NASA TECHNICAL NOTE

NASA TN D-5743

c. 1

NASA TN D-5743

A FREQUENCY DOMAIN APPROACH TO THE DESIGN AND ANALYSIS OF LINEAR MULTIVARIABLE SYSTEMS

by William A. Wolovich

Electronics Research Center
Cambridge, Mass. 02139

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1970

1. Report No. NASA TN D-5743	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A FREQUENCY DOMAIN APPROACH TO THE DESIGN AND ANALYSIS OF LINEAR MULTIVARIABLE SYSTEMS		5. Report Date May 1970
		6. Performing Organization Code
7. Author(s) William A. Wolovich		8. Performing Organization Report No. C-104
9. Performing Organization Name and Address Electronics Research Center Cambridge, Mass., 02139		10. Work Unit No. 125-19-22-05
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration		13. Type of Report and Period Covered Technical Note
		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

We present one rather simple basic result, called the structure theorem, which allows one to combine time-domain and frequency-domain information in a concise expression for the transfer matrix of a linear multivariable system. This transfer matrix expression and the definitions required to formulate it are used to establish several new results, which, for the most part, can be classified as design algorithms, many of which can be implemented either by using pencil and paper methods alone, of the system order is relatively low, or by employing rather simple computer programs.

17. Key Words ·Linear Multivariable Control Systems ·Time Domain ·Frequency Domain ·Design Algorithms		18. Distribution Statement Unclassified — Unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 105	22. Price * $3.00

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151

A FREQUENCY DOMAIN APPROACH TO THE DESIGN AND ANALYSIS OF LINEAR MULTIVARIABLE SYSTEMS

By William A. Wolovich*
Office of Control Theory and Application
Electronics Research Center

SUMMARY

A new approach to the design and analysis of linear multivariable systems is given. Several applications of one rather simple basic result, called the structure theorem, are presented. This theorem allows us to combine time and frequency domain information in a compact, concise expression for the transfer matrix of a linear multivariable system. This expression and the definitions required to formulate it are used to establish new results, which, for the most part, can be classified as design algorithms that can be implemented by using pencil and paper methods alone, if the system order is relatively low, or by employing rather simple computer programs.

In particular, a new technique for obtaining realizations, given the transfer matrix of a system, is introduced. A direct, constructive technique for arbitrarily assigning all eigenvalues (closed loop poles) of a controllable multivariable system by using linear state variable feedback is also given. A simplified expression for the characteristic equation of a system compensated by linear output feedback is discussed from the viewpoint of pole assignment. Questions pertaining to linear optimal control are considered, and improvements are made over prior work in the areas of spectral factorization and the root-square locus. Specifically, a solution to the output regulator problem is formulated in the frequency domain, and spectral factorization is employed to obtain the optimal control law solution. A simplified expression is given for formulating and plotting the root-square locus. Various questions pertaining to the design of noninteracting control systems are also considered, and a procedure is given for determining which poles cannot be altered while decoupling a system using linear feedback. A design algorithm for achieving maximum pole assignment under linear decoupling feedback is presented. All systems which can be decoupled by using input dynamics are characterized, and an algorithm is outlined for achieving an asymptotically stable decoupled design with arbitrary pole placement. Two design algorithms for achieving a noninteractive design via linear state variable feedback are given. The design of a helicopter stability augmentation system based on desired handling qualities is presented. Many of the results are applied, and pencil and paper methods are employed to achieve the final control system used.

*The research reported has been carried out under the supervision of Prof. Peter L. Falb and has also been submitted to Brown University as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Electrical Sciences.

1. INTRODUCTION

We have witnessed a radical change in the methods used for the design and analysis of control systems during the past decade. Control systems engineers, who once relied almost entirely on classical frequency domain techniques such as the root locus, Bode plots, and Nyquist diagrams, are gradually adopting the modern state space approach to system design. This is primarily due to the rapid theoretical advances made over the past few years, especially in the area of optimal control, and to the evolution of the high speed, easily accessible digital computer. Designs which once took weeks of effort involving excessive multiloop analysis and elaborate trial and error simulation can now be obtained in a matter of minutes or even seconds. Today's control literature is filled with applications of modern control theory, via computational algorithms, to the design and analysis of dynamical systems and the end is not in sight. However, we cannot be overly optimistic in evaluating the impact of modern control theory on practical system design. Many systems are still too complex for even the latest theory or computational algorithms to handle, and trial and error simulation and evaluation appear to be the only alternative in achieving a satisfactory design. Furthermore, computational algorithms employing modern control principles do not always lead to a satisfactory design or a design which can easily be altered. Frequency domain techniques still retain certain advantages over state space methods, especially for single input, single output systems. In the case of multivariable systems, however, time domain methods have thus far been dominant.

The primary purpose of this report is to present new insight into the design and analysis of linear multivariable systems. This is accomplished by presenting a number of applications of one rather simple basic result, called the structure theorem. This theorem, which can easily be stated and established, allows us to combine time domain and frequency domain information in one concise expression for the transfer matrix of a linear multivariable system. We use this compact transfer matrix expression to establish several new results, which represent the main contribution of this report. These results, for the most part, can be classified as design algorithms. An important feature of many of these algorithms is that they can be implemented by using pencil and paper methods alone, if the system order is relatively low. For higher order systems, appropriate computer algorithms can be employed to simplify the design process.

We introduce the structure theorem in Section 2. Sections 3, 4, and 5 are, for the most part, mutually independent, and represent a variety of research areas where the structure theorem has been successfully applied. Section 6, which deals with a practical design problem, uses the results obtained in most of the prior sections. Each section contains an introduction and several examples to assist the reader in understanding the various steps taken. A list of symbols and special notations are included as Appendixes A and B respectively.

2. A STRUCTURE THEOREM FOR LINEAR MULTIVARIABLE SYSTEMS

The primary purpose of this section is to state and prove a simple basic result which underlies the remainder of this report. This result, which will be called the structure theorem, enables one to express the transfer matrix of any linear, time-invariant dynamical system in a concise and compact manner. This expression for the transfer matrix is based on a canonical form for the dynamical equations which characterize the system. The structure theorem then provides a means for directly expressing the transfer matrix of the system in terms of the canonical equations of motion. The effect of linear state variable feedback on the transfer matrix of the system is also clarified. All of these points are discussed in reference 1 and will be covered in depth in Sections 2.1, 2.2, and 2.3.

In Section 2.1, we define the class of systems considered and discuss linear state variable feedback and its implications. Section 2.2 serves to introduce the structure theorem for the case of systems which are completely controllable. The general structure theorem is presented in Section 2.3.

2.1 Transfer Matrices and State Variable Feedback

The class of systems which will be considered in this report can be defined in the following manner:

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} \ , \ \underline{y} = \underline{C}\underline{x} \tag{1}$$

where $\underline{x}$ is an n vector, called the state; $\underline{u}$ is an m vector, called the input; $\underline{y}$ is a p vector, called the output; and $\underline{A}$, $\underline{B}$, and $\underline{C}$ are constant matrices of the appropriate dimension. We assume that the m inputs and p outputs are linearly independent, i.e., $\underline{B}$ and $\underline{C}$ are of full rank.* Furthermore, assuming zero initial conditions on the state $\underline{x}$, we can determine and express the open loop transfer matrix, $\underline{T}_o(s)$, relating the Laplace transform of the output $\underline{y}(s)$, to the Laplace transform of the input, $\underline{u}(s)$, of Eq. (1), in terms of the triple $\{\underline{A}, \underline{B}, \underline{C}\}$ as

$$\underline{T}_o(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} \tag{2}$$

We will be interested in the effect of linear state variable feedback on the transfer matrix of the system. In particular, suppose that

$$\underline{u} = \underline{F}\underline{x} + \underline{G}\underline{w} \tag{3}$$

where $\underline{w}$ is an m-dimensional external input, and $\underline{F}$ and $\underline{G}$ are constant matrices of appropriate dimension. Substituting Eq. (3)

*This condition should hold in most cases. If it does not, we merely reduce the dimension of $\underline{B}$ and/or $\underline{C}$ until it does.

for $\underline{u}$ into Eq. (1), and solving for $\underline{y}(s)$ in terms of $\underline{w}(s)$, we obtain the closed loop transfer matrix,

$$\underline{T}_{\underline{F},\underline{G}}(s) = \underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B}\underline{G} \tag{4}$$

Paramount to the development of the structure theorem is the observation that the transfer matrix of a linear system represents an input/output relationship and, consequently, should be independent of the choice of "state" for the system. In particular, if we consider the effect of altering the state, $\underline{x}$, of the system (Eq. (1)) via an (nxn) nonsingular similarity transformation $\underline{Q}$, by setting

$$\underline{z} = \underline{Q}\underline{x} \tag{5}$$

we obtain an "equivalent" system

$$\dot{\underline{z}} = \underline{Q}\underline{A}\underline{Q}^{-1}\underline{z} + \underline{Q}\underline{B}\underline{u} \ , \ \underline{y} = \underline{C}\underline{Q}^{-1}\underline{z} \tag{6}$$

or

$$\dot{\underline{z}} = \hat{\underline{A}}\underline{z} + \hat{\underline{B}}\underline{u} \ , \ \underline{y} = \hat{\underline{C}}\underline{z} \tag{7}$$

where $\hat{\underline{A}} = \underline{Q}\underline{A}\underline{Q}^{-1}$, $\hat{\underline{B}} = \underline{Q}\underline{B}$, and $\hat{\underline{C}} = \underline{C}\underline{Q}^{-1}$. We now establish an important well known proposition (P1):

P1: <u>Transfer matrices of equivalent systems are identical</u>.

<u>Proof</u>: Simply noting that

$$\begin{aligned}\underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} &= \underline{C}\underline{Q}^{-1}\underline{Q}(s\underline{I} - \underline{A})^{-1}\underline{Q}^{-1}\underline{Q}\underline{B} \\ &= \underline{C}\underline{Q}^{-1}(s\underline{I} - \underline{Q}\underline{A}\underline{Q}^{-1})^{-1}\underline{Q}\underline{B} = \hat{\underline{C}}(s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}}\end{aligned} \tag{8}$$

We can actually go one step further and establish the equality of transfer matrices of equivalent systems after the application of linear state variable feedback. In particular, if $\underline{u} = \underline{F}\underline{x} + \underline{G}\underline{w} = \underline{F}\underline{Q}^{-1}\underline{z} + \underline{G}\underline{w} = \hat{\underline{F}}\underline{z} + \underline{G}\underline{w}$, where $\underline{F} = \hat{\underline{F}}\underline{Q}^{-1}$, then

$$\underline{T}_{\underline{F},\underline{G}}(s) = \underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B}\underline{G} = \hat{\underline{C}}(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})\hat{\underline{B}}\underline{G} = \underline{T}_{\hat{\underline{F}},\underline{G}}(s) \quad (9)$$

We note that Eq. (9) is a relatively simple expression for the closed loop transfer matrix of the system in terms of its time domain representation. However, the reduction of Eq. (9) to a (pxm) matrix of transfer functions requires the inversion of the (nxn) matrix $(s\underline{I} - \underline{A} - \underline{B}\underline{F})$ or $(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})$. Furthermore, the effect of linear state variable feedback on the pm terms comprising the closed loop transfer matrix is not at all apparent. The structure theorem, which will be introduced in the next section, significantly corrects these deficiencies.

In developing the structure theorem for multivariable systems, we will first consider single input (m = 1) systems, sometimes referred to as scalar systems. The discussion dealing with this class of systems is not intended to be rigorous, since all of the statements which will be made here either have been established elsewhere or will be established for the more general (multivariable) class of systems in Section 2.2.

For scalar systems, it is well known (ref. 2) that (1) linear state variable feedback ($u = \underline{f}\underline{x} + w$) affects only the closed loop poles of the transfer function, and (2) any closed loop configuration can be achieved via linear state variable feedback if the pair $\{\underline{A},\underline{b}\}$ is completely controllable. In particular, if the pair $\{\underline{A},\underline{b}\}$ is completely controllable, one can find a similarity transformation $\underline{Q}$, which transforms the pair $\{\underline{A},\underline{b}\}$ to a controllable companion form (ref. 3), after which, the effect of state feedback is immediately apparent. This point can easily be illustrated by example. Consider the following completely controllable pair $\{\underline{A},\underline{b}\}$ representing a scalar system whose equations of motion are given by Eq. (1), with

$$\underline{A} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -2 & 7 \\ 1 & 1 & 3 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The output y (and hence $\underline{c}$) can be chosen arbitrarily and plays no important role at this point. Suppose we are required to study the effects of linear state variable feedback on the closed loop transfer function of this system. We note that the closed loop poles (zeros of the denominator of the transfer function) of the system are equal to the eigenvalues of $(\underline{A} + \underline{b}\underline{f})$ (ref. 2). Furthermore, the numerator of the transfer function is unaffected by $\underline{f}$, the feedback vector (ref. 2). The effect on the closed loop poles of the system (the eigenvalues of $(\underline{A} + \underline{b}\underline{f})$), due to variation of $\underline{f}$, is not at all apparent in terms of the given pair $\{\underline{A},\underline{b}\}$. However, suppose we transform the pair $\{\underline{A},\underline{b}\}$ to controllable companion form using the similarity transformation $\underline{Q}$, where

$$\underline{Q} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Techniques for finding a suitable $\underline{Q}$ are given in references 4, 5, 6, and 7, for completely controllable scalar systems. In terms of the new state $\underline{z} = \underline{Q}\underline{x}$, we obtain the differential equations $\dot{\underline{z}} = \underline{Q}\underline{A}\underline{Q}^{-1}\underline{z} + \underline{Q}\underline{b}\underline{u} = \hat{\underline{A}}\underline{z} + \hat{\underline{b}}\underline{u}$, where

$$\hat{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \qquad \hat{\underline{b}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If one now deals with the new state $\underline{z}$ and the corresponding pair $\{\hat{\underline{A}},\hat{\underline{b}}\}$, it is clear that linear state variable feedback,

$u = \hat{\underline{f}}\underline{z} + w$, where $\hat{\underline{f}} = [\hat{f}_1, \hat{f}_2, \hat{f}_3]$, yields the closed loop matrix

$$\hat{\underline{A}} + \hat{\underline{b}}\hat{\underline{f}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hat{f}_1+1, & \hat{f}_2, & \hat{f}_3+2 \end{bmatrix}$$

Clearly, one can select the feedback vector $\hat{\underline{f}}$ to achieve any desired third (final) row of the companion matrix $(\hat{\underline{A}} + \hat{\underline{b}}\hat{\underline{f}})$. Since the characteristic polynominal of a companion matrix is expressed directly in terms of the elements comprising the final row (ref. 8), that is,

$$|s\underline{I} - \hat{\underline{A}} - \hat{\underline{b}}\hat{\underline{f}}| = s^3 - (\hat{f}_3 + 2)s^2 - \hat{f}_2 s - \hat{f}_1 - 1$$

it follows that all three eigenvalues of $(\hat{\underline{A}} + \hat{\underline{b}}\hat{\underline{f}})$ can be arbitrarily chosen. Furthermore, since $(\hat{\underline{A}} + \hat{\underline{b}}\hat{\underline{f}}) = \underline{Q}(\underline{A} + \underline{b}\underline{f})\underline{Q}^{-1}$, where $\underline{f} = \hat{\underline{f}}\underline{Q}$, the eigenvalues of $(\underline{A} + \underline{b}\underline{f})$ are the same as those of $(\hat{\underline{A}} + \hat{\underline{b}}\hat{\underline{f}})$ (ref. 8). Therefore, we conclude that the effect of linear state variable feedback on the eigenvalues of $(\underline{A} + \underline{b}\underline{f})$, actually on the coefficients of the characteristic equation of the given scalar system, can easily be understood by considering the equivalent companion form system. A similar result holds for multivariable systems and, as we will show, forms the basis of the structure theorem.

2.2 A Structure Theorem for Controllable Systems

In the case of multivariable systems, we are interested in finding a similarity transformation $\underline{Q}$, which transforms the given system to a canonical form similar to the companion form for scalar systems. As we will show, an appropriate $\underline{Q}$ always exists and can be found if the pair $\{\underline{A},\underline{B}\}$ is completely controllable by first considering the (nxnm) matrix $\underline{K} = [\underline{B}, \underline{A}\underline{B}, \ldots, \underline{A}^{n-1}\underline{B}]$. Clearly, $\underline{K}$ has rank n if the pair $\{\underline{A},\underline{B}\}$ is completely controllable, and it is possible to define a "lexicographic" basis for R_n consisting of the first n linearly independent columns of $\underline{K}$ possibly reordered (ref. 7). We let $\underline{L}$ be the matrix whose columns are the elements of the "lexicographic" basis so that

$$\underline{L} = \left[\underline{b}_1, \underline{A}\underline{b}_1, \ldots, \underline{A}^{\sigma_1 - 1} \underline{b}_1, \underline{b}_2, \ldots, \underline{A}^{\sigma_2 - 1} \underline{b}_2, \ldots, \underline{A}^{\sigma_m - 1} \underline{b}_m \right] \tag{10}$$

where $\underline{b}_1, \ldots, \underline{b}_m$ are the columns of $\underline{B}$. Setting

$$d_o = 0 \;,\; d_k = \sum_{i=1}^{k} \sigma_i \qquad k = 1,2,\ldots,m \tag{11}$$

and letting $\underline{\ell}_k$ be the d_k-th row of $\underline{L}^{-1}$, we can see that the matrix $\underline{Q}$ given by

$$\underline{Q} = \begin{bmatrix} \underline{\ell}_1 \\ \underline{\ell}_1 \underline{A} \\ \vdots \\ \underline{\ell}_1 \underline{A}^{\sigma_1 - 1} \\ \vdots \\ \underline{\ell}_m \underline{A}^{\sigma_m - 1} \end{bmatrix} \tag{12}$$

represents a similarity transformation for which Eq. (7) is in "companion" form (refs. 7 and 9). More precisely, in terms of the transformed state Eq. (7), $\hat{\underline{A}} = (\hat{a}_{ij})$ is a block matrix of the form

$$\hat{\underline{A}} = \begin{bmatrix} \hat{\underline{A}}_{11} & \cdots & \hat{\underline{A}}_{1m} \\ \hat{\underline{A}}_{21} & \cdots & \hat{\underline{A}}_{2m} \\ \vdots & \ddots & \vdots \\ \hat{\underline{A}}_{m1} & \cdots & \hat{\underline{A}}_{mm} \end{bmatrix} \tag{13}$$

with $\hat{\underline{A}}_{ii}$ a $(\sigma_i x \sigma_i)$ companion matrix given by

$$\hat{\underline{A}}_{ii} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & & 0 & 1 \\ \hat{a}_{d_i,d_{i-1}+1} & \hat{a}_{d_i,d_{i-1}+2} & & \hat{a}_{d_i,d_i-1} & \hat{a}_{d_i,d_i} \end{bmatrix} \tag{14}$$

and $\hat{\underline{A}}_{ij}$ a $(\sigma_i x \sigma_j)$ matrix given by

$$\hat{\underline{A}}_{ij} = \begin{bmatrix} 0 & 0 \,.. & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \hat{a}_{d_i,d_{j-1}+1} & & \cdots & \hat{a}_{d_i,d_j} \end{bmatrix} \tag{15}$$

for $i \neq j$ and with $\hat{\underline{B}} = (\hat{b}_{ij})$ an (nxm) matrix given by

$$\hat{\underline{B}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \hat{b}_{d_1,2} & \hat{b}_{d_1,3} & & \hat{b}_{d_1,m} \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & \hat{b}_{d_2,3} & & \hat{b}_{d_2,m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix} \tag{16}$$

Since $\hat{\underline{B}}$ as given by Eq. (16) has zero rows except for the d_1-th, d_2-th,...,d_mth rows, we need only calculate the corresponding columns of $(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})^{-1}$ in order to obtain the transfer matrix $\underline{T}_F(s) = \underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B} = \hat{\underline{C}}(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})^{-1}\hat{\underline{B}}$. Moreover, $\hat{\underline{B}}\hat{\underline{F}}$ has zero rows except for the d_1-th, d_2-th,...,d_m-th rows and so $\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}}$ is again a block matrix of exactly the same form as $\hat{\underline{A}}$.

In other words, $\hat{A} + \hat{B}\hat{F} = (\phi_{ij})$ is a block matrix of the form

$$\underline{\Phi} = \underline{\hat{A}} + \underline{\hat{B}}\underline{\hat{F}} = \begin{bmatrix} \underline{\Phi}_{11} & \cdots & \underline{\Phi}_{1m} \\ \underline{\Phi}_{21} & \cdots & \underline{\Phi}_{2m} \\ \vdots & & \\ \underline{\Phi}_{m1} & & \underline{\Phi}_{mm} \end{bmatrix} \tag{17}$$

where $\underline{\Phi}_{ii}$ is a$(\sigma_i x \sigma_i)$ companion matrix given by

$$\underline{\Phi}_{ii} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \phi_{d_i,d_{i-1}+1} & \phi_{d_i,d_{i-1}+2} & & \phi_{d_i,d_i-1} & \phi_{d_i,d_i} \end{bmatrix} \tag{18}$$

and $\underline{\Phi}_{ij}$ is a $(\sigma_i x \sigma_j)$ matrix given by

$$\underline{\Phi}_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ \phi_{d_i,d_{j-1}+1} & \phi_{d_i,d_{j-1}+2} & & \phi_{d_i,d_j} \end{bmatrix} \tag{19}$$

for $i \neq j$. These two simple observations are basic to the structure theorem (T1):

T1: <u>Suppose that the pair</u> $(\underline{A},\underline{B})$ <u>is controllable and let</u> $\underline{T}_F(s) = \underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B}$ <u>be the transfer matrix of the system</u> $\dot{\underline{x}} = (\underline{A} + \underline{B}\underline{F})\underline{x} + \underline{B}\underline{w}, \underline{y} = \underline{C}\underline{x}$. <u>Then</u>

$$\underline{T}_F(s) = \underline{\hat{C}}\underline{S}(s)\underline{\delta}_F^{-1}(s)\underline{\hat{B}}_m \tag{20}$$

where $\hat{\underline{C}} = \underline{C}\underline{Q}^{-1}$, $\underline{S}(s)$ is the (nxm) matrix given by

$$\underline{S}(s) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ s & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ s^{\sigma_1 - 1} & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & s^{\sigma_2 - 1} & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & s^{\sigma_m - 1} \end{bmatrix} \qquad (21)$$

$\underline{\delta}_F(s)$ is the (mxm) matrix $(\delta_{F,ij}(s))$ with entries given by

$\delta_{F,ii}(s) = \det(s\underline{I}_{\sigma_i} - \underline{\Phi}_{ii})$ and $\delta_{F,ij}(s) =$

$-\phi_{d_i,d_{j-1}+1} - s\phi_{d_i,d_{j-1}+2} - \cdots - s^{\sigma_i - 1}\phi_{d_i,d_j}$ for $i \neq j$, and

$\hat{\underline{B}}_m$ is the (mxm) matrix given by

$$\hat{\underline{B}}_m = \begin{bmatrix} 1 & \hat{b}_{d_1 2} & \cdots & \hat{b}_{d_1 m} \\ 0 & 1 & \cdots & \hat{b}_{d_2 m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} \qquad (22)$$

where $\hat{\underline{B}} = \underline{Q}\underline{B} = (\hat{b}_{ij})$.

Proof: In view of the proposition (P1), we need only show that $\hat{\underline{C}}(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})^{-1}\hat{\underline{B}} = \hat{\underline{C}}\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m$. To do this, it will be sufficient to show that

$$(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})^{-1}\hat{\underline{B}} = \underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m \tag{23}$$

or, equivalently, that

$$(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})\underline{S}(s) = \hat{\underline{B}}\hat{\underline{B}}_m^{-1}\underline{\delta}_F(s) \tag{24}$$

But Eq. (24) is an immediate consequence of the definitions of $\underline{S}(s)$ and $\underline{\delta}_F(s)$. Thus the theorem is established.

This seemingly innocuous and esaily proved theorem has, as we shall see, a number of significant consequences. For a begining, we have the corollary (C1):

C1: *Let* $\Delta_F(s) = \det(s\underline{I} - \underline{A} - \underline{B}\underline{F})$, *and* $\underline{C}^*(s) = \hat{\underline{C}}\underline{S}(s)$. *Then*, $\Delta_F(s) = \det(\underline{\delta}_F(s))$ *and if* $p = m$, *then*

$$\det \underline{T}_F(s) = (\det \underline{C}^*(s))/\Delta_F(s) \tag{25}$$

where $\underline{T}_F(s) = \underline{N}_F(s)/\Delta_F(s)$ *(i.e.* $\underline{N}_F(s)$ *is the numerator of the transfer matrix).*

Proof: By the definition of $\underline{T}_F(s)$, we have $\underline{T}_F(s) = \underline{N}_F(s)/\Delta_F(s)$. It follows from the theorem that

$$\frac{\underline{N}_F(s)}{\Delta_F(s)} = \frac{\underline{C}^*(s)\underline{D}_F(s)\hat{\underline{B}}_m}{\det(\underline{\delta}_F(s))} \tag{26}$$

where $\underline{\delta}_F^{-1}(s) = \underline{D}_F(s)/\det(\underline{\delta}_F(s))$. However, $\Delta_F(s)$ and $\det(\underline{\delta}_F(s))$ are both monic polynomials of degree (n) and the entries in $\underline{N}_F(s)$

are polynomials of, at most, degree (n-1). It follows that $\Delta_{\underline{F}}(s) = \det(\underline{\delta}_{\underline{F}}(s))$ and, hence, that Eq. (25) holds (since $\det(\underline{\delta}_{\underline{F}}^{-1}(s)) = 1/\det(\underline{\delta}_{\underline{F}}(s))$ and $\det \hat{\underline{B}}_m = 1$).

Now let $(\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}})_m = \hat{\underline{A}}_m + \hat{\underline{B}}_m\hat{\underline{F}}$ be defined as the (mxn) matrix consisting of the m-ordered d_k-th (k = 1,2,...m) rows of $(\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}})$, and define $\text{diag}[s^{\sigma_i}]$ as the (mxm) diagonal matrix whose ii entries are s^{σ_i}. We then have the following corollary (C2):

C2: $\underline{\delta}_{\underline{F}}(s) = \text{diag}[s^{\sigma_i}] - \hat{\underline{A}}_m\underline{S}(s) - \hat{\underline{B}}_m\hat{\underline{F}}\underline{S}(s)$

This corollary can easily be established by direct substitution by using the definitions given above.

