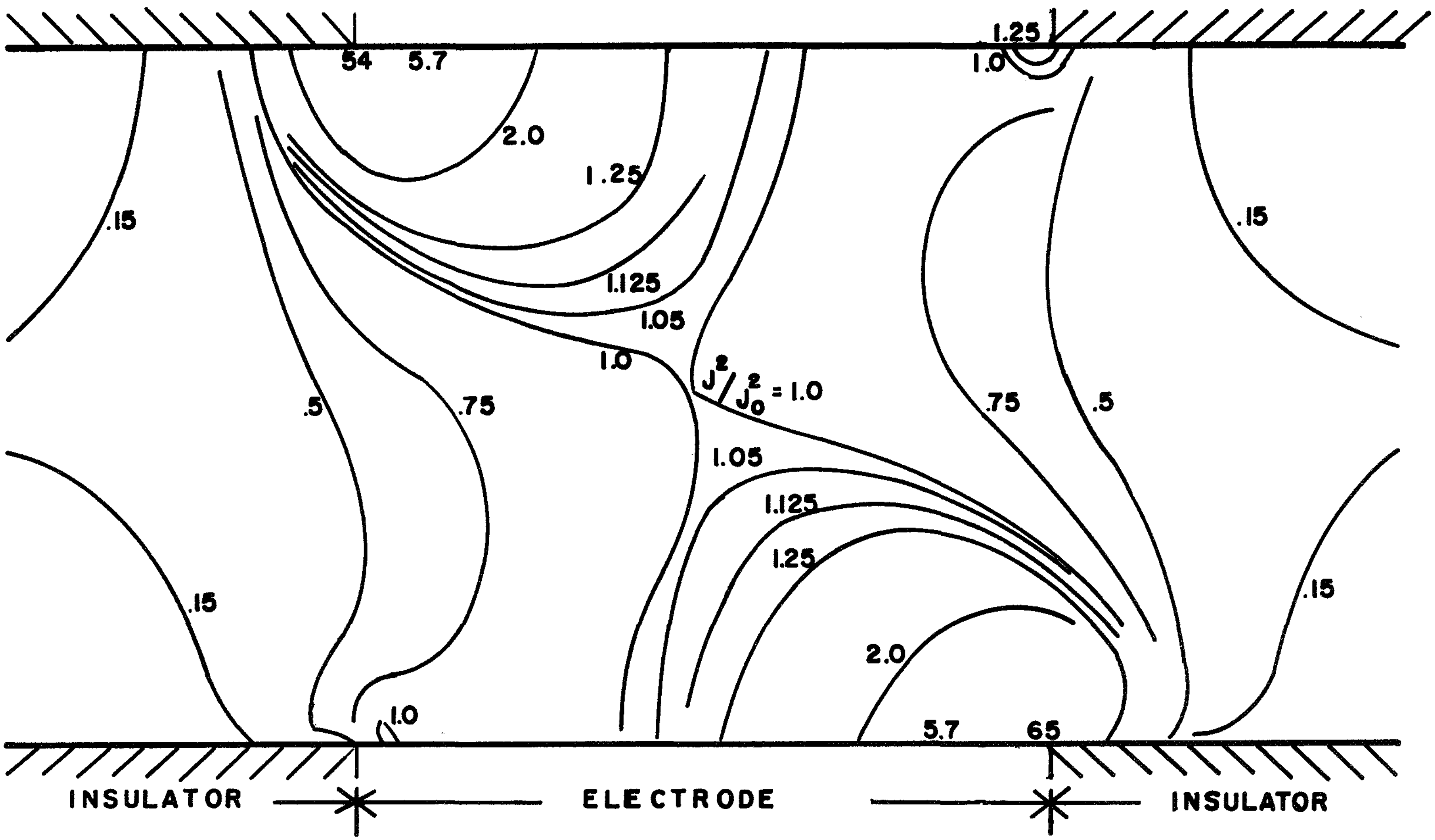

Fig. 14 Calculation of Electron Temperature Distribution by Using Simple Energy Balance Equation and the Obtained Current Distribution at $\beta = 1.0$.

