NONLINEAR FILTERING TECHNIQUES WITH APPLICATIONS TO THE STRAPDOWN ATTITUDE COMPUTATION

Technical Report No. 3

By

J. S. Lee

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ABSTRACT

This report† is devoted to the solution of a special class of nonlinear filtering problems and to its applications to the attitude computational problem in a strapdown inertial navigation system.

A special class of nonlinear filtering problems is formulated in chapter one to study the attitude computational algorithms in a strapdown inertial navigation system. It is shown that the characteristics of this particular mathematical model allows a direct iterative relation be obtained between the successive estimates of the states. A series of filters are defined and algorithms are derived according to the computational load allowed in a real time application.

Chapter two introduce the practical problem of the strapdown inertial navigation system. Monte Carlo simulation results are presented giving comparisons between the use of the derived algorithms and a commonly used algorithm (the Taylor series second order algorithm*).

† This report is a part of a thesis submitted by the author July 1969 to Harvard University in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

* This second order algorithm is a currently used attitude updating algorithm (see section 2.4). It is obtained in a deterministic manner without taking the statistics of the random environment into consideration. To avoid confusion with the second order nonlinear filter to be derived in chapter one, we shall refer to this second order algorithm as Algorithm A.

Chapter three considers the attitude computation algorithms from a more restricted and practical point of view. The approach is to impose a complexity constraint which, in essence, restricts the filtering algorithm to a particular form with a set of parameters which are, then, chosen optimally to minimize an error criterion (the mean square error). Explicit equations are given for the single axis rotation case. The numerical results for the single axis case show that substantial improvement in performance can be obtained over the Taylor series second order algorithm (refer to the footnote of last page), especially in the case where the statistical angular process is biased.

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LIST OF SYMBOLS

Symbol		Page (defined)
x_i	n-state vector to be estimated	3
α_i	m-vector, random forcing function	3
Φ	m x m transition matrix	3
w_i	m-Gaussian vector (white noise)	3
$\overline{w}_i$	$E(w_i)$	3
Q_i	$Cov(w_i)$	3
z_i	k-measurement vector	4
H	k x m matrix associated with the measurement z_i	4
ϵ_i	k-vector, representing Gaussian and white noise	4
R_i	$Cov(\epsilon_i)$	4
Z_i	set of measurements $(z_1, z_2 \text{-----}, z_i)$	4
$\hat{x}_i$	conditional mean of x_i given Z_i	4
$\hat{\alpha}_i$	conditional mean of α_i given Z_i	7
P_{α_i}	$E((\alpha_i - \hat{\alpha}_i)(\alpha_i - \hat{\alpha}_i)^T / Z_i)$	7
$\alpha_{i/i+1}$	smoothed estimate of α_i given Z_{i+1}	8
$K_{i/i+1}$	smoothed gain matrix	9
$P_{i/i+1}$	n x n matrix	9
R_i'	$(= HQ_iH^T + R_i)$	9
H_i'	$(= H\Phi)$	9
$P_{x_i\alpha_i}$	n x m matrix	10
K_i	(as defined by Eq. (1-32))	11

Symbol		Page (defined)
A_i	$(= \Phi - K_{i+1}\dot{H}\Phi)$	11
P_{x_i}	n x n matrix, $E((x_i - \hat{x}_i)(x_i - \hat{x}_i)^T / Z_i)$	13
$T(t)$	3 x 3 direction cosine matrix	21
$\Omega(t)$	3 x 3 skew-symmetrical matrix	21
$\omega(j)$	scalar, body angular rate about jth axis	21
a_I	a vector in inertial coordinates	21
a_b	a vector in body coordinates	21
γ_i	$(= \int_{t_i}^{t_{i+1}} \Omega(t)\, dt)$	24
$\theta_i(j)$	angular increment around jth axis in ith sampling interval	24
η_{i+1}	(defined by Eq. (2-9))	26
C_i	(defined by Eq. (2-19))	30
J_i	scalar, mean square error of estimating T_i	30
$a_i(j)$	weighting coefficient to be chosen optimally-scalar	45
J	cumulative mean square error of $(T_i - \hat{T}_i)$ (scalar)	46
M	scalar, $E(\theta_i(3))$	47
Q	scalar, $\mathrm{Cov}(\theta_i(3))$	47

NOTATION

Matrices will be denoted by upper case Roman letters. The vectors will be denoted by lower case Roman or Greek letters. All vectors are considered to be the column vectors. The row vectors are indicated by the superscript T which stands for transposition. Scalars will be explicitly mentioned. For example, the performance index J is scalar. The element of a matrix A is denoted by a(i, j) and the column vector associated with the matrix A is denoted by A(k). The element of a vector y will be denoted by y(k). The state x at time t_i will be denoted by x_i. δ_{ij} will represent the Kronecker Delta, whereas, $\delta(t - \tau)$ will denote Dirac Delta.

The following symbols will be used to represent probability density function and statistical evaluation of a random function;

p(x) = probability density function of x

p(x/Z) = conditional probability density function of x given the information Z

E( ) = expectation operator

Cov( ) = covariance operator

E(x/Z) = conditional mean of x given Z

CHAPTER ONE

A CLASS OF NONLINEAR FILTERING PROBLEMS

1.1 Introduction

Recursive computation techniques for linear filters as derived by Kalman [1] and others are well-known. The techniques have been successfully applied to many problems; for example, those related to orbit determination, attitude control and process control fields. In contrast to this very little work has been done in applying some of the theoretical work in nonlinear filtering techniques that have been developed during the past few years.

There are basically two approachs in attempting to solve problems in nonlinear filtering. One approach is the use of a linearized model about an estimated state and the successive use of the linear filtering techniques [2, Section 12.6]. The other approach requires the use of the conditional probability density function [3][4]. It has been shown by many that for the continuous signal and measurement case, the conditional probability density function satisfies a partial differential equation [5]. Similar equations have been derived for the case of continuous dynamics with discrete measurements [6]. In all cases, analytic solutions are difficult to obtain. In general, approximations must be made. Examples of the approximation techniques include the use of a Taylor series expansion [6], moment generating sequences [7], and the use of approximate Gaussian distribution functions [4]. There has been very little work on the application of these algorithms to a reasonable problem of practical interest. It is of interest to note that the results of a Monte Carlo simulations of eight different nonlinear filtering algorithms [8] indicate that no one particular approximation scheme is uniformly better than any other.

Most of the nonlinear recursive algorithms were intended for use with very general nonlinear dynamic and measurement models, as such, the algorithms were extremely complex. Moreover, almost all derivations approached the problems in two steps: (1) the propagation and prediction procedure based on all previous measurements, and (2) the procedure of incorporating an additional piece of data, the present measurement. In general, approximation must be used in both steps.

This chapter considers a special class of nonlinear filtering problems where the dynamics are of a special form and where the measurements are linear. This particular model has many practical applications and was motivated from the consideration of a practical problem involving the attitude computation of a strapdown inertial navigation system* [9].

In view of the special characteristics of the mathematical model, it was found that a direct iterative relation between the successive estimates of the state can be developed. In other words, it is no longer necessary to consider the two step approximation as described in the previous paragraph. The net result is a more efficient use of the information at every step.

The problem is formulated mathematically in Section 1.2. The derivation of the optimal filter is given in Section 1.3. Extention to more general filtering problems is considered in Section 1.4.

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* The details concerning the applications to the strapdown navigation system are given in Chapter two.

1.2 Formulation of the problem

The special class of nonlinear filtering problems to be considered in this Chapter can be formulated as follows:

Given:

a) The nonlinear dynamics

$$x_{i+1} = f(\alpha_i, x_i) \tag{1-1}$$

where

x_i = n-vector representing the state

α_i = m-vector forcing function

$$E(x_o) = \bar{x}_o$$
$$\mathrm{Cov}(x_o) = P_{x_o} \tag{1-2}$$

b) The dynamic equation governing the forcing function

$$\alpha_{i+1} = \Phi\alpha_i + w_i \tag{1-3}$$

where

Φ = a known m x m transition matrix

w_i = m-Gaussian vector with

$$E(w_i) = \bar{w}_i$$
$$E((w_i - \bar{w}_i)(w_j - \bar{w}_j)^T) = Q_i\delta_{ij} \tag{1-4}$$

c) Measurements

$$z_i = H\alpha_i + \epsilon_i \tag{1-5}$$

where

z_i = a k-vector representing physical data

H = k x m transformation matrix

ϵ_i = k-Gaussian vector with

$$E(\epsilon_i) = 0$$
$$E(\epsilon_i \epsilon_i^T) = R_i \, \delta_{ij} \tag{1-6}$$

The Problem

To find the best estimate of x_i, denoted by $\hat{x}_i$, given measurements $z_j (j = 1, 2, \ldots, i)$ in the sense that $\hat{x}_i$ is the conditional mean of x_i given all the data up to i. The reason for taking the conditional mean of x_i is that the conditional mean has been proved [10, pp. 217] to be the optimal estimator in the sense of minimizing the mean square error. Fig. 1.1 shows a block diagram describing the above mathematical formulation.

1.3 Filter Structure

Let $Z_{i+1} = (z_1, z_2, z_3, ------, z_{i+1})$ represent the set of measurements up to i+1. The problem is then to evaluate the conditional mean of x_{i+1}, that is

$$\hat{x}_{i+1} = E(x_{i+1}/Z_{i+1}) \tag{1-7}$$

We proceed by deriving an expression for the conditional probability density function $p(x_{i+1}, \alpha_{i+1}/Z_{i+1})$. Using the fact that

$$p(a/b) = \int p(a/b, c) \; p(c/b) \; dc$$

we find

$$p(x_{i+1}, \alpha_{i+1}/Z_{i+1}) = \int\int p(x_{i+1}, \alpha_{i+1}/\alpha_i, x_i, Z_{i+1}) \; p(x_i, \alpha_i/Z_{i+1}) \; dx_i \; d\alpha_i \tag{1-8}$$

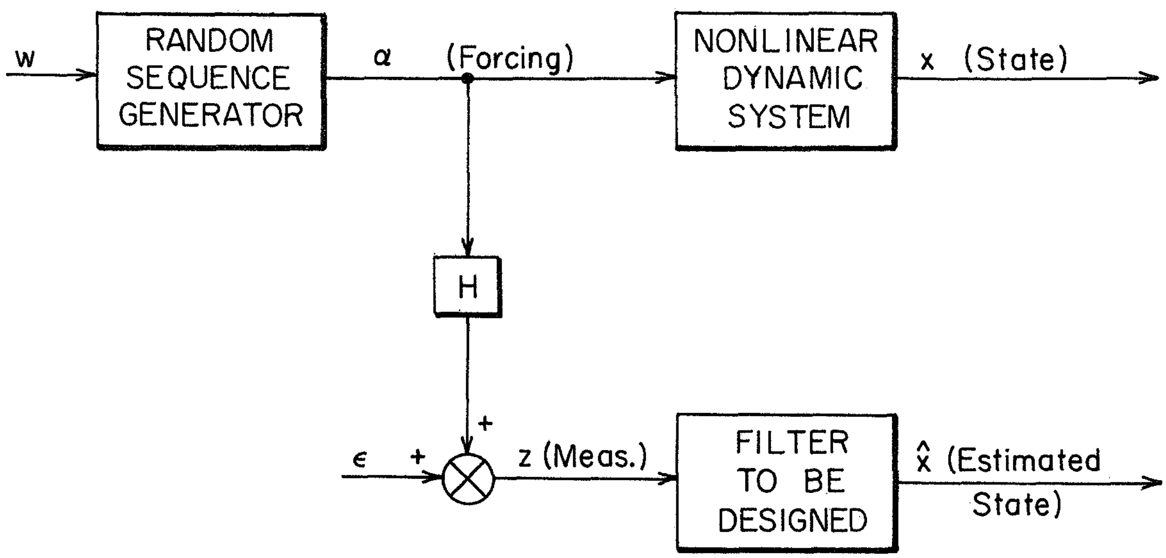

FIG. 1.1 BLOCK DIAGRAM OF THE FILTERING PROBLEM

where the integration is over the m-vector α_i and the n-vector x_i. Now, by applying Bayes formula, we have

$$p(x_{i+1}, \alpha_{i+1}/\alpha_i, x_i, Z_{i+1}) = p(x_{i+1}/\alpha_{i+1}, \alpha_i, x_i, Z_{i+1})\ p(\alpha_{i+1}/\alpha_i, x_i, Z_{i+1})$$

$$= \delta(x_{i+1} - f(\alpha_i, x_i))\ p(\alpha_{i+1}/\alpha_i, Z_{i+1}) \qquad (1\text{-}9)$$

where

$$p(x_{i+1}/\alpha_{i+1}, \alpha_i, x_i, Z_{i+1}) = p(x_{i+1}/\alpha_i, x_i) = \delta(x_{i+1} - f(\alpha_i, x_i))$$

Furthermore, in (1-8),

$$p(x_i, \alpha_i/Z_{i+1}) = p(x_i/\alpha_i, Z_{i+1})\ p(\alpha_i/Z_{i+1}) \qquad (1\text{-}10)$$

The first term of the right hand side of (1-10) can further be decomposed as

$$p(x_i/\alpha_i, Z_{i+1}) = \frac{p(z_{i+1}, x_i/\alpha_i, Z_i)}{p(z_{i+1}/\alpha_i, Z_i)}$$

$$= \frac{p(x_i/\alpha_i, Z_i)\ p(z_{i+1}/x_i, \alpha_i, Z_i)}{p(z_{i+1}/\alpha_i, Z_i)}$$

$$= p(x_i/\alpha_i, Z_i) \qquad (1\text{-}11)$$

since the density function $p(z_{i+1}/x_i, \alpha_i, Z_i)$ is the same as $p(z_{i+1}/\alpha_i, Z_i)$. Combining (1-11) and (1-10) and substituting the remaining and (1-9) into (1-8) give

$$p(x_{i+1}, \alpha_{i+1}/Z_{i+1}) = \int\int \delta(x_{i+1} - f(\alpha_i, x_i))\ p(\alpha_{i+1}/Z_{i+1}, \alpha_i)$$

$$\cdot\ p(x_i/\alpha_i, Z_i)\ p(\alpha_i/Z_{i+1})\ d\alpha_i\ dx_i$$

$$(1\text{-}12)$$

It follows that

$$p(x_{i+1}/Z_{i+1}) = \int\int \delta(x_{i+1} - f(\alpha_i, x_i))\, p(x_i/\alpha_i, Z_i)\, p(\alpha_i/Z_{i+1})\, d\alpha_i\, dx_i \qquad (1\text{-}13)$$

and by definition, we have

$$\hat{x}_{i+1} = \int x_{i+1}\, p(x_{i+1}/Z_{i+1})\, dx_{i+1} \qquad (1\text{-}14)$$

To obtain an explicit expression for the nonlinear filter for an arbitrary $f(\alpha_i, x_i)$, approximations must be made. The approximation technique is based on expanding $f(\alpha_i, x_i)$ in Taylor series around $\hat{\alpha}_i$ and $\hat{x}_i$, where

$$\hat{\alpha}_i = E(\alpha_i/Z_i) \qquad (1\text{-}15)$$

Direct observation of Eq. (1-3) and (1-5) shows that $\hat{\alpha}_{i+1}$ satisfies the solution of the well known Kalman filter. In other words

$$\hat{\alpha}_{i+1} = \Phi\, \hat{\alpha}_i + P_{\alpha_{i+1}} H^T R_i^{-1}(z_{i+1} - H\Phi\, \hat{\alpha}_i) + \overline{w}_i \;; \quad \hat{\alpha}_o = \overline{\alpha}_o \qquad (1\text{-}16)$$

where P_{α_i} is the standard m x m covariance matrix satisfying the well known discrete matrix Riccati Equation.

Consider now the problem of finding $\hat{x}_{i+1}$. From (1-13) and (1-14), we have

$$\boxed{\hat{x}_{i+1} = \int\int f(\alpha_i, x_i)\, p(\alpha_i/Z_{i+1})\, p(x_i/\alpha_i, Z_i)\, d\alpha_i\, dx_i} \qquad (1\text{-}17)$$

Only in very rare cases can an exact solution of Eq. (1-17) be obtained. In general, approximation must be used. One approximation scheme is the Taylor series expansion.

