X-643-70-120
PREPRINT

NASA TM X- 63884

RELATIVE MOTION OF TWO PARTICLES IN ELLIPTIC ORBITS

E. R. LANCASTER

APRIL 1970

GSFC

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

X-643-70-120

PREPRINT

RELATIVE MOTION OF TWO PARTICLES IN ELLIPTIC ORBITS

E. R. Lancaster

April 1970

Goddard Space Flight Center

Greenbelt, Maryland

PRECEDING PAGE BLANK NOT FILMED.

CONTENTS

RELATIVE MOTION OF TWO PARTICLES IN ELLIPTIC ORBITS

E. R. Lancaster*

Goddard Space Flight Center

Greenbelt, Maryland

NOMENCLATURE

$\vec{r}$ = position vector at time t, $r = |\vec{r}|$

$\vec{v}$ = velocity vector at time t, $v = |\vec{v}|$

E = true anomaly at time t

a = semimajor axis, $b = 1/a$, $c = a^{1/2}$

$k^2 = \mu$ = gravitational constant

$\vec{r} \cdot \vec{v}$ = scalar product of $\vec{r}$ and $\vec{v}$

Over the past decade a number of papers[1-6] have presented approximate solutions to the problem of the relative motion of two particles in elliptic orbits in an inverse-square central force field. These papers have assumed that the relative position and velocity vectors are quite small compared to the position and velocity vectors of the particles, and several have further assumed one of the particles to be in a circular[1,2] or nearly circular[3,4] orbit. We derive below an exact solution to this problem, subject to no restrictions. The results

* Aerospace Engineer

have applications to problems of rendezvous, nonlinear error analysis, station keeping, targeting, surveillance, and satellite clustering.

SOLUTION

Kepler's equation for elliptic motion and the updating equations for position and velocity can be written in the form[7]

$$T = C + A\,(1 - \cos C) - B \sin C \tag{1}$$

$$\vec{r} = [1 - H\,(1 - \cos C)]\,\vec{r}_0 + [D\,(1 - \cos C) + N \sin C]\,\vec{v}_0 \tag{2}$$

$$\vec{v} = -\,(P \sin C)\,\vec{r}_0 + [1 - S\,(1 - \cos C)]\,\vec{v}_0 \tag{3}$$

where

$$A = \vec{r}_0 \cdot \vec{v}_0 / k\,c, \quad B = 1 - r_0\,b, \quad T = k\,t\,b/c$$

$$H = a/r_0, \quad D = a\,(\vec{r}_0 \cdot \vec{r}_0)/\mu, \quad N = r_0\,c/k$$

$$P = k\,c/\,r\,r_0, \quad S = a/r, \quad C = E - E_0$$

and a zero subscript indicates the value at time 0.

Let subscript 1 on a symbol designate the value of that symbol for particle 1 in orbit 1 and subscript 2 the value for particle 2 in orbit 2. We define

$$\gamma = C_2 - C_1, \quad \vec{\epsilon} = \vec{r}_2 - \vec{r}_1, \quad \vec{\lambda} = \vec{v}_2 - \vec{v}_1$$

$$\tau = T_2 - T_1, \quad \alpha = A_2 - A_1, \quad \beta = B_2 - B_1, \quad \eta = H_2 - H_1$$

$$\delta = D_2 - D_1, \quad \nu = N_2 - N_1, \quad \rho = P_2 - P_1, \quad \sigma = S_2 - S_1$$

If we place a subscript 1 on all symbols in (1), (2), and (3) and subtract the resulting equations from the set with subscripts 2 on all symbols, we obtain

$$T' = \gamma + A' (1 - \cos \gamma) - B' \sin \gamma \tag{4}$$

$$\begin{aligned} \vec{\epsilon} &= (1 - H_1 F)\, \vec{\epsilon}_0 + (D_1 F + N_1 G)\, \vec{\lambda}_0 \\ &\quad - (H_2 Q + \eta F)\, \vec{r}_{20} + (D_2 Q + \delta F + N_2 R + \nu G)\, \vec{v}_{20} \end{aligned} \tag{5}$$

$$\begin{aligned} \vec{\lambda} &= - P_1 G\, \vec{\epsilon}_0 + (1 - S_1 F)\, \vec{\lambda}_0 \\ &\quad - (P_2 R + \rho G)\, \vec{r}_{20} - (S_2 Q + \sigma F)\, \vec{v}_{20} \end{aligned} \tag{6}$$

