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THE EFFECT OF RECEIVER BANDWIDTH ON LUNAR OCCULTATION OBSERVATIONS— NARROW SYMMETRIC PASSBANDS

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16. Abstract

A general expression is derived for the effect of receiver bandwidth on lunar occultation observations, the phenomenon being considered equivalent to diffraction at a straight edge. The expression is applied to derive the parameters defining "effective beam" due to bandwidth alone for five symmetrical and narrow passbands of common shape. It is shown that the half-power effective beamwidth in seconds of arc (for the mean Moon-Earth distance) varies between 6.2 and 7.2 times the square root of the absolute half-power bandwidth in wavelength expressed in meters. Sensitivity and beam-shape evaluations indicate that in radio-astronomy applications, given a desired resolution, the single-tuned passband is optimal. When the beam-shape is "well behaved" it is unnecessary to preconvolve the theoretical "restoring function" with a gaussian, as is usually done.

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THE EFFECT OF RECEIVER BANDWIDTH ON LUNAR OCCULTATION OBSERVATIONS—NARROW SYMMETRIC PASSBANDS

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INTRODUCTION

In recent years, strip brightness distributions across many radio sources, to very high resolution ($\sim$1 second of arc), have been obtained from observations of their occultation by the Moon. The theory underlying the derivation of the distribution is that first put forward by Scheuer (1962) and applies when the received radiation is purely monochromatic.

The effect of the finite bandwidth of the receiver is important, both from the point of view of effective beam-smearing and from considerations of sensitivity. Von Hoerner (1964) has considered such effects in detail, as produced by computer simulation. Scheuer (1965) has given the expressions for the beam-broadening function $B(\theta)$, where θ is the angular dimension in radians, in two cases: (1) a gaussian passband (in wavelength) and (2) a double-sideband receiver. (He does not give their derivation.) In the first case, he arrives at the very unusual result that the effect of the reception of an appreciable range of wavelengths is to convolve the profile obtained by *restoration*, with a near-gaussian $B(\theta)$ where $B(\theta)$ is dependent only on $\Delta\lambda$, the absolute value of the half-power bandwidth in wavelength, and not on the fractional bandwidth. Sutton (1966)† has briefly derived the function $b(v)$ that is equivalent to $B(\theta)$ in the 'v' or Fresnel integral domain ($v = \theta\sqrt{2D/\lambda}$) and shows the result in the gaussian case to be equivalent to Scheuer's (1965). (For D, see Figure 1; λ is wavelength.) He also gives an expression for $b(v) \equiv B(\theta)$ in the case of a rectangular passband.

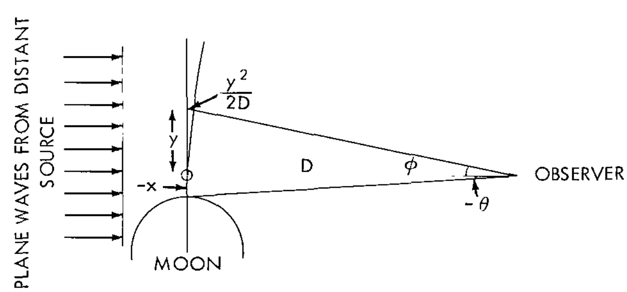

Figure 1—Geometry of a lunar occultation of a distant radio source.

The purpose of this note is to derive an expression for $B(\theta)$—labelled $r(\theta)$ in our analysis—which is quite general, and to apply it to common symmetrical passbands of relatively small fractional width such as those used in

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†The section of this unpublished thesis dealing with bandwidth smoothing was privately communicated by the author.

radio astronomy. The general expression can be used to consider the effects of wide or asymmetric (or both) passbands such as those encountered in optical observations. This will be done in a later paper.

MONOCHROMATIC THEORY

We shall consider Scheuer's theory (1965) for monochromatic diffraction only briefly. He uses x, the distance of a source from the edge of the Moon, as variable; we use θ, the angular distance in radians as observed from the Earth. For the geometry see Figure 1.

The diffraction pattern due to a point source on the Earth's surface is the usual Fresnel diffraction of a straight edge over regions small compared to the Moon's shadow, provided that the lunar mountains are much smaller than the first Fresnel zone (conditions normally satisfied in Earth-based observations).

In general the received power from a point source is proportional to

$$\iint \exp\left[-i\,\frac{\pi}{D\lambda}\left(y^2 + z^2\right)\right] dy\,dz \times \text{complex conjugate} .$$

For small angles, putting $y = \phi D$, $z = \Psi D$, (where D is the earth-moon distance) the power can be expressed as being proportional to

$$\iint \exp\left[-i\,\frac{\pi D}{\lambda}\left(\phi^2 + \Psi^2\right)\right] d\phi\,d\Psi \times \text{complex conjugate} .$$

Under the assumption that the Moon is a straight edge (i.e. the source is much smaller than the moon) the power from a point source

$$p(\theta, \lambda) \propto \int_{-\infty}^{\infty} \exp\left(-i\,\pi b\,\phi^2\right) d\phi \int_{-\theta}^{\infty} \exp\left(i\,\pi b\,\phi^2\right) d\phi , \tag{1}$$

where

$$b = \frac{D}{\lambda} .$$

For a radio source of finite size, the observed power is

$$f(\theta, \lambda) = p(\theta, \lambda) * t(\theta, \lambda) \qquad (* \text{ denotes convolution}) ,$$

where $t(\theta, \lambda)$ is the one-dimensional or strip distribution across the source measured in the direction perpendicular to the Moon's edge. The differentiated occultation curve $g(\theta, \lambda)$, is the true strip distribution $t(\theta, \lambda)$, convolved with $q(\theta, \lambda)$, where

$$q(\theta, \lambda) = p'(\theta, \lambda) ,$$

i.e.,

$$g(\theta, \lambda) = t(\theta, \lambda) * q(\theta, \lambda)$$

By the convolution theorem,

$$G(s, \lambda) = T(s, \lambda)\, Q(s, \lambda) ,$$

where

$$G(s, \lambda) = \int_{-\infty}^{\infty} g(\theta, \lambda) \exp(-i\theta s)\, d\theta$$

$$T(s, \lambda) = \int_{-\infty}^{\infty} t(\theta, \lambda) \exp(-i\theta s)\, d\theta$$

and

$$Q(s, \lambda) = \int_{-\infty}^{\infty} q(\theta, \lambda) \exp(-i\theta s)\, d\theta$$

Note: since we are operating with θ, an angular measure, we use the asymmetric form of the reversibility of the Fourier transform: i.e., the transform of $f(x)$ is defined by

$$F(s) = \int_{-\infty}^{\infty} f(x) \exp(-ixs)\, dx .$$

Generally, it follows from the Fourier Integral Theorem that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \exp(+ixs)\, ds .$$