Expanding $f(\alpha_i, x_i)$ into Taylor series around $\hat{\alpha}_i$ and $\hat{x}_i$*, yields

$$f(\alpha_i, x_i) = f(\alpha_i, x_i)\Big|_{\hat{\alpha}_i, \hat{x}_i} + \frac{\partial f}{\partial \alpha_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} (\alpha_i - \hat{\alpha}_i)$$

$$+ \frac{\partial f}{\partial x_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} (x_i - \hat{x}_i) + \text{(higher order terms)} \qquad (1\text{-}18)$$

A hierarchy of filters can be defined depending on the order of the expansion. For example, the zeroth order filter is defined by retaining only the first term. In which case, we find after substituting $f(\hat{\alpha}_i, \hat{x}_i)$ in place of $f(\alpha_i, x_i)$ in (1-17) that

$$\hat{x}_{i+1} = f(\hat{\alpha}_i, \hat{x}_i) \qquad (1\text{-}19)$$

Consider now the first order filter. It consists of using the first three terms of (1-18). The first term is, obviously, $f(\hat{\alpha}_i, \hat{x}_i)$. The second term is

$$\text{(The 2nd term)} = \iint \frac{\partial f}{\partial \alpha_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} (\alpha_i - \hat{\alpha}_i)\, p(\alpha_i / Z_{i+1})\, p(x_i / \alpha_i, Z_i)\, d\alpha_i\, dx_i$$

$$= \frac{\partial f}{\partial \alpha_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} \int (\alpha_i - \hat{\alpha}_i)\, p(\alpha_i / Z_{i+1})\, d\alpha_i \qquad (1\text{-}20)$$

Let

$$\hat{\alpha}_{i/i+1} = \int \alpha_i\, p(\alpha_i / Z_{i+1})\, d\alpha_i \qquad (1\text{-}21)$$

---

* Based on numerical results in Section 2.3, the approximation of $f(\alpha_i, x_i)$ by its Taylor series expansion as given by (1-18) proves to be a good one, at least in some problems.

Then $\hat{\alpha}_{i/i+1}$ is the smoothed estimate of α_i; namely, the estimate based on data $z_1, -----, z_{i+1}$. It follows that

$$\text{(The 2nd term)} = \frac{\partial f}{\partial \alpha_i}\bigg|_{\hat{\alpha}_i, \hat{x}_i} (\hat{\alpha}_{i/i+1} - \hat{\alpha}_i) \tag{1-22}$$

The smoothed estimate $\hat{\alpha}_{i/i+1}$ is known [11] to satisfy the following equation.

$$\begin{aligned} \hat{\alpha}_{i/i+1} &= \hat{\alpha}_i + K_{i/i+1}(z_{i+1} - H\Phi\,\hat{\alpha}_i) \\ K_{i/i+1} &= P_{i/i+1}\, H_i^{'T} R_i^{'-1} \end{aligned} \tag{1-23}$$

with

$$P_{i/i+1} = P_{\alpha_i} - P_{\alpha_i} H_i^{'T} (H_i' P_{\alpha_i} H_i^{'T} + R_i')^{-1} H_i P_{\alpha_i}$$

$$R_i' = H Q_i H^T + R_i$$

$$H_i' = H\Phi$$

Substituting (1-23) into (1-22) yields

$$\text{(The 2nd term)} = \frac{\partial f}{\partial \alpha_i}\bigg|_{\hat{\alpha}_i, \hat{x}_i} K_{i/i+1} (z_{i+1} - H\,\Phi\,\hat{\alpha}_i) \tag{1-24}$$

It turns out that this additional correction term of (1-24) can be easily incorporated to (1-19). $K_{i/i+1}$ can be precomputed and stored and therefore is applicable for real time application. Numerical results of Section 2.3 seem to indicate that significant improvement can be obtained by the addition of this term.

Let us now consider the contribution of the third term in the expansion of (1-18). Using (1-17), we see that

$$(\text{The 3rd term}) = \frac{\partial f}{\partial x_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} \int\int (x_i - \hat{x}_i)\, p(x_i/\alpha_i, Z_i)\, p(\alpha_i/Z_{i+1})\, dx_i\, d\alpha_i$$

$$= \frac{\partial f}{\partial x_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} \int p(\alpha_i/Z_{i+1}) \left[\int (x_i - \hat{x}_i)\, p(x_i/\alpha_i, Z_i)\, dx_i\right] d\alpha_i \tag{1-25}$$

To evaluate the inner integral on the right hand side of (1-25), it is necessary to make certain assumptions concerning $p(x_i/\alpha_i, Z_i)$. A reasonable assumption appears to be that the random vectors x_i, α_i given Z_i are jointly Gaussian distributed with mean $\hat{x}_i, \hat{\alpha}_i$ and conditional covariances P_{x_i}, P_{α_i} and $P_{x_i\alpha_i}$*. Based on this assumption, it follows that

$$\int (x_i - \hat{x}_i)\, p(x_i/\alpha_i, Z_i)\, dx_i = P_{x_i\alpha_i} P_{\alpha_i}^{-1} (\alpha_i - \hat{\alpha}_i) \tag{1-26}$$

where

$$P_{x_i\alpha_i} = E((x_i - x_i)(\alpha_i - \hat{\alpha}_i)^T / Z_i) \tag{1-27}$$

substituting (1-26) into (1-25) and using (1-23) shows that

$$(\text{The 3rd term}) = \frac{\partial f}{\partial x_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} P_{x_i\alpha_i} P_{\alpha_i}^{-1} K_{i/i+1} (z_{i+1} - H\Phi\hat{\alpha}_i) \tag{1-28}$$

Combining (1-28), (1-24) and (1-19), it is seen that the first order filter can be written as

$$\hat{x}_{i+1} = f(\hat{\alpha}_i, \hat{x}_i)$$

$$+ \left(\frac{\partial f}{\partial \alpha_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} + \frac{\partial f}{\partial x_i}\Big|_{\hat{\alpha}_i, \hat{x}_i} P_{x_i\alpha_i} P_{\alpha_i}^{-1}\right) \cdot K_{i/i+1} (z_{i+1} - H\Phi\hat{\alpha}_i)$$

---

* See Comments at the end of this section.

The only unknown in (1-29) is $P_{x_i\alpha_i}$. An iterative relation for updating $P_{x_i\alpha_i}$ can be derived. The derivation follows essentially the same procedure as that leading to (1-29). Although the derivation is tedious, it is nevertheless straightforward (refer to Appendix A). The final expression is

$$P_{x_{i+1}\alpha_{i+1}} = \left(\frac{\partial f}{\partial \alpha_i}\bigg|_{\hat{\alpha}_i, \hat{x}_i} P_{i/i+1} + \frac{\partial f}{\partial x_i}\bigg|_{\hat{\alpha}_i, \hat{x}_i} P_{x_i\alpha_i} P_{\alpha_i}^{-1} P_{i/i+1}\right) A_i^T \tag{1-30}$$

where

$$A_i^T = \Phi - K_{i+1} H \Phi \tag{1-31}$$

and

$$K_{i+1} = (Q_i^{-1} + H^T R_{i+1}^{-1} H)^{-1} H^T R_i^{-1} \tag{1-32}$$

Comments:

1. In evaluating Eq. (1-25), we assume that $p(x_i, \alpha_i/Z_i)$ is Gaussian distributed. This assumption is not so drastic as it sounds. Taking a close look of (1-26), we find that it is only the conditional mean of x_i is of interest. The conditional mean of x_i is evaluated by

$$\int x_i \, p(x_i/\alpha_i, Z_i) \, dx_i = \int x_i \left[p(x_i, \alpha_i/Z_i)/p(\alpha_i/Z_i)\right] dx_i$$

$$= \frac{1}{p(\alpha_i/Z_i)} \int x_i \, p(x_i, \alpha_i/Z_i) \, dx_i$$

We have approximated $p(x_i, \alpha_i/Z_i)$ by a Gaussian density function $p^*(x_i, \alpha_i/Z_i)$ with mean and variance of $p^*(x_i, \alpha_i/Z_i)$ equivalent to the mean and variance of $p(x_i, \alpha_i/Z_i)$. This Gaussian approximation makes $p^*(x_i/\alpha_i, Z_i)$ a Gaussian density function, since $p(\alpha_i/Z_i)$ is Gaussian distributed. It is shown in Fig. 1.2

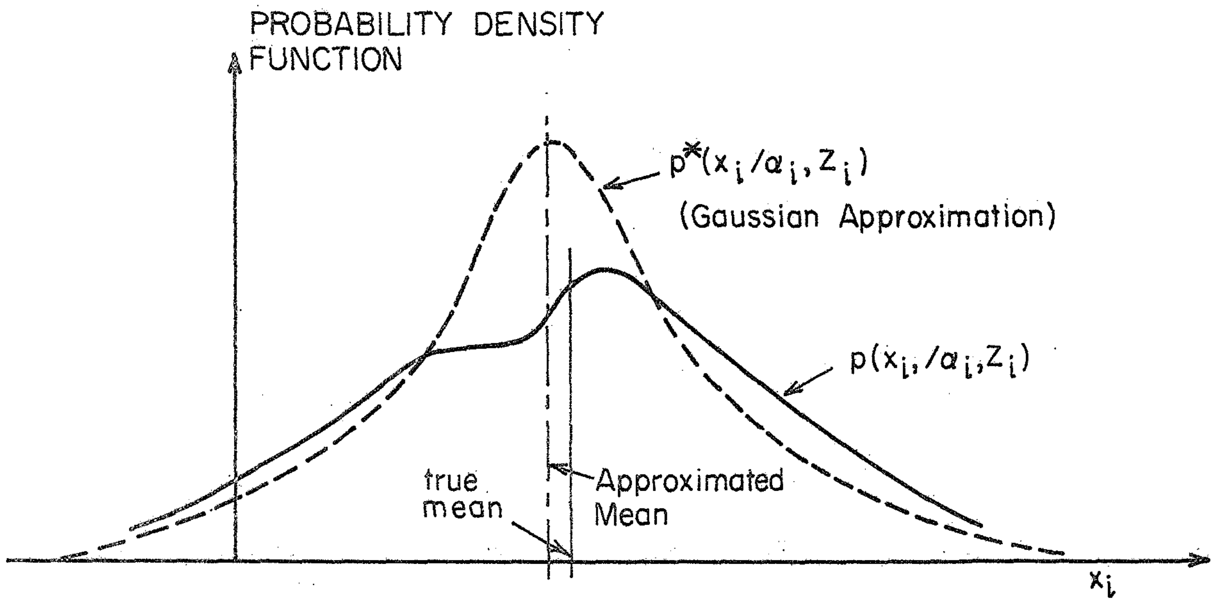

FIG. 1.2 GAUSSIAN APPROXIMATION OF A DENSITY FUNCTION

that $p(x_i/\alpha_i, Z_i)$ may not be close to its Gaussian approximation, but as far as the value of the integral is concerned, the integration tends to smooth out the irregularity of $p(x_i/\alpha_i, Z_i)$ and makes the approximated conditional mean of x_i close to the true conditional mean of x_i.

2. Recursive relations for the zeroth order filter and the first order filter are given by Eqs. (1-19) and (1-29) respectively. The computation of Eq. (1-19) require the knowledge of $\hat{\alpha}_i$ which is given by (1-15) and (1-16). The computation of Eq. (1-29) requires $\hat{\alpha}_i$ as well as $P_{i/i+1}$, $P_{x_i\alpha_i}$ and $K_{i/i+1}$ which can be obtained from (1-23) and (1-30).

3. P_{x_i} which is the covariance of the error in the estimate is not needed in any of these recursive algorithms.

4. Higher order filters can be derived using the same approximation. Eq. (1-33) shows the form of a second order filter. The second order filter is defined by using up to the second order terms in the expansion of $f(\alpha_i, x_i)$. The derivation is straightforward but tedious hence it is omitted.

$$
\begin{aligned}
x_{i+1} = &\,(\text{zero order term}) + (\text{first order term}) \\
&+ \frac{1}{2}\sum_{j,k} \left.\frac{\partial^2 f}{\partial\alpha_i(j)\,\partial\alpha_i(k)}\right|_{\hat{\alpha}_i,\hat{x}_i} \chi_i(j,k) \\
&+ \sum_{j,k} \left.\frac{\partial^2 f}{\partial\alpha_i(j)\,\partial x_i(k)}\right|_{\hat{\alpha}_i,\hat{x}_i} \sum_{\ell,m} P_{x_i\alpha_i}(j,\ell)\, P^{-1}_{\alpha_i}(\ell,m)\, \chi_i(m,k) \\
&+ \frac{1}{2}\sum_{j,k} \left.\frac{\partial^2 f}{\partial x_i(j)\,\partial x_i(k)}\right|_{\hat{\alpha}_i,\hat{x}_i} \left\{ P_{x_i}(j,k) - \sum_{\ell,m} P_{x_i\alpha_i}(j,\ell) P^{-1}_{\alpha_i}(\ell,m) \right. \\
&\qquad \left. \cdot \left[P_{x_i\alpha_i}(m,k) - \chi_i(m,k) \right] \right\}
\end{aligned}
\tag{1-33}
$$

where

$$\chi_i = P_{\alpha_{i/i+1}} + K_{i/i+1}(z_{i+1} - H\Phi\hat{\alpha}_i)(z_{i+1} - H\Phi\hat{\alpha}_i)^T K^T_{i/i+1}$$

5. The first order filter as characterized by Eqs. (1-29) and (1-30) turns out to be equivalent to the approach of using a linearized model along an estimated state and the successive use of the linear filtering techniques [2, Section 12.6].

1.4 Extention to a More General Model

The method which was used to derive the recursive filters in section 1.3. can also be used to obtain solutions related to a more general filtering problem. Consider the following models:

a) The dynamics

$$x_{i+1} = f(x_i) + w_i \tag{1-34}$$

where x_i and w_i are the same as specified by (1-2) and 1-4),

b) The measurements

$$z_i = Hx_i + \epsilon_i \tag{1-35}$$

where the statistics of ϵ_i is given by (1-6).

The problem is to estimate the conditional mean of x_i given measurements up to i.

To obtain the best estimate of x_i, we follow the techniques used in section 1.3 by deriving the conditional probability density and expanding the nonlinear term into Taylor series.

Using the fact that

$$p(a/b) = \int p(a/b, c)\, p(c/b)\, dc$$

we have

$$p(x_{i+1}/Z_{i+1}) = \int p(x_{i+1}/x_i, Z_{i+1})\, p(x_i/Z_{i+1})\, dx_i$$

and

$$\hat{x}_{i+1} = \int x_{i+1}\, p(x_{i+1}/Z_{i+1})\, dx_{i+1}$$

$$= \iint x_{i+1}\, p(x_{i+1}/x_i, Z_{i+1})\, p(x_i/Z_{i+1})\, dx_i\, dx_{i+1} \tag{1-36}$$

The integral $\int x_{i+1}\, p(x_{i+1}/x_i, Z_{i+1})\, dx_{i+1}$ can be analytically carried out because of the fact that $p(x_{i+1}/x_i, Z_{i+1})$ is Gaussian distributed, that is

$$\int x_{i+1}\, p(x_{i+1}/x_i, Z_{i+1})\, dx_{i+1} = f(x_i) + K_i(z_{i+1} - Hf(x_i)) \tag{1-37}$$

where

$$K_i = (Q_i^{-1} + H^T R_i^{-1} H)^{-1} H^T R_i^{-1} \tag{1-38}$$

Substituting (1-37) into (1-36), yields

$$\hat{x}_{i+1} = \int \left[f(x_i) + K_i\, (z_{i+1} - Hf(x_i)) \right] p(x_i/Z_{i+1})\, dx_i \tag{1-39}$$

To obtain an explicit expression of the nonlinear filter for an arbitrary $f(x_i)$, the nonlinear term $f(x_i)$ is expanded into Taylor series around $\hat{x}_i$, or

$$f(x_i) = f(\hat{x}_i) + \left.\frac{\partial f}{\partial x_i}\right|_{\hat{x}_i} (x_i - \hat{x}_i) + \text{(2nd order and higher order terms)} \tag{1-40}$$

As before, a hierarchy of filters can be defined:

1) The zero order filter

Substituting only the first term of (1-40) into (1-39), we have the zero order filter as

$$\hat{x}_{i+1} = f(\hat{x}_i) + K_i\, (z_{i+1} - H f(\hat{x}_i)) \tag{1-41}$$

where K_i is given by (1-38). It is noted from (1-41) that if $f(\hat{x}_i) = f(x_i)$, then the optimal estimate of x_{i+1} is simply $\hat{x}_{i+1}$ given by (1-41).