Proof: It then follows that

$$\underline{\delta}_o(s) = \text{diag}[s^{\sigma_i}] - \hat{\underline{A}}_m\underline{S}(s) \tag{27}$$

and consequently that

$$\underline{\delta}_{\underline{F}}(s) = \underline{\delta}_o(s) - \hat{\underline{B}}_m\hat{\underline{F}}\underline{S}(s) \tag{28}$$

We note, in particular, that these latter three relationships hinge on the fact that only the m d_k-th rows of $\hat{\underline{A}}$ and $\hat{\underline{B}}$ contain the most pertinent information regarding the dynamics of the system. The last equation (Eq. (28)) can be established directly by using Eq. (24).

We observe that entirely analogous results can be obtained for observable systems by a consideration of the dual system (refs. 3 and 10)

$$\dot{\underline{x}} = \underline{A}'\underline{x} + \underline{C}'\underline{v} \ , \ \underline{y} = \underline{B}'\underline{x} \tag{29}$$

which is controllable if and only if Eq. (1) is observable. While we will not derive the results for observable systems here, we will use them without further ado in the sequel.

2.3 A General Structure Theorem

Consider the system, Eq. (1), and again let $\underline{K} = [\underline{B}, \underline{A}\underline{B}, \ldots \underline{A}^{n-1}\underline{B}]$. However, we no longer assume that Eq. (1) is controllable and so, the (nxnm) matrix $\underline{K}$ has rank r with $r \leq n$. To obtain a structure theorem in this general context, we shall consider a controllable extension of Eq. (1) and apply the theorem (T1). With this in mind, we let q = n-r and W be the r-dimensional subspace of R_n spanned by the columns of $\underline{K}$. Denoting the orthogonal complement of W by $W^{\perp}$ so that $R_n = W \oplus W^{\perp}$ and letting $\underline{\beta}_1, \ldots, \underline{\beta}_q$ be a basis of $W^{\perp}$, we consider the system

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}_e\underline{v} \ , \ \underline{y} = \underline{C}\underline{x} \tag{30}$$

where $\underline{B}_e$ is the (n x (m+q)) matrix given by $\underline{B}_e = [\underline{B}, \underline{\beta}_1, \underline{\beta}_q]$. The system, Eq. (7), is controllable and there is a Lyapunov transformation $\underline{Q}_e$ which carries Eq. (7) into block companion form. We note that $\underline{Q}_e$ is a nonsingular (nxn) matrix. It follows that the system

$$\dot{\underline{z}} = \hat{\underline{A}}\underline{z} + \hat{\underline{B}}\underline{u} \ , \ \underline{y} = \hat{\underline{C}}\underline{z} \tag{31}$$

where $\hat{\underline{A}} = \underline{Q}_e^{-1}\underline{A}\underline{Q}_e$, $\hat{\underline{B}} = \underline{Q}_e\underline{B}$, and $\hat{\underline{C}} = \underline{C}\underline{Q}_e^{-1}$ is equivalent to Eq. (1). Moreover, the matrix $\hat{\underline{A}}$ is in block companion form, the last n-r rows of $\hat{\underline{B}}$ are $\underline{0}$, and the lower left-hand block (n-rxr) of $\hat{\underline{A}}$ is $\underline{0}$. Thus, the last n-r rows of $\hat{\underline{A}}$ cannot be altered by state variable feedback of the form $\underline{u} = \hat{\underline{F}}\underline{z} + \underline{w}$. We now have the following theorem (T2):

T2: <u>Let $\underline{T}_{\underline{F}}(s) = \underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B}$ be the transfer matrix of the system</u> $\dot{\underline{x}} = (\underline{A} + \underline{B}\underline{F})\underline{x} + \underline{B}\underline{w}$, $\underline{y} = \underline{C}\underline{x}$.

Then

$$\underline{T}_{\underline{F}}(s) = \frac{\hat{\underline{C}}\underline{S}(s)\Delta_{\underline{F},u}(s)\underline{\delta}^{-1}_{\underline{F},c}(s)\hat{\underline{B}}_m}{\Delta_{\underline{F},u}(s)} \tag{32}$$

where $\hat{\underline{C}} = \underline{C}\underline{Q}_e^{-1}$, $\underline{S}(s)$ is the (nxm) matrix given by

$$\underline{S}(s) = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ s & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ s^{\sigma_1 - 1} & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & s^{\sigma_2 - 1} & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & s^{\sigma_m - 1} \\ \hline \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{array}\right] \tag{33}$$

(with $\underline{b}_1$, $\underline{A}\underline{b}_1$, ... , $\underline{A}^{\sigma_1 - 1}\underline{b}_1$, ... , $\underline{A}^{\sigma_m - 1}\underline{b}_m$ a "lexicographic" basis of the range of K so that

$$\sum_{i=1}^{m}\sigma_i = r), \ \Delta_{\underline{F},u}(s) = \det \underline{\delta}_{\underline{F},u}(s) \ , \ \underline{\delta}_{\underline{F}}(s)$$

is the (m+q) x (m+q) matrix $(\delta_{\underline{F},ij}(s))$ with entries given by $\delta_{\underline{F},ii}(s) = \det(s\underline{I} - \underline{\Phi}_{ii})$ and $\delta_{\underline{F},ij}(s) = -\phi_{d_i,d_{j-1}+1}$ $-\ldots-s^{\sigma_1 - 1}\phi_{d_i,d_j}$ for $i \neq j$ where

$$d_k = \sum_{i=1}^{k}\sigma_i \ , \ \sigma_1 = 1$$

for i = m+1 , ... , m+q , and $\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}} = (\phi_{ij}) = [\underline{\Phi}_{ij}]$,
so that

$$\underline{\delta}_{\underline{F}}(s) = \begin{bmatrix} \delta_{\underline{F},11}(s) & \cdots & \delta_{\underline{F},1m}(s) & \delta_{\underline{F},1,m+1}(s) & \cdots & \delta_{\underline{F},1,m+q}(s) \\ \vdots & & & \vdots & & \vdots \\ \delta_{\underline{F},m1}(s) & \cdots & \delta_{\underline{F},mm}(s) & & & \\ & & & \delta_{\underline{F},m+.,m+1}(s) & \cdots & \delta_{\underline{F},m+1,m+q}(s) \\ & \underline{0} & & \vdots & & \\ & & & \delta_{\underline{F},m+q,m+1}(s) & \cdots & \delta_{\underline{F},m+q,m+q}(s) \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \underline{\delta}_{\underline{F},c}(s) & \underline{\delta}_{\underline{F},cu}(s) \\ \hline \underline{0} & \underline{\delta}_{\underline{F},u}(s) \end{array}\right]^{+} \qquad (34)$$

and where $\hat{\underline{B}}_m$ is the matrix (mxm) consisting of the nonzero rows of $\hat{\underline{B}}$

Proof: Clearly, we now need only show that

$$\hat{\underline{C}}(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})^{-1}\hat{\underline{B}} = \frac{\hat{C}S(s)\Delta_{\underline{F},u}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\hat{\underline{B}}_m}{\Delta_{\underline{F},u}(s)}$$

where $\hat{\underline{F}} = \underline{F}\underline{Q}_e^{-1}$. We shall do this by considering the completely controllable system

$$\dot{\underline{z}} = \hat{\underline{A}}\underline{z} + \hat{\underline{B}}_e\underline{v} \ , \ \underline{y}\ \hat{\underline{C}}\underline{z} \qquad (35)$$

+$\underline{\delta}_{\underline{F},cu}(s)$ involves only constant terms and the off-diagonal terms in $\underline{\delta}_{\underline{F},u}(s)$ are constant.

with $\hat{\underline{B}}_e = \underline{Q}_e \underline{B}_e$ and applying the theorem (T1).

Let $\hat{\underline{F}}_e = \underline{F}_e \underline{Q}_e^{-1}$ where $\underline{F}_e = \begin{bmatrix} \underline{F} \\ 0 \end{bmatrix}$ so that $\hat{\underline{F}}_e = \begin{bmatrix} \hat{\underline{F}} \\ 0 \end{bmatrix}$. Since $\underline{B}_e = [\underline{B}\underline{\beta}_1, \ldots, \underline{\beta}_q]$, we have, by definition of $\underline{Q}_e$,

$$\hat{\underline{B}}_e = \left[\begin{array}{c|c} \hat{\underline{B}} & \begin{array}{c} \underline{0} \\ \hline I_q \end{array} \end{array}\right] \quad (36)$$

and $\hat{\underline{B}}_e \hat{\underline{F}}_e = \hat{\underline{B}}\hat{\underline{F}}$. It follows that $(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}}) = (s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}_e \hat{\underline{F}}_e)$ and, hence, that the transfer matrix of Eq. (35) under the feedback $\underline{v} = \underline{F}_e \underline{x} + \underline{w}$ is given by $\hat{\underline{C}}(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})^{-1}\hat{\underline{B}}_e$. However, the system represented by Eq. (35) is controllable and thus by theorem (T1),

$$\hat{\underline{C}}(s\underline{I} - \hat{\underline{A}} - \hat{\underline{B}}\hat{\underline{F}})^{-1}\hat{\underline{B}}_e = \hat{\underline{C}}\underline{S}_e(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_{e,m+q} \quad (37)$$

where $\underline{S}_e(s)$ is given by

$$\underline{S}_e(s) = \left[\begin{array}{c|c} \underline{S}(s) & \begin{array}{c} \underline{0} \\ \hline \underline{I}_q \end{array} \end{array}\right] \quad (38)$$

and $\underline{B}_{e,m+q}$ is the (m+q x m+q) matrix given by

$$\hat{\underline{B}}_{e,m+q} = \left[\begin{array}{c|c} \hat{\underline{B}}_m & \underline{0} \\ \hline 0 & \underline{I}_q \end{array}\right] \quad (39)$$

By equating the appropriate blocks in Eq. (37) and noting that

$$\underline{\delta}_F^{-1}(s) = \frac{\left[\begin{array}{c|c} (\det \underline{\delta}_{F,u}(s))\,\mathrm{adj}\ \underline{\delta}_{F,c}(s) & -(\mathrm{adj}\ \underline{\delta}_{F,c}(s))\underline{\delta}_{F,cu}(s)(\mathrm{adj}\ \underline{\delta}_{F,u}(s) \\ \hline \underline{0} & (\det \underline{\delta}_{F,c}(s))\,\mathrm{adj}\ \underline{\delta}_{F,u}(s) \end{array}\right]}{\det \underline{\delta}_{F,u}(s)\ \det \underline{\delta}_{F,c}(s)} \quad (40)$$

where adj (·) denotes the adjoint of a martix, we deduce Eq. (32). Thus, the theorem is established.

Corollary C3: $\underline{\Delta}_{\underline{F},u}(s)$ is independent of $\underline{F}$ and the uncontrollable poles of the system $\dot{\underline{x}} = (\underline{A} + \underline{B}\underline{F})\underline{x} + \underline{B}\underline{w}$, $\underline{y} = \underline{C}\underline{x}$ are the zeros of $\underline{\Delta}_{\underline{F},u}(s)$ [= $\underline{\Delta}_{\underline{0},u}(s)$]

C3: is simply a statement of the fact that the uncontrollable poles cannot be altered by state variable feedback. We also note that the factorization (Eq. (32)) involves the well-known pole-zero cancellation of the uncontrollable portion of the system (ref. 11). Then now state corollaries (C4, C5, and C6):

C4: The matrices $\hat{\underline{C}}$, $\underline{S}(s)$ and $\hat{\underline{B}}_m$ are invariant under state variable feedback.

C5: Let p = m and $\underline{C}^*(s) = \hat{\underline{C}}\underline{S}(s)$. Then the inverse system exists if and only if $\underline{C}^*(s)$ is non-singular.

C6: Let p = m and let $\Delta_{\underline{F}}(s) = \det \underline{\delta}_{\underline{F}}(s)$. Then $\det(\underline{T}_{\underline{F}}(s)) = (\det C^*(s))(\Delta_{\underline{F},u}(s))/\Delta_{\underline{F}}(s)$ where $\Delta_{\underline{F}}(s) = \Delta_{\underline{F},u}(s)\Delta_{\underline{F},c}(s)$.

We again observe that entirely analogous results can be obtained for systems which are not observable by a consideration of the dual system (Eq. (29)). We use these results without further ado in the sequel.

3. DIRECT APPLICATIONS OF THE STRUCTURE THEOREM

Now that the structure theorem has been established, we can investigate its application to various questions concerning the analysis and synthesis of linear multivariable systems. In particular, in this section we will discuss three "direct" applications of the structure theorem. We use the term "direct" to imply that the applications are relatively straightforward and require only minor modification of the relationships established in Section 2.

In Section 3.1, we consider the question of realization. In Section 3.2, certain questions related to pole assignment via linear state variable feedback are discussed. The final section (3.3) deals with linear output feedback for pole assignment. An example is given in the last section to clarify certain of the statements made.

3.1 The Problem of Realization

The first application we will discuss involves the use of the structure theorem to obtain an algorithm for solving the problem of realization (refs. 3 and 12). More precisely, we consider the following.

REALIZATION PROBLEM: Let $\underline{T}(s)$ be a (pxm) matrix whose entries, $T_{ij}(s)$, are rational functions of s. Suppose that $T_{ij}(s) = n_{ij}(s)/d_{ij}(s)$ where $n_{ij}(s)$ and $d_{ij}(s)$ are relatively prime and degree $n_{ij}(s)$ < degree $d_{ij}(s)$. Then determine a triple $\{\underline{A},\underline{B},\underline{C}\}$ of matrices such that

$$\underline{T}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} \tag{41}$$

and $\{\underline{A},\underline{B}\}$ is controllable and $\{\underline{A},\underline{C}\}$ is observable. Such a triple is called a minimal realization of $\underline{T}(s)$ (refs. 3 and 12).

Ho and Kalman (ref. 12) proved that the realization problem has a solution and provided a constructive procedure for determining a minimal realization. Here, we present an alternate constructive algorithm for determining minimal realizations analogous to a recent result of Mayne (ref. 13). A computer program has been developed for applying the algorithm.

The basic steps in the algorithm are:

STEP 1: Calculation of the least common multiple of the denominator polynomials $\{d_{1j}(s),\ldots d_{pj}(s)\}$ in each column of $\underline{T}(s)$.

STEP 2: Construction of a standard controllable realization $\{\underline{A}_c,\underline{B}_c,\underline{C}_c\}$ (not necessarily minimal).

STEP 3: Construction of a minimal realization by applying a suitable transformation to $\{\underline{A}_c',\underline{C}_c',\underline{B}_c'\}$.

We shall examine each of these steps, in detail, paying particular attention to step 2.

Now let $g_j(s)$ be the least common multiple of the denominator polynomials $\{d_{1j}(s),\ldots,d_{pj}(s)\}$ (which are assumed, for convenience, to be monic). Let h_j denote the degree of $g_j(s)$ and let $\underline{T}^*(s)$ be the (pxm) matrix given by

$$\underline{T}^*(s) = \begin{bmatrix} n_{11}^*(s)/g_1(s) & \ldots & n_{1m}^*(s)/g_m(s) \\ \vdots & \ddots & \vdots \\ n_{p1}^*(s)/g_1(s) & \ldots & n_{pm}^*(s)/g_m(s) \end{bmatrix} \tag{42}$$

where $n_{ij}^*(s) = n_{ij}(s)g_j(s)/d_{ij}(s)$. In other words, $\underline{T}^*(s)$ is obtained from $\underline{T}(s)$ by multiplying each numerator $n_{ij}(s)$ by $g_j(s)/d_{ij}(s)$ and replacing each denominator $d_{ij}(s)$ by $g_j(s)$. The construction of $\underline{T}^*(s)$ completes step 1.

Let $n_1 = \sum_{j=1}^{m} h_j$ and $p_k = \sum_{1}^{k} h_j$. Since $g_j(s)$ is the least common multiple of $\{d_{1j}(s),\ldots,d_{pj}(s)\}$ and degree $n_{ij}(s)$ < degree $d_{ij}(s)$ and the $d_{ij}(s)$ are assumed monic, we have

$$g_j(s) = s^{h_j} + \gamma_{j1}s^{h_j-1} + \ldots + \gamma_{jh_j} \tag{43}$$

$$n^*_{ij}(s) = \nu_{ij1}s^{h_j-1} + \nu_{ij2}s^{h_j-2} + \ldots + \nu_{ijh_j} \tag{44}$$

for all i,j and suitable constants γ_{jk}, ν_{ijk}. Let $\underline{A}_{c,j}$ be a companion matrix corresponding to $g_j(s)$ so that

$$\underline{A}_{c,j} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ -\gamma_{jh_j} & -\gamma_{jh_j-1} & \cdots & -\gamma_{j1} \end{bmatrix} \tag{45}$$

and let $\underline{A}_c$ be the $(n_1 x n_1)$ block diagonal matrix given by

$$\underline{A}_c = \begin{bmatrix} \underline{A}_{c,1} & & & \underline{0} \\ & \underline{A}_{c,2} & & \\ & & \ddots & \\ \underline{0} & & & \underline{A}_{c,m} \end{bmatrix} \tag{46}$$

If $\underline{B}_c$ is the $(n_1 xm)$ matrix with zero entries in all but the p_k-th rows, each of which is zero except for a 1 in the k-th column, then the pair $\{\underline{A}_c, \underline{B}_c\}$ is controllable. We now have the proposition (P2):

P2: Let $\underline{C}_c$ be the (mxn_1) matrix given by

$$\underline{C}_c = \left[\begin{array}{cccc|ccc|cc} \nu_{11h_1} & \nu_{11h_1-1} & \cdots & \nu_{111} & \nu_{12h_2} & \cdots & \nu_{121} & \cdots & \nu_{1m1} \\ \nu_{21h_1} & \nu_{21h_1-1} & \cdots & \nu_{211} & \nu_{22h_2} & \cdots & \nu_{221} & & \nu_{2m1} \\ \vdots & \vdots & & \vdots & \vdots & & & & \vdots \\ \nu_{p1h_1} & \nu_{p1h_1-1} & & \nu_{p11} & \nu_{p2h_2} & \cdots & \nu_{p21} & & \nu_{pm1} \end{array}\right] \tag{47}$$

then $\{\underline{A}_c, \underline{B}_c, \underline{C}_c\}$ is controllable realization of $\underline{T}(s)$.

Proof: Since $\{\underline{A}_c, \underline{B}_c\}$ is controllable, it follows from the structure theorem (T1) and the definitions of $\underline{A}_c, \underline{B}_c, \underline{C}_c$ that

$$\underline{C}_c(s\underline{I} - \underline{A}_c)^{-1}\underline{B}_c = \underline{C}_c^*(s)\underline{\delta}_c^{-1}(s)\hat{\underline{B}}_{c,m} \tag{48}$$

where $\underline{B}_{c,m} = \underline{I}_m$, $\underline{\delta}_c^{-1}(s) = \text{diag}[1/g_1(s),\ldots,1/g_m(s)]$, and $\underline{C}_c^*(s) = (n_{ij}^*(s))$. Since $n_{ij}^*(s)/g_j(s) = n_{ij}(s)/d_{ij}(s)$, we deduce that $\underline{C}_c(s\underline{I} - \underline{A}_c)^{-1}\underline{B}_c = (n_{ij}(s)/d_{ij}(s)) = \underline{T}(s)$. Thus, the proposition is established. This proposition completes the description of step 2.

As regards step 3, we consider the triple $\{\underline{A}_c', \underline{C}_c', \underline{B}_c'\}$ and apply a similarity transformation $\underline{Q}_e$, of the type used in Section 2.3, to it. Letting n be the rank of $[\underline{C}_c'\underline{A}_c'\underline{C}_c' \ldots \underline{A}_c'^{\,n_1-1}\underline{C}_c']$ and setting $\hat{\underline{A}}_c' = \underline{Q}_e\underline{A}_c'\underline{Q}_e^{-1}$, $\hat{\underline{C}}_c' = \underline{Q}_e\underline{C}_c'$, $\hat{\underline{B}}_c' = \underline{B}_c'\underline{Q}_e^{-1}$, we have

$$\hat{\underline{C}}_c' = \begin{bmatrix} \underline{C}' \\ \hline \underline{0}_{n_1-n,p} \end{bmatrix} , \qquad \hat{\underline{A}}_c' = \left[\begin{array}{c|c} \underline{A}' & * \\ \hline \underline{0}_{n_1-n,n} & * \end{array}\right] \tag{49}$$

and $\underline{B}_c' = [\underline{B}' \,|\, *_{m,n_1-n}]$ where $\underline{C}'$ is (nxp), $\underline{A}'$ is (nxn), and $\underline{B}'$ is (mxn). Since $\underline{T}(s) = \underline{C}_c(s\underline{I} - \underline{A}_c)^{-1}\underline{B}_c$, it follows that $\underline{T}'(s) = \hat{\underline{B}}_c'(s\underline{I} - \hat{\underline{A}}_c')^{-1}\hat{\underline{C}}_c' = \underline{B}'(s\underline{I} - \underline{A}')^{-1}\underline{C}'$ or, equivalently, that $\underline{T}(s) = \underline{C}(s\underline{I} - \underline{A}^{-1}\underline{B}$. Thus, $\{\underline{A}, \underline{B}, \underline{C}\}$ is a realization of T(s). But $\{\underline{A}, \underline{B}, \underline{C}\}$ is both controllable and observable and, hence, is a minimal realization (ref. 12). The triple $\{\underline{A}, \underline{B}, \underline{C}\}$ is in "observable canonical form." The actual available program also produces a minimal realization in "controllable canonical form" as well as all the relevant similarity transformations.

3.2 Pole Assignment via Linear State Variable Feedback

Several recent papers have dealt with the utilization of state variable feedback for closed loop pole assignment in linear multivariable systems (refs. 14, 15, and 16). The main result established in all of these papers was first presented in reference 16 and is summarized below.

Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be an arbitrary set of n complex numbers λ_i , which appear as conjugate pairs whenever $\mathrm{Im}(\lambda_i) \neq 0$. From theorem (T3),

T3: The pair $\{\underline{A}, \underline{B}\}$ is controllable if and only if, for every choice of the set Λ , there is a matrix $\underline{F}$ such that $\underline{A} + \underline{B}\underline{F}$ has Λ for its set of eigenvalues.

The sufficiency portion of the proof was established in reference 16 by using a result of Langenhop (ref. 17), However, a simple constructive procedure for establishing necessity did not appear until Heymann (ref. 15). The purpose of this section will be to present an alternative constructive procedure for arbitrarily assigning all n eigenvalues of $(\underline{A} + \underline{B}\underline{F})$ based on the structure theorem.

The constructive procedure we will employ involves two relatively simple observations: (1) the eigenvalues of similar matrices are identical (ref. 8), and (2) the m d_k-th rows of $(\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}})$ can be completely and arbitrarily specified via $\hat{\underline{F}}$. By (1), we know that if an $\hat{\underline{F}}$ can be found such that all n eigenvalues of $(\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}})$ belong to the set Λ , then we are finished; i.e. by (1), all n eigenvalues of $(\underline{A} + \underline{B}\underline{F})$, where $\underline{F} = \hat{\underline{F}}\underline{Q}$, will also belong to Λ. However, by (2), we note that $\hat{\underline{F}}$ can be chosen such that $(\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}})$ is a companion matrix with any arbitrary final (n-th) row. In particular, we can choose the last row of $(\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}})$ to correspond to the coefficient of the polynominal whose zeros are the numbers of Λ ; i.e., if

$$\prod_{i=1}^{n} (s - \lambda_i) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o \qquad (50)$$

then we can select $\hat{\underline{F}}$ so that

$$\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & 1 \\ -a_o & -a_1 & \cdots & & & -a_{n-1} \end{bmatrix} \tag{51}$$

because all nm terms comprising the m d_k-th rows of $\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}}$ can be completely and arbitrarily selected by using linear state variable feedback. Clearly, the eigenvalues of $\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}}$ are the members of Λ and all that is required is an explicit expression for $\hat{\underline{F}}$. If we define $\hat{\underline{A}}_m$ as the (mxn) matrix consisting of the m-ordered d_k-th rows of $\hat{\underline{A}}$, and $\hat{\underline{A}}^*_m$ an (mxn) matrix consisting of the m-ordered d_k-th rows of $\hat{\underline{A}} + \hat{\underline{B}}\hat{\underline{F}}$, we observe that

$$\hat{\underline{A}}_m + \hat{\underline{B}}_m\hat{\underline{F}} = \hat{\underline{A}}^*_m \tag{52}$$

or that

$$\hat{\underline{F}} = \hat{\underline{B}}_m^{-1}[\hat{\underline{A}}^*_m - \hat{\underline{A}}_m] \tag{53}$$

Necessity in theorem T3 is thus constructively established.

We note at this point that, in general, Eq. (53) represents only one of an infinite number of choices for $\hat{\underline{F}}$, and consequently $\underline{F}$, for pole assignment in multi-input systems. By altering or linearly combining the column vectors of $\underline{B}$ before transforming the system to canonical form via $\underline{Q}$, we can achieve any number of appropriate feedback matrices. While this might appear to present the system designer with an impossible task of selecting the "best" $\underline{F}$ for feedback compensation, we note that this is not generally the case in practice for a number of reasons. In general, practical system constraints are present which prevent arbitrary selection of the first feedback matrix $\underline{F}$ which produces a desirable closed loop pole configuration. For example, two different feedback matrices may yield the same pole configuration, but one of these may be far superior from the point of view of reducing system sensitivity to parameter variations. Along these same lines, the feedback design which reduces sensitivity might be optimal in the sense of a quadratic performance index, while the other need not be, although both designs yield the same closed loop poles. Constraints might also be placed on the allowable magnitudes of feedback gains in order to conserve system power or avoid noise amplification. A certain form for the closed loop transfer matrix, such as being diagonal (decoupled), might be desirable; in which case, the class of allowable feedback matrices would be significantly reduced. If the entire state of the system were not directly measurable, one might want to investigate the feasibility of employing output rather than state feedback to achieve a satisfactory closed loop design. Other constraints, or a combination of constraints, can significantly reduce the allowable class of feedback matrices, perhaps to the point where linear feedback alone would be unsatisfactory. Most of these points will be discussed in more detail in the remainder of this report.

