TIME DEPENDENT MONOENERGETIC NEUTRON TRANSPORT IN A FINITE SLAB WITH INFINITE REFLECTORS

By

Perry Allan Newman

Thesis submitted to the Graduate Faculty of the

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ABSTRACT

The initial-value transport problem of monoenergetic neutrons migrating in a thin slab surrounded by infinitely thick reflectors is solved using the normal-mode expansion technique of Case. The results obtained indicate that the reflector may give rise to a branch-cut integral term typical of a semi-infinite medium while the central slab may contribute a summation over discrete residue terms. Exact expressions are obtained for these discrete time eigenvalues and numerical results are presented showing the behavior of real time eigenvalues as a function of the material properties of the slab and reflector. These eigenvalues are finite in number and all of them may disappear into the branch cut or continuum as the material properties are varied; such disappearing eigenvalues correspond to exponentially time-decaying modes. The two largest eigenvalues can be compared with critical dimensions of slabs and spheres and it is shown that the numerical values agree with criticality results of others. In the limit of purely absorbing reflectors or a bare slab, the present solution has the properties which have been previously reported by others who used the Lehner-Wing technique to solve corresponding problems.

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Finally, the author takes this opportunity to dedicate this thesis to his most patient wife and family.

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I. INTRODUCTION

A few time-dependent, monoenergetic neutron transport problems in plane geometry have been solved recently by applying Case's normal-mode expansion technique (refs. 5, 7) to the equation which results when the time dependence has been removed by a Laplace transformation. In these time-dependent solutions, contributions due to various parts of the spectrum of the transport operator have been indicated by suitably deforming the integration contour of the inverse Laplace transformation. This approach was used by Bowden (refs. 1, 4) for his analysis of time-dependent, one-speed neutron transport in a bare slab of finite thickness with isotropic scattering, a problem which had been treated extensively by Lehner and Wing (refs. 16, 17). Another successful application of this technique was made by Kuščer and Zweifel (ref. 14) to the time-dependent, one-speed albedo problem for a semi-infinite medium which scatters neutrons isotropically. For these two applications, the time-dependent solution involves only a single material medium. In each of these problems, construction of Case's normal-mode expansion in the transform plane depends upon the two material properties which characterize a single uniform medium with isotropic scattering: the total macroscopic cross section, σ, and the average number of secondaries per collision, c. For time-dependent problems in which more than one medium is involved, the transform plane must be taken as the superposition of "single-medium" planes, one for each medium. The situation then for a problem in which material properties vary from point-to-point will be very complicated.

Mika (ref. 18) has studied one such problem: the initial value problem for monoenergetic neutrons in a nonuniform slab surrounded by a vacuum. He used the same approach that Lehner and Wing (refs. 16, 17) had used for a uniform bare slab problem and, as might be expected, his more general hypothesis results in fewer details. In particular, it appears that theorems concerning the reality and number of discrete time eigenvalues cannot be established. At the outset, Mika (ref. 18) indicates that such results would be used, in practice, for a system of uniform slabs. Even for these cases in which there are a limited number of different material media, he states that the most suitable means of calculating discrete time eigenvalues would seem to be the normal-mode expansion approach employed by Bowden (refs. 1, 4). This is the approach used in this thesis to analyze a simple idealized two-media problem in which one would expect to have discrete time eigenvalues in order to obtain some insight concerning their behavior as a function of material properties. Such an approach has been utilized for one two-media time-dependent problem by Erdmann (refs. 8, 9) who investigated the time decay of a plane isotropic burst of monoenergetic neutrons introduced at the interface of two dissimilar semi-infinite media which scatter isotropically. In his solution, contributions due to the continuous spectrum are different for the two media; apparently the continuous spectrum depends on x. There are no discrete eigenvalues in his problem.

Lehner (ref. 15) has demonstrated that the continuous spectrum of the transport operator is very sensitive to the explicit formulation of a physical problem. He considered a slab of finite thickness surrounded by a pure absorber which had the same total macroscopic cross section as the slab. He obtained the same point spectrum as that found for the bare slab (refs. 16, 17) but found the continuous spectrum to be only the imaginary axis instead of the entire left-half plane. Very recently Hintz (ref. 10) has generalized Lehner's problem by allowing the pure absorber to have any cross section and found that when the two total macroscopic cross sections were different ($\sigma_1 \neq \sigma_2$, see Fig. 1) the continuous spectrum is a strip parallel to the imaginary axis of width $|\sigma_1 - \sigma_2|$ and that the point spectrum may be empty. He shows that his results reduce to those of Lehner (ref. 15) when $\sigma_1 = \sigma_2$ but does not indicate how the bare slab results of Lehner and Wing (refs. 16, 17) can be recovered. In the present problem, a finite slab is surrounded with a material which can scatter as well as absorb neutrons. Thus the bare slab and slab surrounded by pure absorbers are special cases and it is shown that the present solution has the proper behavior (refs. 10, 15, 16, 17) for these special cases.

Consider a slab of material which scatters neutrons isotropically, extends from $x = -a$ to $x = a$ and is characterized by the nuclear properties σ_2 and c_2. This uniform slab is surrounded by uniform infinitely-thick reflectors of another material characterized by the properties σ_1 and c_1 (see Fig. 1). For a physically meaningful system, these reflectors should be nonmultiplying media since they extend to infinity. Therefore, we take $c_1 < 1$. For isotropic scattering of monoenergetic neutrons in a sourceless medium and plane geometry, the neutron angular flux, $\psi(x, \mu, t)$, satisfies the equation (ref. 7)

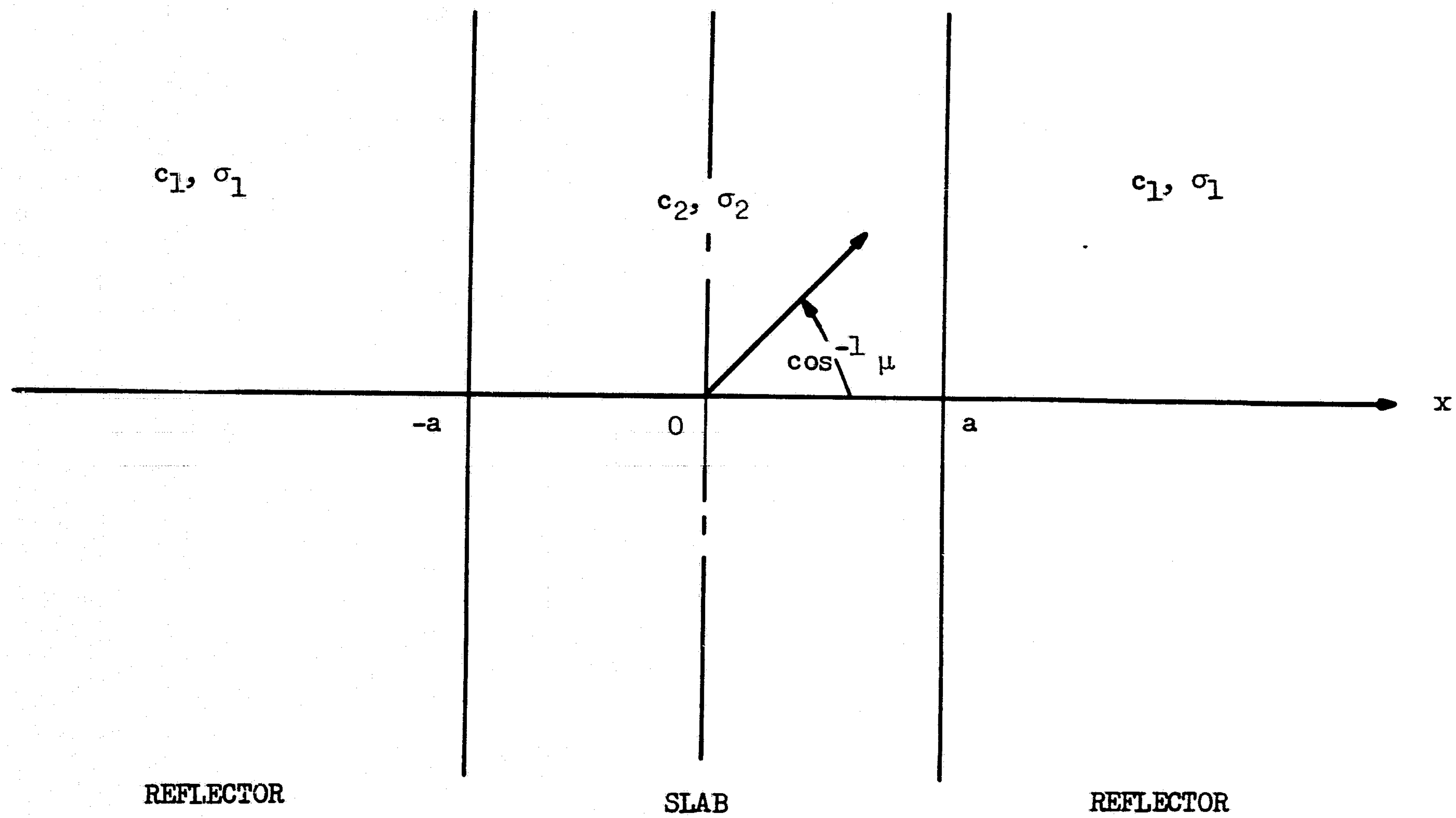

Figure 1.- Geometry of the problem.

$$\frac{\partial\psi(x,\mu,t)}{\partial t} + \mu\,\frac{\partial\psi(x,\mu,t)}{\partial x} + \sigma(x)\psi(x,\mu,t) = \frac{c(x)\sigma(x)}{2}\int_{-1}^{1}\psi(x,\mu',t)d\mu', \tag{1.1}$$

where t is the neutron speed multiplied by the real time, x and μ are shown on Figure 1 while $\sigma(x)$ and $c(x)$ are given by

$$\sigma(x),\ c(x) = \begin{cases} \sigma_1,\ c_1 & \text{for} \quad |x| > a \\ \sigma_2,\ c_2 & \text{for} \quad |x| < a. \end{cases} \tag{1.2}$$

We seek the solution of this equation subject to the boundary conditions

$$\lim_{|x| \to \infty} \psi(\pm x,\mu,t) = 0 \tag{1.3}$$

and the continuity conditions

$$\psi(\pm a+,\mu,t) = \psi(\pm a-,\mu,t), \tag{1.4}$$

given the initial condition

$$\psi(x,\mu,0) = f(x,\mu) \tag{1.5}$$

which we assume satisfies (1.3) and is extendable without poles or branch cuts in the finite μ-plane except perhaps for a discontinuity across the imaginary axis. When the material properties of the reflectors are taken to be those of a vacuum this problem reduces to that of Lehner and Wing (refs. 16, 17) while for a pure absorber it reduces to that considered by Lehner (ref. 15) or Hintz (ref. 10).

The method of attack to be used in solving this problem is the following:

1. Remove the t-dependence with a Laplace transformation.

2. Solve the transformed equation by applying Case's technique.

3. Determine the analytic properties of this transformed solution in some right-half s-plane.

4. Recover the t-dependence and simplify by suitably deforming the integration path of the inverse transformation. Previously cited results (refs. 1, 4, 8, 9, 14) lead us to expect that the reflectors should contribute continuous-spectrum type terms typical of a semi-infinite medium while the central slab should give rise to some point-spectrum type terms and their corresponding discrete time eigenvalues.

5. Calculate real discrete time eigenvalues as a function of material properties if and when they exist.

This is the method which has been successfully employed by Bowden (refs. 1, 4), Kuščer and Zweifel (ref. 14), and Erdmann (refs. 8, 9); we use many of their results in solving the present problem. In fact, our solution contains parts which resemble their solutions. Some preliminary results for the present problem were given in ref. 23.

II. TIME REMOVAL AND ELEMENTARY SOLUTIONS

If we take the Laplace transformation of $\psi(x,\mu,t)$ as

$$\psi_s(x,\mu) = \int_0^\infty e^{-st}\,\psi(x,\mu,t)dt, \tag{2.1}$$

then the inverse transformation required to recover the t-dependence is

$$\psi(x,\mu,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\psi_s(x,\mu)ds, \tag{2.2}$$

where γ is to the right of all singularities and branch cuts of $\psi_s(x,\mu)$ in the transform plane; that is, the s-plane. From previously cited work of others, it is expected that the path of integration in Eq. (2.2) can be deformed to indicate more precisely the character of $\psi(x,\mu,t)$. When we apply the transformation of Eq. (2.1) to Eq. (1.1), integrate by parts in the usual manner and make use of the initial condition (1.5), we obtain

$$\mu\frac{\partial}{\partial x}\psi_s(x,\mu) + \left[s + \sigma(x)\right]\psi_s(x,\mu) = \frac{c(x)\sigma(x)}{2}\int_{-1}^{1}\psi_s(x,\mu')d\mu' + f(x,\mu). \tag{2.3}$$

Equations (1.3) and (1.4) become under this transformation

$$\lim_{|x|\to\infty}\psi_s(\pm x,\mu) = 0 \tag{2.4}$$

and

$$\psi_s(\pm a+,\mu) = \psi_s(\pm a-,\mu). \tag{2.5}$$

Before applying Case's technique to solve Eq. (2.3) subject to conditions (2.4) and (2.5), let us examine some properties of the transformed solution which follow directly from the governing equations. Bowden (refs. 1, 4) introduced these ideas at a later step in his work, but here they aid in the construction of the solution.

An arbitrary function of two variables, $f(x,\mu)$, can be written as the sum of its even and odd parts, namely, $f_+(x,\mu)$ and $f_-(x,\mu)$. They are given, of course, by

$$f_\pm(x,\mu) = \frac{1}{2}\left[f(x,\mu) \pm f(-x,-\mu)\right] \tag{2.6}$$

and have the property

$$f_\pm(-x,-\mu) = \pm f_\pm(x,\mu). \tag{2.7}$$

Since $c(x)$ and $\sigma(x)$ are even functions of x, we can easily show from Eq. (2.3) that the even and odd parts of $\psi_s(x,\mu)$ obey the equation

$$\mu \frac{\partial}{\partial x} \psi_{s\pm}(x,\mu) + \left[s + \sigma(x)\right] \psi_{s\pm}(x,\mu) = \frac{c(x)\sigma(x)}{2}\int_{-1}^{1} \psi_{s\pm}(x,\mu')d\mu' + f_\pm(x,\mu). \tag{2.8}$$

The boundary conditions for $\psi_{s\pm}$ corresponding to Eqs. (2.4) and (2.5) are written as

$$\lim_{|x| \to \infty} \psi_{s\pm}(x,\mu) = 0 \tag{2.9}$$

and

$$\psi_{s\pm}(a+,\mu) = \psi_{s\pm}(a-,\mu), \tag{2.10}$$

where the $\pm$ subscripts denote definite parity parts of a function (see Eqs. (2.6) and (2.7)). Equations (2.8)-(2.10) tell us the following:

1. All solutions of the homogeneous equation associated with (2.8) can be made to have a definite parity.

2. The boundary conditions preserve the parity.

3. The definite parity parts of an initial distribution excite inhomogeneous solutions of corresponding definite parity. Therefore, we can separate this problem into two problems, one for ψ_{s+}, the other for ψ_{s-}, and combine the results at any stage of the calculation.

The functions $f_{\pm}(x,\mu)$ and $\psi_{s\pm}(x,\mu)$ are broken up as

$$f_{\pm}(x,\mu) = \begin{cases} f_{1\pm}(x,\mu), & |x| > a \\ f_{2\pm}(x,\mu), & |x| < a, \end{cases} \tag{2.11}$$

and

$$\psi_{s\pm}(x,\mu) = \begin{cases} \psi_{1\pm}(x,\mu,s), & |x| > a \\ \psi_{2\pm}(x,\mu,s), & |x| < a, \end{cases} \tag{2.12}$$

so that Eqs. (2.8), (2.9), and (2.10) become

$$\mu \frac{\partial}{\partial x} \psi_{j\pm}(x,\mu,s) + (s + \sigma_j)\psi_{j\pm}(x,\mu,s) = \frac{c_j\sigma_j}{2}\int_{-1}^{1} \psi_{j\pm}(x,\mu',s)d\mu'$$

$$+ f_{j\pm}(x,\mu), \quad j = 1,2, \tag{2.13}$$

$$\lim_{|x| \to \infty} \psi_{1\pm}(x,\mu,s) = 0 \tag{2.14}$$

and

$$\psi_{1\pm}(a,\mu,s) = \psi_{2\pm}(a,\mu,s). \tag{2.15}$$

The notation $g_{j\pm}(a,\mu)$ means the limit of $g_{\pm}(x,\mu)$ as $x \to a$ from medium j. Solutions of Eqs. (2.13) will be obtained by constructing even and odd particular solutions, $\psi_{jp\pm}(x,\mu)$, and adding to them solutions of the corresponding homogeneous equations, $\psi_{jc\pm}(x,\mu)$, so that conditions (2.14) and (2.15) can be satisfied. These functions $\psi_{jp\pm}$ and $\psi_{jc\pm}$ will be constructed from Case's elementary solutions which we shall denote here as $\psi_{j\nu}(x,\mu,s)$.

The elementary solutions, $\psi_{j\nu}(x,\mu,s)$, are solutions of the equation

$$\mu \frac{\partial}{\partial x} \psi_{j\nu}(x,\mu,s) + (s + \sigma_j)\psi_{j\nu}(x,\mu,s) = \frac{1}{2} c_j\sigma_j \int_{-1}^{1} \psi_{j\nu}(x,\mu',s)d\mu' \tag{2.16}$$

in the form

$$\psi_{j\nu}(x,\mu,s) = \varphi_{js\nu}(\mu)e^{-(s+\sigma_j)x/\nu} \tag{2.17}$$

where ν is a complex parameter introduced in this separation of variables and $\varphi_{js\nu}(\mu)$ is normalized as

$$\int_{-1}^{1} \varphi_{js\nu}(\mu)d\mu = s + \sigma_j. \tag{2.18}$$

Bowden (refs. 1, 4) and Erdmann (refs. 8, 9) have investigated these solutions; many of their results are given in Appendix A and will be used herein. They show that the solutions $\varphi_{js\nu}(\mu)$ are given by

$$\varphi_{js\nu}(\mu) = \frac{1}{2} c_j\sigma_j\nu P \frac{1}{\nu - \mu} + \lambda_{js}(\nu)\delta(\nu - \mu), \quad \nu \in (-1,+1), \tag{2.19}$$

where P denotes the Cauchy principal value, $\delta(\nu - \mu)$ is the Dirac delta function and

$$\lambda_{js}(\nu) = s + \sigma_j - c_j\sigma_j\nu \tanh^{-1} \nu, \tag{2.20}$$

and two discrete solutions,

$$\varphi_{\pm\nu_{0j}}(\mu) = \frac{1}{2} \frac{c_j\sigma_j\nu_{0j}}{\nu_{0j} \mp \mu}, \quad s \in S_{ji}, \tag{2.21}$$

provided that the function $\Omega_{js}(z)$,

$$\Omega_{js}(z) = s + \sigma_j - c_j\sigma_j z \tanh^{-1} \frac{1}{z}, \tag{2.22}$$

of two complex variables s and z vanishes at the two points $\pm\nu_{0j}$. The condition for this to happen (refs. 1, 4) is that s lie inside the curve C_j ($s \in S_{ji}$, see Fig. 2) defined by

$$C_j = \left\{ \frac{s + \sigma_j}{c_j\sigma_j} = \alpha' + i\beta' \;\middle|\; \alpha' = \frac{2\beta'}{\pi} \tanh^{-1}\left(\frac{2\beta'}{\pi}\right) \right\}. \tag{2.23}$$

We note that ν_{0j} is an analytic function of s for $s \in S_{ji}$ except for a branch cut on the real s-axis between $-\sigma_j$ and $-\sigma_j(1 - c_j)$. We have denoted by $+\nu_{0j}$ that zero of $\Omega_{js}(z)$ for which $\mathrm{Re}(\nu_{0j}) > 0$, $s \notin \left[-\sigma_j, -\sigma_j(1 - c_j)\right]$. The important result is that the general solution of (2.16) can be expressed as the linear combination

$$\psi_j(x,\mu,s) = \left[a_j\psi_{\nu_{0j}}(x,\mu,s) + b_j\psi_{-\nu_{0j}}(x,\mu,s)\right]\delta_j(s) + \int_{-1}^{1} A_j(\nu)\psi_{j\nu}(x,\mu,s)d\nu, \tag{2.24}$$

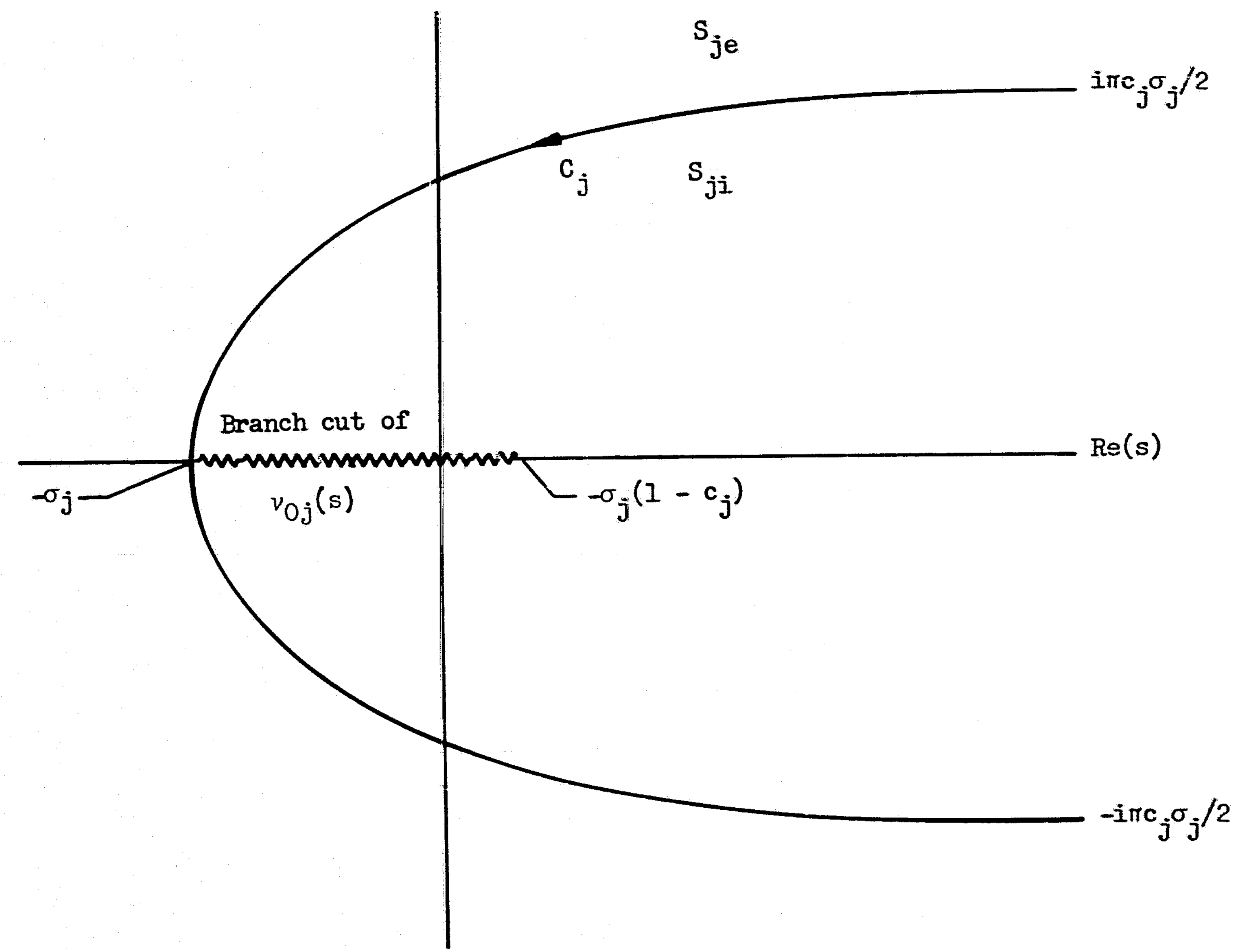

Figure 2.- Regions in a single-medium s-plane.

Location of Im(s) axis depends on whether $c_j \gtrless 1$.

where we define $\delta_j(s)$ as

$$\delta_j(s) \equiv \begin{cases} 1, & s \in S_{ji} \\ 0, & s \in S_{je} \end{cases} \tag{2.25}$$

and the s-dependence of the expansion coefficients has not been indicated. To avoid confusion with our notation $\pm$ for parity, the customary notation for the discrete modes has not been used.

The solution (2.24) does not have definite parity. For a medium which is connected and symmetric about $x = 0$ (such as our slab), even and odd solutions can be written as

$$\psi_{j\pm}(x,\mu,s) = a_{j\pm}\left[\psi_{\nu_{0j}}(x,\mu,s) \pm \psi_{-\nu_{0j}}(x,\mu,s)\right]\delta_j(s) + \int_0^1 A_{j\pm}(\nu)\left[\psi_{j\nu}(x,\mu,s) \pm \psi_{j(-\nu)}(x,\mu,s)\right]d\nu, \tag{2.26}$$

where

$$a_{j\pm} = \frac{1}{2}\left[a_j \pm b_j\right]$$

and

$$A_{j\pm}(\nu) = \frac{1}{2}\left[A_j(\nu) \pm A_j(-\nu)\right]. \tag{2.27}$$

We have used the properties

$$\psi_{\pm\nu_{0j}}(-x,-\mu,s) = \psi_{\mp\nu_{0j}}(x,\mu,s)$$

and

$$\psi_{j(\pm\nu)}(-x,-\mu,s) = \psi_{j(\mp\nu)}(x,\mu,s). \tag{2.28}$$

For a medium which extends to infinity in the x-direction (such as our reflector) the boundary conditions (2.4) require that

$$b_j \equiv A_j(-\nu) \equiv 0, \quad 0 \leq \nu \leq 1 \quad \text{if} \quad x \to +\infty$$

or

$$a_j \equiv A_j(\nu) \equiv 0, \quad 0 \leq \nu \leq 1 \quad \text{if} \quad x \to -\infty \tag{2.29}$$

for the expansion coefficients in Eq. (2.24) when $Re(s) > -\sigma_j$. We use results (2.24), (2.26), and (2.29) now to construct the solutions $\psi_{jc\pm}$ and $\psi_{jp\pm}$.

III. CONSTRUCTION OF TRANSFORMED SOLUTION

The even and odd homogeneous solutions in the slab, $\psi_{2c\pm}$, can be written in the form of Eq. (2.26) with $j = 2$. On the other hand, the homogeneous solution in the reflectors, subject to the boundary condition (2.29), can be written from Eq. (2.24) as

$$\psi_1(x,\mu,s) = \begin{cases} b_1\psi_{-\nu_{01}}(x,\mu,s)\delta_1(s) + \int_{-1}^{0} A_1(\nu)\psi_{1\nu}(x,\mu,s)d\nu, & x < -a \\ a_1'\psi_{\nu_{01}}(x,\mu,s)\delta_1(s) + \int_{0}^{1} A_1'(\nu)\psi_{1\nu}(x,\mu,s)d\nu, & x > a, \end{cases} \tag{3.1}$$

for $\mathrm{Re}(s) > -\sigma_1$. The continuity conditions (2.5) and the parity of the solutions $\psi_{2c\pm}$ can be used to relate the coefficients in Eq. (3.1). We find that an even solution inside the slab requires

$$a_1' = b_1 \quad \text{and} \quad A_1'(\nu) = A_1(-\nu), \quad 0 \leq \nu \leq 1, \tag{3.2a}$$

while an odd solution inside the slab requires

$$a_1' = -b_1 \quad \text{and} \quad A_1'(\nu) = -A_1(-\nu), \quad 0 \leq \nu \leq 1. \tag{3.2b}$$

The explicit forms of $\psi_{2c\pm}$ and $\psi_{1c\pm}$ are therefore

$$\psi_{2c\pm}(x,\mu,s) = a_{2\pm}\left[\psi_{\nu_{02}}(x,\mu,s) \pm \psi_{-\nu_{02}}(x,\mu,s)\right]\delta_2(s) + \int_0^1 A_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right]d\nu \tag{3.3}$$

and

$$\psi_{1c\pm}(x,\mu,s) = \begin{cases} a_{1\pm}\psi_{-\nu_{01}}(x,\mu,s)\delta_1(s) + \int_0^1 A_{1\pm}(-\nu)\psi_{1(-\nu)}(x,\mu,s)d\nu, & x < -a \\ \pm a_{1\pm}\psi_{\nu_{01}}(x,\mu,s)\delta_1(s) \pm \int_0^1 A_{1\pm}(-\nu)\psi_{1\nu}(x,\mu,s)d\nu, & x > a, \end{cases} \tag{3.4}$$

for $\mathrm{Re}(s) > -\sigma_1$.

We turn now to construction of $\psi_{jp\pm}$. Consider a function $g_{js}(x,\mu;x_0)$ which satisfies the equation

$$\mu \frac{\partial}{\partial x} g_{js}(x,\mu;x_0) + (s + \sigma_j)g_{js}(x,\mu;x_0) = \frac{c_j\sigma_j}{2}\int_{-1}^{1} g_{js}(x,\mu';x_0)d\mu' + \delta(x - x_0)f_j(x_0,\mu). \tag{3.5}$$

Upon integrating on x from $x_0 - \epsilon$ to $x_0 + \epsilon$ and taking the limit $\epsilon \to 0$, we obtain the jump condition

$$g_{js}(x_0+,\mu;x_0) - g_{js}(x_0-,\mu;x_0) = \frac{f_j(x_0,\mu)}{\mu}. \tag{3.6}$$

The function $\psi_{jp\pm}(x,\mu,s)$ defined as

$$\psi_{jp\pm}(x,\mu,s) \equiv \frac{1}{2}\left[\psi_{jp}(x,\mu,s) \pm \psi_{jp}(-x,-\mu,s)\right], \tag{3.7a}$$

where

$$\psi_{jp}(x,\mu,s) = \int_{\substack{(\text{medium})\\ j}} g_{js}(x,\mu;x_0)dx_0, \tag{3.7b}$$

is seen to be a solution of Eq. (2.13). It is shown in Appendix B that explicit forms of $\psi_{2p\pm}$ and $\psi_{1p\pm}$ can be written as

$$\psi_{2p\pm}(x,\mu,s) = \Big[F_{2\pm}(x,\nu_{02},s)\psi_{\nu_{02}}(x,\mu,s) \pm F_{2\pm}(-x,\nu_{02},s)\psi_{-\nu_{02}}(x,\mu,s)\Big]\delta_2(s) + \int_0^1 F_{2\pm}(x,\nu,s)\psi_{2\nu}(x,\mu,s)d\nu \pm \int_0^1 F_{2\pm}(-x,\nu,s)\psi_{2(-\nu)}(x,\mu,s)d\nu, \tag{3.8}$$

$$\psi_{1p\pm}(x,\mu,s) = \Big\{F_{1\pm}(x,\nu_{01},s)\psi_{\nu_{01}}(x,\mu,s) + \Big[F_{1\pm}(x,-\nu_{01},s) - \tilde{F}_{\pm}(-a,\nu_{01},s)\Big]\psi_{-\nu_{01}}(x,\mu,s)\Big\}\delta_1(s) + \int_0^1 F_{1\pm}(x,\nu,s)\psi_{1\nu}(x,\mu,s)d\nu + \int_0^1 \Big[F_{1\pm}(x,-\nu,s) - \tilde{F}_{\pm}(-a,\nu,s)\Big]\psi_{1(-\nu)}(x,\mu,s)d\nu, \quad x < -a, \tag{3.9a}$$

for $\mathrm{Re}(s) > -\sigma_1$, and

$$\psi_{1p\pm}(x,\mu,s) = \Big\{\pm\Big[-\tilde{F}_{\pm}(-a,\nu_{01},s) + F_{1\pm}(-x,-\nu_{01},s)\Big]\psi_{\nu_{01}}(x,\mu,s) \pm F_{1\pm}(-x,\nu_{01},s)\psi_{-\nu_{01}}(x,\mu,s)\Big\}\delta_1(s) \pm \int_0^1 \Big[-\tilde{F}_{\pm}(-a,\nu,s) + F_{1\pm}(-x,-\nu,s)\Big]\psi_{1\nu}(x,\mu,s)d\nu \pm \int_0^1 F_{1\pm}(-x,\nu,s)\psi_{1(-\nu)}(x,\mu,s)d\nu, \quad x > a, \tag{3.9b}$$

for $\mathrm{Re}(s) > -\sigma_1$, where

$$\tilde{F}_{\pm}(-a,\omega,s) \equiv F_{1\pm}(-a,-\omega,s) \mp F_{1\pm}(-a,\omega,s),$$

$$F_{2\pm}(x,\omega,s) \equiv \int_{-a}^{x} C_{2\pm}(x_0,\omega)e^{(s+\sigma_2)x_0/\omega}\,dx_0$$

and

$$F_{1\pm}(x,\omega,s) \equiv \int_{-\infty}^{x} C_{1\pm}(x_0,\omega)e^{(s+\sigma_1)x_0/\omega}\,dx_0. \tag{3.10}$$

Here the $C_{j\pm}$ are full-range expansion coefficients of the function $f_{j\pm}(x,\mu)/\mu$ and are given by

$$C_{j\pm}(x_0,\nu) = \frac{1}{\nu\Omega^{+}_{js}(\nu)\Omega^{-}_{js}(\nu)}\int_{-1}^{1} f_{1\pm}(x_0,\mu)\varphi_{js\nu}(\mu)d\mu,$$

and if $s \in S_{ji}$,

$$C_{j\pm}(x_0,\nu_{0j}) = \frac{2}{c_j\sigma_j\nu_{0j}{}^2\Omega'_{js}(\nu_{0j})}\int_{-1}^{1} f_{j\pm}(x_0,\mu)\varphi_{\nu_{0j}}(\mu)d\mu$$

and

$$C_{j\pm}(x_0,-\nu_{0j}) = \frac{2}{c_j\sigma_j\nu_{0j}{}^2\Omega'_{js}(-\nu_{0j})}\int_{-1}^{1} f_{j\pm}(x_0,\mu)\varphi_{-\nu_{0j}}(\mu)d\mu.$$

(3.11)

Throughout, we shall use + and - superscripts to denote the limiting values of a function on its branch cut as the argument approaches the cut from the upper (+) and lower (-) half-planes. The function $\Omega_{js}(z)$ of Eq. (2.22) has a branch cut along the real z-axis (-1,1) such that

$$\Omega^{\pm}_{js}(\nu) = \lambda_{js}(\nu) \pm i\pi c_j\sigma_j\nu/2, \quad -1 \leq \nu \leq 1. \tag{3.12}$$

The functions $\Omega'_{js}(z)$ are defined by

$$\Omega'_{js}(z) \equiv \frac{d}{dz}\,\Omega_{js}(z). \qquad (3.13)$$

Note that the parity of the coefficients $C_{j\pm}$ is opposite that indicated by the $\pm$ subscript. Nevertheless, the solutions $\psi_{jp\pm}$ are easily seen to have the indicated parity.

