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COMPUTATION OF THE POTENTIAL FUNCTION OF A COMPRESSIBLE FLUID NEAR MACH NUMBER 1

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16. Abstract

A procedure is described for finding the potential function of a compressible fluid flow according to Bergman's method of orthogonal functions. In Bergman's formula for the potential function an infinite series occurs, the terms of which are multiple integrals. These multiple integrals have a singularity at Mach number 1. The problem arises of computing the multiple integrals as the Mach number approaches 1.

To calculate the integrals, it is necessary to use a special computational system. A method is described whereby a digital computer equipped with a Formula Manipulation Compiler known as Formac can be used to compute and evaluate the multiple integrals.

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NOMENCLATURE

A_k	homogeneous polynomial of Kth degree in Z, S
A_s	speed of sound where air speed equals zero
A_0, A_1, A_2, A_3	parameters of polynomial P1
a	speed of sound at arbitrary point in the flow
C	parameter of polynomial P1
D, E	infinite series of multiple integrals
h	function of the gas constant k
$I(\ell,m,n)$	symbolic coefficients of integral of quantity E
i,j,ℓ,m	summation indexes
k	ratio of specific heats
M	Mach number
N	degree of polynomial PN
n	iteration number
$\tilde{P}$	polynomial in Z, S
P1	polynomial to be integrated
PN	polynomial of degree N
q	speed of sound
S	numerical value of S
S_0, Z_0	lower limits of integration
S,Z	upper limits of integration
$T_{(n)}$	nth integral of quantity E
V	velocity vector
X,Y	Cartesian coordinates in physical plane
Z	numerical value for Z

θ,λ	Cartesian coordinates in pseudologarithmic plane
θ_0,λ_0	constants in pseudologarithmic plane
θ_ν	numerical value for θ
λ_ν	numerical value for λ
μ,ν,ξ	indexes
ϕ	potential function of a compressible fluid flow

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SUMMARY

A procedure is described for finding the potential function of a compressible fluid flow according to Bergman's method of orthogonal functions. In Bergman's formula for the potential function an infinite series occurs, the terms of which are multiple integrals. These multiple integrals have a singularity at Mach number 1. The problem arises of computing the multiple integrals as the Mach number approaches 1.

To calculate the integrals, it is necessary to use a special computational system. A method is described whereby a digital computer equipped with a Formula Manipulation Compiler known as Formac can be used to compute and evaluate the multiple integrals.

INTRODUCTION

In reference 1 a method is described for computing flows of compressible fluids. Research undertaken with the object of computing flow patterns of compressible fluids has led to large expressions to be manipulated symbolically. The expressions that have arisen in the formula for the potential function are so large that a human could not analyze them mathematically (ref. 1, p. 305). In the present investigation, attention has been directed to the use of a digital computer for performing the mathematical analysis (ref. 2).

The extensive storage capacities of the digital computer, combined with the development of a new programming language known as Formac, enable it to be used to perform the mathematical analysis of problems of the type under consideration (ref. 3). As a consequence of the algorithm for performing symbolic integration of a power series, the operations involved in accomplishing the symbolic integrations of algebraic terms arising in the formula for the potential function of a compressible fluid can be translated into a Formac program. The aspect of this problem which makes it so attractive for computer solution is that the mathematical analysis can be translated into a computer program.

The purpose of this report is to show how the computer can be used to perform the mathematical analysis involved in the evaluation of the formula for the potential function of a compressible fluid as the Mach number approaches 1. With this end result in mind, a Formac program was written to perform symbolic double integrations of polynomials of arbitrary degree combined with the facility for performing an arbitrary number of iterations. To compute the potential function for different Mach numbers, the user need only supply the computer with the appropriate polynomial, and specify the desired number of iterations.