2) The first order filter

The first order filter is obtained by substituting the first and the second terms of (1-40) into (1-39). The first term is obviously given by (1-41), and

$$\text{(The 2nd term)} = (I - K_i H) \left.\frac{\partial f}{\partial x_i}\right|_{\hat{x}_i} \int (x_i - \hat{x}_i)\, p(x_i/Z_{i+1})\, dx_i \tag{1-42}$$

The integral $\int x_i\, p(x_i/Z_{i+1})\, dx_i$ is a smoothed estimate of x_i. Assuming $p(x_i/Z_{i+1})$ and $p(x_i/Z_i)$ are Gaussian, the second term can be approximated as

$$\text{(The 2nd term)} = (I - K_i H) \left.\frac{\partial f}{\partial x_i}\right|_{\hat{x}_i} K_{i/i+1} (z_{i+1} - H f(\hat{x}_i)) \tag{1-43}$$

where

$$K_{i/i+1} = P_{i/i+1} H_i'^T R_i'^{-1}$$

with

$$P_{i/i+1} = P_i - P_i H_i'^T (H_i' P_i H_i'^T + R_i')^{-1} H_i' P_i$$

$$R_i' = H Q_i H^T + R_i$$

$$H_i' = H \left.\frac{\partial f}{\partial x_i}\right|_{\hat{x}_i}$$

and

$$P_i = \int (x_i - \hat{x}_i)(x_i - \hat{x}_i)^T p(x_i/Z_i)\, dx_i$$

Combining (1-43) and (1-41), we obtain the first order filter

$$\hat{x}_{i+1} = f(\hat{x}_i) + K_i (z_{i+1} - H f(\hat{x}_i))$$
$$+ (I - K_i H) \left.\frac{\partial f}{\partial x_i}\right|_{\hat{x}_i} K_{i/i+1} (z_{i+1} - H f(\hat{x}_i)) \tag{1-44}$$

The recursive expression for updating P_i is not available in analytic form, hence, for numerical realization, the approximated version is obtained by expanding $f(x_i)$ into Taylor series up to first order, and we have

$$P_{i+1} = M_{i+1} - M_{i+1}H^T (HM_{i+1}H^T + R_i)^{-1} H M_{i+1}$$
$$M_{i+1} = \frac{\partial f}{\partial x_i}\bigg|_{\hat{x}_i} P_i \frac{\partial f}{\partial x_i}\bigg|_{\hat{x}_i}^T + Q_i \tag{1-45}$$

Comments:

1. (1-41) provides a simple zeroth order filter which explicitly takes the present measurement into consideration and which has the advantage that K_i can be precalculated and stored. In fact, K_i is constant if Q_i and R_i are independent of i (see Eqs. (1-38) and (1-39)).

2. Higher order filters can be derived in the same way as the 1st order. The first order filter given by (1-44) turns out to be equivalent to the usual approach of using the linearized dynamic model applying the recursive filtering techniques derived by Kalman [2, Section 12.6].

3. The filtering algorithms derived in this section do not include the filtering algorithms derived in previous section. The reason is that the general formulation of this section tends to neglect the special characteristics of the class of filtering problems discussed in last section.

1.5 Summary

A class of nonlinear filtering problems has been formulated in this chapter intended to study the attitude computational algorithms in a strap-down inertial navigation system. Recursive filtering algorithms which takes advantage of the special characteristics of this mathematical model have been derived. Extention of the solution to more general dynamic and measurement models has also been considered.

CHAPTER TWO

APPLICATIONS OF NONLINEAR FILTERING ALGORITHMS TO STRAPDOWN ATTITUDE COMPUTATION

2.1 Introduction to the Strapdown Inertial Navigation System

The function of an inertial navigator is to indicate the attitude, velocity and position of a vehicle with respect to a selected reference frame using measurement obtained from onboard inertial instruments. Broadly speaking, inertial navigation systems can be divided into two classes:

1. The gimbaled navigation system (the platform navigation system).
2. The strapdown navigation system.

In the gimbaled navigation system, the reference frame which the measurements are made is a stabilized platform. Mounted on the platform are gyros and accelerometers. The platform is stabilized by means of servomechanisms [12]. A typical servo-stabled platform is shown in Fig. 2.1. The attitude is determined by the reading of the gimbal angles. The accelerometers are used to measure the linear acceleration or the increments of velocity with respect to the reference frame. The velocity and position of the vehicle are obtained by integrating the output of the accelerometers.

In contrast to the gimbaled inertial navigation system, the strapdown navigation system creats the reference frame electronically [13]. The gyros and accelerometers are physically mounted to the vehicle frame. The attitude of the vehicle relative to the inertial space is maintained in a digital computer by solving a set of mathematical equations using gyro output

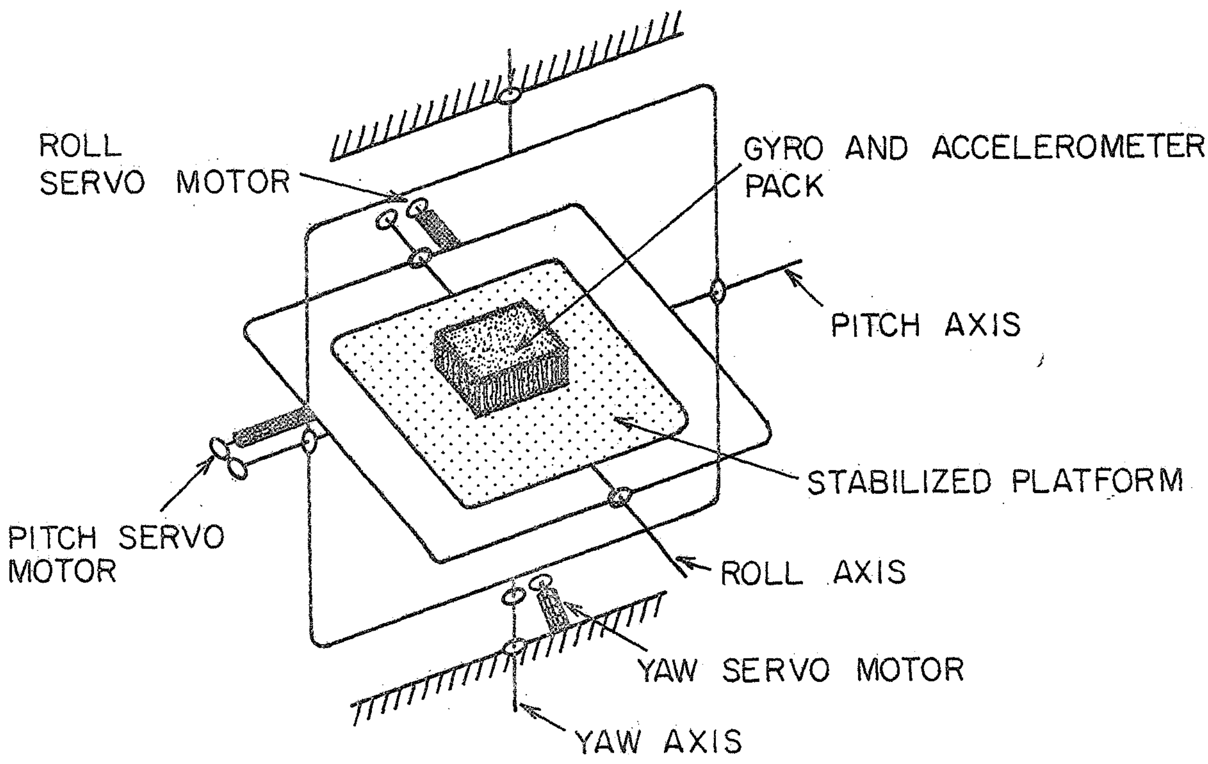

FIG. 2.1 GIMBALED INERTIAL NAVIGATOR

as physical measurements. Many parameters [9] can be used to relate the vehicle frame to the reference frame and many studies have been conducted examining the use of these parameters and the various integration algorithms to update them [9][13][14]. A common set that has been used, for example, in the LEM* Abort Guidance System, is the so called direction cosine matrix. This is an orthonormal matrix of nine elements giving the nine projections of a unit vector along the body coordinates to the inertial coordinates (the reference frame).

Let the 3x3 direction cosine matrix be represented by T(t). Then it is well known that T(t) satisfies the matrix differential equation [13].

$$\frac{d}{dt} T(t) = \Omega(t)\ T(t) \tag{2-1}$$

where $T(t_o) = I$, an identity matrix, and $\Omega(t)$ is a 3x3 matrix whose elements represent the body angular rates with respect to the vehicle frame (or body frame).

$$\Omega(t) = \begin{bmatrix} 0 & \omega(3) & -\omega(2) \\ -\omega(3) & 0 & \omega(1) \\ \omega(2) & -\omega(1) & 0 \end{bmatrix} \tag{2-2}$$

Let a_b be the outputs of the accelerometers mounted on the body. Then, the acceleration with respect to the inertial frame is given by

$$a_I = T\ a_b$$

---

* "LEM" stands for "Lunar Excursion Module".

T and a_I provide the same information as the gimballed angle and the accelerometer output of the gimbaled system. The position and the velocity of a vehicle can be conventionally obtained by double integration. A block diagram comparing these two inertial navigation system is shown in Figure 2.2.

An important problem associated with the strapdown navigation system is to find the computational algorithms for updating the transformation matrix T. This is the problem that will be addressed in this chapter.

The advantages and disadvantages of the strapdown system compared with a gimbaled system are listed in the following*:

1. Advantages
   - i) Minimum mechanical complexity
   - ii) Small size, weight and power
   - iii) Packaging flexibility
   - iv) Reliability and maintainability
2. Disadvantages
   - i) Requires faster onboard guidance computer
   - ii) High angular velocity exaggerats performance errors

2.2 Formulation

As described in section 1.1, the class of filtering problems discussed in Chapter One was motivated from considering the practical problems related to strapdown attitude updating. In this section, we will show how the strapdown attitude updating problem is formulated into the class of filtering problems of Chapter One.

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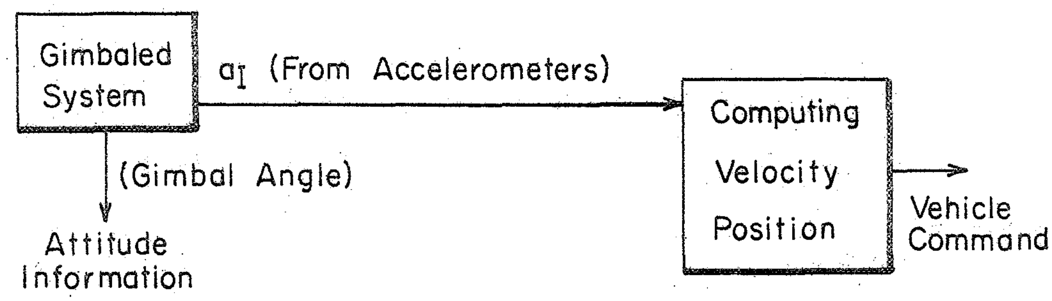
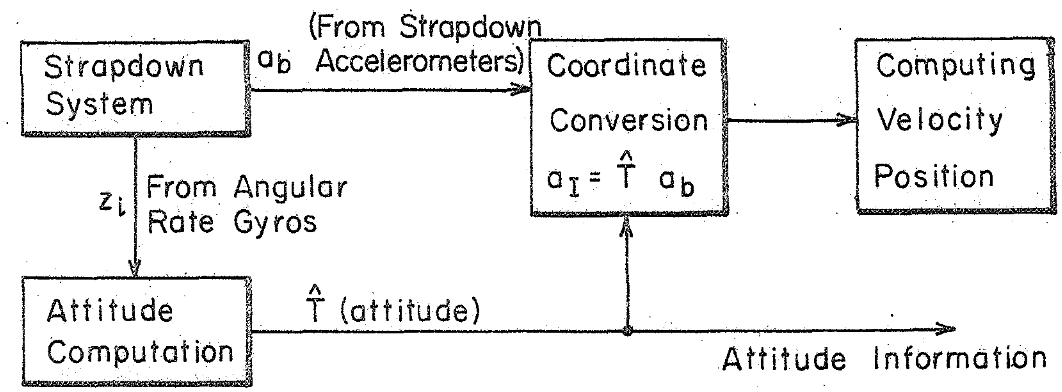
* For detailed discussion please refer to [13].



GIMBALED SYSTEM



STRAPDOWN SYSTEM

FIG. 2.2 COMPARISON OF GIMBALED SYSTEM AND STRAPDOWN SYSTEM

As shown in (2-1), $T(t)$ is updated by a set of differential equations using the body angular rates $\omega(1), \omega(2)$ and $\omega(3)$. In practice, the measurements on the body angular rates are given in terms of incremental angle from which $\Omega(t)$ must be inferred (for instance, measurements from the pulse-torque-rebalanced strapdown gyro [9])

Define

$$\gamma_i = \int_{t_i}^{t_{i+1}} \Omega(t)\, dt = \begin{bmatrix} 0 & \theta_i(3) & -\theta_i(2) \\ -\theta_i(3) & 0 & \theta_i(1) \\ \theta_i(2) & -\theta_i(1) & 0 \end{bmatrix} \tag{2-3}$$

The measurements may be approximated by the equation

$$z_i = \begin{bmatrix} \theta_i(1) \\ \theta_i(2) \\ \theta_i(3) \end{bmatrix} + \begin{bmatrix} \epsilon_i(1) \\ \epsilon_i(2) \\ \epsilon_i(3) \end{bmatrix} \tag{2-4}$$

where ϵ_i is an additive noise term.

The solution* of (2-1) at sampling times can be approximated by

$$T_{i+1} = e^{\gamma_i} T_i \tag{2-5}$$

* (2-5) is the exact solution of (2-1), if $\Omega(t_{i+1})$ commutes with γ_i [15] or more restrictively, if $\omega(j)$ for $j = 1, 2, 3,$ is constant over the sampling period $(t_{i+1}-t_i)$.

This discrete updating equation does not affect the orthonormal condition of the transformation matrix T (see Appendix B). An analytic expression for e^{γ_i} is also given in Appendix B.

From power spectrum analysis or by other statistical methods $\theta_i(j)$, for j = 1, 2, 3, can be modelled as a Gaussian Markov random process. Then, it is clear from (2-3), (2-4) and (2-5) that the problem of finding T_i is equivalent to the nonlinear filtering problem considered in chapter five. T_i corresponds to x_i and θ_i corresponds to α_i.

For example, Eq. (2-5) can be written as a ninth order differential equation

$$T_{i+1}(1) = e^{\gamma_i} T_i(1)$$

$$T_{i+1}(2) = e^{\gamma_i} T_i(2)$$

$$T_{i+1}(3) = e^{\gamma_i} T_i(3)$$

where T(1), T(2) and T(3) are column vectors of the matrix T.

2.3 The optimal Filter for Single Axis Rotation Case

For the case where the angular movement is around a single axis only, an analytical expression exists for the optimal nonlinear filter. It can be used to establish a lower bound for the error variance. The derivation of optimal nonlinear filter is given as follows.

Assuming $\theta_i(1) = 0$ and $\theta_i(2) = 0$, for all i, Eq. (2-5) becomes

$$T_{i+1} = e^{A\theta_i(3)} T_i \qquad (2\text{-}6)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2-7}$$

Successive iteration of (2-6) gives

$$T_{i+1} = e^{A(\theta_i(3) + \theta_{i-1}(3) + ---- + \theta_o(3))} T_o \tag{2-8}$$

Define

$$\eta_{i+1} = \eta_i + \theta_i(3) \qquad ; \; \eta_o = 0 \tag{2-9}$$

Eq. (2-8) can be written as

$$T_{i+1} = e^{A\eta_{i+1}} T_o$$

where

$$e^{A\eta_{i+1}} = \begin{bmatrix} \cos \eta_{i+1} & \sin \eta_{i+1} & 0 \\ -\sin \eta_{i+1} & \cos \eta_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2-10}$$

The optimal nonlinear estimation of T_{i+1} in this case becomes

$$(\hat{T}_{i+1}) \text{ Optimal} = E(e^{A\eta_{i+1}}/Z_{i+1}) \, T_o \tag{2-11}$$

To evaluate Eq. (2-11), it is required to known $\hat{\eta}_{i+1}$ and $P_{\eta_{i+1}}$ which are defined by

$$\begin{aligned} \hat{\eta}_{i+1} &= E[\eta_{i+1}/Z_{i+1}] \\ P_{\eta_{i+1}} &= E[(\eta_{i+1} - \hat{\eta}_{i+1})(\eta_{i+1} - \hat{\eta}_{i+1})^T/Z_{i+1}] \end{aligned} \tag{2-12}$$

Since $\theta_i(3)$ has been assumed to be a Gauss-Markov process, from Eq. (2-9), η_i will be a Gauss-Markov process [2, Chapter 11]. By sequentially applying the Kalman filter, we can update $\hat{\eta}_i$ and P_{η_i}, for all i.