Restrictions must sometimes be imposed on the generality of this statement, but they do not arise in this paper. Then

$$T(s, \lambda) = \frac{1}{Q(s, \lambda)} G(s, \lambda) ,$$

and

$$t(\theta, \lambda) = \left\{ f^n \text{ whose Fourier transform is } \frac{1}{Q(s, \lambda)} \right\} * g(\theta, \lambda) . \tag{2}$$

Now,

$$Q(s, \lambda) = \int_{-\infty}^{\infty} q(\theta, \lambda) \exp(-i\theta s)\, d\theta ,$$

where $q(\theta, \lambda)$ is given by Equation 1 as

$$q(\theta, \lambda) = \exp\left[i\pi b\theta^2\right] \int_{-\theta}^{\infty} \exp\left[-i\pi b\phi^2\right] d\phi + \exp\left[-i\pi b\theta^2\right] \int_{-\theta}^{\infty} \exp\left[i\pi b\phi^2\right] d\phi .$$

Following Scheuer's treatment (1962) identically, we can show that

$$Q(s, \lambda) = \frac{1}{b} \exp\left[\frac{-is^2}{4\pi b} \operatorname{sgn} s\right] , \tag{3)*$$

where

$$\operatorname{sgn} s = \begin{cases} -1 & \text{for } s < 0 \\ +1 & \text{for } s > 0 . \end{cases}$$

Thus $Q(s, \lambda)$ contains all the Fourier components with equal weight, but with different phases. Now, from Equation 2,

$$t(\theta, \lambda) = \text{Fourier transform of } \frac{1}{Q(-s, \lambda)} * q(\theta, \lambda) \qquad \text{(by the Fourier inversion theorem)}$$

$$= b^2 q(-\theta, \lambda) * g(\theta, \lambda) .$$

*Scheuer's expression here does not have a negative sign in the exponent and corresponds to an inversion of sign in the definition of sgn s. This error is trivial for the monochromatic theory, even when passbands are involved, but may be important in other cases.

The true distribution is recoverable in principle by convolution of the differential of the observed curve with $b^2 q(-\theta, \lambda)$. Scheuer (1962) shows that the differentiation can be avoided (in the interests of signal-to-noise ratio) and establishes the identity

$$t(\theta, \lambda) = -b^2 p''(-\theta, \lambda) * f(\theta, \lambda) \tag{4*}$$

The restoring function $c(\theta) = -b^2 p''(-\theta, \lambda)$ is "badly behaved" but can be made "well-behaved" by convolution with a "well-behaved" function such as a gaussian. The "well-behaved" function then defines the "effective" beam of observation.

We obtain $c(\theta)$ from expansions of the Fresnel integrals $C(v)$ and $S(v)$ (see Von Hoerner, 1964), where v is related to θ and λ by the expression

$$v = \theta(2D/\lambda)^{1/2}$$

and D is the topocentric distance to the Moon.

QUASIMONOCHROMATIC THEORY

At any given wavelength λ,

$$g(\theta, \lambda) = \left[t(\theta, \lambda) * q(\theta, \lambda)\right] M(\lambda) ,$$

where $M(\lambda)$ is the response of the receiver at wavelength λ. Now suppose that the spectral index of the source does not change rapidly from strip to strip. Then, as the wavelength changes, the intensity in any strip changes, in the same ratio as in the other strips. Then

$$t(\theta, \lambda) = \left(\frac{\lambda}{\lambda_0}\right)^{-n} t\left(\theta, \lambda_0\right) ,$$

where $\left(\lambda / \lambda_0\right)^{-n}$ is the spectral variation of the whole source.

We shall assume that the spectrum of the source is flat over the narrow range of frequencies considered. Taking Fourier transforms, we obtain

$$G(s) = T\left(s, \lambda_0\right) \int_{\text{passband}} M(\lambda)\, Q(s, \lambda)\, d\lambda$$

$$= T\left(s, \lambda_0\right) \int_{\text{passband}} M(\lambda) \frac{\lambda}{D} \exp\left(-\frac{is^2}{4\pi} \cdot \frac{\lambda}{D} \cdot \operatorname{sgn} s\right) d\lambda \, . \tag{5}$$

*Scheuer errs in saying $t(\theta, \lambda) = b^2 p''(-\theta, \lambda) * f(\theta, \lambda)$.

(from Equation 3) where

$$\dot{G}(s) = \int_{\text{passband}} G(s, \lambda)\, d\lambda \ .$$

Multiplying and dividing Equation 5 by $Q(s, \lambda_0)$ at some reference frequency λ_0, we obtain

$$\dot{G}(s) = T(s, \lambda_0)\, Q(s, \lambda_0) \int_{\text{passband}} M(\lambda) \frac{\lambda}{\lambda_0} \exp\left[- i \frac{s^2}{4\pi} \left(\frac{\lambda - \lambda_0}{D}\right) \operatorname{sgn} s\right] d\lambda \ .$$

Hence

$$\dot{g}(\theta) = t(\theta, \lambda_0) * q(\theta, \lambda_0) * r(\theta) \ , \tag{6}$$

where

$$r(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(s)\, e^{i\pi s\theta}\, ds \tag{7}$$

and

$$R(s) = \int_{\text{passband}} M(\lambda) \frac{\lambda}{\lambda_0} \exp\left[- i \frac{s^2}{4\pi} \left(\frac{\lambda - \lambda_0}{D}\right) \operatorname{sgn} s\right] d\lambda \ . \tag{8}$$

Now, following Scheuer's method, we can state from Equation 4 that

$$t(\theta, \lambda_0) * r(\theta) = - b^2 p''(-\theta, \lambda_0) * f(\theta) * r(\theta) \ , \tag{9}$$

where $f(\theta)$ is the observed curve. If $r(\theta)$ is "well-behaved" it can be considered to be the "effective beam" caused by finite bandwidth.

EVALUATION OF $r(\theta)$

The following treatment is restricted to passband shapes in common use, namely those that

(a) are symmetric in wavelength about a central wavelength,

(b) have $\lambda/\lambda_0 \sim 1$ over the range of wavelengths received.

Then, putting $\ell = \lambda - \lambda_0$, where λ_0 is the central wavelength, we obtain from Equation 8:

$$R(s) = \int_{-\infty}^{\infty} \dot{m}(\ell) \exp\left(-\frac{i s^2}{4\pi D} \cdot \ell \operatorname{sgn} s\right) d\ell \tag{10}$$

noting that within these restrictions the effect of bandwidth on the resultant beam depends only on ℓ and not on the fractional bandwidth.

In general, receiver passbands are described in terms of frequency response. However, we may assume that for the passbands we consider, conditions (a) and (b) lead to identical expressions when stated in wavelengths, at least up to fractional bandwidths of about 20 percent.

