

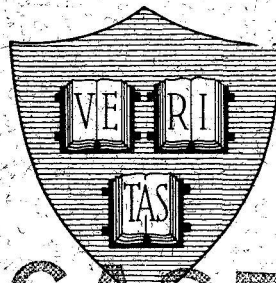
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**NECESSARY AND SUFFICIENT CONDITIONS  
FOR OPTIMALITY FOR SINGULAR CONTROL  
PROBLEMS: A LIMIT APPROACH**



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By

**David H. Jacobson and Jason L. Speyer**

**March 1970**

**Technical Report No. 604**

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Division of Engineering and Applied Physics

Harvard University · Cambridge, Massachusetts

NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY  
FOR SINGULAR CONTROL PROBLEMS: A LIMIT APPROACH

By

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ABSTRACT

Necessary and sufficient conditions for optimality for singular control problems are presented for the case where the extremal path is totally singular. The singular second variation is converted into a nonsingular one by addition of a quadratic functional of the control; a parameter  $\frac{1}{\epsilon}$  multiplies this added functional. By allowing  $\epsilon$  to approach infinity the optimality conditions are deduced for the singular problem from the limiting optimality conditions of the synthesized nonsingular second variation. The resulting conditions are Jacobson's sufficient conditions in slightly modified form. In a companion paper necessity of Jacobson's conditions for a class of singular problems is demonstrated by exploiting the Kelley transformation technique which converts the singular second variation into a nonsingular one in a reduced dimensional state space.

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## I. Preliminaries

### 1. Introduction

In [1] a new necessary condition of optimality for singular control problems is developed and is shown to be nonequivalent to the well known generalized Legendre-Clebsch (or Kelley) condition. In [2] sufficient conditions for non-negativity of the singular second variation are presented; in strengthened form these are sufficient for a weak minimum. The sufficiency conditions, which are in the form of linear algebraic equalities and inequalities, also yield insight into nonsingular control problems and into the behavior of the important matrix Riccati differential equation [3]. Relationship of the above new conditions of optimality to existing necessary conditions is discussed in [1], [2]; thus, we refer the reader to these papers for general references to research in singular control (variational) problems.

In this paper we show that, with slight modification, the sufficiency conditions presented in [2] are also necessary for non-negativity of the singular second variation. Moreover, we show that these conditions are indeed necessary and sufficient for a weak minimum for a class of singular control problems. In certain cases a strong minimum is implied. We prove these results by a limit argument. The singular second variation is converted into a nonsingular one by the addition of a term  $\frac{1}{2\epsilon} \int u^T u dt$ ,  $\epsilon > 0$ . By allowing  $\epsilon$  to approach infinity we deduce the optimality conditions for the singular problem from the limiting optimality conditions of the synthesized nonsingular second variation. This limit approach has been used previously [4] as a computational technique for solving singular optimal control problems.

In a companion paper [5] we prove necessity of the conditions given in [2] by exploiting the Kelley transformation technique [6] which converts the singular second variation into a nonsingular second variation in a state space of reduced dimension. In the reduced state space the equivalence is established of the sufficiency conditions [2] and the existence of the solution of the Riccati differential equation (Jacobi or Conjugate Point Condition) associated with this nonsingular second variation. This proves the necessity of the conditions given in [2]. Disadvantages of the transformation technique are in its algebraic complexity and in the need for the coefficients of the second variation (which depend upon time) to be many times differentiable. Moreover if the problem is singular of order  $p$  (i. e.  $\frac{d^{2p}}{dt^{2p}} H_u$  contains the control  $u$ , where  $H$  is the variational Hamiltonian) the transformation technique must be applied repeatedly,  $p$  times, before a nonsingular problem is obtained.<sup>†</sup> Nevertheless, the transformation technique does give another viewpoint and ties together the conditions given in [1], [2] and the pioneering work of Kelley et al. [6] and Robbins [7]. Independently Goh [8] and McDanell and Powers [9], using Goh's transformation technique [10], have arrived at similar results.

The limit approach presented in this paper has the following advantageous features:

i) A direct proof of necessary and sufficient conditions for optimality is obtained without the need to transform the problem to a reduced state space.

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<sup>†</sup>A nonsingular problem is unattainable if  $u$  does not appear in a time derivative of  $H_u$ .

ii) The concept of "order of the singular arc p" is not required in the proof.

iii) Differentiability requirements are not as severe as those demanded in [5].

## 2. Problem Formulation

We shall consider a class of control problems where the dynamic system is described by the ordinary differential equations

$$\dot{x} = f(x, u, t) \quad ; \quad x(t_o) = x_o \text{ (given)} \quad (1)$$

where

$$f(x, u, t) \triangleq f_1(x, t) + f_u(x, t)u \quad (2)$$

The performance of the system is measured by the cost functional

$$V[u(\cdot)] = \int_{t_o}^{t_f} \{L_1(x, t) + u^T L_2(x, t)\} dt + F(x(t_f)) \quad (3)$$

and the terminal states must satisfy

$$\psi(x(t_f)) = 0 \quad (4)$$

The control function  $u(\cdot)$  is required to satisfy the constraint

$$u(\cdot) \in U \quad (5)$$

where the set  $U$  is defined by

$$U \triangleq U_1 \cap U_2 \quad (6)$$

where

$$U_1 \triangleq \{u(\cdot) : u_{\min} \leq u_i(t) \leq u_{\max}, \quad t \in [t_o, t_f], \quad i = 1, \dots, m, \\ \text{where } u_{\min} > -\infty, \quad u_{\max} < +\infty\} \quad (7)$$

and

$$U_2 \triangleq \{u(\cdot) : u(\cdot) \text{ is piecewise continuous in } [t_o, t_f]\} \quad (8)$$

In the above formulation  $x$  is an  $n$ -dimensional state vector and  $u$  is an  $m$ -dimensional control vector. The function  $f_1$  is an  $n$ -dimensional

vector function of  $x$  at time  $t$ ,  $f_u$  is an  $n \times m$  matrix function of  $x$  at time  $t$  and  $L_2$  is an  $m$ -dimensional vector function of  $x$  at time  $t$ ; the functions  $L_1$  and  $F$  are scalar. The terminal equality constraint function  $\psi$  is an  $s$ -dimensional column vector function of  $x(t_f)$ .

In the sequel we shall use the following assumptions.

Assumption 1. The initial and final times  $t_o, t_f$  are given explicitly,  $-\infty < t_o < t_f < \infty$ .

Assumption 2. The functions  $f$ ,  $L_1$ ,  $L_2$  are three times continuously differentiable in each argument, and the functions  $F$  and  $\psi$  are three times continuously differentiable in  $x(t_f)$ .

With the above formulation and assumptions in mind, we now state the optimal control problem: Determine the control function  $u(\cdot)$  which satisfies (1), (4) and (5) and which minimizes  $V[u(\cdot)]$ .

### 3. Totally Singular Problems

Along an optimal trajectory it is well known that the following necessary conditions (Pontryagin's Principle) hold:

$$-\dot{\lambda} = H_x(\bar{x}, \bar{u}, \lambda, t) \quad ; \quad \lambda(t_f) = \lambda_o F_x(\bar{x}(t_f) + \psi_x^T \nu \quad (9)$$

where

$$\bar{u} = \arg \min_{u_{\min} \leq u_i \leq u_{\max}} H(\bar{x}, u, \lambda, t) \quad (10)$$

and

$$H(x, u, \lambda, t) \triangleq \lambda_o \{L_1(x, t) + u^T L_2(x, t)\} + \lambda^T f(x, u, t) \quad . \quad (11)$$

Here  $\bar{u}(\cdot)$ ,  $\bar{x}(\cdot)$  denote the candidate control and state functions,  $\lambda(\cdot)$  is an  $n$ -vector of Lagrange multiplier functions of time and  $\lambda_o$  is a constant  $\geq 0$ . Associated with the terminal constraint  $\psi$  is an  $s$ -vector of constant Lagrange multipliers  $\nu$ .

Assumption 3. We shall assume that the control problem is normal so that  $\lambda_o$  can be set equal to unity.

Definition 1. A totally singular arc is one along which

$$H_{u_i}(\bar{x}, \lambda, t) = 0 \quad ; \quad t \in [t_0, t_f] \quad , \quad i = 1, \dots, m \quad (12)$$

We now make

Assumption 4.  $\bar{u}(\cdot)$ , the candidate for a minimizing solution is continuous in  $t$ , totally singular, and  $u_{\min} < \bar{u}_i(t) < u_{\max} \quad \forall t \in [t_0, t_f], i = 1, \dots, m$ .

In subsequent sections we develop necessary and sufficient conditions for this totally singular control function to be a minimizing solution (relative minimum).