The solutions $\psi_{s\pm}$ (Eq. (2.12)) of our problem are written in terms of $\psi_{jc\pm}$ and $\psi_{jp\pm}$ as

$$\psi_{s\pm}(x,\mu) = \begin{cases} \psi_{1c\pm}(x,\mu,s) + \psi_{1p\pm}(x,\mu,s), & |x| > a \\ \psi_{2c\pm}(x,\mu,s) + \psi_{2p\pm}(x,\mu,s), & |x| < a. \end{cases} \qquad (3.14)$$

The solutions in medium 1, $|x| > a$, have been constructed so that the boundary condition (2.14) is satisfied. Application of the continuity condition (2.15) allows us to determine the unknown expansion coefficients of $\psi_{jc\pm}$ which appear in Eq. (3.14). That is to say, if we substitute $x = a$ in Eq. (3.14), apply the continuity condition (2.15) and use the explicit forms of $\psi_{jc\pm}$ given by Eqs. (3.3) and (3.4), we obtain a two-media full-range expansion involving the $\varphi_{js\nu}$ which contains unknown coefficients $a_{j\pm}$ and $A_{j\pm}$. The same expansion is, of course, obtained for $x = -a$. This type of expansion and its orthogonality relations are discussed in Appendix C and we show in Appendix D that such an expansion is obtained for the present problem. Erdmann (ref. 8) proved completeness theorems which apply in such time-dependent problems while Kuščer, McCormick and Summerfield (ref. 13) derived orthogonality relations which are applicable to two-media expansions which arise in time-independent problems. In Appendix C, we extend their results to obtain orthogonality relations in a form which are valid for all regions of the transform plane. As usual in problems involving a slab, we cannot obtain closed form solutions for the expansion coefficients. However, we can use the orthogonality relations (App. C) to obtain expressions which give the expansion coefficients implicitly. That is, the continuum coefficients $A_{2\pm}(\nu)$ are given as the solutions of Fredholm integral equations and all of the other coefficients are obtained from the $A_{2\pm}(\nu)$. More specifically, if we define

$$E_{2\pm}(\nu) \equiv A_{2\pm}(\nu)\Omega_{2s}^{+}(\nu)\Omega_{2s}^{-}(\nu)e^{(s+\sigma_2)a/\nu}$$

and

$$E_{1\pm}(\nu) \equiv A_{1\pm}(-\nu)\Omega_{1s}^{+}(\nu)\Omega_{1s}^{-}(\nu)e^{-(s+\sigma_1)a/\nu}\ , \tag{3.15}$$

the use of the orthogonality relations leads, after some algebra, to the following list of equations:

$$E_{2\pm}(\nu) = I_{2\pm}(\nu) \pm \left\{\frac{k_s}{2}\frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)}\, X_0(-\nu,s)\right\}$$
$$\times\left\{\int_0^1 E_{2\pm}(\mu)\,\frac{e^{-2(s+\sigma_2)a/\mu}\, X_0(-\mu,s)\mu\, d\mu}{\Omega_{2s}^{+}(\mu)\Omega_{2s}^{-}(\mu)(\mu+\nu)}\right.$$
$$\left. + \,\delta_2(s)a_{2\pm}e^{-(s+\sigma_2)a/\nu_{02}}\, X_0(-\nu_{02},s)\frac{\nu_{02}}{\nu_{02}+\nu}\right\}, \quad 0 \le \nu \le 1, \tag{3.16}$$

$$\frac{1}{2} c_2\sigma_2\nu_{02}\Omega'_{2s}(\nu_{02})a_{2\pm}e^{(s+\sigma_2)a/\nu_{02}} = J_{2\pm}(\nu_{02}) \pm \left\{\frac{k_s}{2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} X_0(-\nu_{02}, s)\right\}$$

$$\times \left\{\int_0^1 E_{2\pm}(\mu) \frac{e^{-2(s+\sigma_2)a/\mu} X_0(-\mu, s)\mu \, d\mu}{\Omega^+_{2s}(\mu)\Omega^-_{2s}(\mu)(\mu + \nu_{02})}\right.$$

$$\left. + \frac{1}{2} a_{2\pm}e^{-(s+\sigma_2)a/\nu_{02}} X_0(-\nu_{02}, s)\right\},$$

$$s \in S_{2i},$$

(3.17)

$$E_{1\pm}(\nu) = I_{1\pm}(\nu) \pm \frac{c_2\sigma_2}{c_1\sigma_1} \frac{\Omega^+_{1s}(\nu)\Omega^-_{1s}(\nu)}{\Omega^+_{2s}(\nu)\Omega^-_{2s}(\nu)} E_{2\pm}(\nu)e^{-2(s+\sigma_2)a/\nu}$$

$$\pm \left\{\frac{1}{2} k_s \frac{1}{X_0(-\nu, s)}\right\}$$

$$\times \left\{\int_0^1 E_{2\pm}(\mu) \frac{e^{-2(s+\sigma_2)a/\mu} X_0(-\mu, s)\varphi_{1s\nu}(\mu)2\mu \, d\mu}{\Omega^+_{2s}(\mu)\Omega^-_{2s}(\mu)c_1\sigma_1\nu}\right.$$

$$\left. + \delta_2(s)a_{2\pm}e^{-(s+\sigma_2)a/\nu_{02}} X_0(-\nu_{02}, s) \quad \frac{\nu_{02}}{\nu - \nu_{02}}\right\},$$

$$0 \leq \nu \leq 1 \qquad (3.18)$$

and

$$\frac{1}{2} c_1\sigma_1\nu_{01}\Omega'_{1s}(\nu_{01})a_{1\pm}e^{-(s+\sigma_1)a/\nu_{01}} = J_{1\pm}(\nu_{01}) \mp \left\{\frac{1}{2} k_s \frac{1}{X_0(-\nu_{01},s)}\right\}$$

$$\times \left\{\int_0^1 E_{2\pm}(\mu) \frac{e^{-2(s+\sigma_2)a/\mu} X_0(-\mu,s)\mu \, d\mu}{\Omega^+_{2s}(\mu)\Omega^-_{2s}(\mu)(\mu - \nu_{01})}\right.$$

$$\left. + \delta_2(s)a_{2\pm}e^{-(s+\sigma_2)a/\nu_{02}} X_0(-\nu_{02},s) \frac{\nu_{02}}{\nu_{02} - \nu_{01}}\right\},$$

$$s \in S_{1i}. \tag{3.19}$$

The $I_{j\pm}$ and $J_{j\pm}$ terms contain only integrations over the initial distribution and are therefore known functions when $f(x,\mu)$ is specified. They are given by

$$I_{2\pm}(\nu) = \frac{c_1\sigma_1}{c_2\sigma_2} F_{1\pm}(-a,\nu,s)e^{(s+\sigma_1)a/\nu} \Omega^+_{2s}(\nu)\Omega^-_{2s}(\nu) \pm \left\{\frac{1}{2} k_s \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} X_0(-\nu,s)\right\}$$

$$\times \left\{\left[\int_0^1 F_{2\pm}(a,\mu,s)e^{-(s+\sigma_2)a/\mu} X_0(-\mu,s) \frac{\mu \, d\mu}{\mu + \nu}\right.\right.$$

$$\left. + \delta_2(s)F_{2\pm}(a,\nu_{02},s)e^{-(s+\sigma_2)a/\nu_{02}} X_0(-\nu_{02},s) \frac{\nu_{02}}{\nu_{02} + \nu}\right]$$

$$\mp \frac{\Omega_{1s}(\infty)}{\Omega_{2s}(\infty)}\left[\int_0^1 F_{1\pm}(-a,\mu,s)e^{(s+\sigma_1)a/\mu} \frac{\varphi_{2s\nu}(\mu)2\mu \, d\mu}{X_0(-\mu,s)c_2\sigma_2\nu}\right.$$

$$\left.\left. + \delta_1(s)F_{1\pm}(-a,\nu_{01},s)e^{(s+\sigma_1)a/\nu_{01}} \frac{1}{X_0(-\nu_{01},s)} \frac{\nu_{01}}{\nu - \nu_{01}}\right]\right\},$$

$$0 \le \nu \le 1, \tag{3.20}$$

$$J_{2\pm}(\nu_{02}) = \pm \left\{\frac{1}{2} k_s \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} X_0(-\nu_{02}, s)\right\}$$

$$\times \left\{\left[\int_0^1 F_{2\pm}(a,\mu,s)e^{-(s+\sigma_2)a/\mu} X_0(-\mu,s) \frac{\mu \, d\mu}{\mu + \nu_{02}}\right.\right.$$

$$\left. + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)e^{-(s+\sigma_2)a/\nu_{02}} X_0(-\nu_{02},s)\right]$$

$$\pm \frac{\Omega_{1s}(\infty)}{\Omega_{2s}(\infty)} \left[\int_0^1 F_{1\pm}(-a,\mu,s)e^{(s+\sigma_1)a/\mu} \frac{\mu \, d\mu}{X_0(-\mu,s)(\mu - \nu_{02})}\right.$$

$$\left.\left. + \delta_1(s) \; F_{1\pm}(-a,\nu_{01},s)e^{(s+\sigma_1)a/\nu_{01}} \frac{\nu_{01}}{X_0(-\nu_{01},s)(\nu_{01} - \nu_{02})}\right]\right\},$$

$$s \in S_{2i}, \qquad (3.21)$$

$$I_{1\pm}(\nu) = \mp \left[F_{1\pm}(-a,\nu,s)e^{-(s+\sigma_1)a/\nu} - \frac{c_2\sigma_2}{c_1\sigma_1} F_{2\pm}(a,\nu,s)e^{-(s+\sigma_2)a/\nu}\right] \Omega_{1s}^{+}(\nu)\Omega_{1s}^{-}(\nu)$$

$$\pm \left\{\frac{1}{2} k_s \frac{1}{X_0(-\nu,s)}\right\}$$

$$\times \left\{\left[\int_0^1 F_{2\pm}(a,\mu,s)e^{-(s+\sigma_2)a/\mu} X_0(-\mu,s)\varphi_{1s\nu}(\mu) \frac{2\mu \, d\mu}{c_1\sigma_1\nu}\right.\right.$$

$$\left. + \delta_2(s)F_{2\pm}(a,\nu_{02},s)e^{-(s+\sigma_2)a/\nu_{02}} X_0(-\nu_{02},s) \frac{\nu_{02}}{\nu - \nu_{02}}\right]$$

$$\mp \frac{\Omega_{1s}(\infty)}{\Omega_{2s}(\infty)} \left[\int_0^1 F_{1\pm}(-a,\mu,s)e^{(s+\sigma_1)a/\mu} \frac{\mu \, d\mu}{X_0(-\mu,s)(\mu + \nu)}\right.$$

$$\left.\left. + \delta_1(s)F_{1\pm}(-a,\nu_{01},s)e^{(s+\sigma_1)a/\nu_{01}} \frac{\nu_{01}}{X_0(-\nu_{01},s)(\nu_{01} + \nu)}\right]\right\},$$

$$0 \le \nu \le 1, \qquad (3.22)$$

and

$$
\begin{aligned}
J_{1\pm}(\nu_{01}) = &\mp F_{1\pm}(-a,\nu_{01},s)\,\frac{1}{2}\,c_1\sigma_1\nu_{01}\Omega'_{1s}(\nu_{01})e^{-(s+\sigma_1)a/\nu_{01}} \\
&\mp\left\{\frac{1}{2}\,k_s\,\frac{1}{X_0(-\nu_{01},s)}\right\} \\
&\times\left\{\left[\int_0^1 F_{2\pm}(a,\mu,s)e^{-(s+\sigma_2)a/\mu}\,X_0(-\mu,s)\,\frac{\mu\,d\mu}{\mu-\nu_{01}}\right.\right. \\
&\qquad\left.+\,\delta_2(s)F_{2\pm}(a,\nu_{02},s)e^{-(s+\sigma_2)a/\nu_{02}}\,X_0(-\nu_{02},s)\,\frac{\nu_{02}}{\nu_{02}-\nu_{01}}\right] \\
&\qquad\pm\,\frac{\Omega_{1s}(\infty)}{\Omega_{2s}(\infty)}\left[\int_0^1 F_{1\pm}(-a,\mu,s)e^{(s+\sigma_1)a/\mu}\,\frac{\mu\,d\mu}{X_0(-\mu,s)(\mu+\nu_{01})}\right. \\
&\qquad\qquad\left.\left.+\,\frac{1}{2}\,F_{1\pm}(-a,\nu_{01},s)e^{(s+\sigma_1)a/\nu_{01}}\,\frac{1}{X_0(-\nu_{01},s)}\right]\right\}, \\
&\qquad\qquad\qquad s\in S_{1i}. \qquad (3.23)
\end{aligned}
$$

In the above equations, we have used the X_{0j} functions which Kuščer and Zweifel (ref. 14) have shown are continuous across the curves C_j in the s-plane (see App. A and Fig. 2). For two material media, we take the ratio of their single-medium X_{0j} functions,

$$X_0(z,s) = \frac{X_{02}(z,s)}{X_{01}(z,s)}, \qquad (3.24)$$

where

$$X_{0j}(z,s) = \begin{cases}(\nu_{0j}-z)X_{js}(z), & s\in S_{ji}\\ (1-z)X_{js}(z), & s\in S_{je}\end{cases} \qquad (3.25)$$

and

$$X_{js}(z) = \frac{1}{1 - z} \exp\left\{\frac{1}{2\pi i}\int_0^1 \ln\left[\frac{\Omega_{js}^{+}(\nu)}{\Omega_{js}^{-}(\nu)}\right]\frac{d\nu}{\nu - z}\right\} \tag{3.26}$$

For $\mathrm{Re}(z) < 0$, $X_0(z,s)$ given by Eq. (3.24) is a nonvanishing analytic function of z and s provided $s \notin \left[-\sigma_j, -\sigma_j(1 - c_j)\right]$, the branch cut of $\nu_{0j}(s)$, $j = 1,2$. The quantity

$$k_s = s(c_1\sigma_1 - c_2\sigma_2) + \sigma_1\sigma_2(c_1 - c_2) \tag{3.27}$$

is related to the difference between medium 1 and medium 2 continuum solutions; several equivalent expressions for k_s are given in Appendix C.

In Eqs. (3.15) we introduced the coefficients $E_{j\pm}(\nu)$ since they are the forms of the normal-mode expansion coefficients which are extendable to the complex plane (refs. 2, 3). Thus, Eqs. (3.16) through (3.23) can be written in a compact form valid for $\mathrm{Re}(s) > -\sigma_m$. These equations (see App. E) are

$$E_{2\pm}(z,s) = I_{2\pm}(z,s) \pm \frac{k_s}{c_2\sigma_2}\frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)}\frac{X_0(-z,s)}{2\pi i}\int_{C'}\frac{E_{2\pm}(z',s)X_0(-z',s)e^{-2(s+\sigma_2)a/z'}}{\Omega_{2s}(z')(z' + z)}dz', \tag{3.28}$$

$$E_{1\pm}(z,s) = I_{1\pm}(z,s) \pm \frac{c_1\sigma_1}{c_2\sigma_2}E_{2\pm}(z,s)e^{-2(s+\sigma_2)a/z} \mp \frac{k_s}{c_2\sigma_2 X_0(-z,s)2\pi i}\int_{C'}\frac{E_{2\pm}(z',s)X_0(-z',s)e^{-2(s+\sigma_2)a/z'}}{\Omega_{2s}(z')(z' - z)}dz', \tag{3.29}$$

$$I_{2\pm}(z,s) = \frac{c_2\sigma_2}{c_1\sigma_1} L_{1\pm}(-a,z,s) + \left\{\frac{k_s}{2\pi i} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} X_0(-z,s)\right\}$$
$$\times \left\{\pm\int_{C'} \frac{L_{2\pm}(a,z',s)X_0(-z',s)}{c_2\sigma_2\Omega_{2s}(z')(z'+z)} dz'\right.$$
$$\left. + \frac{\Omega_{1s}(\infty)}{\Omega_{2s}(\infty)} \int_{C'} \frac{L_{1\pm}(-a,z',s)\ dz'}{c_1\sigma_1 X_0(-z',s)\Omega_{1s}(z')(z'-z)}\right\} \qquad (3.30)$$

and

$$I_{1\pm}(z,s) = \mp L_{1\pm}(-a,z,s)e^{-2(s+\sigma_1)a/z} \pm \frac{c_1\sigma_1}{c_2\sigma_2} L_{2\pm}(a,z,s) - \left\{\frac{k_s}{2\pi i X_0(-z,s)}\right\}$$
$$\times \left\{\pm\int_{C'} \frac{L_{2\pm}(a,z',s)X_0(-z',s)}{c_2\sigma_2\Omega_{2s}(z')(z'-z)} dz'\right.$$
$$\left. + \frac{\Omega_{1s}(\infty)}{\Omega_{2s}(\infty)} \int_{C'} \frac{L_{1\pm}(-a,z',s)\ dz'}{c_1\sigma_1 X_0(-z',s)\Omega_{1s}(z')(z'+z)} dz'\right\}, \qquad (3.31)$$

where for $\mathrm{Re}(s) > -\sigma_j$,

$$L_{j\pm}(x,z,s) = \int_{l(j)}^{x} e^{-(s+\sigma_j)(x-x_0)/z} \left\{\frac{1}{2} c_j\sigma_j \int_0^1 f_{j\pm}(x_0,-\mu) \frac{d\mu}{\mu+z}\right.$$
$$- \frac{1}{2} c_j\sigma_j \int_0^1 f_{j\pm}(x_0,\mu) \frac{d\mu}{\mu-z}$$
$$\left. + \frac{1}{z} f_{j\pm}(x_0,z)\Omega_{js}(z)\right\} dx_0,$$

$$(3.32)$$

with

$$l(1) = -\infty \quad \text{and} \quad l(2) = -a. \qquad (3.33)$$

In the above equations, z does not lie outside the contour C' which encircles ν_{0j} as shown in Fig. 3 and σ_m is defined as

$$\sigma_m \equiv \min(\sigma_1,\sigma_2). \tag{3.34}$$

The restriction $\mathrm{Re}(s) > -\sigma_m$ is discussed in the next section. The $L_{j\pm}$ functions were introduced as

$$L_{j\pm}(x,\nu,s) = F_{j\pm}(x,\nu,s)\Omega_{js}^{+}(\nu)\Omega_{js}^{-}(\nu)e^{-(s+\sigma_j)x/\nu}, \quad 0 \le \nu \le 1. \tag{3.35}$$

That Eqs. (3.28)-(3.31) reduce to Eqs. (3.16)-(3.23) as all contours C' are collapsed onto the branch cut $\nu \in (0,1)$ due to $\Omega_{js}(z)$ can be seen as follows. If $s \in S_{ji}$, $\left[\Omega_{js}(z)\right]^{-1}$ has a pole at $z = \nu_{0j}$ whose residue leads to a discrete term. When $s \in S_{je}$, $\Omega_{js}(z)$ does not vanish. The continuum terms are simply those due to the integration around the branch cut.

The solutions $\psi_{jc\pm}(x,\mu,s)$ and $\psi_{jp\pm}(x,\mu,s)$ can now be written similarly as

$$\psi_{2c\pm}(x,\mu,s) = \frac{1}{2\pi i}\left\{\int_{C'} \frac{E_{2\pm}(z',s)e^{-(s+\sigma_2)(a+x)/z'}}{\Omega_{2s}(z')(z'-\mu)}\,dz' \pm \int_{C'} \frac{E_{2\pm}(z',s)e^{-(s+\sigma_2)(a-x)/z'}}{\Omega_{2s}(z')(z'+\mu)}\,dz'\right\}, \quad |x| < a, \tag{3.36}$$

for $\mathrm{Re}(s) > -\sigma_m$,

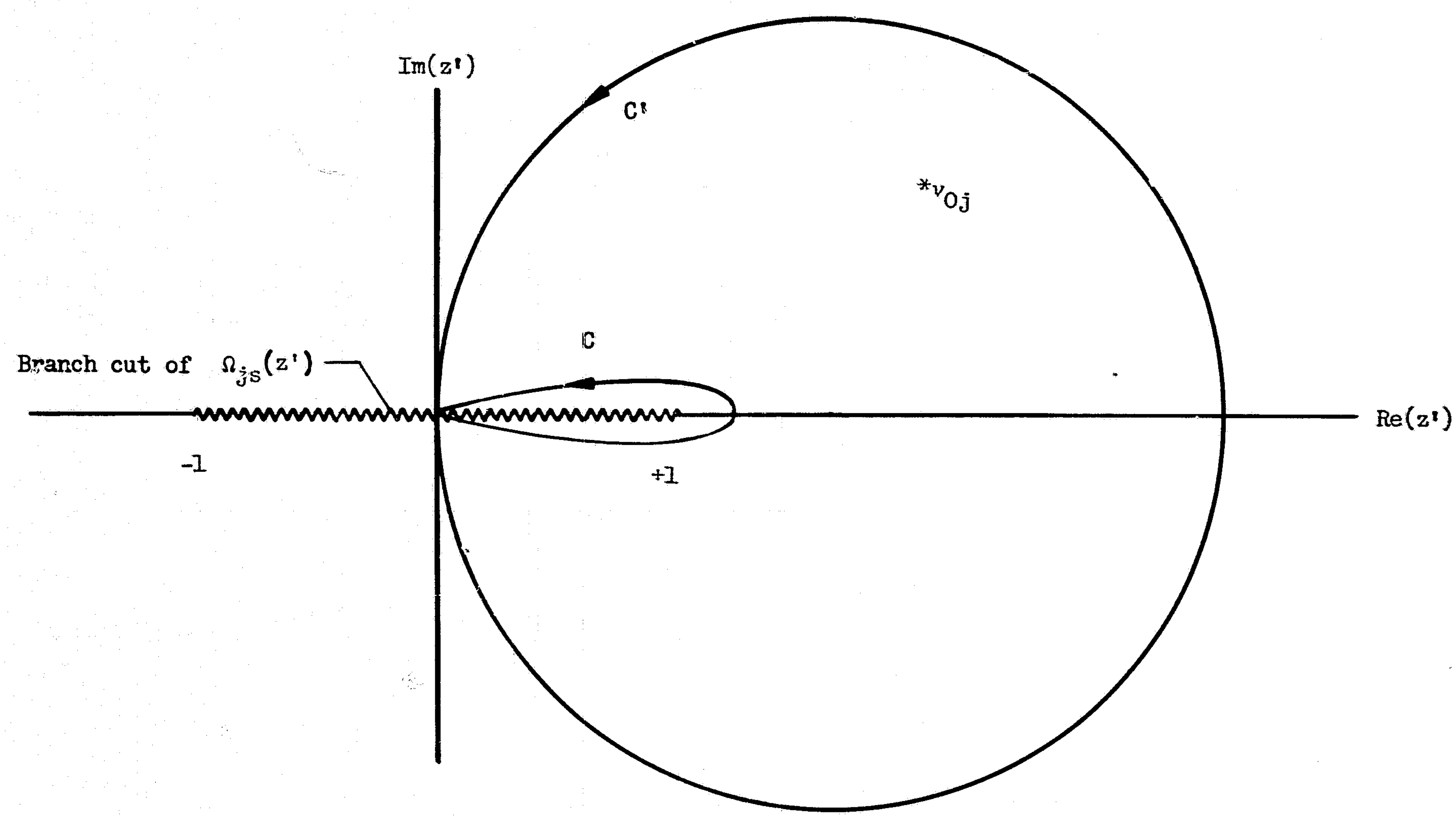

Figure 3.- Contours in the z'-plane.

$$\psi_{1c\pm}(x,\mu,s) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{C'} \frac{E_{1\pm}(z',s)e^{(s+\sigma_1)(x+a)/z'}}{\Omega_{1s}(z')(z'+\mu)}\,dz', & x < -a \\[2ex] \dfrac{\pm 1}{2\pi i}\displaystyle\int_{C'} \frac{E_{1\pm}(z',s)e^{-(s+\sigma_1)(x-a)/z'}}{\Omega_{1s}(z')(z'-\mu)}\,dz', & x > a, \end{cases} \tag{3.37}$$

for $\mathrm{Re}(s) > -\sigma_m$,

$$\psi_{2p\pm}(x,\mu,s) = \frac{1}{2\pi i}\left\{\int_{C'} \frac{L_{2\pm}(x,z',s)}{\Omega_{2s}(z')(z'-\mu)}\,dz' \pm \int_{C'} \frac{L_{2\pm}(-x,z',s)}{\Omega_{2s}(z')(z'+\mu)}\,dz'\right\}, \quad |x| < a, \tag{3.38}$$

for $\mathrm{Re}(s) > -\sigma_2$, and

$$\psi_{1p\pm}(x,\mu,s) = \begin{cases} \dfrac{1}{2\pi i}\left\{\displaystyle\int_{C'} \frac{L_{1\pm}(x,z'',s)}{\Omega_{1s}(z')(z'-\mu)}\,dz' + \int_{C'} \frac{M_{\pm}(x,z',s) \pm L_{1\pm}(-a,z',s)e^{-(s+\sigma_1)(a-x)/z'}}{\Omega_{1s}(z')(z'+\mu)}\,dz'\right\}, & x < -a \\[2ex] \dfrac{1}{2\pi i}\left\{\displaystyle\int_{C'} \frac{L_{1\pm}(-a,z',s)e^{-(s+\sigma_1)(a+x)/z'} \mp M_{\pm}(x,z',s)}{\Omega_{1s}(z')(z'-\mu)}\,dz' \pm \int_{C'} \frac{L_{1\pm}(-x,z',s)}{\Omega_{1s}(z')(z'+\mu)}\,dz'\right\}, & x > a, \end{cases} \tag{3.39}$$

for $\mathrm{Re}(s) > -\sigma_1$.

The functions $M_{\pm}(x,z,s)$ are also integrations over the initial distribution $f_{1\pm}(x,\mu)$ and are given by

$$M_{\pm}(x,z,s) = -\int_{-x}^{-a} e^{-(s+\sigma_1)(x+x_0)/z} \left\{ \frac{1}{2} c_1\sigma_1 \int_0^1 f_{1\pm}(x_0,\mu) \frac{d\mu}{\mu + z} - \frac{1}{2} c_1\sigma_1 \int_0^1 f_{1\pm}(x_0,-\mu) \frac{d\mu}{\mu - z} + \frac{1}{z} f_{1\pm}(x_0,-z)\Omega_{1s}(z) \right\} dx_0, \quad x > a, \tag{3.40}$$

and

$$M_{\pm}(x,z,s) = \int_{x}^{-a} e^{-(s+\sigma_1)(x_0-x)/z} \left\{ \frac{1}{2} c_1\sigma_1 \int_0^1 f_{1\pm}(x_0,\mu) \frac{d\mu}{\mu + z} - \frac{1}{2} c_1\sigma_1 \int_0^1 f_{1\pm}(x_0,-\mu) \frac{d\mu}{\mu - z} + \frac{1}{z} f_{1\pm}(x_0,-z)\Omega_{1s}(z) \right\} dx_0, \quad x < -a, \tag{3.41}$$

for $Re(s) > -\sigma_1$ and z not outside C'. Again, the discrete and continuum terms which appear in Eqs. (3.3), (3.4), (3.8) and (3.9) are due to the zeros and branch cuts of $\Omega_{js}(z)$ which appear in the integrands of Eqs. (3.36)-(3.39).

IV. PROPERTIES OF TRANSFORMED SOLUTION

Analytic properties of $\psi_{s\pm}(x,\mu)$ as a function of s must be investigated before we can recover the time-dependent solution $\psi(x,\mu,t)$ according to the inverse Laplace transformation given by Eq. (2.2). We need to know the behavior of $\psi_{s\pm}$ in some right-half s-plane. Before looking at the details, let us briefly review some results of earlier cited work in which Case's method was used.

In the previously mentioned work of Kuščer and Zweifel (ref. 14) and Erdmann (refs. 8, 9), expansion coefficients could be found explicitly and this aided in the extraction of the s-dependence of their transformed solutions. They find that the branch cuts of $\nu_{0j}(s)$ are inherited by the transformed solution so that the integration contour of the inverse Laplace transformation must be deformed around these branch cuts. For the slab problem solved by Bowden (refs. 1, 4) expansion coefficients could not be found explicitly but theorems of Lehner and Wing (refs. 16, 17) gave the analytic properties of the transformed solution in the s-plane. In that problem, the branch cut of $\nu_0(s)$ is not inherited by the solution. Instead, the transformed solution has a finite number of poles at values of s, say $s_0, \ldots, s_N$, which lie on the branch cut of $\nu_0(s)$, that is, on the real s-axis. These poles contribute a sum of residues as the integration contour is moved to the left of them in the s-plane. Furthermore, in these previously solved time-dependent problems there is a real number, say γ_1, such that the integration contour cannot be deformed into the region $Re(s) < \gamma_1$ for arbitrary values of x. We expect the present transformed solution to exhibit similar properties, that is, $\psi_{s\pm}$ may not be analytic for $\mathrm{Re}(s)$ less than some number γ_1 when x is arbitrary while for $\mathrm{Re}(s)$ greater than γ_1 it should be analytic except for poles and/or branch cuts. Such singularities probably occur where $\nu_{0j}(s)$ has its branch cut.

We first note that for arbitrary initial distributions $f(x,\mu)$, $\psi_{s\pm}(x,\mu)$ is not analytic for $\mathrm{Re}(s) < -\sigma_m$. This is true since each of the inhomogenous terms $I_{j\pm}$ of Eqs. (3.28) and (3.29) contains both $L_{1\pm}$ and $L_{2\pm}$ as can be seen from Eqs. (3.30) and (3.31) and therefore, in general, is not analytic for $\mathrm{Re}(s) < -\sigma_m$, where σ_m is given by Eq. (3.34). In particular, we note that for $|x| > a$, $\psi_{1\pm}(x,\mu,s)$ never appears to be analytic for $\mathrm{Re}(s) < -\sigma_m$. However, for special cases of material properties and initial distributions, $\psi_{2\pm}(x,\mu,s)$ can be shown to be analytic for $-\sigma_2 < \mathrm{Re}(s) < -\sigma_1$ except perhaps for poles.

We now look at the behavior of $\psi_{s\pm}$ for $\mathrm{Re}(s) > -\sigma_m$. Recall that the transform plane for the present problem must be taken as a superposition of two "single-medium" planes, that is, one for each material medium in the problem. The expressions (3.3), (3.4), (3.8) and (3.9) for the transformed solution were not defined for $s\epsilon C_j$ and outwardly appear to be discontinuous at $s\epsilon C_j$. However this is not the case. The complex representation of $E_{j\pm}$ given by Eqs. (3.28) and (3.29) shows that such coefficients are continuous across the curves C_j. Thus it is seen from the representation of $\psi_{s\pm}$ given in Eqs. (3.36)-(3.39) that $\psi_{s\pm}$ is indeed continuous across the curves C_j.

It is convenient to introduce at this time the solution of the associated eigenvalue problem, that is, the solution of Eq. (2.13) subject to the boundary conditions (2.14) and (2.15) with $f_{j\pm}(x,\mu) \equiv 0$. Such solutions, denoted with a bar, have the form

$$\bar{\psi}_{s\pm}(x,\mu) = \begin{cases} = \bar{b}_{1\pm}\psi_{-\nu_{01}}(x,\mu,s)\delta_1(s) + \int_0^1 \bar{B}_{1\pm}(-\nu)\psi_{1(-\nu)}(x,\mu,s)d\nu, & x < -a \\ = \left[\psi_{\nu_{02}}(x,\mu,s) \pm \psi_{-\nu_{02}}(x,\mu,s)\right]\delta_2(s) & \\ \quad + \int_0^1 \bar{B}_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right]d\nu, & |x| < a \\ = \pm\bar{b}_{1\pm}\psi_{\nu_{01}}(x,\mu,s)\delta_1(s) \pm \int_0^1 \bar{B}_{1\pm}(-\nu)\psi_{1\nu}(x,\mu,s)d\nu, & x > a, \end{cases} \tag{4.1}$$

where obviously $\bar{B}_{j\pm}$ and $\bar{b}_{1\pm}$ can be obtained from the $E_{j\pm}$ given by Eqs. (3.28) and (3.29) in the case $f_{j\pm}(x,\mu) \equiv 0$. As we shall see later, the solution $\psi_{s\pm}$ has poles at those values of s for which the associated eigenvalue problem has nontrivial solutions. In Appendix F, it is shown that as the slab thickness becomes very large this eigenvalue problem has only trivial solutions for $Re(s) > -\sigma_2$ except perhaps on the branch cuts of $\nu_{0j}(s)$. When the slab thickness is not large, we still expect that if the eigenvalue problem has nontrivial solutions for $Re(s) > -\sigma_2$, they occur only when s is real. This has been proved rigorously using the method of Lehner and Wing (ref. 25) for several problems which can be obtained as special cases of the present problem: the bare slab considered by Lehner and Wing (refs. 16, 17) and the slab surrounded by pure absorbers considered by Lehner (ref. 15) and Hintz (ref. 10). In all of these problems, there is no scattering in the reflector and, therefore, no branch cut of $\nu_{01}(s)$. As already indicated, the $X_0(z,s)$ function inherits the branch cuts due to both $\nu_{01}(s)$ and $\nu_{02}(s)$ and these branch cuts lie on the real s-axis from $-\sigma_j$ to $-\sigma_j(1-c_j)$ and may or may not overlap depending on the values of material properties. Note that c_1 has been taken less than unity and this insures that the branch cut of ν_{01} lies entirely to the left of $s = 0$. In previously solved time-dependent problems, singularities of the transformed solution always occur where the $\nu_{0j}(s)$ has branch cuts. Since the analysis of Appendix F indicates that for large values of the slab half-thickness, a, the singularities of $\psi_{s\pm}$ for $Re(s) > -\sigma_m$ also occur where the $\nu_{0j}(s)$ have branch cuts, we will assume for all values of a that the singularities of $\psi_{s\pm}$ occur on the branch cuts of $\nu_{0j}(s)$. In any case, we show that the only other singularities of $\psi_{s\pm}$, $Re(s) > -\sigma_m$ which could occur off the branch cuts of $\nu_{0j}(s)$ are poles, whose residue could readily be added to the time-dependent solution.