ANALYSIS

For two dimensions, the potential function ϕ of compressible fluids when considered in the physical plane satisfies the nonlinear partial differential equation

$$\frac{\partial^2\phi}{\partial X^2}\left[1 - \frac{1}{a^2}\left(\frac{\partial\phi}{\partial X}\right)^2\right] - \frac{2}{a^2}\frac{\partial\phi}{\partial X}\frac{\partial\phi}{\partial Y}\frac{\partial^2\phi}{\partial X\,\partial Y} + \frac{\partial^2\phi}{\partial Y^2}\left[1 - \frac{1}{a^2}\left(\frac{\partial\phi}{\partial Y}\right)^2\right] = 0 \qquad (1)$$

where

$$a^2 = a_s{}^2 - \frac{k-1}{2}\left[\left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial Y}\right)^2\right] \quad \text{(ref. 4, pp. 239, 241)} \qquad (2)$$

and a_s and k are constants.

Equation (1) is first linearized by transformation to a pseudologarithmic plane with Cartesian coordinates θ and

$$\lambda = \frac{1}{2}\log\left\{\frac{1 - (1 - M^2)^{1/2}}{1 + (1 - M^2)^{1/2}}\left[\frac{1 + h(1 - M^2)^{1/2}}{1 - h(1 - M^2)^{1/2}}\right]^{1/h}\right\} \qquad (3)$$

The coordinate θ is the angle formed by the velocity vector

$$\vec{V} = qe^{i\theta} \qquad (4)$$

and the positive X axis, and q is the speed (ref. 1, p. 302).

$$h = \left(\frac{k-1}{k+1}\right)^{1/2} \qquad (5)$$

and

$$M = \frac{q}{[a_s{}^2 - (k-1)q^2/2]^{1/2}} \qquad (6)$$

where M is the Mach number (ref. 1, p. 302). The physical and pseudologarithmic planes are compared in figures 1 and 2.

Figure 1.- A flow in a channel.

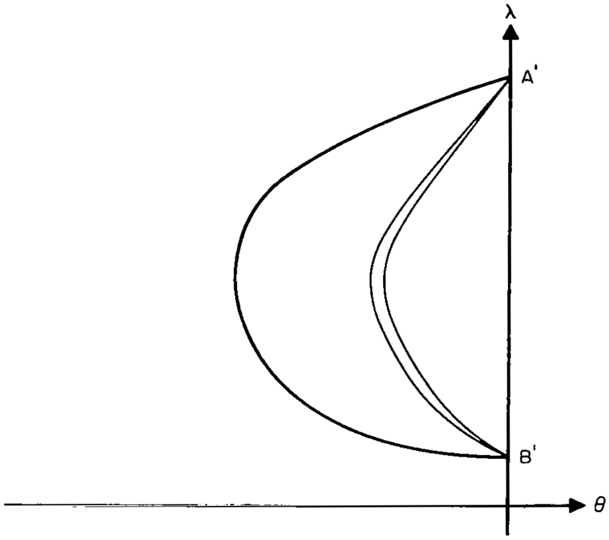

Figure 2.- The image of the flow in figure 1 in the pseudologarithmic plane.

The following formulas are also obtained:

$$q^2 = \left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial Y}\right)^2 \tag{7}$$

$$\left(\frac{\partial\phi}{\partial X}\right)^2 = \frac{q^2 \operatorname{ctn}^2\theta}{1 + \operatorname{ctn}^2\theta} \tag{8}$$

$$\left(\frac{\partial\phi}{\partial Y}\right)^2 = \frac{q^2}{1 + \operatorname{ctn}^2\theta} \qquad \text{(ref. 1, p. 304)} \tag{9}$$

For flows satisfying equations (1) through (9), the potential function has a singularity on the sonic line. In reference 1 (p. 305), an analytic expression for this type of singularity has been determined in the pseudologarithmic plane. In this plane, the formula for the potential function is

$$\phi = \frac{E}{2}\left[\log(Z - Z_0) + \log(S - S_0)\right] + D \tag{10}$$

where

$$\left.\begin{aligned} Z &= \theta + i\lambda \\ S &= \theta - i\lambda \\ Z_0 &= \theta_0 + i\lambda_0 \\ S_0 &= \theta_0 - i\lambda_0 \end{aligned}\right\} \quad \text{ref. 1, p. 305} \tag{11}$$

For the quantity E is obtained an infinite series whose terms are multiple double integrals. In the following section is presented the computational procedure for the formal integration and numerical evaluation of these multiple integrals.