## II. Conditions for a Relative Minimum: Unconstrained Terminal State

### 1. The Second Variation ( $\delta^2 V$ ) and Associated Theorems

Before proceeding with the second variation we make the following definitions:

Definition 2.  $V[u(\cdot)]$  has a weak relative minimum at  $\bar{u}(\cdot)$  if

$$V[u(\cdot)] - V[\bar{u}(\cdot)] \geq 0$$

$$\forall u(\cdot) \in U \ni \sup_{i=1, \dots, n} \sup_{t_0 \leq t \leq t_f} |x_i - \bar{x}_i| + \sup_{j=1, \dots, m} \sup_{t_0 \leq t \leq t_f} |u_j - \bar{u}_j| < w$$

Note. Since  $x(t_0) - \bar{x}(t_0) = 0$  the above restriction on  $u(\cdot)$  is equivalent to

$$\|u(\cdot) - \bar{u}(\cdot)\| < w_1$$

where

$$\|u(\cdot) - \bar{u}(\cdot)\| \triangleq \sup_{j=1, \dots, m} \sup_{t_0 \leq t \leq t_f} |u_j - \bar{u}_j|$$

and where  $w_1 (< w)$  is sufficiently small.

Definition 3.  $V[u(\cdot)]$  has a strong relative minimum at  $\bar{u}(\cdot)$  if

$$V[u(\cdot)] - V[\bar{u}(\cdot)] \geq 0$$

$$\forall u(\cdot) \in U \ni \sup_{i=1, \dots, n} \sup_{t_0 \leq t \leq t_f} |x_i - \bar{x}_i| < w$$

In the sequel we shall denote  $u(\cdot) - \bar{u}(\cdot)$  by  $\delta u(\cdot)$ .



In the absence of terminal constraints (4) an expression for the second variation (for  $\delta u(t)$  sufficiently small,  $t \in [t_0, t_f]$ ) is:

$$\delta^2 V[\delta u(\cdot)] = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x \right\} dt + \frac{1}{2} \delta x^T F_{xx} \delta x \Big|_{t_f} \quad (13)$$

subject to the linearized differential equation

$$\dot{\delta x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad . \quad (14)$$

In (13), (14) partial derivatives are evaluated along  $\bar{u}$ ,  $\bar{x}$ ,  $\lambda$ .

We are led to the study of the second variation because of the following theorem [slightly altered form of Theorem 1, Gelfand and Fomin [11, p. 99]].

Theorem 1. A necessary condition for  $V[u(\cdot)]$  to have a weak relative minimum at  $\bar{u}(\cdot)$  is that  $\delta^2 V[\delta u(\cdot)] \geq 0$   $\forall \delta u(\cdot)$  sufficiently small to justify the expansions (13), (14) and such that  $\bar{u}(\cdot) + \delta u(\cdot) \in U$ .

The condition  $\delta^2 V[\delta u(\cdot)] \geq 0$  for all admissible  $\delta u(\cdot)$  is necessary but not sufficient for the functional  $V[u(\cdot)]$  to have a weak minimum at  $\bar{u}(\cdot)$ . A sufficient condition is provided by the following theorem.

Theorem 2. [Gelfand and Fomin [11, p. 100]] A sufficient condition for  $V[u(\cdot)]$  to have a weak relative minimum at  $\bar{u}(\cdot)$  is that  $\delta^2 V[\delta u(\cdot)]$  be strongly positive.<sup>†</sup>

Unfortunately, it turns out that the singular second variation cannot be strongly positive [12]. Here we prove this using arguments similar to those suggested by Johansen [13]. For simplicity of presentation let us consider the case where  $x$  and  $u$  are scalars; the arguments generalize to the vector case. Suppose that we set  $\delta u(t) = b \sin qt$   $\forall t \in [t_0, t_f]$  and choose  $b$   $\ni \bar{u}(\cdot) + \delta u(\cdot) \in U$ . The solution of (14) when driven by this  $\delta u(\cdot)$  is

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<sup>†</sup>That is,  $\delta^2 V[\delta u(\cdot)] \geq k \|\delta u(\cdot)\|^2$   $\forall$  admissible  $\delta u(\cdot)$ , where  $k > 0$  is a constant.

$$\delta_x(t) = b \int_{t_0}^t \phi(t, \tau) f_u(\tau) \sin q\tau \, d\tau \quad (15)$$

where

$$\frac{\partial}{\partial t} \phi(t, \tau) = f_x(t) \phi(t, \tau) \quad ; \quad \phi(\tau, \tau) = I \quad (16)$$

Integrating (15) by parts yields

$$\begin{aligned} \delta_x(t) = b \{ & \int_{t_0}^t \left[ \frac{\partial \phi}{\partial \tau}(t, \tau) f_u(\tau) + \phi(t, \tau) \frac{\partial f_u}{\partial \tau}(\tau) \right] \frac{1}{q} \cos q\tau \, d\tau \\ & - f_u(t) \frac{1}{q} \cos qt + \phi(t, t_0) f_u(t_0) \frac{1}{q} \cos qt_0 \} \end{aligned} \quad (17)$$

From (13),

$$\begin{aligned} \delta^2 V[\delta_u(\cdot)] \leq & \int_{t_0}^{t_f} \left\{ \frac{1}{2} |H_{xx}| \delta_x^2 + |\delta_u| |H_{ux}| |\delta_x| \right\} dt \\ & + \frac{1}{2} |F_{xx}| \delta_x^2 \Big|_{t_f} \end{aligned} \quad (18)$$

From (17),

$$\begin{aligned} |\delta_x(t)| \leq & \frac{b}{q} \left\{ \int_{t_0}^t \left| \left[ \frac{\partial \phi}{\partial \tau}(t, \tau) f_u(\tau) + \phi(t, \tau) \frac{\partial f_u}{\partial \tau}(\tau) \right] \right| |\cos q\tau| \, d\tau \right. \\ & \left. + |f_u(t)| |\cos qt| + |\phi(t, t_0)| |f_u(t_0)| |\cos qt_0| \right\} \end{aligned} \quad (19)$$

$$\begin{aligned} \leq & \frac{b}{q} \left\{ \int_{t_0}^t \left| \left[ \frac{\partial \phi}{\partial \tau}(t, \tau) f_u(\tau) + \phi(t, \tau) \frac{\partial f_u}{\partial \tau}(\tau) \right] \right| \, d\tau \right. \\ & \left. + |f_u(t)| + |\phi(t, t_0)| |f_u(t_0)| \right\} \end{aligned} \quad (20)$$

$$\triangleq \frac{b}{q} \cdot \beta(t) \quad (21)$$

Thus

$$\delta^2 V[\delta u(\cdot)] \leq \int_{t_0}^{t_f} \left\{ \frac{1}{2} |H_{xx}| \left(\frac{b}{q}\right)^2 \beta^2 + \frac{b^2}{q} \beta |H_{ux}| \right\} dt$$

$$+ \frac{1}{2} |F_{xx}| \left(\frac{b}{q}\right)^2 \beta^2 \Big|_{t_f} \quad (22)$$

$$\triangleq b^2 \left\{ \frac{d_1}{2} + \frac{d_2}{q} \right\} \quad (23)$$

Now if  $\delta^2 V[\delta u(\cdot)]$  is strongly positive then  $\exists k$  (constant)  $> 0$  such that

$$b^2 \left\{ \frac{d_1}{2} + \frac{d_2}{q} \right\} \geq k \|\delta u(\cdot)\|^2, \quad \forall q; \quad 0 < q < \infty \quad (24)$$

Clearly (24) cannot hold for  $k$  a constant ( $> 0$ ) since  $q$  can be made arbitrarily large ( $q$  is directly proportional to the frequency of the sine input). Thus  $\delta^2 V[\delta u(\cdot)]$  cannot be strongly positive.

In view of the above failure of Theorem 2 for singular problems we offer the following more useful sufficiency theorem.