In order to see the behavior of $\psi_{s\pm}$ on the branch cuts of $\nu_{0j}(s)$ we first look at $\overline{\psi}_{s\pm}$ in the region $s \epsilon S_{1i} \cap S_{2i}$. For this region, the expansion coefficients are given by the equations (see App. G)

$$\bar{B}_{2\pm}(\mu) = \pm \frac{ks}{2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{(\nu_{02}{}^2 - \mu^2)}{(\nu_{01}{}^2 - \mu^2)} \frac{h_2(\mu)}{g_2(\mu)}$$

$$\times \left\{ \frac{h_2(\nu_{02})}{\mu + \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\mu - \nu_{02}} + \int_0^1 \bar{B}_{2\pm}(\nu) h_2(\nu) \frac{d\nu}{\nu + \mu} \right\}, \ 0 \leq \mu \leq 1, \tag{4.2}$$

$$\bar{B}_{1\pm}(-\mu) \mp \frac{c_2\sigma_2}{c_1\sigma_1} \bar{B}_{2\pm}(\mu) \ e^{(\sigma_1 - \sigma_2)a/\mu}$$

$$= \pm \frac{ks}{2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{h_1(\mu)}{g_1(\mu)} \left\{ \frac{h_2(\nu_{02})}{\mu - \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\mu + \nu_{02}} \right.$$

$$\left. + \int_0^1 B_{2\pm}(\nu) h_2(\nu) \frac{2\varphi_{1s\,\mu}(\nu)}{c_1\sigma_1\mu} \, d\nu \right\} \tag{4.3}$$

and

$$\mp h_1(-\nu_{01}) \bar{b}_{1\pm} = h_2(\nu_{02}) \pm h_2(-\nu_{02})$$

$$+ (\nu_{02}{}^2 - \nu_{01}{}^2) \int_0^1 \bar{B}_{2\pm}(\nu) h_2(\nu) \frac{d\nu}{\nu^2 - \nu_{01}{}^2}, \tag{4.4}$$

where

$$h_2(\omega) = \omega \frac{X_{2s}(-\omega)}{X_{1s}(-\omega)} e^{-(s+\sigma_2)a/\omega}$$

$$h_1(\omega) = \frac{\Omega_{1s}(\infty)}{\Omega_{2s}(\infty)} \omega \frac{X_{1s}(-\omega)}{X_{2s}(-\omega)} e^{(s+\sigma_1)a/\omega}$$

and

$$g_j(\mu) = \mu \, \Omega_{js}^{+}(\mu) \, \Omega_{js}^{-}(\mu). \tag{4.5}$$

In addition the eigenvalue condition

$$0 = \frac{h_2(\nu_{02})}{\nu_{01} + \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\nu_{01} - \nu_{02}} + \int_0^1 \bar{B}_{2\pm}(\nu) h_2(\nu) \frac{d\nu}{\nu + \nu_{01}} \tag{4.6}$$

must be satisfied. Since the eigenvalue condition (4.6) has different limiting values as s approaches the branch cut of $\nu_{01}(s)$, we conclude that there are only trivial solutions of the associated eigenvalue problem on the $\nu_{01}(s)$ cut. When s belongs to the branch cut of $\nu_{02}(s)$, which is not also part of the $\nu_{01}(s)$ cut, that is, when $\mathrm{Re}(\nu_{02}) = \mathrm{Im}(\nu_{01}) = 0$, it appears that nontrivial solutions of the associated eigenvalue problem may exist. From Bowden's results (refs. 1, 4) for the bare slab, it is expected that Eqs. (4.2) and (4.6) are satisfied only at isolated points, $\{s_n\}$. In the limit $c_2\sigma_2 a \to \infty$ these points lie on the branch cut of $\nu_{02}(s)$, that is, the s_n are real. The "thick-slab" eigenvalue condition is seen from Eqs. (4.2) and (4.6) to be Eq. (4.6) with $\bar{B}_{2\pm}(\mu) = 0$.

If material properties are such that $-\sigma_2 < -\sigma_1$, then a portion of the branch cut of $\nu_{02}(s)$ lies in $s \epsilon S_{2i} \cap S_{1e}$. In this region however, $s < -\sigma_m = -\sigma_1$ and for such values, the solution $\bar{\Psi}_{s\pm}(x,\mu)$, $|x| > a$, that is $\bar{\Psi}_{1\pm}$, is not bounded as $|x| \to \infty$. However, $\bar{\Psi}_{2\pm}$ may have nontrivial solutions on such a portion of the branch cut of $\nu_{02}(s)$. The equation for $\bar{B}_{2\pm}$ and the additional constraint for this region are (see App. G)

$$\overline{B}_{2\pm}(\mu) = \pm \frac{ks}{2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\alpha)} (\nu_{02}^2 - \mu^2) \frac{h_2(\mu)}{g_2(\mu)} \frac{X_{1s}(-\mu)}{X_{01s}(-\mu)}$$

$$\times \left\{ \frac{h_2(\nu_{02})}{\mu + \nu_{02}} \frac{X_{1s}(-\nu_{02})}{X_{01s}(-\nu_{02})} \pm \frac{h_2(-\nu_{02})}{\mu - \nu_{02}} \frac{X_{1s}(\nu_{02})}{X_{01s}(\nu_{02})} \right.$$

$$\left. + \int_0^1 \overline{B}_{2\pm}(\nu) h_2(\nu) \frac{X_{1s}(-\nu)}{X_{01s}(-\nu)} \frac{d\nu}{\nu + \mu} \right\}, \quad 0 \leq \mu \leq 1 \tag{4.7}$$

and

$$0 = h_2(\nu_{02}) \frac{X_{1s}(-\nu_{02})}{X_{01s}(-\nu_{02})} \pm h_2(-\nu_{02}) \frac{X_{1s}(\nu_{02})}{X_{01s}(\nu_{02})}$$

$$+ \int_0^1 \overline{B}_{2\pm}(\nu) h_2(\nu) \frac{X_{1s}(-\nu)}{X_{01s}(-\nu)} d\nu. \tag{4.8}$$

As we shall see later, the zeros of Eq. (4.8) can, under some conditions, be poles of $\psi_{2\pm}$ and therefore may contribute discrete modes in $\psi(x,\mu,t)$, $|x| < a$. For this reason we are interested in where these zeros lie and shall refer to them as pseudo-eigenvalues.

We now show how the solution of the associated eigenvalue problem $\overline{\psi}_{s\pm}$, is contained in the inhomogeneous solution, $\psi_{s\pm}$, by following a procedure of Bowden and Williams (ref. 4). In Appendix H, it is shown that the original expansion coefficients of Eqs. (3.3) and (3.4) can be written as

$$A_{j\pm}(\mu) = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right] \bar{B}_{j\pm}(\mu) + B_{j\pm}(\mu)$$

and

$$a_{1\pm} = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right] \bar{b}_{1\pm} + b_{1\pm}, \quad s\epsilon S_{1i} \cap S_{2i} \tag{4.9}$$

where $\bar{B}_{j\pm}$ and $\bar{b}_{1\pm}$ are given by Eqs. (4.2) - (4.4). The coefficients $B_{j\pm}$ and $b_{1\pm}$ are given by

$$\begin{aligned}
B_{2\pm}(\nu) = {} & \frac{c_1\sigma_1}{c_2\sigma_2} F_{1\pm}(-a,\nu,s)e^{(\sigma_1-\sigma_2)a/\nu} \\
& \pm \frac{ks}{2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{(\nu_{02}{}^2 - \nu^2)}{(\nu_{01}{}^2 - \nu^2)} \frac{h_2(\nu)}{g_2(\nu)} \\
& \times \left\{ \int_0^1 B_{2\pm}(\mu)h_2(\mu) \frac{d\mu}{\mu + \nu} \right. \\
& + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[\frac{h_2(\nu_{02})}{\nu + \nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu - \nu_{02}}\right] \\
& + \int_0^1 F_{2\pm}(a,\mu,s)h_2(\mu) \frac{d\mu}{\mu + \nu} \\
& \left. \mp \int_0^1 F_{1\pm}(-a,\mu,s)h_1(\mu) \frac{\nu_{01}{}^2 - \mu^2}{\nu_{02}{}^2 - \mu^2} \frac{2\varphi_{2s\nu}(\mu)}{c_2\sigma_2\nu} d\mu \right\},
\end{aligned} \tag{4.10}$$

$$B_{1\pm}(-\nu) \mp \frac{c_2\sigma_2}{c_1\sigma_1} B_{2\pm}(\nu)e^{(\sigma_1-\sigma_2)a/\nu} = \pm \frac{ks}{2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{h_1(\nu)}{g_1(\nu)}$$

$$\times \left\{ \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[\frac{h_2(\nu_{02})}{\nu - \nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu + \nu_{02}}\right] \right.$$

$$+ \int_0^1 \left[B_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu) \frac{2\varphi_{1s\nu}(\mu)}{c_1\sigma_1\nu} d\mu$$

$$\left. \mp \int_0^1 F_{1\pm}(-a,\mu,s)h_1(\mu) \frac{(\nu_{01}^2 - \mu^2)}{(\nu_{02}^2 - \mu^2)} \frac{d\mu}{\mu + \nu} \right\}$$

$$\mp \left[F_{1\pm}(-a,\nu,s) - \frac{c_2\sigma_2}{c_1\sigma_1} F_{2\pm}(a,\nu,s)e^{(\sigma_1-\sigma_2)a/\nu}\right] \tag{4.11}$$

and

$$\mp h_1(-\nu_{01})\left[b_{1\pm} - \tilde{F}_{\pm}(-a,\nu_{01},s)\right] = \beta_{1\pm}. \tag{4.12}$$

The coefficient $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]$ is given by

$$\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right] = \frac{-\nu_{01}\beta_{1\pm} + \beta_{2\pm}}{(\nu_{01}\alpha_{1\pm} - \alpha_{2\pm})} . \tag{4.13}$$

In these equations, $\alpha_{j\pm}$ and $\beta_{j\pm}$ are

$$\alpha_{1\pm} = h_2(\nu_{02}) \pm h_2(-\nu_{02}) + (\nu_{02}^2 - \nu_{01}^2) \int_0^1 \bar{B}_{2\pm}(\mu)h_2(\mu) \frac{d\mu}{\mu^2 - \nu_{01}^2} ,$$

$$\alpha_{2\pm} = \nu_{02}h_2(\nu_{02}) \mp \nu_{02}h_2(-\nu_{02}) + (\nu_{02}^2 - \nu_{01}^2) \int_0^1 \bar{B}_{2\pm}(\mu)h_2(\mu) \frac{\mu d\mu}{\mu^2 - \nu_{01}^2}, \tag{4.14}$$

$$\beta_{1\pm} = \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[h_2(\nu_{02}) \mp h_2(-\nu_{02})\right]$$

$$+ (\nu_{02}{}^2 - \nu_{01}{}^2)\int_0^1 \left[B_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu)\,\frac{d\mu}{\mu^2 - \nu_{01}{}^2}$$

$$\pm (\nu_{01}{}^2 - \nu_{02}{}^2)\int_0^1 F_{1\pm}(-a,\mu,s)h_1(\mu)\,\frac{d\mu}{\mu^2 - \nu_{02}{}^2}$$

$$\pm \left[F_{1+}(-a,\nu_{01},s)h_1(\nu_{01}) + F_{1\pm}(-a,-\nu_{01},s)h_1(-\nu_{01})\right] \qquad (4.15a)$$

and

$$\beta_{2\pm} = \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\nu_{02}\left[h_2(\nu_{02}) \pm h_2(-\nu_{02})\right]$$

$$+ (\nu_{02}{}^2 - \nu_{01}{}^2)\int_0^1 \left[B_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu)\,\frac{\mu\, d\mu}{\mu^2 - \nu_{01}{}^2}$$

$$\mp (\nu_{01}{}^2 - \nu_{02}{}^2)\int_0^1 F_{1\pm}(-a,\mu,s)h_1(\mu)\,\frac{\mu\, d\mu}{\mu^2 - \nu_{02}{}^2}$$

$$\mp \left[\nu_{01}F_{1\pm}(-a,\nu_{01},s)h_1(\nu_{01}) - \nu_{01}F_{1\pm}(-a,-\nu_{01},s)h_1(-\nu_{01})\right].$$

(4.15b)

In terms of these quantities, the solutions $\psi_{j\pm}$ can be written as

$$\psi_{2\pm}(x,\mu,s) = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\overline{\psi}_{2\pm}(x,\mu,s)$$

$$+ \int_0^1 B_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right] d\nu$$

$$+ \int_0^1 \left[F_{2\pm}(x,\nu,s)\psi_{2\nu}(x,\mu,s) \pm F_{2\pm}(-x,\nu,s)\psi_{2(-\nu)}(x,\mu,s)\right] d\nu$$

$$+ \frac{1}{2}\left[F_{2\pm}(x,\nu_{02},s) \pm F_{2\pm}(-x,-\nu_{02},s)\right]\psi_{\nu_{02}}(x,\mu,s)$$

$$+ \frac{1}{2}\left[F_{2\pm}(x,-\nu_{02},s) \pm F_{2\pm}(-x,\nu_{02},s)\right]\psi_{-\nu_{02}}(x,\mu,s), \quad |x| < a, \tag{4.16}$$

and

$$\psi_{1\pm}(x,\mu,s) = \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\overline{\psi}_{1\pm}(x,\mu,s)$$

$$\pm \left[b_{1\pm} - \tilde{F}_{\pm}(-a,\nu_{01},s) + F_{1\pm}(-x,-\nu_{01},s)\right]\psi_{\nu_{01}}(x,\mu,s)$$

$$\pm F_{1\pm}(-x,\nu_{01},s)\psi_{-\nu_{01}}(x,\mu,s)$$

$$\pm \int_0^1 \left[B_{1\pm}(-\nu) - \tilde{F}_{\pm}(-a,\nu,s) + F_{1\pm}(-x,-\nu,s)\right]\psi_{1\nu}(x,\mu,s)$$

$$\pm \int_0^1 F_{1\pm}(-x,\nu,s)\psi_{1(-\nu)}(x,\mu,s)d\nu, \quad x > a. \tag{4.17}$$

The solution $\psi_{1\pm}(x,\mu,s)$ for $x < -a$ has a similar form. In these equations, $\bar{\psi}_{j\pm}(x,\mu,s)$ are the parts of $\bar{\psi}_{s\pm}(x,\mu)$ which are given by Eq. (4.1). Equation (4.4) is written in terms of $\alpha_{1\pm}$ as

$$\mp h_1(-\nu_{01})\bar{b}_{1\pm} = \alpha_{1\pm}. \tag{4.18}$$

Consider now what happens on the branch cut of $\nu_{01}(s)$ where $\nu_{01} = i\,|\nu_{01}|$ for $\mathrm{Im}(s) = 0^-$ and $\nu_{01} = -i\,|\nu_{01}|$ for $\mathrm{Im}(s) = 0^+$. From the above equations it can be seen that the quantities $\bar{B}_{2\pm}$, $\bar{B}_{1\pm}$, $B_{2\pm}$, $B_{1\pm}$, $\alpha_{1\pm}$, $\alpha_{2\pm}$, $\beta_{1\pm}$ and $\beta_{2\pm}$ do not inherit the branch cut of $\nu_{01}(s)$. Equations (4.18) and (4.12) show that $\bar{b}_{1\pm}$ and $b_{1\pm}$ have branch cuts due to that of $\nu_{01}(s)$. Equation (4.13) indicates that $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]$ has the branch cut due to $\nu_{01}(s)$ unless $\alpha_{1\pm}/\alpha_{2\pm}$ is equal to $\beta_{1\pm}/\beta_{2\pm}$. In general, this will not be true since $\beta_{1\pm}/\beta_{2\pm}$ depends on the arbitrary initial distribution $f_\pm(x,\mu)$ whereas $\alpha_{1\pm}/\alpha_{2\pm}$ does not. Therefore, it is concluded that both $\psi_{1\pm}$ and $\psi_{2\pm}$ inherit the branch cut of $\nu_{01}(s)$.

On the branch cut of $\nu_{02}(s)$, the quantities $B_{2\pm}$, $B_{1\pm}$, $b_{1\pm}$, $\beta_{1\pm}$ and $\beta_{2\pm}$ are single-valued. Since the quantities $\alpha_{1\pm}$ and $\alpha_{2\pm}$ of Eq. (4.14) are related above and below the branch cut of $\nu_{02}(s)$ by

$$\left[\alpha_{j\pm}\right]^+ = \pm\left[\alpha_{j\pm}\right]^-, \tag{4.19}$$

it follows then from Eq. (4.13) that on that part of the branch cut of $\nu_{02}(s)$ which is not also part of the $\nu_{01}(s)$ cut; that is, for $\mathrm{Re}(\nu_{02}) = \mathrm{Im}(\nu_{01}) = 0$; we have

$$\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^{+} = \pm\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^{-}, \tag{4.20}$$

if the denominator on the RHS of Eq. (4.13) does not vanish. It is seen from Eqs. (4.1), (4.2), (4.3), and (4.4) that for this same region,

$$\left[\overline{\psi}_{s\pm}(x,\mu)\right]^{+} = \pm\left[\overline{\psi}_{s\pm}(x,\mu)\right]^{-}. \tag{4.21}$$

Hence the product

$$\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right] \overline{\psi}_{s\pm}(x,\mu), \tag{4.22}$$

which appears in $\psi_{s\pm}$ does not inherit the branch cut of $\nu_{02}(s)$. However, the denominator of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]$, namely $(\nu_{01}\alpha_{1\pm}-\alpha_{2\pm})$ is equivalent to the eigenvalue condition, Eq. (4.6). Thus, if the associated eigenvalue problem has a nontrivial solution at $s = s_n$, $\mathrm{Re}(s) > -\sigma_m$, then $\psi_{s\pm}$ has a pole there.

We briefly summarize the analytic properties of the transformed solution $\psi_{s\pm}(x,\mu)$. For arbitrary initial distributions $f_{\pm}(x,\mu)$, $\psi_{s\pm}$ is not analytic to the left of $\mathrm{Re}(s) = -\sigma_m$ in the s-plane, whereas to the right of $\mathrm{Re}(s) = -\sigma_m$ it is analytic except for the branch cut along $\left[-\sigma_m, -\sigma_1(1-c_1)\right]$ (due to the branch cut of $\nu_{01}(s)$) if $\sigma_m > \sigma_1(1-c_1)$ and poles at the values of s at which the associated eigenvalue problem has nontrivial solutions, $\overline{\psi}_{s\pm}$. We have assumed that for arbitrary slab thicknesses, a, these poles, if they exist, lie on the branch cut of $\nu_{02}(s)$ since this is the rigorous result obtained by others for several special cases of the present problem and obtained herein for the case when $c_2\sigma_2 a$ is large. For special values of material properties and initial data, $\psi_{s\pm}(x,\mu)$ for $|x| < a$ (that is $\psi_{2\pm}$) may be analytic in the region $-\sigma_2 < \mathrm{Re}(s) < -\sigma_1$ except perhaps for poles.

V. RECOVERY OF TIME-DEPENDENT SOLUTION

The time-dependent solution $\psi(x,\mu,t)$ is obtained from the inverse Laplace transformation Eq. (2.2), where γ is to the right of all singularities of $\psi_s(x,\mu)$ in the s-plane. From the analysis of the preceding section we expect that we can choose any $\gamma > \max\left[-\sigma_1(1-c_1), -\sigma_2(1-c_2)\right]$. In order to show the time dependence of the solution $\psi(x,\mu,t)$ more explicitly, we deform the inversion contour as far as possible to the left in the s-plane by making use of the analytic properties of $\psi_s(x,\mu)$ obtained in section IV.

We first look at the behavior of $\psi_{s\pm}$ on the contour $\mathrm{Re}(s) = \gamma$. This contour crosses both of the curves C_j and it has been shown that $\psi_{s\pm}$ is continuous across these curves. As $|s| \to \infty$ on such a contour, $s \epsilon S_{1e} \cap S_{2e}$ and we show in Appendix 1 that $\psi_{s\pm}$ behaves as follows

$$\psi_{2\pm}(x,\mu,s) \to \frac{1}{\mu}\int_{-a}^{x} e^{-(s+\sigma_2)(x-x_0)/\mu}\left[f_{2\pm}(x_0,\mu) + O\left(\frac{1}{s}\right)\right]dx_0$$
$$+ \frac{e^{-(\sigma_2-\sigma_1)\left(\frac{a+x}{\mu}\right)}}{\mu}\int_{-\infty}^{-a} e^{-(s+\sigma_1)\left(\frac{x-x_0}{\mu}\right)}\left[f_{1\pm}(x_0,\mu) + O\left(\frac{1}{s}\right)\right]dx_0 \tag{5.1}$$

for $|x| < a$ and $\mu > 0$;

$$\psi_{2\pm}(x,\mu,s) \to \frac{1}{|\mu|}\int_{x}^{a} e^{-(s+\sigma_2)\left(\frac{x_0-x}{|\mu|}\right)}\left[f_{2\pm}(x_0,-|\mu|) + O\left(\frac{1}{s}\right)\right]dx_0$$
$$+ \frac{e^{-(\sigma_2-\sigma_1)\left(\frac{a-x}{|\mu|}\right)}}{|\mu|}\int_{a}^{\infty} e^{-(s+\sigma_1)\left(\frac{x_0-x}{|\mu|}\right)}\left[f_{1\pm}(x_0,-|\mu|) + O\left(\frac{1}{s}\right)\right]dx_0 \tag{5.2}$$

for $|x| < a$ and $\mu < 0$;

$$\psi_{1\pm}(x,\mu,s) \to \frac{e^{-(\sigma_2-\sigma_1)\left(\frac{a-x}{\mu}\right)}}{\mu}\int_{-a}^{a} e^{-(s+\sigma_2)\left(\frac{x-x_0}{\mu}\right)}\left[f_{2\pm}(x_0,\mu) + O\left(\frac{1}{s}\right)\right]dx_0$$

$$+ \frac{e^{-(\sigma_2-\sigma_1)\frac{2a}{\mu}}}{\mu}\int_{-\infty}^{-a} e^{-(s+\sigma_1)\left(\frac{x-x_0}{\mu}\right)}\left[f_{1\pm}(x_0,\mu) + O\left(\frac{1}{s}\right)\right]dx_0$$

$$+ \frac{1}{\mu}\int_{a}^{x} e^{-(s+\sigma_1)\left(\frac{x-x_0}{\mu}\right)}\left[f_{1\pm}(x_0,\mu) + O\left(\frac{1}{s}\right)\right]dx_0 \qquad (5.3)$$

for $x > a$ and $\mu > 0$ and

$$\psi_{1\pm}(x,\mu,s) \to \frac{1}{|\mu|}\int_{x}^{\infty} e^{-(s+\sigma\ )\left(\frac{x_0-x}{|\mu|}\right)}\left[f_{1\pm}(x_0,-|x|) + O\left(\frac{1}{s}\right)\right]dx_0 \qquad (5.4)$$

for $x > a$ and $\mu < 0$. Expressions similar to Eqs. (5.3) and (5.4) are obtained for $x < -a$. It is seen that $\psi_{s\pm}$ is not necessarily $O\left(\frac{1}{s}\right)$. However, the parts which are not can be easily inverted as follows. Define <u>for all s</u> the function $\psi_{us\pm}(x,\mu)$ as that part of each of Eqs. (5.1) - (5.4) which is not $O\left(\frac{1}{s}\right)$. We show in Appendix I that upon making the substitution

$$\begin{aligned} x - x_0 &= \mu t, \quad \mu > 0 \\ x_0 - x &= |\mu| t, \ \mu < 0, \end{aligned} \qquad (5.5)$$

that $\psi_{us\pm}(x,\mu)$ can be written as

$$\psi_{us\pm}(x,\mu) = \int_0^{\infty} e^{-st}\left[\psi_{u\pm}(x,\mu,t)\right] dt. \qquad (5.6)$$

That is, the parts of $\psi_{s\pm}$ which do not behave as $O\left(\frac{1}{s}\right)$ as $|s| \to \infty$, $\mathrm{Re}(s) = \gamma$ can be inverted by inspection. The solution $\psi_{u\pm}(x,\mu,t)$ is given by

$$\psi_{u\pm}(x,\mu,t) = \begin{cases} e^{-\sigma_2 t} \; f_{2\pm}(x-\mu t,\mu) & , \quad t < \dfrac{a+x}{\mu} \\ e^{-\sigma_1 t} \; e^{-(\sigma_2-\sigma_1)\frac{a+x}{\mu}} \; f_{1\pm}(x-\mu t,\mu), & \quad t > \dfrac{a+x}{\mu}, \end{cases} \tag{5.7}$$

for $|x| < a$ and $\mu > 0$;

$$\psi_{u\pm}(x,\mu,t) = \begin{cases} e^{-\sigma_2 t} \; f_{2\pm}(x-\mu t,\mu) & , \quad t < \dfrac{a-x}{|\mu|} \\ e^{-\sigma_1 t} \; e^{(\sigma_2-\sigma_1)\frac{a-x}{\mu}} \; f_{1\pm}(x-\mu t,\mu), & \quad t > \dfrac{a-x}{|\mu|}, \end{cases} \tag{5.8}$$

for $|x| < a$ and $\mu < 0$;

$$\psi_{u\pm}(x,\mu,t) = \begin{cases} e^{-\sigma_1 t} \; f_{1\pm}(x-\mu t,\mu) & , \quad t < \dfrac{x-a}{\mu} \\ e^{-\sigma_2 t} \; e^{(\sigma_2-\sigma_1)\left(\frac{x-a}{\mu}\right)} \; f_{2\pm}(x-\mu t,\mu), & \quad \dfrac{x-a}{\mu} < t < \dfrac{x+a}{\mu} \\ e^{-\sigma_1 t} \; e^{-(\sigma_2-\sigma_1)\frac{2a}{\mu}} \; f_{1\pm}(x-\mu t,\mu), & \quad t > \dfrac{x+a}{\mu}, \end{cases} \tag{5.9}$$

for $x > a$ and $\mu > 0$ and

$$\psi_{u\pm}(x,\mu,t) = e^{-\sigma_1 t} \; f_{1\pm}(x-\mu t,\mu) \tag{5.10}$$

for $x > a$ and $\mu < 0$. That $\psi_{u\pm}$ describes the motion of uncollided neutrons from the initial distribution can be seen by direct substitution, that is, $\psi_{u\pm}$ satisfies the uncollided equation

$$\frac{\partial\psi_{u\pm}}{\partial t} + \mu\,\frac{\partial\psi_{u\pm}}{\partial x} + \sigma(x)\psi_{u\pm} = 0.$$

In the limit $t \to 0$, we note that

$$\psi_{u\pm}(x,\mu,0) = f_{\pm}(x,\mu). \tag{5.11}$$

For arbitrary $f(x,\mu)$ which vanishes as $|x| \to \infty$, $\psi_{us\pm}(x,\mu)$ given by Eqs. (5.6) and (5.7) - (5.10) is an analytic function of s for $\mathrm{Re}(s) > -\sigma_m$ for almost all x and μ. If $f_{1\pm} \equiv 0$ $(f_{2\pm} \equiv 0)$, then $\psi_{us\pm}$ is an analytic function of s for $\mathrm{Re}(s) > -\sigma_2$ $(\mathrm{Re}(s) > -\sigma_1)$. Therefore the function $\Phi_{s\pm}(x,\mu)$ defined as

$$\Phi_{s\pm}(x,\mu) \equiv \psi_{s\pm}(x,\mu) - \psi_{us\pm}(x,\mu), \quad \mathrm{Re}(s) > -\sigma_m, \tag{5.12}$$

has the same analytic properties as $\psi_{s\pm}$ in the right-half plane $\mathrm{Re}(s) > -\sigma_m$ except that it is $O\left(\frac{1}{s}\right)$ as $|s| \to \infty$. If $\psi_{s\pm}$ has a branch cut along $\left[-\sigma_m, -\sigma_1(1-c_1)\right]$; i.e., if $\sigma_m > \sigma_1(1-c_1)$; then

$$\left[\Phi_{s\pm}\right]^- - \left[\Phi_{s\pm}\right]^+ = \left[\psi_{s\pm}\right]^- - \left[\psi_{s\pm}\right]^+. \tag{5.13}$$

Similarly if $\psi_{s\pm}$ has a pole at $s = s_n$, then

$$\text{Residue}\left(\Phi_{s\pm}\right)_{s_n} = \text{Residue}\left(\psi_{s\pm}\right)_{s_n}. \tag{5.14}$$

The definite parity parts of the time-dependent solution therefore can be written from Eq. (2.2) as

$$\psi_{\pm}(x,\mu,t) = \psi_{u\pm}(x,\mu,t) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Phi_{s\pm}(x,\mu)e^{st}\, ds. \qquad (5.15)$$

Now using the analytic properties, we can deform the contour to the left and obtain in general

$$\psi_{\pm}(x,\mu,t) = \psi_{u\pm}(x,\mu,t) + \sum_{s=s_n} \text{Residue}\left[\psi_{s\pm}(x,\mu)e^{st}\right]$$

$$+ \frac{1}{2\pi i}\int_{-\sigma_m}^{-\sigma_1(1-c_1)} \left\{\left[\psi_{s\pm}(x,\mu)\right]^{-} - \left[\psi_{s\pm}(x,\mu)\right]^{+}\right\} e^{st}\, ds$$

$$+ \frac{1}{2\pi i}\int_{-\sigma_m-i\infty}^{-\sigma_m+i\infty} \left[\psi_{s\pm}(x,\mu) - \psi_{us\pm}(x,\mu)\right] e^{st}\, ds$$

$$+ \frac{1}{2\pi i}\lim_{\rho\to 0}\int_{C_\rho} \psi_{s\pm}(x,\mu)e^{st}\, ds, \quad -\sigma_m < -\sigma_1(1-c_1) < s_n, \qquad (5.16)$$

where C_ρ is a small circular contour of radius ρ with center at $s = -\sigma_1(1-c_1)$. Generally the point $s = -\sigma_1(1-c_1)$ will not satisfy the eigenvalue conditions, Eq. (4.6), and the contribution from the contour C_ρ vanishes as $\rho \to 0$. If however $s = -\sigma_1(1-c_1)$ happens to satisfy Eq. (4.6), the contribution from the contour C_ρ has the form of a discrete residue term. Details concerning this point are discussed in Appendices I and K.

Equation (5.16) is the solution of the time-dependent problem written in a form in which the uncollided portion of the initial distribution $f(x,\mu)$ has been separated. For arbitrary $f(x,\mu)$ the contour cannot be deformed further to the left. In the final section it will be shown how this solution reduces to those obtained previously by others for special cases of the present problem.

We close this section by indicating the form of some parts of Eq. (5.16). The uncollided term, $\psi_{u\pm}(x,\mu,t)$ is given explicitly by Eqs. (5.7) - (5.10). The form of $\psi_{s\pm}(x,\mu)$ on the branch cut $\left[-\sigma_m, -\sigma_1(1-c_1)\right]$ was given in section IV. From those results, it is seen that on this branch cut $\left[\psi_{s\pm}(x,\mu)\right]^- - \left[\psi_{s\pm}(x,\mu)\right]^+$ can be written from Eqs. (4.16) and (4.17) as

$$\left[\psi_{s\pm}(x,\mu)\right]^- - \left[\psi_{s\pm}(x,\mu)\right]^+ = \left\{\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^- - \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^+\right\} \overline{\psi}_{2\pm}(x,\mu,s) \tag{5.17}$$

for $|x| < a$ and

$$\begin{aligned}\left[\psi_{s\pm}(x,\mu)\right]^- - \left[\psi_{s\pm}(x,\mu)\right]^+ = {}& \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^- \left[\overline{\psi}_{1\pm}(x,\mu,s)\right]^- \\ & - \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]^+ \left[\overline{\psi}_{1\pm}(x,\mu,s)\right]^+ \\ & \pm \left\{\left[b_{1\pm} - \tilde{F}_{\pm}(-a,\nu_{01},s)\right]^- \psi_{\nu_{01}}(x,\mu,s) \right. \\ & \left. - \left[b_{1\pm} - \tilde{F}_{\pm}(-a,\nu_{01},s)\right]^+ \psi_{-\nu_{01}}(x,\mu,s)\right\}\end{aligned} \tag{5.18}$$

for $x > a$, where $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right]$ is given by Eq. (4.13), $\overline{\Psi}_{j\pm}(x,\mu,s)$ by Eq. (4.1) and $\left[b_{1\pm} - \tilde{F}_{\pm}(-a, \nu_{01}, s)\right]$ by Eq. (4.12). The solution $\Psi_{s\pm}(x,\mu)$ has poles at $s = s_0, \ldots, s_N$ due to the poles of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right] \nu_{02}^{(1\mp 1)/2}$. Again, from the results given in section IV, it follows that

$$\text{Residue} \left[\Psi_{s\pm}(x,\mu)e^{st}\right]_{s_n} = e^{s_n t} \left\{\overline{\Psi}_{s\pm}(x,\mu)\left[\nu_{02}^{-(1\mp 1)/2}\right]\right\}_{s_n}$$

$$\times \text{Residue} \left\{\nu_{02}^{(1\mp 1)/2}\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right]\right\}_{s_n} . \tag{5.19}$$

Note that the factor $\nu_{02}^{(1\mp 1)/2}$ is introduced so that $\overline{\Psi}_{s\pm}\nu_{02}^{-(1\mp 1)/2}$ and $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right] \nu_{02}^{(1\mp 1)/2}$ are single-valued on the branch cut of ν_{02} [cf., Eqs. (4.20) and (4.21)]. These terms have an exponential time dependence, $e^{s_n t}$, and we have obtained the implicit Eqs., namely, (4.2) and (4.6), from which the eigenvalues $\{s_n\}$ can be computed. Since information concerning the behavior of eigenvalues (i.e., number, location, etc.) as a function of material properties is not readily obtained analytically from such expressions, we have made a numerical study of real time eigenvalues and the results are discussed in the next section.