Computational Procedure

For the quantity E appearing in equation (10) are obtained multiple integrals of the form

$$\left.\begin{aligned}
\tilde{T}_{(1)} &= \int_{Z_0}^{Z}\int_{S_0}^{S} PN(Z,S)dZ\,dS \\
\tilde{T}_{(2)} &= \int_{Z_0}^{Z}\int_{S_0}^{S} \tilde{T}_{(1)}PN(Z,S)dZ\,dS \\
\tilde{T}_{(3)} &= \int_{Z_0}^{Z}\int_{S_0}^{S} \tilde{T}_{(2)}\,PN(Z,S)dZ\,dS \\
&\vdots \\
\tilde{T}_{(n)} &= \int_{Z_0}^{Z}\int_{S_0}^{S} \tilde{T}_{(n-1)}PN(Z,S)dZ\,dS \qquad \text{(ref. 1, p. 305)}
\end{aligned}\right\} \tag{12}$$

The subscript of $\tilde{T}$ in equations (12) denotes the iteration number referred to later. An expression similar to equations (12) is obtained for the quantity D in equation (10) (ref. 1, p. 305). In this report is discussed the formal symbolic integration and evaluation of equations (12). The results presented were obtained by using the IBM 360/50 computer and the Formac programming language. The Formac programming language is a practical system for doing nonnumeric mathematics on the computer.

In reference 5 a method is given for determining the polynomial PN. If PN is a polynomial of degree N in Z,S, then, as formal computation shows, $\tilde{T}_{(n)}$ is also a polynomial in Z,S; namely,

$$\tilde{T}_{(n)} = \sum_{\ell,m} I(\ell,m,n)Z^{\ell}S^{m} \tag{13}$$

where $I(\ell,m,n)$ are symbolic coefficients dependent on Z_0 and S_0 and ℓ and m are the powers of Z and S, respectively, and n is the iteration number. In equation (13) the products $Z^{\ell}S^{m}$ are ordered in such a way that at first those terms appear for which $\ell + m = p =$ maximum. Then the process is repeated for terms of degree (p - 1), (p - 2), etc. According to this procedure, $T_{(n)}$ can be expressed as a sum

$$\tilde{T}_{(n)} = A_p + A_{p-1} + \dots + A_0 \tag{14}$$

where A_k is a homogeneous polynomial of the Kth degree in Z,S. Then in every A_k the terms are ordered in descending powers of S.

A recursion procedure for determining the coefficients $I(\ell,m,n+1)$ from the known coefficient $I(\ell,m,n)$ will be discussed next. From the definition of equations (12), it follows that

$$\tilde{T}_{(n+1)} = \int_{Z_0}^{Z} \int_{S_0}^{S} \tilde{T}_{(n)} PN \, dZ \, dS \tag{15}$$

The first step in determining $I(\ell,m,n+1)$ is to form the polynomial

$$\tilde{T}_{(n)} PN = \left[\sum_{\ell,m} \tilde{T}_{(n)}(\ell,m,n) Z^{\ell} S^{m} \right] \left[\sum_{i,j} I(i,j,1) Z^{i} S^{j} \right] \tag{16}$$

where PN has been denoted by

$$PN = \sum_{i,j} I(i,j,1) Z^{i} S^{j} \tag{17}$$

The polynomial represented by equation (16) can be determined by the use of Formac, and the result written in the form

$$\tilde{T}_{(n)} PN = \sum_{\ell,m} \tilde{P}(\ell,m,n) Z^{\ell} S^{m} \tag{18}$$

The next step in determining $I(\ell,m,n+1)$ is to perform a double integration on the right member of equation (18).

$$\tilde{T}_{(n+1)} = \int_{Z_0}^{Z} \int_{S_0}^{S} \left[\sum_{\ell,m} \tilde{P}(\ell,m,n) Z^{\ell} S^{m} \right] dZ \, dS \tag{19}$$

Formac can be used to perform this double integration if the right side of equation (19) is replaced by

$$\tilde{T}_{(n+1)} = \sum_{\ell,m} \frac{\tilde{P}(\ell,m,n)}{(\ell + 1)(m + 1)} \left(Z^{\ell+1}S^{m+1} - Z_0^{\ell+1}S^{m+1} - Z^{\ell+1}S_0^{m+1} + Z_0^{\ell+1}S_0^{m+1} \right) \tag{20}$$

and the terms are arranged in descending powers of $Z^{\ell+1}S^{m+1}$.