Theorem 3. A sufficient condition for  $V[u(\cdot)]$  to have a minimum at  $\bar{u}(\cdot)$  is that  $\delta^2 V[\delta u(\cdot)] \geq 0$  for all admissible  $\delta u(\cdot)$  and that

$$\delta^2 V[\delta u(\cdot)] \geq |a[\delta u(\cdot)]| \|\delta u(\cdot)\|^2 \quad (25)$$

for  $\|\delta u(\cdot)\|$  sufficiently small, where

$$a[\delta u(\cdot)] \triangleq \frac{V[\bar{u}(\cdot) + \delta u(\cdot)] - V[\bar{u}(\cdot)] - \delta^2 V[\delta u(\cdot)]}{\|\delta u(\cdot)\|^2} \quad (26)$$

Proof. Let

$$\Delta V[\delta u(\cdot)] \triangleq V[\bar{u}(\cdot) + \delta u(\cdot)] - V[\bar{u}(\cdot)]$$

then, by Assumption 2,

$$\Delta V[\delta u(\cdot)] = \delta^2 V[\delta u(\cdot)] + a[\delta u(\cdot)] \|\delta u(\cdot)\|^2 \quad (27)$$

where  $a[\delta u(\cdot)] \rightarrow 0$  as  $\|\delta u(\cdot)\| \rightarrow 0$ . From (27) and (25),

$$\Delta V[\delta u(\cdot)] - a[\delta u(\cdot)] \|\delta u(\cdot)\|^2 = \delta^2 V[\delta u(\cdot)] \geq |a[\delta u(\cdot)]| \|\delta u(\cdot)\|^2$$

for  $\|\delta u(\cdot)\|$  sufficiently small. Thus,

$$\Delta V[\delta u(\cdot)] \geq \|\delta u(\cdot)\|^2 \{ |\alpha[\delta u(\cdot)]| + \alpha[\delta u(\cdot)] \} \quad (28)$$

so that

$$\Delta V[\delta u(\cdot)] \geq 0 \quad \forall \delta u(\cdot) \ni \|\delta u(\cdot)\| \text{ sufficiently small.} \quad (29)$$

However, (29) is just the definition that  $V[u(\cdot)]$  has a weak relative minimum at  $\bar{u}(\cdot)$  so that the theorem is proved.

Note. If, as in the nonsingular control problem, the conditions of **Theorem 2** are satisfied then clearly so are those of Theorem 3. However, the important point is that Theorem 3 can be satisfied without  $\delta^2 V[\delta u(\cdot)]$  being strongly positive.

## 2. Principal Results

Before stating our main theorems we make the following assumption and definition.

Assumption 5. The linearized system  $\delta \dot{x} = f_x \delta x + f_u \delta u$  is controllable from time  $t_0$  to time  $\tau$ ,  $\forall \tau \in (t_0, t_f]$ ; that is,

$$\int_{t_0}^{\tau} \phi(\tau, \sigma) f_u(\sigma) f_u^T(\sigma) \phi^T(\tau, \sigma) d\sigma > 0 \quad , \quad \forall \tau \in (t_0, t_f] \quad (30)$$

where

$$\frac{\partial}{\partial \tau} \phi(\tau, \sigma) = f_x(\tau) \phi(\tau, \sigma) \quad ; \quad \phi(\sigma, \sigma) = I \quad (31)$$

Definition 4. A real symmetric  $n \times n$  matrix function of time  $\Theta(t)$  is said to be monotone increasing in  $t$  if the scalar  $J(x, t) \triangleq x^T \Theta(t) x$  is monotone increasing in  $t$  for all constant  $n$ -vectors  $x$ .

### Theorem A1.

(i) Necessary Condition. Under Assumptions 1 through 5 a necessary condition for  $\delta^2 V[\delta u(\cdot)] \geq 0$  for all admissible  $\delta u(\cdot)$  is that  $\exists \forall t \in (t_0, t_f]$  a real symmetric matrix function of time  $\hat{P}(\cdot)$  which is monotone increasing in  $t$  such that

$$H_{ux} + f_u^T P = 0 \quad \forall t \in (t_o, t_f] \quad (32)$$

$$F_{xx}(t_f) - P(t_f) = -\hat{P}(t_f) \geq 0 \quad (33)$$

where

$$P = Q + \Phi^T \hat{P} \Phi \quad (34)$$

$$\dot{\Phi} = -\Phi f_x \quad ; \quad \Phi(t_f) = I \quad (35)$$

and

$$-\dot{Q} = H_{xx} + f_x^T Q + Q f_x \quad ; \quad Q(t_f) = F_{xx}(t_f) \quad (36)$$

(ii) Sufficient Condition. In addition to the above stated condition  $\hat{P}(\cdot)$  exists  $\forall t \in [t_o, t_f]$  (strengthened existence condition).

Note. The gap between the necessary and the sufficient condition is minimal.

Corollary A1.1. If  $L_1$  and  $F$  are quadratic functions of  $x$ , if  $f_1$  and  $L_2$  are linear in  $x$ , and if  $f_u$  is independent of  $x$ , then condition (i) of Theorem A1 is necessary and together with the strengthened existence condition (ii) of Theorem A1 is sufficient for  $V[u(\cdot)]$  to have a strong minimum at  $\bar{u}(\cdot)$ .

Corollary A1.2. [2] A sufficient condition for  $\delta^2 V[\delta u(\cdot)] \geq 0$  for all admissible  $\delta u(\cdot)$  is that  $\exists \forall t \in [t_o, t_f]$  a real symmetric matrix function of time  $P(\cdot)$  which is continuously differentiable, such that

$$H_{ux} + f_u^T P = 0 \quad \forall t \in [t_o, t_f] \quad (37)$$

$$\dot{P} + P f_x + f_x^T P + H_{xx} = M(t) \geq 0 \quad \forall t \in [t_o, t_f] \quad (38)$$

$$F_{xx}(t_f) - P(t_f) \geq 0 \quad . \quad (39)$$

Proof. Integration of (38) and the use of (34), (35) and (36) yields

$$\frac{d}{dt} \hat{P} \triangleq [\Phi^{-1}(t)]^T M(t) [\Phi^{-1}(t)] \geq 0 \implies \hat{P} \text{ monotone increasing.}$$

Note. The gap between this sufficient condition and the sufficient condition of Theorem A1 appears to be minimal since  $\hat{P}$  is, a priori, differentiable almost everywhere (because it is monotone [14]).

Theorem A2. Sufficient conditions for  $V[u(\cdot)]$  to have a weak relative minimum at  $\bar{u}(\cdot)$  are that the strengthened condition (ii) of Theorem A1 holds and that

$$\delta^2 V[\delta u(\cdot)] \leq 0 \implies V[\bar{u}(\cdot) + \delta u(\cdot)] \geq V[\bar{u}(\cdot)]$$

See Theorem 3 for a condition which ensures this.

### 3. Useful Lemmas

We need the following Lemmas in order to prove Theorem A1.

Lemma 1. The second variation (13), (14) is expressible in the equivalent canonical form:

$$\delta^2 V[\delta u(\cdot)] = \delta^2 \hat{V}[\delta u(\cdot)] \triangleq \int_{t_0}^{t_f} \delta u^T C(t) \delta y dt \quad (40)$$

subject to

$$\frac{d}{dt} (\delta y(t)) = B(t) \delta u(t) \quad ; \quad \delta y(t_0) = 0 \quad (41)$$

where

$$C(t) = [H_{ux}(t) + f_u^T(t)Q(t)]\Phi^{-1}(t) \quad (42)$$

$$B(t) = \Phi(t)f_u(t) \quad (43)$$

$$\dot{\Phi}(t) = -\Phi(t)f_x(t) \quad ; \quad \Phi(t_f) = I \quad (44)$$

and where

$$-\dot{Q}(t) = H_{xx}(t) + Q(t)f_x(t) + f_x^T(t)Q(t) \quad ; \quad Q(t_f) = F_{xx}(\bar{x}(t_f)) \quad (45)$$

$$\delta y(t) = \Phi(t)\delta x(t) \quad (46)$$

Proof.