2

VI. CALCULATION OF REAL TIME EIGENVALUES

We first note that the eigenvalues and pseudo-eigenvalues depend on five parameters (c_1, σ_1, c_2, σ_2, and a) and therefore many numerical computations would be required in order to see the specific dependence on each parameter. As we shall see, the bare-slab results of Bowden (refs. 1, 4), the theorems of Hintz (ref. 10) for slabs surrounded by pure absorbers and some observations of the present numerical results for a few reflected slab cases allow us to draw some conclusions about the behavior of eigenvalues for reflected slabs as a function of the slab half-thickness a. However, rather than compute eigenvalues $\{s_n\}$ in terms of c_1, σ_1, c_2, σ_2, and a we define a nondimensional variable ζ and nondimensional parameters σ_R, σ_D, and A as

$$\zeta = \frac{s + \sigma_2}{c_2\sigma_2}, \quad \sigma_R = \frac{c_1\sigma_1}{c_2\sigma_2}, \quad \sigma_D = \frac{\sigma_1 - \sigma_2}{c_2\sigma_2} \quad \text{and} \quad A = c_2\sigma_2 a. \tag{6.1}$$

In terms of these quantities, the branch cut of ν_{02} becomes the real interval $(0,1)$ and the branch cut of ν_{01} becomes the real interval $(-\sigma_D, -\sigma_D + \sigma_R)$. Since σ_j and c_j are non-negative, it follows that

$$-\sigma_D \leq \frac{1}{c_2}, \tag{6.2}$$

where the equality holds only if $\sigma_1 = 0$. Also we have restricted $c_1 < 1$ so that $-\sigma_D + \sigma_R \geq 1$ implies that $c_2 < 1$. Obviously, $\sigma_R = 0$ when the reflector is a pure absorber or a vacuum and $\sigma_D = 0$ when the total macroscopic cross sections of the two media are the same. We have seen from the last section that in general the inversion contour can be deformed to the left only as far as $\mathrm{Re}(s) = -\sigma_m$, which corresponds to $\mathrm{Re}(\zeta) = \max(-\sigma_D, 0)$. However, there are no eigenvalues on the branch cut of ν_{01} so the region of the real ζ axis where the eigenvalues $\{\zeta_n\}$ should appear is

$$\max(-\sigma_D + \sigma_R, 0) < \zeta_n < 1. \tag{6.3}$$

This interval corresponds to $s \in S_{1i} \cap S_{2i}$ and Eqs. (4.2) and (4.6), written in terms of the quantities of Eq. (6.1), are solved numerically to obtain the real eigenvalues $\{\zeta_n\}$ for specified σ_R, σ_D and A. In addition the pseudo-eigenvalues are obtained numerically by solving Eqs. (4.7) and (4.8) also written in terms of the quantities of Eq. (6.1). Numerical results are also obtained in the thick-slab approximation, that is, Eq. (4.6) with $\bar{B}_{2\pm}(\mu) = 0$. Details concerning numerical procedures and computational equations are given in Appendix J.

The calculations were done on a Control Data 6600 computer system at NASA Langley Research Center.

The time dependence of discrete modes is seen from Eqs. (5.16) and (5.19) to be

$$e^{s_n t} = e^{(c_2\zeta_n - 1)\sigma_2 t}. \tag{6.4}$$

Now $\zeta_n = -\sigma_D + \sigma_R$ implies that $s_n = -\sigma_1(1 - c_1) \leq 0$ since $c_1 < 1$ and the equality holds only if $\sigma_1 = 0$. Therefore such ζ_n correspond to time-decaying modes regardless of the value of c_2. For values of ζ_n within the interval (6.3), the time decay or growth depends on whether $c_2\zeta_n$ is less than or greater than unity as can be seen from Eq. (6.4). A discrete mode represents a critical system if $c_2\zeta_n = 1$.

The largest eigenvalue ζ_0 with an even parity eigenfunction corresponds to a critical slab problem with

$$c_{slab} = \frac{1}{\zeta_0},$$

$$c_{reflector} = \frac{\sigma_R}{\zeta_0 + \sigma_D}$$

and

$$\sigma_{slab}\, a_{critical} = \zeta_0 A, \qquad (6.5)$$

where $a_{critical}$ is the critical slab half-thickness. For a bare sphere ($\sigma_R = 0$) the largest eigenvalue ζ_1 with an odd parity eigenfunction gives the critical sphere radius, $a_{critical}$, when it is used in Eqs. (6.5) in place ζ_0 (ref. 21).

Many different combinations of material parameters could be considered, but here we restrict our study of the eigenvalue behavior to the case of overlapping branch cuts. As σ_R departs from zero, we would like to see how the eigenvalues depart from those previously reported (refs. 1, 4) for a bare slab. A comparison of the present eigenvalues $\{\zeta_n\}$ for vacuum reflectors, i.e., $\sigma_R = 0$, with those of Bowden (ref. 1) is given in Tables I and II. Results generally agree to three figures for slab half-thicknesses A from 0.4 to 20. In Table II, eigenvalues calculated in the thick-slab approximation are also shown for bare slabs. For slabs with half-thicknesses $A > 1$, the thick-slab approximation generally agrees with the numerical solution of the exact eigenvalue condition to three figures. This can be seen from Table III where we compare such results as σ_R departs from zero with $\sigma_D = 0$. From the bare slab results ($\sigma_R = 0$) of Tables I, II,

TABLE I.- EIGENVALUES $\{\zeta_n\}$ FOR BARE SLABS

n	A = 1		A = 5		A = 10		A = 15		A = 20	
	Present	Bowden	Present	Bowden	Present	Bowden	Present	Bowden	Present	Bowden
0	0.703	0.705	0.975	0.975	0.993	0.993	0.997	0.997	0.998	0.998
1			.897	.897	.971	.971	.987	.987	.992	.992
2			.762	.762	.935	.935	.970	.970	.983	.983
3			.560	.560	.883	.883	.946	.946	.969	.969
4			.276	.276	.814	.816	.915	.915	.952	.951
5					.728	.727	.877	.877	.930	.930
6					.621	.621	.831	.831	.904	.905
7					.493	.493	.777	.777	.874	.874
8					.340	.340	.714	.714	.840	.840
9					.157	.157	.642	.644	.800	.800
10							.560	.560	.756	.756
11							.467	.467	.707	.707
12							.362	.361	.653	.653
13							.243	.243	.593	.593
14							.110	.110	.526	.526
15									.453	.453
16									.373	.373
17									.286	.286
18									.190	.190
19									.084	.084

TABLE II.- EIGENVALUE ζ_0 FOR THIN BARE SLABS

Slab thickness A	Thick-slab approximation Eq. (J. 19)	Present Eq. (J.6)	Bowden (ref. 1)
1.0	0.702	0.703	0.705
.8	.612	.615	.615
.6	.473	.483	.483
.4	.244	.282	.282
.2	*	.043	.048

*No solution found for $\zeta > 0.001$

TABLE III.- EIGENVALUES ζ_0 AND ζ_1 FOR THIN REFLECTED SLABS, $\sigma_D = 0$

σ_R	A = 0.4		A = 0.7		A = 1.0		A = 1.4				A = 2.0			
	ζ_0		ζ_0		ζ_0		ζ_1		ζ_0		ζ_1		ζ_0	
	TSA*	Pres.**	TSA	Pres.	TSA	Pres.	TSA	Pres.	TSA	Pres.	TSA	Pres.	TSA	Pres.
0	0.244	0.282	0.550	0.556	0.702	0.703	0.142	0.132	0.808	0.808	0.508	0.508	0.885	0.885
0.2	.376	.398	.600	.604	.727	.728	.252	.247	.820	.820	.540	.539	.89.	.891
.4	.512	.522	.661	.663	.759	.759	.403	.402	.836	.836	.585	.584	.898	.898
.6	.656	.660	.739	.740	.803	.803	***	***	.860	.860	.656	.656	.909	.909
.8	.816	.817	.843	.844	.870	.870	***	***	.900	.900	***	***	.930	.930

* Thick-slab approximation, Eq. (J.19)
** Present numerical solution, Eq. (J.6)
*** $\sigma_R > \zeta_1$ or ζ_1 in branch cut of ν_{01}

and III, critical slab half-thicknesses are obtained from ζ_0 using Eqs. (6.5). These are compared with critical slab half-thickness results of Mitsis (taken from ref. 7) in Figure 4 (open symbols). Closed symbols give critical sphere quarter-diameters obtained from Eqs. (6.5) and ζ_1 while Mitsis' critical sphere results are taken from ref. 20. The agreement is good to the scale of the Figure. For $\sigma_R = 0$ the eigenvalues ζ_0 and ζ_1 have also been compared directly with numerical bounds computed by Mullikin (ref. 21) for bare slabs and spheres and again the agreement is good. Critical half-thicknesses of slabs with infinite reflectors have been recently computed by Kowalska (ref. 12) for a number of combinations of c_{slab} and $c_{reflector}$. Some present results ζ_0 for $\sigma_R \neq 0$ can be compared with her critical slab half-thicknesses. Her parameters are given in terms of ζ_0 and present input quantities σ_R, σ_D, and A by Eqs. (6.5). Figure 5 gives a few present cases (circles) for which c_{slab} was close to some of Kowalska's points (diamonds) (ref. 12). No attempt has been made yet to compute points which lie on Kowalska's curves. The present cases for $c_{slab} \sim 1.11$ are from A = 2 and 1.4 in Table III.

The remainder of the results have been computed for A = 5. For a bare slab with A = 5, it can be seen from Table I that there are five eigenvalues. We have studied the behavior of these eigenvalues as σ_R departs from zero for several values of σ_D. In Figure 6, results are given for $\sigma_D = 0$. Our calculations show that the largest eigenvalue, ζ_0, is present up to $\sigma_R = 0.9999$. Apparently this eigenvalue remains up to $\sigma_R = 1$, which is only obtained for $c_2 < 1$. All other

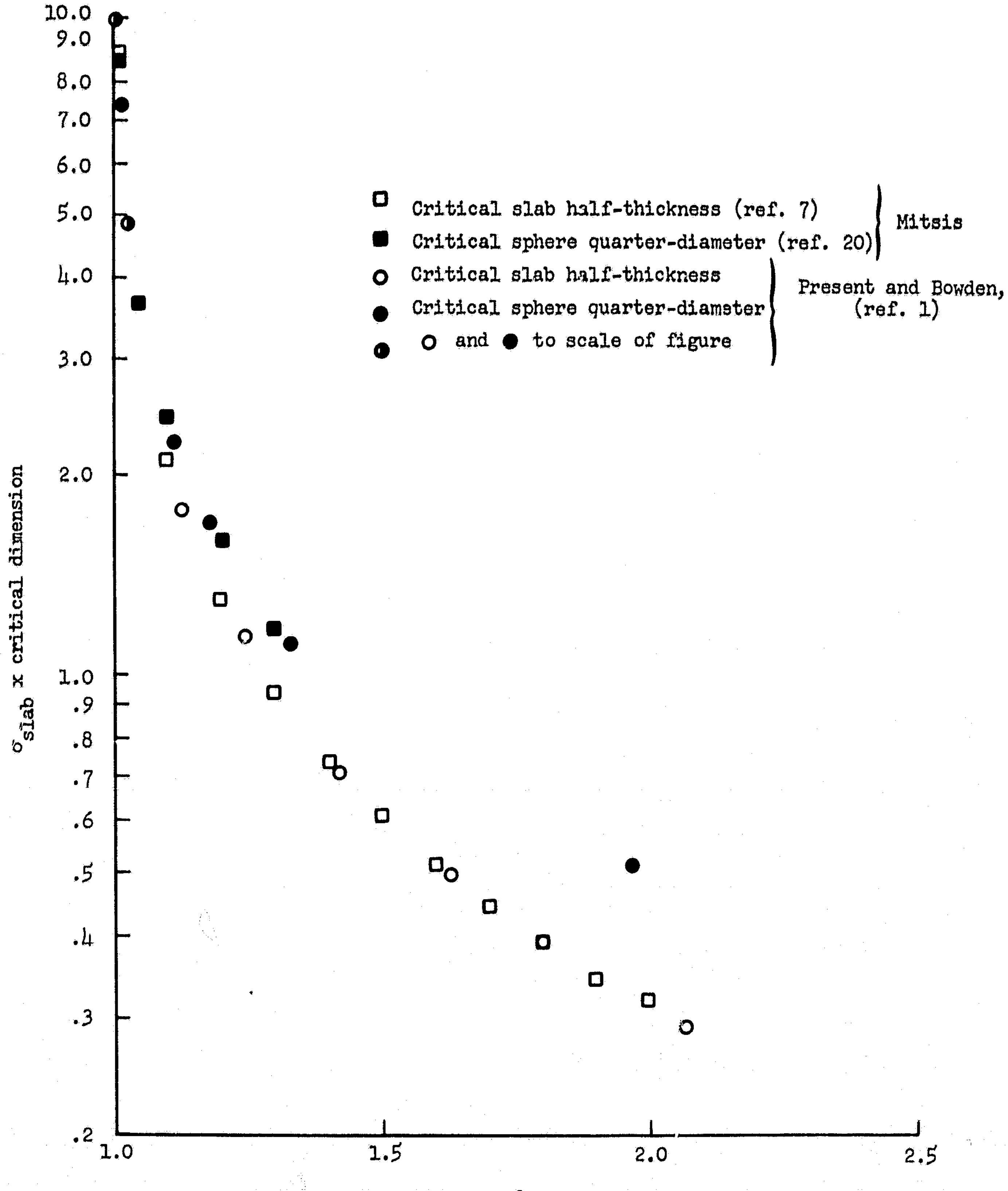

Figure 4.- Critical dimension of bare systems.

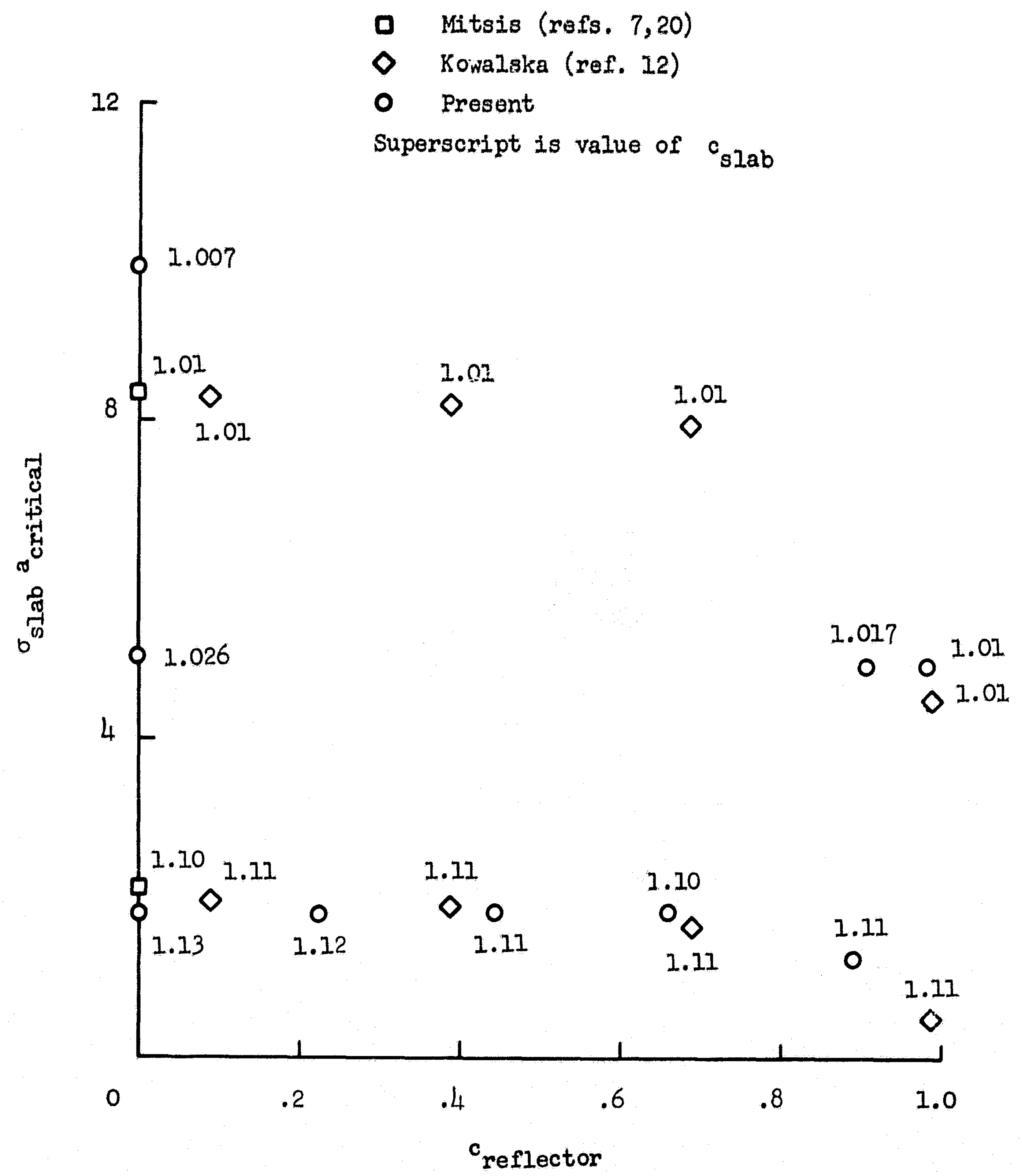

Figure 5.- Critical half-thickness for finite slabs with infinite reflectors.

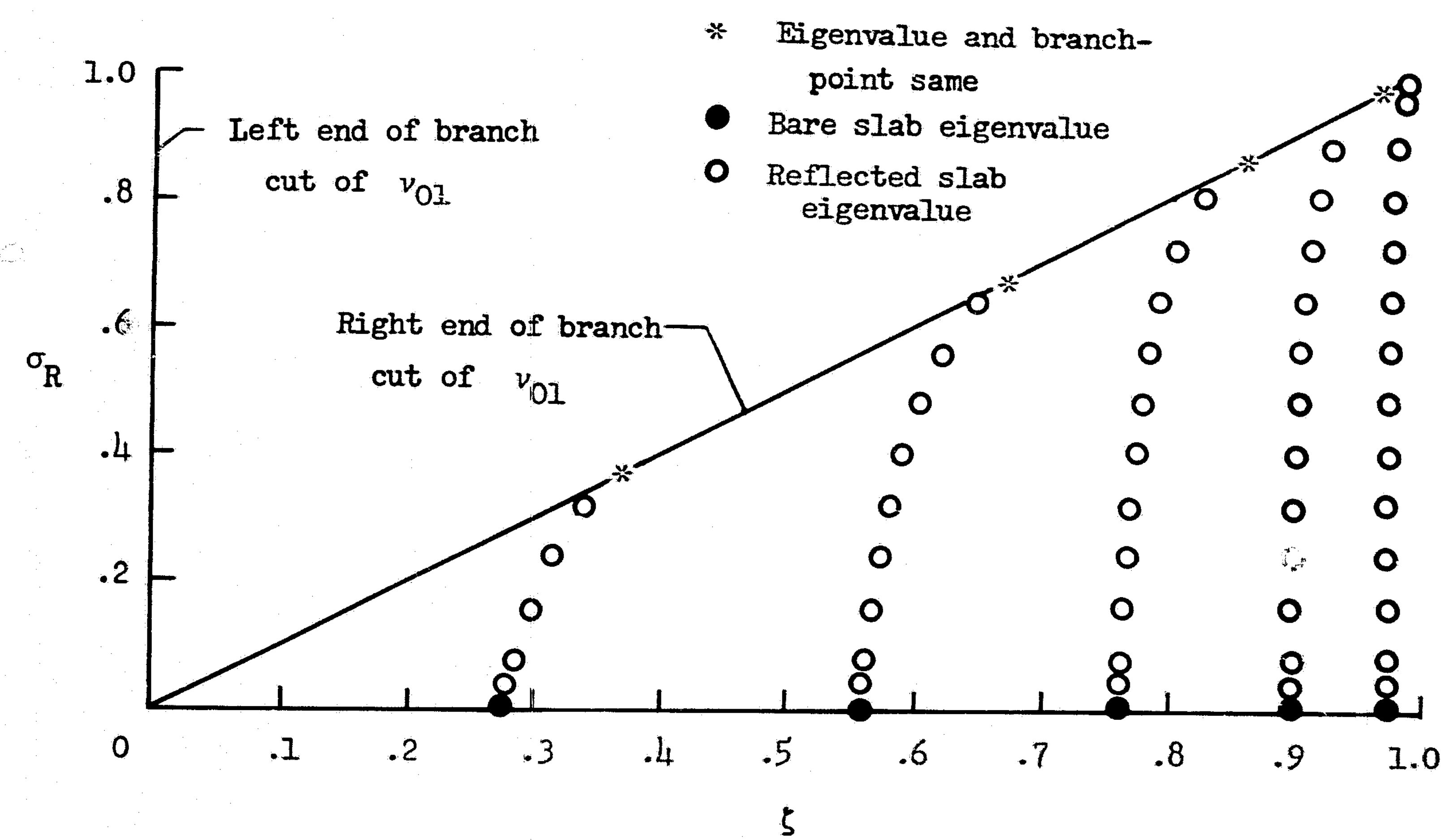

Figure 6.- Dependence of eigenvalues ζ_n on σ_R; $\sigma_D = 0$, $A = 5$.

eigenvalues disappear into the branch cut of ν_{01} at $\zeta_n = \sigma_R$, labeled with *, which corresponds to a time-decaying mode, regardless of the value of c_2. In fact, on Figures 6 - 9, we indicate the points at which an eigenvalue or pseudo-eigenvalue coincides with the branch points of ν_{01} by an asterisk, *. Even though such points appear to have a discrete eigenvalue type of time dependence, we feel that they are properly part of the branch-cut integral contribution. We note that the branch points of ν_{01} are located at $\zeta = -\sigma_D$ and $\zeta = -\sigma_D+\sigma_R$ and find that the limiting form of the condition which determines whether or not such points are eigenvalues (or pseudo-eigenvalues) no longer depends explicitly on σ_R or σ_D. (See Appendices J and K.) The theorems of Lehner (ref. 15) apply for $\sigma_R = 0$ in this Figure.

In Figure 7, results are presented for $\sigma_D = -0.65 + 0.5\,\sigma_R$. These represent what happens for $-\sigma_D$ in the range between zero and $[\zeta_0]_{\sigma_R=0}$, where the notation $[\zeta_n]_{\sigma_R=0}$ means bare-slab eigenvalue, which we note depends on c_2, σ_2 and a. The open and closed circles represent eigenvalues as in Figure 6 while the half-closed circles are pseudo-eigenvalues corresponding to $s < -\sigma_m = -\sigma_1$. Again the largest eigenvalue, ζ_0, appears to remain provided that $c_2 > 1$. Here, as in the next two figures, results for $\sigma_R = 0$ agree with the theorems of Hintz (ref. 10) which apply only for $c_1 = 0$. Basically his result is that the strip $\mathrm{Re}(\zeta)$ between 0 and $-\sigma_D$ belongs to the continuous spectrum and that the bare-slab eigenvalues lying in this interval are not eigenvalues of the slab surrounded by perfect absorbers. He finds that there are no eigenvalues if $-\sigma_D > [\zeta_0]_{\sigma_R=0}$, but says nothing about

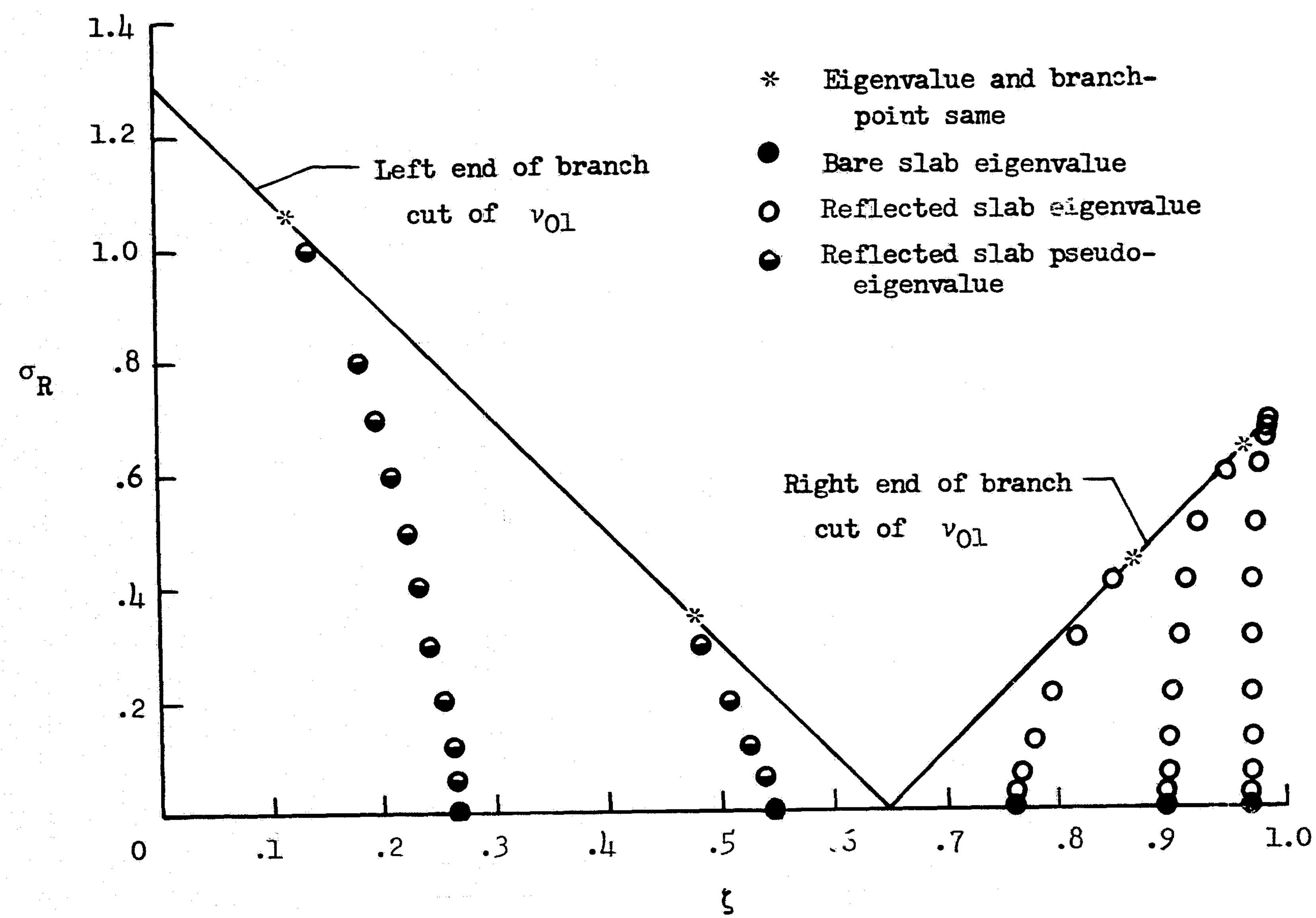

Figure 7.- Dependence of eigenvalues ζ_n on σ_R; $\sigma_D = 0.5\ \sigma_R - 0.65$, A = 5.

the physical significance. It is seen from Eq. (6.2) that for such cases $\zeta_0 < 1/c_2$ and corresponds therefore to a time-decaying mode. In other words, stationary (critical) or time-increasing modes cannot disappear into the continuous spectrum as material properties are varied. In fact, we have seen that when $\sigma_R \neq 0$ such modes cannot disappear into the branch cut of ν_{01} either. In Figure 8, results are given for $-\sigma_D + \sigma_R = 1$ which we recall implies $c_2 < 1$. For this case, all of the bare-slab eigenvalues lie in the continuous spectrum found by Hintz (ref. 10) when $\sigma_R = 0$. In both Figures 7 and 8, $s = -\sigma_m$ corresponds to $\zeta = -\sigma_D$. Figure 9 shows the behavior of the eigenvalues for $\sigma_D = 1$ and it is seen to be similar to that of Figure 6. For $\sigma_R = 0$, the continuous spectrum found by Hintz (ref. 10) lies in the strip $-\sigma_D = -1 < \mathrm{Re}(\zeta) < 0$. Here $s = -\sigma_m$ corresponds to $\zeta = 0$.

All numerical results indicate that real time eigenvalues $\{\zeta_n\}$ for material reflectors are finite in number and tend to eigenvalues previously obtained for a vacuum as $\sigma_R \to 0$, as do the pseudo-eigenvalues for $s < -\sigma_m$. If the set $\{\zeta_n\}$ is empty, then the neutron density is necessarily decaying in time. Conversely, if the neutron density is stationary or increasing in time, then the set $\{\zeta_n\}$ is <u>not</u> empty. One also expects that if $c_2 > 1$, then a critical thickness can be found. That is, the largest eigenvalue ζ_0 must be present for large enough slab thicknesses for the given c_2. This can be seen from Table I as follows: For example, if $-\sigma_D = 0.8$, then the eigenvalue ζ_0 for $A = 1$ is not present, while that for $A = 5$ would be and represents a mode whose amplitude increases exponentially with time

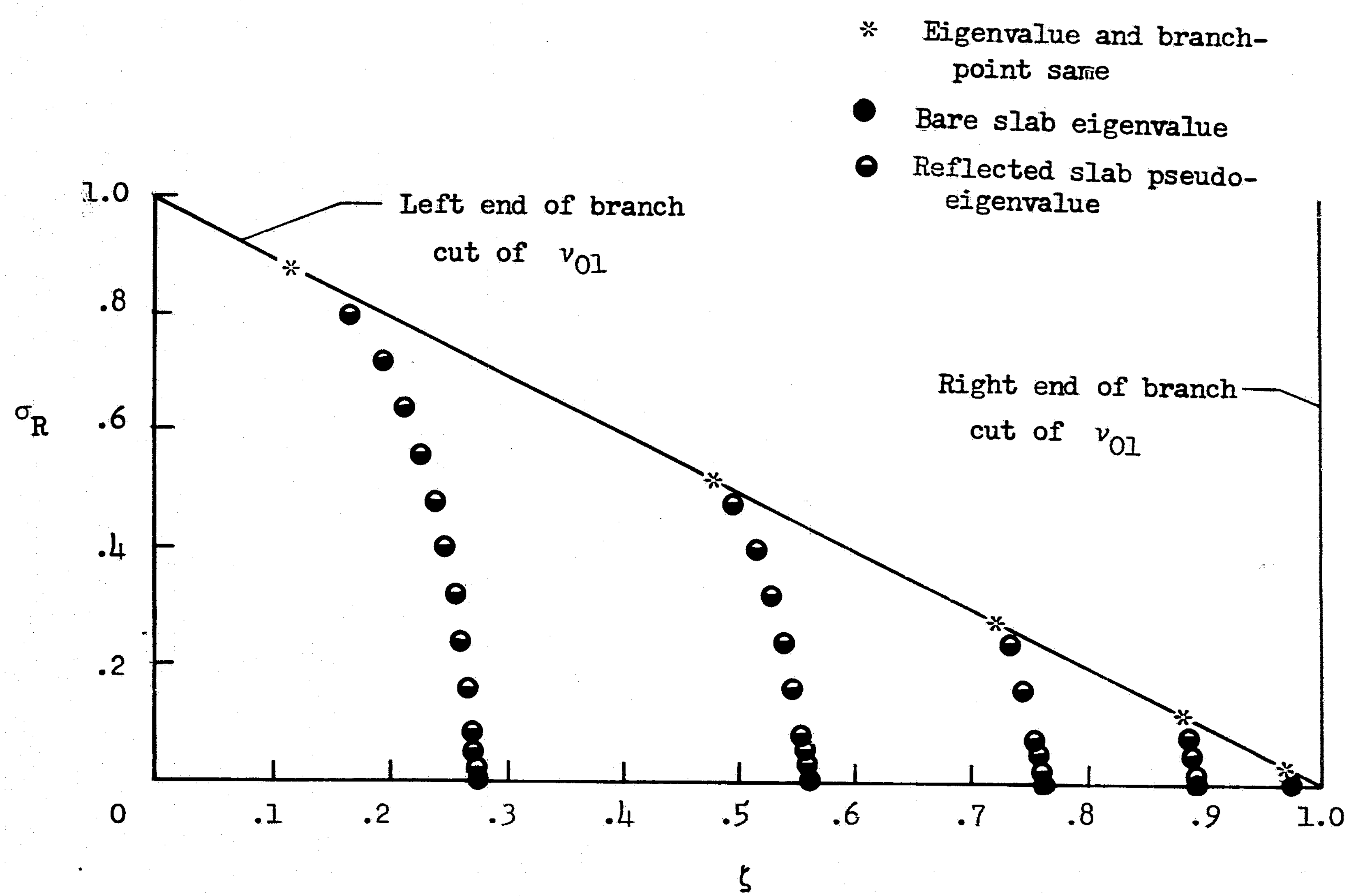

Figure 8.- Dependence of eigenvalues ζ_n on σ_R; $\sigma_D = \sigma_R - 1$, $A = 5$.

Note that $c_2 < 1$ for this figure.