3.3 Pole Assignment via Output Feedback

At the conclusion of the previous section, we noted that the entire state of a given system might not be directly measurable. In particular, in many practical systems, various sensors must be used to obtain the physical "state" of the system or some portion of it. Feedback designs which involve the entire state of the system are often difficult to achieve in practice unless additional sensors or state estimators (filters) are employed. This, of course, involves additional costs which may be avoided if alternative designs are used. The purpose of this section is to discuss a procedure for specifying closed loop pole locations by using linear output feedback. The technique we will employ involves a direct application of the structure theorem and, in particular, corollaries C1 and C2. It should be noted that the question we are considering here has a rather elegant solution in the scalar case, namely the familiar root locus. Unfortunately, there does not appear to be a simple pictorial solution to the question of output feedback compensation in the multivariable case due to the dimensionality of the problem; i.e. instead of feedback from one output variable to one input terminal, we consider feedback from p outputs to m inputs and, consequently, a total of pm individual gains, instead of one. We remark here that recent results pertaining to this question have been obtained by using a pictorial nested root locus approach (ref. 18). The approach taken here, however, will not involve any graphs or plots. In particular, consider the multivariable system, Eq. (1), and suppose we wish to calculate the effect on the closed loop eigenvalue if output feedback is used, i.e., if

$$\underline{u} = \underline{H}\underline{C}\underline{x} + \underline{G}\underline{w} \tag{54}$$

The choice of $\underline{G}$ does not affect the eigenvalue of $(\underline{A} + \underline{B}\underline{H}\underline{C})$ and can be neglected. Also, as in the case of state feedback, we will consider the canonical equations of motion (Eq. (7)) rather than the actual system; i.e., the eigenvalue of $(\hat{\underline{A}} + \hat{\underline{B}}\underline{H}\hat{\underline{C}})$. Note that $\underline{H}$, the (mxp) output feedback matrix is unaffected by a transformation of coordinates. The eigenvalue of $(\hat{\underline{A}} + \hat{\underline{B}}\underline{H}\hat{\underline{C}})$ and, consequently, of $(\underline{A} + \underline{B}\underline{H}\underline{C})$ can therefore be determined directly in terms of the actual $\underline{H}$ which is employed. According to corollaries C1 and C2, these eigenvalues are equal to the zeros of $\det(\underline{\delta}_{\underline{HC}}(s))$, where

$$\underline{\delta}_{HC}(s) = \underline{\delta}_{o}(s) - \hat{\underline{B}}_m\underline{H}\hat{\underline{C}}\underline{S}(s) \tag{55}$$

The evaluation of the determinant of Eq. (55) directly produces the characteristic polynomial of the output feedback state matrix $(\underline{A} + \underline{B}\underline{H}\underline{C})$ in terms of the pm elements comprising $\underline{H}$.

This point will now be demonstrated by example. In particular, consider the following triple $\{A, B, C\}$ representing the system (Eq. (1)).

$$
\underset{\sim}{A} = \begin{bmatrix} -1 & 0 & 0 & -6 & 3 & -1 \\ 1 & -2 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 6 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 2 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \qquad \underset{\sim}{B} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \\ 0 & -1 \\ 0 & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}
$$

$$
\underset{\sim}{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

If one transforms the above triple via the similarity transformation $\underset{\sim}{Q}$, where

$$
\underset{\sim}{Q} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{bmatrix}
$$

the resulting triple $\{\hat{\underset{\sim}{A}}, \hat{\underset{\sim}{B}}, \hat{\underset{\sim}{C}}\}$ is in companion canonical form as desired; i.e.

$$
\hat{\underset{\sim}{A}} = \left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -2 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -4 & -1 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \qquad \hat{\underset{\sim}{B}} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 2 \\ \hline 0 & 0 \\ \hline 0 & 1 \\ \hline 0 & 0 \end{array}\right]
$$

and

$$\hat{\underline{C}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We note that this system is not completely controllable; i.e. $\Delta_{\underline{F},u}(s) = \Delta_{\underline{HC},u}(s) = (s + 1)$. Also,

$$\underline{\delta}_o(s) = \begin{bmatrix} s^3 - 2s + 1, & 2 \\ -3s^2, & s^2 + s + 4 \end{bmatrix}$$

$$\hat{\underline{B}}_m = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \hat{\underline{C}}\underline{S}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}$$

We can simplify the computations required to compute the determinant of $\underline{\delta}_{\underline{HC}}(s) = \underline{\delta}_o(s) - \hat{\underline{B}}_m\underline{H}\hat{\underline{C}}\underline{S}(s)$ still further by defining

$$\underline{H}^* = -\hat{\underline{B}}_m\underline{H} = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}$$

i.e.

$$\underline{\delta}_{\underline{HC}}(s) = \begin{bmatrix} s^2 + h_1 s^2 - 2s + 1, & h_2 s + 2 \\ (h_3 - 3)s^2, & s^2 + (h_4 + 1)\ s + 4 \end{bmatrix}$$

Solving for the determinant of $\underline{\delta}_{\underline{HC}}(s)$ in terms of the unspecified elements of $\underline{H}^*$, we obtain

$$\begin{aligned} \det \underline{\delta}_{\underline{HC}}(s) = s^5 &+ (h_1 + h_4 + 1)s^4 \\ &+ (h_1 + h_1 h_4 + h_2(3 - h_3) + 2)s^3 \\ &+ (4h_1 - 2h_3 - 2h_4 + 5)s^2 + (h_4 - 7)s + 4 \end{aligned}$$

By inspection, we note that h_4, h_1, and h_3 can be chosen successively, in order, to arbitrarily specify the coefficients of s, s^4, and s^2. Furthermore, if $h_3 \neq 3$, h_2 can then be chosen to specify the coefficient of the s^3 term. We also note that the constant term is 4, and it cannot be altered by $\underline{H}^*$; i.e., the product of the five controllable poles of the output feedback system must equal -4. Let us carry the analysis to its conclusion by assuming that we wish to choose closed loop poles (eigenvalues of $(\underline{A} + \underline{B}\underline{H}\underline{C})$) at s = -1, -2, -1/2, and -3. The fifth controllable pole must be -4/3 in order to satisfy the requirement that the product of the five controllable poles be -4. The desired closed loop characteristic polynomial is thus $s^5 + \frac{47}{6}s^4 + \frac{68}{3}s^3 + \frac{181}{6}s^2 + \frac{55}{3}s + 4$. Solving for h_4, h_1, and h_3 in succession, we obtain: $h_4 = \frac{76}{3}$, $h_1 = -\frac{111}{6}$, and $h_3 = -\frac{599}{12}$. Since $h_3 \neq 3$, we can solve for h_2 to choose $\frac{68}{3}$ as the coefficient of the s^3 term; i.e. $h_2 = \frac{6094}{635}$. The actual output feedback gain matrix, $\underline{H}$, can now be determined since $\hat{\underline{B}}_m$ is nonsingular; i.e. $\underline{H} = \hat{\underline{B}}_m^{-1}\underline{H}^*$, and in terms of the elements of $\underline{H}^*$ which we have computed,

$$\underline{H} = \begin{bmatrix} -81.33 & 41.07 \\ 49.92 & -25.33 \end{bmatrix}$$

This choice for the output feedback gain matrix will yield closed loop poles at s = -1, -2, -1/2, -3, and -4/3, in addition to the uncontrollable pole at s = -1.

4. LINEAR OPTIMAL CONTROL

Over the past few years, optimal control theory has proven to be a useful tool for designing linear feedback control systems. In this section, we will discuss many of the implications of a feedback design based on linear optimal control. We state here that few new results will be presented. What we will emphasize are certain computational simplifications which can be made over previous work, if one formulates this problem in the frequency domain and then employs the structure theorem.

In Section 4.1, we state the optimal control problem considered. We then present a known time domain solution and discuss certain characteristics of an optimal feedback design. In the next section (4.2) we employ the time domain solution of Section (4.1) to derive a frequency domain solution based on the structure theorem. We then employ this solution to establish certain frequency domain relationships implied by an optimal design. In the final section (4.3) we discuss a frequency domain solution to the optimal control problem by using spectral factorization, and conclude with a discussion of the root-square locus.

4.1 Problem Formulation and Prior Results

The particular problem which we will consider in this section is the so-called output regulator problem (ref. 19). That is, given the linear multivariable system,

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} \ , \ \underline{y} = \underline{C}\underline{x} \tag{56}$$

find a control $\underline{u}^*$, which minimizes the quadratic performance index J, where

$$J = \int_0^\infty (\underline{x}^T\underline{C}^T\underline{C}\underline{x} + \underline{u}^T\underline{R}\underline{u})dt \tag{57}$$

The assumptions are that (1) the pair $\{\underline{A},\underline{B}\}$ is controllable, (2) the pair $\{\underline{A},\underline{C}\}$ is observable, (3) $\underline{B}$ is of full rank $m \leq n$, and (4) $\underline{R}$ is a positive definite matrix; i.e. $\underline{R} = \underline{T}^T\underline{T}$ for some nonsingular matrix $\underline{T}$. The output regulator problem, thus formulated, is a time domain (state space) problem. It was initially presented and solved by Kalman (ref. 20) for scalar systems, and later extended, in part, to include multivariable systems by Anderson (ref. 21), who generalized many of Kalman's original results. It should be noted that the determination of $\underline{u}^*$, the optimal control which minimizes J, was not the only factor discussed by Kalman and Anderson. In particular, as we will show, a feedback design based on linear optimal control theory exhibits certain desirable properties which can easily be expressed in terms of a frequency domain representation for the system. For this reason, linear optimal control theory provides an important link between the classical (frequency domain) and modern (time domain) approaches to linear system analysis and design.

We will now summarize the main contributions of Kalman and Anderson relative to the stated output regulator problem, and later offer an alternative approach and solution via the structure theorem. In particular, we state, without proof, the following theorem (T4) and two corollaries (C7 and C8).

T4 (ref. 20): Consider the system $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $y = \underline{C}\underline{x}$, where (1) the pair $\{\underline{A},\underline{B}\}$ is controllable, (2) the pair $\{\underline{A},\underline{C}\}$ is observable, (3) $\underline{B}$ is of full rank $m \leq n$, and (4) $\underline{R}$ is a positive definite matrix. The control $\underline{u}^*$, which minimizes $J = \int_0^\infty \{\underline{x}^T\underline{C}^T\underline{C}\underline{x} + \underline{u}^T\underline{R}\underline{u}\}dt$, exists and is unique, and can be expressed as a linear function of the state $\underline{x}$; i.e.

$$\underline{u}^* = -\underline{R}^{-1}\underline{B}^T\underline{K}\underline{x} \tag{58}$$

where $\underline{K}$ is the unique, positive definite solution to the matrix Riccati equation,

$$\underline{K}\underline{A} + \underline{A}^T\underline{K} - \underline{K}\underline{B}\underline{R}^{-1}\underline{B}^T\underline{K} = -\underline{C}^T\underline{C} \tag{59}$$

Furthermore, the closed loop (optimal) poles of the system are the zeros of $\det(s\underline{I} - \underline{A} + \underline{B}\underline{R}^{-1}\underline{B}^T\underline{K})$, and lie in the half-plane $Re(s) < 0$; i.e. the optimal system is asymptotically stable.

C7 (refs. 20 and 21): If $\underline{R} = \underline{I}$, then

$$\begin{aligned}&[\underline{I} + \underline{B}^T(-s\underline{I} - \underline{A})^{-T}\underline{K}^T\underline{B}][\underline{I} + \underline{B}^T\underline{K}(s\underline{I} - \underline{A})^{-1}\underline{B}] = \\ &\underline{I} + [\underline{B}^T(-s\underline{I} - \underline{A})^{-T}\underline{C}^T][\underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B}]\end{aligned} \tag{60}$$

i.e., optimality implies that

$$[\cdot]^*[\underline{I} + \underline{B}^T\underline{K}(j\omega\underline{I} - \underline{A})^{-1}\underline{B}] - \underline{I} \geq 0 \tag{61}$$

for all real ω .

$[\cdot]^*$ is used here to denote the conjugate transpose of the postmultiplying matrix $[\underline{I} + \underline{B}^T\underline{K}(j\omega\underline{I} - \underline{A})^{-1}\underline{B}]$, and ≥ 0 is a shorthand notation for nonnegative definite.

An additional frequency domain relationship implied by an optimal design is the following:

C8 (ref. 20): Let $\Delta(s) = \det(s\underline{I} - \underline{A})$ and $\Delta^*(s) = \det(s\underline{I} - \underline{A} + \underline{B}\underline{R}^{-1}\underline{B}^T\underline{K})$ denote the open and closed loop (optimal) transfer matrix denominator polynomials respectively. Then,

$$\left\{\frac{\Delta^*(j\omega)}{\Delta(j\omega)}\right\}^2 = \left\{\frac{\Delta^*(-j\omega)\Delta^*(j\omega)}{\Delta(-j\omega)\Delta(j\omega)}\right\} \geq 1 \tag{62}$$

for all real ω .

Equations (61) and (62) represent two important relationships which all linear feedback designs satisfy. It should be noted that Eq. (62) was established for the scalar case only in reference 20 (when m = 1).

The quantity $\underline{I} + \underline{B}^T\underline{K}(s\underline{I} - \underline{A})^{-1}\underline{B}$ represents the return difference in multivariable systems (refs. 20, 21, and 22). Furthermore, Eq. (61), an expression involving the return difference, has been shown to represent a necessary condition for optimality. We can now give a heuristic interpretation of Eq. (61); i.e., optimal state variable feedback diminishes the effect of plant parameter variations (sensitivity) on the closed loop performance of the system. This point is covered in detail in reference 20 and will not be dwelled on here. We point out, however, that Eq. (61), with strict inequality sign, represents a sufficient condition for optimality of a linear feedback design (ref. 21). The theorem and corollaries presented in this section summarize some important aspects of linear optimal control. We can now use theorem T4 to derive an alternative (frequency domain) solution to the output regulator problem. This solution will then be used to extend corollary C7 to include cases when $\underline{R} \neq \underline{I}$ and to extend corollary C8 to include multivariable as well as scalar systems.

4.2 Optimal Control via the Structure Theorem

In order to present a frequency domain solution to the output regulator problem, we will formulate an "equivalent" optimization problem in terms of the canonical state $\underline{z}(t)$ (Section 2.2). A frequency domain solution will then be given for the "equivalent" problem. This solution will then be related to the original output regulator problem, defined in terms of the state $\underline{x}(t)$.

In particular, if Eq. (7) is used to define the system dynamics, then

$$J = \int_0^{\infty} (\underline{z}^T \hat{\underline{C}}^T \hat{\underline{C}} \underline{z} + \underline{u}^T \underline{R} \underline{u}) dt \tag{63}$$

where $\hat{\underline{C}} = \underline{C}\underline{Q}^{-1}$ and $\underline{z} = \underline{Q}\underline{x}$. The optimal control, $\underline{u}^*(\underline{z})$, which minimizes (Eq. (63)), subject to the constraint (Eq. (7)) is unique (ref. 20). Consequently, it must be related to $\underline{u}^*(\underline{x})$, as given by Eq. (58), since the two optimization problems are "equivalent." This is indeed the case, as we will now show. In particular, we use theorem T4, and note that in the case of minimizing (Eq. (63)),

$$\underline{u}^* = \underline{u}^*(\underline{z}) = -\underline{R}^{-1} \hat{\underline{B}}^T \hat{\underline{K}} \underline{z} \tag{64}$$

where $\hat{\underline{K}}$ is the unique positive definite solution to the matrix Riccati equation

$$\hat{\underline{K}}\hat{\underline{A}} + \hat{\underline{A}}^T \underline{K} - \hat{\underline{K}}\hat{\underline{B}}\underline{R}^{-1} \hat{\underline{B}}^T \hat{\underline{K}} = -\hat{\underline{C}}^T \hat{\underline{C}} \tag{65}$$

It can be shown (ref. 23) that $\hat{\underline{K}}$ and $\underline{K}$ (given by Eq. (59)) are related through the transformation matrix $\underline{Q}$; i.e.

$$\underline{K} = \underline{Q}^T \hat{\underline{K}} \underline{Q} \tag{66}$$

Substituting Eq. (66) in Eq. (64), and recalling that $\hat{\underline{B}}^T = \underline{B}^T \underline{Q}^T$, establishes the desired relationship; i.e.

$$\underline{u}^*(\underline{x}) = -\underline{R}^{-1}\underline{B}^T\underline{K}\underline{x} = -\underline{R}^{-1}\hat{\underline{B}}^T\underline{Q}^{-T}\underline{Q}^T\hat{\underline{K}}\underline{Q}\underline{Q}^{-1}\underline{z} = -\underline{R}^{-1}\hat{\underline{B}}^T\hat{\underline{K}}\underline{z} = \underline{u}^*(\underline{z}) \tag{67}$$

For convenience, define $\underline{F}^* = \underline{R}^{-1}\underline{B}^T\underline{K}$ and $\hat{\underline{F}}^* = \underline{R}^{-1}\hat{\underline{B}}^T\hat{\underline{K}}$. Clearly, if $\underline{u}^*(\underline{z}) = -\hat{\underline{F}}^*\underline{z}$ can be determined, then $\underline{u}^*(\underline{x}) = -\underline{F}^*\underline{x}$ can easily be obtained since $\underline{F}^* = \hat{\underline{F}}^*\underline{Q}$. With this in mind, the main result of this section can now be stated and established, as follows in theorem T5.

T5: Consider the completely controllable and observable canonical system, $\dot{\underline{z}} = \hat{\underline{A}}\underline{z} + \hat{\underline{B}}\underline{u}$, $\underline{y} = \hat{\underline{C}}\underline{z}$, and performance index $J = \int_0^\infty (\underline{z}^T\hat{\underline{C}}^T\hat{\underline{C}}\underline{z} + \underline{u}^T\underline{R}\underline{u})dt$, where $\underline{R}$ is positive definite. Let $\underline{u}^*(\underline{z}) = -\hat{\underline{F}}^*\underline{z}$ represent the optimal feedback control vector which minimizes J . $\underline{\delta}_{\underline{F}^*}(s) = \underline{\delta}(s) + \hat{\underline{B}}_m\hat{\underline{F}}^*\underline{S}(s)$ and then satisfies the relationship (note: $\delta(s)$ is shorthand for $\delta_o(s)$):

$$\underline{\delta}^T_{\underline{F}^*}(-s)\hat{\underline{B}}_m^{-T}\underline{R}\hat{\underline{B}}_m^{-1}\underline{\delta}_{\underline{F}^*}(s) = \underline{\delta}^T(-s)\hat{\underline{B}}_m^{-T}\underline{R}\hat{\underline{B}}_m^{-1}\underline{\delta}(s) + \underline{S}^T(-s)\hat{\underline{C}}^T\hat{\underline{C}}\underline{S}(s) \tag{68}$$

Proof: We will use theorem T4 to establish this frequency domain characterization for the optimal system.

Rewrite Eq. (65) in a more convenient form--namely

$$-\hat{\underline{K}}\hat{\underline{A}} - \hat{\underline{A}}^T\hat{\underline{K}} = \hat{\underline{C}}^T\hat{\underline{C}} - \hat{\underline{K}}\hat{\underline{B}}\underline{R}^{-1}\hat{\underline{B}}^T\hat{\underline{K}} \tag{69}$$

Add and subtract $\hat{\underline{K}}s$.

$$\hat{\underline{K}}(s\underline{I} - \hat{\underline{A}}) + (-s\underline{I} - \hat{\underline{A}}^T)\hat{\underline{K}} = \hat{\underline{C}}^T\hat{\underline{C}} - \hat{\underline{K}}\hat{\underline{B}}\underline{R}^{-1}\hat{\underline{B}}^T\hat{\underline{K}} \tag{70}$$

Premultiply and postmultiply Eq. (70) by $\hat{\underline{B}}^T(-s\underline{I} - \hat{\underline{A}}^T)^{-1}$ and $(s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}}$, respectively;

$$\hat{\underline{B}}^T(s\underline{I} - \hat{\underline{A}}^T)^{-1}\hat{\underline{K}}\hat{\underline{B}} + \hat{\underline{B}}^T\hat{\underline{K}}(s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}} =$$

$$\hat{\underline{B}}^T(-s\underline{I} - \hat{\underline{A}}^T)^{-1}[\hat{\underline{C}}^T\hat{\underline{C}} - \hat{\underline{K}}\hat{\underline{B}}\underline{R}^{-1}\hat{\underline{B}}^T\hat{\underline{K}}](s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}} \tag{71}$$

Now use Eq. (23), noting that if $\hat{\underline{F}} = \underline{0}$, $(-s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}} = \underline{S}(-s)\underline{\delta}^{-1}(-s)\hat{\underline{B}}_m$, and consequently, that

$$[(-s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}}]^T = \hat{\underline{B}}^T(-s\underline{I} - \hat{\underline{A}}^T)^{-1} = \hat{\underline{B}}_m^T\underline{\delta}^{-T}(-s)\underline{S}^T(-s) \tag{72}$$

Using Eqs. (15) and (72), we can rewrite Eq. (71) as

$$\underline{B}_m^T\underline{\delta}^T(-s)\underline{S}^T(-s)\hat{\underline{K}}\hat{\underline{B}} + \hat{\underline{B}}^T\hat{\underline{K}}\underline{S}(s)\underline{\delta}^{-1}(s)\hat{\underline{B}}_m =$$

$$\hat{\underline{B}}_m^T\underline{\delta}^{-T}(-s)\underline{S}^T(-s)[\hat{\underline{C}}^T\hat{\underline{C}} - \hat{\underline{K}}\hat{\underline{B}}\underline{R}^{-1}\hat{\underline{B}}^T\hat{\underline{K}}]\underline{S}(s)\underline{\delta}^{-1}(s)\hat{\underline{B}}_m \tag{73}$$

Premultiply and postmultiply Eq. (73) by $\underline{\delta}^T(-s)\underline{B}_m^{-T}$ and $\underline{B}_m^{-1}\underline{\delta}(s)$, respectively, to obtain

$$\underline{S}^T(-s)\hat{\underline{K}}\hat{\underline{B}}\underline{B}_m^{-1}\underline{\delta}(s) + \underline{\delta}^T(-s)\hat{\underline{B}}_m^{-T}\hat{\underline{B}}^T\hat{\underline{K}}\hat{\underline{S}}(s) =$$

$$\underline{S}^T(-s)[\hat{\underline{C}}^T\hat{\underline{C}} - \hat{\underline{K}}\hat{\underline{B}}\underline{R}^{-1}\hat{\underline{B}}^T\hat{\underline{K}}]\underline{S}(s) \tag{74}$$

Noting that

$$[\underline{\delta}^T(-s)\hat{\underline{B}}_m^{-T}\underline{T}^T + \underline{S}^T(-s)\hat{\underline{K}}\hat{\underline{B}}\underline{T}^{-1}][\underline{T}\hat{\underline{B}}_m^{-1}\underline{\delta}(s) + \underline{T}^{-T}\hat{\underline{B}}^T\hat{\underline{K}}\underline{S}(s)] =$$

$$\underline{\delta}^T(-s)\hat{\underline{B}}_m^{-T}\underline{R}\hat{\underline{B}}_m^{-1}\underline{\delta}(s) + \underline{\delta}^T(-s)\hat{\underline{B}}_m^{-T}\underline{B}^T\underline{K}\underline{S}(s) +$$

$$\underline{S}^T(-s)\hat{\underline{K}}\hat{\underline{B}}\hat{\underline{B}}_m^{-1}\underline{\delta}(s) + \underline{S}^T(-s)\hat{\underline{K}}\hat{\underline{B}}\underline{R}^{-1}\underline{B}^T\hat{\underline{K}}\underline{S}(s) \tag{75}$$

substitute Eq. (74) into Eq. (75) to obatin

$$[\underline{\delta}^T(-s)\underline{\hat{B}}_m^{-T}\underline{T}^T + \underline{S}^T(-s)\underline{\hat{K}}\underline{\hat{B}}\underline{T}^{-1}][\underline{T}\underline{\hat{B}}_m^{-1}\underline{\delta}(s) + \underline{T}^{-T}\underline{\hat{B}}^T\underline{\hat{K}}\underline{S}(s)] =$$

$$\underline{\delta}^T(-s)\underline{\hat{B}}_m^{-T}\underline{R}\underline{\hat{B}}_m^{-1}\underline{\delta}(s) + \underline{S}^T(-s)\underline{\hat{C}}^T\underline{\hat{C}}\underline{S}(s) \tag{76}$$

Now recall that

$$\underline{\delta}_{F*}(s) = \underline{\delta}(s) + \underline{\hat{B}}_m\underline{T}^{-1}\underline{T}^{-T}\underline{\hat{B}}^T\underline{\hat{K}}\underline{S}(s) \tag{77}$$

or

$$\underline{T}\underline{\hat{B}}_m^{-1}\underline{\delta}_{F*}(s) = \underline{T}\underline{\hat{B}}_m^{-1}\underline{\delta}(s) + \underline{T}^{-T}\underline{\hat{B}}^T\underline{\hat{K}}\underline{S}(s). \tag{78}$$

Substituting Eq. (78) into Eq. (76) yields Eq. (68) and thus establishes the theorem.