Now to compute $E[e^{A\eta_{i+1}}/Z_{i+1}]$, we make use of the characteristic function [7, pp. 159] of η_{i+1}, then the elements of $E[e^{A\eta_{i+1}}/Z_{i+1}]$, namely, $E[\text{Cos } \eta_{i+1}/Z_{i+1}]$ and $E[\text{Sin } \eta_{i+1}/Z_{i+1}]$, can be easily obtained. The characteristic function of η_{i+1} given Z_{i+1} is (refer to [10, pp. 159])

$$E\left[e^{j\sigma\eta_{i+1}}/Z_{i+1}\right] = e^{j\hat{\eta}_{i+1}\sigma - \frac{1}{2}P_{\eta_{i+1}}\sigma^2} \tag{2-13}$$

where $j = \sqrt{-1}$ and σ is a dummy variable (scalar). Using Eq. (2-13), we have

$$\begin{aligned} E[\text{Cos } \eta_{i+1}/Z_{i+1}] &= \text{Real Part of } \left(E[e^{j\eta_{i+1}}/Z_{i+1}]\right) \\ &= e^{-\frac{1}{2}P_{\eta_{i+1}}} \text{Cos } \hat{\eta}_{i+1} \end{aligned} \tag{2-14}$$

and similarly,

$$E[\text{Sin } \eta_{i+1}/Z_{i+1}] = e^{-\frac{1}{2}P_{\eta_{i+1}}} \text{Sin } \hat{\eta}_{i+1} \tag{2-15}$$

Using Eqs. (2-10), (2-14) and (2-15), we have

$$E[e^{A\eta_{i+1}}/Z_{i+1}] = e^{-\frac{1}{2}P_{\eta_{i+1}}} \begin{bmatrix} \cos \hat{\eta}_{i+1}, & \sin \hat{\eta}_{i+1} & 0 \\ -\sin \hat{\eta}_{i+1}, & \cos \hat{\eta}_{i+1}, & 0 \\ 0 & 0 & e^{\frac{1}{2}P_{\eta_{i+1}}} \end{bmatrix} \tag{2-16}$$

Eq. (2-11) and Eq. (2-16) give the optimal nonlinear estimate of T_{i+1}.

2.4 Numerical Results

For purpose of illustration, numerical work based on Monte Carlo simulations were carried out using the recursive algorithms developed in Chapter one. Figure 2.3 is a block diagram describing the simulation procedures. For simplicity, it is assumed that the vehicle is subject to a wide band noise fluctuation (i.e., purely random noise). In this case, the dynamic of the gyro output can be approximated as the output of the first order difference equation

$$\theta_{i+1}(j) = \beta(j)\ \theta_i(j) + w_i(j) \tag{2-17}$$

$$\theta_o(j) = 0 \quad \text{with probability 1}$$

$$\text{for } j = 1, 2, 3$$

where

$$E(w_i(j)) = 0$$

and

$$E(w_i(j)w_k(j)) = Q\ \delta_{ik} \tag{2-18}$$

The measurements are given by equation (2-4) and the attitude matrix satisfies equations (2-5) where γ_i is given by (2-3).

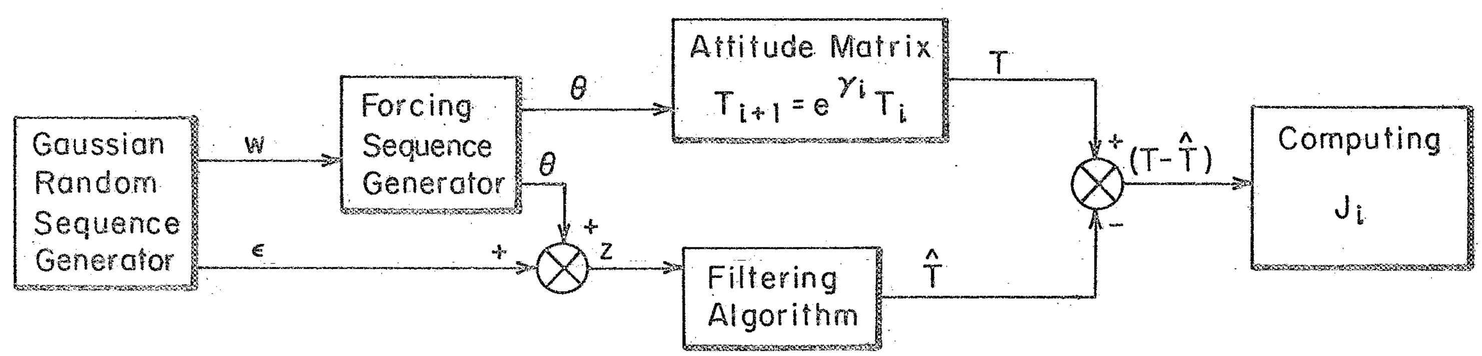


FIG. 2.3 SIMULATION DIAGRAM

For comparison purposes, we included in our numerical work a typical strapdown attitude computation algorithm being used in current LEM system based on Taylor series expansions of e^{γ_i}, with θ_i replaced by z_i [16]. This algorithm which is generally called "the second order algorithm" has the form

$$T^*_{i+1} = (I + C_i + \frac{1}{2} C_i^2)\, T^*_i \tag{2-19}$$

where

$$C_i = \begin{bmatrix} 0 & z_i(3) & -z_i(2) \\ -z_i(3) & 0 & z_i(1) \\ z_i(2) & -z_i(1) & 0 \end{bmatrix}$$

To avoid confusion with the second order nonlinear filter derived in Chapter one, we shall refer to (2-19) as ALGORITHM A.

The mean square error is adopted as the criterion for the evaluation of the various algorithms. Let T_i be the true solution and let $\hat{T}_i$ be the solution using a particular algorithm. Define

$$J_i = \text{Trace } E((T_i - \hat{T}_i)(T_i - \hat{T}_i)^T) \tag{2-20}$$

then J_i gives the mean square error at time t_i. Physically, J_i is a measure of the expected magnitude square of the difference vector between the vectors which are transferred from the body coordinates to the inertial coordinates by using T_i and $\hat{T}_i$ respectively (see Appendix B).

In the following, simulation results are categorized into two classes: class A, the single axis rotation case, and class B, the three axes rotation case.

Class A: Single Axis Rotation Case

Four simulation results will be presented. The first two simulations are to show the advantage of applying the filters derived in Chapter one. The third shows a contradictive result with explanations given in Appendix C. The fourth simulation considers a practical problem associated with the test of a strapdown gyro. It will be seen that in the case of higher noise levels (e. i., Q and R) and lower correlation (e. i., $\beta(3) << 1.$) the performance of the nonlinear filtering algorithms is much better than that of Algorithm A. The reason for this is given in Appendix C. The sample means of square error in the first and second simulations are averaged over 100 runs, whereas the third and fourth simulations are averaged over 50 runs and 15 runs respectively (see Simulation 4 for the explanation of 15 runs).

Simulation 1 (Figure 2.4)

This simulation demostrates the advantage of applying the second order nonlinear filter.

The set of parameters is

$$Q = 0.1 \quad \text{radian}^2$$

$$R = 0.1 \quad \text{radian}^2$$

$$\beta(3) = 0.3$$

Errors are squared and averaged over 100 runs. As shown in Figure 2.4, the averaged square errors of applying the zero order filter and the first order filter are very close. It is seen that substantial improvement is obtained by the use of the second order filter which is also seen to be very close to the true optimal filter.* The mean square

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* Refer to Section 2.3.

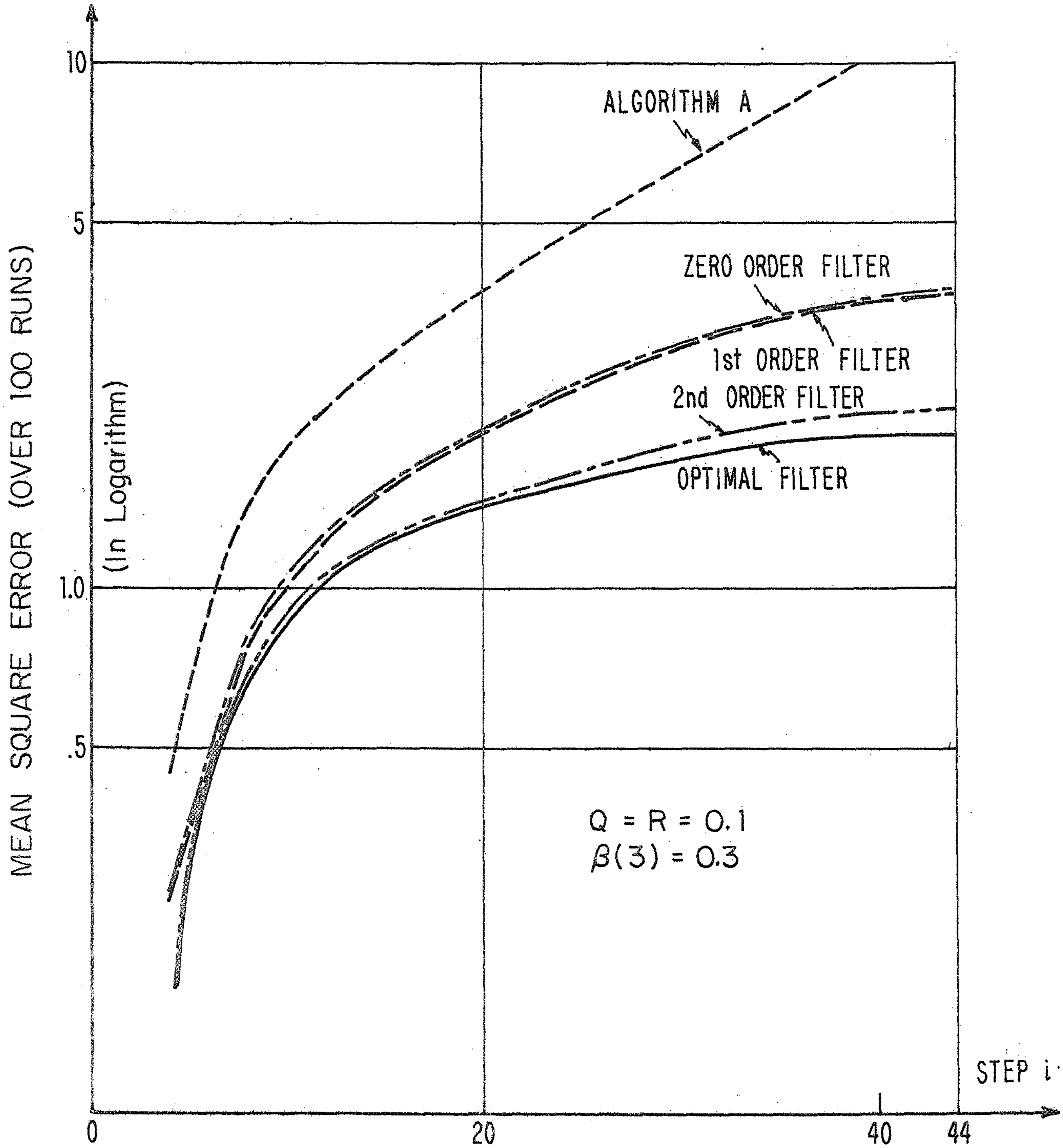

FIG. 2.4 SIMULATION 1 (SINGLE AXIS ROTATION)

error generated by the use of Algorithm A is comparatively larger which is caused mainly by the truncation and the measurement errors.

Simulation 2 (Figure 2.5)

It has been mentioned in Chapter one that the zeroth order filter with a correction term as given by (2-24) will show significant improvement over the zeroth order filter in many cases. Figure 2.5 shows the results of such a case. It is seen that the addition of this simple correction term provides a considerable improvement over the zeroth order filter. Very little additonal computations are needed for the inclusion of this additonal term (refer to section 1.3). In this simulation, we take

$$Q = 0.0001 \text{ radians}^2$$

$$R = 0.0001 \text{ radinas}^2$$

$$\beta(3) = 0.5$$

Simulation 3 (Figure 2.6)

In general, better results can be expected from applying the zeroth order filter than from applying algorithm A. There are cases, however, where this is not true. Such a case is shown in Figure 2.6. In this simulations, we take

$$Q = 0.5856 \times 10^{-17} \text{ radians}^2$$

$$R = 0.5856 \times 10^{-17} \text{ radians}^2$$

$$\beta(3) = 1.$$

Detailed explanation of this contradictive simulation results is given in Appendix C.

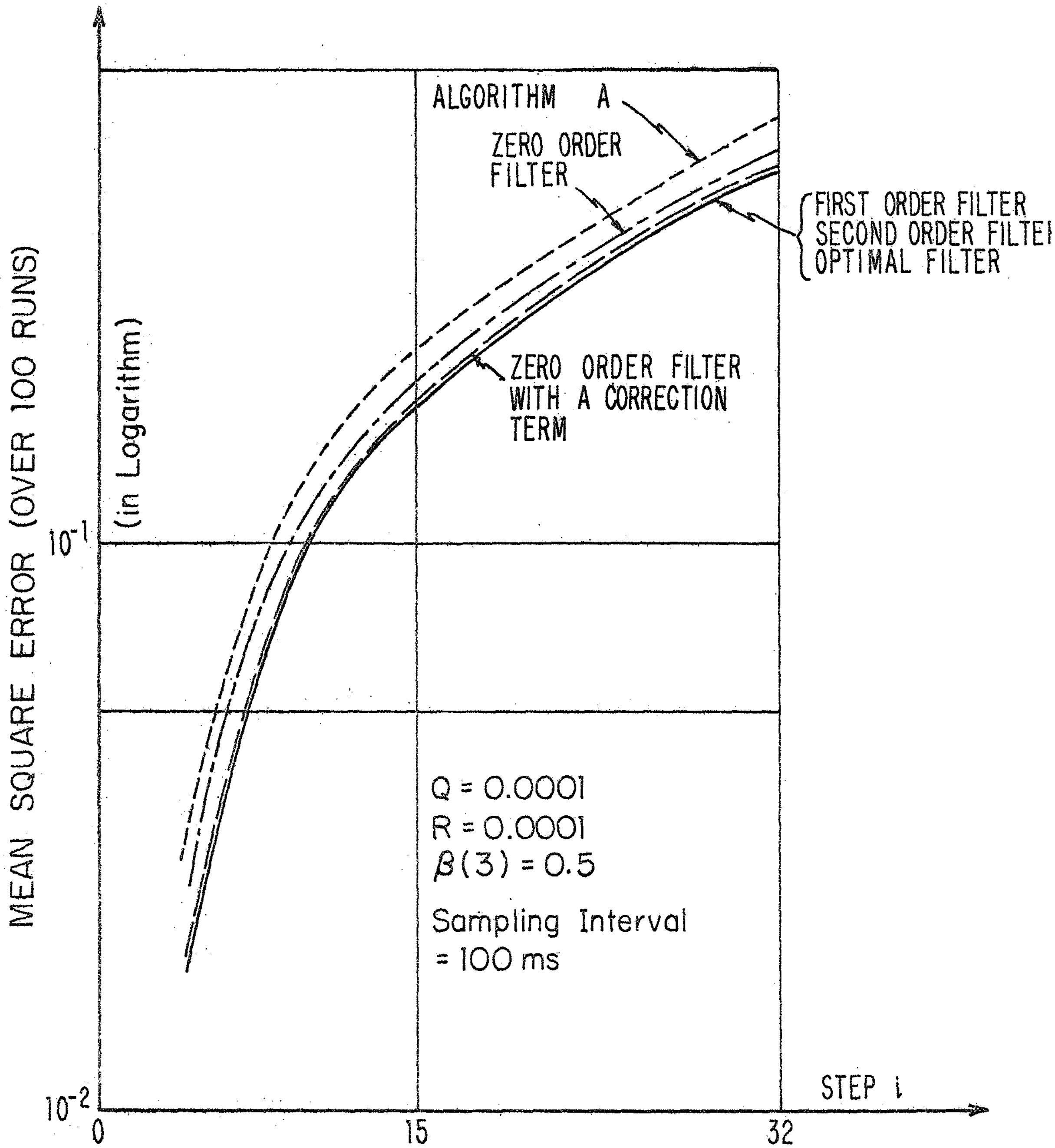

FIG. 2.5 SIMULATION 2 (SINGLE AXIS ROTATION)