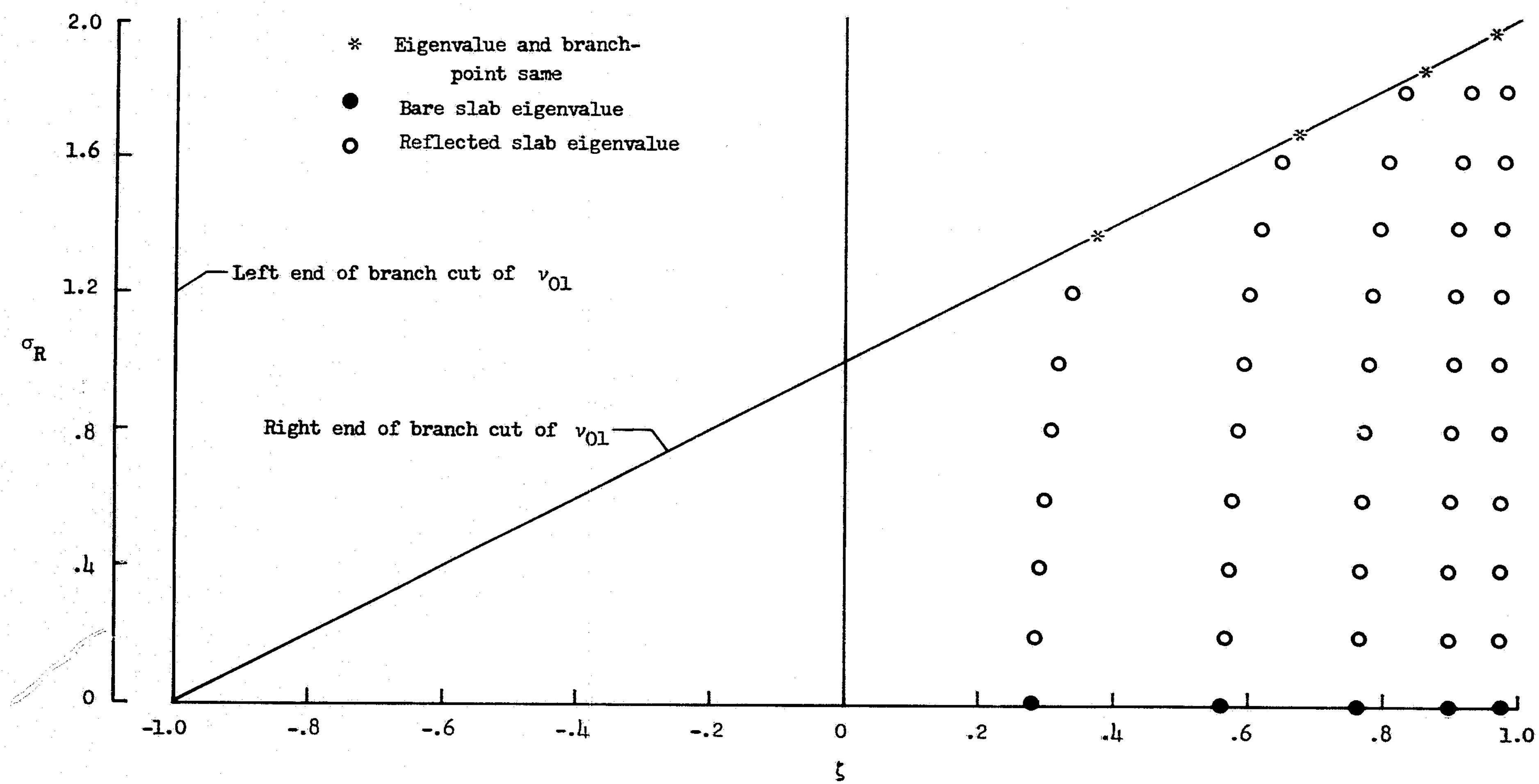

Figure 9.- Dependence of eigenvalues ζ_n on σ_R; $\sigma_D = 1$, $A = 5$.

for $c_2 > 1/0.975$. That for $A = 20$ needs only $c_2 > 1/0.998$ in order to represent a time-increasing mode.

As pointed out at the beginning of this section, we can draw some conclusions concerning the behavior of $\{\zeta_n\}$ for reflected slabs as a function of the slab half-thickness a. That is, given c_j and σ_j what can be said about $\{\zeta_n\}$ as a function of a. We base the following conclusions on the observation that if ζ_0 at $\sigma_R = 0$ lies to the right of $-\sigma_D$, then it appears to remain to the right of $-\sigma_D + \sigma_R$ as σ_R increases until $-\sigma_D + \sigma_R = 1$. (See Figs. 6 and 7.) The dependence of $[\zeta_0]_{\sigma_R=0}$ on slab half-thickness is given in Tables I-III and many more points are given in reference 1. First, if $-\sigma_D + \sigma_R \geq 1$ (recall that this implies $c_2 < 1$) then the set $\{\zeta_n\}$ is empty for all a. However, there may be pseudo-eigenvalues if $-\sigma_D > 0$. Next, if $-\sigma_D + \sigma_R < 1$ then two cases arise, depending on the value of σ_D.

(a) When $-\sigma_D > 0$ then regardless of the value of c_2, we can find an a^* such that $a < a^*$ implies that the set $\{\zeta_n\}$ is empty, whereas $a > a^*$ implies that the set $\{\zeta_n\}$ is not empty. The number a^* is obtained from the bare-slab result $[\zeta_0]_{\sigma_R=0}$ as

$$\left[\zeta_0(c_2,\sigma_2,a^*)\right]_{\sigma_R=0} = -\sigma_D. \tag{6.6}$$

(b) When $-\sigma_D \leq 0$, the set $\{\zeta_n\}$ is never empty. Thus, given c_j, σ_j, a and the bare-slab eigenvalues corresponding to c_2, σ_2 and a, we can say whether or not the set $\{\zeta_n\}$ is empty. Furthermore, the number of eigenvalues $\{\zeta_n\}$ will not exceed the number of bare-slab eigenvalues $\left\{[\zeta_n]_{\sigma_R=0}\right\}$ which are greater than $-\sigma_D$. Finally, the number of real reflected-slab eigenvalues and pseudo-eigenvalues does not exceed the number of bare-slab eigenvalues.

VII. CONCLUDING REMARKS

It has been shown using Case's method that the solution to the initial-value problem of monoenergetic neutrons migrating in a finite slab (properties c_2, σ_2) with infinite reflectors (properties c_1, σ_1) can be written in the form

$$\psi(x,\mu,t) = \psi_u(x,\mu,t) + \sum_{s=s_n} \text{Residue}\left[\psi_s(x,\mu)\right]_{s_n} e^{s_n t}$$

$$+ \frac{1}{2\pi i}\int_{-\sigma_m}^{-\sigma_1(1-c_1)}\left\{\left[\psi_s(x,\mu)\right]^- - \left[\psi_s(x,\mu)\right]^+\right\} e^{st}\, ds$$

$$+ \frac{1}{2\pi i}\int_{-\sigma_m-i\infty}^{-\sigma_m+i\infty}\left[\psi_s(x,\mu) - \psi_{us}(x,\mu)\right] e^{st}\, ds, \quad -\sigma_m < -\sigma_1(1 - c_1) < s_n. \tag{7.1}$$

In this equation, t is the neutron speed multiplied by the real time, σ_m is the minimum of σ_1 and σ_2 and each ψ function is the sum of its definite parity parts $\psi_\pm$. Some terms of the solution (7.1) will not be present if $-\sigma_m \nless -\sigma_1(1 - c_1) \nless s_n$. That is, if $-\sigma_1(1 - c_1) < -\sigma_m$ then the branch-cut integral does not appear. Likewise, if all $s_n < -\sigma_1(1 - c_1)$, then there are no residue terms. These discrete eigenvalue terms are characteristic of a finite slab (refs. 1, 4) while the branch-cut integral term is typical of a semi-infinite medium (ref. 14). The term $\psi_u(x,\mu,t)$ describes the behavior of neutrons from the initial distribution, $f(x,\mu)$, which have not suffered a scattering collision and its definite parity parts are given in Eqs. (5.7)-(5.10). The discrete eigenvalue terms in Eq. (7.1) are given by Eq. (5.19) while the integrand of the branch-cut integral is given by Eqs. (5.17)-(5.18).

The definite parity parts of the last integrand are given by Eqs. (5.6) and Eqs. (3.36)-(3.39). The eigenvalues $\{s_n\}$ can be computed as was demonstrated in the last section; thus, everything which appears in Eq. (7.1) can be readily calculated.

In all special cases of the present problem which have been solved using the Lehner-Wing technique (refs. 10, 15, 16, 17), $c_1 = 0$. In these cases, there is no branch cut due to $\nu_{01}(s)$; therefore the branch-cut integral is not present in Eq. (7.1). It was shown that as $c_1 \to 0$ the eigenvalues, $\{s_n\}$, which are greater than $-\sigma_m$ approach those for a bare slab as do the pseudo-eigenvalues for $s < -\sigma_m$. The solution $\psi_{s\pm}$ has the proper behavior as $c_1 \to 0$ since those terms of Eq. (3.30) and (3.31) which appear to blow up in such a limit actually cancel when the contour C' is collapsed onto the portion of the branch cut of $\Omega_{Js}(z')$, $0 \leq z' \leq 1$. When the uncollided term is combined with the last integral it is then seen that the solution (7.1) and the eigenvalues $\{s_n\}$ have the behavior required by the theorems of Lehner (ref. 15) and Hintz (ref. 10). The present problem reduces to those considered by Lehner and Hintz when

$$c_1 = 0, \quad \sigma_1 = \sigma_2; \quad \text{Lehner (ref. 15)}$$

and

$$c_1 = 0, \quad \sigma_1 \neq \sigma_2; \quad \text{Hintz (ref. 10).} \tag{7.2}$$

Hintz shows that for $\sigma_1 = \sigma_2$, his spectral results reduce to those of Lehner.

In order to describe the same physical problem in the slab as that solved by Lehner and Wing (refs. 16, 17) we must not only have

$$c_1 = 0 \quad \text{and} \quad \sigma_1 = 0, \tag{7.3}$$

but also

$$f(x,\mu) = 0, \begin{cases} x < -a, & \mu > 0 \\ x > a, & \mu < 0. \end{cases} \tag{7.4}$$

In other words, neutrons from the initial distribution outside the slab cannot impinge on the slab faces at times $t > 0$. Lehner and Wing solved the time-dependent problem with boundary conditions

$$\psi(\pm a,\mu,t) = 0; \quad \mu \lesseqgtr 0, \quad t > 0. \tag{7.5}$$

Restrictions (7.3) and (7.4) in the present solution make $I_{2\pm}(\mu,s)$ and therefore $A_{2\pm}(\mu,s)$ depend only on slab properties. Then, in looking for solutions inside the slab ($|x| < a$), the inversion contour along $\mathrm{Re}(s) = -\sigma_m$ can be deformed back to $\mathrm{Re}(s) = -\sigma_2$, and we pick up a residue contribution from any pseudo-eigenvalue in the region and thus obtain the Lehner-Wing results. That is, the solution has the proper form and all bare-slab eigenvalues are recovered. Hintz (ref. 10) did not indicate how the Lehner-Wing solution for the bare slab could be obtained from his results. Here we emphasize that he is not solving the same physical problem inside the slab unless both conditions (7.3) and (7.4) are satisfied.

The analogous problem for $c_1 \neq 0$ in which the inversion contour can be deformed to the left of $\mathrm{Re}(s) = -\sigma_m$ for $|x| < a$ is obtained when $\sigma_2 > \sigma_1$ and $f_1(x,\mu) \equiv 0$. That is, if

$$f(x,\mu) \equiv 0, \quad |x| > a \quad \text{and} \quad \sigma_2 > \sigma_1, \tag{7.6}$$

then all terms in $I_{2\pm}(\mu,s)$ which contain $(s + \sigma_1)$ factors in the exponentials are identically zero and this allows us to deform the contour along $Re(s) = -\sigma_m$ back to $Re(s) = -\sigma_2$ when $|x| < a$. Such a deformation is not possible for $|x| > a$; for this latter range of x we must stop at $Re(s) = -\sigma_m = -\sigma_1$. If there are pseudo-eigenvalues in $-\sigma_2 < Re(s) < -\sigma_1 = -\sigma_m$ (see, for example, Fig. 7) they will appear in the solution for $|x| < a$ as residue terms which have the exponential time dependence. They are not eigenvalues for the reflected slab though, since such terms do not appear for $|x| > a$. Erdmann (refs. 8, 9) solved the time-dependent problem for two semi-infinite media where an isotropic pulse of neutrons was introduced at the interface and found that the inversion contour for $x \in$ medium j could be deformed to the left as far as $Re(s) = -\sigma_j$. In the present problem, such deformations can be made only when conditions (7.6) are satisfied. It appears that the contour $Re(s) = -\sigma_m$ cannot be deformed to the left of $Re(s) = -\sigma_2$, since the implicit equation which determines $A_{2\pm}(\mu,s)$ (see Eq. (I.3)) requires $Re(s) \geq -\sigma_2$. Apparently $Re(s) = -\sigma_2$ is the edge of a continuous spectrum in all cases for the reflected slab.

We briefly summarize the results which have been obtained. The present solution has been shown to have the required properties in all special cases which have been solved previously by others using the Lehner-Wing technique. However, in all of these rigorous solutions, there was no scattering outside the slab. We have seen that with infinite reflectors on the slab and neutrons anywhere outside the slab initially that it is possible for some neutrons which have spent their entire history in the reflector to impinge on the slab faces at times $t > 0$.

Such neutrons have a collision rate which is characteristic of reflector properties and this, in general, restricts us from deforming the inversion contours to the left of $\mathrm{Re}(s) = -\sigma_m$. We have illustrated two cases in which a further deformation is possible for $|x| < a$, by eliminating neutrons outside the slab initially which can later impinge on the slab faces. This is equivalent to a further restriction on the Hilbert space which has been used in some of the above-mentioned rigorous solutions. The exact eigenvalue condition has been obtained and real time eigenvalues have been calculated for a number of combinations of material parameters. The largest eigenvalues have been shown to agree with criticality results of others. Our calculations show that eigenvalues can disappear into the branch cut or continuum as material properties are varied and we point out that all such disappearing eigenvalues correspond to exponentially time-decaying modes regardless of the value of c_2 since we have taken $c_1 < 1$. We expect (but have not shown) that there is no drastic change in the shape of the solution given by Eq. (7.1) when this happens; we conjecture that one of the integrals in Eq. (7.1) probably has resonance-like terms due perhaps to zeros of the eigenvalue condition on the next Riemann sheet. We have made the assumption that the eigenvalues are real for arbitrary slab half-thicknesses. We have shown this to be true for thick slabs and it has been proved rigorously by others for the above-mentioned special cases. On the basis of our sample calculations, we conclude that if one is given the material properties c_j, σ_j and slab half-thickness, a, as well as the bare-slab eigenvalues corresponding to c_2, σ_2 and a then he can conclude whether or not the set $\{s_n\}$ is empty and the maximum number of s_n in $\{s_n\}$.

Perhaps the present results can serve as a guide for a rigorous Lehner-Wing type analysis of the reflected-slab problem. If the eigenvalues are all real, then one might be able to prove it in such an analysis of the present problem.

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IX. VITA

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X. APPENDICES

A. Summary of Elementary Solution Properties

In this Appendix, the elementary solution properties derived by others (refs. 1, 4, 8, 9), following the lead of Case (refs. 5, 7), are summarized. These solutions are obtained from Eqs. (2.16) - (2.18) of the text and are given by Eqs. (2.19) - (2.21). Such solutions are complete and orthogonal in the following sense. A function, say $g(\mu)$, satisfying very weak restrictions (see for example ref. 7, Appendix G) for $-1 \leq \alpha \leq \mu \leq \beta \leq 1$ can be expanded as follows:

1. Full Range; $\alpha = -1$, $\beta = 1$

$$g(\mu) = \left[a_j \varphi_{\nu_{0j}}(\mu) + b_j \varphi_{-\nu_{0j}}(\mu)\right]\delta_j(s) + \int_{-1}^{1} A_j(\nu)\varphi_{js\nu}(\mu)d\nu, \tag{A.1}$$

where the notation $\delta_j(s)$ was defined by Eq. (2.25). The orthogonality relations used to determine the expansion coefficients in Eq. (A.1) are

$$\int_{-1}^{1} \mu\varphi_{js\nu'}(\mu)d\mu \int_{-1}^{1} A_j(\nu)\varphi_{js\nu}(\mu)d\nu = A_j(\nu')\nu'\Omega_{js}^{+}(\nu')\Omega_{js}^{-}(\nu')$$

and for $s \epsilon S_{ji}$

$$\int_{-1}^{1} \mu\varphi_{js\nu'}(\mu)\varphi_{\pm\nu_{0j}}(\mu)d\mu = 0,$$

$$\int_{-1}^{1} \mu\varphi_{\nu_{0j}}(\mu)\varphi_{-\nu_{0j}}(\mu)d\mu = 0$$

and

$$\int_{-1}^{1} \mu\varphi_{\pm\nu_{0j}}^{2}(\mu)d\mu = \frac{1}{2}\, c_j\sigma_j\nu_{0j}^{2}\Omega_{js}'(\pm\nu_{0j}), \tag{A.2}$$

where

$$\Omega'_{js}(\nu_{0j}) = \frac{d}{dz}\,\Omega_{js}(z)\Big|_{z=\nu_{0j}}$$

and

$$\Omega^{\pm}_{js}(\nu) = \lambda_{js}(\nu) \pm i\pi c_j\sigma_j\nu/2. \tag{A.3}$$

2. Half Range; $\alpha = 0,\ \beta = 1$

$$g(\mu) = a_j\varphi_{\nu_{0j}}(\mu)\delta_j(s) + \int_0^1 A_j(\nu)\varphi_{js\nu}(\mu)d\nu. \tag{A.4}$$

Here the orthogonality relations for $s\epsilon S_{ji}$ are

$$\int_0^1 W_{js}(\mu)\varphi_{js\nu'}(\mu)d\mu \int_0^1 A_j(\nu)\varphi_{js\nu}(\mu)d\nu = A_j(\nu')W_{js}(\nu')\Omega^+_{js}(\nu')\Omega^-_{js}(\nu'),$$

$$\int_0^1 W_{js}(\mu)\varphi_{js\nu}(\mu)\varphi_{\nu_{0j}}(\mu)d\mu = 0,$$

$$\int_0^1 W_{js}(\mu)\varphi_{js\nu}(\mu)\varphi_{-\nu_{0j}}(\mu)d\mu = \nu c_j\sigma_j\nu_{0j}X_{js}(-\nu_{0j})\varphi_{-\nu_{0j}}(\nu),$$

$$\int_0^1 W_{js}(\mu)\varphi_{\nu_{0j}}(\mu)\varphi_{\pm\nu_{0j}}(\mu)d\mu = \mp\left(\frac{c_j\sigma_j\nu_{0j}}{2}\right)^2 X_{js}(\pm\nu_{0j}),$$

$$\int_0^1 W_{js}(\mu)\varphi_{\nu_{0j}}(\mu)\varphi_{js(-\nu)}(\mu)d\mu = \left(\frac{c_j\sigma_j}{2}\right)^2 \nu\nu_{0j}X_{js}(-\nu),$$

$$\int_0^1 W_{js}(\mu)\varphi_{js\nu'}(\mu)\varphi_{js(-\nu)}(\mu)d\mu = \frac{c_j\sigma_j}{2}\,\nu'(\nu_{0j} + \nu')X_{js}(-\nu)\varphi_{js(-\nu)}(\nu')$$

and

$$\int_0^1 W_{js}(\mu)\varphi_{js\nu}(\mu)d\mu = \frac{1}{2}\,c_j\sigma_j\nu, \tag{A.5a}$$

where

$$W_{js}(\mu) = \frac{c_j\sigma_j\mu}{2\Omega_{js}(\infty)(\nu_{0j} + \mu)X_{js}(-\mu)}, \quad 0 \le \mu \le 1, \tag{A.6a}$$

and

$$\frac{X^{+}_{js}(\mu)}{X^{-}_{js}(\mu)} \equiv \frac{\Omega^{+}_{js}(\mu)}{\Omega^{-}_{js}(\mu)}, \quad 0 \leq \mu \leq 1, \tag{A.7a}$$

with $X_{js}(z)$ given by

$$\begin{aligned} X_{js}(z) &= \frac{1}{1-z} \exp\left\{\frac{1}{2\pi i}\int_0^1 \ln\left[\frac{\Omega^{+}_{js}(\nu)}{\Omega^{-}_{js}(\nu)}\right]\frac{d\nu}{\nu-z}\right\} \\ &= \frac{\Omega_{js}(z)}{(\nu_{0j}^2 - z^2)\Omega_{js}(\infty)X_{js}(-z)} \\ &= \frac{c_j\sigma_j}{2\Omega_{js}(\infty)}\int_0^1 \frac{\mu d\mu}{(\nu_{0j}^2 - \mu^2)X_{js}(-\mu)(\mu-z)} . \end{aligned} \tag{A.8a}$$

The orthogonality relations for $s\epsilon S_{je}$ for the expansion (A.4) are

$$\int_0^1 W_{js}(\mu)\varphi_{js\nu'}(\mu)d\mu \int_0^1 A_j(\nu)\varphi_{js\nu}(\mu)d\nu = A_j(\nu')W_{js}(\nu')\Omega^{+}_{js}(\nu')\Omega^{-}_{js}(\nu'),$$

$$\int_0^1 W_{js}(\mu)\varphi_{js\nu'}(\mu)\varphi_{js(-\nu)}(\mu)d\mu = \frac{1}{2}\, c_j\sigma_j\nu' X_{0js}(-\nu)\varphi_{js(-\nu)}(\nu')$$

and

$$\int_0^1 W_{js}(\mu)\varphi_{js\nu}(\mu)d\mu = \frac{1}{2}\, c_j\sigma_j\nu, \tag{A.5b}$$

where

$$W_{js}(\mu) = \frac{c_j\sigma_j\mu}{2\Omega_{js}(\infty)X_{0js}(-\mu)}, \quad 0 \leq \mu \leq 1, \tag{A.6b}$$

and

$$\frac{X^{+}_{0js}(\mu)}{X^{-}_{0js}(\mu)} \equiv \frac{\Omega^{+}_{js}(\mu)}{\Omega^{-}_{js}(\mu)}, \quad 0 \leq \mu \leq 1, \tag{A.7b}$$

with $X_{0js}(z)$ given by

$$X_{0js}(z) = \exp\left\{\frac{1}{2\pi i}\int_0^1 \ln\left[\frac{\Omega^+_{js}(\nu)}{\Omega^-_{js}(\nu)}\right]\frac{d\nu}{\nu - z}\right\}$$

$$= \frac{\Omega_{js}(z)}{\Omega_{js}(\infty)X_{0js}(-z)}$$

$$= 1 + \frac{c_j\sigma_j}{2\Omega_{js}(\infty)}\int_0^1 \frac{\mu d\mu}{X_{0js}(-\mu)(\mu - z)} . \tag{A.8b}$$

These half-range orthogonality relations and identities are obtained by extending the time-independent results of Kuščer, McCormick and Summerfield (ref. 13).

A result, due to Kuščer and Zweifel (ref. 14), which we shall need to analytically continue solutions follows from Eqs. (A.8). For a fixed value of z, $X_{js}(z)$ does not become $X_{0js}(z)$ as s crosses C_j. However, it follows from the middle expressions in (A.8a) and (A.8b) that

$$(\nu_{0j} - z)X_{js}(z)\Big|_{\substack{s \to C_j \\ s\in S_{ji}}} = X_{0js}(z)\Big|_{\substack{s \to C_j \\ s\in S_{je}}} . \tag{A.9}$$

Following Kuščer and Zweifel (ref. 14) then, we define a function $X_{0j}(z,s)$ which is continuous as $s \to C_j$ by Eq. (3.25). Such a function of the two complex variables z and s has the following analytical properties (ref. 14).

Fixed s: no singularity in z-plane cut along (0, 1);

one simple zero at $z = \nu_{0j}(s)$, $\mathrm{Re}(\nu_{0j}) \geq 0$,

only if $s\in S_{ji}$.

Fixed z: no singularity in the s-plane cut along $(-\sigma_j, -\sigma_j(1-c_j))$;
one simple zero at $s = -\sigma_j + c_j\sigma_j z \tanh^{-1}(1/z)$
for $\mathrm{Re}(z) > 0$.

We note here that $X_{0j}(z,s)$ is a nonvanishing analytic function of z and s for $\mathrm{Re}(z) < 0$ and $s \notin (-\sigma_j, -\sigma_j(1-c_j))$, the branch cut of $\nu_{0j}(s)$.

B. Derivation of $\psi_{jp\pm}(x,\mu,s)$

In this Appendix, explicit forms of $\psi_{jp\pm}(x,\mu,s)$ are obtained. Following Bowden (ref. 1) we take, for medium j, the function $g_{js}(x,\mu;x_0)$ as

$$g_{js}(x,\mu;x_0) = \begin{cases} -D_j(x_0)\varphi_{-\nu_{0j}}(\mu)e^{(s+\sigma_j)(x-x_0)/\nu_{0j}}\delta_j(s) \\ -\int_{-1}^{0} C_j(x_0,\nu)\varphi_{js\nu}(\mu)e^{-(s+\sigma_j)(x-x_0)/\nu}d\nu, \quad x < x_0 \\ \\ C_j(x_0)\varphi_{\nu_{0j}}(\mu)e^{-(s+\sigma_j)(x-x_0)/\nu_{0j}}\delta_j(s) \\ +\int_{0}^{1} C_j(x_0,\nu)\varphi_{js\nu}(\mu)e^{-(s+\sigma_j)(x-x_0)/\nu}d\nu, \quad x > x_0. \end{cases} \tag{B.1}$$

The expansion coefficients in Eq. (B.1) are to be determined so that $g_{js}(x,\mu;x_0)$ satisfies Eqs. (3.5) and (3.6). That is, on putting the expansion (B.1) into Eq. (3.6), we obtain in the limit $x \to x_0$

$$f_j(x_0,\mu)/\mu = \left[C_j(x_0)\varphi_{\nu_{0j}}(\mu) + D_j(x_0)\varphi_{-\nu_{0j}}(\mu)\right]\delta_j(s) + \int_{-1}^{1} C_j(x_0,\nu)\varphi_{js\nu}(\mu)d\nu. \tag{B.2}$$

This is a full-range expansion (see Eq. (A.1)) of the function $f_j(x_0,\mu)/\mu$ and use of the orthogonality relations (A.2) gives the coefficients as

$$C_j(x_0,\nu) = \frac{1}{\nu\Omega^+_{js}(\nu)\Omega^-_{js}(\nu)}\int_{-1}^{1} f_j(x_0,\mu)\varphi_{js\nu}(\mu)d\mu$$

and, if $s\epsilon S_{ji}$

$$C_j(x_0) = \frac{2}{c_j\sigma_j\nu_{0j}^2\Omega'_{js}(\nu_{0j})}\int_{-1}^{1} f_j(x_0,\mu)\varphi_{\nu_{0j}}(\mu)d\mu$$

and

$$D_j(x_0) = \frac{2}{c_j\sigma_j\nu_{0j}^2\Omega'_{js}(-\nu_{0j})}\int_{-1}^{1} f_j(x_0,\mu)\varphi_{-\nu_{0j}}(\mu)d\mu. \qquad (B.3)$$

However, we shall need expansion coefficients for the even and odd parts of $f_j(x_0,\mu)/\mu$ separately. It follows from Eqs. (B.3) that

$$C_{j\pm}(x_0,\nu) \equiv \frac{1}{2}\left[C_j(x_0,\nu) \mp C_j(-x_0,-\nu)\right]$$

and, if $s\epsilon S_{ji}$

$$C_{j\pm}(x_0,\nu_{0j}) \equiv \frac{1}{2}\left[C_j(x_0) \mp D_j(-x_0)\right]$$

and

$$C_{j\pm}(x_0,-\nu_{0j}) \equiv \frac{1}{2}\left[D_j(x_0) \mp C_j(-x_0)\right], \qquad (B.4)$$

are the expansion coefficients of $f_{j\pm}(x_0,\mu)/\mu$; that is, Eqs. (3.11) of the text.

In order to construct $\psi_{jp\pm}(x,\mu,s)$ according to Eq. (3.7), we note that for $j = 2$

$$\int_{\substack{(\text{medium})\\2}} \cdot\, dx_0 = \int_{-a}^{a} \cdot\, dx_0 = \int_{-a}^{x} \cdot\, dx_0 + \int_{x}^{a} \cdot\, dx_0. \qquad (B.5)$$

Upon using Eq. (B.1), we obtain $\psi_{2p}(x,\mu,s)$ as

$$
\begin{aligned}
\psi_{2p}(x,\mu,s) = &\left[\int_{-a}^{x} C_2(x_0)e^{(s+\sigma_2)x_0/\nu_{02}}dx_0\right]\psi_{\nu_{02}}(x,\mu,s)\delta_2(s) \\
&+\int_0^1\left[\int_{-a}^{x} C_2(x_C,\nu)e^{(s+\sigma_2)x_0/\nu}dx_0\right]\psi_{2\nu}(x,\mu,s)d\nu \\
&-\left[\int_{x}^{a} D_2(x_0)e^{-(s+\sigma_2)x_0/\nu_{02}}dx_0\right]\psi_{-\nu_{02}}(x,\mu,s)\delta_2(s) \\
&-\int_0^1\left[\int_{x}^{a} C_2(x_0,-\nu)e^{-(s+\sigma_2)x_0/\nu}dx_0\right]\psi_{2(-\nu)}(x,\mu,s)d\nu .
\end{aligned}
\tag{B.6}
$$

The definite parity particular solution $\psi_{2p\pm}(x,\mu,s)$ is then obtained using Eq. (B.6) as

$$
\begin{aligned}
\psi_{2p\pm}(x,\mu,s) = &\left\{\left[\int_{-a}^{x} C_{2\pm}(x_0,\nu_{02})e^{(s+\sigma_2)x_0/\nu_{02}}dx_0\right]\psi_{\nu_{02}}(x,\mu,s)\right. \\
&\left.\pm\left[\int_{-a}^{-x} C_{2\pm}(x_0,\nu_{02})e^{(s+\sigma_2)x_0/\nu_{02}}dx_0\right]\psi_{-\nu_{02}}(x,\mu,s)\right\}\delta_2(s) \\
&+\int_0^1\left[\int_{-a}^{x} C_{2\pm}(x_0,\nu)e^{(s+\sigma_2)x_0/\nu}dx_0\right]\psi_{2\nu}(x,\mu,s)d\nu \\
&\pm\int_0^1\left[\int_{-a}^{-x} C_{2\pm}(x_0,\nu)e^{(s+\sigma_2)x_0/\nu}dx_0\right]\psi_{2(-\nu)}(x,\mu,s)d\nu
\end{aligned}
\tag{B.7}
$$

That (B.7) is a solution of Eq. (2.13) for $j = 2$ can be seen by direct substitution as follows. The $\psi_{j\nu}(x,\mu,s)$ in Eq. (B.7) are solutions of Eq. (2.16), the homogeneous equation corresponding to Eq. (2.13).

However, their coefficients in Eq. (B.7) are functions of x so that some additional terms are obtained from the $\frac{\partial}{\partial x}$ operation. Thus we get

$$\mu\left\{\left[C_{2\pm}(x,\nu_{02})\varphi_{\nu_{02}}(\mu) + C_{2\pm}(x,-\nu_{02})\varphi_{-\nu_{02}}(\mu)\right]\delta_2(s) + \int_{-1}^{1} C_{2\pm}(x,\nu)\varphi_{2s\nu}(\mu)d\nu\right\} = f_{2\pm}(x,\mu), \qquad \text{(B.8)}$$

which is an identity since according to Eq. (3.11) the $C_{2\pm}$ are the full-range expansion coefficients of $f_{2\pm}(x,\mu)/\mu$.

To get $\psi_{1p}(x,\mu,s)$ according to Eq. (3.7), we first note that

$$\int_{\substack{(\text{medium})\\ 1}} \cdot\, dx_0 = \int_{-\infty}^{a} \cdot\, dx_0 + \int_{a}^{\infty} \cdot\, dx_0$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{x} \cdot\, dx_0 + \int_{x}^{-a} \cdot\, dx_0 + \int_{a}^{\infty} \cdot\, dx_0, & x < -a \\ \displaystyle\int_{-\infty}^{-a} \cdot\, dx_0 + \int_{a}^{x} \cdot\, dx_0 + \int_{x}^{\infty} \cdot\, dx_0, & x > a. \end{cases} \qquad \text{(B.9)}$$

We follow the same procedure as before and get $\psi_{1p\pm}(x,\mu,s)$ as

$$\begin{aligned}
\Psi_{1p\pm}(x,\mu,s) = {} & \left[\int_{-\infty}^{x} C_{1\pm}(x_0,\nu_{01}) e^{(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right] \Psi_{\nu_{01}}(x,\mu,s)\delta_1(s) \\
& + \int_0^1 \left[\int_{-\infty}^{x} C_{1\pm}(x_0,\nu) e^{(s+\sigma_1)x_0/\nu}\, dx_0\right] \Psi_{1\nu}(x,\mu,s)\, d\nu \\
& + \left[-\int_{x}^{-a} C_{1\pm}(x_0,-\nu_{01}) e^{-(s+\sigma_1)x_0/\nu_{01}}\, dx_0 \right. \\
& \qquad \left. \pm \int_{-\infty}^{-a} C_{1\pm}(x_0,+\nu_{01}) e^{+(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right] \Psi_{-\nu_{01}}(x,\mu,s)\delta_1(s) \\
& + \int_0^1 \left[-\int_{x}^{-a} C_{1\pm}(x_0,-\nu) e^{-(s+\sigma_1)x_0/\nu}\, dx_0 \right. \\
& \qquad \left. \pm \int_{-\infty}^{-a} C_{1\pm}(x_0,\nu) e^{(s+\sigma_1)x_0/\nu}\, dx_0\right] \Psi_{1(-\nu)}(x,\mu,s)\, d\nu, \quad x < -a
\end{aligned} \tag{B.10a}$$

and

$$\begin{aligned}
\Psi_{1p\pm}(x,\mu,s) = {} & \left[\int_{-\infty}^{-a} C_{1\pm}(x_0,\nu_{01}) e^{(s+\sigma_1)x_0/\nu_{01}}\, dx_0 \right. \\
& \qquad \left. \mp \int_{-x}^{-a} C_{1\pm}(x_0,-\nu_{01}) e^{-(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right] \Psi_{\nu_{01}}(x,\mu,s)\delta_1(s) \\
& + \int_0^1 \left[\int_{-\infty}^{-a} C_{1\pm}(x_0,\nu) e^{(s+\sigma_1)x_0/\nu}\, dx_0 \right. \\
& \qquad \left. \mp \int_{-x}^{-a} C_{1\pm}(x_0,-\nu) e^{-(s+\sigma_1)x_0/\nu}\, dx_0\right] \Psi_{1\nu}(x,\mu,s)\, d\nu \\
& \pm \left[\int_{-\infty}^{-x} C_{1\pm}(x_0,\nu_{01}) e^{(s+\sigma_1)x_0/\nu_{01}}\, dx_0\right] \Psi_{-\nu_{01}}(x,\mu,s)\delta_1(s) \\
& \pm \int_0^1 \left[\int_{-\infty}^{-x} C_{1\pm}(x_0,\nu) e^{(s+\sigma_1)x_0/\nu}\, dx_0\right] \Psi_{1(-\nu)}(x,\mu,s)\, d\nu, \quad x > a.
\end{aligned} \tag{B.10b}$$

Again, it is easily shown by direct substitution that Eqs. (B.10a) and (B.10b) are solutions of Eq. (2.13) for $j = 1$. We introduce the F functions of Eqs. (3.10) and by allowing x to take on negative and positive values, it follows that Eqs. (B.7), (B.10a) and (B.10b) can be written as Eqs. (3.8) and (3.9) of the text.