Gaussian Passbands

We use the probability ordinate definition of a Gaussian, i.e.

$$\dot{m}(\ell) = \frac{1}{\sigma_g \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ell}{\sigma_g}\right)^2\right] ,$$

which has an area of unity and half-power bandwidth $(8\ln 2)^{1/2}\,\sigma_g$. Then

$$R_g(s) = \frac{1}{\sigma_g\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{\ell^2}{\sigma_g^2}\right) \exp(-i\mu\ell)\, d\ell ,$$

where $\mu = s^2/4\pi D \operatorname{sgn} s$

$$= \frac{1}{\sigma_g\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\, 2\pi\, \frac{\ell^2}{2\pi\sigma_g^2}\right) \exp(-i\mu\ell)\, d\ell .$$

Let us put

$$m = \ell\Big/\left(\sigma_g \sqrt{2\pi}\right) .$$

Then

$$R_g(s) = \int_{-\infty}^{\infty} \exp\left(-\pi m^2\right) \exp\left(-i\mu\sigma_g \sqrt{2\pi} \cdot m\right) dm .$$

(We have here changed the variable so that the passband is described by a gaussian of unit area and unit central ordinate.) Then

$$R_g(s) = \int_{-\infty}^{\infty} \exp\left[-\pi\left(m^2 + i\mu\sigma_g\sqrt{2\pi}\cdot m\right)\right] dm$$

$$= \exp\left(\frac{-\pi\mu^2\sigma_g^2}{2\pi}\right)\int_{-\infty}^{\infty} \exp\left[-\pi\left(m + \frac{i\mu\sigma_g\sqrt{2\pi}}{2\pi}\right)^2\right] d\left(m + \frac{i\mu\sigma_g\sqrt{2\pi}}{2\pi}\right)$$

$$= \exp\left[\frac{-\mu^2\sigma_g^2}{2}\right].$$

Substituting for μ and putting the half-width in wavelength

$$\Delta\ell_g = (8\ln 2)^{1/2}\sigma_g ,$$

we have

$$R_g(s) = \exp\left[-\left(\Delta\ell_g/D\right)^2 s^4/256\pi^2 \ln 2\right].$$

Thus the effective beam

$$r_g(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_g(s)\, e^{i\theta s}\, ds .$$

Hence

$$r_g(\theta) = \frac{1}{\pi}\int_0^{\infty} \exp\left[-\left(\Delta\ell_g/D\right)^2 s^4/256\pi^2 \ln 2\right] \cos(s\theta)\, ds$$

since $R_g(s)$ is symmetric and even. This is identical to the solution obtained by Scheuer (1965), except that there is a misprint; for the coefficient of the exponent he has

$$\left[-\left(\Delta\ell_g/D\right)^4/256\pi^2 \ln 2\right].$$

We may write, as Scheuer (1965) does

$$r_g(\theta) = \frac{1}{\pi}\int_0^{\infty} \exp\left[-\beta s^4\right] \cos(s\theta)\, ds ,$$

where

$$\beta = \left(\Delta\ell_g/D\right)^2 \Big/ 256\pi^2 \ln 2 \ .$$

Then, putting

$$t = \beta^{1/4} s \ , \qquad dt = \beta^{1/4} ds \ ,$$

we have

$$\beta^{1/4}\, r_g(\theta) = \frac{1}{\pi} \int_0^{\infty} \exp\left(-t^4\right) \cos\left(t \frac{\theta}{\beta^{1/4}}\right) dt$$

and the integral can be computed for various values of $\theta/\beta^{1/4}$ as shown by Scheuer (1965).

For the sake of uniformity with equivalent expressions for other bandshapes, we put instead $\gamma_g = \Delta\ell_g / 8\pi D$, an expression that occurs for all the passbands considered. Then putting

$$\omega = \gamma_g^{1/2} s \ ,$$

$$d\omega = \gamma_g^{1/2} ds \ ,$$

we have

$$\gamma_g^{1/2}\, r_g(\theta)$$

$$= \frac{1}{\pi} \int_0^{\infty} \exp\left(-0.7752\omega\right)^4 \cos\left(\omega \frac{\theta}{\gamma_g^{1/2}}\right) d\omega \ .$$

This integral was computed for the limits 0 to 10.0 of ω, taking values of ω at intervals of 10.0/64.0 for $\theta/\gamma_g^{1/2}$ in steps of 0.1 and is shown by the crosses in Figure 2. The integral converges quite rapidly and yields an exact

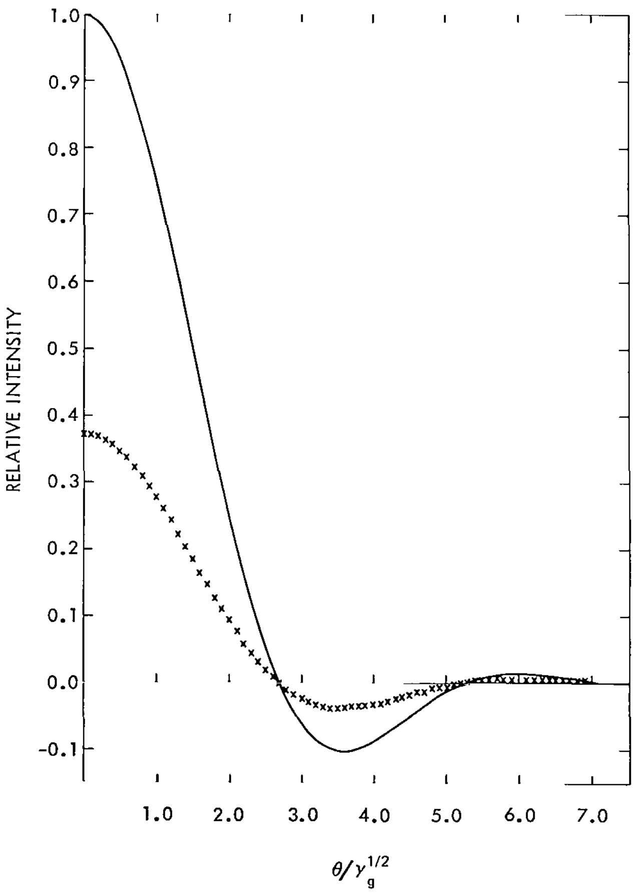

Figure 2—Relative intensity versus $\theta/\gamma_g^{1/2}$ for gaussian passband.

value for

$$\int_0^{\infty} \exp\left(-0.7752\omega^4\right) d\omega$$

at $\theta/\gamma_g^{1/2} = 0$ as obtained analytically. Figure 2, and Figures 4-7 (discussed later on), show $\gamma^{1/2}\, r(\theta)$ plotted against $\theta/\gamma^{1/2}$, for all five passbands considered. Both the actual curves (marked by crosses) and the responses normalized to unity at $\theta/\gamma^{1/2} = 0$ (full lines) are shown. θ is in radians, and $\gamma = (\Delta\ell/8\pi D)^{1/2}$ where $\Delta\ell$ and D are in meters.