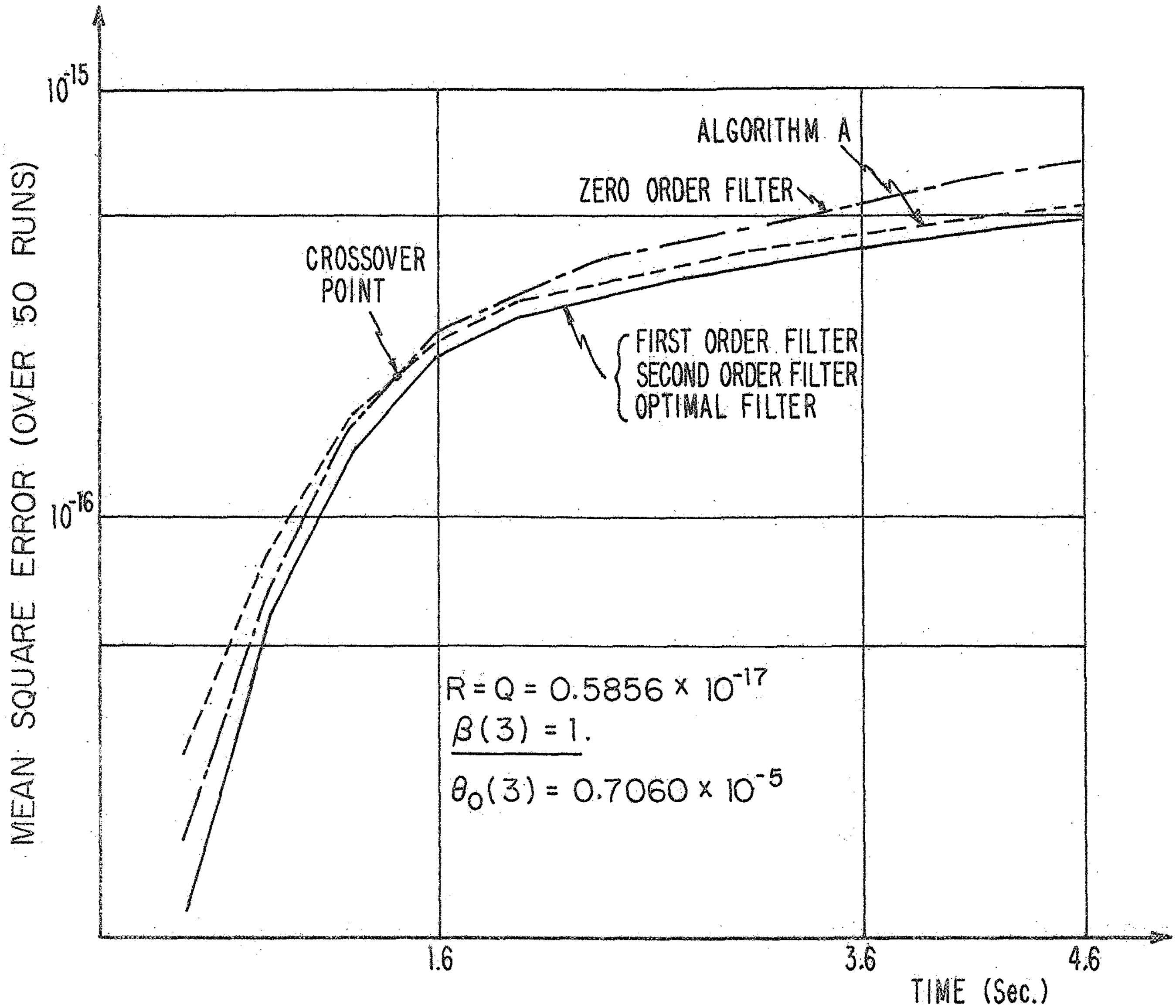

FIG. 2.6 SIMULATION 3 (SINGLE AXIS ROTATION)

Simulation 4 (Figure 2.7)

Consider the problem associated with the test of a strapdown pulse-rebalanced gyro [14]. The test is performed on a table which rotates at the rate of rotation of the earth. The gyro frame is fixed to the table which is assumed to be subject to random disturbances. In this case, the input rate $\omega(3)$ can be modelled as an exponentially correlated random process. The discrete form is

$$\omega_{i+1}(3) = \beta(3)\omega_i(3) + w_i'$$

with

$$\beta(3) = 0.9999$$

$$E[\omega_o(3)] = 10 \text{ degree/hr} \qquad Cov[\omega_o(3)] = (0.05 \text{ degree/hr})^2$$

$$E[w_i'] = 0 \qquad E[w_k' w_i'] = (0.005 \text{ degree/hr})^2 \, \delta_{ik}$$

where $E[\omega_o(3)] = 10$ degree/hr corresponds to be earth angular rate at the latitude of 45° and the uncertainty on the earth rate is expressed by the covariance $Cov(\omega_o(3)) = (0.05 \text{ degree/hr})^2$.

A perfect gyro is assumed and the only measurement error is the quantization error (1 sec) and the sampling interval is 0.1 second.

The purpose of this simulation is again to compare the results of using Algorithm A with the filtering algorithms derived in Chapter one. For saving computing time, only 15 runs are simulated and averaged to obtain the sample means and sample variances of error square. The simulation results are presented in Fig. 2.7 and Table 2.1. The sample means of square error which are computed over 15 runs may not be representative of the true mean square errors. But this simulation result does indicate

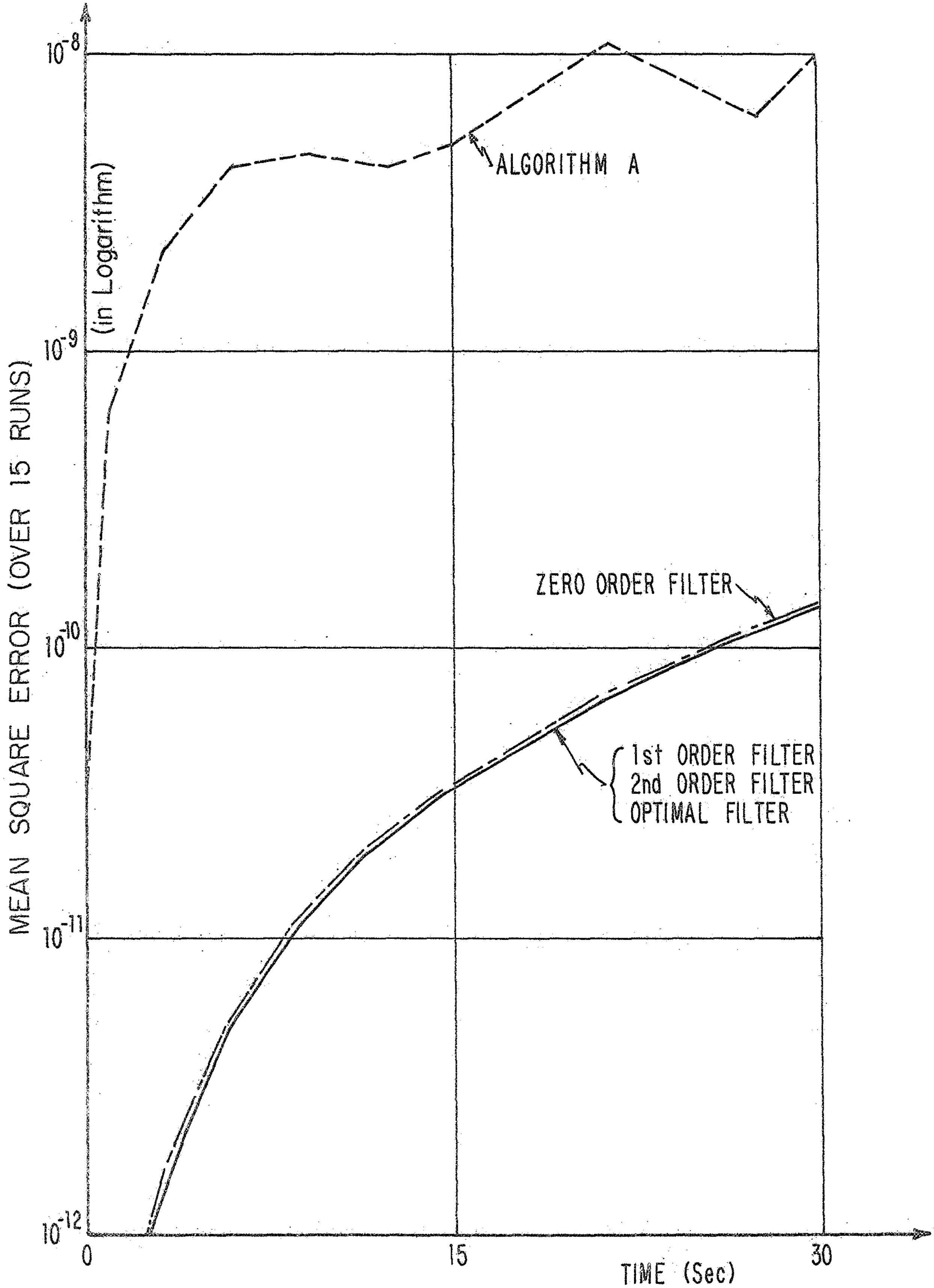

FIG. 2.7 SIMULATION 4 (SINGLE AXIS ROTATION)

the tendency that the filtering algorithm is superior over Algorithm A. Two reasons are given below to support the above statement.

1. Simulation results showed that the square errors of applying these filtering algorithms are smaller than that of applying Algorithm A in every run. This is because that in the simulation process, the same samples are used for both filtering algorithms and Algorithm A. A bad random sample (e. i., the value of the random sample is far away from its mean) will increase the sample mean square errors of applying the filtering algorithms as well as that of applying Algorithm A. In other words, bad random samples will generally hurt the performance of both Algorithm A and the filtering algorithms.

2. Fig. 2.7 shows that the sample means of square error generated by using the filtering algorithms are much smaller than that of using Algorithm A. The sample variances of terminal square error as given by Table 2.1 indicate that in this simulation the sample trajectories of square error obtained by applying the filtering algorithms are close to its sample mean as compared with those obtained by applying Algorithm A.

	Algorithm A	Zeroth Order Filter	First Order Filter	Second Order Filter	Optimal Filter
Sample Mean of Square Error	0.8756×10^{-8}	0.1381×10^{-9}	0.1350×10^{-9}	0.1350×10^{-9}	0.1350×10^{-20}
Sample Variance of Square Error	0.1300×10^{-15}	0.8578×10^{-20}	0.7644×10^{-20}	0.7644×10^{-20}	0.7642×10^{-20}

Table 2.1 Numerical Results of Simulation 4.

Class B: Three Axes Rotation Case

In the following simulations, the attitude updating algorithms are simulated under the circumstance that all three body axes of the vehicle are under random disturbances. One should note that no optimal filter in analytical form is available in this case.

Simulation 5 (Figure 2.8)

This simulation is concerned with a space vehicle in its orbit. The statistical models for input rates $\omega(1), \omega(2)$ and $\omega(3)$ are identical but statistically independent. The statistical values concerning the angular rate models are given in the following

$$E(\omega_o) = 1 \text{ deg/sec}$$

$$Cov(\omega_o) = (1 \text{ deg/hr})^2$$

$$E(w_i') = 0$$

$$E(w_i' w_j') = (5 \text{ deg/sec})^2 \, \delta_{ij}$$

The sampling interval is taken as 100ms and $\beta = 0.9048$ corresponding to 1 see in time constant of the continuous model. The simulation is carried out over a period of 30 sec and the sampling interval is 0.1 sec. For saving computing time, the sample means and sample variances of square error are computed over 10 runs. The significance of this 10 run simulation has been discussed in Simulation 4. Simulation result is presented in Fig. 2.8 and Table 2.2. It is seen from Fig. 2.8 that improvement is obtained by the use of the zeroth order filter over the use of Algorithm A. The first order and second order filter do not show much improvement over the zeroth order filter. The sample variances are shown in Table 2.2. For comparison purpose,

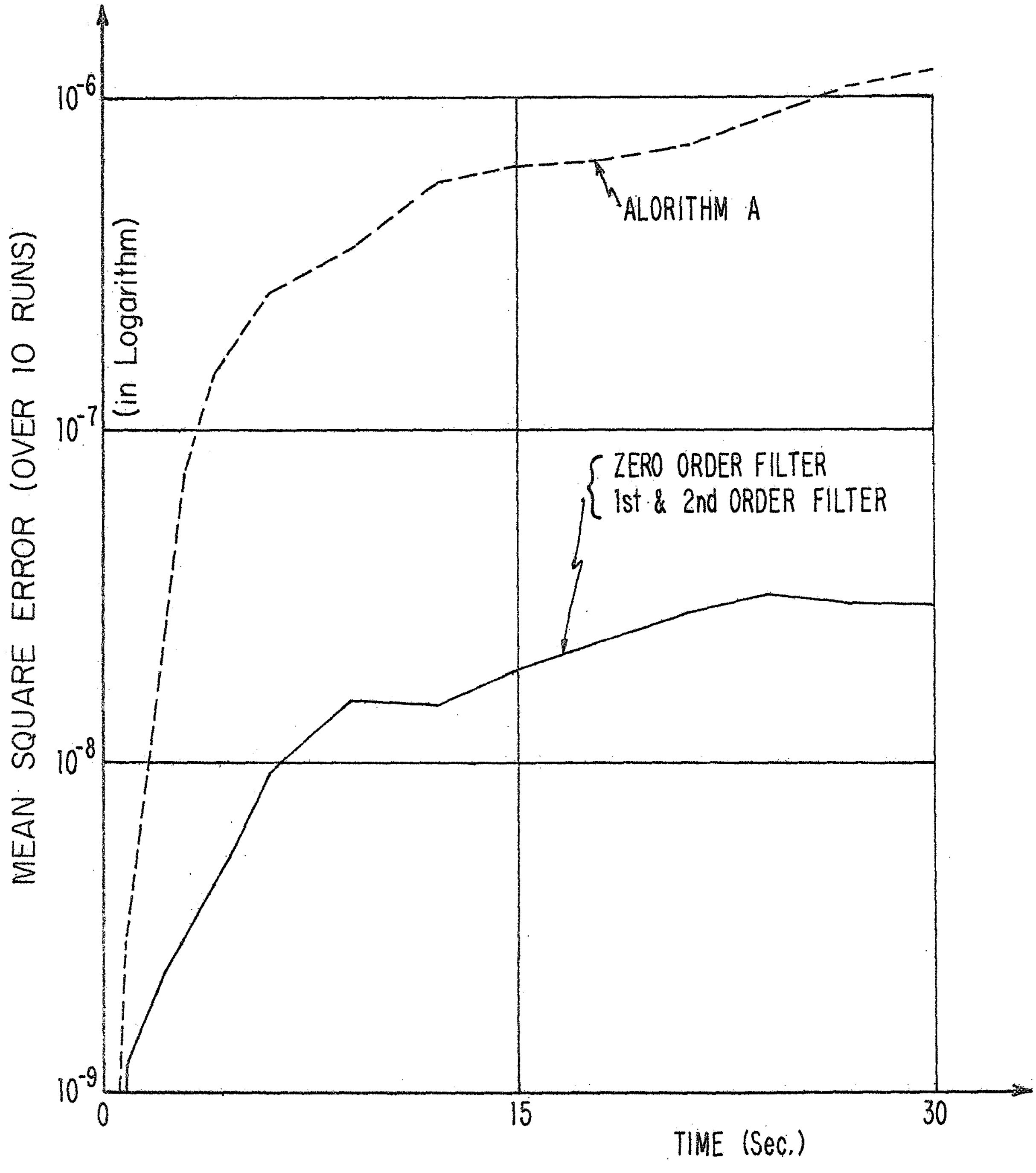

FIG. 2.8 SIMULATION 5 (THREE AXIS ROTATION)

we also include in Table 2.2 the square root of sample variances and sample means. It is seen that the square roots of sample variances are smaller as compared with their sample means. Since the sample variance is the estimate of the true variance and since the sample mean trajectory of Algorithm A is far apart from those of filtering algorithms, based on these, we are more confident to conclude that the filtering algorithms are superior to Algorithm A.

Simulation 6 (Figure 2.9)

In simulation 3, we showed that in the case of very small but highly correlated noise, Algorithm A may perform better than the zeroth order filter in single axis rotation case. In this three axes rotation case, we have pretty much similar results. In Fig. 2.9, it is clearly seen that the algorithm A is superior to the zeroth order filter and the application of the first order filter is desirable. In this simulation, we take

$$R = Q = 0.5856 \times 10^{-17} \text{ radian}^2$$

$$\beta(1) = \beta(2) = \beta(3) = 1.$$

2.5 Discussion

1. The simulation results presented in this chapter indicated that simulation results would be helpful in choosing an appropriate filtering algorithm for a certain random circumstance.