We also note here that the $C_{j\pm}$ coefficients of Eqs. (B.4) have the property

$$C_{j\pm}(-x_0, -\nu) = \mp C_{j\pm}(x_0, \nu)$$

and

$$C_{j\pm}(-x_0, -\nu_{0j}) = \mp C_{j\pm}(x_0, \nu_{0j}), \tag{B.11}$$

so that it then follows from Eqs. (3.10) that

$$F_{2\pm}(a, -\omega, s) = \mp F_{2\pm}(a, \omega, s)$$

and

$$\tilde{F}_{\pm}(-a, -\omega, s) = \mp \tilde{F}_{\pm}(-a, \omega, s). \tag{B.12}$$

C. Two-Media Full-Range Expansions and Orthogonality Relations

In this Appendix, we first summarize some results of Erdmann (refs. 8, 9) and Kuščer, McCormick and Summerfield (ref. 13). Erdmann (refs. 8, 9) shows that a function, say $h(\mu)$, satisifying very weak restrictions for μ on the interval $-1 \leq \mu \leq 1$ can be expanded as

$$h(\mu) = a_1 \varphi_{\nu_{01}}(\mu)\delta_1(s) + b_2 \varphi_{-\nu_{02}}(\mu)\delta_2(s) + \int_0^1 A_1(\nu)\varphi_{1s\nu}(\mu)d\nu + \int_{-1}^0 A_2(\nu)\varphi_{2s\nu}(\mu)d\nu. \quad (C.1)$$

This is a two-media full-range expansion of the function $h(\mu)$ and the expansion coefficients in it can be determined using orthogonality relations which are easily determined from the time-independent ones of Kuščer, McCormick, and Summerfield (ref. 13). For $\delta_1(s) = \delta_2(s) = 1$, that is $s \in S_{1i} \cap S_{2i}$, these relations are

$$\int_{-1}^1 W_s(\mu)\Phi_{s\nu'}(\mu)d\mu \int_{-1}^1 A(\nu)\Phi_{s\nu}(\mu)d\nu = A(\nu')W_s(\nu')\Omega_s^+(\nu')\Omega_s^-(\nu'),$$

$$\int_{-1}^1 W_s(\mu)\Phi_{s\nu}(\mu)\varphi_{\nu_{01}}(\mu)d\mu = 0,$$

$$\int_{-1}^1 W_s(\mu)\Phi_{s\nu}(\mu)\varphi_{-\nu_{02}}(\mu)d\mu = 0,$$

$$\int_{-1}^1 W_s(\mu)\varphi_{\nu_{01}}(\mu)\varphi_{-\nu_{02}}(\mu)d\mu = 0,$$

$$\int_{-1}^1 W_s(\mu)\Phi_{s\nu}(\mu)\varphi_{-\nu_{01}}(\mu)d\mu = \nu c(\nu)\sigma(\nu)\nu_{01}(\nu_{02} - \nu_{01})X_s(-\nu_{01})\varphi_{-\nu_{01}}(\nu),$$

$$\int_{-1}^{1} W_s(\mu)\Phi_{s\nu}(\mu)\varphi_{\nu_{02}}(\mu)d\mu = \nu c(\nu)\sigma(\nu)\nu_{02}(\nu_{01} - \nu_{02})X_s(\nu_{02})\varphi_{\nu_{02}}(\nu),$$

$$\int_{-1}^{1} W_s(\mu)\varphi_{\nu_{01}}(\mu)\varphi_{\pm\nu_{01}}(\mu)\, d\mu = -\left(\frac{c_1\sigma_1\nu_{01}}{2}\right)^2 (\nu_{01} \pm \nu_{02})X_s(\pm\nu_{01}),$$

$$\int_{-1}^{1} W_s(\mu)\varphi_{-\nu_{02}}(\mu)\varphi_{\pm\nu_{02}}(\mu)d\mu = \left(\frac{c_2\sigma_2\nu_{02}}{2}\right)^2 (\nu_{02} \mp \nu_{01})X_s(\pm\nu_{02}),$$

$$\int_{-1}^{1} W_s(\mu)\varphi_{\nu_{01}}(\mu)\varphi_{\nu_{02}}(\mu)d\mu = -\frac{1}{2}\, c_1\sigma_1 c_2\sigma_2\nu_{01}\nu_{02}^2\, X_s(\nu_{02})$$

and

$$\int_{-1}^{1} W_s(\mu)\varphi_{-\nu_{02}}(\mu)\varphi_{-\nu_{01}}(\mu)d\mu = \frac{1}{2}\, c_1\sigma_1 c_2\sigma_2\nu_{01}^2\nu_{02}X_s(-\nu_{01}), \tag{C.2}$$

where

$$c(\nu),\sigma(\nu) = \begin{cases} c_1,\sigma_1, & \nu > 0 \\ c_2,\sigma_2, & \nu < 0, \end{cases}$$

$$\Phi_{s\nu}(\mu) = \begin{cases} \varphi_{1s\nu}(\mu), & \nu > 0 \\ \varphi_{2s\nu}(\mu), & \nu < 0, \end{cases}$$

$$\Omega_s^{\pm}(\nu) = \begin{cases} \Omega_{1s}^{\pm}(\nu), & \nu > 0 \\ \Omega_{2s}^{\pm}(\nu), & \nu < 0, \end{cases}$$

$$X_s(z) = X_{1s}(z)X_{2s}(-z),$$

and

$$W_s(\nu) = \begin{cases} (\nu_{02} + \nu)X_{2s}(-\nu)W_{1s}(\nu), & \nu > 0 \\ -(\nu_{01} - \nu)X_{1s}(\nu)W_{2s}(-\nu), & \nu < 0. \end{cases} \tag{C.3}$$

All remaining quantities have been defined in Appendix A.

Rather than write out explicit orthogonality relations for the other three regions of the transform plane in the present notation, we introduce a function which is continuous as $s \to C_j$. From the results of Kuščer and Zweifel (ref. 14) quoted in Appendix A, we see that one such function is given by Eq. (3.24) and can be written using Eq. (3.25) as

$$X_0(z,s) = \begin{cases} \dfrac{(\nu_{02} - z)X_{2s}(z)}{(\nu_{01} - z)X_{1s}(z)}, & s \in S_{1i} \cap S_{2i} \\[2ex] \dfrac{(\nu_{02} - z)X_{2s}(z)}{X_{01s}(z)}, & s \in S_{1e} \cap S_{2i} \\[2ex] \dfrac{X_{02s}(z)}{(\nu_{01} - z)X_{1s}(z)}, & s \in S_{1i} \cap S_{2e} \\[2ex] \dfrac{X_{02s}(z)}{X_{01s}(z)}, & s \in S_{1e} \cap S_{2e}. \end{cases} \tag{C.4}$$

In terms of this function, $W_s(\nu)$ can be written as

$$W_s(\nu) = \begin{cases} \dfrac{c_1\sigma_1\nu}{2\Omega_{1s}(\infty)} X_0(-\nu,s), & \nu > 0 \\[2ex] \dfrac{c_2\sigma_2\nu}{2\Omega_{2s}(\infty)} \dfrac{1}{X_0(\nu,s)}, & \nu < 0. \end{cases} \tag{C.5}$$

The function $X_s(z)$ is expressed as

$$X_s(z) = \begin{cases} \dfrac{X_0(-z,s)}{(\nu_{02} + z)} \dfrac{\Omega_{1s}(z)}{(\nu_{01} - z)\Omega_{1s}(\infty)}, & \mathrm{Re}(z) > 0 \\[3ex] \dfrac{1}{(\nu_{01} - z)X_0(z,s)} \dfrac{\Omega_{2s}(z)}{(\nu_{02} + z)\Omega_{2s}(\infty)}, & \mathrm{Re}(z) < 0. \end{cases} \tag{C.6}$$

In order to obtain a two-media expansion in the form (C.1), we generally have to switch some continuum solutions in one medium to those in the other. From the explicit form of $\varphi_{js\nu}(\mu)$ (Eqs. (2.19)), we have that

$$c_1\sigma_1\varphi_{2s\nu}(\mu) - c_2\sigma_2\varphi_{1s\nu}(\mu) = k_s\delta(\mu - \nu),$$

where

$$k_s \equiv s(c_1\sigma_1 - c_2\sigma_2) + \sigma_1\sigma_2(c_1 - c_2). \tag{C.7}$$

We see that when the two media are the same, $k_s \equiv 0$. This quantity can be expressed in a number of different ways, and several that we shall use are

$$k_s = \begin{cases} c_1\sigma_1\lambda_{2s}(\nu) - c_2\sigma_2\lambda_{1s}(\nu) \\ c_1\sigma_1\Omega_{2s}(\nu_{01}) \\ -c_2\sigma_2\Omega_{1s}(\nu_{02}). \end{cases} \tag{C.8}$$

The orthogonality relations (C.2) can now be written in terms of $X_0(z,s)$ and k_s as

$$\int_{-1}^{1} W_s(\mu)\Phi_{s\nu'}(\mu)d\mu \int_{-1}^{1} A(\nu)\Phi_{s\nu}(\mu)d\nu = A(\nu')W_s(\nu')\Omega_s^+(\nu')\Omega_s^-(\nu'),$$

$$\int_{-1}^{1} W_s(\mu)\Phi_{s\nu}(\mu)\varphi_{\nu_{01}}(\mu)d\mu = 0,$$

$$\int_{-1}^{1} W_s(\mu)\Phi_{s\nu}(\mu)\varphi_{-\nu_{02}}(\mu)d\mu = 0,$$

$$\int_{-1}^{1} W_s(\mu)\varphi_{\nu_{01}}(\mu)\varphi_{-\nu_{02}}(\mu)d\mu = 0,$$

$$\int_{-1}^{1} W_s(\mu)\Phi_{s\nu}(\mu)\varphi_{-\nu_{01}}(\mu)d\mu = \frac{\nu c(\nu)\sigma(\nu)\nu_{01}k_s}{4\Omega_{2s}(\infty)X_0(-\nu_{01},s)(\nu_{01}+\nu)},$$

$$\int_{-1}^{1} W_s(\mu)\Phi_{s\nu}(\mu)\varphi_{\nu_{02}}(\mu)d\mu = -\frac{\nu c(\nu)\sigma(\nu)\nu_{02}k_s X_0(-\nu_{02},s)}{4\Omega_{1s}(\infty)(\nu_{02}-\nu)},$$

$$\int_{-1}^{1} W_s(\mu)\varphi_{\nu_{01}}(\mu)\varphi_{\pm\nu_{01}}(\mu)d\mu = \begin{cases} \left(\frac{1}{2}c_1\sigma_1\nu_{01}\right)^2 \frac{\Omega'_{1s}(\nu_{01})}{\Omega_{1s}(\infty)} X_0(-\nu_{01},s) \\ \frac{c_1\sigma_1\nu_{01}k_s}{8\Omega_{2s}(\infty)X_0(-\nu_{01},s)} \end{cases},$$

$$\int_{-1}^{1} W_s(\mu)\varphi_{-\nu_{02}}(\mu)\varphi_{\pm\nu_{02}}(\mu)d\mu = \begin{cases} \frac{c_2\sigma_2\nu_{02}k_s X_0(-\nu_{02},s)}{8\Omega_{1s}(\infty)} \\ -\left(\frac{1}{2}c_2\sigma_2\nu_{02}\right)^2 \frac{\Omega'_{2s}(\nu_{02})}{\Omega_{2s}(\infty)} \frac{1}{X_0(-\nu_{02},s)}, \end{cases}$$

$$\int_{-1}^{1} W_s(\mu)\varphi_{\nu_{01}}(\mu)\varphi_{\nu_{02}}(\mu)d\mu = \frac{c_1\sigma_1\nu_{01}\nu_{02}k_s X_0(-\nu_{02},s)}{4\Omega_{1s}(\infty)(\nu_{01}-\nu_{02})}$$

and

$$\int_{-1}^{1} W_s(\mu)\varphi_{-\nu_{02}}(\mu)\varphi_{-\nu_{01}}(\mu)d\mu = \frac{c_2\sigma_2\nu_{01}\nu_{02}k_s}{4\Omega_{2s}(\infty)(\nu_{02}-\nu_{01})} \frac{1}{X_0(-\nu_{01},s)}. \tag{C.9}$$

These expressions appear more complicated than the corresponding ones in Eqs. (C.2); however, the orthogonality relations needed for all regions of the transform plane are given by Eqs. (C.9). That is, for $s \in S_{1e} \cap S_{2i}$, the proper orthogonality relations are the first, third, sixth, and eighth equations of (C.9) with $X_0(z,s)$ given by Eqs. (C.4). We note here that $X_0(z,s)$ always appears in Eqs. (C.9) with $Re(z) < 0$. It follows then from Appendix A that for $Re(z) < 0$, $X_0(z,s)$ is a nonvanishing analytic function of both z and s except for the branch cuts in the s-plane due to $\nu_{01}(s)$ and $\nu_{02}(s)$.

D. The Two-Media Full-Range Expansion for This Problem

In this Appendix we show that application of the continuity condition, Eq. (2.15), results in a two-media full-range expansion of the type discussed in the last Appendix. For $x = a$, we readily obtain from Eqs. (3.14) and (2.15) upon using the explicit forms of $\psi_{jc\pm}$ and $\psi_{jp\pm}$ given by Eqs. (3.3), (3.4), (3.8) and (3.9) that

$$
\begin{aligned}
0 = {} & a_{2\pm}\left[\psi_{\nu_{02}}(a,\mu,s) \pm \psi_{-\nu_{02}}(a,\mu,s)\right]\delta_2(s) \\
& + \int_0^1 A_{2\pm}(\nu)\left[\psi_{2\nu}(a,\mu,s) \pm \psi_{2(-\nu)}(a,\mu,s)\right] d\nu \\
& \mp \left[a_{1\pm}\psi_{\nu_{01}}(a,\mu,s)\delta_1(s) + \int_0^1 A_{1\pm}(-\nu)\psi_{1\nu}(a,\mu,s)d\nu\right] \\
& + F_{2\pm}(a,\nu_{02},s)\psi_{\nu_{02}}(a,\mu,s)\delta_2(s) + \int_0^1 F_{2\pm}(a,\nu,s)\psi_{2\nu}(a,\mu,s)d\nu \\
& - F_{1\pm}(-a,\nu_{01},s)\left[\psi_{\nu_{01}}(a,\mu,s) \pm \psi_{-\nu_{01}}(a,\mu,s)\right]\delta_1(s) \\
& - \int_0^1 F_{1\pm}(-a,\nu,s)\left[\psi_{1\nu}(a,\mu,s) \pm \psi_{1(-\nu)}(a,\mu,s)\right] d\nu. \qquad \text{(D.1)}
\end{aligned}
$$

We have indicated in Appendix C that according to Erdmann (ref. 8) the functions $\varphi_{\nu_{01}}(\mu)$; $\varphi_{-\nu_{02}}(\mu)$; $\varphi_{1s\nu}(\mu)$, $0 \leq \nu \leq 1$ and $\varphi_{2s\nu}(\mu)$, $-1 \leq \nu \leq 0$ form a complete orthogonal set of basis functions for the expansion of $h(\mu)$, $-1 \leq \mu \leq 1$ for $s \in S_{1i} \cap S_{2i}$ (see Eq. (C.1)). However, Eq. (D.1) also contains terms in which $\varphi_{2s\nu}(\mu)$, $0 \leq \nu \leq 1$, and $\varphi_{1s\nu}(\mu)$, $-1 \leq \nu \leq 0$ appear. These continuum solutions must be replaced by corresponding continuum solutions for the other media. We use the relationship (C.7) to do this; that is,

$$\varphi_{2s\nu}(\mu) = \left[\frac{c_2\sigma_2}{c_1\sigma_1}\varphi_{1s\nu}(\mu) + \frac{k_s}{c_1\sigma_1}\delta(\nu - \mu)\right] H(\nu)$$

and

$$\varphi_{1s\nu}(\mu) = \left[\frac{c_1\sigma_1}{c_2\sigma_2}\varphi_{2s\nu}(\mu) - \frac{k_s}{c_2\sigma_2}\delta(\nu - \mu)\right] H(-\nu), \tag{D.2}$$

where

$$H(\nu) = \begin{cases} 1, & \nu > 0 \\ 0, & \nu < 0. \end{cases} \tag{D.3}$$

When explicit forms of the elementary solutions and Eqs. (D.2) and (D.3) are used in Eq. (D.1), we obtain the two-media full-range expansion

$$\begin{aligned} h(\mu) = {} & \left[F_{1\pm}(-a,\nu_{01},s) \pm a_{1\pm}\right] e^{-(s+\sigma_1)a/\nu_{01}}\, \varphi_{\nu_{01}}(\mu)\delta_1(s) \\ & \mp a_{2\pm} e^{(s+\sigma_2)a/\nu_{02}}\, \varphi_{-\nu_{02}}(\mu)\delta_2(s) \\ & + \int_0^1 \left\{\left[F_{1\pm}(-a,\nu,s) \pm A_{1\pm}(-\nu)\right] e^{-(s+\sigma_1)a/\nu} \right. \\ & \qquad \left. - \frac{c_2\sigma_2}{c_1\sigma_1}\left[F_{2\pm}(a,\nu,s) + A_{2\pm}(\nu)\right] e^{-(s+\sigma_2)a/\nu}\right\} \varphi_{1s\nu}(\mu)\,d\nu \\ & \pm \int_{-1}^0 \left[\frac{c_1\sigma_1}{c_2\sigma_2} F_{1\pm}(-a,-\nu,s) e^{-(s+\sigma_1)a/\nu} - A_{2\pm}(-\nu) e^{-(s+\sigma_2)a/\nu}\right] \varphi_{2s\nu}(\mu)\,d\nu, \end{aligned} \tag{D.4a}$$

where $h(\mu)$ is given by

$$h(\mu) = \frac{k_s}{c_1\sigma_1} H(\mu)\left[F_{2\pm}(a,\mu,s) + A_{2\pm}(\mu)\right] e^{-(s+\sigma_2)a/\mu}$$

$$\pm \frac{k_s}{c_2\sigma_2} H(-\mu)F_{1\pm}(-a,-\mu,s)e^{-(s+\sigma_1)a/\mu}$$

$$+ \left[F_{2\pm}(a,\nu_{02},s) + a_{2\pm}\right]e^{-(s+\sigma_2)a/\nu_{02}} \varphi_{\nu_{02}}(\mu)\delta_2(s)$$

$$\mp F_{1\pm}(-a,\nu_{01},s)e^{(s+\sigma_1)a/\nu_{01}} \varphi_{-\nu_{01}}(\mu)\delta_1(s). \tag{D.4b}$$

The orthogonality relations, Eqs. (C.9), can be used on the expansion (D.4) to obtain equations which determine the remaining unknown coefficients implicitly. However, it is convenient to introduce first the $E_{j\pm}$ coefficients given by Eqs. (3.15). We then have, after some algebra, the equations listed in the text as Eqs. (3.16)-(3.23).

E. Complex Representation of $\psi_{s\pm}(x,\mu)$

In this Appendix we outline how Eqs. (3.15) - (3.23) are extended to the complex plane $(\mu \to z)$ with s considered a parameter. As shown in references 2 and 3, functions such as those introduced in Eq. (3.15) are extendable. The particular grouping of terms in Eqs. (3.16) - (3.23) indicates some integrals and residues which go together.

The first functions to be considered are the $F_{j\pm}(x,\omega,s)$ functions given by Eqs. (3.10). In Eqs. (3.20) - (3.23), these functions appear with $\mathrm{Re}(\omega) > 0$ so we consider the functions $L_{j\pm}(x,\nu,s)$ given by Eq. (3.35). When the explicit expressions of Eqs. (3.10) and (3.11) are used, we can show that for $f_{j\pm}(x_0,\mu)$ extendable $\mu \to z$ without singularities in the finite z-plane then $L_{j\pm}(x,\nu,s)$ can be extended to $L_{j\pm}(x,z,s)$ given by Eqs. (3.32) and (3.33). Now as $z \to \nu \in (0,1)$ it can be seen that the limiting values of $L_{j\pm}$, namely $L^+_{j\pm}(x,\nu,s)$ and $L^-_{j\pm}(x,\nu,s)$, are identical. Thus, $L_{j\pm}$ does not inherit the branch cut of $\Omega_{js}(z)$ as one might be led to expect from Eq. (3.32). There appear to be no other singularities of $L_{j\pm}$ in the finite z-plane, $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > \sigma_j$. It follows from Eq. (3.32) that

$$L_{j\pm}(x,\nu_{0j},s) = \frac{1}{2}\, c_j\sigma_j\nu_{0j}\Omega'_{js}(\nu_{0j})F_{j\pm}(x,\nu_{0j},s)e^{-(s+\sigma_j)x/\nu_{0j}}, \quad s \in S_{ji}. \tag{E.1}$$

In order to extend the functions $I_{j\pm}(\nu)$ to the complex z-plane, we need the identity

$$\frac{c_1\sigma_1\Omega_{2s}^{+}(\nu)\Omega_{2s}^{-}(\nu)}{c_2\sigma_2\Omega_{1s}^{+}(\nu)\Omega_{1s}^{-}(\nu)} = \frac{c_2\sigma_2}{c_1\sigma_1} + \frac{ks}{c_1\sigma_1c_2\sigma_2}\left[\frac{c_1\sigma_1\lambda_{2s}(\nu) + c_2\sigma_2\lambda_{1s}(\nu)}{\Omega_{1s}^{+}(\nu)\Omega_{1s}^{-}(\nu)}\right], \tag{E.2}$$

which can be verified directly. We use this identity and find that $I_{j\pm}(\nu)$, given by Eqs. (3.20) and (3.22), can be written respectively as Eqs. (3.30) and (3.31). The restriction $\mathrm{Re}(s) > -\sigma_m$ on these equations comes from the fact that $L_{j\pm}$ for both $j = 1$ and 2 occur in each $I_{j\pm}$. More will be said about this restriction later. The contours C' are given in Figure 3. By letting $z = \nu_{02}$ in Eq. (3.30) and $z = \nu_{01}$ in Eq. (3.31), it can be seen that

$$I_{2\pm}(\nu_{02}, s) \equiv J_{2\pm}(\nu_{02})$$

and

$$I_{1\pm}(\nu_{01}, s) \equiv J_{1\pm}(\nu_{01}). \tag{E.3}$$

Thus, the inhomogeneous terms of Eqs. (3.16) - (3.19) are seen to be extendable and related as shown in Eq. (E.3). For $z \to \nu_{01}$ in $I_{2\pm}$ (see Eq. (3.30)) and $z \to \nu_{02}$ in $I_{1\pm}$ (see Eq. (3.31)), these functions might seem to be singular. However, it is seen upon examining the residues that this is not the case. Thus, the $I_{j\pm}(z,s)$ appear to be analytic in the finite z-plane, $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > -\sigma_m$.

In Eq. (3.16), we now let $\nu \to z$ and for $\mathrm{Re}(s) > -\sigma_m$ and $\mathrm{Re}(z) > 0$ in the finite z-plane, $E_{2\pm}(z)$ is given by the inhomogeneous term $I_{2\pm}(z)$, a term involving $a_{2\pm}$ if $s \in S_{2i}$ and an integral over $E_{2\pm}(\mu)$, $0 \leq \mu \leq 1$. A singularity occurs in the integrand when either $\Omega_{2s}^{+}(\mu)$ or $\Omega_{2s}^{-}(\mu)$ vanishes and this happens for $s \in C_2$. However, for this case, it is seen that we obtain from Eqs. (3.16) and (3.17) that

$E_{2\pm}(\nu_{02})$ is related to $a_{2\pm}$. It appears that $E_{2\pm}(z)$ is analytic in the finite z-plane, $\mathrm{Re}(z) > 0$, $\mathrm{Re}(s) > -\sigma_m$ and can be written as Eq. (3.28). We follow the same procedure with Eq. (3.17), and obtain an equation which is easily seen to be Eq. (3.28) evaluated at $z = \nu_{02}$; that is, $E_{2\pm}(\nu_{02}, s)$ and $a_{2\pm}$ are related as

$$E_{2\pm}(\nu_{02}, s) = \frac{1}{2} c_2\sigma_2\nu_{02}\Omega'_{2s}(\nu_{02})a_{2\pm}e^{(s+\sigma_2)a/\nu_{02}}, \quad s \in S_{2i}. \tag{E.4}$$

In a similar manner, we obtain from Eqs. (3.18) and (3.19) on letting $\nu \to z$ and making use of Eq. (E.4), Eq. (3.29) of the text and again it follows that $E_{1\pm}(\nu_{01}, s)$ and $a_{1\pm}$ are related as

$$E_{1\pm}(\nu_{01}, s) = \frac{1}{2} c_1\sigma_1\nu_{01}\Omega'_{1s}(\nu_{01})a_{1\pm}e^{-(s+\sigma_1)a/\nu_{01}}, \quad s \in S_{1i}. \tag{E.5}$$

It also appears that $E_{1\pm}(z,s)$ is analytic in the finite z-plane, $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > -\sigma_m$.

The solutions $\Psi_{jc\pm}$ and $\Psi_{jp\pm}$ can now be written in terms of the $E_{j\pm}$ as shown in Eqs. (3.36) - (3.39) of the text.

F. Investigation of the Associated Eigenvalue Problem

We consider in this Appendix, the associated eigenvalue problem; that is, the problem for which $f(x,\mu) \equiv 0$. The inhomogeneous terms $I_{j\pm}$ (given by Eqs. (3.30) - (3.32)) can be seen to be identically zero when $f(x,\mu)$ is zero everywhere. Solutions for $I_{j\pm} \equiv 0$ will be denoted with a bar, i.e., $\bar{E}_{j\pm}$. The unknown expansion coefficients for the eigenvalue problem, $\bar{E}_{j\pm}$, are given by Eqs. (3.28) and (3.29) with $I_{j\pm} = 0$. It is seen from such equations that $\bar{E}_{j\pm}$ can be determined only to within an arbitrary factor independent of z and that $\bar{E}_{1\pm}$ depends on $\bar{E}_{2\pm}$. Furthermore, the original normal-mode expansion coefficients for the eigenvalue problem are given by $\bar{E}_{j\pm}(\mu,s)$, $0 \leq \mu \leq 1$, $j = 1,2$; $\bar{E}_{1\pm}(\nu_{01},s)$, $s \in S_{1i}$ and $\bar{E}_{2\pm}(\nu_{02},s)$, $s \in S_{2i}$. Therefore we must examine solutions of such equations as a function of the transform variable s for $z \to \mu$ with the contour C' collapsed onto the branch cut $(0,1)$ due to $\Omega_{2s}(z')$, and for $z = \nu_{0j}$ when $s \in S_{ji}$. This will be done for all s in some right-half s-plane and it is convenient to divide the plane into three regions: $s \in S_{2e}$, $s \in S_{2i}$ and $s \in C_2$.

When $s \in S_{2e}$, $\Omega_{2s}(z')$ does not vanish within C' so that Eq. (3.28) with $I_{2\pm} = 0$ can be written as

$$\bar{E}_{2\pm}(\mu,s) = \pm\int_0^1 K_s(\mu,\nu)\bar{E}_{2\pm}(\nu,s)d\nu, \quad s \in S_{2e}, \quad 0 \leq \mu \leq 1, \tag{F.1}$$

where

$$K_s(\mu,\nu) = \frac{k_R}{2}\frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)}\frac{X_0(-\mu,s)X_0(-\nu,s)\nu}{\Omega_{2s}^+(\nu)\Omega_{2s}^-(\nu)(\nu+\mu)}e^{-2(s+\sigma_2)a/\nu}, \quad 0 \leq \mu,\nu \leq 1, \quad \mu = \nu_{02}. \tag{F.2}$$

When $s \in S_{2i}$, $\Omega_{2s}(z')$ vanishes inside C' but not on the real interval $(0,1)$. As C' is collapsed onto $(0,1)$, a residue term appears so that Eq. (3.28) with $I_{2\pm} = 0$ takes the form

$$\bar{E}_{2\pm}(\mu,s) = \pm \frac{k_s}{c_2\sigma_2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{X_0(-\mu,s)X_0(-\nu_{02},s)}{\Omega'_{2s}(\nu_{02})(\nu_{02}+\mu)} e^{-2(s+\sigma_2)a/\nu_{02}} \bar{E}_{2\pm}(\nu_{02},s)$$

$$\pm \int_0^1 K_s(\mu,\nu)\bar{E}_{2\pm}(\nu,s)d\nu, \quad s \in S_{2i}, \quad 0 \le \mu \le 1, \qquad \text{(F.3)}$$

for $z = \mu$ where $K_s(\mu,\nu)$ is given by Eq. (F.2). However, Eq. (3.28) with $I_{2\pm} = 0$ must also hold at $z = \nu_{02}$ and this gives an additional constraint on solutions of (F.3), namely

$$\bar{E}_{2\pm}(\nu_{02},s) = \pm \frac{k_s}{c_2\sigma_2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{X_0^2(-\nu_{02},s)}{2\nu_{02}\Omega'_{2s}(\nu_{02})} e^{-2(s+\sigma_2)a/\nu_{02}} \bar{E}_{2\pm}(\nu_{02},s)$$

$$\pm \int_0^1 K_s(\nu_{02},\nu)\bar{E}_{2\pm}(\nu,s)d\nu, \quad s \in S_{2i}. \qquad \text{(F.4)}$$

When $s \in C_2$, the curve separating S_{2i} and S_{2e}, $\Omega^+_{2s}(\nu)\, \Omega^-_{2s}(\nu)$ vanishes for some ν on the interval $(0,1)$. That is, ν_{02} is real and lies on $(0,1)$. Setting $\nu_{02} = \eta$, we can put Eq. (3.28) with $I_{2\pm} = 0$ in the form

$$\bar{E}_{2\pm}(\mu,s) = \pm \frac{1}{2} \frac{k_s}{c_2\sigma_2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{X_0(-\mu,s)X_0(-\eta,s)}{\Omega'_{2s}(\eta)(\eta+\mu)} e^{-2(s+\sigma_2)a/\eta} \bar{E}_{2\pm}(\eta,s)$$

$$\pm \frac{k_s}{2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} X_0(-\mu,s) \underset{\nu=\eta}{P} \int_0^1 \frac{\bar{E}_{2\pm}(\nu,s)X_0(-\nu,s)}{\Omega^+_{2s}(\nu)\Omega^-_{2s}(\nu)} e^{-2(s+\sigma_2)a/\nu} \frac{\nu d\nu}{\nu+\mu},$$

$$s \in C_2, \quad 0 \le \mu, \eta \le 1. \qquad \text{(F.5)}$$

Note that this equation would be obtained from either Eq. (F.1) or Eq. (F.3) for $s \to C_2$ from $s \in S_{2e}$ or $s \in S_{2i}$ respectively.

For arbitrary complex values of s, the kernel $K_s(\mu,\nu)$ which appears in Eqs. (F.1) and (F.3) is not symmetric since

$$K_s(\mu,\nu) \neq \left[K_s(\nu,\mu)\right]^*, \quad \mathrm{Im}(s) \neq 0, \tag{F.6}$$

where $*$ denotes complex conjugation. Note, however, that when $\mathrm{Im}(s) = 0$, the unknown functions $\overline{E}_{2\pm}(\mu,s)$ can be redefined so that a symmetric kernel is obtained. Solutions of Eqs. (F.1) and (F.3) depend on the behavior of $K_s(\mu,\nu)$ and we shall look at a quantity $B^2(s)$ given by

$$B^2(s) = \int_0^1 \int_0^1 \left| K_s(\mu,\nu)\right|^2 d\mu d\nu. \tag{F.7}$$

To do this, we introduce the nondimensional parameters ζ, σ_R, σ_D and A given by Eq. (6.1) with $\zeta = \alpha + i\beta$. Note that α, β, σ_R, σ_D and A are real while σ_R and A are nonnegative. In terms of these quantities, we have

$$\left|K_s(\mu,\nu)\right|^2 = \frac{\left\{\left[\alpha(\sigma_R-1)-\sigma_D\right]^2 + \beta^2(\sigma_R-1)^2\right\}\left[(\alpha-1)^2+\beta^2\right]}{4\left[(\alpha+\sigma_D-\sigma_R)^2+\beta^2\right]}$$

$$\times \frac{\left|X_0(-\mu,s)\right|^2 \left|X_0(-\nu,s)\right|^2 \nu^2}{\left|\dfrac{\Omega_{2s}^+(\nu)}{c_2\sigma_2}\right|^2 \left|\dfrac{\Omega_{2s}^-(\nu)}{c_2\sigma_2}\right|^2 (\nu+\mu)^2}\, e^{-4A\alpha/\nu}, \quad 0 \le \mu,\nu \le 1. \tag{F.8}$$

To make estimates of the function $\left|X_0(-\mu,s)\right|^2$, we use an integral representation of the single-medium X-function given by Kuščer and Zweifel (ref. 14), namely,

$$X_{0j}(-\mu,s) = \exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ln\left[\frac{\Omega_{js}(z')}{\Omega_{js}(\infty)}\right]\frac{dz'}{z'+\mu}\right\}. \qquad \text{(F.9)}$$

Upon letting $z' = iy$ and using $\Omega_{js}(z') = \Omega_{js}(-z')$, we see that Eq. (F.9) becomes

$$X_{0j}(-\mu,s) = \exp\left\{\frac{\mu}{\pi}\int_0^{\infty} \ln\left[\frac{\Omega_{js}(iy)}{\Omega_{js}(\infty)}\right]\frac{dy}{y^2+\mu^2}\right\}, \qquad \text{(F.10)}$$

which is seen to be real for s real. In terms of the quantities of Eq. (6.1) we find

$$\left|X_{01}(-\mu,s)\right|^2 = \exp\left\{\frac{2\mu}{\pi}\int_0^{\infty} \ln\sqrt{\frac{\left[\alpha+\sigma_D-\sigma_R g(y)\right]^2+\beta^2}{(\alpha+\sigma_D-\sigma_R)^2+\beta^2}}\;\frac{dy}{y^2+\mu^2}\right\}$$

and

$$\left|X_{02}(-\mu,s)\right|^2 = \exp\left\{\frac{2\mu}{\pi}\int_0^{\infty} \ln\sqrt{\frac{\left[\alpha-g(y)\right]^2+\beta^2}{(\alpha-1)^2+\beta^2}}\;\frac{dy}{y^2+\mu^2}\right\}, \qquad \text{(F.11)}$$

where

$$g(y) = y\tan^{-1} 1/y. \qquad \text{(F.12)}$$

It follows from Eq. (F.12) that $0 \leq g(y) \leq 1$ for $0 \leq y \leq \infty$ and that it is a monotonic increasing function of y on this interval. Furthermore, since

$$\frac{2\mu}{\pi}\int_0^{\infty}\frac{dy}{y^2+\mu^2} = 1, \qquad \text{(F.13)}$$

the following bounds (perhaps rather loose) are obtained for $\left|X_{0j}(-\mu,s)\right|^2$:

$$1 \leq \left|X_{02}(-\mu,s)\right|^2 \leq \sqrt{\frac{\alpha^2+\beta^2}{(\alpha-1)^2+\beta^2}}$$

and

$$1 \leq \left|X_{01}(-\mu,s)\right|^2 \leq \sqrt{\frac{(\alpha+\sigma_D)^2+\beta^2}{(\alpha+\sigma_D-\sigma_R)^2+\beta^2}} \tag{F.14}$$

for $\alpha > \max\,(1,-\sigma_D+\sigma_R)$, and

$$\sqrt{\frac{\beta^2}{(\alpha-1)^2+\beta^2}} \leq \left|X_{02}(-\mu,s)\right|^2 \leq \max\left(1, \sqrt{\frac{\alpha^2+\beta^2}{(\alpha-1)^2+\beta^2}}\right)$$

and

$$\sqrt{\frac{\beta^2}{(\alpha+\sigma_D-\sigma_R)^2+\beta^2}} \leq \left|X_{01}(-\mu,s)\right|^2 \leq \max\left(1, \sqrt{\frac{(\alpha+\sigma_D)^2+\beta^2}{(\alpha+\sigma_D-\sigma_R)^2+\beta^2}}\right) \tag{F.15}$$

for $\alpha < \max\,(1,-\sigma_D+\sigma_R)$ and $\beta \neq 0$. We note that in the ζ-plane, the points $\zeta = 1$ and $\zeta = -\sigma_D + \sigma_R$ are the right ends of the branch cuts of ν_{02} and ν_{01} respectively and these cuts lie on the real ζ-axis. The left ends are at $\zeta = 0$ and $\zeta = -\sigma_D$, respectively.