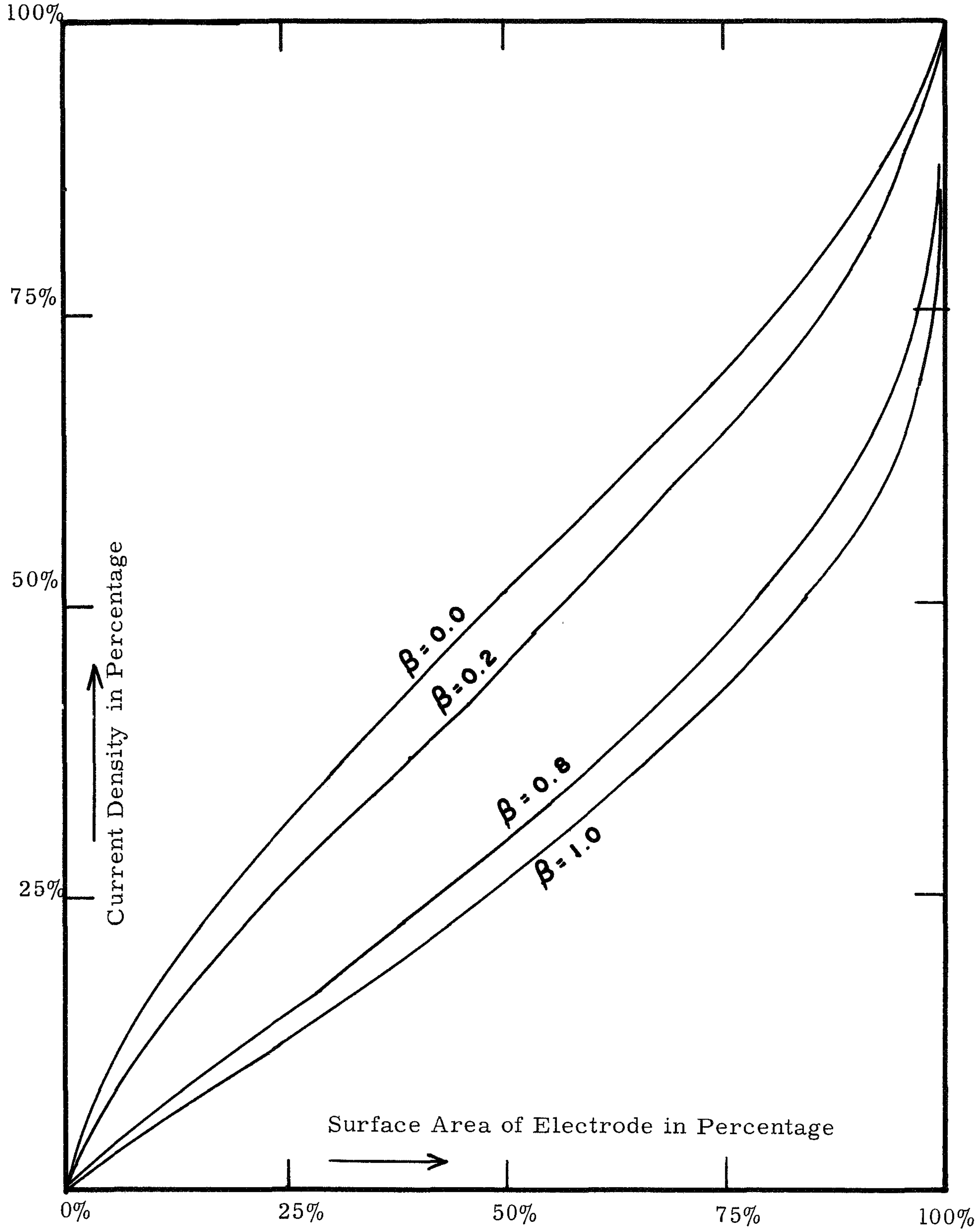

Fig. 15 Current Distribution Along the Electrode Surface