2. The statistical models which describe the angular motion and the measurements are not necessary restricted as those specified by (2-4) and (2-17). The gyro dynamics can be taken into consideration for designing a more sophisticated filter.

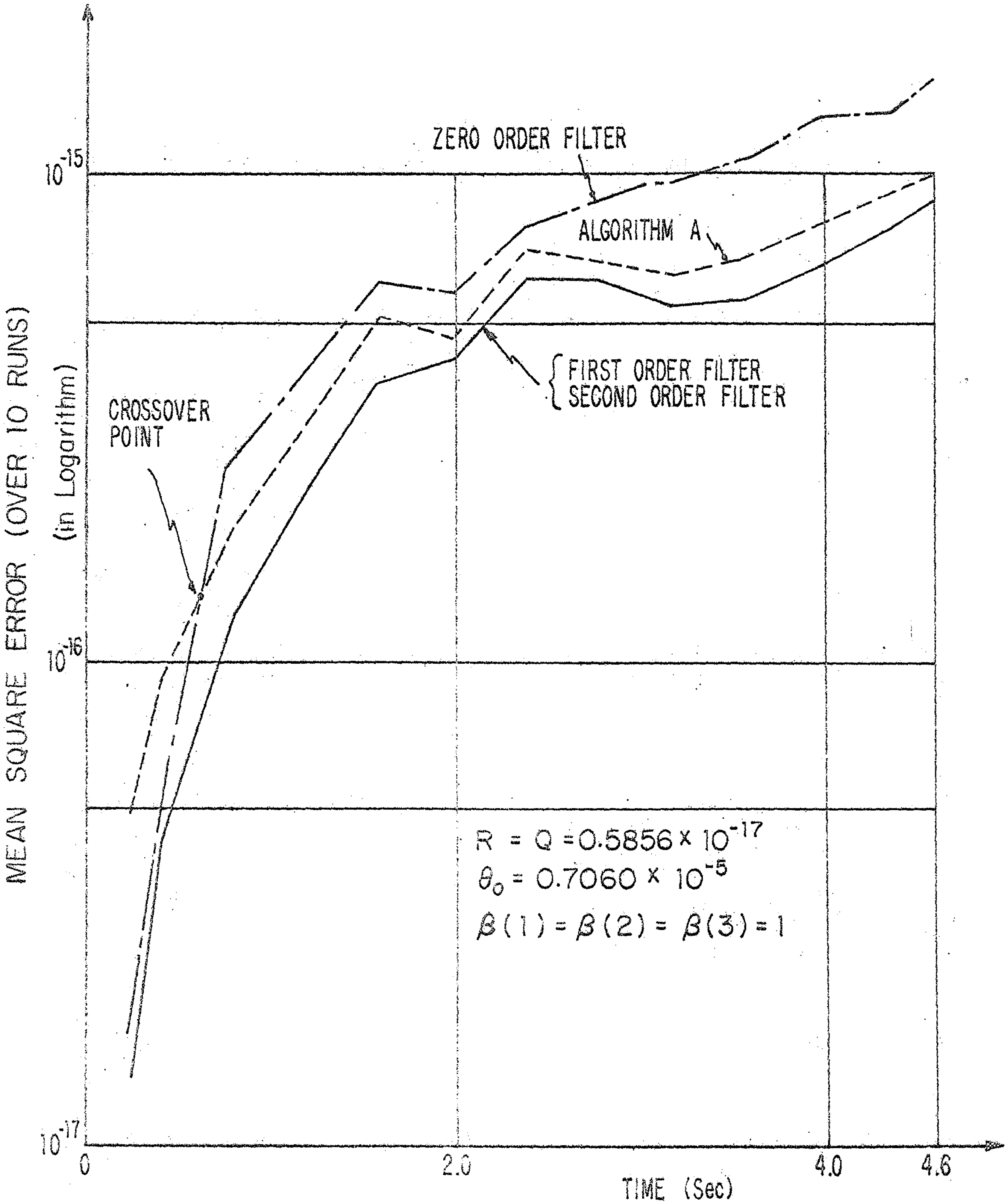

FIG. 2.9 SIMULATION 6 (THREE AXIS ROTATION)

3. Another advantage in applying the filtering algorithms of Chapter one to the strapdown attitude computation is that the angular rate which is needed for attitude control is obtained in the process of computing $\hat{T}_{i+1}$ from the output of the Kalman filter (see Eq. (1-16)).

	Algorithm A	Zeroth Order Filter	First Order Filter	Second Order Filter
Sample Variance of Square Error	0.1054×10^{-11}	0.8452×10^{-15}	0.8460×10^{-15}	0.6460×10^{-15}
Square Root of Sample Variance	0.1029×10^{-6}	0.2906×10^{-8}	0.2906×10^{-8}	0.2908×10^{-8}
Sample Mean of Square Error	0.1131×10^{-5}	0.3249×10^{-7}	0.3249×10^{-7}	0.3249×10^{-7}

Table 2.2 Numerical Results of Simulation 5

CHAPTER THREE

FILTERS WITH COMPUTATIONAL CONSTRAINTS

3.1 Introduction

The filtering algorithms of Chapter one are derived mainly for improving the accuracy of estimation; the computational complexity is of secondary importance. The structures of these filters are optimally chosen by minimizing the criterion of mean square error. However, many practical applications may require the filtering algorithm to be of a form that it can be computed efficiently, as such, many simplified filters have been developed for the case of linear dynamic and measurement models. Johnson [17] modified the Kalman filter into a form which has a structure of smaller dimension than that of the signal process. Athans, et al. [18][19] restricted the Kalman gain matrix to be piece-wise constant or to be approximated by a simple function. There seems to be a scarcity of work dealing with the nonlinear filtering problems with complexity constraints. The approach used in this chapter is to fix the structure of the filter to a particular form with a set of parameters which are, then, chosen optimally to minimize an error criterion (the mean square error). For illustration, the filters for the problem formulated in section 1.2 is taken to be

$$\hat{x}_{i+1} = g(\hat{x}_i, \hat{\alpha}_i, a_i) \quad ; \quad \hat{x}_o = \bar{x}_o \tag{3-1}$$

where g is a fixed sample function which requires only a limited amount of computations and a_i's are coefficients which are to be optimally chosen to minimize the mean square error. In particular, the application of (3-1) to the strapdown attitude computation is considered in this chapter.

3.2 Formulation of the Problem

Consider the strapdown attitude updating problem of Chapter two, where

$$T_{i+1} = e^{\gamma_i} T_i \quad ; \quad T_o = I \tag{3-2}$$

and measurements

$$z_i = \begin{bmatrix} \theta_i(1) \\ \theta_i(2) \\ \theta_i(3) \end{bmatrix} + \begin{bmatrix} \epsilon(1) \\ \epsilon(2) \\ \epsilon(3) \end{bmatrix} \tag{3-3}$$

where θ_i is a vector vandom process. The structure of the constrained filter is chosen as that of Algorithm A (the second order algorithm, refer to Chapter two) but with the weighting coefficients to be chosen optimally, that is, the constrained filter is of the form

$$\hat{T}_{i+1} = (a_i(1)\, I + a_i(2)\, C_i + a_i(3)\, C_i^2)\, T_i \quad ; \quad \hat{T}_o = I$$

where C_i is defined by (2-15).

The set of $a_i(1), a_i(2)$ and $a_i(3)$ are to be chosen to minimize the cumulative mean square error

$$J = \sum_{i=0}^{N} E\left(\mathrm{Trace}\, [(T_i - \hat{T}_i)(T_i - \hat{T}_i)^T]\right) \tag{3-5}$$

The approach which is used in solving this problem is to transform the stochastic optimization problem into deterministic problem by working with mean and variance [17]. In next section, the constraint filter for the single axis rotation case is designed. Numerical results are presented in section 3.4 showing the advantage of this filtering algorithm. Extention to general case (three axes rotation case) is straightforward in principle but very complicate in derivation. Hence the general three axes rotation case will not be considered in this paper.

3.3 Single-axis Rotation with Statistically Uncorrelated Angular Increments

Consider the single axis rotation case as specified in section 2.3. Let us assume that $\theta_i(3)$ is taken to be a purely Gaussian random process with mean M and variance Q. The problem is to find $a_i(j)$ of the constrained filter Eq. (3-4), for $i = 0, 1, 2, \ldots\ldots$, and $j = 1, 2, 3$, such that the performance index J as given by Eq. (3-5) is a minimum with T_i and z_i given by Eq. (3-2) and Eq. (3-3) respectively. In general, analytic solution is not available. The derivation for reaching a numerical solution is straightforward but tedious, hence it is given in Appendix D for interested readers. In next section, the numerical results of applying the constrained filter will be given and comparison are made with Algorithm A.

3.4 Numerical Results

The optimization procedure in finding the optimal coefficients $a_i(1), a_i(2)$ and $a_i(3)$ is carried out by applying the well-known steepest descent gradient method to the equivalent optimization problem as derived in Appendix D. Here, we consider a case where

$$M = 0. \text{ radian}$$
$$Q = 0.1 \text{ radian}^2$$
$$R = 0.001 \text{ radian}^2$$

The initial guess $a_i(1), a_i(2)$ and $a_i(3)$ for $i = 1, \text{---------}, N$ is chosen to be those used in algorithm A, that is

$$a_i^{(0)} = 1.$$

$$a_i^{(0)} = 1. \qquad ; \qquad \text{for } i = 0, \text{----------}, N$$

$$a_i^{(0)} = \frac{1}{2}$$

After optimization has been done, the set of optimal coefficients $a_i(1), a_i(2)$ and $a_i(3)$, for all i are tested in a Monte Carlo simulation. Also included in this simulation are the filtering algorithms derived in Chapter one, optimal filtering algorithm of Chapter two, and Algorithm A. The results are plotted in Figure 3.1. It is seen that the constrained filter gives a much lower J than the Algorithm A, although the computational load is the same for both algorithms. Further improvement can be obtained by the use of the zeroth order filter but this is done at the expense of higher computational load. It is noted that the first order filter and the zeroth order filter for this case are identical and is expressed by

$$\hat{T}_{i+1} = e^{A\hat{\theta}_i} \hat{T}_i \quad ; \quad \hat{T}_o = I$$

where $\hat{\theta}_i$ is obtained through the use of Kalman-Bucy filter.

It is interesting to point out that J obtained in the process of finding the optimal $a_i(k)$'s is very close to the simulated J. Specifically,

$$J_{\text{Simulation}} = 0.665600 \ \text{radian}^2$$

$$J_{\text{Optimal}} = 0.654863 \ \text{radian}^2$$

This fact leads credence to the mathematical derivation developed in Appendix D.

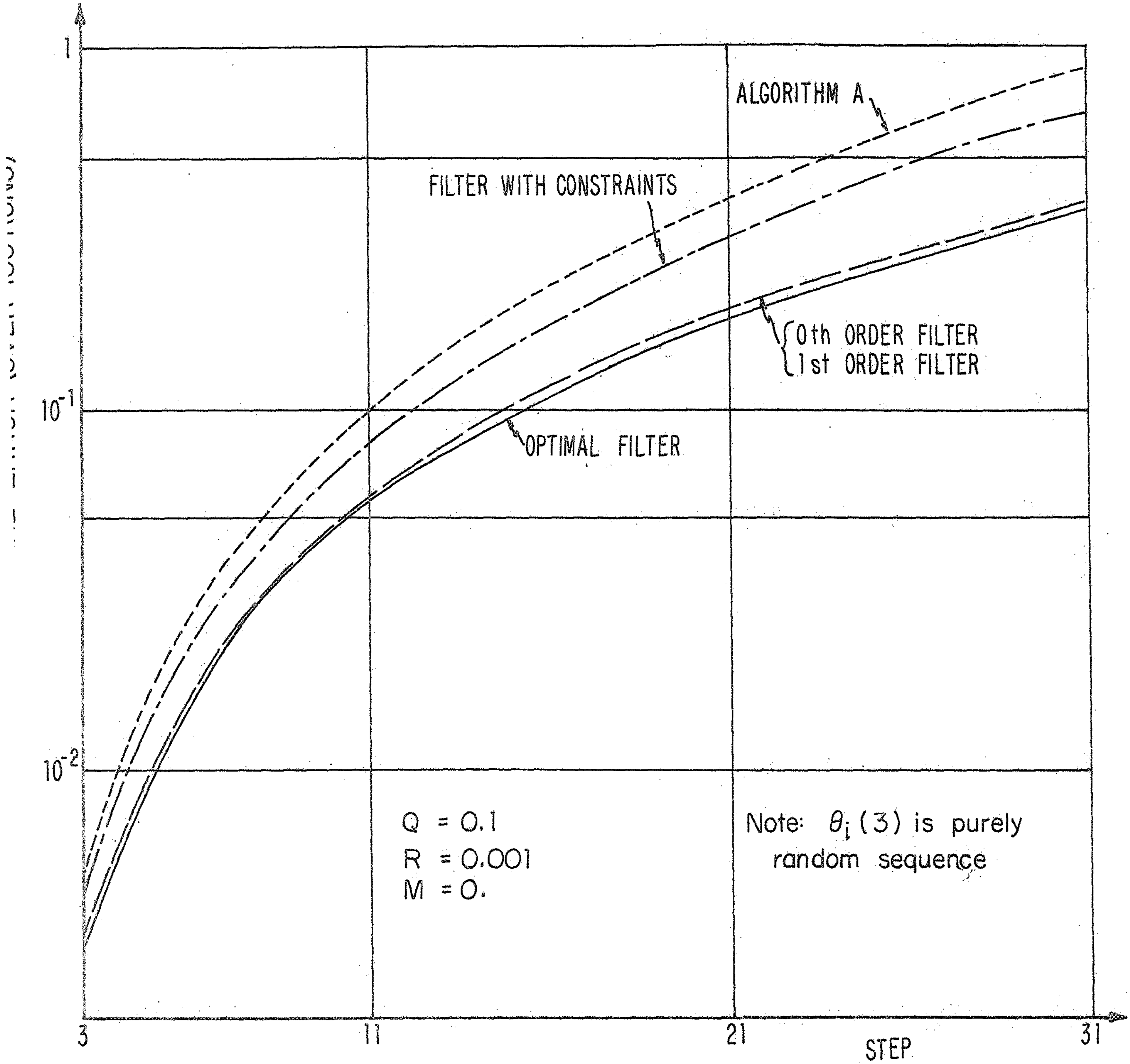

FIG. 3.1 SIMULATION OF FILTERING ALGORITHMS (SINGLE AXIS CASE)

In the case that the mean of $\theta_i(3)$ is non-zero ($M \neq 0$), the use of the constrainted filter is much more favorable than the use of Algorithm A as shown in Figure 3.2.

3.5 Discussion

i) Section 3.3 shows a case of designing the filtering algorithm under computational constraints. This development is ready to extend to the rotation around three axes case. However it is a very messy problem, hence it will not be carried out here.

ii) The coefficient $a_i(1)$, $a_i(2)$ and $a_i(3)$, in general, will be time-varying. In the case of limited storages, the design of piece-wise constant coefficients is desirable. The technique for designing piece-wise constant control as given in [20] can be employed for this purpose.