The functions $\left|\frac{\Omega_{2s}^{\pm}(\nu)}{c_2\sigma_2}\right|^2$ are easily found to be

$$\left|\frac{\Omega_{2s}^{\pm}(\nu)}{c_2\sigma_2}\right|^2 = (\alpha-\nu\tanh^{-1}\nu)^2 + \left(\beta \pm \frac{\pi\nu}{2}\right)^2. \tag{F.16}$$

Recall from Appendix A that the curve C_2 (see Figure 2) is given by

$$\alpha' = \frac{2\beta'}{\pi}\tanh^{-1}\frac{2\beta'}{\pi}\,. \tag{F.17}$$

The parametric form of this equation is

$$\left|\beta'\right| = \frac{\pi\nu}{2}, \quad \alpha' = \nu \tanh^{-1} \nu, \quad 0 \leq \nu \leq 1. \tag{F.18}$$

We see from Eq. (F.16) that $\left|\frac{\Omega^{\pm}_{2s}(\nu)}{c_2\sigma_2}\right|^2$ are the squares of the distance (in the ζ-plane) from the point (α,β) to the points $(\alpha'(\nu), \mp \beta'(\nu))$ respectively which lie on the curve C_2. Since these functions appear in the denominator of $\left|K_s(\mu,\nu)\right|^2$ the integral (F.7) will not be bounded when α and β' are related as in Eq. (F.17). We define $D_{min}(\alpha,\beta)$ as the minimum distance from the point (α,β) to the curve C_2 for $0 \leq \nu \leq 1$. That is,

$$D_{min}(\alpha,\beta) \equiv \sqrt{\min\left(\left|\frac{\Omega^{+}_{2s}(\nu)}{c_2\sigma_2}\right|^2, \left|\frac{\Omega^{-}_{2s}(\nu)}{c_2\sigma_2}\right|^2\right)}, \quad 0 \leq \nu \leq 1 \tag{F.19}$$

and $D_{min}(\alpha,\beta) \neq 0$ for $(\alpha,\beta) \notin C_2$. Therefore, we have from Eqs. (F.16) and (F.19) that

$$\frac{1}{\left|\frac{\Omega^{+}_{2s}(\nu)}{c_2\sigma_2}\right|^2 \left|\frac{\Omega^{-}_{2s}(\nu)}{c_2\sigma_2}\right|^2} \leq \frac{1}{D^4_{min}(\alpha,\beta)}, \quad 0 \leq \nu \leq 1. \tag{F.20}$$

Analytical bounds for the above function are not as easy to get. For $\beta = 0$, $\Omega^{+}_{2s}(\nu) = \left[\Omega^{-}_{2s}(\nu)\right]^*$ and

$$\left|\frac{c_2\sigma_2}{\Omega^{\pm}_{2s}(\nu)}\right|^2_{\beta=0} = \frac{1}{\alpha^2}\, g\left(\frac{1}{\alpha}, \nu\right), \tag{F.21}$$

where $g\left(\frac{1}{\alpha}, \nu\right)$ has been investigated and tabulated by Case, de Hoffmann, and Placzek (ref. 6). They show that $g\left(\frac{1}{\alpha}, \nu\right)\Big|_{max}$ occurs at $\nu = 0$

for $\alpha < \pi^2/8$ whereas for $\alpha > \pi^2/8$ it occurs for ν between 0 and 1. For α very large they have $g_{max} \to 4\alpha^2/\pi^2$. The present geometric interpretation is consistent with all of these characteristics. The radius of curvature of the curve C_2 given by Eq. (F.17) is $\pi^2/8$ at $(\alpha',\beta') = (0,0)$. For α' very large, $\beta' \to \pi/2$ so that the minimum squared distance from $(\alpha,0)$ to (α',β') approaches $\pi^2/4$, in agreement with Eq. (F.21) and $g_{max} \to 4\alpha^2/\pi^2$.

Note that the exponential factor $e^{-4\alpha A/\nu}$ in Eq. (F.8) requires $\alpha > 0$ in order for $B^2(\alpha,\beta)$ to be bounded since both ν and A are nonnegative. On using estimates (F.14), (F.15) and (F.20), we obtain from Eq. (F.7) a bound for $B^2(\alpha,\beta)$ which we denote as $B^2_{max}(\alpha,\beta)$:

$$B^2_{max}(\alpha,\beta) = \begin{cases} \dfrac{e^{-4\alpha A}}{4D^4_{min}(\alpha,\beta)} \dfrac{\left\{\left[\alpha(\sigma_R-1)-\sigma_D\right]^2 + \beta^2(\sigma_R-1)^2\right\}(\alpha^2+\beta^2)}{(\alpha+\sigma_D-\sigma_R)^2+\beta^2}, \\ \qquad\qquad \alpha > \max(1,-\sigma_D+\sigma_R) \\ \dfrac{e^{-4\alpha A}}{4D^4_{min}(\alpha,\beta)} \left[\max\left(\dfrac{\alpha^2+\beta^2}{(\alpha-1)^2+\beta^2}, \; 1\right)\right] \\ \quad \times \dfrac{\left\{\left[\alpha(\sigma_R-1)-\sigma_D\right]^2 + \beta^2(\sigma_R-1)^2\right\}\left[(\alpha-1)^2+\beta^2\right]}{\beta^2}, \quad \beta \neq 0, \\ \qquad\qquad 0 < \alpha < \max(1,-\sigma_D+\sigma_R). \end{cases} \tag{F.22}$$

It can be seen that B^2_{max} depends not only on α and β but also the nondimensional material parameters σ_D and σ_R as well as the slab thickness parameter A. The estimate (F.22) for B^2_{max} is not bounded for the following regions in the s-plane:

$$\mathrm{Re}(s) < -\sigma_2 \ (\alpha < 0),$$

$$s \in C_2$$

and

$$s \in \text{branch cuts of } \nu_{01}(s) \cup \nu_{02}(s). \tag{F.23}$$

These regions must be handled separately. Even for the general case, where s does not belong to any of the regions (F.23), it appears difficult to say whether or not the eigenvalue problem has nontrivial solutions. We suspect that it has only trivial solutions for such regions since that is the result which has been found for certain special cases by others. Lehner and Wing (refs. 16, 17) have shown this for the bare slab, while Lehner (ref. 15) and Hintz (ref. 10) have obtained this result for the slab surrounded by pure absorbers. We can show that this result is also obtained for the special case $A \to \infty$; that is, a thick slab.

Since the slab thickness parameter A appears only in the exponential term of Eq. (F.22), it is seen that $B^2_{max}(\alpha,\beta)$ can be made as small as one likes as $A \to \infty$ if s does not belong to any of the regions given in (F.23). For $|B_{max}(\alpha,\beta)| < 1$, the Neumann series solution of the inhomogeneous integral Eq. (F.3) converges to a unique solution. (See ref. 19, for example.) Fredholm's Alternative Theorem (ref. 19) then guarantees that the corresponding homogeneous equation, namely (F.1), has only the trivial solution. Thus, for $s \in S_{2e}$ the eigenvalue problem has only the trivial solution as $A \to \infty$. When $s \in S_{21}$, the unique Neumann series solution of Eq. (F.3) must satisfy the additional constraint, Eq. (F.4). Using the condition $\Omega_{2s}(\nu_{02}) = 0$, we obtain

$$e^{-2(s+\sigma_2)a/\nu_{02}} = \left[\rho(s)e^{i\theta(s)}\right]^A, \tag{F.24}$$

where

$$\rho^2(s) = \frac{\left[\mathrm{Re}(\nu_{02})-1\right]^2 + \mathrm{Im}^2(\nu_{02})}{\left[\mathrm{Re}(\nu_{02})+1\right]^2 + \mathrm{Im}^2(\nu_{02})}$$

and

$$\theta(s) = \tan^{-1}\left[\frac{\mathrm{Im}(\nu_{02})}{\mathrm{Re}(\nu_{02}) - 1}\right] - \tan^{-1}\left[\frac{\mathrm{Im}(\nu_{02})}{\mathrm{Re}(\nu_{02}) + 1}\right]. \tag{F.25}$$

Now since $\mathrm{Re}(\nu_{02}) \geq 0$, we have

$$\left[\rho(s)\right]^A \xrightarrow[A\to\infty]{} 0, \quad \mathrm{Re}(\nu_{02}) \neq 0. \tag{F.26}$$

Therefore, the Neumann series solution is seen to converge to zero as $A \to \infty$ when $\mathrm{Re}(\nu_{02}) \neq 0$. Note that $\mathrm{Re}(\nu_{02}) = 0$ is the branch cut of $\nu_{02}(s)$ which is one of the regions given by (F.23) which we must consider separately. When $s \in C_2$, $\nu_{02} = \eta$, $0 \leq \eta \leq 1$ so that $\rho^2(s)$ of Eq. (F.25) becomes

$$\rho^2(s) = \left(\frac{\eta - 1}{\eta + 1}\right)^2 \leq 1 \tag{F.27}$$

and $\rho = 1$ occurs only at $s = -\sigma_2$ (that is, $(\alpha,\beta) = (0,0)$).

We use Eq. (F.27) in Eq. (F.5) and on taking the limit $A \to \infty$ we find that $\tilde{E}_{2\pm}(\mu,s) \to 0$ for $s \in C_2$, $s \neq -\sigma_2$.

Summarizing the results then for $A \to \infty$, we find that the eigenvalue problem has only the trivial solution for $\mathrm{Re}(s) > -\sigma_2$ unless s belongs to either the branch cut of $\nu_{01}(s)$ or $\nu_{02}(s)$. In order to see what happens on these cuts, we must write Eqs. (3.28) and (3.29) with $I_{j\pm} = 0$ in terms of the $X_{js}(-z)$ functions rather than the $X_0(-z,s)$ function. This will be done in the next Appendix. When A is not large, one has for the problems of Lehner and Wing (refs. 16, 17), Lehner (ref. 15) and Hintz (ref. 10) that if the eigenvalue problem has nontrivial discrete solutions, they occur on the real s-axis. For the bare slab, Bowden (refs. 1, 4) has shown that these solutions lie on the branch cut of $\nu_0(s)$. In view of these results, it is assumed that the eigenvalue problem has nontrivial solutions for $\mathrm{Re}(s) > -\sigma_2$ only if s belongs to either the branch cut of $\nu_{01}(s)$ or $\nu_{02}(s)$.

G. Solution of the Associated Eigenvalue Problem for $s \in S_{2i}$

In this appendix we look at solutions of Eqs. (3.28) and (3.29) with $I_{j\pm} \equiv 0$ for s on the branch cuts of $\nu_{0j}(s)$. It is convenient to use coefficients related to the original expansion coefficients $\bar{A}_{j\pm}(\nu)$ and $\bar{a}_{j\pm}$ (the bar indicates that we are considering the associated eigenvalue problem). Recall that the $\bar{E}_{j\pm}$ are related to such coefficients by Eqs. (3.15), (E.4) and (E.5). We also noted in Appendix F that the coefficients can be determined only to within an arbitrary factor independent of ν. Following Bowden (refs. 1, 4), we introduce coefficients $\bar{B}_{j\pm}$ as

$$\bar{A}_{j\pm}(\nu) = \bar{a}_{2\pm}\bar{B}_{j\pm}(\nu), \quad \bar{a}_{1\pm} = \bar{a}_{2\pm}\bar{b}_{1\pm} \,. \tag{G.1}$$

The estimate $B^2_{max}(\alpha,\beta)$, Eq. (F.22) of Appendix F, was not bounded on the branch cuts of $\nu_{0j}(s)$. In that estimate we used the $X_0(-z,s)$ function so that the behavior for s inside, on, and outside the curve C_2 could be seen. To investigate what happens on the branch cuts of $\nu_{0j}(s)$, we should use the $X_{js}(z)$ functions (Appendix A) which do not inherit the branch cuts of ν_{0j}. Also, when $\nu_{01}(s)$ becomes pure imaginary (that is, on its branch cut), we cannot include its contribution (the pole at $z' = \nu_{01}$) in the integral over the contour C' of equation (3.29). We note again that the material properties c_j and σ_j determine where on the real s-axis the branch cuts of $\nu_{0j}(s)$ lie. The only restriction which has been made is $c_1 < 1$ and this alone does not specify how the cuts overlap. It does, however, guarantee that the branch cut of $\nu_{01}(s)$ lies entirely to the left of $s = 0$.

We consider $s \in S_{2i} \cap S_{1i}$ first. When the relationships (G.1), (3.15), (E.4) and (E.5) are used in Eqs. (F.3) and (F.4), we obtain after some algebra and use of the X-identities of Appendix A, equations for $\bar{B}_{2\pm}(\mu)$ and the additional constraint, namely Eqs. (4.2) and (4.6) of the text. Recall that Eqs. (F.3) and (F.4) were obtained from Eq. (3.28) with $I_{2\pm} \equiv 0$. Equations for $\bar{B}_{1\pm}(-\mu)$ and $\bar{b}_{1\pm}$ are obtained in a similar manner from Eq. (3.29) with $I_{1\pm} \equiv 0$ when the contour C' is collapsed onto the interval (0,1) of the branch cut of $\Omega_{2s}(z')$. These equations are given as Eqs. (4.3) and (4.4) of the text. The normal-mode expansion of the solution of the associated eigenvalue problem is given in terms of the $\bar{B}_{j\pm}$ coefficients by Eq. (4.1). We note that Eq. (4.6) is the exact eigenvalue condition since all material properties have been assumed known. It determines the values of s, $\langle s_n \rangle$, for which the eigenvalue problem has nontrivial solutions. When s belongs to the branch cut of ν_{01}, Eq. (4.6) takes on different values above and below the ν_{01} cut. Therefore, it is concluded that the eigenvalue problem has only the trivial solution on the branch cut of ν_{01}. On that portion of the branch cut of ν_{02} which is not also part of the ν_{01} cut, Eqs. (4.2)-(4.6) require that the limiting values of the coefficients above and below the ν_{02} cut be related as

$$\left[\bar{B}_{j\pm}(\mu)\right]^+ = \pm\left[\bar{B}_{j\pm}(\mu)\right]^-$$

and

$$\left[\bar{b}_{1\pm}\right]^+ = \pm\left[\bar{b}_{1\pm}\right]^-, \quad \mathrm{Re}(\nu_{02}) = \mathrm{Im}(\nu_{01}) = 0, \qquad \text{(G.2)}$$

that is, where s is real and is given by $\max\left[-\sigma_2, -\sigma_1(1 - c_1)\right] < s < -\sigma_2(1 - c_2)$. It then follows from Eqs. (4.1) and (G.2) that the limiting values of $\bar{\psi}_{s\pm}(x,\mu)$ for the same region are given by Eq. (4.21). From Bowden's results (refs. 1, 4) for the bare slab, it is expected that the eigenvalue problem has nontrivial solutions only at isolated points, $\{s_n\}$, which lie on the branch cut of ν_{02} but not on the branch cut of ν_{01}.

In the limit $c_2\sigma_2 a \to \infty$ which was discussed in Appendix F, we see that Eq. (4.2) gives $\bar{B}_{2\pm}(\mu) \to 0$ while Eq. (4.6), the eigenvalue condition, becomes

$$0 = \frac{X_{2s}(-\nu_{02})}{X_{1s}(-\nu_{02})} \frac{e^{-(s+\sigma_2)a/\nu_{02}}}{\nu_{01} + \nu_{02}} \mp \frac{X_{2s}(\nu_{02})}{X_{1s}(\nu_{02})} \frac{e^{(s+\sigma_2)a/\nu_{02}}}{\nu_{01} - \nu_{02}}, \quad c_2\sigma_2 a \to \infty,$$

$$\mathrm{Re}(\nu_{02}) = \mathrm{Im}(\nu_{01}) = 0. \tag{G.3}$$

Equation (G.3) is the "thick-slab" eigenvalue condition and for the region of the s-plane where it is valid, it can be seen that we have an even eigenvalue s_n if

$$\mathrm{Im}\left\{\frac{X_{2s}(\nu_{02})}{X_{1s}(\nu_{02})} \frac{e^{(s_n+\sigma_2)a/\nu_{02}}}{(\nu_{01} - \nu_{02})}\right\} = 0$$

and an odd eigenvalue s_n if

$$\mathrm{Re}\left\{\frac{X_{2s}(\nu_{02})}{X_{1s}(\nu_{02})} \frac{e^{(s_n+\sigma_2)a/\nu_{02}}}{(\nu_{01} - \nu_{02})}\right\} = 0, \quad c_2\sigma_2 a \to \infty. \tag{G.4}$$

We note that Eq. (G.3) has the same form as the zero-order approximation of the critical condition given by Case and Zweifel (ref. 7), except here we have both even and odd parity solutions. Numerical solutions of the eigenvalue conditions will be discussed in Appendix J.

In the region of the s-plane $s \in S_{1e} \cap S_{2i}$ we are specifically interested in the solution on the branch cut of $\nu_{02}(s)$ which lies to the left of $s = -\sigma_1$. That is, for s real and $-\sigma_2 < s \leq -\sigma_1$. For such values of s the solution (4.1) outside the slab is not bounded as $x \to \infty$, since

$$\psi_{1\nu}(x,\mu,s) = \varphi_{1s\nu}(\mu)\; e^{-(s+\sigma_1)x/\nu}, \quad 0 \leq \nu \leq 1. \tag{G.5}$$

In addition, the restriction $\mathrm{Re}(s) > -\sigma_m$ on both inhomogeneous terms $I_{j\pm}$ (see Eqs. (3.30) and (3.31)) also indicates that we cannot deform the inversion contour to the left of $\mathrm{Re}(s) = -\sigma_1$ in general. However, when one is looking for the solution inside the slab, $|x| \leq a$, perhaps the inversion contour can be deformed to the left of $\mathrm{Re}(s) = -\sigma_1$ for special values of material properties and/or initial data. For $s \in S_{2i} \cap S_{1e}$, expansion coefficients for the solution inside the slab are obtained as Eqs. (4.7) and (4.8). We note that Eq. (4.8) is exactly Eq. (4.6) with $X_{01s}(z)$ replacing $(\nu_{01} - z)X_{1s}(z)$. Recall from Eq. (A.9) that these are the X-functions which are continuous as $s \to C_1$. Under the same replacement of $X_{01s}(z)$ with $(\nu_{01} - z)X_{1s}(z)$, Eq. (4.7) reduces to the equation from which Eq. (4.2) was obtained. Equation (4.8), which corresponds to the eigenvalue condition Eq. (4.6), determines the pseudo-eigenvalues. That is, the values of s, $-\sigma_2 < s < -\sigma_1$, where $\overline{\psi}_{2\pm}(x,\mu,s)$ has nontrivial solutions.

H. Form of $\psi_{s\pm}(x,\mu)$ on the Branch Cuts of $\nu_{0j}(s)$

In this appendix, we show how the transformed solution $\psi_{s\pm}(x,\mu)$ is put in a form where one can see how it behaves on the branch cuts of $\nu_{0j}(s)$. We expect that $\psi_{s\pm}$ inherits the branch cut of $\nu_{01}(s)$ since only one of the two discrete modes appears for $|x| > a$. Such branch cuts appeared in the half-space albedo problem solved by Kuščer and Zweifel (ref. 14) as well as the two dissimilar semi-infinite media problem solved by Erdmann (refs. 8, 9). We also expect that the branch cut of $\nu_{02}(s)$ is not inherited by $\psi_{s\pm}$ but instead one should find poles at $s = s_n$, the place where the associated eigenvalue problem has nontrivial solutions. This is what Lehner and Wing (refs. 16, 17) and Bowden (refs. 1, 4) found for the bare slab.

It is not obvious upon looking at the equations of section III which determine the expansion coefficients implicitly how we should group terms to show what we expect. We start by looking at $\psi_{2\pm}(x,\mu,s)$. From Eqs. (3.3) and (3.8) of the text we have that

$$\begin{aligned}
\psi_{2\pm}(x,\mu,s) = {} & a_{2\pm}\left[\psi_{\nu_{02}}(x,\mu,s) \pm \psi_{-\nu_{02}}(x,\mu,s)\right] \\
& + \int_0^1 A_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right] d\nu \\
& + F_{2\pm}(x,\nu_{02},s)\psi_{\nu_{02}}(x,\mu,s) \\
& \pm F_{2\pm}(-x,\nu_{02},s)\psi_{-\nu_{02}}(x,\mu,s) \\
& + \int_0^1 \left[F_{2\pm}(x,\nu,s)\psi_{2\nu}(x,\mu,s) \pm F_{2\pm}(-x,\nu,s)\psi_{2(-\nu)}(x,\mu,s)\right] d\nu .
\end{aligned} \tag{H.1}$$

Note that this equation can be obtained of course from Eqs. (3.36) and (3.38). It is readily shown from the definition of the $F_{j\pm}$ functions, Eqs. (3.10), and the properties of the $C_{j\pm}$, Eqs. (B.11), that

$$F_{2\pm}(x,\nu_{02},s) = F_{2\pm}(a,\nu_{02},s) \pm F_{2\pm}(-x,-\nu_{02},s)$$

and

$$F_{2\pm}(x,-\nu_{02},s) = F_{2\pm}(a,-\nu_{02},s) \pm F_{2\pm}(-x,\nu_{02},s). \tag{H.2}$$

It follows from Eqs. (H.2) then that two coefficients in Eq. (H.1) can be written as

$$F_{2\pm}(x,\nu_{02},s) = \frac{1}{2}\left[F_{2\pm}(x,\nu_{02},s) \pm F_{2\pm}(-x,-\nu_{02},s)\right] + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)$$

and

$$\pm F_{2\pm}(-x,\nu_{02},s) = \frac{1}{2}\left[F_{2\pm}(x,-\nu_{02},s) \pm F_{2\pm}(-x,\nu_{02},s)\right] \pm \frac{1}{2}F_{2\pm}(a,\nu_{02},s), \tag{H.3}$$

where we have used Eq. (B. 12) to replace $F_{2\pm}(a,-\nu_{02},s)$. Equation (H.1) becomes then

$$\begin{aligned}\psi_{2\pm}(x,\mu,s) = &\left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]\left[\psi_{\nu_{02}}(x,\mu,s) \pm \psi_{-\nu_{02}}(x,\mu,s)\right]\\ &+\int_0^1 A_{2\pm}(\nu)\left[\psi_{2\nu}(x,\mu,s) \pm \psi_{2(-\nu)}(x,\mu,s)\right]d\nu\\ &+\frac{1}{2}\left[F_{2\pm}(x,\nu_{02},s) \pm F_{2\pm}(-x,-\nu_{02},s)\right]\psi_{\nu_{02}}(x,\mu,s)\\ &+\frac{1}{2}\left[F_{2\pm}(x,-\nu_{02},s) \pm F_{2\pm}(-x,\nu_{02},s)\right]\psi_{-\nu_{02}}(x,\mu,s)\\ &+\int_0^1 \left[F_{2\pm}(x,\nu,s)\psi_{2\nu}(x,\mu,s) \pm F_{2\pm}(-x,\nu,s)\psi_{2(-\nu)}(x,\mu,s)\right]d\nu.\end{aligned} \tag{H.4}$$

When $s \in$ branch cut of $\nu_{02}(s)$, we have $\nu_{02} = i\,|\nu_{02}|$ for $\mathrm{Im}(s) = 0^-$ and $\nu_{02} = -i\,|\nu_{02}|$ for $\mathrm{Im}(s) = 0^+$. Therefore, on going from below the ν_{02} cut to above it, we see that the third and fourth terms in the RHS of Eq. (H.4) simply interchange while those containing $F_{2\pm}(x,\nu,s)$ and $F_{2\pm}(-x,\nu,s)$ are unaffected since these functions do not depend on ν_{02}. The coefficient of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]$ changes sign however for odd-parity solutions and we do not yet know how $A_{2\pm}(\nu)$ behaves. By comparing Eq. (H.4) with Eq. (4.1) for $|x| < a$, we suspect that $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]$ is the coefficient which excites the associated eigen-solution $\bar{\psi}_{s\pm}(x,\mu)$. This is the information we needed to see how to group terms in the implicit equations for the expansion coefficients.

Now we look at the equations which determine the expansion coefficients. We obtain from Eq. (3.17) upon using the X-identities, the definition of the h_j functions (Eqs. (4.5)), and the relationship between the $E_{j\pm}$ and the original expansion coefficients that

$$\begin{aligned} 0 = &\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\left[\frac{h_2(\nu_{02})}{\nu_{01} + \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\nu_{01} - \nu_{02}}\right] \\ &+ \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[\frac{h_2(\nu_{02})}{\nu_{01} + \nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu_{01} - \nu_{02}}\right] \\ &+ \int_0^1 \left[A_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu)\, \frac{d\mu}{\mu + \nu_{01}} \\ &\pm \int_0^1 F_{1\pm}(-a,\mu,s) h_1'(\mu)\, \frac{\mu + \nu_{01}}{\mu^2 - \nu_{02}^2}\, d\mu \\ &\pm F_{1\pm}(-a,\nu_{01},s) h_1(\nu_{01})\, \frac{2\nu_{01}}{\nu_{01}^2 - \nu_{02}^2}. \end{aligned} \tag{H.5}$$

Following the same procedure with Eq. (3.16), we obtain after several partial fractionings and use of Eq. (H.5)

$$
\begin{aligned}
A_{2\pm}(\nu) = {} & \frac{c_1\sigma_1}{c_2\sigma_2} F_{1\pm}(-a,\nu,s)e^{(\sigma_1-\sigma_2)a/\nu} \\
& \pm \frac{k_s}{2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{\nu_{02}^2 - \nu^2}{\nu_{01}^2 - \nu^2} \frac{h_2(\nu)}{g_2(\nu)} \\
& \times \left\{ \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\left[\frac{h_2(\nu_{02})}{\nu + \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\nu - \nu_{02}}\right] \right. \\
& \quad + \frac{1}{2} F_{2\pm}(a,\nu_{02},s) \left[\frac{h_2(\nu_{02})}{\nu + \nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu - \nu_{02}}\right] \\
& \quad + \int_0^1 \left[A_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu) \frac{d\mu}{\mu + \nu} \\
& \quad \left. \mp \int_0^1 F_{1\pm}(-a,\mu,s)h_1(\mu) \frac{\nu_{01}^2 - \mu^2}{\nu_{02}^2 - \mu^2} \frac{2\varphi_{2s\nu}(\mu)}{c_2\sigma_2\nu} d\nu \right\} .
\end{aligned}
\tag{H.6}
$$

From Eq. (3.19), we get

$$
\begin{aligned}
\mp h_1(-\nu_{01})a_{1\pm} = {} & \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\left[h_2(\nu_{02}) \pm h_2(-\nu_{02})\right] \\
& + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[h_2(\nu_{02}) \mp h_2(-\nu_{02})\right] \\
& + (\nu_{02}^2 - \nu_{01}^2) \int_0^1 \left[A_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu) \frac{d\mu}{\mu^2 - \nu_{01}^2} \\
& \mp (\nu_{02}^2 - \nu_{01}^2) \int_0^1 F_{1\pm}(-a,\mu,s)h_1(\mu) \frac{d\mu}{\mu^2 - \nu_{02}^2} \\
& + F_{1\pm}(-a,\nu_{01},s)\left[h_1(-\nu_{01}) \pm h_1(\nu_{01})\right] .
\end{aligned}
\tag{H.7}
$$

Finally, we have from Eq. (3.18) that

$$A_{1\pm}(-\nu) \mp \frac{c_2\sigma_2}{c_1\sigma_1} A_{2\pm}(\nu)e^{(\sigma_1-\sigma_2)a/\nu} = \pm\left\{\frac{k_s}{2}\,\frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)}\,\frac{h_1(\nu)}{g_1(\nu)}\right\}$$

$$\times \Bigg\{\left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]\left[\frac{h_2(\nu_{02})}{\nu - \nu_{02}} \pm \frac{h_2(-\nu_{02})}{\nu + \nu_{02}}\right]$$

$$+ \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\left[\frac{h_2(\nu_{02})}{\nu - \nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu + \nu_{02}}\right]$$

$$+ \int_0^1 \left[A_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu)\,\frac{2\varphi_{1s\nu}(\mu)}{c_1\sigma_1\nu}\,d\mu$$

$$\mp \int_0^1 F_{1\pm}(-a,\mu,s)h_1(\mu)\,\frac{\nu_{01}^2 - \mu^2}{\nu_{02}^2 - \mu^2}\,\frac{d\mu}{\mu + \nu}\Bigg\}$$

$$\mp \left[F_{1\pm}(-a,\nu,s) - \frac{c_2\sigma_2}{c_1\sigma_1} F_{2\pm}(a,\nu,s)e^{(\sigma_1-\sigma_2)a/\nu}\right]. \qquad \text{(H.8)}$$

We now follow the procedure of Bowden and Williams (ref. 4) and write the expansion coefficients $A_{j\pm}(\mu)$ and $a_{1\pm}$ in the form

$$A_{j\pm}(\mu) = \left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]A'_{j\pm}(\mu) + B_{j\pm}(\mu)$$

and

$$a_{1\pm} = \left[a_{2\pm} + \frac{1}{2}F_{2\pm}(a,\nu_{02},s)\right]a'_{1\pm} + b_{1\pm}(\mu). \qquad \text{(H.9)}$$

When Eqs. (H.9) are used in Eqs. (H.6)-(H.8) it follows that

$$A'_{j\pm}(\mu) = \bar{B}_{j\pm}(\mu) \quad \text{and} \quad a'_{1\pm} = \bar{b}_{1\pm}, \qquad \text{(H.10)}$$

where $\bar{B}_{j\pm}(\mu)$ and $\bar{b}_{1\pm}$ are the expansion coefficients of the associated eigenvalue problem given by Eqs. (4.2)-(4.4). The coefficients $B_{j\pm}(\mu)$ and $b_{1\pm}$ are found to be given by Eqs. (4.10)-(4.12).

The coefficient $a_{2\pm}$ is obtained from Eq. (H.5) as

$$\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]\left[\frac{h_2(\nu_{02})}{\nu_{01}+\nu_{02}} \pm \frac{h_2(-\nu_{02})}{\nu_{01}-\nu_{02}} + \int_0^1 \bar{B}_{2\pm}(\mu)h_2(\mu)\frac{d\mu}{\mu+\nu_{01}}\right]$$

$$= -\frac{1}{2} F_{2\pm}(a,\nu_{02},s)\left[\frac{h_2(\nu_{02})}{\nu_{01}+\nu_{02}} \mp \frac{h_2(-\nu_{02})}{\nu_{01}-\nu_{02}}\right]$$

$$-\int_0^1 \left[B_{2\pm}(\mu) + F_{2\pm}(a,\mu,s)\right] h_2(\mu)\frac{d\mu}{\mu+\nu_{01}}$$

$$\mp \int_0^1 F_{1\pm}(-a,\mu,s)h_1(\mu)\frac{\mu+\nu_{01}}{\mu^2-\nu_{02}^2}\,d\mu$$

$$\mp F_{1\pm}(-a,\nu_{01},s)h_1(\nu_{01})\frac{2\nu_{01}}{\nu_{01}^2-\nu_{02}^2}. \tag{H.11}$$

It can be seen from Eq. (4.6) that the coefficient of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right]$ in Eq. (H.11) is the eigenvalue condition, and it will be zero at the places where the associated eigenvalue problem has nontrivial solutions. Equation (H.11) appears in the text as Eq. (4.13). The solutions $\psi_{j\pm}(x,\mu,s)$ can now be written as Eqs. (4.16) and (4.17) of the text.

I. Behavior of $\psi_{s\pm}(x,\mu)$ on Inversion Contours

In this appendix, several points concerning the behavior of $\psi_{s\pm}(x,\mu)$ on the integration contour of the inverse Laplace transformation and some portions of related deformed contours are discussed. First, we look at the behavior of $\psi_{s\pm}(x,\mu)$ as $|s| \to \infty$ with $\mathrm{Re}(s) = \gamma$, a large finite positive number. It will be seen that $\psi_{s\pm}(x,\mu)$ is not necessarily $O\left(\frac{1}{s}\right)$. Such parts of $\psi_{s\pm}(x,\mu)$ are inverted separately and the resulting solutions are shown to satisfy the uncollided transport equation. Then we consider how $\psi_{s\pm}(x,\mu)$ minus the uncollided term, $\psi_{u\pm}(x,\mu,s)$, can be deformed around the poles and branch cut of $\psi_{s\pm}$ which were discussed in section IV.