(i) (13), (14)  $\implies$  (40)-(46). Adjoin the linearized system equation (14) to the second variation (13) using a continuously differentiable vector multiplier function of time  $\delta\lambda(\cdot)$ :

$$\begin{aligned} \delta^2 \hat{V}[\delta u(\cdot)] = & \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x + \delta \lambda^T (f_x \delta x + f_u \delta u - \dot{\delta x}) \right\} dt \\ & + \frac{1}{2} \delta x^T F_{xx} \delta x \Big|_{t_f} \end{aligned} \quad (47)$$

Integrating  $\delta\lambda \dot{\delta x}$  by parts yields

$$\begin{aligned} \delta^2 \hat{V}[\delta u(\cdot)] = & \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x + \delta \lambda^T (f_x \delta x + f_u \delta u) \right. \\ & \left. + \delta \dot{\lambda} \delta x \right\} dt + \frac{1}{2} \delta x^T F_{xx} \delta x \Big|_{t_f} - \delta \lambda^T \delta x \Big|_{t_0}^{t_f}. \end{aligned} \quad (48)$$

Now set

$$\delta \lambda(t) = \frac{1}{2} Q(t) \delta x(t) \quad (49)$$

where  $Q(t)$  is an  $n \times n$  symmetric matrix (continuously differentiable) function of time. Noting that  $\delta x(t_0) \triangleq 0$ , the second variation becomes

$$\begin{aligned} \delta^2 \hat{V}[\delta u(\cdot)] = & \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T (\dot{Q} + Q f_x + f_x^T Q + H_{xx}) \delta x \right. \\ & \left. + \delta u^T (H_{ux} + f_u^T Q) \delta x \right\} dt + \frac{1}{2} \delta x^T (F_{xx} - Q) \delta x \Big|_{t_f}. \end{aligned} \quad (50)$$

Now we let

$$-\dot{Q} = H_{xx} + f_x^T Q + Q f_x \quad ; \quad Q(t_f) = F_{xx}(t_f) \quad (51)$$

Clearly  $Q(\cdot)$  is a well defined continuously differentiable function of time. With this choice of  $Q(\cdot)$  the second variation is

$$\delta^2 \hat{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \delta u^T (H_{ux} + f_u^T Q) \delta x \, dt \quad (52)$$

subject to

$$\dot{\delta x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad (53)$$

Define

$$\delta y = \Phi \delta x \quad (54)$$

where

$$\dot{\Phi} = -\Phi f_x \quad ; \quad \Phi(t_f) = I \quad (55)$$

Clearly  $\Phi(t)$  is invertible  $\forall t \in [t_0, t_f]$  so that the second variation becomes

$$\delta^2 \hat{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \delta u^T (H_{ux} + f_u^T Q) \Phi^{-1} \delta y \, dt \quad (56)$$

subject to

$$\dot{\delta y} = \Phi f_u \delta u \quad ; \quad \delta y(t_f) = 0 \quad (57)$$

which is the required form of the second variation.

(ii) (40)-(46)  $\implies$  (13), (14). Since  $\Phi$  is invertible, (54), (56), (57) yield (52), (53). Adjoin (53) to (52) with a Lagrange multiplier function  $\delta \tilde{\lambda}(\cdot)$  and set

$$\delta \tilde{\lambda}(t) = \frac{1}{2} \tilde{Q}(t) \delta x(t) \quad (58)$$

The second variation becomes

$$\begin{aligned} \delta^2 \tilde{V}[\delta u(\cdot)] = & \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T (\tilde{Q} + \tilde{Q} f_x + f_x^T \tilde{Q}) \delta x \right. \\ & + \delta u^T (H_{ux} + f_u^T Q + f_u^T \tilde{Q}) \delta x \} \, dt \\ & - \left. \frac{1}{2} \delta x^T \tilde{Q} \delta x \right|_{t_f} \end{aligned} \quad (59)$$



Letting

$$\tilde{Q}(t) = -Q(t) \quad \forall t \in [t_o, t_f] \quad (60)$$

yields

$$\delta^2 \tilde{V}[\delta u(\cdot)] = \int_{t_o}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x \right\} dt + \frac{1}{2} \delta x^T F_{xx} \delta x \Big|_{t_f} \quad (61)$$

so that the Lemma is proved.

Lemma 2. Condition (i) of Theorem A1 is equivalent to the existence

$\forall t \in (t_o, t_f]$  of a real symmetric monotone increasing matrix function of time  $\hat{P}(\cdot)$  such that

$$C + B^T \hat{P} = 0 \quad \forall t \in (t_o, t_f] \quad (62)$$

$$-\hat{P}(t_f) \geq 0 \quad (63)$$

Proof. Let

$$P = \Phi^T \hat{P} \Phi + Q \quad (64)$$

Substituting this into (62), (63) yields (32), (33) and vice versa. Note the coordinate independence of conditions (32), (33).

#### 4. A Related Nonsingular Second Variation

Consider the nonsingular quadratic functional

$$\delta^2 V_N[\delta u(\cdot), \epsilon] = \delta^2 \hat{V}[\delta u(\cdot)] + \frac{1}{2\epsilon} \int_{t_o}^{t_f} \delta u^T \delta u dt \quad (65)$$

$$= \int_{t_o}^{t_f} \left\{ \delta u^T C \delta u + \frac{1}{2\epsilon} \delta u^T \delta u \right\} dt \quad (66)$$

where  $\epsilon > 0$  is a scalar and

$$\delta \dot{y} = B \delta u \quad ; \quad \delta y(t_o) = 0 \quad (67)$$

Lemma 3. If  $\delta^2 \hat{V}[\delta u(\cdot)]$  is nonnegative  $\forall \delta u(\cdot)$  then so is  $\delta^2 V_N[\delta u(\cdot), \epsilon]$  nonnegative. Moreover  $\delta^2 V_N[\delta u(\cdot), \epsilon]$  is positive-definite.

Proof.  $\frac{1}{2\epsilon} \int_{t_0}^{t_f} \delta u^T \delta u dt > 0$  unless  $\delta u(\cdot) = 0$ .

Theorem 4. [Gelfand and Fomin [11, p. 123], Breakwell and Ho [15]].

If the quadratic functional  $\delta^2 \hat{V}[\delta u(\cdot)]$  is nonnegative then the matrix Riccati differential equation

$$-\dot{S}_\epsilon = -(C + B^T S_\epsilon)^T (C + B^T S_\epsilon) \epsilon \quad ; \quad S_\epsilon(t_f) = 0 \quad (68)$$

associated with  $\delta^2 V_N[\delta u(\cdot), \epsilon]$  has a solution which exists  $\forall t \in [t_0, t_f]$ .

Proof. By Lemma 3,  $\delta^2 V_N[\delta u(\cdot), \epsilon]$  is positive-definite if  $\delta^2 \hat{V}[\delta u(\cdot)]$  is nonnegative  $\forall \delta u(\cdot)$ ; Gelfand and Fomin's theorem for positive-definite functionals then applies.

Note. A special case of (66), (67) is treated in [11]; see [15] for the general case.

Theorem 5. If the matrix Riccati differential equation (68) has a solution which exists in the interval  $[\tau, t_f]$ ;  $t_0 \leq \tau \leq t_f$  then the control function that minimizes

$$\delta^2 V_N^\tau[\delta y(\tau), \delta u(\cdot), \epsilon, \tau] \triangleq \int_\tau^{t_f} \{ \delta u^T C \delta y + \frac{1}{2\epsilon} \delta u^T \delta u \} dt \quad (69)$$

subject to

$$\delta \dot{y} = B \delta u \quad ; \quad \delta y(\tau) \text{ given} \quad (70)$$

is

$$\delta u^0(t) = -\epsilon [C(t) + B^T(t) S_\epsilon(t)] \delta y(t) \quad ; \quad t \in [\tau, t_f] \quad (71)$$

and moreover

$$\begin{aligned} \min_{\delta u(\cdot)} \delta^2 V_N^T[\delta y(\tau), \delta u(\cdot), \epsilon, \tau] &\triangleq \delta^2 V^O[\delta y(\tau), \epsilon, \tau] \\ &= \frac{1}{2} \delta y^T(\tau) S_\epsilon(\tau) \delta y(\tau) \end{aligned} \quad (72)$$

Proof. Substitute (72) into the Bellman equation

$$\begin{aligned} -\frac{\partial}{\partial t} \{ \delta^2 V^O[\delta y(t), \epsilon, t] \} &= \min_{\delta u} [ \delta u^T C \delta y + \frac{1}{2\epsilon} \delta u^T \delta u \\ &\quad + \{ \frac{\partial}{\partial(\delta y)} \delta^2 V^O[\delta y(t), \epsilon, t] \}^T B \delta u ] \quad . \end{aligned} \quad (73)$$

Minimizing with respect to  $\delta u$  yields (71) and causes the right hand side of (73) to be quadratic in  $\delta y(t)$ . Equating coefficients of quadratic terms in (73) yields equation (68) for  $\dot{S}_\epsilon$ . If  $S_\epsilon(t)$  exists  $\forall t \in [t_o, t_f]$  Bellman's equation is valid and the theorem is proved.

Lemma 4.  $S_\epsilon(\cdot)$  is a continuous function of the parameter  $\epsilon$ .

Proof.  $C$  and  $B$  are continuous functions of time in  $[t_o, t_f]$  and the right hand side of (68) is analytic in  $S_\epsilon$  and  $\epsilon$  [Coddington and Levinson [16]].

Lemma 5.  $\delta^2 V^O[\delta y(\tau), \epsilon, \tau] = \frac{1}{2} \delta y^T(\tau) S_\epsilon(\tau) \delta y(\tau)$  is a monotone decreasing function of  $\epsilon$  ( $\epsilon$  increasing);  $\tau \in [t_o, t_f]$ . Moreover,  $S_\epsilon(\tau)$  is a monotone decreasing matrix function of  $\epsilon$ ;  $\tau \in [t_o, t_f]$ .