where we have defined

$$F = 1 - \cos C_1, \quad G = \sin C_1 \tag{7}$$

$$T' = \tau + \beta G - \alpha F \tag{8}$$

$$A' = A_2 \cos C_1 + B_2 \sin C_1 \tag{9}$$

$$B' = B_2 \cos C_1 - A_2 \sin C_1 \tag{10}$$

$$Q = \cos C_1 (1 - \cos \gamma) + \sin C_1 \sin \gamma \tag{11}$$

$$R = \cos C_1 \sin \gamma - \sin C_1 (1 - \cos \gamma) \tag{12}$$

To obtain equations for $\alpha, \beta, \tau, \eta, \delta, \nu, \rho$, and σ which do not suffer a loss of significant digits due to the subtraction of nearly equal numbers, we proceed as follows.

$$k\, c\, A = \vec{r}_0 \cdot \vec{v}_0$$

$$k\, (c_2 A_2 - c_1 A_1) = \vec{r}_{20} \cdot \vec{v}_{20} - \vec{r}_{10} \cdot \vec{v}_{10}$$

$$k\, [c_2 (A_2 - A_1) + A_1 (c_2 - c_1)] = \vec{r}_{20} \cdot \vec{v}_{20} - \vec{r}_{10} \cdot \vec{v}_{10}$$

The last equation can be solved for α. Equations for the other quantities can be obtained in a similar manner. The full set follows.

$$k\, c_2\, \alpha = \vec{r}_{20} \cdot \vec{v}_{20} - \vec{r}_{10} \cdot \vec{v}_{10} - k\, A_1 (c_2 - c_1) \tag{13}$$

$$\beta = r_{10} (b_1 - b_2) - b_2 (r_{20} - r_{10}) \tag{14}$$

$$c_2 \tau = -k\, t\, (b_1 - b_2) - T_1 (c_2 - c_1) \tag{15}$$

$$r_{10}\, \eta = a_2 - a_1 - H_2 (r_{20} - r_{10}) \tag{16}$$

$$\mu\, \delta = a_2 (\vec{r}_{20} \cdot \vec{v}_{20} - \vec{r}_{10} \cdot \vec{v}_{10}) + (a_2 - a_1)\, \vec{r}_{10} \cdot \vec{v}_{10} \tag{17}$$

$$k\, \nu = c_2 (r_{20} - r_{10}) + r_{10} (c_2 - c_1) \tag{18}$$

$$r_{10}\, r_1\, \rho = k (c_2 - c_1) - P_2 [r_2 (r_{20} - r_{10}) + r_{10} (r_2 - r_1)] \tag{19}$$

$$r_1\, \sigma = a_2 - a_1 - S_2 (r_2 - r_1) \tag{20}$$

$$\vec{r}_{20} \cdot \vec{v}_{20} - \vec{r}_{10} \cdot \vec{v}_{10} = \vec{\lambda}_0 \cdot \vec{r}_{10} + \vec{\epsilon}_0 \cdot \vec{v}_{20} \tag{21}$$

$$r_{20} - r_{10} = \vec{\epsilon}_0 \cdot (\vec{r}_{10} + \vec{r}_{20}) / (r_{10} + r_{20}) \tag{22}$$

$$r_2 - r_1 = \vec{\epsilon} \cdot (\vec{r}_1 + \vec{r}_2) / (r_1 + r_2) \tag{23}$$

$$c_2 - c_1 = (a_2 - a_1) / (c_1 + c_2) \tag{24}$$

$$a_2 - a_1 = a_1\, a_2 (b_1 - b_2) \tag{25}$$

$$b_1 - b_2 = 2 (r_{20} - r_{10}) / r_{10}\, r_{20} + \vec{\lambda}_0 \cdot (\vec{v}_{10} + \vec{v}_{20}) / \mu \tag{26}$$

In the derivation of (26) use was made of

$$b = 2/r - v^2/\mu. \tag{27}$$

SUMMARY

Given $\vec{r}_{10}$, $\vec{v}_{10}$, $\vec{\epsilon}_0$, $\vec{\lambda}_0$ at time 0, the steps for finding $\vec{\epsilon}$ and $\vec{\lambda}$ at time t follow.