We are interested in the behavior of $\psi_{s\pm}$ on the contour $\mathrm{Re}(s) = \gamma$ as $|s| \to \infty$, where γ is finite. For such cases, $s \in S_{1e} \cap S_{2e}$ and the solutions $\psi_{j\pm}$ can be seen from Eqs. (3.3), (3.4), (3.8) and (3.9) to be

$$\psi_{2\pm}(x,\mu,s) = \int_0^1 \left[A_{2\pm}(\nu) + F_{2\pm}(x,\nu,s)\right] e^{-(s+\sigma_2)x/\nu} \varphi_{2s\nu}(\mu)\, d\nu$$
$$\pm \int_0^1 \left[A_{2\pm}(\nu) + F_{2\pm}(-x,\nu,s)\right] e^{(s+\sigma_2)x/\nu} \varphi_{2s\nu}(-\mu)\, d\nu \tag{I.1}$$

and, for $x > a$,

$$\psi_{1\pm}(x,\mu,s) = \pm \int_0^1 \left[A_{1\pm}(-\nu) - \tilde{F}_{\pm}(-a,\nu,s) + F_{1\pm}(-x,-\nu,s)\right]$$
$$\times\, e^{-(s+\sigma_1)x/\nu} \varphi_{1s\nu}(\mu)\, d\nu$$
$$\pm \int_0^1 F_{1\pm}(-x,\nu,s) e^{(s+\sigma_1)x/\nu} \varphi_{1s\nu}(-\mu)\, d\nu, \tag{I.2}$$

with an equation similar to (I.2) for $x < -a$. We see then that we need the coefficients $A_{1\pm}(-\nu)$, $A_{2\pm}(\nu)$ and the $F_{j\pm}$ functions. The expansion coefficients are given implicitly in terms of the $F_{j\pm}$ as

$$A_{2\pm}(\mu) = \frac{I_{2\pm}(\mu,s)}{\Omega_{2s}^{+}(\mu)\Omega_{2s}^{-}(\mu)} e^{-(s+\sigma_2)a/\mu}$$

$$\pm \frac{k_s}{c_2\sigma_2} \frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)} \frac{X_0(-\mu,s)e^{-(s+\sigma_2)a/\mu}}{\Omega_{2s}^{+}(\mu)\Omega_{2s}^{-}(\mu)} \int_0^1 A_{2\pm}(\nu)X_0(-\nu,s)e^{-(s+\sigma_2)a/\nu} \varphi_{2s\nu}(-\mu)d\nu \qquad \text{(I.3)}$$

and

$$A_{1\pm}(-\mu) = \frac{I_{1\pm}(\mu,s)e^{(s+\sigma_1)a/\mu}}{\Omega_{1s}^{+}(\mu)\Omega_{1s}^{-}(\mu)} \pm \frac{c_1\sigma_1}{c_2\sigma_2} \frac{\Omega_{2s}^{+}(\mu)\Omega_{2s}^{-}(\mu)}{\Omega_{1s}^{+}(\mu)\Omega_{1s}^{-}(\mu)} e^{-(\sigma_2-\sigma_1)a/\mu} A_{2\pm}(\mu)$$

$$\mp \frac{k_s e^{(s+\sigma_1)a/\mu}}{X_0(-\mu,s)\Omega_{1s}^{+}(\mu)\Omega_{1s}^{-}(\mu)} \int_0^1 \frac{A_{2\pm}(\nu)X_0(-\nu,s)e^{-(s+\sigma_2)a/\nu}}{c_2\sigma_2} \varphi_{2s\nu}(\mu)d\nu, \qquad \text{(I.4)}$$

where the $I_{j\pm}(\mu,s)$ are given by

$$I_{2\pm}(u,s) = \frac{c_2\sigma_2}{c_1\sigma_1}\,\Omega^+_{1s}(\mu)\Omega^-_{1s}(\mu)F_{1\pm}(-a,\mu,s)e^{(s+\sigma_1)a/\mu}$$

$$\pm \frac{k_s}{c_2\sigma_2}\,\frac{\Omega_{2s}(\infty)}{\Omega_{1s}(\infty)}\,X_0(-\mu,s)\int_0^1 F_{2\pm}(a,\nu,s)e^{-(s+\sigma_2)a/\nu}\,X_0(-\nu,s)\varphi_{2s\nu}(-\mu)d\nu$$

$$+ \frac{k_s}{c_1\sigma_1}\,X_0(-\mu,s)\int_0^1 \frac{F_{1\pm}(-a,\nu,s)e^{(s+\sigma_1)a/\nu}}{X_0(-\nu,s)}\,\varphi_{1s\nu}(\mu)d\nu \qquad \text{(I.5)}$$

and

$$I_{1\pm}(\mu,s) = \mp\Omega^+_{1s}(\mu)\Omega^-_{1s}(\mu)F_{1\pm}(-a,\mu,s)e^{-(s+\sigma_1)a/\mu}$$

$$\pm \frac{c_1\sigma_1}{c_2\sigma_2}\,\Omega^+_{2s}(\mu)\Omega^-_{2s}(\mu)F_{2\pm}(a,\mu,s)e^{-(s+\sigma_2)a/\mu}$$

$$\mp \frac{k_s}{c_2\sigma_2}\,\frac{1}{X_0(-\mu,s)}\int_0^1 F_{2\pm}(a,\nu,s)e^{-(s+\sigma_2)a/\nu}\,X_0(-\nu,s)\varphi_{2s\nu}(\mu)d\nu$$

$$- \frac{k_s}{c_1\sigma_1}\,\frac{\Omega_{1s}(\infty)}{\Omega_{2s}(\infty)}\,\frac{1}{X_0(-\mu,s)}\int_0^1 \frac{F_{1\pm}(-a,\nu,s)e^{(s+\sigma_1)a/\nu}}{X_0(-\nu,s)}\,\varphi_{1s\nu}(-\mu)d\nu. \qquad \text{(I.6)}$$

The behavior of various functions which appear in Eqs. (I.1)-(I.6) as $|s| \to \infty$, $\mathrm{Re}(s) = \gamma$ is

$$\Omega_{js}(z),\quad \lambda_{js}(\nu),\quad k_s \to O(s),$$

3

$$X_0(z,s) \to O(1),$$

and, for $0 \leq \mu, \nu \leq 1$

$$\varphi_{jsv}(\mu) \to O(s)$$

and

$$\varphi_{jsv}(-\mu) \to O(1). \qquad (I.7)$$

The $F_{j\pm}$ functions appear with an exponential factor and its behavior in the same limit is

$$F_{j\pm}(x,\nu,s)e^{-(s+\sigma_j)x/\nu} \to \frac{1}{\nu(s+\sigma_j)}\int_{l(j)}^{x} e^{-(s+\sigma_j)(x-x_0)/\nu}\left[f_{j\pm}(x_0,\nu) + O\left(\frac{1}{s}\right)\right]dx_0$$
$$\to O\left(\frac{1}{s}\right). \qquad (I.8)$$

On using Eqs. (I.7) and (I.8) in Eqs. (I.5) and (I.6) we find that

$$\frac{I_{2\pm}(\mu,s)}{\Omega_{2s}^{+}(\mu)\Omega_{2s}^{-}(\mu)} \to F_{1\pm}(-a,\mu,s)e^{(s+\sigma_1)a/\mu} \to O\left(\frac{1}{s}\right) \qquad (I.9)$$

and

$$\frac{I_{1\pm}(\mu,s)}{\Omega_{1s}^{+}(\mu)\Omega_{1s}^{-}(\mu)} \to \mp F_{1\pm}(-a,\mu,s)e^{-(s+\sigma_1)a/\mu} \pm F_{2\pm}(a,\mu,s)e^{-(s+\sigma_2)a/\mu}$$
$$\to O\left(\frac{1}{s}\right). \qquad (I.10)$$

The coefficient $A_{2\pm}(\mu)$ is obtained from the integral equation (I.3). Since the kernel of this equation is also $O\left(\frac{1}{s}\right)$, the first term of the Neumann series solution will give the behavior of $A_{2\pm}(\mu)$ as $|s| \to \infty$. It follows then from Eqs. (I.3) and (I.9) that

$$A_{2\pm}(\mu) \to F_{1\pm}(-a,\mu,s)e^{-(\sigma_2-\sigma_1)a/\mu} \to O\left(\frac{1}{s}\right). \tag{I.11}$$

Using Eqs. (I.10) and (I.11) in Eq. (I.4), we obtain

$$\begin{aligned}\left[A_{1\pm}(-\mu) \pm F_{1\pm}(-a,\mu,s)\right] &\to \pm F_{2\pm}(a,\mu,s)e^{-(\sigma_2-\sigma_1)a/\mu} \\ &\quad \pm F_{1\pm}(-a,\mu,s)e^{-2(\sigma_2-\sigma_1)a/\mu} \\ &\to O\left(\frac{1}{s}\right).\end{aligned} \tag{I.12}$$

These last two results are used in Eqs. (I.1) and (I.2) to get

$$\psi_{2\pm}(x,\mu,s) \to \begin{cases} (s+\sigma_2)e^{-(s+\sigma_2)x/\mu}\left[F_{2\pm}(x,\mu,s) + F_{1\pm}(-a,\mu,s)e^{-(\sigma_2-\sigma_1)a/\mu}\right], & \mu > 0 \\ \pm(s+\sigma_2)e^{-(s+\sigma_2)x/\mu}\left[F_{2\pm}(-x,-\mu,s) + F_{1\pm}(-a,-\mu,s)e^{(\sigma_2-\sigma_1)a/\mu}\right], & \mu < 0 \end{cases} \tag{I.13}$$

and

$$\psi_{1\pm}(x,\mu,s) \to \begin{cases} \pm(s+\sigma_1)e^{-(s+\sigma_1)x/\mu}\Big[F_{1\pm}(-x,-\mu,s) - F_{1\pm}(-a,-\mu,s) \\ \qquad \pm F_{2\pm}(a,\mu,s)e^{-(\sigma_2-\sigma_1)a/\mu} \\ \qquad \pm F_{1\pm}(-a,\mu,s)e^{-2(\sigma_2-\sigma_1)a/\mu}\Big], & \mu > 0 \\ \pm(s+\sigma_1)e^{-(s+\sigma_1)x/\mu}F_{1\pm}(-x,-\mu,s), & \mu < 0, \end{cases} \tag{I.14}$$

when $x > a$. For $x < -a$, $\psi_{1\pm}$ has a similar form. Upon using Eq. (I.8) for the $F_{j\pm}$ functions, we find that Eqs. (I.13) and (I.14) can be written as Eqs. (5.1)-(5.4) of the text, where we have used the symmetry properties of $f_{j\pm}(x,\mu)$. It can be seen from Eqs. (5.1)-(5.4) that $\psi_{j\pm}(x,\mu,s)$ is not necessarily $O\left(\frac{1}{s}\right)$. In fact if $f_{j\pm}(x_0,\mu)$ contains $\delta(x - x_0)$ then $\psi_{s\pm}(x,\mu)$ is $O(1)$ as $|s| \to \infty$, $\mathrm{Re}(s) = \gamma$. The parts of $\psi_{s\pm}(x,\mu)$ which are not $O\left(\frac{1}{s}\right)$ can be inverted by inspection after a change of variables is made.

We define $\psi_{us\pm}(x,\mu)$ for $x > a$, $\mu > 0$ and <u>all s</u> as

$$\begin{aligned}\psi_{us\pm}(x,\mu) \equiv \frac{1}{\mu}\int_a^x e^{-(s+\sigma_1)(x-x_0)/\mu}\, f_{1\pm}(x_0,\mu)dx_0 \\ + \frac{e^{-(\sigma_2-\sigma_1)(a-x)/\mu}}{\mu}\int_{-a}^{a} e^{-(s+\sigma_2)(x-x_0)/\mu}\, f_{2\pm}(x_0,\mu)dx_0 \\ + \frac{e^{-(\sigma_2-\sigma_1)2a/\mu}}{\mu}\int_{-\infty}^{-a} e^{-(s+\sigma_1)(x-x_0)/\mu}\, f_{1\pm}(x_0,\mu)dx_0,\end{aligned} \tag{I.15}$$

which gives the portions of Eq. (5.3) which are not $O\left(\frac{1}{s}\right)$ for $|s| \to \infty$, $\mathrm{Re}(s) = \gamma$. Now we make the change of variables

$$x - x_0 = \mu t, \tag{I.16}$$

where $t \geq 0$ since $x \geq x_0$ and $\mu > 0$. Equation (I.15) then becomes

$$\psi_{us\pm}(x,\mu) = \int_0^{(x-a)/\mu} e^{-(s+\sigma_1)t} f_{1\pm}(x - \mu t,\mu)dt$$

$$+ \int_{(x-a)/\mu}^{(x+a)/\mu} e^{-(s+\sigma_2)t} e^{-(\sigma_2-\sigma_1)(a-x)/\mu} f_{2\pm}(x - \mu t,\mu)dt$$

$$+ \int_{(x+a)/\mu}^{\infty} e^{-(s+\sigma_1)t} e^{-(\sigma_2-\sigma_1)2a/\mu} f_{1\pm}(x - \mu t,\mu)dt, \tag{I.17}$$

which is easily seen to be Eq. (5.6) of the text with $\psi_{u\pm}(x,\mu,t)$ given by Eq. (5.9). For $\mu < 0$, we use

$$x_0 - x = |\mu|t. \tag{I.18}$$

It is seen then that all of the results given as Eqs. (5.6)-(5.10) follow.

Another point to be discussed in this appendix is the contribution from the contour C_ρ (see Eq. (5.16)) around the right-hand end of the branch cut of $\nu_{01}(s)$ as the radius ρ goes to zero. This branch point is located at $s = -\sigma_1(1 - c_1)$ so we define

$$s + \sigma_1(1 - c_1) \equiv \rho e^{i\varphi}. \tag{I.19}$$

Here $\nu_{01}(s) \to \infty$ as $\rho \to 0$ as

$$\nu_{01}^2 \xrightarrow[\rho\to 0]{} \frac{c_1\sigma_1}{3} \frac{1}{\rho e^{i\varphi}}, \quad -\pi < \varphi < \pi. \tag{I.20}$$

The branch cut has already been picked so that $\nu_{01}(s)$ is real when s is real and greater than $-\sigma_1(1 - c_1)$. The integral

$$\frac{1}{2\pi i}\int_{C_\rho} \psi_{s\pm}(x,\mu)e^{st}\, ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho\left[\psi_{s\pm}(x,\mu)e^{st}\right]e^{i\varphi}\, d\varphi, \tag{I.21}$$

with s given by Eq. (I.19), is zero in the limit $\rho \to 0$ if

$$\lim_{\rho\to 0}\left[\rho\psi_{s\pm}(x,\mu)\right] = 0, \tag{I.22}$$

independent of φ. As pointed out in section V, the point $s = -\sigma_1(1 - c_1)$ may happen to satisfy the eigenvalue condition, Eq. (4.6). We assume for the moment that it does not and show later what changes are required if it does. The function $\Omega_{1s}(\infty) \to 0$ as $\rho \to 0$ as

$$\Omega_{1s}(\infty) \xrightarrow[\rho\to 0]{} \rho e^{i\varphi}, \tag{I.23}$$

so that

$$\nu_{01}{}^2\Omega_{1s}(\infty)\Big|_{s = -\sigma_1(1 - c_1)} = \frac{c_1\sigma_1}{3}. \tag{I.24}$$

At this branch point $s \in S_{1i} \cap S_{2i}$ so we need to show the behavior of all functions given in section IV as $\rho \to 0$. This behavior can be given in terms of the behavior of ν_{01} and $\Omega_{1s}(\infty)$. In the relationships which follow, quantities which are functions of s will be given as $O(\nu_{01})$, $O(1/\nu_{01})$, $O(\Omega_{1s})$, $O(1)$, etc. as $s \to -\sigma_1(1 - c_1)$. For example,

$$\Omega_{1s}(\infty)\nu_{01}{}^2 \to \text{finite} \to O(1), \tag{2.22}$$

where we have given the equation number as that from which the relationship can be seen.

$$x_{js}(-\mu) \to 0(1),$$
$$x_{js}(\pm\nu_{02}) \to 0(1),$$

and

$$x_{js}(\pm\nu_{01}) \to 0(1/\nu_{01}). \qquad \text{(A.8a)}$$

$$k_s \to 0(1). \qquad \text{(C.7)}$$

$$h_2(\omega) \to 0(1), \quad \omega = \pm\nu_{02}, \quad \mu(0 \leq \mu \leq 1),$$
$$g_j(\mu) \to 0(1),$$
$$h_1(\mu) \to 0(\Omega_{1s})$$

and

$$h_1(\pm\nu_{01}) \to 0(1/\nu_{01}). \qquad (4.5)$$

$$\overline{B}_{2\pm}(\mu) \to 0(1). \qquad (4.2)$$

$$\overline{b}_{1\pm} \to 0(\nu_{01}). \qquad (4.4)$$

$$\overline{B}_{1\pm}(-\mu) \to 0(1). \qquad (4.3)$$

$$\overline{\Psi}_{2\pm}(x,\mu,s) \to 0(1)$$

and

$$\overline{\Psi}_{1\pm}(x,\mu,s) \to 0(\nu_{01}). \qquad (4.1)$$

$$F_{j\pm}(x,\mu,s) \to 0(1),$$
$$F_{2\pm}(x,\pm\nu_{02},s) \to 0(1)$$

and

$$F_{1\pm}(x,\pm\nu_{01},s) \to 0(1/\nu_{01}). \qquad (3.10)$$

$$B_{2\pm}(\nu) \to O(1). \tag{4.10}$$

$$B_{1\pm}(-\nu) \to O(1). \tag{4.11}$$

$$\alpha_{1\pm} \to O(1)$$

and

$$\alpha_{2\pm} \to O(1). \tag{4.14}$$

$$\beta_{1\pm} \to O(1)$$

and

$$\beta_{2\pm} \to O(1). \tag{4.15}$$

$$\tilde{F}_{\pm}(-a,\nu_{01},s) \to O(1/\nu_{01})$$

and

$$\tilde{F}_{\pm}(-a,\nu,s) \to O(1). \tag{3.10}$$

$$\left[b_{1\pm} - \tilde{F}_{\pm}(-a,\nu_{01},s)\right] \to O(\nu_{01}). \tag{4.12}$$

$$\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a,\nu_{02},s)\right] \to O(1). \tag{4.13}$$

$$\psi_{2\pm}(x,\mu,s) \to O(1). \tag{4.16}$$

$$\psi_{1\pm}(x,\mu,s) \to O(\nu_{01}). \tag{4.17}$$

We have from these last two relationships and Eq. (I.24) that

$$\rho\psi_{s\pm}(x,\mu) \to \sqrt{\rho}\ O(1), \tag{I.25}$$

so that Eq. (I.22) is satisfied. Therefore, there is no contribution from the integral (I.21) for the case when $s = -\sigma_1(1 - c_1)$ does not satisfy the eigenvalue condition, Eq. (4.6).

If the point $s = -\sigma_1(1 - c_1)$ happens to satisfy the eigenvalue condition, then the denominator of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right]$, which is equivalent to the eigenvalue condition Eq. (4.6), vanishes. It can be seen from Eq. (4.13) that the limiting form of this condition at the branch point is $\alpha_{1\pm} = 0$ and we shall say something about it in the last two appendices. If we consider, for such cases, the function

$$\psi_{s\pm}(x,\mu) - \left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right] \bar{\psi}_{s\pm}(x,\mu), \qquad \text{(I.26)}$$

instead of $\psi_{s\pm}(x,\mu)$ as the integrand of the integral (I.21), then it follows that in the limit $\rho \to 0$, the contribution from such an integral vanishes. The part which has been subtracted from $\psi_{s\pm}(x,\mu)$ in (I.26) is considered separately and would appear to have a pole, due to the zero in the denominator of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right]$. Its contribution therefore does not vanish in the limit $\rho \to 0$; in fact, its contribution looks like a discrete residue term. However, the point is not isolated (remember that we are considering the branch point of ν_{01} at $s = -\sigma_1(1 - c_1)$) so we shall understand that its contribution is included in the branch-cut integral term of Eq. (7.1). We shall see from the numerical results that such points occur.

J. Equations and Procedures for Computation of Eigenvalues

The equations from which the time eigenvalues, s_n are determined for $s \in S_{1\ell} \cap S_{2\ell}$ are Eqs. (4.2) and (4.6) of section IV. When $s \in S_{1e} \cap S_{2\ell}$, the corresponding equations are Eqs. (4.7) and (4.8) and they determine what we have called the pseudo-eigenvalues. All of these equations are solved numerically using the procedure of references 1 and 4. As pointed out in section VI, the above equations can be written in terms of the nondimensional quantities introduced in Eqs. (6.1). By making the substitution

$$B_\pm(\mu) = \frac{\bar{B}_{2\pm}(\mu) h_2(\mu)}{(\nu_{01} + \mu)\nu_{02}} \sqrt{\frac{\Omega_{1\zeta}(\nu_{02})}{\Omega_{1\zeta}(\infty)}}\, (i)^{-\left(\frac{1\pm 1}{2}\right)}, \tag{J.1}$$

it follows that Eq. (4.2) can be written for ζ real and $\max(-\sigma_D + \sigma_R, 0) < \zeta < 1$ (that is, on that part of the branch cut of ν_{02} which is not also part of the branch cut of ν_{01}) as

$$B_\pm(\mu) = -g(\mu)\left[\frac{\mu g'_\pm \pm |\nu_{02}| g_\mp}{\mu^2 + |\nu_{02}|^2} \pm \frac{1}{2}\int_0^1 B_\pm(\nu)\, \frac{d\nu}{\nu + \mu}\right], \tag{J.2}$$

where

$$g(\mu) = \frac{\Omega_{1\zeta}(\nu_{02})}{\Omega_{1\zeta}(\infty)} \left[\frac{X_{2\zeta}(-\mu)}{X_{1\zeta}(-\mu)}\right]^2 \frac{\mu(1 - \zeta)e^{-2\zeta A/\mu}}{\left(\dfrac{\Omega^+_{2\zeta}(\mu)\, \Omega^-_{2\zeta}(\mu)}{c_2^2 \sigma_2^2}\right)} \frac{\mu^2 + |\nu_{02}|^2}{(\mu + \nu_{01})^2}$$

and

$$g_\pm = -\sqrt{\frac{\Omega_{1\zeta}(\nu_{02})}{\Omega_{1\zeta}(\infty)}}\; \begin{matrix}\mathrm{Im}\\ \mathrm{Re}\end{matrix} \left\{\frac{X_{2\zeta}(\nu_{02})}{X_{1\zeta}(\nu_{02})}\, \frac{e^{\zeta A/\nu_{02}}}{\nu_{01} - \nu_{02}}\right\}. \tag{J.3}$$

Now we define $\Delta_{\zeta\pm}$ as

$$\Delta_{\zeta\pm} \equiv -2g_\pm \mp \int_0^1 B_\pm(\nu)d\nu, \tag{J.4}$$

which is the eigenvalue condition, Eq. (4.6), if $\Delta_{\zeta\pm} = 0$.

Equations (J.2) are reduced to two sets ($\pm$) of N equations in the N unknowns $B_+(\mu_i)$ and $B_-(\mu_i)$, $i = 1,\ldots,N$ (see for example, ref. 24), given by

$$B_\pm(\mu_i) = -g(\mu_i)\left[\frac{\mu_i g_\pm \pm |\nu_{02}|g_\mp}{\mu_i^2 + |\nu_{02}|^2} \pm \frac{1}{2}\sum_{j=1}^{N} R_j \frac{B_\pm(\mu_j)}{\mu_j + \mu_i}\right], \tag{J.5}$$

where R_j is the weighting function for the numerical integration scheme which is used. Equation (J.4) is written as

$$\Delta_{\zeta\pm} = -2g_\pm \mp \sum_{j=1}^{N} R_j\, B_\pm(\mu_j). \tag{J.6}$$

Since we must search for values of ζ for which $\Delta_{\zeta\pm} = 0$, it is seen that we must be able to compute all quantities which appear in Eqs. (J.3) for any value of ζ in the range (6.3). These quantities are computed as follows.

$$\frac{\Omega_{1\zeta}(\nu_{02})}{\Omega_{1\zeta}(\infty)} = \frac{\zeta + \sigma_D - \zeta\sigma_R}{\zeta + \sigma_D - \sigma_R}. \tag{J.7}$$

$$\frac{\Omega^+_{2\zeta}(\mu)\Omega^-_{2\zeta}(\mu)}{c_2^2\sigma_2^2} = \left[\zeta - \frac{\mu}{2}\ln\left(\frac{1+\mu}{1-\mu}\right)\right]^2 + \left(\frac{\pi\mu}{2}\right)^2. \tag{J.8}$$

The functions ν_{0j} are determined by $\Omega_{j\zeta}(\nu_{0j}) = 0$ and they are computed numerically using the Newton-Raphson iteration (ref. 24) on the nonlinear equations

$$|\nu_{02}| \tan^{-1} \frac{1}{|\nu_{02}|} = \zeta \tag{J.9}$$

and

$$\nu_{01} \tanh^{-1} \frac{1}{\nu_{01}} = \frac{\zeta + \sigma_D}{\sigma_R} . \tag{J.10}$$

The X-functions are computed from the first relationship in Eqs. (A.8a); namely,

$$X_{j\zeta}(z) = \frac{1}{1 - z} \exp \left\{ \frac{1}{2\pi i} \int_0^1 \ln \left[\frac{\Omega^+_{j\zeta}(\nu)}{\Omega^-_{j\zeta}(\nu)} \right] \frac{d\nu}{\nu - z} \right\} . \tag{J.11}$$

For ζ and z real where $z = -\mu$, $0 \leq \mu \leq 1$, we have from Eq. (J.11) that

$$\frac{X_{2\zeta}(-\mu)}{X_{1\zeta}(-\mu)} = \exp \left\{ \frac{1}{\pi} \int_0^1 \left[\theta_2(\zeta,\nu) - \theta_1(\zeta,\sigma_R,\sigma_D,\nu) \right] \frac{d\nu}{\nu + \mu} \right\} , \tag{J.12}$$

where

$$\theta_2(\zeta,\nu) = \tan^{-1} \left[\frac{\pi\nu/2}{\zeta - \nu \tanh^{-1} \nu} \right]$$

and

$$\theta_1(\zeta,\sigma_R,\sigma_D,\nu) = \tan^{-1} \left[\frac{\sigma_R \pi\nu/2}{\zeta + \sigma_D - \sigma_R \nu \tanh^{-1} \nu} \right] . \tag{J.13}$$

For ζ real and $z = \nu_{02}$, we calculate the real and imaginary parts of $\dfrac{X_{2\zeta}(\nu_{02})}{X_{1\zeta}(\nu_{02})}$ as

$$\mathrm{Re}\left[\frac{X_{2\zeta}(\nu_{02})}{X_{1\zeta}(\nu_{02})}\right] = e^{\Gamma_1/\pi} \cos(\Gamma_2/\pi)$$

and

$$\mathrm{Im}\left[\frac{X_{2\zeta}(\nu_{02})}{X_{1\zeta}(\nu_{02})}\right] = e^{\Gamma_1/\pi} \sin(\Gamma_2/\pi), \qquad (J.14)$$

where

$$\Gamma_1 = \int_0^1 \left[\theta_2(\zeta,\nu) - \theta_1(\zeta,\sigma_R,\sigma_D,\nu)\right] \frac{\nu d\nu}{\nu^2 + |\nu_{02}|^2}$$

and

$$\Gamma_2 = |\nu_{02}| \int_0^1 \left[\theta_2(\zeta,\nu) - \theta_1(\zeta,\sigma_R,\sigma_D,\nu)\right] \frac{d\nu}{\nu^2 + |\nu_{02}|^2}. \qquad (J.15)$$

Integrals in Eqs. (J.12) and (J.15) are computed as

$$\int_0^1 \left[\theta_2(\zeta,\nu) - \theta_1(\zeta,\sigma_R,\sigma_D,\nu)\right] f(\nu)d\nu = \sum_{i=1}^{M} R_i \left[\theta_2(\zeta,\nu_i) - \theta_1(\zeta,\sigma_R,\sigma_D,\nu_i)\right] f(\nu_i), \qquad (J.16)$$

where R_i is again the weighting function for the numerical integration scheme.

In all numerical integrations, we used Gauss' Method (ref. 24). For integrations in Eqs. (J.5) and (J.6), the interval (0,1) was split into four intervals,

$$(0,1) = (0,0.05) + (0.05,0.1) + (0.1,0.9) + (0.9,1.0), \qquad (J.17)$$

and we used a 10-point Gaussian formula in each subinterval. For integrations in Eqs. (J.12) and (J.15) the interval (0.1) was divided as

$$(0,1) = (0,0.1) + (0.1,0.9) + (0.9,0.99) + (0.99 + 0.999) + (0.999,1.0), \qquad (J.18)$$

and in each of these subintervals we also used a 10-point Gaussian formula. The subdivision (J.18) is the same as that used by Kowalska (ref. 11) and the X-functions calculated here agree with those she gives to all figures which she quotes except for the real and imaginary parts of $X_{j\zeta}(\nu_{02})$. She apparently used Γ_2 instead of Γ_2/π in Eqs. (J.14) to obtain the numerical values for the real and imaginary parts given in Part II of reference 11. Since her later published critical-slab results (ref. 12) agree with those of Mitsis (ref. 20) for a bare slab, we expect that this oversight was corrected.

Conditions (4.7) and (4.8) which determine the pseudo-eigenvalues for $s \in S_{1e} \cap S_{2i}$ lead to very similar equations which will not be written down. In this region, the real s-axis corresponds to $0 \leq \zeta \leq -\sigma_D$ and such equations need be considered only if $-\sigma_D > 0$.

The procedure we use to calculate the eigenvalues ζ_n is as follows. For fixed values of A, σ_R and σ_D, we select a number of ζ values in the interval given by (6.3). For each of these values we obtain $|\nu_{02}|$ and ν_{01} from Eqs. (J.9) and (J.10) by iteration (Newton-Raphson). Equations (J.13) are evaluated at each of the

50 Gaussian integration points, ν_i, $0 < \nu_i < 1$. Next, the $\frac{X_{2\zeta}(-\mu_j)}{X_{1\zeta}(-\mu_j)}$ are calculated for each of the 40 Gaussian integration points, μ_j, $0 < \mu_j < 1$ by using Eqs. (J.16) in Eq. (J.19). The real and imaginary parts of $\frac{X_{2\zeta}(\nu_{02})}{X_{1\zeta}(\nu_{02})}$ are computed in the same way from Eqs. (J.14) - (J.16). Now we can compute $g(\mu_j)$ from Eq. (J.3) at each of the 40 points μ_j and evaluate all of the coefficients in the two sets ($\pm$) of N equations in the N unknowns $B_+(\mu_j)$ and $B_-(\mu_j)$ (Eqs. (J.5)). These two sets of simultaneous equations are solved numerically for $B_\pm(\mu_j)$ which are then used to compute $\Delta_{\zeta\pm}$ from Eq. (J.6) at the selected values of ζ. In this way we locate the zeros of $\Delta_{\zeta\pm}$ approximately. A new set of ζ values, located about each approximate ζ_n, is selected and the process is repeated. For the present computations, the ζ_n were located to three figures. Discussion of computed results is given in section VI.

In Appendix G, the thick-slab eigenvalue condition was given as Eq. (G.4). We note that $g_\pm$ given by Eq. (J.3) are, within a factor, exactly the quantities needed in Eq. (G.4). Therefore the thick-slab approximation eigenvalues are obtained from

$$g_\pm = 0, \tag{J.19}$$

as would be expected from Eq. (J.4).

The bare slab eigenvalues are obtained when $\sigma_R = 0$ and it is easily shown that in this case Eqs. (J.5) and (J.6) no longer depend on σ_D. That is, for $\sigma_R = 0$ these equations do not contain σ_D.

We have noted in section V and Appendix I that the branch point of ν_{01} located at $s = -\sigma_1(1 - c_1)$ may happen to satisfy the eigenvalue condition, which can be seen from Eq. (4.13) to be

$$\alpha_{1\pm} = 0 \tag{J.20}$$

when $\nu_{01} \to \infty$. This point corresponds to $\zeta = -\sigma_D + \sigma_R$ and it can be shown that Eq. (J.20) then determines values of $\zeta = \zeta_n$ which do not depend on σ_D or σ_R. That is, if we use $\zeta = -\sigma_D + \sigma_R$ to eliminate σ_D from the condition (J.20), then σ_R drops out of the equations. Equation (J.20) determines the values of ζ at which eigenvalues disappear into the right end of the branch cut of ν_{01}. We also note that the limiting form of the pseudo-eigenvalue condition for $s = -\sigma_1$, which corresponds to $\zeta = -\sigma_D$, determines the values of ζ where the pseudo-eigenvalues disappear into the left end of the branch cut of ν_{01}. Such points, as well as those given by Eq. (J.20), are labeled with $*$ in Figures 6-9.

K. Remarks on Eigenvalue - Branch-Point Coincidence

In this Appendix we make a few remarks concerning the situation when the eigenvalues (or pseudo-eigenvalues) disappear into the branch cut of ν_{01}. This situation is somewhat analogous to that encountered by Hintz (ref. 10) for the slab surrounded by pure absorbers. He could not say whether a bare-slab eigenvalue (which does not depend on σ_D) that happened to coincide with $-\sigma_D$ belonged to the point spectrum or the continuous spectrum for his problem. In the present problem, the eigenvalues coincide with a branch point as they disappear into the branch cut of ν_{01}. We have not made a numerical study of the branch-cut integral in Eq. (7.1) nor have we looked at the eigenvalue condition on another Riemann sheet. We suspect that there is no drastic change in the shape of the solution given by Eq. (7.1) when an eigenvalue disappears into the branch cut of ν_{01} and such studies would resolve this point. We pointed out in Appendix J that the condition (J.20), which determines whether or not the point $s = -\sigma_1(1 - c_1)$, $(\zeta = -\sigma_D + \sigma_R)$, is a zero of the denominator of $\left[a_{2\pm} + \frac{1}{2} F_{2\pm}(a, \nu_{02}, s)\right]$ given by Eq. (4.13), does not depend on σ_D or σ_R explicitly. In Appendix I, we indicated that the contribution from such points should be included in the branch-cut integral since it arises from the integration around the branch point. We understand then that such a contribution is included in Eq. (7.1) if $s = -\sigma_1(1 - c_1)$ happens to satisfy Eq. (J.20). We do not know how such zeros of Eq. (J.20) behave or appear in the solution after passing through the branch point as the material properties are varied.

If one considered the problem of a finite slab with symmetric reflectors of finite thickness then he might be able to see what is happening at the places where the eigenvalues coincide with $\nu_{01} = \infty$. In such a problem, the solution probably does not inherit the branch cut of ν_{01}, but instead has discrete eigenvalues along it. Even though there is another parameter in the problem, the reflector thickness, one might be able to do a numerical study of all the eigenvalues.