Proof. For some arbitrary  $\delta y(\tau)$ ,  $\tau$  and  $\epsilon_1$ , we have

$$\delta^2 V^O[\delta y(\tau), \epsilon_1, \tau] = \min_{\delta u(\cdot)} \int_\tau^{t_f} \{ \delta u^T C \delta y + \frac{1}{2\epsilon_1} \delta u^T \delta u \} dt \quad . \quad (74)$$

Let the control function that minimizes (74) be  $\delta u_1^O(\cdot)$  and its associated state path be  $\delta y_1^O(\cdot)$ . Now for any  $\epsilon_2 \geq \epsilon_1$  it is clear that

$$\begin{aligned} \int_\tau^{t_f} \{ (\delta u_1^O)^T C \delta y_1^O + \frac{1}{2\epsilon_2} (\delta u_1^O)^T \delta u_1^O \} dt &\leq \int_\tau^{t_f} \{ (\delta u_1^O)^T C \delta y_1^O \\ &\quad + \frac{1}{2\epsilon_1} (\delta u_1^O)^T \delta u_1^O \} dt \end{aligned} \quad (75)$$

and, by definition

$$\delta^2 V^0[\delta y(\tau), \epsilon_2, \tau] \leq \int_{\tau}^{t_f} \{(\delta u_1^0)^T C \delta y_1^0 + \frac{1}{2\epsilon_2} (\delta u_1^0)^T \delta u_1^0\} dt \quad (76)$$

Thus for any  $\epsilon_2 \geq \epsilon_1$  we have

$$\delta^2 V^0[\delta y(\tau), \epsilon_2, \tau] \leq \delta^2 V^0[\delta y(\tau), \epsilon_1, \tau] \quad . \quad (77)$$

Since  $\delta y(\tau)$  and  $\tau$  are arbitrary, the first part of the theorem is proved.

That  $S_{\epsilon_2}(\tau) \leq S_{\epsilon_1}(\tau)$ ,  $\tau \in [t_o, t_f]$ ,  $\epsilon_2 \geq \epsilon_1$ , follows from (77) and Theorem 5.

Lemma 6. Under Assumptions 1 through 5, if  $\delta^2 \hat{V}[\delta u(\cdot)] \geq 0$  for all admissible  $\delta u(\cdot)$  then  $S_{\infty}(t) \triangleq \lim_{\epsilon \rightarrow \infty} S_{\epsilon}(t)$  exists  $\forall t \in (t_o, t_f]$  and is negative semi-definite.

Proof. From (68)  $S_{\epsilon}(\tau) \leq 0 \quad \forall \tau \in [t_o, t_f]$ ,  $\forall \epsilon$ ;  $0 < \epsilon < \infty$ . By Lemma 5  $S_{\epsilon}(\tau)$  is a monotone decreasing function of  $\epsilon$  ( $\epsilon$  increasing) so that it has a limit (possibly  $-\infty$ ).

Given an arbitrary time  $\tau$  in the interval  $(t_o, t_f)$  we can by Assumption 5 construct a variation  $\delta u_3(t)$ ;  $t \in [t_o, \tau)$  such that

$$\int_{t_o}^{\tau} \{ \delta u_3^T C \delta y_3 + \frac{1}{2\epsilon} \delta u_3^T \delta u_3 \} dt$$

is finite and such that

$$\delta y_3(\tau) = \gamma(\tau) \quad ; \quad \gamma(\tau) \text{ arbitrary} \quad . \quad (78)$$

Suppose that  $S_{\infty}(\tau) = -\infty$ . Then, by Lemmas 4, 5,  $S_{\epsilon}(\tau)$  can be made arbitrarily large and negative by increasing  $\epsilon$ . This implies that for some  $\gamma(\tau)$ ,  $\delta^2 V^0[\delta y_3(\tau), \epsilon, \tau]$  can be made arbitrarily large and negative which implies that

$$\int_{t_o}^{\tau} \{ \delta u_3^T C \delta y_3 + \frac{1}{2\epsilon} \delta u_3^T \delta u_3 \} dt + \delta^2 V^0[\delta y_3(\tau), \epsilon, \tau] < 0 \quad (79)$$

for some  $\epsilon$ ;  $0 < \epsilon < \infty$ . By Lemma 3 this contradicts the assumption that  $\delta^2 \hat{V}[\delta u(\cdot)] \geq 0$  so that  $S_\infty(\tau)$  cannot be infinite. Since  $\tau$  is arbitrary, and since  $S_\epsilon(t_f) = 0 \quad \forall \epsilon$ , we conclude that

$$S_\infty(\tau) ; \tau \in (t_o, t_f] \text{ is finite} \quad (80)$$

Note. We have not shown that as  $\epsilon$  becomes large the control

$\{\bar{u}(t) + \delta u_3(t), t \in [t_o, \tau]; \bar{u}(t) + \delta u^0(t), t \in [\tau, t_f]\} \in U$ . However, we have shown that if  $S_\infty(\cdot)$  is not finite in the interval  $(t_o, t_f]$  then  $\delta^2 V_N[\delta y(t_o), \delta u(\cdot), \epsilon, t_o]$ ,  $\epsilon$  sufficiently large but finite, does not have a minimum at  $\delta u(\cdot) = 0$ . Since for the quadratic functional there is no distinction between a weak and a strong minimum it follows that  $\delta^2 V_N[\delta y(t_o), \delta u(\cdot), \epsilon, t_o]$  can be made negative by a weak variation which satisfies  $\bar{u}(\cdot) + \delta u(\cdot) \in U$ . We conclude, then, that the Lemma is proved.

Lemma 7.  $S_\infty(\tau)$  is a monotone increasing matrix function of  $\tau$ .

Proof. From (68)

$$S_\infty(\tau) = \lim_{\epsilon \rightarrow \infty} \left[ \int_{t_o}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon) \epsilon dt \right] \quad (81)$$

$$\begin{aligned} &= \lim_{\epsilon \rightarrow \infty} \left[ \int_{t_o}^{\tau - \Delta} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon) \epsilon dt \right] \\ &\quad + \lim_{\epsilon \rightarrow \infty} \left[ \int_{\tau - \Delta}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon) \epsilon dt \right] \end{aligned} \quad (82)$$

From (82)

$$S_\infty(\tau) = S_\infty(\tau - \Delta) + \lim_{\epsilon \rightarrow \infty} \left[ \int_{\tau - \Delta}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon) \epsilon dt \right]. \quad (83)$$

The validity of the Lemma follows immediately from (83) since for all  $\epsilon > 0$ ,

$$\left[ \int_{\tau-\Delta}^{\tau} (C + B^T S_{\epsilon})^T (C + B^T S_{\epsilon}) \epsilon dt \right] \geq 0 \quad \forall \Delta \geq 0 \quad . \quad (84)$$

Lemma 8. If  $S_{\infty}(\tau)$  exists for  $\tau \in (t_o, t_f]$  then

$$C(\tau) + B^T(\tau) S_{\infty}(\tau) = 0 \quad \text{a.e. in } [t_o, t_f] \quad . \quad (85)$$

Proof. Suppose the contrary. Then, for some  $\tau \in (t_o, t_f)$ , by Assumption 2 and Lemma 4,  $\exists \epsilon^* > 0 \ni$

$$-\int_{t_f}^{\tau} (C + B^T S_{\epsilon})^T (C + B^T S_{\epsilon}) dt > 0, \quad \forall \epsilon \geq \epsilon^* \quad (86)$$

so that

$$\lim_{\epsilon \rightarrow \infty} \left[ -\int_{t_f}^{\tau} (C + B^T S_{\epsilon})^T (C + B^T S_{\epsilon}) \epsilon dt \right] = \infty \quad (87)$$

contradicting the fact that  $S_{\infty}(\tau)$  exists for  $\tau \in (t_o, t_f]$ .

##### 5. Proof of Theorem A1.

(i) Necessary Condition. If Assumptions 1 through 5 are satisfied and if  $\delta^2 \hat{V}[\delta_u(\cdot)] \geq 0$  for all admissible  $\delta_u(\cdot)$ , then by Lemmas 6-8  $\exists S_{\infty}(t); t \in (t_o, t_f]$  which is a real symmetric monotone increasing matrix function of time such that

$$C + B^T S_{\infty} = 0 \quad \text{a.e. in } [t_o, t_f] \quad (88)$$

$$S_{\infty}(t_f) = 0 \quad . \quad (89)$$

We show now that (88), (89) imply (62), (63) which by Lemma 2 yield the conditions of Theorem A1.