Equation (68) represents a concise frequency domain solution to the optimal output regulator problem. As we will show in the next section, Eq. (68) can be used to directly solve for the optimal feedback matrix $\underline{\hat{F}}^*$, and consequently for $\underline{\hat{F}}$, via spectral factorization. Furthermore, variations in the optimal closed loop pole locations, as $\underline{C}$ and $\underline{R}$ are varied, can easily be determined by using Eq. (68) without first solving the entire optimization problem. Before considering these applications of the theorem, however, we will present an alternative statement and proof of corollary C7, and establish corollary C8 for the general multivariable case. In terms of the structure theorem and the results derived thus far, corollary C7 can be extended to include the case when $\underline{R} \neq \underline{I}$. In particular, we have the corollary (C9):

C9: <u>An optimal feedback design (output regulator problem) implies that the following relationship holds for all real</u> ω .

$$[\cdot]^*[\underline{I} + \underline{T}^{-T}\underline{\hat{B}}^T\underline{\hat{K}}\underline{S}(j\omega)\underline{\delta}^{-1}(j\omega)\underline{\hat{B}}_m\underline{T}^{-1}] - \underline{I} \geq 0 \tag{79}$$

Proof: We premultiply and postmultiply Eq. (71) by $\underline{T}^{-T}$ and $\underline{T}^{-1}$, respectively. We then substitute the resulting expression into the following relationship:

$$
\begin{aligned}
(\underline{I} + \underline{T}^{-T}\hat{\underline{B}}^T(-s\underline{I} - \hat{\underline{A}}^T)^{-1}\hat{\underline{K}}\hat{\underline{B}}\underline{T}^{-1})(\underline{I} + \underline{T}^{-T}\hat{\underline{B}}^T\hat{\underline{K}}(s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}}\underline{T}^{-1}) = \\
\underline{I} + \underline{T}^{-T}\hat{\underline{B}}^T(-s\underline{I} - \hat{\underline{A}}^T)^{-1}\hat{\underline{K}}\hat{\underline{B}}\underline{T}^{-1} + \underline{T}^{-T}\hat{\underline{B}}^T\hat{\underline{K}}(s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}}\underline{T} + \\
\underline{T}^{-T}\hat{\underline{B}}^T(-s\underline{I} - \hat{\underline{A}}^T)^{-1}\hat{\underline{K}}\hat{\underline{B}}\underline{R}^{-1}\hat{\underline{B}}^T\hat{\underline{K}}(s\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}}\underline{T}^{-1}
\end{aligned}
\tag{80}
$$

We then employ Eq. (23), which yields the result

$$
\begin{aligned}
(\underline{I} + \underline{T}^{-T}\hat{\underline{B}}_m^T\underline{\delta}^{-T}(-s)\underline{S}^T(-s)\hat{\underline{K}}\hat{\underline{B}}\underline{T}^{-1})(\underline{I} + \underline{T}^{-T}\hat{\underline{B}}^T\hat{\underline{K}}\underline{S}(s)\underline{\delta}^{-1}(s)\hat{\underline{B}}_m\underline{T}^{-1} = \\
(\underline{I} + \underline{T}^{-1}\hat{\underline{B}}_m^T\underline{\delta}^{-T}(-s)\underline{S}^T(-s)\hat{\underline{C}}^T\hat{\underline{C}}\underline{S}(s)\underline{\delta}^{-1}(s)\hat{\underline{B}}_m T^{-1})
\end{aligned}
\tag{81}
$$

If we then let $s = j\omega$, we obtain

$$
[\cdot]^*(\underline{I} + \underline{T}^{-T}\hat{\underline{B}}^T\hat{\underline{K}}\underline{S}(j\omega)\underline{\delta}^{-1}(j\omega)\hat{\underline{B}}_m\underline{T}^{-1}] - \underline{I} = [\cdot]^*[\hat{\underline{C}}\underline{S}(j\omega)\underline{\delta}^{-1}(j\omega)\hat{\underline{B}}_m\underline{T}^{-1}]
\tag{82}
$$

Clearly, the right side of Eq. (82) is a nonnegative definite matrix. The corollary is thus established. When $\underline{R} = \underline{I}$ and $\underline{T} = \underline{T}^T = \underline{I}$, Eq. (82) exactly corresponds to the results given in references 20 and 21.

We can now establish corollary C8 for the general multivariable case. In particular, we first write Eq. (77) in a more convenient form, namely

$$
\underline{\delta}_{F^*}(s) = \hat{\underline{B}}_m\underline{T}^{-1}(\underline{I} + \underline{T}^{-T}\hat{\underline{B}}^T\hat{\underline{K}}\underline{S}(s)\underline{\delta}^{-1}(s)\hat{\underline{B}}_m\underline{T}^{-1})\underline{T}\hat{\underline{B}}_m^{-1}\underline{\delta}(s)
\tag{83}
$$

We then take the determinant of both sides of Eq. (83), obtaining

$$
|\underline{I} + \underline{T}^{-T}\hat{\underline{B}}^T\hat{\underline{K}}\underline{S}(s)\underline{\delta}^{-1}(s)\hat{\underline{B}}_m\underline{T}^{-1}| = |\underline{\delta}_{F^*}(s)| \div |\underline{\delta}(s)|
\tag{84}
$$

If we now take the determinant of Eq. (81), substituting Eq. (84)

where appropriate and letting $s = j\omega$, the resulting expression is

$$\frac{|\underline{\delta}^T_{F^*}(-j\omega)||\underline{\delta}_{F^*}(j\omega)|}{|\underline{\delta}^T(-j\omega)||\underline{\delta}(j\omega)|} = |\underline{I} + \underline{M}(\omega)| \tag{85}$$

where $\underline{M}(\omega) = [\cdot]^*\hat{\underline{C}}\underline{S}(j\omega)\underline{\delta}^{-1}(j\omega)\hat{\underline{B}}_m\underline{T}^{-1}$, a nonnegative definite matrix. Since the determinant of the sum of $\underline{I}$ and a nonnegative definite matrix is always greater than or equal to 1 (ref. 8), Eq. (85) implies Eq. (62) directly, and thus establishes corollary C8 for the general multivariable case.

A final point to be clarified in this section concerns certain erroneous statements made in reference 20 regarding strict inequality of Eqs. (61) and (62) representing a necessary condition for optimality in the scalar case. In particular, Condition (II) in reference 20 states that a stable control law may be optimal only if the return difference is greater than 1 for all real ω. The following example refutes this claim as well as other erroneous statements which were based on Condition (II).

Consider the scalar system, $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}u$, $y = \underline{c}\underline{x}$, where $\underline{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $c = [0,2]$. Suppose we now wish to find the optimal control $u^*(\underline{x})$, which minimizes $J = \int_0^\infty (4x_2^2 + u^2)dt$. We note that the pair $\{\underline{A},\underline{b}\}$ is controllable and that $\underline{R} = 1$ is a scalar. Theorem T4 can therefore be applied directly. The reader can easily verify that $\underline{K}$, the solution to the matrix Riccati equation, $\underline{K}\underline{A} + \underline{A}^T K - \underline{K}\underline{b}\underline{b}^T\underline{K} = -\underline{c}^T\underline{c}$ is equal to $2\underline{I}$. Hence, $u^*(\underline{x}) = -\underline{b}^T\underline{K}\underline{x} = -2x_2$. According to corollary C7 and, in particular, Eq. (61), optimality implies that $[\cdot]^*[\underline{I} + \underline{B}^T\underline{K}(j\omega\underline{I} - \underline{A})^{-1}\underline{B}] - \underline{I} \geq 0$. For the example, this condition implies that

$$\left\{1 - \frac{2j\omega}{1-\omega^2}\right\}\left\{1 + \frac{2j\omega}{1-\omega^2}\right\} - 1 \geqq 0$$

or

$$\frac{4\omega^2}{(1-\omega^2)^2} \geqq 0 \text{ for all real } \omega$$

Clearly, $\frac{4\omega^2}{(1-\omega^2)^2}$ is indeed $\geqq 0$ for all real ω . However, we note that $\frac{4\omega^2}{(1-\omega^2)^2} = 0$ at $\omega = 0$, or the return difference $\left\{1 + \frac{2s}{s^2+1}\right\}_{s=j\omega} = \left\{1 + \frac{2j\omega}{1-\omega^2}\right\}$ is equal to 1 at $\omega = 0$. Hence, Condition (II), Eqs. (34) and (17) of reference 20 fail to hold in general.

4.3 Spectral Factorization and the Root-Square Locus

Perhaps the most significant aspect of theorem T5 is that $\underline{u}^*(x)$ can be obtained directly from Eq. (68) via spectral factorization (ref. 24). This fact had been noted previously by Chang (ref. 25), Kalman (ref. 20), and Rynaski and Whitbeck (ref. 26). However, their expressions, corresponding to Eq. (68) in this report, were not in as compact a form as Eq. (68) and, in the multivariable case, included the unaltered term $(s\underline{I} - \underline{A})^{-1}\underline{B}$. We note that the right side of Eq. (68) represents the sum of two (mxm) matrices. Furthermore, the entries of each of these two matrices are only polynomials in the Laplace operator s (no ratios of polynomials appear, as in the term $(s\underline{I} - \underline{A})^{-1}\underline{B}$). It would therefore appear that a spectral factorization solution to Eq. (68) would be easier to achieve than prior spectral factorization solutions. Indeed, Rynaski and Whitbeck (ref. 26) dismiss spectral factorization as a "tedious chore" and, instead, adopt a "direct" method for obtaining $\underline{u}^*(\underline{x})$. This "direct" method, later refined by Whitbeck (ref. 27), involves some rather cumbersome transfer matrix manipulations. The optimal control $\underline{u}^*(\underline{s})$, is also expressed in terms of the initial state $\underline{x}(0)$ instead of $\underline{F}^*\underline{x}$. Furthermore, their technique involves an initial spectral factorization in order to determine the optimal closed loop poles of the system. It appears that their "direct" method is not totally amenable to automatic computation and involves more information than necessary to obtain $u^*(\underline{x})$. Nevertheless, it has been demonstrated that their method does produce the optimal feedback matrix, $\underline{F}^*$, and the merits of their technique, relative to other solutions to the output regulator problem, remain a debatable subject.

We observed earlier that Eq. (68) represents a compact frequency domain solution to the output regulator problem. We will now establish this fact by demonstrating that $\underline{\delta}_{\underline{F}^*}(s)$ and, therefore, $\hat{\underline{F}}^*$ and $\underline{F}^* = \hat{\underline{F}}^*\underline{Q}$ can be determined by a spectral factorization of Eq. (68). Let us define the right side of Eq. (68) as $\Phi(s)$;

$$\underline{\Phi}(s) = \underline{\delta}^T(-s)\hat{\underline{B}}_m^{-T}\underline{T}^T\underline{T}\hat{\underline{B}}_m^{-1}\underline{\delta}(s) + \underline{S}^T(-s)\hat{\underline{C}}^T\hat{\underline{C}}\underline{S}(s) \tag{86}$$

We will now employ two results due to Youla (ref. 24).

(I) Consider the matrix $\underline{\Phi}(s)$. If (1) $\underline{\Phi}^T(-j\omega) = \underline{\Phi}(j\omega)$ and (2) $\underline{\Phi}(j\omega)$ is nonnegative definite for all real ω, then $\underline{\Phi}(s)$ can be written as the product of two matrices, $W^T(-s)$ and $W(s)$; i.e.

$$\underline{\Phi}(s) = \underline{W}^T(-s)\underline{W}(s) \tag{87}$$

where $\underline{W}(s)$ and its enverse are both analytic in the half-plane Re $s \geq 0$, and $\underline{W}^T(-s)$ and its inverse are both analytic in the half-plane Re $s \geq 0$.

(II) Consider that any two solutions, $\underline{W}_1(s)$ and $\underline{W}_2(s)$, of the "spectral factorization" of $\underline{\Phi}(s)$ which satisfy Eq. (87), are orthogonally equivalent (ref. 8); i.e.

$$\underline{W}_1(s) = \underline{U}\underline{W}_2(s) \tag{88}$$

for some orthogonal matrix $\underline{U}$ (ref. 18), where $\underline{U}^T\underline{U} = \underline{I}$. $\underline{\Phi}(s)$, the right side of Eq. (68), clearly satisfies (1) and (2) of (I). Furthermore, since $|\underline{\delta}_{F*}(s)| = 0$ only in the half-plane Re $s < 0$ (i.e., the closed loop optimal system is asymptotically stable (refs. 19 and 20), it follows that $\underline{\delta}_{F*}(s)$ and its inverse are both analytic in the half-plane Re $s \geq 0$. The matrix $\underline{T}\hat{\underline{B}}_m^{-1}\underline{\delta}_{F*}(s)$ thus qualifies as a solution of the spectral factorization of $\Phi(s)$; i.e., if one determines a solution $\underline{W}(s)$ of Eq. (87) via spectral factorization, then by (II),

$$\underline{\delta}_{F*}(s) = \hat{\underline{B}}_m\underline{T}^{-1}\underline{U}\underline{W}(s) \tag{89}$$

for some orthogonal matrix $\underline{U}$. The problem of solving Eq. (68) for $\underline{\delta}_{F*}(s)$ and, consequently, $\hat{\underline{F}}^*$ and $\hat{\underline{F}}^* = \hat{\underline{F}}^*\underline{Q}$, thus reduces to two distinct steps: (1) the spectral factorization of $\underline{\Phi}(s)$, the right side of Eq. (68); and (2) the implementation of Eq. (89) once $\underline{W}(s)$, a solution to (1), has been found. The first of these steps is, by far, the more difficult. The spectral factorization problem is important in other related research areas, such as filtering (ref. 28) and network synthesis (ref. 29), and has been the subject of a number of technical reports over the past few years. Several investigators have proposed various solutions to the spectral factorization problem (refs. 24, 30, and 39), and we will not dwell on their relative merits in this report. It should be noted, however, that computer programs which perform the factorization have been developed (ref. 31).

Once $\underline{W}(s)$, a solution to the spectral factorization of $\underline{\Phi}(s)$, has been obtained, the task of implementing Eq. (89) to determine $\underline{\delta}_{F*}(s)$ remains. In other words, $\hat{\underline{B}}_m\underline{T}^{-1}\underline{U}$ must be determined in order to solve Eq. (89) for $\underline{\delta}_{F*}(s)$. $\underline{F}^*$ can then be determined by employing the relations, $\underline{\delta}_{F*}(s) = \underline{\delta}(s) + \hat{\underline{B}}_m\hat{\underline{F}}^*S(s)$, or $\hat{\underline{F}}^*\underline{S}(s) = \hat{\underline{B}}_m^{-1}(\underline{\delta}_{F*}(s) - \underline{\delta}(s))$, and $\underline{F}^* = \hat{\underline{F}}^*\underline{Q}$. The determination of $\hat{\underline{B}}_m\underline{T}^{-1}\underline{U}$, however, is surprisingly simple, and does not depend on the facts that $\hat{\underline{B}}_m$ and $\underline{T}^{-1}$ are already known, and $\underline{U}$ is an orthogonal matrix. We simply recall that, by corollary C2 and Eq. (27),

$\underline{\delta}_{F*}(s) = \underline{\delta}_o(s) + \hat{\underline{B}}_m\hat{\underline{F}}^*\underline{S}(s)$, where $\underline{\delta}_o(s) = \underline{\delta}(s) = \text{diag}[s^{\sigma_i}] - \hat{\underline{A}}_m\underline{S}(s)$. Substituting these expressions for $\underline{\delta}_{F*}(s)$ in Eq. (89) and solving for $\underline{W}(s)$, we obtain

$$\underline{W}(s) = \underline{U}^T\underline{T}\hat{\underline{B}}_m^{-1}\text{diag}[s^{\sigma_i}] - (\hat{\underline{A}}_m - \hat{\underline{B}}_m\hat{\underline{F}}^*)\underline{S}(s) \tag{90}$$

We now note that the coeffecients of s^{σ_i} appearing in each of the m i-th columns of $\underline{W}(s)$ are precisely the entries which appear in each of the m i-th columns of $\underline{U}^T\underline{T}\hat{\underline{B}}_m^{-1}$. Consequently, $\underline{U}^T\underline{T}\hat{\underline{B}}_m^{-1}$ can be determined directly from $\underline{W}(s)$ by inspection, since σ_i is the highest power of s which appears in each of the m i-th columns of $\underline{W}(s)$. Once $\underline{U}^T\underline{T}\hat{\underline{B}}_m^{-1}$ has been determined, its inverse $\hat{\underline{B}}_m\underline{T}^{-1}\underline{U}$, can then be used to obtain $\underline{\delta}_{F*}(s)$ via Eq. (89). $\hat{\underline{F}}^*\underline{S}(s)$ is then given by

$$\hat{\underline{F}}^*\underline{S}(s) = \hat{\underline{B}}_m^{-1}[\underline{\delta}_{F*}(s) - \underline{\delta}(s)] \tag{91}$$

$\hat{\underline{F}}^*$ can then be obtained by inspection and $\underline{F}^* = \hat{\underline{F}}^*\underline{Q}$. An example which demonstrates these procedures will now be presented. In particular, consider the system $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{y} = \underline{C}\underline{x}$, and performance index $J = \int_0^\infty (\underline{x}^T\underline{C}^T\underline{C}\underline{x} + \underline{u}^T\underline{R}\underline{u})dt$, where

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -3 \\ 1 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 3 & 2 & 0 & 0 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -1 & -3 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

We must now find $\underline{u}^*(\underline{x}) = \underline{F}^*\underline{x}$ which minimizes J, using spectral factorization. The first step is to find a $\underline{Q}$ which transforms this given system to controllable canonical form as represented by Eqs. (7) and (13) through (16). Using the technique outlined in Section 2.2 gives

$$\underline{Q} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\underline{QAQ}^{-1} = \hat{\underline{A}} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 & -3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 2 & -2 & 1 & 0 & 0 \end{array}\right] \qquad \underline{QB} = \hat{\underline{B}} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 3 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right]$$

$\hat{\underline{C}} = \underline{CQ}^{-1}$ is given by

$$\hat{\underline{C}} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \hat{\underline{S}}(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix}$$

since $\sigma_1 = 3$, $\sigma_2 = 2$, $d_1 = 3$, and $d_2 = 5 = n$. By inspection,

$$\hat{\underline{B}}_m = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad \hat{\underline{A}}_m = \begin{bmatrix} -1 & 0 & 0 & 2 & -3 \\ 2 & -2 & 1 & 0 & 0 \end{bmatrix}$$

$\delta(s)$ is then determined by using Eq. (27);

$$\underline{\delta}(s) = \begin{bmatrix} s^3+1, & , & 3s-2 \\ -s^2+2s-2, & & s^2 \end{bmatrix}$$

Using the above relationships, we can now also use Eq. (86) to determine $\underline{\Phi}(s)$; i.e.

$$\underline{\Phi}(s) = \begin{bmatrix} -2s^6+44s^4+8s^2+102, & 6s^5-25s^4-16s^3-20s^2+18s-28 \\ -6s^5-25s^4+16s^3-20s^2-18s-28, & 19s^4+5s^2+8 \end{bmatrix}$$

$\underline{\Phi}(s)$ can then be used as the input to Tuel's program (ref. 31), modified to yield $\underline{\delta}_{\underline{F}*}(s)$ directly via spectral factorization of $\underline{\Phi}(s)$ and multiplication by $\hat{\underline{B}}_m \underline{T}^{-1} \underline{U}$; i.e.

$$\underline{\delta}_{\underline{F}*}(s) = \begin{bmatrix} s^3+5.748s^2+17.764s+31.018, & 1.676s-8.370 \\ .417s^2+6.203s+9.6, & s^2+.82s-2.461 \end{bmatrix}$$

$\hat{\underline{F}}*S(s)$ is determined next by using Eq. (91);

$$\hat{\underline{F}}*\underline{S}(s) = \begin{bmatrix} 1.497s^2+5.155s-4.782, & -3.784s+1.013 \\ 1.417s^2+4.203s+11.6, & .82s-2.461 \end{bmatrix}$$

and $\hat{\underline{F}}*$can then be determined by inspection since $\underline{S}(s)$ is known.

$$\hat{\underline{F}}* = \begin{bmatrix} -4.782 & 5.155 & 1.497 & 1.013 & -3.784 \\ 11.6 & 4.203 & 1.417 & -2.461 & .82 \end{bmatrix}$$

and since $\underline{F}* = \hat{\underline{F}}*\underline{Q}$

$$\underline{F}* = \begin{bmatrix} 5.155 & -3.285 & -4.782 & 1.013 & -3.784 \\ 4.203 & 13.017 & 11.6 & -2.461 & .82 \end{bmatrix}$$

Therefore, $\underline{F}*$, the feedback gain matrix associated with the optimal system has been determined via spectral factorization. It should be clear that $\underline{F}* = \underline{R}^{-1}\underline{B}^T\underline{K}$, where $\underline{K}$ is the solution to Eq. (59).

The final topic we will discuss in this section is the "root-square locus." In particular, it is desirable, in certain applications of linear optimal control theory, to readily determine the effect on the closed loop optimal poles caused by changes in the elements comprising the performance index weighting matrices $\underline{C}$ and $\underline{R}$ (or $\underline{T}$). A plot of optimal pole location changes caused by weighting matrix changes is known as the root-square locus. It was first introduced by Chang (ref. 25) to study optimal scalar systems. The root-square locus was refined somewhat by Kalman (ref. 20), and later extended to include multivariable systems by Rynaski and Whitbeck (ref. 26). In all cases mentioned, the root-square locus was a frequency domain tool, and its implementation depended on the ability to formulate a solution to the regulator problem in the frequency domain and then to find the characteristic polynomial representing the closed loop optimal system without first determining $\underline{u}^*(\underline{x})$; i.e. $\underline{F}^*$. Rynaski and Whitbeck (ref. 26) appear to be the only investigators who have sought an expression for the root-square locus in the general (multivariable) case. In particular, they show (Eq. (7-33) in reference 26) that if $\underline{W}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B}$, then

$$\det(\underline{R} + \underline{W}^T(-s)\underline{W}(s)) = \frac{\Delta_{F^*}(s)\Delta_{F^*}(-s)}{\Delta(s)\Delta(-s)} \tag{92}$$

where $\Delta_{F^*}(s)$ is the characteristic polynomial associated with the optimal system; i.e., note that $\Delta_{\underline{F}^*}(s) = |\underline{\delta}_{\underline{F}^*}(s)| = |s\underline{I} - \underline{A} + \underline{B}\underline{F}^*|$, and $\Delta(s)$ is the open loop characteristic polynomial. We point out also that the evaluation of the determinant of $\underline{R} + \underline{W}^T(-s)\underline{W}(s)$ involves some rather cumbersome polynomial manipulations which can be significantly reduced if the structure theorem is employed. In particular, by simply taking the determinant of both sides of Eq. (68), we obtain the following relationship:

$$|\underline{R}|\Delta_{\underline{F}^*}(-s)\Delta_{\underline{F}^*}(s) = \det(\underline{\delta}^T(-s)\hat{\underline{B}}_m^{-T}\underline{R}\hat{\underline{B}}_m^{-1}\underline{\delta}(s) + \underline{S}^T(-s)\hat{\underline{C}}^T\hat{\underline{C}}\underline{S}(s)) \tag{93}$$

Clearly, Eq. (93) also represents a means of obtaining the root-square locus, since the characteristic polynomial of the optimal system, $\Delta_{\underline{F}*}(s)$, is expressed directly in terms of known quantities which include the performance index weighting matrices $\underline{R}$ and $\underline{C} = \underline{\hat{C}}\underline{Q}$. The latter expression, moreover, is significantly easier to evaluate, as the reader can verify. Equation (93) thus represents an alternative expression which can be employed to plot the root-square locus in the general multivariable case. We finally note that Eq. (93) directly reduces to the scalar root-square locus expression given by Kalman (Eq. (45) in reference 20) for the case when m = 1.

We will now present an example to illustrate the application of Eq. (93) in determining closed loop optimal pole variations corresponding to variations in the weighting matrices $\underline{C}$ and $\underline{R}$. In particular, consider the system $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{y} = \underline{C}\underline{x}$, and performance index $J = \int_0^\infty (\underline{x}^T\underline{C}^T\underline{C}\underline{x} + \underline{u}^T\underline{R}\underline{u})dt$, where

$$\underline{A} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] \qquad \underline{B} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array}\right] \qquad \underline{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\underline{R} = \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$$

It has been shown that if $r > 0$, the optimal control law, $\underline{u}*(\underline{x})$, which minimizes J, is given by $\underline{u}*(\underline{x}) = -\underline{R}^{-1}\underline{B}^T\underline{K}\underline{x} = -\underline{F}*\underline{x}$, where $\underline{K}$ is the unique positive definite solution to the matrix Riccati equation (Eq. (59)). The closed loop poles associated with the optimal system are therefore equal to the zero of $|s\underline{I} - \underline{A} + \underline{B}\underline{R}^{-1}\underline{B}^T\underline{K}| = \Delta*(s)$. However, Eq. (93) can be used to determine the closed loop optimal poles without first solving for $u*(x)$; i.e.

$\underline{F}^*$ need not be known. For this example, an expression for the optimal poles, in terms of the positive scalar r, will now be obtained.

We first note that the system is already in controllable companion form (no transformation of state is required). Therefore, by inspection,

$$\underline{S}(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\delta}(s) = \begin{bmatrix} s^2 & 0 \\ -1 & s \end{bmatrix} \qquad \underline{C}\underline{S}(s) = \underline{I} \qquad \hat{\underline{B}}_m = \underline{I}$$

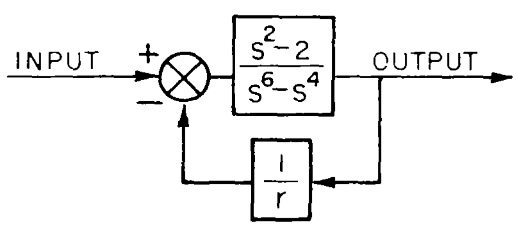
If we now use Eq. (93), we obtain

$$r\Delta^*(-s)\Delta^*(s) = \det\left\{\begin{bmatrix} s^2 & -1 \\ 0 & -s \end{bmatrix}\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ -1 & s \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

or

$$\Delta^*(-s)\Delta^*(s) = s^4 - s^6 + \frac{1}{r}(-s^2 + 2)$$

This expression can now be used to plot the root-square locus by using the conventional root locus (ref. 32). In particular, consider the following block diagram:



EQUIVALENT BLOCK DIAGRAM.

The root locus (ref. 32) can be employed to plot the variation of the closed loop poles of this system with respect to $\frac{1}{r}$. Furthermore, note that the poles of this closed loop system are equal to the zeros of $(s^6 - s^4 + \frac{1}{r}(s^2 - 2))$ which, in turn, equal the zeros of $\Delta^*(-s)\Delta^*(s)$. For the example, the optimal poles and their mirror images are thus equal to poles of the system shown in the block diagram. An approximate root-locus plot of this system is given in Figure 1. The arrows indicate the direction of increasing r; i.e., as $r \to \infty$, $\Delta^*(s) \to s^2(s + 1)$. Note further, that the optimal poles display a Butterworth configuration, as noted previously in reference 20.