Computer Results

If a set of values $Z_\mu = \theta_\nu + i\lambda_\xi$, $S_\mu = \theta_\nu - i\lambda_\xi$, $\mu = 1, 2, \ldots, p$ is given, the values of $\tilde{T}_{(n)}$ at these points can be determined by the use of Formac. The integrated expressions, as well as the expressions to be integrated, become very long. Therefore, only a few typical integrals and integrands will be presented, together with some numerical values of the integrated expressions.

In equations (21) a third degree polynomial is shown to be integrated. The name of the polynomial is P1.

```
P1(3,0,1) = 1/8 A3
P1(2,1,1) = 3/8 A3
P1(2,0,1) = 1/4 A2 + 3/8 A3 C
P1(1,2,1) = 3/8 A3
P1(1,1,1) = 1/2 A2 + 3/4 A3 C
                                        2
P1(1,0,1) = 1/2 A1 + 1/2 A2 C + 3/8 A3 C
P1(0,3,1) = 1/8 A3
P1(0,2,1) = 1/4 A2 + 3/8 A3 C
                                        2
P1(0,1,1) = 1/2 A1 + 1/2 A2 C + 3/8 A3 C
                                    2            3
P1(0,0,1) = A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C
```
(21)

The first subscript of P1 indicates the power of Z, the second subscript the power of S, and the third subscript the iteration number.

In equations (22) are shown the integrals of equations (21). The name of the integral is I. The first subscript of I indicates the power of Z, the second subscript the power of S, and the third subscript the iteration number.