Define

$$\hat{P}(t_f) = S_{\infty}(t_f^-) \quad (90)$$

Since by Lemma 7,  $S_{\infty}(t)$  is a monotone increasing function of  $t$  it follows that

$$-\hat{P}(t_f) \geq 0 \quad (91)$$

Defining

$$\hat{P}(t) = S_{\infty}(t) \quad , \quad t \in (t_o, t_f) \quad (92)$$

yields

$$C + B^T \hat{P} = 0 \quad \text{a.e. in } [t_o, t_f] \quad (93)$$

where

$$\hat{P}(t) \text{ exists } \forall t \in (t_o, t_f] \quad (94)$$

and is monotone increasing in  $t$ . In order to establish necessity of (62)

we need only show that (93) implies

$$C + B^T \hat{P} = 0 \quad \forall t \in (t_o, t_f] \quad (95)$$

Suppose that for some  $t \in (t_o, t_f)$

$$C(t) + B^T(t) \hat{P}(t) \neq 0 \quad . \quad (96)$$

Since  $C$  and  $B$  are continuous functions of time (96) can only occur if a jump in  $\hat{P}$  occurs at time  $t$  that does not lie in the null space of  $B^T(t)$ .

Moreover, since  $\hat{P}$  is monotone increasing in  $t$  it follows from (96) that

$$C(t)B(t) + B^T(t)\hat{P}(t)B(t) > 0 \quad . \quad (97)$$

Now since  $\hat{P}$  is monotone increasing in  $t$ , and  $C$  and  $B$  are continuous functions of time, (97) must hold during the time interval  $[t, t + \Delta]$

( $\Delta > 0$  and sufficiently small), contradicting (93). Thus, we are led

to the conclusion that

$$C + B^T \hat{P} = 0 \quad \forall t \in (t_o, t_f) \quad . \quad (98)$$

Equation (95) follows from (98) since  $\hat{P}(t_f^-) \equiv \hat{P}(t_f)$ .

(ii) Sufficient Condition. Suppose that (91) and (95) are satisfied and that (94) is satisfied in strengthened form (i.e.  $\hat{P}(t)$  exists  $\forall t \in [t_o, t_f]$ ).

Adjoin the linearized dynamics to the second variation as follows:

$$\begin{aligned}\delta^2 \hat{V}[\delta u(\cdot)] &= \delta^2 \tilde{V}[\delta u(\cdot)] \triangleq \int_{t_o}^{t_f} \{ \delta u^T C \delta y + \delta y^T \hat{P} (B \delta u - \delta \dot{y}) \} dt \quad (99) \\ &= \int_{t_o}^{t_f} \delta u^T (C + B^T \hat{P}) \delta y dt - \int_{t_o}^{t_f} \delta y^T \hat{P} \delta \dot{y} dt \quad .\end{aligned}\quad (100)$$

The first integral is zero since (95) holds. The remaining integral can be written in Stieltjes form as

$$-\int_{t_o}^{t_f} \delta y^T \hat{P} d(\delta y) \quad (101)$$

which, upon integration by parts [14, p. 118] becomes

$$\int_{t_o}^{t_f} \frac{1}{2} \delta y^T d\hat{P}(t) \delta y - \frac{1}{2} \delta y^T \hat{P} \delta y \Big|_{t_o}^{t_f} \quad (102)$$

Thus

$$\delta^2 \tilde{V}[\delta u(\cdot)] = \int_{t_o}^{t_f} \frac{1}{2} \delta y^T d\hat{P}(t) \delta y - \frac{1}{2} \delta y^T \hat{P} \delta y \Big|_{t_o}^{t_f} \quad (103)$$

since  $\delta y(t_o) = 0$ . From (91) the term evaluated at  $t_f$  is nonnegative, and the integral is nonnegative since  $\hat{P}(\cdot)$  is a monotone increasing matrix function of time. Sufficiency is proved.

### III. Conditions for a Relative Minimum: Constrained Terminal State

#### 1. The Second Variation $\delta^2 V^*$

Here we treat the class of totally singular problems where equality (4) is present. In this case the second variation (for  $\delta u(t)$  sufficiently small,  $t \in [t_o, t_f]$ ) has the form



$$\begin{aligned} \delta^2 V^*[\delta u(\cdot)] = & \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x \right\} dt \\ & + \frac{1}{2} \delta x^T (F_{xx} + v^T \psi_{xx}) \delta x \Big|_{t_f} \end{aligned} \quad (104)$$

subject to the linearized differential equation

$$\dot{\delta x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad (105)$$

and the linearized terminal constraint

$$\psi_x \delta x \Big|_{t_f} = 0 \quad (106)$$

We now introduce

Assumption 6. The  $s \times n$  matrix  $\psi_x(\bar{x}(t_f))$  has full rank  $s$ .

Lemma 9. By Assumption 6,  $s$  components of  $\delta x(t_f)$  -- referred to as  $\delta x^s(t_f)$  -- can be solved for in terms of the remaining  $n - s$  components  $\delta x^{n-s}(t_f)$ .

For example,

$$\delta x^s(t_f) = -A_1^{-1} A_2 \delta x^{n-s}(t_f) \quad (108)$$

where<sup>†</sup>

$$s \updownarrow \begin{bmatrix} \xrightarrow{n} \\ A_1 \quad A_2 \\ \xleftarrow{s} \quad (n-s) \end{bmatrix} = \psi_x \quad (109)$$

so that

$$\delta x(t_f) = \begin{bmatrix} -A_1^{-1} A_2 \delta x^{n-s}(t_f) \\ \delta x^{n-s}(t_f) \end{bmatrix} = \begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix} \delta x^{n-s}(t_f) \triangleq Z \delta x^{n-s}(t_f) \quad (110)$$

where  $Z$  is  $n \times (n - s)$ .

<sup>†</sup>If  $A_1$  is singular then differently partitioned  $\psi_x$  and  $\delta x(t_f)$  must be used.

## 2. Principal Results

### Theorem B1.

(i) Necessary Condition. Under Assumptions 1 through 6 a necessary condition for  $\delta^2 V^*[\delta u(\cdot)] \geq 0$  for all admissible  $\delta u(\cdot)$  is that  $\exists \forall t \in (t_0, t_f)$  a real symmetric matrix function of time  $\hat{P}^*(\cdot)$  which is monotone increasing in  $t$  such that

$$H_{ux} + f_u^T P^* = 0 \quad \forall t \in (t_0, t_f) \quad (111)$$

$$Z^T (F_{xx} + v^T \psi_{xx} - P^*(t_f^-)) Z = -Z^T \hat{P}^*(t_f^-) Z \geq 0 \quad (112)$$

where

$$P^* = Q + \Phi^T \hat{P}^* \Phi, \quad (113)$$

$$\dot{\Phi} = -\Phi f_x; \quad \Phi(t_f) = I \quad (114)$$

$$-\dot{Q} = H_{xx} + f_x^T Q + Q f_x; \quad Q(t_f) = F_{xx} + v^T \psi_{xx} \quad (115)$$

and where  $Z$  is defined in (110).

(ii) Sufficient Condition. In addition to the above stated condition  $\hat{P}^*(t)$  exists  $\forall t \in [t_0, t_f]$  and

$$Z^T (F_{xx} + v^T \psi_{xx} - P^*) Z \Big|_{t_f} \geq 0$$

Corollary B1.1. If  $L_1$  and  $F$  are quadratic functions of  $x$ , if  $f_1$  and  $L_2$  are linear in  $x$ , if  $f_u$  is independent of  $x$  and if  $\psi$  is linear in  $x(t_f)$  then condition (i) of Theorem B1 is necessary, and together with the strengthened existence condition (ii) of Theorem B1 is sufficient, for  $V[u(\cdot)]$  to have a strong minimum at  $\bar{u}(\cdot)$ .

Corollary B1.2. [2] A sufficient condition for  $\delta^2 V^*[\delta u(\cdot)] \geq 0$  is that  $\exists \forall t \in [t_0, t_f]$  a real, symmetric, continuously differentiable matrix function of time  $P^*(\cdot)$  such that

$$H_{ux} + f_u^T P^* = 0 \quad \forall t \in [t_o, t_f] \quad (116)$$

$$\dot{P}^* + P^* f_x + f_x^T P^* + H_{xx} = M'(t) \geq 0, \quad \forall t \in [t_o, t_f] \quad (117)$$

$$Z^T (F_{xx} + v^T \psi_{xx} - P^*) Z \Big|_{t_f} \geq 0 \quad (118)$$

### 3. Useful Lemmas

Lemma 10. The second variation (104)-(106) is expressible in the equivalent canonical form

$$\delta^2 V^*[\delta u(\cdot)] = \delta^2 \hat{V}^*[\delta u(\cdot)] \triangleq \int_{t_o}^{t_f} \delta u^T C \delta y \, dt \quad (119)$$

subject to

$$\delta \dot{y} = B \delta u \quad ; \quad \delta y(t_o) = 0 \quad (120)$$

and

$$\psi_x \delta y \Big|_{t_f} = 0 \quad (121)$$

where C and B are given by (42), (43).