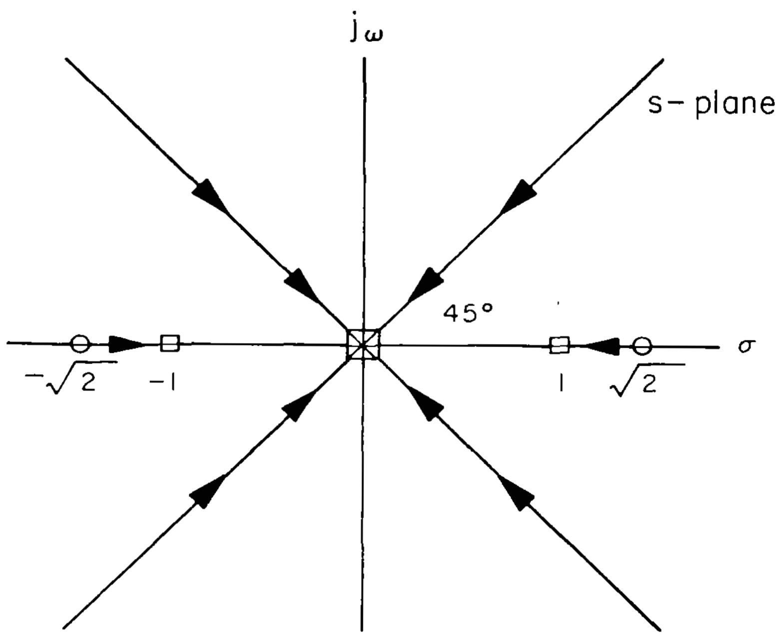


Figure 1.- Root-square locus plot of the example

5. TRANSFER MATRIX FORMS AND NONINTERACTION

In this section, we apply the structure theorem to obtain some results pertaining to transfer matrix forms. We provide answers to certain questions dealing with the noninteraction of various input/output pairs. The motivation for this work should be rather obvious to those who have dealt with the analysis and design of linear multivariable systems. It is almost always desirable to isolate various segments of a multivariable system in order to deal with the smaller and less complex subsystems individually. Furthermore, a noninteractive or decoupled system is usually easier to control. The control literature of the past several years contains a rather significant number of articles dealing with noninteractive design and additional results still appear quite frequently.

In this section, we will combine certain prior results with the structure theorem in order to obtain answers to a number of questions dealing with noninteraction which have, thus far, remained unresolved. In Section 5.1, we will consider various questions pertaining to decoupling multivariable systems through the use of linear state variable feedback. In terms of the transfer matrix of the system, decoupling will be used to denote complete noninteraction of off-diagonal input/output pairs; i.e. the transfer matrix of a decoupled system is strictly diagonal and nonsingular. In Section 5.2, we extend the results of Section 5.1 and discuss decoupling via "input dynamics" as well as state variable feedback. Roughly speaking, "input dynamics" involves the addition of a dynamical system to the input of the system we wish to decouple.

In Section 5.3, a more general type of noninteraction is discussed. Here, we are not necessarily interested in complete off-diagonal noninteraction, but rather, in achieving a closed loop transfer matrix which displays a significant number of zero entries. We will present a simple example here and carry it throughout the remainder of this section. In particular, consider the system $\dot{\underline{x}} = \underline{\underline{A}}\underline{x} + \underline{\underline{B}}\underline{u}$, $\underline{y} = \underline{\underline{C}}\underline{x}$, where

$$\underline{\underline{A}} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -2 \end{array}\right] \qquad \underline{\underline{B}} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right]$$

$$\underline{C} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 1 & 2 & 0 \end{bmatrix}$$

Clearly, the pair {A,B} is in multi-input companion form (Section 2.1) and, consequently, the structure theorem can be employed directly (without) transformation of coordinates); i.e., $\sigma_1 = \sigma_2 = 2$, $\underline{B}_m = \underline{I}$,

$$\underline{S}(s) = \begin{bmatrix} 1 & 1 \\ s & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix} \qquad \underline{CS}(s) \doteq \begin{bmatrix} s+1, 1 \\ \\ s-2, 2 \end{bmatrix}$$

$$\det(CS(s)) = s + 4 \qquad \underline{\delta}(s) = \text{diag}[s^{\sigma_i}] - \underline{A}_m\underline{S}(s)$$

or

$$\underline{\delta}(s) = \begin{bmatrix} s^2 & 0 \\ \\ 0 & s^2 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 & 0 \\ \\ 1 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} s^2+3, & 0 \\ \\ -s-1, & (s+1)^2 \end{bmatrix}$$

Consequently,

$$\underline{\delta}^{-1}(s) = \begin{bmatrix} \dfrac{1}{s^2+3}, & 0 \\ \\ \dfrac{1}{(s+1)(s^2+3)}, & \dfrac{1}{(s+1)^2} \end{bmatrix}$$

The open loop transfer matrix of this system, $\underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} = \underline{CS}(s)\underline{\delta}^{-1}(s)\underline{B}_m$, is given by

$$\underline{T}(s) = \underline{CS}(s)\underline{\delta}^{-1}(s) = \begin{bmatrix} \dfrac{s^2+2s+2}{(s+1)(s^2+3)}, & \dfrac{1}{(s+1)^2} \\ \\ \dfrac{s^2-s}{(s+1)(s^2+3)}, & \dfrac{2}{(s+1)^2} \end{bmatrix}$$

Note that the open loop transfer matrix of this system displays complete interaction of all input/output terms; i.e., no elements, $t_{ij}(s)$, of $\underline{T}(s)$ are zero.

5.1 Decoupling via Linear State Variable Feedback

We now apply the structure theorem to obtain some results related to the problem of decoupling using linear state variable feedback. This problem has been examined previously by a number of authors (refs. 33 through 36) and a number of relevant questions have been resolved. More precisely, consider the following.

DECOUPLING PROBLEM: Let $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{y} = \underline{C}\underline{x}$ be an m input, m output system. Does there exist a pair of matrices {F,G} such that the transfer matrix

$$\underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B}\underline{G} = \underline{T}_{\underline{F},\underline{G}}(s) \tag{94}$$

is diagonal and nonsingular? (In other words, does the state variable feedback $\underline{u} = \underline{F}\underline{x} + \underline{G}\underline{w}$ "decouple" the system?)

A necessary and sufficient condition for the existence of a decoupling pair was first given in reference 34. In particular, it has been shown that the system

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} \ , \ \underline{y} = \underline{C}\underline{x} \tag{95}$$

can be decoupled by using linear state variable feedback if and only if $\underline{B}^*$ is nonsingular, where $\underline{B}^*$ is the (mxm) matrix given by

$$\underline{B}^* = \begin{bmatrix} \underline{c}_1\underline{A}^{f_1}\underline{B} \\ \vdots \\ \underline{c}_m\underline{A}^{f_m}\underline{B} \end{bmatrix} \tag{96}$$

with $\underline{c}_i$, the i-th row of $\underline{C}$, and $f_i = \min[\{j: \underline{c}_i\underline{A}^j\underline{B} \neq 0\}, n - 1]$. $\underline{B}^*$ and the f_i can also be characterized in the following way (ref. 35): let $\underline{T}_{\underline{F},\underline{G},i}(s)$ be the i-th row of the transfer matrix $\underline{T}_{\underline{F},\underline{G}}(s)$; then $f_i = \min[\{j; \lim_{s\to\infty} s^{j+1}\underline{T}_{\underline{F},\underline{G},i}(s) \neq 0\}, n - 1]$ and $\underline{B}^*\underline{G} = \lim_{s\to\infty}\underline{\Delta}(s)\underline{T}_{\underline{F},\underline{G}}(s)$ where $\underline{\Delta}(s)$ is a diagonal matrix with entries s^{f_i+1}. It can be shown (refs. 34 and 35) that $\underline{B}^*$ and the f_i are invariant under state variable feedback.

Here, we shall use the structure theorem to answer the following questions:

1. Assuming that the system represented by Eq. (95) can be decoupled, what is the maximum number of closed loop poles which can be arbitrarily specified while simultaneously decoupling the system?
2. Assuming that the system represented by Eq. (95) can be decoupled, which closed loop poles are invariant under decoupling state variable feedback?
3. How can a decoupling pair which specifies the maximum number of closed loop poles be implemented?

While these questions are to some degree resolved in references 34, 35, and 36, we provide a complete and elementary answer to them here.

Let $\underline{T}(s)$ be the transfer matrix of Eq. (95). Then $\underline{T}(s) = \underline{C}^*(s)\,\frac{\Delta_u(s)}{\Delta_u(s)}\,\underline{\delta}_{\underline{0},c}^{-1}(s)\hat{\underline{B}}_m$ where $\underline{C}^*(s) = \hat{\underline{C}}\underline{S}(s)$ by the structure theorem T2. We recall that $\underline{C}^*(s)$ and $\Delta_u(s)$ are invariant under state variable feedback. Now we let $p_i(s)$ be the greatest common divisor of the polynomials which are the entries in the i-th row $\underline{C}_i^*(s)$ of $\underline{C}^*(s)$. We note that $p_i(s)$ is invariant under state variable feedback. We let r_i be the degree of $p_i(s)$ and we use the notation ∂_p to denote the degree of a polynomial (thus, $r_i = \partial_{p_i}$). We now have the theorem (T6):

T6: Suppose that the system (Eq. (95)) can be decoupled. Then (1) the maximum number ν of the closed loop poles which can be arbitrarily specified while decoupling is given by

$$\nu = \sum_{i=1}^{m} (r_i + f_i + 1) \qquad (97)$$

and (2) the invariant poles under decoupling feedback are the zeros of $\Delta_u(s)$ and $\{\det C^*(s)\}/\prod_{i=1}^{m}(p_i(s)$.

Proof: Let $\{\underline{F},\underline{G}\}$ be any decoupled pair. Then $\underline{T}_{\underline{F},\underline{G}}(s) = \underline{C}(s\underline{I} - \underline{A} - \underline{BF})^{-1}\underline{BG}$ is a diagonal matrix with entries $n_{ii}(s)/d_{ii}(s)$ where $n_{ii}(s)$ and $d_{ii}(s)$ are relatively prime. We note that, since $f_i = \min\{j:\lim_{s\to\infty} s^{j+1}\underline{T}_{\underline{F},\underline{G},i}(s) \neq \underline{0}\}$,* $\partial_{n_{ii}} = \partial_{d_{ii}} - f_i - 1$.

It follows from corollary C6 and the definition of the $p_i(s)$ that

$$\prod_{i=1}^{m} \frac{n_{ii}(s)}{d_{ii}(s)} = \prod_{i=1}^{m} p_i(s) \det \underline{C}^*_{II}(s) \frac{\Delta_u(s)}{\Delta_{\underline{F}}(s)} \det \underline{G} \qquad (98)$$

where $C^*_{II}(s)$ is the matrix with rows $\underline{C}^*_{II}(s) = \frac{1}{p_i(s)} \underline{C}^*_i(s)$. Since $\Delta_{\underline{F}}(s) = \Delta_u(s)\Delta_{\underline{F},c}(s)$, we have

$$\partial_{F,c} = \sum_{i=1}^{m} (r_i + f_i + 1) + \partial^*_{II} \qquad (99)$$

where ∂^*_{II} is the degree of $\det \underline{C}^*_{II}(s)$ and $\partial_{\underline{F},c}$ is the degree of $\Delta_{\underline{F},c}(s)$. Now, it is clear from theorem T2 that

$$\underline{T}_{\underline{F},\underline{G},i}(s)\underline{G}^{-1}\underline{\hat{B}}_m^{-1}\underline{\delta}_{\underline{F},c}(s) = \underline{C}^*_i(s) \qquad (100)$$

and, hence, that $n_{ii}(s)$ is a common divisor of the entries in

---

*Note that $\underline{B}^*$ is nonsingular.

$\underline{C}_i^*(s)$ (since $n_{ii}(s)$ and $d_{ii}(s)$ are relatively prime). In other words, $n_{ii}(s)$ <u>must divide</u> $p_i(s)$ and so $\partial_{n_{ii}} \leq r_i$. Since no more than $\sum_{i=1}^{m} \partial_{d_{ii}}$ poles are assignable through $\{\underline{F},\underline{G}\}$ and $\sum_{i=1}^{m} \partial_{d_{ii}} = \sum_{i=1}^{m} (\partial_{n_{ii}} + f_i + 1)$, we deduce that at most $\nu = \sum_{i=1}^{m} (r_i + f_i + 1)$ poles are assignable while decoupling.

Writing $\underline{T}_{\underline{F},\underline{G}}(s)$ as a diagonal matrix with entries $q_{ii}(s)/\Delta_{\underline{F}}(s) = n_{ii}(s)/d_{ii}(s)$, we deduce that $q_{ii}(s)$ must divide $p_i(s)\Delta_{\underline{F}}(s)$ or, equivalently, that

$$\frac{q_{ii}(s)}{\Delta_{\underline{F}}(s)} = \frac{p_i(s)}{q_i(s)} \tag{101}$$

for $i = 1,\ldots,m$ and polynomials $q_i(s)$ with $\partial_{q_i} = r_i + f_i + 1$. It follows that $\det \underline{T}_{\underline{F},\underline{G}}(s) = \prod_{i=1}^{m} (p_i(s)) / \prod_{i=1}^{m} (q_i(s))$ and, hence, from Eq. (98) that

$$\begin{aligned} \Delta_{\underline{F}}(s) &= \det \underline{C}_{II}^*(s)\Delta_u(s)\det \underline{G} \prod_{i=1}^{m} q_i(s) \\ &= \frac{\det C^*(s)}{\prod_{i=1}^{m} p_i(s)} \Delta_u(s)\det \underline{G} \prod_{i=1}^{m} q_i(s) \end{aligned} \tag{102}$$

Since $\underline{C}_{II}^*(s)$ is invariant under decoupling feedback, it follows that the zeros of $\Delta_u(s)$ and $\det \underline{C}_{II}^*(s)$ are invariant poles under decoupling feedback.

Thus, to complete the proof we need only construct a decoupling pair $\{\underline{F},\underline{G}\}$ such that the resulting polynomials $q_i(s)$ are arbitrary polynomials of degree $r_i + f_i + 1$. To begin with, we note that transfer matrix

$$\underline{T}(s) = \underline{C}^*(s)\,\frac{\Delta_u(s)}{\Delta_u(s)}\,\underline{\delta}_{0,c}^{-1}(s)\hat{\underline{B}}_m = \underline{P}(s)\underline{C}_{II}^*(s)\,\frac{\Delta_u(s)}{\Delta_u(s)}\,\underline{\delta}_{0,c}^{-1}(s)\hat{\underline{B}}_m$$

where $\underline{P}(s)$ is a diagonal matrix with entries $p_i(s)$. Setting

$$\underline{T}_{II}(s) = \underline{C}_{II}^*(s)\,\frac{\Delta_u(s)}{\Delta_u(s)}\,\underline{\delta}_{0,c}^{-1}(s)\hat{B}_m \tag{103}$$

we can easily see that $r_i + f_i = \min\{j:\lim_{s\to\infty} s^{j+1}\underline{T}_{II,i}(s) \neq 0\}$ and that $\underline{B}_{II}^* = \lim_{s\to\infty}\underline{\Delta}_{II}(s)\underline{T}_{II}(s) = \underline{B}^*$ where $\underline{\Delta}_{II}(s)$ is a diagonal matrix with entries $s^{r_i+f_i+1}$ (note that the $p_i(s)$ are monic). Moreover, as $\underline{C}^*(s)$ is given by $\hat{\underline{C}}\underline{S}(s)$ and $p_i(s)$ is the greatest common divisor of the entries in $\underline{C}_i^*(s)$, we can write $\underline{C}_{II}^*(s)$ in the form $\hat{\underline{C}}_{II}(s)$ for some constant matrix $\hat{\underline{C}}_{II}$ (where $\underline{S}(s)$ is given by Eq. (33)). In other words, $\underline{T}_{II}(s)$ is the transfer matrix of the system $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{y}_{II} = \underline{C}_{II}\underline{x}$ where $\underline{C}_{II} = \hat{\underline{C}}_{II}\underline{Q}$ and $\underline{Q}$ is the Lyapunov transformation corresponding to Eq. (95). Since $\underline{P}(s)$ is diagonal it will be sufficient to construct a decoupling pair $\{\underline{F},\underline{G}\}$ for the system

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}\ ,\ \underline{y}_{II} = \underline{C}_{II}\underline{x} \tag{104}$$

such that the closed loop poles are arbitrarily placed. However, letting $d_i = r_i + f_i$ and applying the synthesis procedure of reference 34 (p. 655), we find that Eq. (104) can be decoupled and all its closed loop poles assigned.

To be more explicit, if $q_i(s) = s^{d_i+1} - \sum_{j=0}^{d_i} m_j^i s^j$,* then the decoupling pair is given by

$$\underline{F} = \underline{B}^{*-1}[\sum_0^d \underline{M}_k\underline{C}_{II}\underline{A}^k - \underline{A}^*]\ ,\ \underline{G} = \underline{B}^{*-1} \tag{105}$$

where $d = \max d_i$, the $\underline{M}_k$ are diagonal matrices with entries m_k^i

*Clearly, it is enough to consider the case of a monic $q_i(s)$.

(i.e. $\underline{M}_k = \mathrm{diag}[m_k^1,\ldots,m_k^m]$) and $\underline{A}^* = (\underline{C}_{II,i}\underline{A}^{d_i+1})$ (i.e. the i-th row of $\underline{A}^*$ is given by $\underline{C}_{II,i}\underline{A}^{d_i+1}$). This completes the proof.

We are now in a position to determine whether or not the system presented at the beginning of this section can be decoupled via linear state variable feedback. If the answer is yes, several related questions can also be resolved. We first note that $\underline{B}^*$, as defined by Eq. (96), for the example is $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ and, consequently, is singular. Hence the system cannot be decoupled by using linear state variable feedback alone. This provides some motivation for the next section.

5.2 Decoupling via Input Dynamics

In the previous section, we considered the question of decoupling, using linear state variable feedback. We may now raise some pertinent questions related to decoupling via state variable feedback. In particular, suppose $\underline{B}^*$ were singular. Would it still be possible to decouple the system by using some alternative technique? Also, suppose that certain of the poles which could not be altered by decoupling state variable feedback were unstable. Could the system still be decoupled and simultaneously stabilized by using an alternative decoupling technique? The purpose of this section is to provide answers to these two questions.

Thus far, we have defined decoupling in a somewhat restricted sense, namely decoupling via linear state variable feedback. Recently, it has been shown that it is possible to decouple a larger class of systems than those which can be decoupled via state variable feedback alone, through the use of "input dynamics" (ref. 37). Roughly speaking, we will now permit the addition of dynamics to the input of the system (replace $\underline{G}$ in the closed loop transfer matrix by $\underline{G}(s)$, a k-th order dynamical system). It will be shown that certain systems which cannot be decoupled via state variable feedback can be decoupled by appropriate selection of $\underline{G}(s)$. In particular, the example presented at the beginning of this section falls into this category. Furthermore, in certain cases, poles which are invariant under decoupling state variable feedback can be altered if an appropriate $\underline{G}(s)$ is employed. It should be stressed that decoupling via input dynamics involves state variable feedback as well as $\underline{G}(s)$ in the most general case. This point will be clarified in the remainder of this section.

We start by defining the class of systems we will consider and exactly what we mean by "input dynamics." In particular, we will consider systems whose dynamics can be expressed by Eq. (95). $\underline{B}$ and $\underline{C}$ are assumed to be of full rank m, where $m \leq n$.

We next define "input dynamics" as the k-dimensional linear system whose dynamical equations can be expressed in the time domain as

$$\begin{aligned} \dot{\underline{g}} &= \underline{K}\underline{g} + \underline{L}\underline{v} \\ \underline{u}_e &= \underline{M}\underline{g} + \underline{N}\underline{v} \end{aligned} \tag{106}$$

where $\underline{g}$ is a k vector called the state, $\underline{v}$ an m vector called the input, and $\underline{u}_e$ an m vector called the output. $\underline{K}$, $\underline{L}$, $\underline{M}$, and $\underline{N}$ are constant matrices of the appropriate dimensions. The transfer matrix relating the Laplace transform of the output, $\underline{u}_e(s)$, to the Laplace transform of the input, $\underline{v}(s)$, of Eq. (106) will be defined as $\underline{G}(s)$, the input dynamics; i.e.

$$\underline{G}(s) = \underline{M}(s\underline{I} - \underline{K})^{-1}\,\underline{L} + \underline{N} \tag{107}$$

We can now define a composite system by combining systems (Eqs. (95) and (107)) via the relationship

$$\underline{u} = \underline{F}\underline{x} + \underline{u}_e \tag{108}$$

The composite system of total order (n+k) can now be expressed in time domain as

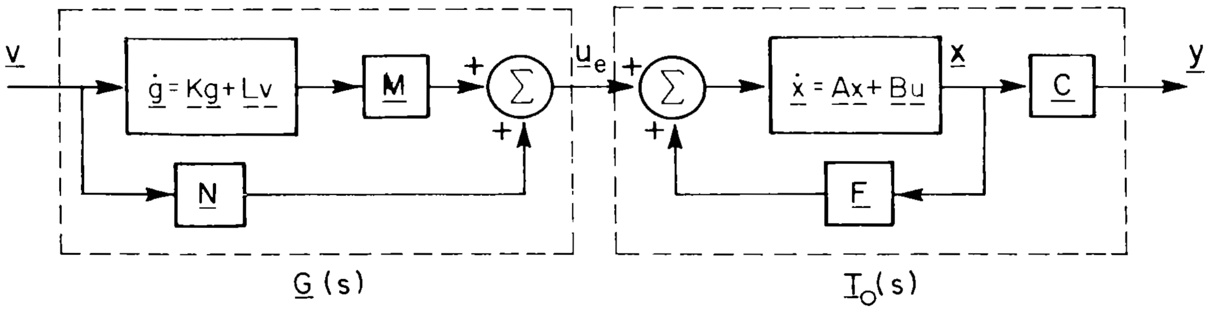
$$\begin{aligned}
\dot{\underline{x}} &= \underline{A}\underline{x} + \underline{B}\underline{u} \\
\underline{y} &= \underline{C}\underline{x} \\
\underline{u} &= \underline{F}\underline{x} + \underline{u}_e \\
\dot{\underline{g}} &= \underline{K}\underline{g} + \underline{L}\underline{v} \\
\underline{u}_e &= \underline{M}\underline{z} + \underline{N}\underline{v}
\end{aligned} \tag{109}$$

If we define $\dot{\underline{x}}_c = \begin{bmatrix} \underline{x} \\ \underline{g} \end{bmatrix}$ as the composite state, Eq. (109) may be written more succinctly as

$$\dot{\underline{x}}_c = \begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{g}} \end{bmatrix} = \begin{bmatrix} \underline{A}+\underline{B}\underline{F}, & \underline{B}\underline{M} \\ \underline{0} \quad , & \underline{K} \end{bmatrix} \underline{x}_c + \begin{bmatrix} \underline{B}\underline{N} \\ \underline{L} \end{bmatrix} \underline{v}$$

$$\underline{y}_c = \underline{y} = [\underline{C}, \underline{0}]\underline{x}_c \tag{110}$$

Pictorially, we have the following block diagram representation (Figure 2) for the composite system.



$$\underline{u}_e(s) = \{\underline{M}(s\underline{I} - \underline{K})^{-1}\underline{L} + \underline{N}\}\underline{v}(s) \qquad \underline{y}(s) = \hat{\underline{C}}\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{u}_e$$

Figure 2.- Block diagram of composite system

Clearly, the composite transfer matrix $\underline{T}_c(s)$ relating the output, $\underline{y}(s)$, to the new input, $\underline{v}(s)$, can be written as

$$\underline{T}_c(s) = \hat{\underline{C}}\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{G}(s)\ \frac{\Delta_u(s)}{\Delta_u(s)} \tag{111}$$

where $\underline{G}(s)$ is given by Eq. (107).

We can now define the concept of "decoupling via input dynamics" as follows. Consider the composite system, Eq. (110). We will say that the system (Eq.(95)) can be decoupled via input dynamics if there exists a pair of matrices $\{\underline{F},\underline{G}(s)\}$, where $\underline{G}(s)$ is given by Eq. (107), such that the composite system, Eq. (110), can be decoupled via linear state variable feedback.

Some observations will now be made. First, it should be noted that if system, Eq. (95), can be decoupled by using linear state variable feedback (if $\underline{B}^*$ is nonsingular), then input dynamics are not required to decouple the system. However, we recall that certain eigenvalues of $(\underline{A} + \underline{B}\underline{F})$ are invariant under decoupling state variable feedback, namely the zeros of $\Delta_u(s)$ and $\det(\underline{C}^*_{II}(s))$. These eigenvalues (poles) appear in "cancelled" pole-zero pairs in the final decoupled transfer matrix. If one or more of these poles lie in the half-plane Re $s \geq 0$, the resulting decoupled system will be unstable. Therefore, although $\underline{B}^*$ is nonsingular, decoupling via linear state variable feedback may produce an unstable system. One might therefore ask if it is possible to employ input dynamics in these situations in order to decouple the system and also ensure stability. The answer to this question is yes, provided $\Delta_u(s)$ is Hurwitz (i.e., the unstable roots are zeros of $\det(\underline{C}^*_{II}(s))$ only). Clearly, if $\Delta_u(s)$ is non-Hurwitz, any linear feedback system will fail to achieve stability. These points will be covered in more detail in the remainder of this section.

A pictorial representation of the definition of decoupling via input dynamics is now in order. In particular, according to the definition, linear state variable feedback may be employed twice in order to achieve decoupling via input dynamics. First, we seek a pair $\{\underline{F},\underline{G}(s)\}$, which defines the open loop composite system (Eq. (110)), a system which can then be decoupled via linear state variable feedback. Once the pair $\{\underline{F},\underline{G}(s)\}$ is chosen, an additional pair $\{\underline{F}_c,\underline{G}_c\}$ can then be used to actually decouple the system (Eq. (110)). The employment of the second pair, $\{\underline{F}_c,\underline{G}_c\}$, is not always required, as we will show. Figure 3 is a pictorial representation for decoupling via input dynamics by modifying Figure 2 to include the additional pair $\{\underline{F}_c,\underline{G}_c\}$. The subscript c denotes composite feedback and feedforward for the (n+k) dimensional system (Eq. (110)).

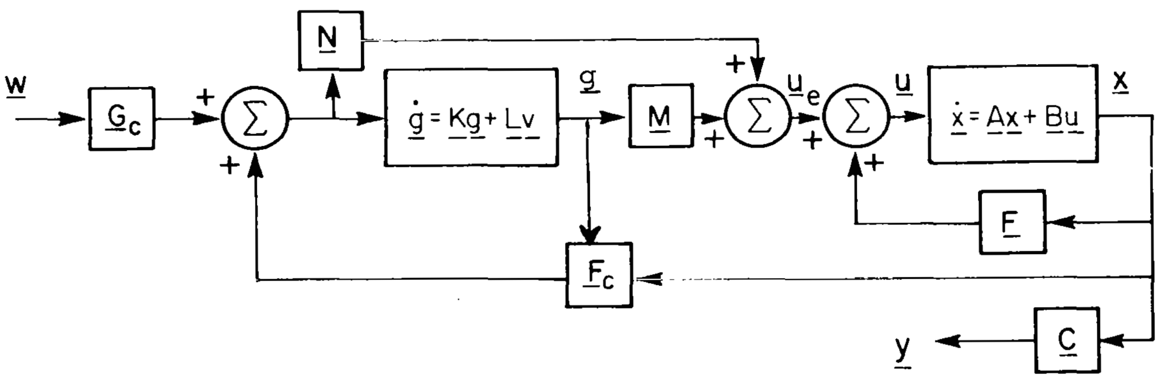


Figure 3.- Decoupled composite system

Summarizing, we note that decoupling via input dynamics involves two steps: (1) selection of a pair $\{\underline{F},\underline{G}(s)\}$ which increases the dimension of the state, resulting in a composite system of dimension (n+k), which can then be decoupled via composite linear state variable feedback, and (2) selection of the constant decoupling pair $\{\underline{F}_c,\underline{G}_c\}$ which then decouples the composite system. We also remark that once step (1) has been accomplished, the techniques outlined in Section 5.1 can be employed to accomplish step (2).