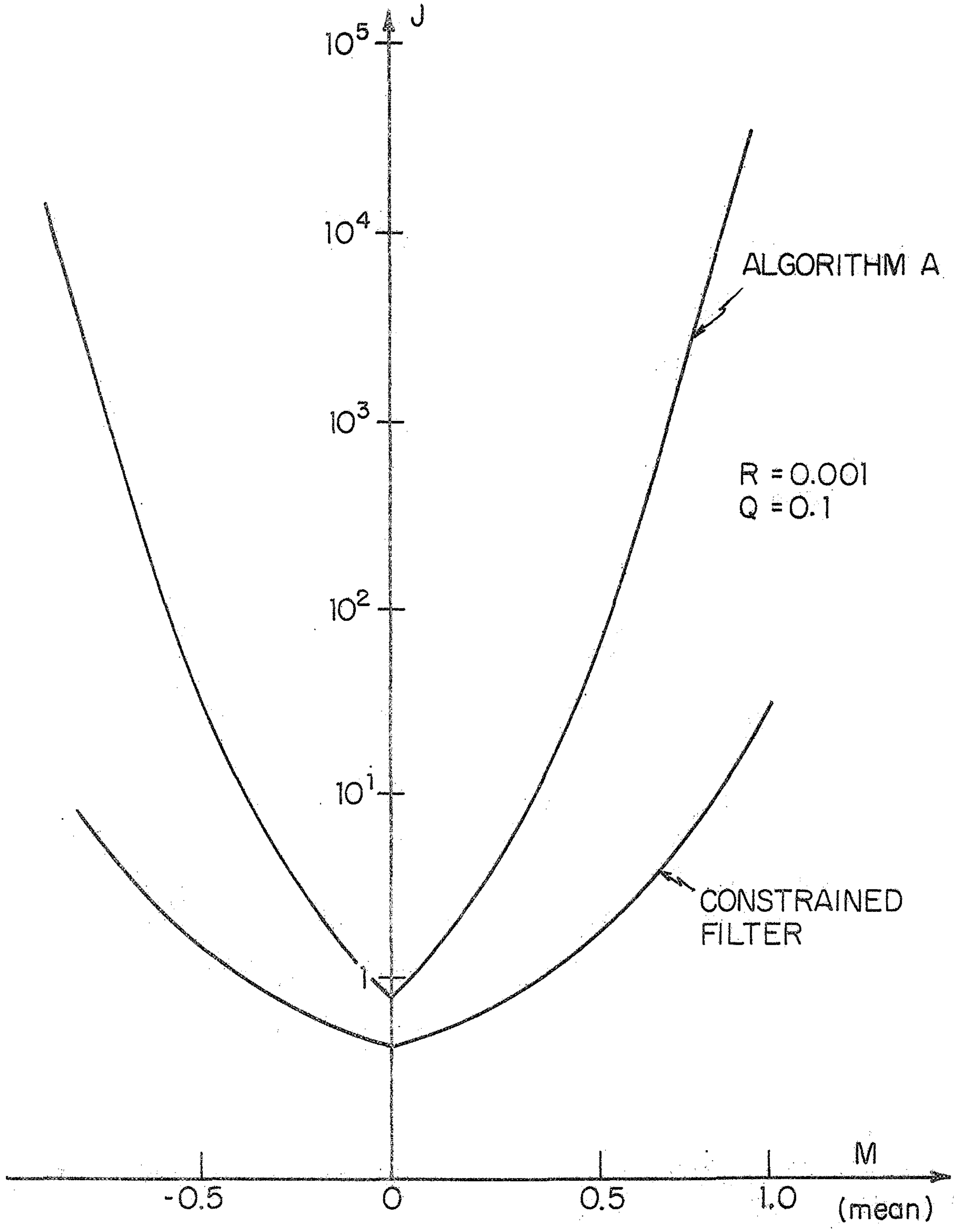

FIG. 3.2 PLOT OF J vs. M FOR SINGLE AXIS ROTATION CASE

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APPENDIX A

DERIVATION OF AN UPDATING EQUATION FOR $P_{x_i \alpha_i}$

In this appendix, an iterative expression for updating $P_{x_i \alpha_i}$ will be derived, where $P_{x_i \alpha_i}$ is defined by (1-27).

By definition

$$P_{x_i \alpha_i} = E\left[(x_i - \hat{x}_i)(\alpha_i - \hat{\alpha}_i)^T / Z_i\right] \tag{A-1}$$

Using (1-12), we have

$$\begin{aligned} P_{x_{i+1}\alpha_{i+1}} = & \iiiint x_{i+1}\alpha_{i+1}^T \, \delta(x_{i+1} - f(\alpha_i, x_i)) \, p(\alpha_{i+1}/Z_{i+1}, \alpha_i) \\ & \cdot p(x_i/\alpha_i, Z_i) \, p(\alpha_i/Z_{i+1}) \, d\alpha_i \, dx_i \, d\alpha_{i+1} \, dx_{i+1} \\ & - \hat{x}_{i+1}\hat{\alpha}_{i+1}^T \end{aligned}$$

or after manipulations, we have

$$\begin{aligned} P_{x_{i+1}\alpha_{i+1}} = & \iint f(\alpha_i, x_i) \, p(x_i/\alpha_i, Z_i) \, p(\alpha_i/Z_{i+1}) \\ & \cdot \left[\int \alpha_{i+1}^T \, p(\alpha_{i+1}/Z_{i+1}, \alpha_i) \, d\alpha_{i+1}\right] dx_i \, d\alpha_i \\ & - \hat{x}_{i+1}\hat{\alpha}_{i+1}^T \end{aligned} \tag{A-2}$$

The last integral on the right hand side of (A-2) can be written as

$$\int \alpha_{i+1}^T \, p(\alpha_{i+1}/Z_{i+1}, \alpha_i) \, d\alpha_{i+1} = \left[\Phi\alpha_i + K_{i+1}(z_{i+1} - H\Phi\alpha_i)\right]^T \tag{A-3}$$

where

$$K_{i+1} = (Q_i^{-1} + H^T R_{i+1}^{-1} H)^{-1} H^T R_i^{-1}$$

Substituting (A-3) into (A-2), yields

$$P_{x_{i+1}\alpha_{i+1}} = \iint (\Phi\alpha_i + K_{i+1}(z_{i+1} - H\Phi\alpha_i))^T f(\alpha_i, x_i) \cdot p(x_i/\alpha_i, Z_i)p(\alpha_i/Z_{i+1})\, d\alpha_i\, dx_i - \hat{x}_{i+1}\hat{\alpha}_{i+1}^T \tag{A-4}$$

Defining $A_i = \Phi - K_{i+1}H\Phi$, (A-4) can be written as

$$P_{x_{i+1}\alpha_{i+1}} = \left[\iint f(\alpha_i, x_i)(\alpha_i - \hat{\alpha}_i)^T p(\alpha_i/Z_{i+1})\, p(x_i/\alpha_i, Z_i)\, dx_i\, d\alpha_i\right] A_i^T + \hat{x}_{i+1}(\Phi\hat{\alpha}_i + K_{i+1}(z_{i+1} - H\Phi\hat{\alpha}_i))^T - \hat{x}_{i+1}\hat{\alpha}_{i+1}^T \tag{A-5}$$

Expanding $f(\alpha_i, x_i)$ along $\hat{x}_i, \hat{\alpha}_i$ in Taylor series, we have

$$f(\alpha_i, x_i) = f(\hat{\alpha}_i, \hat{x}_i) + \left.\frac{\partial f}{\partial \alpha_i}\right|_{\hat{\alpha}_i, \hat{x}_i}(\alpha_i - \hat{\alpha}_i) + \left.\frac{\partial f}{\partial x_i}\right|_{\hat{\alpha}_i, \hat{x}_i}(x_i - \hat{x}_i) + \text{(Second order and higher order terms)} \tag{A-6}$$

Substituting (A-6) up to first order into (A-5) and using the integration results in Chapter one, (A-5) is written into

$$P_{x_{i+1}\alpha_{i+1}} = \left[\left.\frac{\partial f}{\partial \alpha_i}\right|_{\hat{\alpha}_i, \hat{x}_i} P_{\alpha_{i/i+1}} + \left.\frac{\partial f}{\partial x_i}\right|_{\hat{\alpha}_i, \hat{x}_i} P_{x_i\alpha_i} P_{\alpha_i}^{-1} P_{\alpha_{i/i+1}}\right] A_i^T + \hat{x}_{i+1}(K_{i/i+1}(z_{i+1} - H\Phi\hat{\alpha}_i))^T A_i^T + \hat{x}_{i+1}(\Phi\hat{\alpha}_i + K_{i+1}(z_{i+1} - H\Phi\hat{\alpha}_i))^T - \hat{x}_{i+1}\hat{\alpha}_{i+1}^T \tag{A-7}$$

The last three terms on the right hand side of (A-7) are equal to

$$\hat{x}_{i+1}\left[(\Phi\hat{\alpha}_i + (A_iK_{i/i+1} + K_{i+1})(z_{i+1} - H\Phi\hat{\alpha}_i))^T - \hat{\alpha}_{i+1}^T\right].$$

In the following, we are going to show that

$$\hat{\alpha}_{i+1} = \Phi\hat{\alpha}_i + (A_iK_{i/i+1} + K_{i+1})(z_{i+1} - H\Phi\hat{\alpha}_i) \qquad \text{(A-8)}$$

This will make the last three terms of (A-7) equivalent to zero.

By definition,

$$\hat{\alpha}_{i+1} = E(\alpha_{i+1}/Z_{i+1}) \qquad \text{(A-9)}$$

or

$$\hat{\alpha}_{i+1} = \int \alpha_{i+1} p(\alpha_{i+1}/Z_{i+1})\, d\alpha_{i+1}$$

we have

$$\begin{aligned}\hat{\alpha}_{i+1} &= \iint \alpha_{i+1} p(\alpha_{i+1}/\alpha_i, Z_{i+1})\, p(\alpha_i/Z_{i+1})\, d\alpha_i\, d\alpha_{i+1} \\ &= \int \left[\int \alpha_{i+1} p(\alpha_{i+1}/\alpha_i, Z_{i+1})\, d\alpha_{i+1}\right] p(\alpha_i/Z_{i+1})\, d\alpha_i\end{aligned} \qquad \text{(A-10)}$$

The inner integral on the right hand side of (A-10) is given by the transposition of (A-3), hence (A-10) can be written as

$$\hat{\alpha}_{i+1} = \int \left[\Phi\alpha_i + K_{i+1}(z_{i+1} - H\Phi\alpha_i)\right] p(\alpha_i/Z_{i+1})\, d\alpha_i \qquad \text{(A-11)}$$

From the definition of A_i, (A-11) is written into the form

$$\hat{\alpha}_{i+1} = \int A_i\alpha_i\, p(\alpha_i/Z_{i+1})\, d\alpha_i + K_{i+1}z_{i+1} \qquad \text{(A-12)}$$

From Eq. (1-23), we have

$$\begin{aligned}\hat{\alpha}_{i+1} &= A_i\left[\hat{\alpha}_i + K_{i/i+1}(z_{i+1} - H\Phi\hat{\alpha}_i)\right] + K_{i+1}z_{i+1} \\ &= \Phi\hat{\alpha}_i + (A_iK_{i/i+1} + K_{i+1})(z_{i+1} - H\Phi\hat{\alpha}_i)\end{aligned} \qquad \text{(A-13}$$

Substituting (A-13) into (A-7), we have the iterative equation for updating $P_{x_i \alpha_i}$, that is,

$$P_{x_{i+1}\alpha_{i+1}} = \left[\frac{\partial f}{\partial \alpha_i} \bigg|_{\hat{\alpha}_i, \hat{x}_i} P_{\alpha_{i/i+1}} + \frac{\partial f}{\partial x_i} \bigg|_{\hat{\alpha}_i, \hat{x}_i} P_{x_i \alpha_i} P_{\alpha_i}^{-1} P_{\alpha_{i/i+1}} \right] A_i^T \qquad \text{(A-14)}$$

APPENDIX B

SOME CHARACTERISTICS OF DIRECTION COSINE MATRIX

Some characteristics of the direction cosine matrix are considered in the following.

1. The analytical expression of e^{γ_i} is given by

$$e^{\gamma_i} = I + \frac{1}{\sigma_i}(\mathrm{Sin}\,\sigma_i)\gamma_i + \frac{1}{\sigma_i^2}(1 - \mathrm{Cos}\,\sigma_i)\gamma_i^2 \tag{B-1}$$

where γ_i is given by (2-3) and

$$\sigma_i = \sqrt{\theta_i(1)^2 + \theta_i(2)^2 + \theta_i(3)^2} \tag{B-2}$$

Proof:

From the definition of e^{γ_i},

$$e^{\gamma_i} = I + \gamma_i + \frac{1}{2!}\gamma_i^2 + \frac{1}{3!}\gamma_i^3 + \ldots\ldots\ldots\ldots \tag{B-3}$$

Since γ_i is a skew-symmetrical matrix, it can be shown that

$$\gamma_i^3 = -\sigma_i^2\gamma_i \tag{B-4}$$

where σ_i is given by (B-2).

Using (B-4) and collecting even and odd terms in (B-3), after manipulation, (B-3) is converted into,

$$e^{\gamma_i} = I + \left(\frac{1}{2!} - \frac{\sigma_i^2}{4!} + \frac{\sigma_i^4}{6!} - \ldots\ldots\ldots\ldots\right)\gamma_i^2$$
$$+ \left(1 - \frac{\sigma_i^2}{3!} + \frac{\sigma_i^4}{5!} - \frac{\sigma_i^6}{7!} + \ldots\ldots\ldots\right)\gamma_i$$

or

$$e^{\gamma_i} = I + \frac{1}{\sigma_i^2}\left(\frac{\sigma_i^2}{2!} - \frac{\sigma_i^4}{4!} + \frac{\sigma_i^6}{6!} - \cdots\cdots\cdots\right)\gamma_i^2$$

$$+ \frac{1}{\sigma_i}\left(\sigma_i - \frac{\sigma_i^3}{3!} + \frac{\sigma_i^5}{5!} - \frac{\sigma_i^7}{7!} + \cdots\cdots\right)\gamma_i \tag{B-5}$$

By summing up the series, we have

$$e^{\gamma_i} = I + \frac{1}{\sigma_i^2}\,(1 - \mathrm{Cos}\,\sigma_i)\,\gamma_i^2 + \frac{1}{\sigma_i}(\mathrm{Sin}\,\sigma_i)\,\gamma_i$$

Q. E. D.

It is interesting to note that if $\theta_i(j)$ for $j = 1, 2, 3$ is small, or mathematically, $\sigma_i \longrightarrow 0$, then the weighting coefficients of γ_i and γ_i^2 in (B-1) approach 1 and $\frac{1}{2}$, respectively. In this case, (B-1) is of the form

$$e^{\gamma_i} = I + \gamma_i + \gamma_i^2/2 \tag{B-6}$$

In essence, (B-6) is the Algorithm A which is given by (2-15).

2. e^{γ_i} is an orthonormal matrix

Proof:

Since γ_i is a skew-symmetrical matrix, we have

$$e^{\gamma_i}[e^{\gamma_i}]^T = e^{\gamma_i}e^{-\gamma_i} = I \tag{B-7}$$

Q. E. D.

3. If T_i is an orthonormal matrix, then T_{i+1} is also an orthonormal matrix, where $T_{i+1} = e^{\gamma_i}\,T_i$.

Proof:

$$T_{i+1}T_{i+1}^T = e^{\gamma_i}T_iT_i^T[e^{\gamma_i}]^T \tag{B-8}$$

Since

$$T_i T_i^T = I$$

(E-8) is converted into

$$T_{i+1} T_{i+1}^T = e^{\gamma_i} [e^{\gamma_i}]^T$$

By (E-7), we have

$$T_{i+1} T_{i+1}^T = I$$

Q. E. D.

4. Physical interpretation of the performance index given by (2-16).

Let R_I and R_b be a unit vector with respect to the inertial coordinates and body coordinates respectively, then

$$R_I = T\, R_b$$

where T is the true direction cosine matrix.

If the computed direction cosine matrix $\hat{T}$ is used to transform R_b into inertial coordinates, then we have

$$\begin{aligned} R_I' &= \hat{T}\, R_b \\ &= \hat{T}\, T^T R_I \end{aligned}$$

The difference vector between R_I and R_I' is

$$R_I' - R_I = (\hat{T}\, T^T - I) R_I$$

and the magnitude of the difference vector is

$$M = |R_I' - R_I| .$$

or

$$M^2 = R_I^T (\hat{T}\, T^T - I)^T (\hat{T}\, T^T - I)\, R_I \tag{B-9}$$

Assuming R_I being uniformly distributed over a unit sphere and $\hat{T}$ being randomly distributed, then

$$E(M^2) = \text{Trace}\left[E((\hat{T}\,T^T - I)^T(\hat{T}\,T^T - I))\right] \tag{B-10}$$

Using $T\,T^T = I$, (B-10) is converted into

$$E(M^2) = \text{Trace}\left[E(I - \hat{T}\,T^T - T\,\hat{T}^T + \hat{T}\,\hat{T}^T)\right] \tag{B-11}$$

or

$$E(M^2) = \text{Trace}\left[E((T - \hat{T})(T - \hat{T})^T\right]$$

$$= J$$

where J is the performance index given by (2-16).

We conclude that the performance index J is a measure of the expected magnitude square of the difference vector between the vectors which are transformed from the body coordinates to the inertial coordinates by using T and $\hat{T}$ respectively.

APPENDIX C

EXPLANATION OF THE CONTRADICTIVE RESULTS IN SIMULATION 3 OF CHAPTER TWO

Referring to Simulation 3 of chapter two and Figure 2.6, the contradictive results of simulation 3 are explained by the following two reasons:

1. For the numerical values used in simulation 3, the noise level on both measurement and process are extremely small that the magnitude of $z_i(3)$ is of the order of 10^{-3} rad. This small angular measurement makes Algorithm A a good approximation of (2-5), since Algorithm A is the second order Taylor series expansion of (2-5) with $\theta_i(3)$ replaced by $z_i(3)$. In other words, in the small angle situation, Algorithm A performs as well as (C-1), which is given by

$$T^*_{i+1} = e^{Az_i(3)} T^*_i \quad ; \quad T^*_o = T_o \tag{C-1}$$

   where A is specified by (2-7).