Proof. See Lemma 1. In addition we have (121) which follows from the fact that  $\Phi(t_f) = I$ , see (44).

Lemma 11. Condition (i) of Theorem B1 is equivalent to the existence  $\forall t \in (t_o, t_f)$  of a real symmetric monotone increasing matrix function of time  $\hat{P}^*(\cdot)$  such that

$$C + B^T \hat{P}^* = 0 \quad \forall t \in (t_o, t_f) \quad (122)$$

$$-Z^T \hat{P}^*(t_f^-) Z \geq 0 \quad (123)$$

Proof. See Lemma 2.

#### 4. A Related Nonsingular Second Variation

Consider the nonsingular variational problem

$$\delta^2 V_N^*[\delta u(\cdot), \epsilon] = \delta^2 \hat{V}^*[\delta u(\cdot)] + \frac{1}{2\epsilon} \int_{t_0}^{t_f} \delta u^T \delta u dt \quad (124)$$

$$= \int_{t_0}^{t_f} \left\{ \delta u^T C \delta y + \frac{1}{2\epsilon} \delta u^T \delta u \right\} dt \quad (125)$$

subject to

$$\delta \dot{y} = B \delta u \quad ; \quad \delta y(t_0) = 0 \quad (126)$$

and the linearized terminal constraint

$$\psi_x \delta y \Big|_{t_f} = 0 \quad . \quad (127)$$

Theorem 6. [Gelfand and Fomin [11, p. 123], Breakwell and Ho [15]].

If the quadratic functional  $\delta^2 \hat{V}^*[\delta u(\cdot)]$  is nonnegative then the matrix Riccati differential equation

$$-\dot{S}_\epsilon = -(C + B^T S_\epsilon)^T (C + B^T S_\epsilon) \epsilon \quad (128)$$

associated with  $\delta^2 V_N^*[\delta u(\cdot), \epsilon]$  has a solution  $S_\epsilon(t)$  which exists  $\forall t \in [t_0, t_f]$ . In a neighborhood of  $t_f$ ,  $S_\epsilon$  is given by [17]

$$S_\epsilon(t) = W_\epsilon(t) - N_\epsilon(t) M_\epsilon^{-1}(t) N_\epsilon^T(t) \quad (129)$$

where

$$-\dot{W}_\epsilon = -(C + B^T W_\epsilon)^T (C + B^T W_\epsilon) \epsilon \quad ; \quad W(t_f) = 0 \quad (130)$$

$$\dot{M}_\epsilon = N_\epsilon^T B B^T N_\epsilon \epsilon \quad ; \quad M_\epsilon(t_f) = 0 \quad (131)$$

and

$$\dot{N}_\epsilon = (C + B^T W_\epsilon)^T B^T N_\epsilon \epsilon \quad ; \quad N_\epsilon(t_f) = \psi_x^T \quad . \quad (132)$$

Note that  $M_\epsilon(t)$ ,  $t < t_f$ , is invertible by Assumption 3.

Theorem 7. If the matrix Riccati differential equation (128) has a solution  $S_\epsilon(\tau)$  which exists for all  $\tau \in [t_o, t_f]$  then

$$\begin{aligned} \min_{\delta u(\cdot)} \delta^2 V_N^*[\delta y(\tau), \delta u(\cdot), \epsilon, \tau] &\triangleq \delta^2 V^{*0}[\delta y(\tau), \epsilon, \tau] \\ &= \frac{1}{2} \delta y^T(\tau) S_\epsilon(\tau) \delta y(\tau) \end{aligned} \quad (133)$$

Proof. See [17, p. 183-184].

Lemma 12.  $S_\infty(t) \triangleq \lim_{\epsilon \rightarrow \infty} S_\epsilon(t)$  exists  $\forall t \in (t_o, t_f)$  and is a real symmetric monotone increasing matrix function of time such that

$$C + B^T S_\infty = 0 \quad \text{a. e. in } [t_o, t_f] \quad (134)$$

Proof. The proof is very similar to the proof of Lemmas 4-8 and so is not described here.

$$\text{Lemma 13. } \lim_{\epsilon \rightarrow \infty} \lim_{t \rightarrow t_f} Z^T S_\epsilon(t) Z = Z^T S_\infty(t_f) Z = 0 \quad (135)$$

where  $Z$  is defined in (110).

Proof. We have

$$S_\epsilon(t) = W_\epsilon(t) - N_\epsilon(t) M_\epsilon^{-1}(t) N_\epsilon^T(t) \quad (136)$$

Since  $N_\epsilon$  obeys a linear homogeneous differential equation we have

$$S_\epsilon(t) = W_\epsilon(t) - \psi_x^T \theta_\epsilon^T(t) M_\epsilon^{-1}(t) \theta_\epsilon(t) \psi_x \quad (137)$$

where

$$\dot{\theta}_\epsilon = (C + B^T W_\epsilon)^T B^T \theta_\epsilon \quad ; \quad \theta_\epsilon(t_f) = I \quad (138)$$

From (110), (137) it is clear that

$$Z^T S_\epsilon(t) Z = Z^T W_\epsilon(t) Z \quad (139)$$

because

$$\psi_x^T Z \equiv 0 \quad (140)$$

Thus, from (139),

$$Z^T S_\infty(t_f) Z = Z^T W_\infty(t_f) Z = 0 \quad (141)$$

since

$$W_{\epsilon}(t_f) = 0, \quad \forall \epsilon \quad (142)$$

Lemma 14.

$$Z^T S_{\infty}(t_f^-) Z \leq 0 \quad . \quad (143)$$

Proof. From Lemma 12  $S_{\infty}(t)$  is monotone increasing. Lemma 14 then follows from Lemma 13.

### 5. Proof of Theorem B1.

(i) Necessary Condition. If Assumptions 1 through 7 are satisfied and if  $\delta^2 \hat{V} [\delta u(\cdot)] \geq 0$  for all admissible  $\delta u(\cdot)$  then by Lemmas 12-14  $\exists \forall t \in (t_o, t_f)$  a real symmetric monotone increasing matrix function of time such that

$$C + B^T S_{\infty} = 0 \quad \text{a.e. in } [t_o, t_f] \quad (144)$$

$$Z^T S_{\infty}(t_f^-) Z \leq 0 \quad (145)$$

Defining

$$\hat{P}^*(t) = S_{\infty}(t) \quad , \quad t \in (t_o, t_f) \quad (146)$$

yields

$$\hat{P}^*(t_f^-) = S_{\infty}(t_f^-) \quad (147)$$

and

$$C + B^T \hat{P}^* = 0 \quad \text{a.e. in } [t_o, t_f] \quad , \quad (148)$$

$$Z^T \hat{P}^*(t_f^-) Z \leq 0 \quad . \quad (149)$$

Using the same argument on (148) as was used in Section II. 5 on equation (93) shows that (148), (149) imply (122), (123) which by Lemma 11 imply the necessary conditions of Theorem B1.

(ii) Sufficient Condition. Suppose that (122) is satisfied by a  $\hat{P}^*(t)$  which exists  $\forall t \in [t_o, t_f]$  (strengthened existence condition) such that  $-Z^T \hat{P}^* Z \Big|_{t_f} \geq 0$ .

Adjoin the linearized dynamics to the canonical second variation as follows

$$\delta^2 \hat{V}^*[\delta u(\cdot)] = \delta^2 \tilde{V}^*[\delta u(\cdot)] \triangleq \int_{t_0}^{t_f} \{ \delta u^T C \delta y + \delta y^T \hat{P}^* (B \delta u - \delta \dot{y}) \} dt \quad (150)$$

$$= - \int_{t_0}^{t_f} \delta y^T \hat{P}^* \delta \dot{y} dt \quad (151)$$

because of (122). Integrating (151) by parts yields

$$\delta^2 \hat{V}^*[\delta u(\cdot)] = \int_{t_0}^{t_f} \frac{1}{2} \delta y^T d\hat{P}^*(t) \delta y - \frac{1}{2} \delta y^T \hat{P}^* \delta y \Big|_{t_f} \quad (152)$$

where the integral is in the Stieltjes sense.