We will now characterize the class of systems (Eq. (95)) which can be decoupled via input dynamics. Clearly, this class must include those systems which can be decoupled by using linear state variable feedback. Fortunately, this class can be characterized readily by using the structure theorem. Furthermore, we can also say something about the stability of systems which can be decoupled via input dynamics. In particular, we can now state and prove the following theorem (T7):

T7: Let $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{y} = \underline{C}\underline{x}$ denote an m-input, m-output system whose transfer matrix is (in terms of the structure theorem) $\hat{\underline{C}}\underline{S}(s)\underline{\delta}^{-1}(s)\hat{\underline{B}}_m\Delta_u(s)\div\Delta_u(s)$. There exists a pair of matrices $\{\underline{F},\underline{G}(s)\}$ such that this system can be decoupled via input dynamics if and only if $\hat{\underline{C}}\underline{S}(s)$ is nonsingular; i.e. if and only if the system is invertible. Furthermore, if $\Delta_u(s)$ is Hurwitz, the pair $\{\underline{F},\underline{G}(s)\}$ can always be chosen to ensure the existence of a pair $\{\underline{F}_c,\underline{G}_c\}$ which decouples the composite system and simultaneously ensures that all (n+k) closed loop poles lie in the half-plane Re s<0.

Proof: In order to establish necessity of the first statement, we will assume that a suitable pair $\{\underline{F},\underline{G}(s)\}$ exists. The transfer matrix of the open loop composite system is then given by Eq. (111); i.e.

$$\underline{T}_c(s) = \hat{\underline{C}}\underline{S}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\hat{\underline{B}}_m\underline{G}(s)\ \frac{\Delta_u(s)}{\Delta_u(s)}$$

$\underline{T}_c(s)$ represents the transfer matrix of a system which can now be decoupled via linear state variable feedback. Consequently, $\underline{T}_c(s)$ is nonsingular and invertible. Clearly, this implies that $\hat{\underline{C}}\underline{S}(s)$ is also nonsingular and invertible. In order to establish the fact that the nonsingularity of $\hat{\underline{C}}\underline{S}(s)$ is sufficient to achieve an asymptotically stable decoupled system via input dynamics, a constructive algorithm will be employed. Since the algorithm is rather involved, no interpretation of the various steps taken will be given until the theorem has been established. At that time, we will demonstrate its application by example and comment on the more important steps which were employed. The steps will be numbered for convenience and later interpretation.

STEP (1): We begin by considering $\underline{T}_{\underline{C}}(s)$, the transfer matrix of the composite system. We then factor $\hat{\underline{C}}\underline{S}(s)$ as in Section 5.1; i.e.

$$\hat{\underline{C}}\underline{S}(s) = \underline{P}(s)\underline{C}^*_{II}(s) \tag{112}$$

where $\underline{P}(s)$ is a diagonal matrix consisting of the ordered greatest common divisors, $p_i(s)$, of each row of $\hat{\underline{C}}S(s)$. We now consider the altered transfer matrix, $\underline{T}_{\underline{C}_{II}}(s)$, obtained by factoring $\underline{P}(s)$ out of $\underline{T}_c(s)$ or, equivalently, premultiplying $\underline{T}_c(s)$ by $\underline{P}^{-1}(s)$; i.e.

$$\underline{T}_{\underline{C}_{II}}(s) = \underline{C}^*_{II}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\hat{\underline{B}}_m\underline{G}(s)\ \frac{\Delta_u(s)}{\Delta_u(s)} \tag{113}$$

STEP (2): We will now obtain a pair $\{\underline{F},\underline{G}(s)\}$ which diagonalizes Eq. (113). Solving Eq. (113) for $\hat{\underline{B}}_m\underline{G}(s)$, we obtain

$$\hat{\underline{B}}_m\underline{G}(s) = \underline{\delta}_{\underline{F}}(s)(\underline{C}^*_{II}(s))^{-1}\ \underline{T}_{\underline{C}_{II}}(s)\ \frac{\Delta_u(s)}{\Delta_u(s)} \tag{114}$$

Now, $(\underline{C}^*_{II}(s))^{-1} = \mathrm{adj}(\underline{C}^*_{II}(s) \div |\underline{C}^*_{II}(s)|$, where $\mathrm{adj}(\cdot)$ denotes the adjoint, and $|\cdot|$ the determinant of a square matrix. Note that this step is possible since $\hat{\underline{C}}\underline{S}(s)$ was assumed to be nonsingular.

STEP (3): Next, we factor $|\underline{C}^*_{II}(s)|$ as the product of two monic polynomials, $p_s(s)$ and $p_u(s)$, and some constant c, where $p_s(s)$ and $p_u(s)$ represent the stable and unstable (or conditionally stable) zeros of $|\underline{C}^*_{II}(s)|$ respectively; i.e.

$$|\underline{C}^*_{II}(s)| = cp_s(s)p_u(s) \tag{115}$$

Substituting this expression into Eq. (114), we obtain

$$\underline{G}(s) = \frac{\hat{\underline{B}}_m^{-1}\underline{\delta}_{\underline{F}}(s)\,\mathrm{adj}(\underline{C}^*_{II}(s))\,\underline{T}_{\underline{C}_{II}}(s)\,\Delta_u(s)}{cp_s(s)\,p_u(s)\,\Delta_u(s)} \tag{116}$$

STEP (4): We now set $\underline{\delta}_{\underline{F}}(s)$ equal to a diagonal matrix whose entries are $(s+\lambda)^{\sigma_i}$; i.e.

$$\underline{\delta}_{\underline{F}}(s) = \mathrm{diag}\,[(s+\lambda)^{\sigma_i}] \tag{117}$$

where λ is a positive constant equal to one of the zeros of $p_s(s)$, if possible. This choice for $\underline{\delta}_{\underline{F}}(s)$ ensures the asymptotic stability of the original portion of the composite system, provided $\Delta_u(s)$ is Hurwitz. Simultaneously, a certain amount of pole-zero cancellation will be possible in reducing Eq. (116) to simplest terms. Pole-zero cancellation is desirable if we expect to keep k, the dimension of $\underline{G}(s)$, as low as possible.

STEP (5): Since $\underline{G}(s)$ must also represent the transfer matrix of a stable system, $\underline{T}_{\underline{C}_{II}}(s)$ will now be set equal to a diagonal matrix which cancels the unstable polynomial, $p_u(s)$, now appearing in the denominator of $\underline{G}(s)$; i.e., we let

$$\underline{T}_{\underline{C}_{II}}(s) = \mathrm{diag}\left[\frac{cp_u(s)}{(s+\lambda)^{q_i}}\right] \tag{118}$$

The integers, q_i, $i = 1,2,\ldots,m$, have yet to be determined.

STEP (6): Equations (117) and (118) are now substituted into Eq. (116) in order to obtain $\underline{G}(s)$ in terms of the unspecified integers, q_i. $\underline{G}(s)$ can then be written as

$$\underline{G}(s) = \hat{\underline{B}}_m^{-1}\mathrm{diag}\left[\frac{(s+\lambda)^{\sigma_i}}{p_s(s)}\right]\mathrm{adj}(\underline{C}^*_{II}(s))\;\mathrm{diag}\left[\frac{cp_u(s)}{(s+\lambda)^{q_i}cp_u(s)}\right]\frac{\Delta_u(s)}{\Delta_u(s)} \tag{119}$$

STEP (7): The q_i's are chosen as small as possible, consistent with the requirement that $\underline{G}(s)$ represent the transfer matrix of a physically realizable system, namely Eq. (106). The actual input dynamics we use to decouple the system can now be chosen to be a minimal realization of the transfer matrix (Eq. (119)). The algorithm has now been carried far enough to establish the theorem. In particular, we note that after all pole-zero cancellations are made in Eq. (119), the only poles remaining are either equal to $-\lambda$, or are zeros of the Hurwitz polynomials $p_s(s)$. Since $|\underline{\delta}_F(s)|$ and $\Delta_u(s)$ are Hurwitz polynomials, the latter by assumption, the composite system (Eq. (110)) defined by Eq. (117) and a minimal realization (Section 3.1) of Eq. (119) is asymptotically stable. Furthermore, the transfer matrix, $\underline{T}_c(s)$, of the open loop composite system is given by

$$\underline{T}_c(s) = \underline{P}(s)\mathrm{diag}\left[\frac{cp_u(s)}{(s+\lambda)^{q_i}}\right] \tag{120}$$

a diagonal matrix (a decoupled system) with poles in the half-plane Re $s<0$. The theroem is thus established.

In general, we do not want a multiple root at $s = -\lambda$. Rather, we would like to arbitrarily specify as many closed loop poles of the decoupled composite system as possible. This is where the composite state variable feedback, $\{\underline{F}_c,\underline{G}_c\}$, enters into consideration. The open loop system, obtained by selecting $\underline{F}$ and $\underline{G}(s)$ according to the algorithm just outlined, is already decoupled. Therefore, composite state variable feedback, $\{\underline{F}_c,\underline{G}_c\}$ can now be used to adjust $(\sum_1^m q_i)$ of the closed loop poles of the composite system (see Section 5.1).

The example presented at the beginning of this section can now be used to illustrate this point and the various steps employed in outlining the algorithm. In particular, we noted at the conclusion of Section 5.1 that the example represented a system which could not be decoupled by using linear state variable feedback alone. However, for the example, we note that $|\hat{\underline{C}}\underline{S}(s)| = |\underline{C}\underline{S}(s)| = (s + 4)$; i.e., since $\underline{C}\underline{S}(s)$ is nonsingular, input dynamics can be employed to decouple the system. We will now go through the seven steps outlined in the algorithm in order to derive a feedback pair, $\{\underline{F},\underline{G}(s)\}$, which decouples the given system.

STEP (1): No further factorization of $\underline{C}\underline{S}(s)$ is possible; i.e. $\underline{P}(s) = \underline{I}$ and $\underline{C}^*_{II}(s) = \underline{C}\underline{S}(s)$.

STEP (2): Here, we calculate the inverse of $\underline{C}^*_{II}(s)$, or $\underline{C}\underline{S}(s)$ in the case of the example under consideration: i.e.

$$[\underline{C}\underline{S}(s)]^{-1} = \frac{\begin{bmatrix} 2 & , -1 \\ -s+2, & s+1 \end{bmatrix}}{s+4} = \frac{\mathrm{adj}(\underline{C}\underline{S}(s))}{\underline{C}\underline{S}(s)}$$

Furthermore, note that in the example $\Delta_u(s) = 1$ and $\hat{\underline{B}}_m = \underline{I}$. Hence, according to Eq. (114),

$$\underline{G}(s) = \underline{\delta}_{\underline{F}}(s)[\underline{C}\underline{S}(s)]^{-1}\underline{T}_{\underline{C}_{II}}(s)$$

STEP (3): For example, $p_s(s) = (s + 4)$ and $p_u(s) = c = 1$. This step is necessary so that any unstable roots of $|\underline{C}^*_{II}(s)|$ can be isolated and later cancelled by $\underline{T}_{\underline{C}_{II}}(s)$.

STEP (4): For the example, the choice of λ is obvious; i.e., since $p_s(s) = (s + 4)$, λ should be chosen equal to 4 and

$$\underline{\delta}_{\underline{F}}(s) = \begin{bmatrix} (s+4)^2, & 0 \\ 0 & , (s+4)^2 \end{bmatrix}$$

This particular choice for $\underline{\delta}_{\underline{F}}(s)$ accomplishes two things. First, it partially assures us of a stable composite system. It also results in eventual pole-zero cancellations in the computation of $\underline{G}(s)$, the added dynamics. This maintains k, the dimension of

$\underset{\sim}{G}(s)$, at a relatively low value.

STEP (5): We now employ Eq. (118); i.e.

$$\underset{\sim}{T}_{\underset{\sim}{C}_{II}}(s) = \begin{bmatrix} \frac{1}{(s+4)^{q_1}} & 0 \\ 0 & \frac{1}{(s+4)^{q_2}} \end{bmatrix}$$

If $p_u(s) \neq 1$, this step would have resulted in pole-zero cancellation of unstable roots which otherwise would have appeared in the denominator polynomial of $\underset{\sim}{G}(s)$.

STEP (6): We now employ Eq. (119) to obtain a frequency-domain description of $\underset{\sim}{G}(s)$; i.e.

$$\underset{\sim}{G}(s) = \begin{bmatrix} s+4 & 0 \\ 0 & s+4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -s+2 & s+1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+4)^{q_1}} & 0 \\ 0 & \frac{1}{(s+4)^{q_2}} \end{bmatrix}$$

or

$$\underset{\sim}{G}(s) = \begin{bmatrix} \frac{2}{(s+4)^{q_1-1}} & \frac{-1}{(s+4)^{q_2-1}} \\ \frac{-s+2}{(s+4)^{q_1-1}} & \frac{s+1}{(s+4)^{q_2-1}} \end{bmatrix}$$

STEP (7): We now select the q_i's as small as possible, consistent with the requirement that $\underset{\sim}{G}(s)$ represent a physically realizable system. By inspection, this implies that $q_1 = q_2 = 2$ or that

$$\underset{\sim}{G}(s) = \begin{bmatrix} \frac{2}{s+4} & \frac{-1}{s+4} \\ \frac{-s+2}{s+4} & \frac{s+1}{s+4} \end{bmatrix} = \frac{\begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix}}{s+4} + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

Note that with $\underline{G}(s)$ equal to the above asymptotically stable system and

$$\underline{\delta}_{\underline{F}}(s) = \begin{bmatrix} (s+4)^2 & 0 \\ 0 & (s+4)^2 \end{bmatrix} \text{; i.e.} \quad \underline{C}\underline{S}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\underline{B}_m = \begin{bmatrix} s+1 & 1 \\ s-2 & 2 \end{bmatrix} \div (s+4)^2$$

another asymptotically stable system, the open loop transfer matrix of the composite system, $\underline{C}\underline{S}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\underline{B}_m\underline{G}(s)$, is

$$\underline{T}_{\underline{C}_{II}}(s) = \underline{T}_{\underline{C}}(s) = \begin{bmatrix} \frac{1}{(s+4)^2} & 0 \\ 0 & \frac{1}{(s+4)^2} \end{bmatrix}$$

a result which can be obtained either by performing the multiplication indicated above or simply employing Eq. (118).

We will now continue our analysis of this example by computing k, the dimension of a minimal realization of $\underline{G}(s)$ and then describe how composite state variable feedback can be used to arbitrarily reassign certain of the closed loop poles of this decoupled system.

Employing the results of Section 3.1, we can readily obtain a minimal realization of $\underline{G}(s)$; i.e.

$$\underline{G}(s) = \underline{M}(s\underline{I} - \underline{K})^{-1}\underline{L} + \underline{N}$$

where

$$\underline{M} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \underline{K} = -4$$

a scalar

$$\underline{L} = [2 \quad -1]$$

and

$$\underline{N} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

as noted previously. $\underline{G}(s)$ thus represents the transfer matrix of a first order system; i.e. $k = 1$. A time domain representation for the composite decoupled fifth order system can now be written directly:

$$\underline{A}_C = \begin{bmatrix} \underline{A}+\underline{B}\underline{F}, & \underline{B}\underline{M} \\ \underline{0}, & \underline{K} \end{bmatrix} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -16 & -8 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & -8 & 3 \\ \hline 0 & 0 & 0 & 0 & -4 \end{array}\right]$$

$$\underline{B}_C = \begin{bmatrix} \underline{B}\underline{N} \\ \underline{L} \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ \hline 2 & -1 \end{array}\right]$$

and

$$\underline{C}_C = [\underline{C} \quad \underline{0}] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 \end{array}\right]$$

Using the results of the previous section (5.1), we can easily verify that this composite fifth order system can be decoupled via linear state variable feedback; i.e.

$$\underline{B}_C^{*} = \underline{C}_C\underline{A}_C\underline{B}_C = \underline{I}$$

Consequently, four of the five closed loop poles of the decoupled system can be arbitrarily specified via the algorithm given in reference 34 and repeated in Section 5.1 (Eq. (105)). It can also be shown that the single pole which cannot be altered while decoupling is $s = -4$. However, this pole will not appear in the final closed loop transfer matrix because of pole-zero cancellation.

5.3 Two Noninteraction Algorithms

In this section, we will outline two algorithms which can be used to achieve a degree of noninteraction in linear multivariable systems. We have just presented a fairly comprehensive review of decoupling (complete off-diagonal noninteraction). In certain systems, however, we might not be willing to add dynamics in order to achieve decoupling or to stabilize a system which can be decoupled via linear state variable feedback. However, we might want to zero certain closed loop inout-output pairs, if possible, using linear state variable feedback. The structure theorem can be used to provide partial answers to questions involving the noninteraction of various input-output pairs. The reader will note that this section lacks the formalism of previous sections. We will merely present two design algorithms for noninteraction and demonstrate them via the example presented earlier in Section 5. Both of the algorithms employ the structure theorem and, consequently, depend on the ability to express the closed loop transfer matrix of a system as

$$\underline{T}_{\underline{F},\underline{G}}(s) = \hat{\underline{C}}\underline{S}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\hat{\underline{B}}_m\underline{G}\,\frac{\Delta_u(s)}{\Delta_u(s)} \tag{121}$$

We still assume that the system has m linearly independent inputs and m outputs.

The purpose of the first algorithm (single row altered) is to demonstrate a technique which can be employed to achieve almost diagonal noninteraction (decoupling). The exception is that a certain row of the closed loop transfer matrix will not contain just one nonzero entry but could, in general, contain all nonzero entries. To demonstrate this algorithm, we observe the effect of altering a single row of $\hat{\underline{C}}$; i.e., we let

$$\hat{\underline{C}}_q = \hat{\underline{C}} + \underline{R} \tag{122}$$

where $\underline{R}$ is an (mxn) constant matrix with (m-1) zero rows and only one nonzero row (the q-th row). We would like to choose $\underline{R}$ in such a way that $\hat{\underline{C}}_q\underline{S}(s)$ has certain desirable properties from the point of view of the decoupling Section 5.1. In particular, we would like to choose $\underline{R}$ such that (1) the polynomial $|\hat{\underline{C}}_q\underline{S}(s)|$ is Hurwitz, and (2) the matrix $\underline{B}_q^*$ (defined in the same way as $\underline{B}^*$ (Section 5.1), but with $\underline{C}_q$ replacing $\underline{C}$) is nonsingular. If this is possible, the system $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{Y}_q = \underline{C}_q\underline{x}$, where $\underline{C}_q = \underline{Q}\hat{\underline{C}}_q$, can be decoupled via linear state variable feedback; i.e., by appropriate choice of the pair $\{\underline{F},\underline{G}\}$, $\hat{\underline{C}}_q\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{G}$ will be diagonal and nonsingular, and all zeros of $\lceil\underline{\delta}_F(s)\rceil$ will lie in the half-plane Re s<0.

Now, by Eq. (122), the closed loop transfer matrix of the actual system we are dealing with can be expressed as

$$\hat{\underline{C}}\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{G} = \hat{\underline{C}}_q\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{G} - \underline{R}\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{G} \tag{123}$$

where $\underline{R}\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{G}$ is an (mxm) matrix of transfer functions with zero entries everywhere but the q-th row. Consequently, the closed loop transfer matrix of the given system displays complete off-diagonal noninteraction with the exception of the q-th row. Our design objective has thus been met provided $\underline{R}$ can be suitably chosen.

We will now demonstrate this algorithm, using the example given at the beginning of Section 5. In pariticular, in the example,

$$\underline{C} = \hat{\underline{C}} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 1 & 2 & 0 \end{bmatrix}$$

If we now alter the first row ($q = 1$) of $\underline{C}$ by setting

$$\underline{R} = \begin{bmatrix} -1 & 5/4 & -1/2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then

$$\hat{\underline{C}}_q = \underline{C}_q = \begin{bmatrix} 0 & 9/4 & 1/2 & -1 \\ -2 & 1 & 2 & 0 \end{bmatrix}$$

In other words,

$$\underline{C}_q\underline{S}(s) = \begin{bmatrix} 9/4s & -s + 1/2 \\ s-2 & 2 \end{bmatrix}$$

and

$$|\underline{C}_q\underline{S}(s)| = (s+1)^2$$

a Hurwitz polynomial. Furthermore, $\underline{B}^*_q = \underline{C}_q\underline{B}$ and, in this case, is equal to

$$\begin{bmatrix} 9/4 & -1 \\ 1 & 0 \end{bmatrix}$$

a nonsingular matrix. The altered system can therefore be decoupled via linear state variable feedback. Two of the four closed loop poles of the decoupled system can be arbitrarily selected (the other two are $s = -1$ since $|\underline{C}_q\underline{S}(s)| = (s+1)^2$). If we select both of these poles at $s = -2$, Eq. (105) can then be used to find the required feedback pair $\{\underline{F},\underline{G}\}$; for this example, $\underline{F} = \underline{B}^{*-1}_q[\underline{M}_o\underline{C}_q - A^*]$, where

$$\underline{B}_q^{*-1} = \begin{bmatrix} 0 & 1 \\ -1 & 9/4 \end{bmatrix}, \quad \underline{M}_o = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad \underline{A}^* = \begin{bmatrix} -31/4 & -1 & 1 & 5/2 \\ -3 & -2 & 0 & 2 \end{bmatrix}$$

Solving for $\underline{F}$, we obtain

$$\underline{F} = \begin{bmatrix} 7 & 0 & -4 & -2 \\ 8 & 7/8 & -7 & -4 \end{bmatrix}$$

Therefore,

$$\underline{\delta}_{\underline{F}}(s) = \underline{\delta}(s) - \underline{B}_m\underline{F}\underline{S}(s) = \begin{bmatrix} s^2-4 & 2s+4 \\ -9/2\ s-9 & s^2+6s+8 \end{bmatrix}$$

and $|\underline{\delta}_{\underline{F}}(s)| = (s+1)^2(s+2)^2$ as required. Since $\underline{G} = \underline{B}_q^{*-1}$, we can write the closed loop transfer matrix of the altered system;

$$\underline{C}_q\underline{S}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\underline{B}_m\underline{B}_q^{*-1} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \frac{(s+1)^2}{(s+1^2}$$

We now use Eq. (123) to determine the closed loop transfer matrix of the actual system we are dealing with;

$$\underline{R}\underline{S}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\hat{\underline{B}}_m\underline{G} = \frac{\begin{bmatrix} s^2+s-3, & -s^2 - 11/4\ s + 1/2 \\ 0 & , \ 0 \end{bmatrix}}{(s+1)^2(s+2)} \frac{s+2}{s+2}$$

and

$$\underline{C}\underline{S}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\underline{B}_m\underline{B}_q^{*-1} = \frac{\begin{bmatrix} s+4, & s^2 + 11/4\ s - 1/2 \\ 0 & , \ (s+1)^2 \end{bmatrix}}{(s+1)^2(s+2)} \frac{s+2}{s+2}$$

We have thus succeeded in obtaining noninteraction in all off-diagonal elements of the closed loop transfer matrix with the exception of those in the first row.

ALGORITHM 2 (Row Combination)

The second algorithm is a modification of the first. It also relies on an alteration of $\hat{\underline{C}}$ and utilization of the results in Section 5.1. To demonstrate this algorithm, we observe the effect of premultiplying $\hat{\underline{C}}$ by a nonsingular matrix $\underline{P}$; i.e., linearly combining various rows of $\hat{\underline{C}}$. In particular, we let

$$\hat{\underline{C}}_p = \underline{P}\hat{\underline{C}} \tag{124}$$

where $\underline{P}$ is a nonsingular (mxm) matrix. We would like to choose $\underline{P}$ in such a way that the system $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{y}_p = \underline{C}_p s$, where $\underline{C}_p = \underline{Q}\hat{\underline{C}}_p$, can be decoupled via linear state variable feedback. In other words, we would like the matrix $\underline{B}_p^*$ (defined as in Section 5.1, but with $\underline{C}_p$ replacing $\underline{C}$) to be nonsingular. Suppose we can decouple the altered system via linear state variable feedback. This means that the closed loop transfer matrix of the altered system, $\underline{P}\hat{\underline{C}}\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{G}$, will be diagonal and nonsingular. We obtain the closed loop transfer matrix of the actual system under consideration by premultiplying $\underline{P}\hat{\underline{C}}\underline{S}(s)\underline{\delta}_F^{-1}(s)\hat{\underline{B}}_m\underline{G}$ by $\underline{P}^{-1}$. Note that the nonzero entries of $\underline{P}^{-1}$ will produce corresponding nonzero entries in the closed loop transfer matrix of the actual system, since $\underline{P}^{-1}$ premultiplies the diagonal transfer matrix. Hence the choice of $\underline{P}$, and therefore $\underline{P}^{-1}$, directly determines whether the various entries of $\underline{T}_{\underline{F},\underline{G}}(s)$ will be zero or nonzero.