The same limits of integration (0 to 10.0) were taken in the next two passbands considered, both of which represent similar continuous functions.

Now, we find from the curve that $\gamma_g^{1/2}\, r(\theta)$ is a near-gaussian, of full-width at half-maximum given by $\theta/\gamma_g^{1/2} = 2 \times 1.4923$. This corresponds to an effective full-beamwidth at half-maximum (FWHM):

$$\Delta\theta_g = 2 \times 1.4923 \times \sqrt{\Delta\ell_g/8\pi D}$$

radians, where λ and D are in meters. For the mean Moon-Earth distance of 3.794×10^8 m,

$$\Delta\theta_g = 6.23 \sqrt{\Delta\ell_g} \text{ arc sec} .$$

If the actual distance is kD,

$$\Delta\theta_g = 6.23 \sqrt{\Delta\ell_g/k} \text{ arc sec} .$$

This relationship has been tested on a computer. The Fresnel integral response $\dot{p}(v)$ for a point source was evaluated at numerous points along a gaussian passband (width $\Delta\ell_g/\lambda_0 = 0.10$) with appropriate weighting and re-normalized and is illustrated in Figure 3(a) showing that the fringes are smeared out for large v. The curve was then "restored" by direct convolution with $-p''(-v, \lambda_0)$ as shown in Figure 3(b), to give the strip brightness distribution, $r(v)$. By dealing with the theory throughout in v and λ, it can be shown that the half-width of the equivalent function $r(v)$ (Sutton 1966) should be given by

$$0.83\sqrt{\frac{\Delta\ell_g}{\lambda_0}} \text{ arc sec} \left(\cdot\ 0.263\, v, \quad \text{for } \frac{\Delta\ell_g}{\lambda_0} = 0.10 \right) .$$

Figure 3(c) shows the power response $r_g(v)$ (unnormalized) versus v. Within the accuracy with which the curve can be read, it has a half-width, close to the above value, confirming our analytical result.

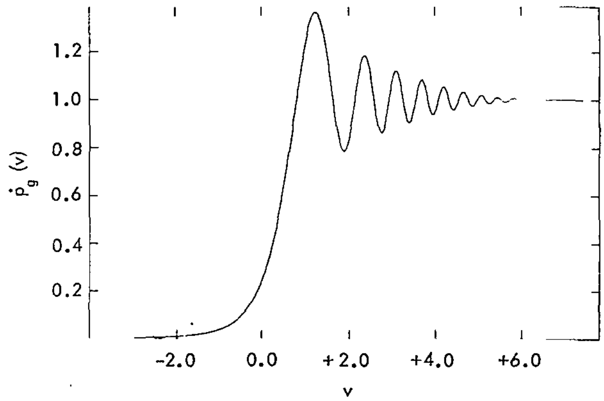

Figure 3(a)—Computed power response, versus v, of a point source due to a gaussian passband of fractional bandwidth 10 percent.

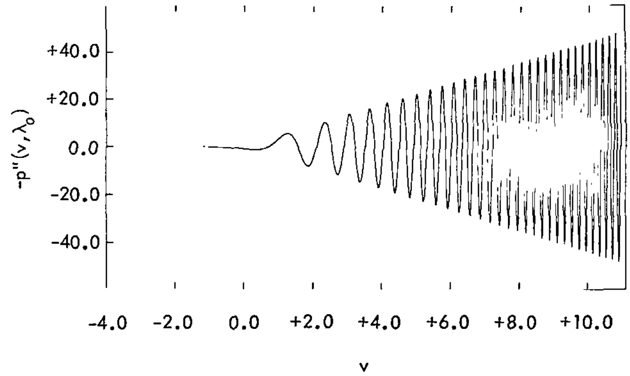

Figure 3(b)—"Restoring function" for power response of Figure 3(a), plotted with sign of v reversed.

It has become customary in 'restoring' strip distributions to convolve $-b^2 p''(-\nu, \lambda_0)$ with a gaussian in ν, $\ell(\nu, \lambda_0)$ (Von Hoerner, 1964), in order to make the $f^n\left[-b^2 p''(-\nu, \lambda_0)\right]$, which is rapidly divergent a "well-behaved" one. From our result it is clear that a passband shape which is "well-behaved" and smooth (such as a gaussian) already performs this function automatically, by convolution with $r(\nu)$. Smoothing by a further gaussian, say $\ell(\nu)$, is quite unnecessary; when this is done, it leads to an effective beam which has the half-width of the function

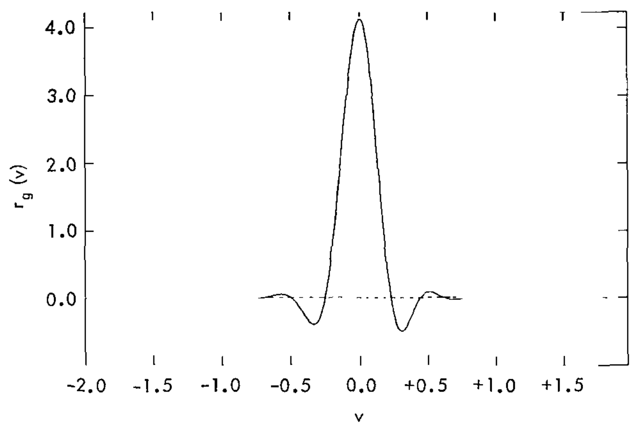

Figure 3(c)—"Restored distribution".

$$h(\nu) = \ell(\nu, \lambda_0) * r(\nu) .$$

Single-Tuned Passbands

For the purposes of this analysis, we define the passband as

$$m(\ell) = \frac{1}{\sigma_s \pi} \cdot \frac{1}{1 + \frac{\ell^2}{\sigma_s^2}} .$$

As in the previous case this function has an area of unity. However, its half-power bandwidth $\Delta\ell_s = 2\sigma_s$. Then

$$R_s(s) = \frac{1}{\sigma_s \pi} \int_{-\infty}^{\infty} \frac{1}{\left(1 + \ell^2/\sigma_s^2\right)} \exp(-i\mu\ell)\, d\ell$$

(where, as before $\mu = s^2/4\pi D \cdot \mathrm{sgn}\, s$)

$$R_s(s) = \frac{1}{\pi\sigma_s} \int_{-\infty}^{\infty} \frac{1}{\left[1 + \pi^2 \left(\ell/\pi\sigma_s\right)^2\right]}\, d\ell \,.$$

Putting $m = \ell/(\pi\sigma_s)$, we obtain

$$R_s(s) = \int_{-\infty}^{\infty} \frac{1}{1 + \pi^2 m^2} \exp\left(-i\mu\sigma_s \cdot \pi m\right) dm$$

$$= \exp - \left(\mu\sigma_s\right) \,.$$

Substituting for μ and putting $\sigma_s = \Delta\ell_s/2$, we have

$$R_s(s) = \exp\left(- \frac{\Delta\ell_s}{8\pi D} \cdot s^2\right) = \exp\left(-\gamma_s s^2\right)$$