$$
\left.
\begin{aligned}
I(4,1,1) &= \tfrac{1}{32} A_3 \\
I(4,0,1) &= -\tfrac{1}{32} A_3 S_0 \\
I(3,2,1) &= \tfrac{1}{16} A_3 \\
I(3,1,1) &= \tfrac{1}{3} \left(\tfrac{1}{4} A_2 + \tfrac{3}{8} A_3 C \right) \\
I(3,0,1) &= -\tfrac{1}{3} \left(\tfrac{1}{4} A_2 + \tfrac{3}{8} A_3 C \right) S_0 - \tfrac{1}{16} A_3 S_0^2 \\
I(2,3,1) &= \tfrac{1}{16} A_3 \\
I(2,2,1) &= \tfrac{1}{4} \left(\tfrac{1}{2} A_2 + \tfrac{3}{4} A_3 C \right) \\
I(2,1,1) &= \tfrac{1}{2} \left(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 C + \tfrac{3}{8} A_3 C^2 \right) \\
I(2,0,1) &= -\tfrac{1}{2} \left(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 C + \tfrac{3}{8} A_3 C^2 \right) S_0 - \tfrac{1}{4} \left(\tfrac{1}{2} A_2 + \tfrac{3}{4} A_3 C \right) S_0^2 - \tfrac{1}{16} A_3 S_0^3 \\
I(1,4,1) &= \tfrac{1}{32} A_3 \\
I(1,3,1) &= \tfrac{1}{3} \left(\tfrac{1}{4} A_2 + \tfrac{3}{8} A_3 C \right) \\
I(1,2,1) &= \tfrac{1}{2} \left(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 C + \tfrac{3}{8} A_3 C^2 \right) \\
I(1,1,1) &= A_0 + \tfrac{1}{2} C A_1 + \tfrac{1}{4} A_2 C^2 + \tfrac{1}{8} A_3 C^3 \\
I(1,0,1) &= -\left(A_0 + \tfrac{1}{2} C A_1 + \tfrac{1}{4} A_2 C^2 + \tfrac{1}{8} A_3 C^3 \right) S_0 - \tfrac{1}{2} \left(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 C + \tfrac{3}{8} A_3 C^2 \right) S_0^2 - \tfrac{1}{3} \left(\tfrac{1}{4} A_2 + \tfrac{3}{8} A_3 C \right) S_0^3 - \tfrac{1}{32} A_3 S_0^4 \\
I(0,4,1) &= -\tfrac{1}{32} A_3 Z_0 \\
I(0,3,1) &= -\tfrac{1}{3} \left(\tfrac{1}{4} A_2 + \tfrac{3}{8} A_3 C \right) Z_0 - \tfrac{1}{16} A_3 Z_0^2 \\
I(0,2,1) &= -\tfrac{1}{2} \left(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 C + \tfrac{3}{8} A_3 C^2 \right) Z_0 - \tfrac{1}{4} \left(\tfrac{1}{2} A_2 + \tfrac{3}{4} A_3 C \right) Z_0^2 - \tfrac{1}{16} A_3 Z_0^3 \\
I(0,1,1) &= -\left(A_0 + \tfrac{1}{2} C A_1 + \tfrac{1}{4} A_2 C^2 + \tfrac{1}{8} A_3 C^3 \right) Z_0 - \tfrac{1}{2} \left(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 C + \tfrac{3}{8} A_3 C^2 \right) Z_0^2 - \tfrac{1}{3} \left(\tfrac{1}{4} A_2 + \tfrac{3}{8} A_3 C \right) Z_0^3 - \tfrac{1}{32} A_3 Z_0^4 \\
I(0,0,1) &= \left(A_0 + \tfrac{1}{2} C A_1 + \tfrac{1}{4} A_2 C^2 + \tfrac{1}{8} A_3 C^3 \right) Z_0 S_0 + \tfrac{1}{2} \left(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 C + \tfrac{3}{8} A_3 C^2 \right) Z_0^2 S_0 + \tfrac{1}{3} \left(\tfrac{1}{4} A_2 + \tfrac{3}{8} A_3 C \right) Z_0^3 S_0 + \tfrac{1}{32} A_3 Z_0^4 S_0 \\
&\quad + \tfrac{1}{2} \left(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 C + \tfrac{3}{8} A_3 C^2 \right) Z_0 S_0^2 + \tfrac{1}{4} \left(\tfrac{1}{2} A_2 + \tfrac{3}{4} A_3 C \right) Z_0^2 S_0^2 + \tfrac{1}{16} A_3 Z_0^3 S_0^2 \\
&\quad + \tfrac{1}{3} \left(\tfrac{1}{4} A_2 + \tfrac{3}{8} A_3 C \right) Z_0 S_0^3 + \tfrac{1}{16} A_3 Z_0^2 S_0^3 + \tfrac{1}{32} A_3 Z_0 S_0^4
\end{aligned}
\right\} \quad (22)
$$

For the numerical evaluation of equations (22) the parameters of the polynomial P1 are assigned the following values:

$$
\left.
\begin{aligned}
A_0 &= 37290.0 \\
A_1 &= -0.8426261.10^{7} \\
A_2 &= -0.56560455.10^{10} \\
A_3 &= -0.33684736.10^{13} \\
C &= 0.0045 \\
Z_0 &= -0.00225 \\
S_0 &= -0.00225
\end{aligned}
\right\} \quad (23)
$$

For convenience, the formulas for Z and S are reproduced.

$$Z_\mu = \theta_\nu + i\lambda_\xi \tag{24}$$

$$S_\mu = \theta_\nu - i\lambda_\xi \tag{25}$$

Five distinct values are assigned to θ_ν as represented by formula (26).

$$\theta_\nu = 0.1(\nu - 1) \ , \qquad 1 \leq \nu \leq 5 \tag{26}$$

For each of the five values of θ_ν, four values are assigned to λ_ξ as represented by formula (27).

$$\lambda_\xi = -0.003 + 0.0005(\xi - 1) \ , \qquad 1 \leq \xi \leq 4 \tag{27}$$

Twenty values of Z_μ, S_μ are obtained, yielding twenty values for equations (22).