We will now demonstrate this algorithm, using the example given at the beginning of Section 5. We recall again that $\underline{B}^*$ is singular and that

$$\hat{\underline{C}}\underline{S}(s) = \underline{C}\underline{S}(s) = \begin{bmatrix} s+1 & 1 \\ s-2 & 2 \end{bmatrix}$$

Suppose we choose $\underline{P} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ so that $\underline{P}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. By Eq. (124), $\hat{\underline{C}}_p = \underline{P}\hat{\underline{C}} = \begin{bmatrix} 3 & 0 & -1 & 0 \\ -2 & 1 & 2 & 0 \end{bmatrix}$. Consequently, $\underline{P}\hat{\underline{C}}\underline{S}(s) = \hat{\underline{C}}_p\underline{S}(s) = \begin{bmatrix} 3 & -1 \\ s-2 & 2 \end{bmatrix}$. Note that $\underline{P}$ cannot alter the zeros associated with $|\hat{\underline{C}}\underline{S}(s)|$ i.e. $|\hat{\underline{C}}_p\underline{S}(s)| = |\underline{P}||\hat{\underline{C}}\underline{S}(s)|$, where $|\underline{P}|$ is, of course, a scalar. $\underline{B}^*_p$ is thus equal to

$$\begin{bmatrix} \underline{C}_{p_1} & \underline{A}\underline{B} \\ \underline{C}_{p_2} & \underline{B} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

a nonsingular matrix. The altered system can therefore be decoupled via linear state variable feedback, and three of the four closed loop poles of the decoupled system can be arbitrarily specified (Section 5.1). We will choose these poles at s = -1, -2, and -3 in order to demonstrate an alternate method of determining $\underline{F}$, the feedback matrix. If

$$\underline{G} = \underline{B}^{*-1}_p = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

we can write the final closed loop transfer matrix of the altered system directly;

$$\underline{C}_p\underline{S}(s)\underline{\delta}_F^{-1}(s)\underline{B}_m\underline{B}^{*-1}_p = \begin{bmatrix} \frac{1}{s^2+3s+2} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix}\frac{s+4}{s+4}$$

Since all quantities are known with the exception of $\underline{\delta}_F(s)$, we solve the above equation for $\underline{\delta}_F(s)$;

$$\underline{\delta}_F(s) = \underline{B}_m\underline{B}_p^{*-1}\begin{bmatrix} s^2+3s+2, & 0 \\ 0, & s+3 \end{bmatrix}\underline{C}_p\underline{S}(s)$$

or

$$\underline{\delta}_F(s) = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} s^2+3s+2, & 0 \\ 0, & s+3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ s-2 & 2 \end{bmatrix}$$

which yields

$$\underline{\delta}_F(s) = \begin{bmatrix} s^2+s-6 & 2s+6 \\ -6s-24 & s^2+9s+20 \end{bmatrix}$$

Since $\underline{\delta}_F(s) = \underline{\delta}(s) - \hat{\underline{B}}_m\hat{\underline{F}}\underline{S}(s) = \underline{\delta}(s) - \underline{B}_m\underline{F}\underline{S}(s)$, in this example, and $\underline{B}_m = \underline{I}$, $\underline{F}\underline{S}(s) = \underline{\delta}(s) - \underline{\delta}_F(s)$, or

$$\underline{F}\underline{S}(s) = \begin{bmatrix} -s+9 & -2s-6 \\ 5s+23, & -7s-19 \end{bmatrix}$$

or

$$\underline{F} = \begin{bmatrix} 9 & -1 & -6 & -2 \\ 23 & 5 & -19 & -7 \end{bmatrix}$$

The closed loop transfer matrix of the altered system can be premultiplied by $\underline{P}^{-1}$ to yield the closed loop transfer matrix of the given system;

$$\underline{C}\underline{S}(s)\underline{\delta}_F^{-1}(s)\underline{B}_m\underline{B}_p^{*-1} = \begin{bmatrix} \frac{1}{(s+1)(s+2} & \frac{1}{s+3} \\ 0 & \frac{1}{s+3} \end{bmatrix}\frac{s+4}{s+4}$$

We have therefore succeeded in obtaining a transfer matrix whose nonzero entries correspond to those of $\underline{P}^{-1}$.

The two algorithms we have now presented do not, of course, represent the totality of structure theorem algorithms which can be employed to achieve closed loop noninteraction. The reader will undoubtedly note that certain basic modifications can be made. For example, two rows of $\hat{\underline{C}}$ can be altered instead of one and $\underline{P}$ can be replaced by $\underline{P}(s)$ in certain cases. The two algorithms can also be combined, and the combination modified in a number of ways. Space limitations, however, prevent us from covering these points in more detail.

6. A HELICOPTER STABILITY AUGMENTATION SYSTEM

In this section of the report, we will apply several of the results presented earlier in order to design a stability augmentation system, for a hovering helicopter, based on desired handling qualities. In particular, we consider the linearized equations of motion (ninth order) of a hovering helicopter, and proceed to design a linear, state variable feedback control system which satisfies certain conditions. These conditions are expressed in terms of the transfer matrix of the system; i.e., in terms of the desired closed loop transfer functions which comprise the (6x4) transfer matrix of the system. The steps taken to achieve the final feedback design are enumerated here in order to facilitate the review of the design method employed.

1. A brief discussion of the vehicle which will be considered is given and our overall design objective is compared with an alternative design.

2. A corollary to the structure theorem is presented, which enables us to deal directly with the physical state of the system throughout the entire design process. In this way, our engineering insight can be employed at various stages in the design to motivate the various mathematical steps taken.

3. The linearized equations of motion of the vehicle are presented, and the physical state of the system is defined.

4. A qualitative discussion of the properties which we desire in the final closed loop system is presented

and a quantitative interpretation of these properties is given in terms of a desired closed loop transfer matrix.

5. The practical considerations which constrain our design are presented--first qualitatively and then quantitatively.

6. A design of an initial feedback control system, based on the corollary to the structure theorem, is presented. Use is made of certain characteristics of the particular vehicle considered.

7. The deficiencies of this initial design are discussed and corrected in order to satisfy the constraints imposed.

8. A final design is then obtained. This design is discussed on its own merits, and then compared with the feedback system we were hoping to achieve.

The helicopter we consider is the Sikorsky SH-3D Sea King, whose primary mission is to detect submerged submarines through the use of a sonar ball, which is lowered into the water while the helicopter is hovering at an altitude of approximately 40 ft. The linearized equations of motion of this vehicle, as well as an alternate feedback stabilization system, are given in reference 38. We point out that the stabilization system obtained in reference 38 is based on linear optimal control, the objective there being to maintain the vehicle about a fixed point in the presence of disturbances. Our design objective, however, is based on desired handling qualities; i.e., the ability to design a feedback control system which ensures "acceptable manual control" of the helicopter about the equilibrium point (the hovering position). The primary design tool we employ is the structure theorem. As we will show, however, certain practical constraints, such as limited feedback gains and the inability to combine mechanical inputs, prevent the direct application of the structure theorem. Furthermore, when dealing with practical systems, it is desirable to avoid any transformation of coordinates (linear combinations of state variables). It is usually preferable to work with the actual state variables which define the system. For this reason, we will introduce a corollary to the structure theorem. The corollary is most useful when a significant number of state variables are derivatives of other state variables, a condition which occurs quite frequently in practice. In particular, we have the following corollary (C10):

C10: Consider the system $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{y} = \underline{C}\underline{x}$. Suppose $\underline{A}$, an (nxn) matrix, is in multi-input companion form as defined by Eqs. (13), (14), and (15); i.e., $\underline{A}$ can be partitioned into m diagonal blocks, each a companion matrix of dimension σ_i, where i = 1, 2, ... m. Furthermore, suppose that $\underline{B}$, an (nxq) matrix, has zero entries everywhere but the m d_k rows, where $d_K = \sum_{1}^{K}\sigma_i$, K = 1, 2, ... m. If $\underline{S}(s)$ is now defined by Eq. (21);

$$\underline{S}(s) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ s & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ s^{\sigma_1 - 1} & 0 & & 0 \\ 0 & 1 & \cdots & \cdot \\ \cdot & s & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & s^{\sigma_2 - 1} & \cdots & 0 \\ \cdot & 0 & & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & & s^{\sigma_m - 1} \end{bmatrix}$$

and $\underline{\delta}(s) = \text{diag}[s^{\sigma_i}] - \underline{A}_m\underline{S}(s)$, where $\underline{A}_m$ is the (mxn) matrix consisting of the m-ordered d_K-th rows of $\underline{A}$, and $\underline{B}_m$ as the (mxq) matrix consisting of the m-ordered d_K-th rows of $\underline{B}$, then

$$\underline{T}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} = \underline{C}\underline{S}(s)\underline{\delta}^{-1}(s)\underline{B}_m$$

Furthermore, if $\underline{u} = \underline{F}\underline{x} + \underline{G}\underline{w}$, then

$$\underline{T}_{\underline{F},\underline{G}}(s) = \underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B}\underline{G} = \underline{C}\underline{S}(s)\underline{\delta}_{\underline{F}}^{-1}(s)\underline{B}_m\underline{G}$$

where

$$\underline{\delta}_{\underline{F}}(s) = \underline{\delta}(s) - \underline{B}_m\underline{F}\underline{S}(s)$$

Proof: The proof is a direct consequence of theorem T1 and corollary C2.

We now present the linearized equations of motion of the SH-3D helicopter. The state variables defining the system have been reordered and numbered differently than in reference 38 in order to allow direct application of the above corollary. In particular, we consider the system $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}$, $\underline{y} = \underline{C}\underline{x}$, where

$$\underline{A} = \begin{bmatrix} -.016 & -.05 & .0025 & 0 & 0 & -.0001 & 0 & 0 & -.0047 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.97 & 0 & -.542 & 1 & 0 & .548 & 0 & 0 & .736 \\ 0 & 0 & .00018 & -.3242 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2.61 & 0 & -1.94 & -.163 & 0 & -1.96 & 0 & .01 & -7.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ .016 & 0 & -.0083 & -.193 & 0 & -.0043 & 0 & -.303 & 5.59 \\ .0047 & 0 & -.0024 & -.0007 & .05 & -.0025 & 0 & .0009 & -.033 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} .05 & .005 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6.15 & .69 & 0 & 0 \\ 0 & -.424 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2.13 & 21.81 & .3475 \\ 0 & 0 & 0 & 0 \\ 0 & 5.12 & .174 & -7.48 \\ 0 & .01 & .05 & .022 \end{bmatrix}$$

and

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The normalized state $\underline{x}$ and control $\underline{u}$ are defined as follows:

x_1 - longitudinal velocity

x_2 - pitch

x_3 - pitch rate

x_4 - vertical velocity

x_5 - roll

x_6 - roll rate

x_7 - yaw

x_8 - yaw rate

x_9 - lateral velocity

u_1 - longitudinal cyclic pitch

u_2 - main rotor collective

u_3 - lateral cyclic pitch

u_4 - tail rotor collective

Given this open loop system, our objective is to design a linear feedback control system based on certain desired handling qualities and constrained by certain practical considerations. We first discuss the desired handling qualities.

In particular, we would like to achieve a final "decoupled" feedback design where longitudinal velocity and pitch (x_1, x_2, and x_3) are affected by longitudinal cyclic pitch (u_1) alone. Simultaneously, it would be desirable if vertical velocity (x_4) were affected by main rotor collective input (u_2) alone, roll and lateral velocity (x_5, x_6, and x_9) were affected by lateral cyclic pitch input (u_3) alone, and yaw (x_7 and x_8) were affected by tail rotor collective (u_4) alone. In addition to these requirements, we would also like to select the poles associated with various input/output transfer functions as follows. Longitudinal velocity (y_1), lateral velocity (y_6), and yaw (y_5) should be affected by u_1, u_3, and u_4 via pure integration. Furthermore, the transfer functions relating pitch (y_2) and roll (y_4) to longitudinal cyclic pitch (u_1) and lateral cyclic pitch (u_3) should represent second order systems with critical damping (ref. 32) (ξ = .707) and an undamped natural frequency (ref. 32) of 3 rad/sec (ω_n = 3). All other specified poles should lie in the half-plane Re s<0. These conditions represent the handling qualities which we hope to achieve, using a linear feedback control system. In terms of the closed loop transfer matrix, $\underline{C}(s\underline{I} - \underline{A} - \underline{BF})^{-1}\underline{BG}$, relating the six outputs to the four inputs, these conditions would be satisfied if

$$\underline{C}(s\underline{I} - \underline{A} - \underline{BF})^{-1}\underline{BG} = \begin{bmatrix} \frac{n_{11}(s)}{s(s^2+4.24s+9)} & 0 & 0 & 0 \\ \frac{n_{21}(s)}{s^2+4.24s+9} & 0 & 0 & 0 \\ 0 & t_{32}(s) & 0 & 0 \\ 0 & 0 & \frac{n_{43}(s)}{s^2+4.24s+9} & 0 \\ 0 & 0 & 0 & \frac{n_{54}}{s} \\ 0 & 0 & \frac{n_{63}(s)}{s(s^2+4.24s+9)} & 0 \end{bmatrix}$$

where the poles associated with $t_{32}(s)$ lie in the half-plane Re $s<0$ and the numerator polynomials $n_{11}(s)$, $n_{21}(s)$, $n_{43}(s)$, and $n_{63}(s)$ contain no roots at $s = 0$.

As stated earlier, we also have some constraints associated with the problem. Actually, there are two constraints, both of which are given in reference 38 and repeated here for convenience. As stated in reference 38, the amplitude of the controller output (feedback gains) is limited to ±10% of the total available range of the mechanical input (external input $\underline{w}$). Furthermore, physical coupling of the mechanical inputs is not permitted ($\underline{G}$ must equal $\underline{I}$). The first of these two constraints will ensure that the pilot can recover successfully from a hardover failure in the augmentation system. The second, although not explicitly stated as a constraint in reference 38, ensures a consistent open loop mechanical system throughout the entire range of operation of the helicopter. The requirement for additional hardware design is also eliminated. Mathematically stated, these two constraints imply that $\underline{u}$ must be of the form, $\underline{u} = \underline{Fx} + \underline{w}$, and all elements comprising the feedback matrix $\underline{F}$ must be less than or equal to 2 in absolute values. This latter mathematical constraint is the author's own interpretation of the ±10% controller output constraint and is based on the final feedback design given in reference 38.

The problem has now been completely formulated, and our task is to design a feedback control system which matches the desired closed loop transfer matrix as closely as possible. We note here that an exact match in impossible, as we will later show.

We now recall corollary C10; i.e. $\underline{T}_F(s) = \underline{C}\underline{S}(s)\underline{\delta}_F^{-1}(s)\underline{B}_m$, where $\underline{\delta}_F(s) = \underline{\delta}(s) - \underline{B}_m\underline{F}\underline{S}(s)$. For this example, $\underline{C}\underline{S}(s) = \underline{I}$ and

$$\underline{B}_m = \begin{bmatrix} .05 & .005 & 0 & 0 \\ -6.15 & .69 & 0 & 0 \\ 0 & -.424 & 0 & 0 \\ 0 & -2.13 & 21.81 & .3475 \\ 0 & 5.12 & .174 & -7.48 \\ 0 & .01 & .05 & .022 \end{bmatrix}$$

Note that the elements comprising the first and last rows of $\underline{B}_m$ are considerably smaller than those contained in the remaining rows. Also recall that $\underline{F}$, the feedback matrix, is constrained in magnitude. Consequently, we conclude that the effect of feedback will be considerably more significant on the middle four rows of $\underline{\delta}_F(s)$ than on the first and last rows. Our design will therefore be based on properly altering these middle four rows of $\underline{\delta}_F(s)$. The effect on the other two rows should be relatively insignificant.

Previously, we stated that in view of the constraints, the desired transfer matrix could not be achieved via the feedback $\underline{u} = \underline{F}\underline{x} + \underline{w}$. We will now be more specific regarding this fact. In particular, since $\underline{C}\underline{S}(s) = \underline{I}$, the closed loop transfer matrix,

$\underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B} = \underline{\delta}_F^{-1}(s)\underline{B}_m$. If we now recall the definition of $\underline{B}^*$ given in Section 5.1, i.e.

$$\underline{B}^* = \begin{bmatrix} \underline{C}_1 & \underline{A}^{f_1} & \underline{B} \\ \underline{C}_2 & \underline{A}^{f_2} & \underline{B} \\ & \vdots & \\ \underline{C}_m & \underline{A}^{f_m} & \underline{B} \end{bmatrix}$$

where $f_i = \min[\{j:\underline{C}_i\underline{A}^j\underline{B} \neq 0\}, n-1]$, we note that for this example, $\underline{B}^* = \underline{B}_m$. Furthermore, $\underline{B}^*$ is invariant under linear state variable feedback (refs. 34 and 35). Since $\underline{G}$ is constrained to equal $\underline{I}$, every nonzero entry of $\underline{B}^*$ or $\underline{B}_m$ represents a nonzero entry in the closed loop transfer matrix, $\underline{C}(s\underline{I} - \underline{A} - \underline{B}\underline{F})^{-1}\underline{B}$, of the system. However, we can achieve a certain number of zeros in the closed loop transfer matrix (partial decoupling) by judicious choice of the feedback matrix $\underline{F}$. In particular, if we select $\underline{F}$ in such a way that the middle four rows of $\underline{\delta}_F(s)$ are diagonal, then seven entries in $\underline{T}_F(s)$ will be zero--namely, those entries corresponding to the zero entries in the middle four rows of $\underline{B}_m$. In particular, suppose $\underline{F}$ were chosen so that

$$\underline{A} + \underline{B}\underline{F} = \begin{bmatrix} v & v & v & v & v & v & v & v & v \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -9 & -4.24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -9 & -4.24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 \\ v & v & v & v & v & v & v & v & v \end{bmatrix}$$

where the v's denote unimportant (for the moment) entries corresponding to the $\underline{F}$ selected. This $(\underline{A} + \underline{B}\underline{F})$ is possible to achieve since the middle four rows of $\underline{B}_m$ constitute a nonsingular matrix. The corresponding $\underline{\delta}_F(s)$ would then be

$$\underline{\delta}_F(s) = \begin{bmatrix} v & v & v & v & v & v \\ 0 & s^2+4.24s+9 & 0 & 0 & 0 & 0 \\ 0 & 0 & s+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^2+4.24s+9 & 0 & 0 \\ 0 & 0 & 0 & 0 & s(s+10) & 0 \\ v & v & v & v & v & v \end{bmatrix}$$

If we now compute the closed loop transfer matrix $\underline{T}_F(s)$: $\underline{\delta}_F^{-1}(s)\underline{B}_m$, it is clear that the (2,3), (2,4), (3,1), (3,3), (3,4), (4,1), and (5,1) entries of $\underline{T}_F(s)$ will be zero. There are two things wrong with this choice of $\underline{F}$, however. First, if we actually compute $\underline{F}$, the feedback matrix corresponding to this choice of $\underline{\delta}_F(s)$, we would discover that certain entries of $\underline{F}$ have an absolute value greater than 2, a violation of one of the two design constraints. Secondly, this choice of $\underline{\delta}_F(s)$ would produce a closed loop transfer matrix with s(s+10) as the denominator polynomial of the (5,2), (5,3), and (5,4) transfer functions; i.e., the transfer functions relating yaw to main rotor collective and lateral cyclic pitch would contain pure integrators with relatively high gain. This would be an undesirable condition, since the helicopter would yaw at constant rate if either of these two controls were displaced from their equilibrium positions. At this point in the design, the structure theorem proves most useful. We first reduce the entries of $\underline{F}$ by altering the initial requirement that the transfer functions relating y_2 and y_4 to u_1 and u_3, respectively, contain the denominator polynomial $s^2 + 4.24s + 9$. We still maintain critical damping ($\xi = .707$), but reduce the undamped natural frequency to 2.5 instead of 3 rad/sec. This still represents an acceptable frequency for the human operator to control. We now concentrate on that portion of the $\underline{\delta}_F(s)$ block which produces the undesirable integrator transfer functions relating yaw to main rotor collective and lateral cyclic pitch. In particular, suppose that

$$\underline{\delta}_F(s) = \begin{bmatrix} v & v & v & v & v & v \\ 0 & s^2+3.53s+6.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & s+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^2+3.53s+6.25 & 0 & 0 \\ 0 & 0 & c & as+b & s^2+10s & 0 \\ v & v & v & v & v & v \end{bmatrix}$$

where a, b, and c are, as yet, unspecified constants. The inverse of the dotted (3x3) submatrix of $\underline{\delta}_F(s)$ is now determined;

$$\boxed{\underline{\delta}_F(s)}^{-1}_{(3,3)-(5,5)} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s^2+3.53s+6.25} & 0 \\ \frac{-c}{s(s+1)(s+10)} & \frac{-as-b}{s(s+10)(s^2+3.53s+6.25)} & \frac{1}{s(s+10)} \end{bmatrix}$$

Postmultiply this expression by the appropriate portion of $\underline{B}_m$, namely

$$\boxed{\underline{B}_m}_{(3,2)-(5,4)} = \begin{bmatrix} -.424 & 0 & 0 \\ -2.13 & 21.81 & .3475 \\ 5.12 & .174 & -7.48 \end{bmatrix}$$

yields the appropriate block (submatrix) of the closed loop transfer matrix, $\underline{T}_F(s)$;

$$\boxed{\underline{T}_{\underline{F}}(s)}_{(3,2)-(5,4)} = \begin{bmatrix} \dfrac{-.424}{s+1} & 0 & 0 \\ \dfrac{-2.13}{s^2+3.53s+6.25} & \dfrac{21.81}{s^2+3.53s+6.25} & \dfrac{.3475}{s^2+3.53s+6.25} \\ \underline{T}_{\underline{F}_{5,2}}(s) & \underline{T}_{\underline{F}_{5,3}}(s) & \underline{T}_{\underline{F}_{5,4}}(s) \end{bmatrix}$$

where

$$\underline{T}_{\underline{F}_{5,2}}(s) = \frac{(.424c+5.12s+5.12)(s^2+3.53s+6.25) + 2.13(s+1)(as+b)}{s(s+1)(s+10)(s^2+3.535s+6.25)}$$

The constant term in the numerator of $\underline{T}_{\underline{F}_{5,2}}(s)$ is equal to 2.65c + 2.13b + 32. If this quantity is set equal to zero we will achieve a pole-zero cancellation at s = 0 and, hence, eliminate the pure integrator in this transfer function. Similarly,

$$\underline{T}_{\underline{F}_{5,3}}(s) = \frac{-21.81(as+b) + .174(s^2+3.53s+6.25)}{s(s+10)(s^2+3.53s+6.25)}$$

and the integrator can be cancelled by selecting -21.81b + 1.0875 = 0. Therefore, to avoid pure integration in the transfer matrix elements relating yaw to both main rotor collective and lateral cyclic pitch, we set b = .0499 and c =-12.115. Furthermore, if a is set equal to zero, $\underline{T}_{\underline{F}_{5,4}} = \frac{-7.48}{s(s+10)}$, a desirable transfer function (includes pure integration) relating yaw to tail rotor collective. In summary, thus far we have concentrated on the middle four rows of the closed loop transfer matrix $\underline{T}_{\underline{F}}(s) = \underline{\delta}_{\underline{F}}^{-1}(s)\underline{B}_m$. We have shown that if

$$\underline{A} + \underline{BF} = \begin{bmatrix} v & v & v & v & v & v & v & v & v \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6.25 & -3.535 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6.25 & -3.535 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 12.115 & -.0499 & 0 & 0 & -10 & 0 \\ v & v & v & v & v & v & v & v & v \end{bmatrix}$$

then the resulting closed loop transfer matrix will possess certain desirable features. Before discussing this point any further, we will calculate $\underline{F}$, the feedback matrix in order to ensure that all elements have an absolute value less than or equal to 2 and also to evaluate the effect of $\underline{F}$ on the first and last rows of $\underline{T}_F(s)$. To obtain $\underline{F}$, we note that the third, fourth, sixth, and eighth rows of both $\underline{A}$ and $(\underline{A} + \underline{BF})$ are known. Denoting these rows of $\underline{A}$, $(\underline{A} + \underline{BF})$, and $\underline{B}$ as $\underline{A}_r$, $(\underline{A} + \underline{BF})_r$, and $\underline{B}_r$, respectively, we note that $(\underline{A} + \underline{BF})_r = \underline{A}_r + \underline{B}_r\underline{F}$; i.e. $(\underline{A} + \underline{BF})_r - \underline{A}_r = \underline{B}_r\underline{F}$, and since $\underline{B}_r$ is nonsingualr, $\underline{F} = \underline{B}_r^{-1}[(\underline{A} + \underline{BF})_r - \underline{A}_r]$. Now,

$$(\underline{A} + \underline{BF})_r - \underline{A}_r =$$

$$\begin{bmatrix} -1.97 & -6.25 & -2.993 & -1 & 0 & -.548 & 0 & 0 & -.736 \\ 0 & 0 & -.00018 & -.6758 & 0 & 0 & 0 & 0 & 0 \\ -2.61 & 0 & 1.94 & .163 & -6.25 & -1.575 & 0 & -.01 & 7.25 \\ -.016 & 0 & .0083 & 12.308 & -.0499 & .0043 & 0 & 9.607 & -5.59 \end{bmatrix}$$

and

$$\underline{B}_r^{-1} = \begin{bmatrix} -.1626 & -.2646 & 0 & 0 \\ 0 & -2.3585 & 0 & 0 \\ 0 & -.2045 & .0458 & .0021 \\ 0 & -1.62 & .001 & -.1336 \end{bmatrix}$$

Therefore,

$$\underline{F} = \begin{bmatrix} .3203 & 1.016 & .4867 & .3414 & 0 & .0891 & 0 & 0 & .12 \\ 0 & 0 & .0004 & 1.594 & 0 & 0 & 0 & 0 & 0 \\ -.1195 & 0 & .0888 & .1719 & -.2863 & -.0721 & 0 & -.02 & .3202 \\ -.0005 & 0 & .0012 & -.5495 & .0004 & -.0011 & 0 & 1.28 & .754 \end{bmatrix}$$

and we note that all elements of $\underline{F}$ satisfy the constraint that $|f_{ij}| \leqq 2$. In fact, the largest is 1.594. Once $\underline{F}$ has been determined, we can compute $\underline{A} + \underline{B}\underline{F}$ explicitly; i.e.

$$\underline{A} + \underline{B}\underline{F} = \begin{bmatrix} 0 & 0 & .0275 & .017 & 0 & .0044 & 0 & 0 & .0013 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6.25 & -3.535 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6.25 & -3.535 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 12.115 & -.0499 & 0 & 0 & -10 & 0 \\ -.0013 & 0 & .002 & .012 & .04 & -.006 & 0 & .0281 & 0 \end{bmatrix}$$

and, by inspection,

$$
\underline{\delta}_F(s) = \begin{bmatrix}
s & -.0275s & -.017 & -.0044s & 0 & -.0013 \\
0 & s^2+3.535s+6.25 & 0 & 0 & 0 & 0 \\
0 & 0 & s+1 & 0 & 0 & 0 \\
0 & 0 & 0 & s^2+3.535s+6.25 & 0 & 0 \\
0 & 0 & -12.115 & .0499 & s(s+10) & 0 \\
.0013 & -.002s & -.012 & .006s-.04 & -.0281s & s
\end{bmatrix}
$$

and

$$
\underline{\delta}_F^{-1}(s) \cong \begin{bmatrix}
\frac{1}{s} & \frac{.0275}{s^2+3.535s+6.25} & \frac{.017}{s(s+1)} & \frac{.0044}{s^2+3.535s+6.25} & 0 & \frac{.0013}{s^2} \\
0 & \frac{1}{s^2+3.535s+6.25} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{s+1} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{s^2+3.535s+6.25} & 0 & 0 \\
0 & 0 & \frac{12.115}{s(s+1)(s+10)} & \frac{-.0499}{s(s+10)(s^2+3.535s+6.25} & \frac{1}{s(s+10)} & 0 \\
\frac{-.0013}{s^2} & \frac{.002}{s^2+3.535s+6.25} & \frac{.012s+.46}{s(s+1)(s+10)} & \frac{-.006s+.04}{s(s^2+3.535s+6.25} & \frac{.0281}{s(s+10)} & \frac{1}{s}
\end{bmatrix}
$$