Using (110) and the fact that  $\delta x(t_f) = \delta y(t_f)$  yields

$$\delta^2 \hat{V}^*[\delta u(\cdot)] = \int_{t_0}^{t_f} \frac{1}{2} \delta y^T d\hat{P}^*(t) \delta y - \frac{1}{2} [\delta y^{n-s}]^T Z^T \hat{P}^* Z [\delta y^{n-s}] \Big|_{t_f} \quad (153)$$

Since  $\hat{P}^*(t)$  is monotone increasing the integral is nonnegative and since  $-Z^T \hat{P}^* Z \Big|_{t_f} \geq 0$  the boundary term is nonnegative. This concludes the

sufficiency proof.

Note. This sufficient condition for the constrained terminal state problem may appear to be stringent because a  $\hat{P}^*(t_f)$  is required to exist, but  $S_{\infty}(t_f)$  is undefined. However, as in the free terminal state case it is the authors' opinion that the gap between the necessary and the sufficient condition is minimal. This is supported by the following (slightly altered) sufficiency theorem due to Brockett [18, p. 140] (a proof of necessity of this theorem is not given in [18] but appears to be straightforward).

Theorem 7. [c.f. Theorem 6]. If there exists a symmetric boundary condition  $S_\epsilon(t_f)$  such that the solution of (128) exists  $\forall t \in [t_o, t_f]$  then  $\delta^2 V_N^*[\delta u(\cdot), \epsilon]$  is positive-definite.<sup>†</sup>

Of course the solution of (128) with finite boundary condition is not the same as the weighting matrix (136) of the quadratic optimal value function. This is consistent with the fact that  $\hat{P}^*(t_f)$  in our sufficient condition cannot be identified with  $S_\infty(t_f)$ ; i.e., in our sufficient condition for the fixed terminal state problem we lose the identification of  $\hat{P}^*(\cdot)$  being the limit of the weighting matrix of the quadratic optimal value function as  $\epsilon \rightarrow \infty$ .

#### IV. Relation to Existing Necessary Conditions of Optimality

Known necessary conditions for singular problems can be deduced from our theorems. Here we give the most important ones.

Theorem C1. [Robbins [7], Goh [10]]. Under Assumptions 1 through 5 a necessary condition for  $V[u(\cdot)]$  to have a minimum at  $\bar{u}(\cdot)$  is that  $H_{ux} f_u$  be symmetric for all  $t \in [t_o, t_f]$ .

Proof. From (32)

$$H_{ux} f_u + f_u^T P f_u = 0 \quad \forall t \in (t_o, t_f] \quad (154)$$

Since  $P$  is symmetric it follows that  $H_{ux} f_u$  is symmetric in  $(t_o, t_f]$ .

Assumption 2 implies that indeed  $H_{ux} f_u$  is symmetric in  $[t_o, t_f]$ .

Theorem C2. [Jacobson [1]]. Under Assumptions 1 through 5 a necessary condition for  $V[u(\cdot)]$  to have a minimum (unconstrained terminal state case) at  $\bar{u}(\cdot)$  is that

$$(H_{ux} + f_u^T Q) f_u \geq 0 \quad \forall t \in [t_o, t_f] \quad (155)$$

where  $Q$  satisfies (36).

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<sup>†</sup>Brockett treats the case where  $\psi_x = I$ .



Proof. From (32) and (34)

$$H_{ux} f_u + f_u^T Q f_u + f_u^T \Phi^T \hat{P} \Phi f_u = 0 \quad \forall t \in (t_o, t_f] \quad (156)$$

Since  $\hat{P}(t) \leq 0 \quad \forall t \in [t_o, t_f]$  it follows from (156) that

$$(H_{ux} + f_u^T Q) f_u \geq 0 \quad \forall t \in (t_o, t_f] \quad (157)$$

Since  $H_{ux}$ ,  $f_u$ ,  $Q$  are continuous in time (155) follows.

Theorem C3. [Kelley [6], Robbins [7] generalized Legendre-Clebsch condition]. Under Assumptions 1 through 5 a necessary condition for  $V[u(\cdot)]$  to have a minimum at  $\bar{u}(\cdot)$  is that

$$(-1) \frac{\partial}{\partial u} \ddot{H}_u \geq 0 \quad \forall t \in [t_o, t_f] \quad (158)$$

Proof. From (32), and post multiplying by  $f_u$ , we obtain<sup>†</sup>

$$[(\dot{H}_{ux} + \dot{f}_u^T P) f_u + (H_{ux} + f_u^T P) \dot{f}_u] dt + f_u^T dP f_u = 0 \quad (159)$$

where, from (34)-(36)

$$dP = (-H_{xx} - f_x^T P - P f_x) dt + \Phi^T d\hat{P} \Phi \quad (160)$$

Using (32) and (160) in (159) yields

$$\begin{aligned} & (\dot{H}_{ux} f_u + H_{ux} \dot{f}_u - \dot{f}_u^T H_{xu} - H_{ux} \dot{f}_u - f_u^T H_{xx} f_u + f_u^T f_x^T H_{xu} \\ & + H_{ux} f_x f_u) dt = -f_u^T \Phi^T d\hat{P} \Phi f_u \end{aligned} \quad (161)$$

The left hand side of (161) is just  $(\frac{\partial}{\partial u} \ddot{H}_u) dt$ . Since  $\hat{P}$  is monotone increasing in  $t$  we obtain from (161)

$$(-1) \frac{\partial}{\partial u} \ddot{H}_u \geq 0 \quad \forall t \in (t_o, t_f] \quad (162)$$

By the assumed continuity of  $\frac{\partial}{\partial u} \ddot{H}_u$  (162) implies

$$(-1) \frac{\partial}{\partial u} \ddot{H}_u \geq 0 \quad \forall t \in [t_o, t_f] \quad (163)$$

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<sup>†</sup>Here  $dP$  is the increment in  $P$  in time  $dt$ .

Note. If  $H_{xx}$ ,  $f_x$ ,  $f_u$ ,  $H_{ux}$  are assumed smooth in  $t$  then the general form of the generalized Legendre-Clebsch condition can be deduced in the manner outlined above

$$(-1)^p \frac{\partial}{\partial u} \left[ \frac{d^{2p}}{dt^{2p}} H_u \right] \geq 0 \quad (164)$$

#### V. Relation to Generalized Jacobi (Riccati equation) Necessary Condition

In [5] Kelley's transformation technique is used to transform the singular second variation into a nonsingular one in a reduced state space. A condition for this to succeed is that (163) be satisfied with strict inequality. It is then shown that the Riccati differential equation associated with this nonsingular problem implies the conditions of Corollary A1.2 i.e., these conditions are necessary as well as sufficient for a large class of problems. (It should be possible to prove this via our limit approach; one would only have to show that  $S_{\infty}(t)$  is continuously differentiable with respect to  $t$ ,  $t \in (t_0, t_f)$ .)

In [5] sufficient conditions are given to ensure that Theorem A2 holds. We state one set of these conditions here;

Theorem D1. If the conditions of Corollary A1.2 are satisfied then the following are sufficient to ensure that

$$\delta^2 V[\delta u(\cdot)] \geq 0 \implies V[\bar{u}(\cdot) + \delta u(\cdot)] \geq V[\bar{u}(\cdot)] \quad (165)$$

a) Strengthened Generalized Legendre-Clebsch Condition

$$-\frac{\partial}{\partial u} \ddot{H}_u > 0 \quad \forall t \in [t_0, t_f] \quad (166)$$

b) Strengthened Jacobson Condition at the Terminal Time

$$(H_{ux} + f_u^T F_{xx}) f_u \Big|_{t_f} > 0 \quad (167)$$

In fact (166), (167) together with the conditions of Corollary A1.2 are sufficient to ensure that the transformed second variation is strongly positive with respect to the control variable in the transformed space.

## VI. Conclusion

Necessary and sufficient conditions for optimality for singular control problems are obtained by studying the limiting behavior of a nonsingular second variation. This nonsingular second variation is constructed in such a way that it tends to the singular second variation as a parameter approaches infinity. Optimality conditions for both unconstrained and constrained terminal-state problems are obtained.

The optimality conditions derived in this paper are very similar to certain sufficient conditions of Jacobson [2].<sup>†</sup> In a companion paper [5] it is shown that Jacobson's sufficient conditions are also necessary for a large class of singular optimal control problems; moreover, satisfaction of these conditions is shown to be equivalent to the existence of a solution of a certain matrix Riccati differential equation.

The closing sections of the present paper relate the necessary and sufficient conditions to known necessary conditions. In particular the important necessary conditions of Robbins [7], Goh [8], Kelley et al. [6] and Jacobson [1] follow easily from these new results.

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<sup>†</sup>Control problem examples which illustrate the use of these conditions are given in [2].

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