The numerical values of equations (22) are shown in equations (28). The names of the numerical values are denoted as SUM(ν,ξ,1), where ν,ξ are as defined above, and the third subscript 1 denotes the first iteration. Each of the SUM(ν,ξ,1) in equations (28) represents a numerical value of equations (22) for a given θ_ν, λ_ξ. To the right of the first equal signs in equations (28) are shown the numerical values of equations (22). A second equal sign appears in equations (28) beginning with the fifth SUM. This second equal sign, together with the symbol I which follows it, represent $\sqrt{-1}$. Therefore, the first parts of the numerical values are complex, and the second parts real.

```
SUM(1,1,1) = .02238094
----------------------
SUM(1,2,1) = .00239062
----------------------
SUM(1,3,1) = .00225773
----------------------
SUM(1,4,1) = .01852636

----------------------
SUM(2,1,1) =  - .13633494E-09 =I + 8175.47963
---------------------------------------------
SUM(2,2,1) =  - .26372815E-09 =I + 18686.2335
---------------------------------------------
SUM(2,3,1) =  - .64614535E-10 =I + 29191.5161
---------------------------------------------
SUM(2,4,1) =  - .11833733E-09 =I + 39688.1661
---------------------------------------------
SUM(3,1,1) =  - .15423315E-08 =I + 126027.486
---------------------------------------------
SUM(3,2,1) =  - .41026835E-08 =I + 294388.321
---------------------------------------------
SUM(3,3,1) = .69750161E-09 =I + 462727.212
------------------------------------------

SUM(3,4,1) = .10666321E-08 =I + 631031.523
------------------------------------------
SUM(4,1,1) =  - .72820669E-08 =I + 633532.947
---------------------------------------------
SUM(4,2,1) =  - .10358456E-08 =I + 1486036.45
---------------------------------------------
SUM(4,3,1) = .31876594E-08 =I + 2338490.55
------------------------------------------
SUM(4,4,1) =  - .68060566E-08 =I + 3190866.83
---------------------------------------------
SUM(5,1,1) = .42211181E-07 =I + 1997320.03
-----------------------------------------
SUM(5,2,1) =  - .51793813E-07 =I + 4691847.6
--------------------------------------------
SUM(5,3,1) =  - .78278551E-07 =I + 7386287.32
---------------------------------------------
SUM(5,4,1) =  - .54500974E-07 =I + 1.00805886E+07
-------------------------------------------------
```

(28)

The polynomial for the second iteration is obtained by multiplying equations (22) by equations (21). The multiplied expressions become so large that only two typical terms will be shown in equations (29).

```
P1(3,1,2) = 3/8 (  - ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) SO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) SO  - 1/3
-----------------------------------------------------------------------------------------------------------------------
                          3           4                                            2         3
 ( 1/4 A2 + 3/8 A3 C ) SO   - 1/32 A3 SO   ) A3 + 1/8 (  - ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) ZO - 1/2 ( 1/2
-----------------------------------------------------------------------------------------------------------------------
                        2      2                             3            4
A1 + 1/2 A2 C + 3/8 A3 C   ) ZO   - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO   - 1/32 A3 ZO   ) A3 + 4/3 ( 1/4 A2 + 3/8 A3 C ) ( A0
-----------------------------------------------------------------------------------------------------------------------
                        2          3                                                              2
 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) + ( 1/2 A2 + 3/4 A3 C ) (  - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) SO - 1/4 (
-----------------------------------------------------------------------------------------------------------------------
                         2            3                               2
1/2 A2 + 3/4 A3 C ) SO   - 1/16 A3 SO   ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) (  - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO - 1/16
-----------------------------------------------------------------------------------------------------------------------
     2                                         2  2
 A3 SO   ) + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   )
-----------------------------------------------------
                                              2         3                                         2       2
P1(3,0,2) = 1/8 ( ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) ZO SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) ZO   SO +
-----------------------------------------------------------------------------------------------------------------------
                                 3                   4                                        2       2
1/3 ( 1/4 A2 + 3/8 A3 C ) ZO   SO + 1/32 A3 ZO   SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) ZO SO   + 1/4 ( 1/2 A2 + 3
-----------------------------------------------------------------------------------------------------------------------
             2    2             3   2                                3            2    3                4
/4 A3 C ) ZO   SO   + 1/16 A3 ZO   SO   + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO   + 1/16 A3 ZO   SO   + 1/32 A3 ZO SO   ) A3
-----------------------------------------------------------------------------------------------------------------------
                                                2                          2          3
 + (  - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO   ) ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) + ( 1/4 A2 + 3/8 A3 C
-----------------------------------------------------------------------------------------------------------------------
                                  2          3                                     2     2
 ) (  - ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) SO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) SO   - 1/3 ( 1/4 A2 + 3
-----------------------------------------------------------------------------------------------------------------------
              3           4                                  2                                        2
/8 A3 C ) SO   - 1/32 A3 SO   ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) (  - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) SO - 1
-----------------------------------------------------------------------------------------------------------------------
                          2            3
/4 ( 1/2 A2 + 3/4 A3 C ) SO   - 1/16 A3 SO   )
-----------------------------------------------
```