Once $\underline{\delta}_F^{-1}(s)$ has been calculated, the overall (6x4) closed loop transfer matrix $\underline{T}_F(s) = \underline{\delta}_F^{-1}(s)\underline{B}_m$ can be determined, as follows:

$$
\underline{T}_F(s) = \begin{array}{cc}
\begin{bmatrix}
\frac{.05s^2+.0077s+.3175}{s(s^2+3.535s+6.25)} & 0 & \frac{.096}{s^2+3.535s+6.25} & 0 \\
\frac{-6.15}{s^2+3.535s+6.25} & \frac{.69}{s^2+3.535s+6.25} & 0 & 0 \\
0 & \frac{-.424}{s+1} & 0 & 0 \\
0 & \frac{-2.13}{s^2+3.535s+6.25} & \frac{21.81}{s^2+3.535s+6.25} & \frac{.3475}{s^2+3.535s+6.25} \\
0 & \frac{5.12s^2+28.36s+68.37}{(s+1)(s+10)(s^2+3.535s+6.25)} & \frac{.174s+.615}{(s+10)(s^2+3.535s+6.25)} & \frac{-7.48}{s(s+10)} \\
\frac{-.0123}{s^2+3.535s+6.25} & \frac{.01s^4+.297s^3+1.05s^2+.804s-.55}{s(s+1)(s+10)(s^2+3.535s+6.25} & \frac{.05s^2+.046s+1.19}{s(s^2+3.535s+6.25)} & \frac{.022s+.01}{s(s+10)}
\end{bmatrix}
&
\begin{array}{l}
\text{LONG. VEL.} \\
\text{PITCH} \\
\text{VERT. VEL.} \\
\text{ROLL} \\
\text{YAW} \\
\text{LAT. VEL.}
\end{array}
\\
\begin{array}{cccc}
\text{LONG.CYCLIC PITCH} & \text{MAIN ROTOR COLLECTIVE} & \text{LATERAL CYCLIC PITCH} & \text{TAIL ROTOR COLLECTIVE}
\end{array}
&
\end{array}
$$

This $\underline{T}_F(s)$ is proposed as the final closed loop transfer matrix. If we compare this transfer matrix with the original desired closed loop transfer matrix, we can make several observations. First, the requirement that pure integrations appear in the transfer functions relating longitudinal velocity to longitudinal cyclic pitch, lateral velocity to lateral cyclic pitch, and yaw to tail rotor collective has been met. Furthermore, by appropriate selection of the feedback matrix $\underline{F}$, we have eliminated pure integrations in almost all of the other transfer functions. The only exceptions are the transfer functions relating lateral velocity to the main rotor and tail rotor collective inputs. We note, however, that the gains associated with the integrator portion of these two transfer functions are quite low, relative to the lateral cyclic pitch input. We were unable to achieve the undamped natural frequency of 3 rad/sec relating pitch and roll to longitudinal cyclic pitch and lateral cyclic pitch, respectively, because of the constraint placed on the magnitude of the elements comprising the feedback matrix $\underline{F}$. A reduction of this desired frequency to 2.5 rad/sec, however, was found to be obtainable without excessive feedback gains. Continuing our analysis, we note that we have achieved as much decoupling as possible, consistent with the constraint that $\underline{G} = \underline{I}$. (This point was discussed in considerable detail earlier.) We also note that longitudinal cyclic pitch affects pitch and longitudinal velocity almost exclusively. Vertical velocity is affected by main rotor collective only. Lateral cyclic pitch affects lateral velocity and roll, as desired, with some unavoidable cross-coupling into yaw. The main input affecting yaw is tail rotor collective, as desired.

Although not all design objectives were met, it appears that this design is about as close to the desired design as possible, consistent with the constraints on $\underline{F}$ and $\underline{G}$. If there were no constraints on $\underline{F}$ or $\underline{G}$, we could have achieved almost a perfect match between desired and actual closed loop transfer matrices. The only exceptions would be nonzero entries appearing in all elements of the first and last rows of the closed loop transfer matrix, in general.

7. CONCLUDING REMARKS

We have introduced the structure theorem, a new and rather basic idea which was shown to have broad implications in the analysis and design of linear multivariable control systems. The underlying idea behind the structure theorem is the ability to combine time and frequency domain information in a compact and concise expression through a transformation (restructuring) of the dynamical equations of motion which characterize the system.

A significant number of new results were obtained, and certain prior results were given a new interpretation. In particular, in Section 3 we presented a new method for obtaining realizations, given the transfer matrix of a linear multivariable system. We also derived a direct, constructive technique for arbitrarily assigning all n eigenvalues of a controllable multivariable system using linear state variable feedback. The question of pole placement via output feedback was then considered. We presented a compact expression for the closed loop characteristic equation (closed under linear output feedback) of a multivariable system, and then demonstrated how this result could be used to obtain certain desired closed loop poles.

In Section 4, various questions pertaining to linear optimal control were considered. Few new ideas were presented here, but a number of improvements were made over previous work. In particular, we demonstrated how a solution to the output regulator problem could be formulated in the frequency domain. This result led to two new expressions for characterizing an optimal feedback design. We then demonstrated how spectral factorization could be used to obtain a solution to the regulator problem, and concluded by discussing a new technique for formulating the root-square locus.

We considered the design of noninteracting control systems in Section 5. A number of new results and design algorithms were presented. We gave a procedure for determining what poles could not be altered while decoupling via linear state variable feedback. We also presented a design algorithm for achieving maximum pole assignment under linear decoupling feedback. We then characterized those systems which can be decoupled via input dynamics, and presented an algorithm for achieving an asymptotically stable decoupled design with arbitrary pole placement. Two design algorithms for achieving a noninteractive design via linear state variable feedback were then given and demonstrated by example.

In Section 6, we considered the design of a helicopter stability augmentation system based on desired handling qualities. Many of the results obtained in the earlier sections were applied. Here, it was shown that pencil and paper methods can be employed to achieve an acceptable feedback control system.

Preliminary computer programs have been devised to implement much of the theory developed in this report. Some of these programs were mentioned briefly. Currently, there is an effort underway to combine these preliminary programs in a highly interactive man-computer system which will facilitate the analysis and design of complex multivariable systems.

Although a significant amount of work has been accomplished, much remains to be done. Answers to additional questions are required in most of the areas already mentioned as well as in related research areas such as model matching, state estimation, sensitivity reduction, and nondeterministic systems. The structure theorem, which we have introduced in this report, appears to offer a valuable design and analysis technique for achieving further results in these areas and others.

REFERENCES

1. Wolovich, W. A. and Falb, P. L.: On the Structure of Multivariable Systems. SIAM J. Control, vol. 7, no. 3, August 1969.

2. Brockett, R. W.: Poles, Zeros, and Feedback: State Space Interpretation. IEEE Trans. Auto. Control, vol. AC-10, April 1965, pp. 129-135.

3. Kalman, R. E.: The Mathematical Description of Linear Dynamical Systems. SIAM J. Control, vol. 1, 1963, pp. 152-192.

4. Chidambra, M. R.: The Transformation to (phase-variable) Canonical Form. IEEE Trans. Auto. Control, vol. AC-10, October 1965, pp. 492-495.

5. Mufti, I. H.: On the Reduction of a System to Canonical (phase-variable) Form. IEEE Trans. Auto. Control, vol. AC-10, April 1965, pp. 206-207.

6. Silverman, L. M.: Transformation of Time Variable Systems to Canonical (phase-variable) Form. IEEE Trans. Auto. Control, vol. AC-11, April 1966, pp. 300-303.

7. Wolovich, W. A.: On the Stabilization of Controllable Systems. IEEE Trans. Auto. Control, vol. AC-13, 1968, pp. 569-572.

8. Hoffman, K. and Kunze, R.: Linear Algebra. Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1961.

9. Luenberger, D. G.: Canonical Forms for Linear Multivariable Systems. IEEE Trans. Auto. Control, vol. AC-12, 1967, pp. 290-293.

10. Kalman, R. E., Falb, P. L., and Arbib, M. A.: Topics in Mathematical System Theory. McGraw-Hill Book Co., Inc., New York, N. Y., 1968.

11. Gilbert, E. G.: Controllability and Observability in Multivariable Control Systems. SIAM J. Control, vol. 2, no. 1, 1963, pp. 128-151.

12. Ho, B. L. and Kalman, R. E.: Effective Construction of Linear State Variable Models from Input/Output Functions. Proc. Third Allerton Conference, 1965, pp. 449-459.

13. Mayne, D. Q.: Computational Procedure for the Minimal Realization of Transfer Function Matrices. Proc. IEEE, 115, 1968, pp. 1363-1368.

14. Davison, E. J. and Wonham, W. H.: On Pole Assignment in Multivariable Linear Systems. IEEE Trans. Auto. Control, vol. AC-13, no. 6, 1968, pp. 747-748.

15. Heymann, M. and Wonham, W. H.: Comments on Pole Assignment in Multi-Input Controllable Systems. IEEE Trans. Auto. Control, vol. AC-13, no. 6, 1968, pp. 748-749.

16. Wonham, W. H.: On Pole Assignment in Multi-Input Controllable Linear Systems. IEEE Trans. Auto. Control, vol. AC-12, no. 6, 1967, pp. 660-665.

17. Langenhop, C. E.: On the Stabilization of Linear Systems. Proc. Am. Math. Soc., vol. 15, no. 5, 1964, pp. 735-742.

18. Bar-Ness, Y.: On the Discrete Control Problem. Ph.D. Dissertation, Dept. of App. Math., Brown University, June 1969.

19. Athans, M. and Falb, P. L.: Optimal Control. McGraw-Hill Book Co., Inc., New York, N. Y., 1966.

20. Kalman, R. E.: When is a Linear Control System Optimal? ASME J. Basic Engineering, March 1964.

21. Anderson, B. D. O.: The Inverse Problem of Optimal Control. Stanford Electronics Lab. tech. report no. 6560-3, April 1966.

22. Bode, H. W.: Network Analysis and Feedback Amplifier Design. D. Van Nostrand Co., Inc., Princeton, N. J., 1945

23. Athans, M: The Relationship of Alternate State-Space Representations in Linear Filtering Problems. IEEE Trans. Auto. Control, vol. AC-12, December 1967, pp. 775-776.

24. Youla, D. C.: On the Factorization of Rational Matrices. IRE Trans. Info. Theory IT-7, July 1961.

25. Chang, S. S. L.: Synthesis of Optimum Control Systems. McGraw-Hill Book Co., Inc., New York, N. Y., 1961.

26. Rynaski, E. G. and Whitbeck, R. F.: The Theory and Application of Linear Optimal Control. Cornell Aero. Lab. TR No. IH-1943-F-1, October 1965.

27. Whitbeck, R. F.: A Frequency Domain Approach to Linear Optimal Control. Preprints 1968 JACC, pp. 726-737.

28. Wiener, N. and Masani, L.: The Prediction Theory of Multivariable Stochastic Processes. ACTA Mathematics, vol. 98, June 1958.

29. Newcomb, R. W.: Linear Multiport Synthesis. McGraw-Hill Book Co., Inc., New York, N. Y., 1966.

30. Davis, M. C.: Factoring the Spectral Matrix. IEEE Trans. Auto. Control, vol. AC-8, October 1963, pp. 296-305.

31. Tuel, W. G.: Computer Algorithm for Spectral Factorization of Rational Matrices. IBM Research and Development, vol. 12, no. 2, March 1968, pp. 163-170.

32. Truxal, J. G.: Control System Synthesis. McGraw-Hill Book Co., Inc., New York, N. Y., 1955.

33. Morgan, B. S.: The Synthesis of Linear Multivariable Systems by State Variable Feedback. Preprints 1964 JACC, pp. 468-472.

34. Falb, P. L. and Wolovich, W. A.: Decoupling in the Design and Synthesis of Multivariable Control Systems. IEEE Trans. Auto. Control, vol. AC-12, 1967, pp. 651-659.

35. Gilbert, E. G.: The Decoupling of Multivariable Systems by State Feedback. SIAM J. Control, vol. 7, no. 1, February 1969, pp. 50-63.

36. Wonham, W. H. and Morse, A. S.: Decoupling and Pole Assignment in Linear Multivariable Systems: A Geometric Approach. NASA ERC PM-66, October 1968; SIAM J. Control, vol. 7, no. 4, 1969.

37. Morse, A. S. and Wonham, W. H.: Decoupling and Pole Assignment by Dynamic Compensation. NASA PM-86, July 1969.

38. Murphy, R. D. and Narendra, K. S.: Design of Helicopter Stabilization Systems Using Optimal Control Theory. J. Aircraft, vol. 6, no.2, March-April 1969, pp. 129-136.

39. Anderson, B. D. O.: An Algebraic Solution to the Spectral Factorization Problem. IEEE Trans. Auto. Control, vol. AC-12, August 1967, pp. 410-414.

Appendix A
LIST OF SYMBOLS

Symbol	Page	Definition
$\underline{x}$	4	The n-dimensional state of a linear system
$\underline{A}$	4	The constant (nxn) state matrix
$\underline{B}$	4	The constant (nxm) input matrix
$\underline{C}$	4	The constant (pxn) output matrix
$\underline{y}$	4	The p-dimensional output of a linear system
$\underline{u}$	4	The m-dimensional input of a linear system
$\underline{T}_o(s)$	4	The open loop transfer matrix
$\underline{y}(s)$	4	The Laplace transform of the output $\underline{y}$
$\underline{u}(s)$	4	The Laplace transform of the input $\underline{u}$
s	4	The Laplace operator $s = \sigma+jw$
$\underline{w}$	4	An m-dimensional external input
$\underline{F}$	4	A constant (mxn) state variable feedback matrix
$\underline{G}$	4	A constant (mxm) feedforward matrix
$\underline{T}_{\underline{F},\underline{G}}(s)$	5	The closed loop transfer matrix
$\underline{Q}$	5	An (nxn) nonsingular similarity transformation matrix
$\underline{z}$	5	The altered state under $\underline{Q}$, $\underline{Q}\underline{x}$
$\hat{\underline{A}}$	5	The altered state matrix under $\underline{Q}$, $\underline{Q}\underline{A}\underline{Q}^{-1}$
$\hat{\underline{B}}$	5	The altered input matrix under $\underline{Q}$, $\underline{Q}\underline{B}$
$\hat{\underline{C}}$	5	The altered output matrix under $\underline{Q}$, $\underline{C}\underline{Q}^{-1}$
$\hat{\underline{F}}$	6	The altered feedback matrix under $\underline{Q}$, $\underline{F}\underline{Q}^{-1}$
$\hat{\underline{T}}_{\hat{\underline{F}},\underline{G}}$	6	The altered closed loop transfer matrix under $\underline{Q}$

Symbol	Page	Definition
$\underline{K}$	8	The (nxnm) controllability matrix (see $\underline{K}$, pp. 32 and 58 also)
$\underline{L}$	9	The (nxn) nonsingular matrix obtained by a "lexicographic" ordering of the first (n) independent columns of $\underline{K}$
$\underline{\Phi}$	11	The (nxn) "companion" form closed loop state matrix
$\underline{T}_{\underline{F}}(s)$	11	The closed loop transfer matrix when $\underline{G} = \underline{I}$
$\underline{S}(s)$	12	An (nxm) matrix of monic single term polynomials in s used in defining the closed loop transfer matrix via the structure theorem.
$\underline{\delta}_{\underline{F}}(s)$	12	An (mxm) matrix of polynomials in s used in defining the closed loop transfer matrix via the structure theorem.
$\hat{\underline{B}}_m$	12	An (mxm) upper-triangular matrix obtained from $\hat{\underline{B}}$ and used in defining the transfer matrix via the structure theorem
$\Delta_{\underline{F}}(s)$	13	The characteristic polynomial of the closed loop system
$\underline{C}^*(s)$	13	Shorthand notation for $\hat{\underline{C}}\underline{S}(s)$
$\underline{N}_{\underline{F}}(s)$	13	A (pxm) matrix of polynomials in s--the numerator of the closed loop transfer matrix
$\underline{D}_{\underline{F}}(s)$	13	An (mxm) matrix of polynomials in s equal to the adjoint of $\underline{\delta}_{\underline{F}}(s)$
$\hat{\underline{A}}_m$	14	An (nxm) matrix obtained from $\hat{\underline{A}}$ and used in obtaining $\underline{\delta}_{\underline{F}}(s)$
$\underline{\delta}_o(s)$	14	An (mxm) matrix of polynomials in s used in defining the open loop transfer matrix via the

Symbol	Page	Definition
		structure theorem; i.e. $\underline{\delta}_o(s) = \underline{\delta}_{\underline{F}}(s)$ when $\underline{F} = \underline{0}$
W	15	An r-dimensional subspace of R_N
R_N	15	The n-dimensional Euclidean space
$W^{\perp}$	15	The orthogonal complement of W
$\underline{\beta}_i$	15	One of the basis vectors of $W^{\perp}$
$\underline{B}_e$	15	An extension of $\underline{B}$ which includes the basis vectors, $\underline{\beta}_i$, of $W^{\perp}$
$\underline{Q}_e$	15	An (nxn) similarity transformation matrix defined in terms of the extended input matrix $\underline{B}_e$
$\Delta_{\underline{F},u}(s)$	16	The characteristic polynomial associated with the "uncontrollable" part of the closed loop state matrix
$\underline{\delta}_{\underline{F},c}(s)$	17	An (mxm) matrix of polynomials in s associated with the "controllable" part of the closed loop state matrix
$\hat{\underline{F}}_e$	18	An (m+qxn) extension of the feedback matrix $\hat{\underline{F}}$
$\hat{\underline{B}}_{e,m+q}$	18	An (m+q)x(m+q) extension of the input matrix $\hat{\underline{B}}_m$
$\underline{S}_e(s)$	18	An nx(m+q) extension of $\underline{S}(s)$
$\underline{T}(s)$	20	A given (pxm) transfer matrix--also shorthand notation for $\underline{T}_o(s)$
$\{\underline{A}_c, \underline{B}_c, \underline{C}_c\}$	21	A controllable realization of the transfer matrix
$\underline{T}^*(s)$	21	A modified version of $\underline{T}(s)$ used in obtaining a controllable realization
Λ	24	An arbitrary set of (n) complex numbers
$\hat{\underline{A}}^*_m$	25	A special form of $\hat{\underline{A}}_m+\hat{\underline{B}}_m\hat{\underline{F}}$ used for arbitrary pole placement via state variable feedback

Symbol	Page	Definition
$\underline{H}$	27	An (mxp) constant output feedback matrix
J	31	A quadratic performance index
$\underline{R}$	31	An (mxm) positive definite input weighting matrix (see p. 71 also)
$\underline{T}$	31	The "square root" of $\underline{R}$; i.e. $\underline{R} = \underline{T}^T\underline{T}$
$\underline{u}^*$	31	The optimal control which minimizes J
$\underline{K}$	32	The positive definite solution to the matrix Riccati equation (see $\underline{K}$, pp. 8 and 58 also)
ω	33	A frequency in radians/second
$\Delta(s)$	33	The characteristic polynomial of the open loop system
$\Delta^*(s)$	33	The characteristic polynomial of the closed loop optimal system
$\underline{F}^*(s)$	35	The optimal state variable feedback matrix
$\underline{\delta}(s)$	35	Shorthand notation for $\underline{\delta}_o(s)$
$\underline{\Phi}(s)$	41	A symmetric (mxm) matrix of polynomials in s
$\underline{W}(s)$	41	A "spectral factorization" of $\underline{\Phi}(s)$; i.e. $\underline{\Phi}(s) = \underline{W}^T(-s)\underline{W}(s)$
$\underline{U}$	41	An orthogonal matrix $\underline{U}^T\underline{U} = \underline{I}$
$\underline{B}^*$	52	An (mxm) matrix used in decoupling multivariable systems
∂_p	53	The degree of the polynomial p(s)
ν	54	The maximum number of specifiable poles under decoupling state variable feedback
$\underline{C}^*_{II}(s)$	54	An (mxm) matrix of polynomials in s obtained by factoring $\underline{C}^*(s)$

Symbol	Page	Definition
$\underline{P}(s)$	56	An (mxm) diagonal matrix of polynomials in s obtained by factoring $\underline{C}^*(s)$; i.e. $\underline{C}^*(s) = \underline{P}(s)\underline{C}^*_{II}(s)$
$\underline{T}_{II}(s)$	56	The open loop transfer matrix of the system $\dot{\underline{x}} = \underline{A}\underline{x}+\underline{B}\underline{u}$, $\underline{y}_{II} = \underline{C}_{II}\underline{x}$
$\underline{C}_{II}$	56	A modified output matrix $\underline{C}_{II}\underline{Q}^{-1}\underline{S}(s) = \underline{C}^*_{II}(s)$
$\underline{y}_{II}$	56	The output $\underline{C}_{II}\underline{x}$
$\underline{M}_k$	56	A diagonal matrix used in obtaining a decoupling state variable feedback matrix for arbitrary pole placement
$\underline{A}^*$	56	An (mxn) matrix used in obtaining a decoupling state variable feedback matrix
$\underline{G}(s)$	57	The (mxm) transfer matrix of a dynamical system used for decoupling via "input dynamics"
$\underline{g}$	58	The k-dimensional state of the system whose transfer matrix is $\underline{G}(s)$
$\underline{u}_e$	58	The output of the system whose transfer matrix is $\underline{G}(s)$
$\underline{v}$	58	The input to the system whose transfer matrix is $\underline{G}(s)$
$\underline{x}_c$	59	The state of the composite system consisting of $\underline{G}(s)$ in series with the given system
$\{\underline{K},\underline{L},\underline{M},\underline{N}\}$	59	The quadruple of constant matrices defining the k-th order system whose transfer matrix is $\underline{G}(s)$; i.e. $\underline{G}(s) = \underline{M}(s\underline{I}-\underline{K})^{-1}\underline{L}+\underline{N}$ (see K, pp. 8 and 32 also)
$\underline{y}_c$	59	The output of the composite system--equivalent to $\underline{y}$

Symbol	Page	Definition
$\underline{T}_c(s)$	60	The transfer matrix of the composite system
$\{\underline{F}_c, \underline{G}_c\}$	61	A constant composite feedback pair used to decouple and specify poles
$\underline{T}_{\underline{C}_{II}}(s)$	63	The transfer matrix associated with a factored version of the composite system
$p_s(s)$	63	The "stable" polynomial factor of $\lvert\underline{C}^*_{II}(s)\rvert$
$p_u(s)$	63	The "unstable" polynomial factor $p_u(s)$ of $\lvert\underline{C}^*_{II}(s)\rvert$; $\lvert\underline{C}^*_{II}(s)\rvert = \underline{C}p_s(s)p_u(s)$
$\underline{B}^*_c$	69	The matrix $\underline{B}^*$ (see p. 52) associated with the composite system
$\hat{\underline{C}}_q$	71	An alteration of the matrix $\hat{\underline{C}}$ used in achieving noninteraction
$\underline{R}$	71	An (mxn) matrix added to $\hat{\underline{C}}$ to yield $\hat{\underline{C}}_q$; i.e. $\hat{\underline{C}}_q = \hat{\underline{C}}+\underline{R}$ (see p. 31 also)
$\underline{B}^*_q$	71	The matrix $\underline{B}^*$ (see p. 52) defined in terms of $\hat{\underline{C}}_q$ rather than $\hat{\underline{C}}$
$\hat{\underline{C}}_p$	74	An alteration of the matrix $\hat{\underline{C}}$ used in achieving noninteraction
$\underline{P}$	74	An (mxm) matrix which premultiplies $\hat{\underline{C}}$ to produce $\hat{\underline{C}}_p$; i.e. $\hat{\underline{C}}_p = \underline{P}\hat{\underline{C}}$
$\underline{B}^*_p$	74	The matrix $\underline{B}^*$ (see p. 52) defined in terms of $\hat{\underline{C}}_p$ rather than $\hat{\underline{C}}$
ξ	82	The damping ratio associated with a second order linear system
ω_n	82	The undamped natural frequency associated with a second order linear system

Appendix B
SPECIAL NOTATION

NOTE: Underlined letters refer to vectors or matrices

$\underline{I}_p$	The (pxp) identity matrix
$\dot{\underline{x}}$	The derivative of $\underline{x}$ with respect to time
$\underline{a}_i$ or $\underline{A}_i$	The i-th row of the matrix $\underline{A}$
a_{ij}	The ij-th element of the matrix $\underline{A}$
$\underline{a}_{ij}$ or $\underline{A}_{ij}$	The ij-th block of a portioned matrix $\underline{A}$
$\underline{A}'$ or $\underline{A}^T$	The transpose of the matrix $\underline{A}$
$\underline{A}^{-1}$	The inverse of the matrix $\underline{A}$
$\underline{A}^{-T}$	The transpose of $\underline{A}^{-1}$ or the inverse of $\underline{A}^T$
$\mathrm{adj}(\underline{A})$	The adjoint of the matrix $\underline{A}$: i.e. $\underline{A}^{-1} = \mathrm{adj}(\underline{A}) \div \lvert\underline{A}\rvert$
$\det(\underline{A})$ or $\lvert\underline{A}\rvert$	The determinant of the matrix $\underline{A}$
$\mathrm{diag}[a_i]$	A diagonal matrix whose entries are a_i
$[\cdot]^*\underline{A}(s)$	The product $\underline{A}^T(-s)\underline{A}(s)$
Re s	The real part of s; i.e. σ if $s = \sigma + jw$
Im λ	The imaginary part of the complex number λ
$\underline{A} \geq 0$	A nonnegative definite matrix $\underline{A}$
$\boxed{\underline{A}}_{(i,j)-(k,l)}$	The (k-i+1)x(l-j+1) submatrix of $\underline{A}$ consisting of those elements common to the i through k-th rows and j through l-th columns of $\underline{A}$