2. In the following, we present a mathematical analysis to explain the result of simulation 3. This analysis is based on the previous argument and will show that in the case of highly correlated $\theta_i(3)$ process ($\beta(3) = 1.$), Algorithm A tends to work better than the zero order filter.

Throughout the following derivation, $|z_i(3)|$ is assumed to be small that we can replace Algorithm A by (C-1). Terminal mean square error produced by using Algorithm A as well as that produced by using the zeroth order filter will be analytically derived to explain the contradictive simulation results.

a) Derivation of J_N^* generated by applying Algorithm A

Iteratively using (C-1), we have

$$T_N^* = e^{A\sum_{i=0}^{N-1} z_i(3)} T_o^* \quad ; \quad T_o^* = I \tag{C-2}$$

Similarly, form (2-6), we have

$$T_N = e^{A\sum_{i=0}^{N-1} \theta_i(3)} T_o \quad ; \quad T_o = I \tag{C-3}$$

The error criterion given by (2-16) can be written into

$$\begin{aligned} J_N^* &= \text{Trace } E\left[2(I - T_N^* T_N^T)\right] \\ &= \text{Trace } E\left[2(I - e^{A\sum_{i=0}^{N-1} \epsilon_i(3)})\right] \end{aligned} \tag{C-4}$$

where ϵ_i is the measurement noise (white and Gaussian with zero mean and R variance). Taking expectation of (C-4), we end up with

$$J_N^* = \text{Trace }\left[2(1 - e^{-\frac{1}{2}(N-1)R})I\right] \tag{C-5}$$

It is noted that in (C-5) if $R = 0$, $J_N^* = 0$.

b) Derivation of J_N generated by applying the zeroth order filter

The zeroth order filter can be written as

$$\hat{T}_{i+1} = e^{A\hat{\theta}_i(3)}\hat{T}_i \quad ; \quad \hat{T}_o = I \tag{C-6}$$

where $\hat{\theta}_i(3)$ is the best estimate of $\theta_i(3)$ obtained from applying Kalman-Bucy filter. Following the same derivation used in a). We have

$$J_N = \text{Trace}\left[2(1 - e^{-\frac{1}{2}S_N})I\right] \tag{C-7}$$

where the scalar S_N is given by (C-8)

$$S_N = E\left[\sum_{i=0}^{N-1} (\hat{\theta}_i(3) - \theta_i(3))\right]^2 \tag{C-8}$$

Now, we are in a position to explain this contradictive simulation results by comparing J_N^* and J_N as given by (C-6) and (C-7) respectively. In the case of highly correlated $\theta_i(3)$ process ($\beta(3) = 1.$), the correlations between $(\hat{\theta}_i(3) - \theta_i(3))$ and $(\hat{\theta}_j(3) - \theta_j(3))$, for i, j=1,................., N-1, would make $S_n > (N-1)R$, for larger N, and hence

$$J_N > J_N^*$$

However, in the early steps (N is small), the estimated error is smaller than the measurement error or $J_N < J_N^*$ as is shown in Figure 2.6.

APPENDIX D

DERIVATION OF THE CONSTRAINED FILTER FOR SINGLE-AXIS ROTATION CASE

Consider the stochastic optimization problem as formulated in Section 3.3. For clarity, we restate the problem as follows.

1. The Attitude are updated by

$$T_{i+1} = e^{\gamma_i} T_i \quad ; T_o = I \tag{D-1}$$

Since the vehicle is rotating around one single body axis, only four elements in T are affected by this rotation. All other elements remain unchanged. From Eq. (D-1), these four changing elements in T are physically updated by the following equation.

$$\begin{bmatrix} T_{i+1}(1,1) & T_{i+1}(1,2) \\ T_{i+1}(2,2) & T_{i+1}(2,2) \end{bmatrix} = \begin{bmatrix} \text{Cos } \theta_i(3) & \text{Sin } \theta_i(3) \\ -\text{Sin } \theta_i(3) & \text{Cos } \theta_i(3) \end{bmatrix} \cdot \begin{bmatrix} T_i(1,1) & T_i(1,2) \\ T_i(2,1) & T_i(2,2) \end{bmatrix}$$

or simply

$$T_{i+1} = \Phi_i T_i \quad ; \quad T_o = I \tag{D-2}$$

where we have defined

$$T_i = \begin{bmatrix} T_i(1,1) & T_i(1,2) \\ T_i(2,1) & T_i(2,2) \end{bmatrix}$$

and

$$\Phi_i = \begin{bmatrix} \text{Cos } \theta_i(3) & \text{Sin } \theta_i(3) \\ -\text{Sin } \theta_i(3) & \text{Cos } \theta_i(3) \end{bmatrix}$$

This notation is defined exclusively for use in this appendix.

2. Measurement

$$z_i(3) = \theta_i(3) + \epsilon_i(3) \tag{D-3}$$

3. $\theta_i(3)$ is a Gaussian white sequence with mean M and variance Q.

4. Constrained filter

$$\hat{T}_{i+1} = (a_i(1)\, I + a_i(2)\, C_i + a_i(3)\, C_i^2)\hat{T}_i \quad ; \quad \hat{T}_o = I \tag{D-4}$$

where

$$C_i = \begin{bmatrix} 0 & z_i(3) \\ -z_i(3) & 0 \end{bmatrix}$$

5. Performance index

$$J = \sum_{i=0}^{N} E\left[(T_i - \hat{T}_i)(T_i - \hat{T}_i)^T\right] \tag{D-5}$$

The problem is to choose the set of weighting coefficients $a_i(1), a_i(2)$ and $a_i(3)$ such that J is a minimum.

For the convenience of later developement, Eq. (D-5) is rewritten into

$$J = J^{(1)} + J^{(2)} \tag{D-6}$$

where

$$J^{(k)} = \sum_{i=0}^{N} E\left[T_i(k) - \hat{T}_i(k))^T (T_i(k) - \hat{T}_i(k))\right] \quad ; \quad \text{for } k = 1, 2 \qquad \text{(D-7)}$$

and $T_i(k)$ is the kth column vector of $T_i(k)$, so is $\hat{T}_i(k)$.

Defining

$$\begin{aligned} P_i^{(k)} &= E(T_i(k)\ T_i(k)^T) \\ P_i^{(k)} &= E(\hat{T}_i(k)\ \hat{T}_i(k)^T) \quad ; \quad \text{for } k = 1, 2 \\ S_i^{(k)} &= E(\hat{T}_i(k)\ T_i(k)^T) \end{aligned} \qquad \text{(D-8)}$$

we convert the stochastic optimization problem defined by Eqs. (D-2), (D-3), (D-4) and (D-6) into a deterministic optimization problem as follows. Using (D-8), (D-7) becomes

$$J^{(k)} = \sum_{i=0}^{N} \text{Trace}\ (P_i^{(k)} + \hat{P}_i^{(k)} - 2S_i^{(k)}) \quad ; \quad \text{for } k = 1, 2 \qquad \text{(D-9)}$$

where $P_i^{(k)}, \hat{P}_i^{(k)}$ and $S_i^{(k)}$ are computed by the iterative equations to be derived.

For the sake of convenience, $\theta_i(3)$ will be simply denoted by θ_i. In the following, the iterative equations for updating $P_i^{(k)}, \hat{P}_i^{(k)}$ and $S_i^{(k)}$, for $k = 1, 2$, will be derived. These iterative equations together with Eq. (D-9) constitute a deterministic optimization problem.

1. From Eqs. (D-2) and (D-8), we have

$$P_{i+1}^{(k)} = E(T_{i+1}(k)\ T_{i+1}(k)^T)$$

$$= E(\Phi_i T_i(k)\ T_i(k)^T \Phi_i^T)$$

$$= E(\Phi_i P_i^{(k)} \Phi_i^T) \qquad \text{(D-10)}$$

The three distinct elements in $P_i^{(k)}$ are updated by

$$\begin{bmatrix} P_{i+1}^{(k)}(1,1) \\ P_{i+1}^{(k)}(1,2) \\ P_{i+1}^{(k)}(2,2) \end{bmatrix} = E\left\{ \begin{bmatrix} \mathrm{Cos}^2\theta_i & 2\mathrm{Sin}\ \theta_i\ \mathrm{Cos}\ \theta_i & \mathrm{Sin}^2\theta_i \\ -\mathrm{Sin}\ \theta_i\ \mathrm{Cos}\ \theta_i & \mathrm{Cos}\ 2\theta_i & \mathrm{Cos}\ \theta_i\ \mathrm{Sin}\ \theta_i \\ \mathrm{Sin}^2\theta_i & -2\mathrm{Sin}\ \theta_i\ \mathrm{Cos}\ \theta_i & \mathrm{Cos}^2\theta_i \end{bmatrix} \right\} \cdot \begin{bmatrix} P_i^{(k)}(1,1) \\ P_i^{(k)}(1,2) \\ P_i^{(k)}(2,2) \end{bmatrix} \qquad \text{(D-11)}$$

where Eq. (D-11) is obtained directly from Eq. (D-10).

The initial condition of Eq. (D-11) is

$$\begin{bmatrix} P_o^{(1)}(1,1) \\ P_o^{(1)}(1,2) \\ P_o^{(1)}(2,2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad \begin{bmatrix} P_o^{(2)}(1,1) \\ P_o^{(2)}(1,2) \\ P_o^{(2)}(2,2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(D-12)}$$

(D-11) is obtained by using the fact that θ_i is a purely random sequence. The expected values of the elements on the right hand side of (D-11) are obtained by using the characteristic function [10, pp. 160] of the Gaussian random variable $2\theta_i$. They are

$$E(\cos^2\theta_i) = \frac{1}{2} + \frac{1}{2}(\cos 2M)\, e^{-2Q}$$

$$E(\sin^2\theta_i) = \frac{1}{2} - \frac{1}{2}(\cos 2M)\, e^{-2Q}$$

and

$$E(\sin\theta_i \cos\theta_i) = \frac{1}{2}(\sin 2M)\, e^{-2Q}$$

2. From (D-2), (D-4) and (D-8), and defining

$$D_i = a_i(1)\, I + a_i(2)\, C_i + a_i(3)\, C_i^2 \qquad \text{(D-13)}$$

we have

$$\hat{P}_{i+1}^{(k)} = E(D_i\, \hat{P}_i^{(k)}\, D_i^T)$$

The three distinct elements are updated by

$$\begin{bmatrix} \hat{P}_{i+1}^{(k)}(1,1) \\ \hat{P}_{i+1}^{(k)}(1,2) \\ \hat{P}_{i+1}^{(k)}(2,2) \end{bmatrix} = E \begin{bmatrix} D_i(1,1)^2 & 2\, D_i(1,1)\, D_i(1,2) & D_i(1,2)^2 \\ -D_i(1,1)\, D_i(1,2) & (D_i(1,1)^2 - D_i(1,2)^2) & D_i(1,1) D_i(1,2) \\ D_i(1,2)^2 & -2D_i(1,1)\, D_i(1,2) & D_i(1,1)^2 \end{bmatrix} \cdot \begin{bmatrix} \hat{P}_i^{(k)}(1,1) \\ \hat{P}_i^{(k)}(1,2) \\ \hat{P}_i^{(k)}(2,2) \end{bmatrix} \qquad \text{(D-14)}$$

with initial conditions

$$\begin{bmatrix} \hat{P}_o^{(1)}(1,1) \\ \hat{P}_o^{(1)}(1,2) \\ \hat{P}_o^{(1)}(2,2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \begin{bmatrix} \hat{P}_o^{(2)}(1,1) \\ \hat{P}_o^{(2)}(1,2) \\ \hat{P}_o^{(2)}(2,2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(D-15)}$$

The expected values of the elements on the right hand side of Eq. (D-14) are

$$E(D_i(1,1)^2) = a_i(1)^2 - 2a_i(1)a_i(3)(R + Q + M^2)$$

$$+ a_i(3)^2(3(R + Q)^2 + 6M^2(R + Q) + M^4)$$

$$E(D_i(1,2)^2) = a_i(2)^2(R + Q + M^2)$$

and

$$E(D_i(1,1)D_i(1,2)) = a_i(1)a_i(2)M - a_i(2)a_i(3)(3M(R + Q) + M^3)$$

3. From Eqs. (D-4) and (D-8), we have

$$S_{i+1}^{(k)} = E(\hat{T}_{i+1}(k)\ T_{i+1}(k)^T)$$

$$= E(D_i S_i \Phi_i^T)$$

The four distinct elements are updated by

$$\begin{bmatrix} S_{i+1}^{(k)}(1,1) \\ S_{i+1}^{(k)}(1,2) \\ S_{i+1}^{(k)}(2,1) \\ S_{i+1}^{(k)}(2,2) \end{bmatrix} = E\left\{\begin{bmatrix} D_i(1,1)\ \mathrm{Cos}\ \theta_i & D_i(1,2)\ \mathrm{Cos}\ \theta_i & D_i(1,1)\ \mathrm{Sin}\ \theta_i & D_i(1,2)\ \mathrm{Sin}\ \theta_i \\ -D_i(1,2)\ \mathrm{Cos}\ \theta_i & D_i(1,1)\ \mathrm{Cos}\ \theta_i & -D_i(1,2)\ \mathrm{Sin}\ \theta_i & D_i(1,1)\ \mathrm{Sin}\ \theta_i \\ -D_i(1,1)\ \mathrm{Sin}\ \theta_i & -D_i(1,2)\ \mathrm{Sin}\ \theta_i & D_i(1,1)\ \mathrm{Cos}\ \theta_i & D_i(1,2)\ \mathrm{Cos}\ \theta_i \\ -D_i(1,2)\ \mathrm{Sin}\ \theta_i & -D_i(1,1)\ \mathrm{Sin}\ \theta_i & -D_i(1,2)\ \mathrm{Cos}\ \theta_i & D_i(1,1)\ \mathrm{Cos}\ \theta_i \end{bmatrix}\right\} \begin{bmatrix} S_i^{(k)}(1,1) \\ S_i^{(k)}(1,2) \\ S_i^{(k)}(2,1) \\ S_i^{(k)}(2,2) \end{bmatrix} \qquad \text{(D-16)}$$

with initial condition

$$\begin{bmatrix} S_o^{(1)}(1,1) \\ S_o^{(1)}(1,2) \\ S_o^{(1)}(2,1) \\ S_o^{(1)}(2,2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} S_o^{(2)}(1,1) \\ S_o^{(2)}(1,2) \\ S_o^{(2)}(2,1) \\ S_o^{(2)}(2,2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(D-17)}$$

By using the characteristic function [10, pp. 160] of θ_i and its derivatives the expected values of the elements of Eq. (D-16) are

$$E(D_i(1,1)\ \mathrm{Cos}\ \theta_i) = (a_i(1) - a_i(3)R)\ \mathrm{Cos}\ M\ e^{-\frac{1}{2}Q}$$

$$+ a_i(3)[(Q^2 - Q - M^2) + 2MQ\ \mathrm{Sin}\ M]e^{-\frac{1}{2}Q}$$

$$E(D_i(1,2)\ \mathrm{Cos}\ \theta_i) = a_i(2)[M\ \mathrm{Cos}\ M - Q\ \mathrm{Sin}\ M]e^{-\frac{1}{2}Q}$$

$$E(D_i(1,1)\ \mathrm{Sin}\ \theta_i) = (a_i(1) - a_i(3)R)\ \mathrm{Sin}\ M\ e^{-\frac{1}{2}Q}$$

$$+ a_i(3)[(Q^2 - Q - M^2)\ \mathrm{Sin}\ M - 2MQ\ \mathrm{Cos}\ M]e^{-\frac{1}{2}Q}$$

and

$$E(D_i(1,2)\ \mathrm{Sin}\ \theta_i) = a_i(2)[Q\ \mathrm{Cos}\ M + M\ \mathrm{Sin}\ M]e^{-\frac{1}{2}Q}$$

The equivalent optimization problem:

Now the problem is to choose $a_i(1), a_i(2)$ and $a_i(3)$ for $i = 1, \text{--------}, N$ such that J given by (D-6) is optimized subject to (D-11), (D-14) and (D-16) as constraints. Variational calculus can be applied to the solution of this problem which results in a two point boundary value problem. However, no solution in analytical form is available. Numerical solution can be obtained through applying the numerical optimization techniques.

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