The effective beam is given by

$$r_s(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_s(s) \exp^{i\theta s}\, ds \,,$$

i.e.,

$$r_s(\theta) = \frac{1}{\pi} \int_{0}^{\infty} \exp\left(-\gamma_s s^2\right) \cos(s\theta)\, ds \,.$$

This expression for $r_s(\theta)$ represents the inverse Fourier transform of a gaussian; this is also a gaussian and is easily determined:

$$r_s(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\gamma_s s^2\right) \cos(s\theta)\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\gamma_s s^2\right) \cos\left(2\pi s \frac{\theta}{2\pi}\right) ds$$

$$= \frac{1}{2\pi} \frac{\pi^{1/2}}{\gamma_s^{1/2}} \exp\left[-\frac{1}{4}\left(\frac{\theta}{\gamma_s^{1/2}}\right)^2\right] .$$

Thus

$$\gamma_s^{1/2}\, r_s(\theta) = \frac{1}{2\sqrt{\pi}} \exp\left[-\frac{1}{2}\left(\frac{1}{\sqrt{2}} \cdot \frac{\theta}{\gamma_s^{1/2}}\right)^2\right],$$

which has a full width at half-maximum of

$$\frac{\theta}{\gamma_s^{1/2}} = 3.3302 .$$

For purposes of testing the computational program $\theta/\gamma_s^{1/2}$ was obtained from an integration identical to that performed for the gaussian passband and yielded

$$\frac{\theta}{\gamma_s^{1/2}} = 3.3308 .$$

The FWHM in this case is given by

$$\Delta\theta_s = 7.03\sqrt{\Delta\ell_s/k}\ \text{arc sec} ,$$

where $\Delta\ell_s$ is in meters. The computed curve is plotted in Figure 4 as well as its normalized version.

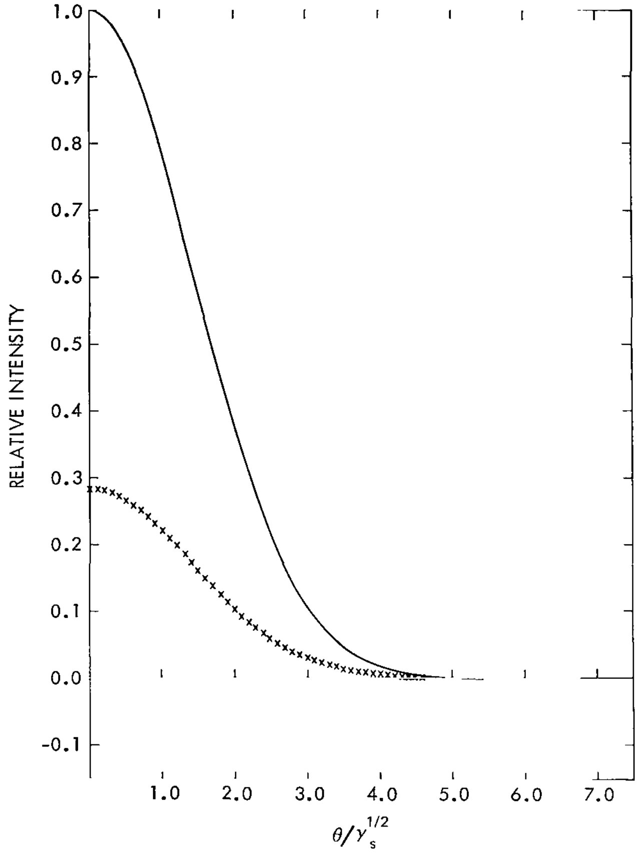

Figure 4—Relative intensity versus $\theta/\gamma^{1/2}$ for single-tuned passband.

The "effective beam" due to bandwidth is so "well-behaved" that there is no reason why restoration with $b^2 p''(-\nu, \lambda_0)$ should not again be attempted directly. In both the gaussian and the single-tuned cases, the lengths of the restoring function and the observed records should be adequate. Sutton (1966) has shown that a restoring function extending to $\pm z$ units of v will enable a narrowest beam $1.2/z$ units of v in width to be obtained.

Negative-Exponential Passbands

Though this is not a passband in common use, it is instructive to calculate its effect on beam width and shape.

Let the passband be expressed as

$$\dot{m}(\ell) = \frac{1}{2\sigma_e} \exp\left(-|\ell/\sigma_e|\right) .$$

Its area is unity and it has a full bandwidth at half-maximum $\Delta\ell_e = 2\ln^2 \sigma_e$. Then

$$R_e(s) = \frac{1}{2\sigma_e}\int_{-\infty}^{\infty} \exp\left(-\left|\frac{\ell}{\sigma_e}\right|\right) \exp(-i\mu\ell)\, d\ell \cdot \left(\mu = \frac{s^2}{4\pi D} \operatorname{sgn} s\right)$$

Proceeding as before and putting $\ell/\sigma_e = m$, we obtain

$$R_e(s) = \frac{1}{2}\int_{-\infty}^{\infty} \exp(-|m|) \exp(-i\mu\sigma_e m)\, dm$$

$$= \int_0^{\infty} \exp(-m) \cos \mu\sigma_e m\, dm$$

$$= \mathrm{Re} \int_0^{\infty} \exp\left[(i\mu\sigma_e - 1)\, m\right] dm$$

$$= \mathrm{Re}\left(\frac{-1}{i\mu\sigma_e - 1}\right)$$

$$= \frac{1}{\mu^2 \sigma_e^2 + 1} \ .$$

Now

$$\mu^2 \sigma_e^2 = \frac{s^4}{16\pi^2 D^2} \cdot \frac{(\Delta\ell_e)^2}{(2\ln 2)^2}$$

$$= \left(\frac{\Delta\ell_e}{8\pi D}\right)^2 \cdot \frac{1}{(\ln 2)^2} \cdot s^4$$

$$= \left[\gamma_e^{1/2}\, s(1.2011)\right]^4 \ .$$

Hence

$$R_e(s) = \frac{1}{1 + \left(1.2011\, \gamma_e^{1/2}\, s\right)^4} \ .$$

Then, as in the Gaussian case, putting $\omega = \gamma_e^{1/2}\, s$, we have

$$\gamma_e^{1/2}\, r_e(\theta) = \frac{1}{\pi}\int_0^{\infty} \frac{1}{1 + (1.2011\omega)^4} \cos\left(\omega \frac{\theta}{\gamma_e^{1/2}}\right) d\omega \ .$$

Figure 5 shows the actual and normalized curves of this integral. Function $\{\gamma_e^{1/2}\, r_e(\theta)\}$ is a near-gaussian closely resembling $\{\gamma_g^{1/2} \times r_g(\theta)\}$ in shape. The FWHM of $\gamma_e^{1/2}\, r_e(\theta)$, however, is broader and is 2×1.7024 units of $\theta/\gamma_e^{1/2}$. This corresponds to a FWHM of $r_e(\theta)$ given by

$$\Delta_e \theta = 7.18\sqrt{\Delta\ell_e / k}\ \text{arc sec}$$

where, again, $\Delta\ell_e$ is in meters.