1. Compute $\vec{r}_{20} = \vec{r}_{10} + \vec{\epsilon}_0$, $\vec{v}_{20} = \vec{v}_{10} + \vec{\lambda}_0$, $r_{10} = (\vec{r}_{10} \cdot \vec{r}_{10})^{1/2}$, $v_{10}^2 = \vec{v}_{10} \cdot \vec{v}_{10}$.
2. Compute A_1, B_1, T_1, H_1, D_1, N_1 (equations after (3)), a_1 and b_1 from (27).
3. Solve Kepler's equation (1) for C_1. If C_1 is given rather than t, (1) is used to compute T_1.
4. Compute $\vec{r}_1$ from (2) and $r_1 = (\vec{r}_1 \cdot \vec{r}_1)^{1/2}$.
5. Compute P_1, S_1, A_2, B_2, H_2, D_2, N_2 (equations after (3)), a_2 and b_2 from (27).
6. Compute in order $\vec{r}_{20} \cdot \vec{v}_{20} - \vec{r}_{10} \cdot \vec{v}_{10}$, $r_{20} - r_{10}$, $b_1 - b_2$, $a_2 - a_1$, $c_2 - c_1$ from (21), (22), (26), (25), (24).
7. Compute α, β, τ, η, δ, ν from (13), (14), (15), (16), (17), (18).
8. Compute F, G, T', A', B' from (7) (8), (9), (10).
9. Solve (4) for γ.
10. Compute Q and R from (11) and (12).
11. Compute $\vec{\epsilon}$ from (5).
12. Compute $\vec{r}_2 = \vec{r}_1 + \vec{\epsilon}$, $r_2 = (\vec{r}_2 \cdot \vec{r}_2)^{1/2}$.

13. Compute P_2 and S_2 (equations after (3)).

14. Compute ρ and σ from (23), (19), (20).

15. Compute $\vec{\lambda}$ from (6).

Since γ will usually be small, $1 - \cos\gamma$ should be replaced by $2\sin^2(\gamma/2)$ for computational purposes.

NUMERICAL EXAMPLE

We assume the two particles to be in coplanar circular orbits with units chosen such that $a_1^3 = \mu$. The initial conditions are

$$x_{10} = 1, \quad y_{10} = 0, \quad \dot{x}_{10} = 0, \quad \dot{y}_{10} = 1$$

$$\epsilon_{x0} = 0.001, \quad \epsilon_{y0} = 0, \quad \lambda_{x0} = 0, \quad \lambda_{y0} = -.0004996253122$$

If we let $t = \pi/4$, solve for $\vec{r}_1, \vec{r}_2, \vec{v}_1, \vec{v}_2$ at time t and compute $\vec{\epsilon}$ and $\vec{\lambda}$ from the differences $\vec{r}_2 - \vec{r}_1$ and $\vec{v}_2 - \vec{v}_1$, we obtain

$$\epsilon_x = .0015394491, \quad \epsilon_y = -.0001262154,$$

$$\lambda_x = .0011853623, \quad \lambda_y = .0004778069.$$

Using the formulas developed in this paper we obtain

$\epsilon_x = .001539449086, \quad \epsilon_y = -.0001262154558,$

$\lambda_x = .001185362260, \quad \lambda_y = .0004778069038.$

A ten-digit calculator was used for these calculations.

REFERENCES

1. Clohessy, W. H. and Wiltshire, R. S., "Terminal guidance system for satellite rendezvous," J. Aerospace Sci. 27, 653-658 and 674 (1960).

2. London, H. S., "Second approximation to the solution of rendezvous equations," AIAA J. 1, 1691-1693 (1963).

3. de Vries, J. P., "Elliptic elements in terms of small increments of position and velocity components," AIAA J. 1, 2626-2629 (1963).

4. Anthony, M. L. and Sasaki, F. T., "Rendezvous problem for nearly circular orbits," AIAA J. 3, 1666-1673 (1965).

5. Tschauner, J. and Hempel, P., "Rendezvous zu einem in elliptischer Bahn umlaufenden Ziel," Astronaut. Acta 11, 104-109 (1965).

6. Euler, E. A. and Shulman, Y., "Second-Order Solution to the Elliptical Rendezvous Problem," AIAA J. 5, 1033-1035 (1967).

7. Battin, R. H., Astronautical Guidance (McGraw-Hill, New York, 1964), p. 46-47.