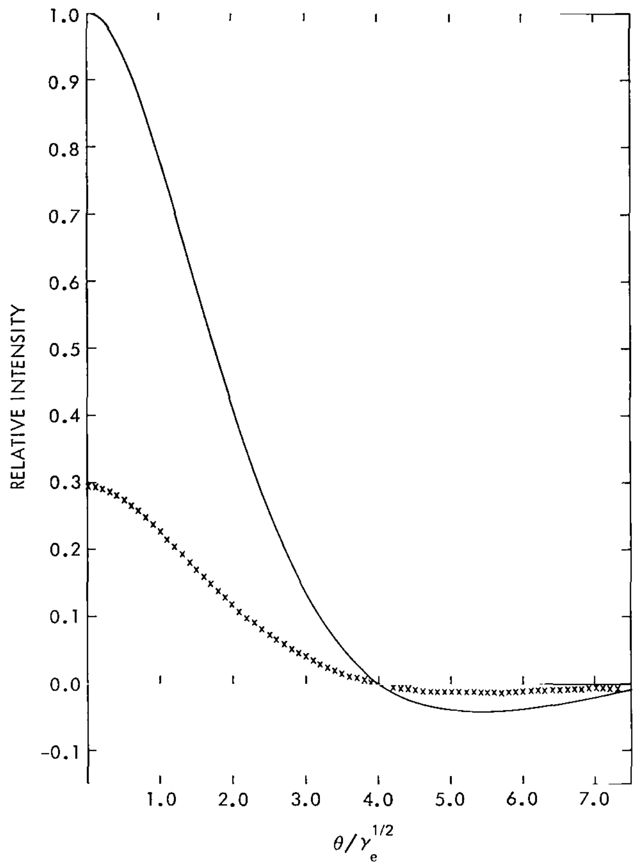

Figure 5—Relative intensity versus $\theta/\gamma^{1/2}$ for negative-exponential passband.

Rectangular Passband

The rectangular passband differs in character from those we have considered insofar that it has sharp discontinuities. Care must therefore be taken in the numerical inverse Fourier transformation of $R(s)$. The rectangular passband is defined as

$$\dot{m}(\ell) = \frac{1}{\sigma_r}\, \Pi\left(\frac{\ell}{\sigma_r}\right)$$

$$\Pi\left(\frac{\ell}{\sigma_r}\right) = \begin{cases} 0 & \ell/\sigma_r > 1/2 \\ 1/2 & \ell/\sigma_r = 1/2 \\ 1 & \ell/\sigma_r < 1/2 \end{cases},$$

whose area is unity and whose full-bandwidth-at-half-maximum $\Delta\ell_r = \sigma_r$. For most practical purposes it is unnecessary to dwell on the behavior of the function at $\ell/\sigma_r = \pm 1/2$, and this neglect is implicit in our treatment. Then

$$R_r(s) = \frac{1}{\sigma_r}\int_{-\infty}^{\infty} \left(\frac{\ell}{\sigma_r}\right) \cdot \exp(-i\ell\mu)\, d\ell \qquad \left(\mu = \frac{s^2}{4\pi D}\, \mathrm{sgn}\, s\right)$$

$$= \int_{-\infty}^{\infty} \Pi(m) \exp\left(-i\mu\sigma_r m\right) sm \qquad \left(\text{putting } m = \frac{\ell}{\sigma_r}\right)$$

$$= \int_{-1/2}^{1/2} \cos\left(\mu\sigma_r m\right) dm$$

$$= \left[\frac{\sin \mu\sigma_r m}{\mu\sigma_r}\right]_{-1/2}^{+1/2}$$

$$= \frac{2}{\mu\sigma_r} \sin \frac{\mu\sigma_r}{2}\ .$$

Substituting for μ and σ_r, we obtain

$$R_r(s) = \frac{8\pi D}{\Delta\ell\, s^2} \sin\left(\frac{\Delta\ell}{8\pi D} s^2\right),$$

$$r_r(\theta) = \frac{1}{\pi}\int_0^\infty \frac{1}{\gamma_r s^2} \sin\left(\gamma_r s^2\right) \cos(s\theta)\, ds .$$

Putting, as before,

$$\omega = \gamma_r^{1/2} s ,$$

we have

$$\gamma_r^{1/2} r_r(\theta) = \frac{1}{\pi}\int_0^\infty \frac{1}{\omega^2} \sin\omega^2 \cos\left(\omega \frac{\theta}{\gamma_r^{1/2}}\right) d\omega .$$

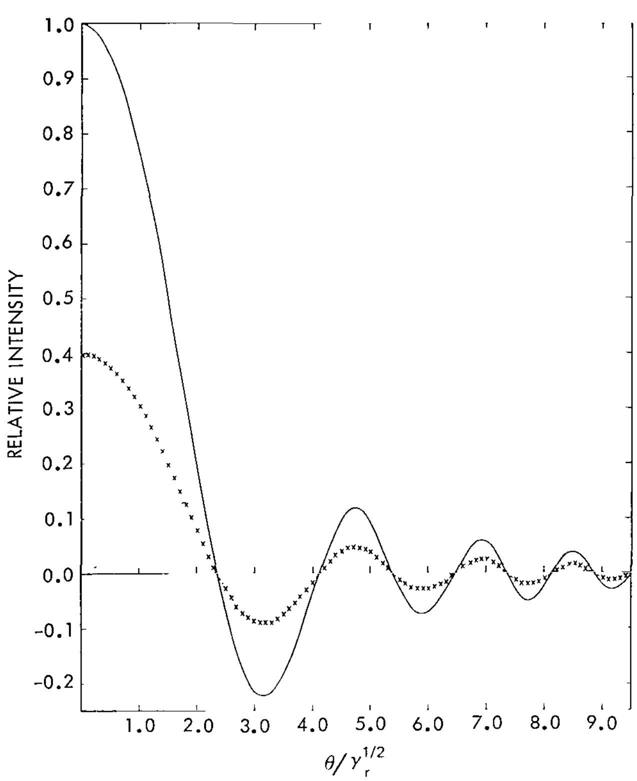

Figure 6—Relative intensity versus $\theta/\gamma^{1/2}$ for rectangular passband.

Problems arise in the numerical computation of this integral. We have taken the limits of integration as $\omega = 0$ and $\omega = \sqrt{9\pi}$. The upper limit of $\sqrt{9\pi}$ is chosen so as to evaluate the integral accurately without "aliasing," given the requirements of adequate sampling of the function $\sin\omega^2/\omega^2$. This upper limit also makes the function terminate at zero at the truncation point. The curve that is shown as $\gamma_r^{1/2} r_r(\theta)$ in Figure 6, is in fact

$$\frac{1}{\pi}\int_0^{\sqrt{9\pi}} \frac{1}{\omega^2} \sin\omega^2 \cos\left(\omega \frac{\theta}{\gamma_r^{1/2}}\right) d\omega$$

and represents

$$\gamma_r^{1/2} r_r(\theta) * \frac{\sqrt{9\pi}}{\pi} \cdot \frac{\sin\sqrt{9\pi}\,\theta/\gamma_r^{1/2}}{\sqrt{9\pi}\,\theta/\gamma_r^{1/2}}$$

The convolving sinc function falls to its first zero at $\theta/\gamma_r^{1/2} = 0.6$, while the full-width at half-maximum of the resulting curve is of the order of 3.0 in units of $\theta/\gamma_r^{1/2}$. We therefore expect the truncation to have negligible effect on the derived beam.

The curve, as would be expected, has large sidelobes. It has a FWHM of $\theta/\gamma_r^{1/2} = 2 \times 1.4895$, yielding a value for the angular FWHM of

$$\Delta\theta_r = 6.29\sqrt{\Delta\ell_r/k}\ \text{arc sec}$$

where $\Delta\ell_r$ is in meters.

In order to reduce the sidelobes, the observed curve would have to be restored by the smoothing of $-b^2 p''(-\theta, \lambda_0)$ with a well-behaved function such as a gaussian, further reducing the angular resolution.

Triangular Passband

We next consider a triangular passband defined by

$$\dot{m}(\ell) = \frac{1}{\sigma_T} \Lambda\left(\frac{\ell}{\sigma_T}\right),$$

where

$$\Lambda\left(\frac{\ell}{\sigma_T}\right) = \begin{cases} 0 & \ell/\sigma_T > 1 \\ (1 - \ell/\sigma_T) & \ell/\sigma_T < 1 \end{cases}.$$

The triangular passband is closely related to the rectangular passband; it results from the convolution of the rectangle function with itself. Hence, we may write

$$\frac{1}{\sigma_T} \cdot \Lambda\left(\frac{\ell}{\sigma_T}\right) = \frac{1}{\sigma_T} \Pi\left(\frac{\ell}{\sigma_T}\right) * \Pi\left(\frac{\ell}{\sigma_T}\right).$$

Function $\dot{m}(\ell)$ has area unity and a full bandwidth at half-maximum $\Delta\ell_T = \sigma_T$.

It follows directly from the convolution theorem that

$$R_T(s) = R_R(s) \cdot R_R(s)$$

(substituting, however, σ_T for σ_R),

$$R_T(s) = \left[\frac{1}{\gamma_T s^2} \sin\gamma_T s^2\right]^2,$$

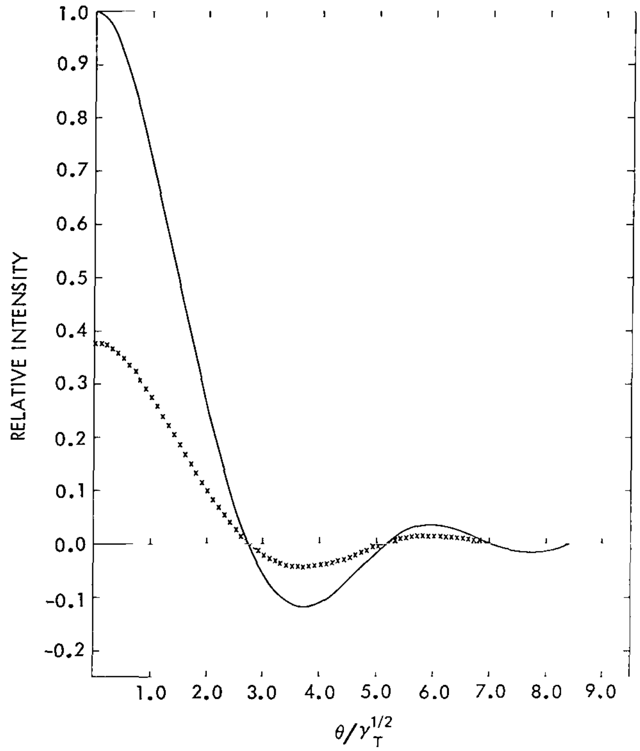

Figure 7—Relative intensity versus $\theta/\gamma^{1/2}$ for triangular passband.

where $\beta = \Delta\ell_T/8\pi D$, and

$$r_T(\theta) = \frac{1}{\pi}\int_0^\infty \left[\frac{1}{\gamma_T s^2}\sin\gamma_T s^2\right]^2 \cos(s\theta)\,ds .$$

Making the usual substitution, $\omega = \gamma_T^{1/2} s$, we have

$$\gamma_T^{1/2}\, r_T(\theta) = \frac{1}{\pi}\int_0^\infty \frac{\sin^2\omega^2}{\omega^2}\cos\left(\omega\frac{\theta}{\gamma_T^{1/2}}\right)d\omega .$$

Integrating for $\omega = 0$ to $\sqrt{9\pi}$ (the effects of truncation are thus the same as in the rectangular case) we obtain the curves shown in Figure 7.

The FWHM corresponds to a value of $\theta/\gamma_T^{1/2} = 2 \times 1.48$, and the full-width to half-maximum of the effective beam is given by

$$\Delta\theta_T = 6.25\sqrt{\Delta\ell_T/k}\ \text{arc sec} .$$

The curve has smaller sidelobes than the rectangular passband.

SENSITIVITY

The limiting sensitivity of a radio-astronomy receiver with a square-law detector in responding to a noise-signal is set by the power response, in frequency, of the reception filter $\dot{m}(f)$ and the smoothing low-pass output filter $s(f)$. As derived by Bracewell (1965, pp. 337-339),

$$\frac{\text{rms fluctuation}}{\text{mean}} = \frac{1}{\sqrt{\tau\beta_f}} ,$$

where

$$\beta_f = 1/2 \cdot W_{\dot{m}\times\dot{m}}\ (\times \text{ denoting correlation}) ,$$

$\tau = 1/W_s$, $W_{\dot{m}\times\dot{m}}$ is the auto-correlation width of $\dot{m}(f)$, including the negative frequency part of the response, and W_s is the equivalent width of the power-transfer function of the smoothing filter.

Without considering details of the smoothing filter, we can say that, given a particular smoothing filter,

$$\frac{\text{rms fluctuation}}{\text{mean}} \propto \frac{1}{\left[(1/2)\, W_{\dot{m}\times\dot{m}}\right]^{1/2}},$$

$$W_{\dot{m}\times\dot{m}} = \frac{\int_{-\infty}^{\infty} \dot{m}(f) \times \dot{m}(f)\, df}{\dot{m}(f) \times \dot{m}(f)\big|_0} .$$

The passbands we have considered are all narrow and symmetrical in wavelength. They were made sufficiently narrow (< 20 percent) to make λ/λ_0 in Equation 8 equal to 1. In such cases the passband expressed in wavelength is identical in form with the passband expressed in frequency. This is true when $f^2 \simeq f f_o{}^2$. It may be that this identity does not hold at high fractional widths (> 20 percent), the symmetry of the bandpass in frequency being impaired. However, the equivalent width of the frequency passband as well as its autocorrelation function would even then be very nearly proportional to the corresponding expressions of the wavelength passband, and we can say that

$$\frac{\text{rms fluctuation}}{\text{mean}} \propto \frac{1}{\left[(1/2)\, W_{\ell_{\dot{m}\times\dot{m}}}\right]^{1/2}},$$

where $W_{\ell_{\dot{m}\times\dot{m}}}$ is the equivalent width of $\dot{m}(\ell) \times \dot{m}(\ell)$. Since $\dot{m}(\ell)$ in all the cases we have considered is even, the autocorrelation is equivalent to self-convolution, and $\rho \propto \left(1/2\, W_{\ell_{\dot{m}*\dot{m}}}\right)^{1/2}$. The larger $\left[(1/2)\, W_{\ell_{\dot{m}*\dot{m}}}\right]^{1/2}$ is, the greater is the sensitivity.

Table 1 presents the results for the five passbands considered (for $k = 1$). We find that the beams obtained all have widths lying between 6.2 and 7.2 times $\sqrt{\Delta\ell}$ seconds of arc, where $\Delta\ell$ is expressed in meters, but have different shapes.

Thus from the point of view of resolution, it is the half-width to full maximum of the passband that determines the resolution in angle.

However, $1/\left[(1/2)\, W_{\ell_{\dot{m}*\dot{m}}}\right]^{1/2}$, which is a measure of the sensitivity, is differingly dependent on the shape of the passband. Thus, if $\Delta\ell$ is equalized for all the passbands (defining a specified half-power beam-width due to broadening), the single-tuned passband seems to offer the best sensitivity; also, it leads to a gaussian beam with low sidelobes. At first this result is surprising, but not when we consider how finite passbands affect the diffraction pattern. All the other passbands considered have more weighting for the responses at wavelengths away from the central wavelength than the single-turned passband, which is narrow in the central part of its power-response but has a broad, low amplitude response in the outer parts. It is interesting to note that the rectangular passband is the most undesirable from all points of view—width, shape, and

Table 1

Summarized Data for Five Passbands.

$\dot{m}(\ell)$	Halfwidth $\Delta\ell$	Beamwidth $\Delta\theta$ (arc sec)	$1/2\, W_{\ell_{\dot{m}*\dot{m}}}$	$1/2\, W_{\ell_{\dot{m}*\dot{m}}}$	$1/2\, W'_{\ell_{\dot{m}*\dot{m}}}$	n
$\sigma_g \frac{1}{\sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}\left(\frac{\ell}{\sigma_g}\right)^2\right]$	$(8 \ln 2)^{1/2}\, \sigma_g$	$6.23\sqrt{\Delta\ell_g}$	$2\pi^{1/2}\,\sigma_g$	$1.5053\,\Delta\ell_g$	$1.5344\,\Delta\ell_g'$	1.24
$\frac{1}{\sigma_s \pi} \cdot \frac{1}{1+\ell^2 \sigma_s^2}$	$2\sigma_s$	$7.03\sqrt{\Delta\ell_s}$	$2\pi \cdot \sigma_s$	$3.1416\,\Delta\ell_s$	$2.5150\,\Delta\ell_s'$	1.59
$\frac{1}{\sigma_e} \exp\left(-\left\lvert\frac{\ell}{\sigma_e}\right\rvert\right)$	$2 \ln 2\sigma_e$	$7.18\sqrt{\Delta\ell_e}$	$4\sigma_e$	$2.8854\,\Delta\ell_e$	$2.2144\,\Delta\ell_e'$	1.49
$\frac{1}{\sigma_r} \Pi\left(\frac{\ell}{\sigma_r}\right)$	σ_r	$6.29\sqrt{\Delta\ell_r}$	σ_r	$1.0000\,\Delta\ell_r$	$1.0000\,\Delta\ell_r'$	1.00
$\frac{1}{\sigma_T} \Lambda\left(\frac{\ell}{\sigma_T}\right)$	σ_T	$6.25\sqrt{\Delta\ell_T}$	$1.5\sigma_T$	$1.5000\,\Delta\ell_T$	$1.5193\,\Delta\ell_T'$	1.23

sensitivity. As noted earlier, it leads to a beam of about the same width as that due to a gaussian, but still needs further smoothing without any gain in sensitivity.

The relative sensitivity can be normalized, relating all widths to that of the rectangular passband. The value of $\Delta\ell$ for any passband can be changed to a new value $\Delta\ell'$, so that its beamwidth is given by $6.29\sqrt{\Delta\ell_r}$ arc sec. The resulting measures $1/2\, W'_{\ell_{\dot{m}*\dot{m}}}$ are shown in the table as well as the ratio

$$n = \sqrt{\frac{\frac{1}{2} W'_{\ell_{\dot{m}*\dot{m}}}}{\frac{1}{2} W_{\ell_{\dot{m}*\dot{m}}} \text{ (for the rectangle)}}},$$

which is a direct measure of the sensitivity in relation to that of the rectangular passband.

Though the output time constant is ignored in this treatment, it can be stated that it must be $\tau \leq 2.0\, \Delta\theta$ sec, where $\Delta\theta$ is the beamwidth in radians, this value being appropriate for a central occultation.

CONCLUSIONS

For passbands symmetrical in wavelength whose fractional bandwidth is less than 20 percent:

1. The resultant "effective beam" depends only on $\sqrt{\Delta\ell}$, where $\Delta\ell$ is the half-power bandwidth.

2. This result leads to an increase in attainable sensitivity at shorter wavelengths (or higher frequencies).

3. Of the passbands considered, the single-tuned circuit gives maximum sensitivity for a given angular resolution, while all the others except the rectangle are almost as suitable in other respects.

4. "Restoration" of observations can be undertaken by direct convolution with $-b^2 p''(-\theta, \lambda_0)$ of appropriate length, in all cases except the rectangle, without further smoothing.

No comparison is made here with Von Hoerner (1964) for the sensitivity and resolution due to a gaussian and rectangular passband, as his expressions do not lend themselves to easy comparison.

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National Aeronautics and Space Administration
Greenbelt, Maryland, October 1969
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