(29)

The integral of equations (29) for the second iteration becomes so large that only one term will be shown by equation (30).

```
                                              2         3                                        2       2
I(0,0,2) = ( ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) ZO SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) ZO   SO + 1/3
-----------------------------------------------------------------------------------------------------------------------
                              3                  4                                        2       2
( 1/4 A2 + 3/8 A3 C ) ZO   SO + 1/32 A3 ZO   SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) ZO SO   + 1/4 ( 1/2 A2 + 3/4
-----------------------------------------------------------------------------------------------------------------------
          2    2             3   2                                3             2    3              4
A3 C ) ZO   SO   + 1/16 A3 ZO   SO   + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO   + 1/16 A3 ZO   SO   + 1/32 A3 ZO SO   ) ( A0
-----------------------------------------------------------------------------------------------------------------------
                      2          3                                                 2         3
 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) ZO SO + 1/2 ( (  - ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) SO - 1/2 ( 1/2
-----------------------------------------------------------------------------------------------------------------------
                        2     2                              3            4                                 2
A1 + 1/2 A2 C + 3/8 A3 C   ) SO   - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO   - 1/32 A3 SO   ) ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8
-----------------------------------------------------------------------------------------------------------------------
    3                                         2                                  2          3
A3 C   ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) ( ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) ZO SO + 1/2 ( 1/2 A1 + 1/2
-----------------------------------------------------------------------------------------------------------------------
                  2      2                                3                 4                                          2
A2 C + 3/8 A3 C   ) ZO   SO + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO   SO + 1/32 A3 ZO   SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C
-----------------------------------------------------------------------------------------------------------------------
         2                                      2    2            3   2                                   3           2
 ) ZO SO   + 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO   SO   + 1/16 A3 ZO   SO   + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO   + 1/16 A3 ZO
-----------------------------------------------------------------------------------------------------------------------
   3                4        2                                                2
 SO   + 1/32 A3 ZO SO   ) ) ZO   SO + 1/3 ( (  - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) SO - 1/4 ( 1/2 A2 + 3/4 A3 C )
-----------------------------------------------------------------------------------------------------------------------
  2            3                          2          3                                  2
SO   - 1/16 A3 SO   ) ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) (  - ( A0 + 1/2
-----------------------------------------------------------------------------------------------------------------------
               2          3                                   2     2                                3
C A1 + 1/4 A2 C   + 1/8 A3 C   ) SO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) SO   - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO   - 1/
-----------------------------------------------------------------------------------------------------------------------
         4                                                             2          3
32 A3 SO   ) + ( 1/4 A2 + 3/8 A3 C ) ( ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) ZO SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/
-----------------------------------------------------------------------------------------------------------------------
       2      2                               3                4                                         2
8 A3 C   ) ZO   SO + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO   SO + 1/32 A3 ZO   SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   ) ZO SO
-----------------------------------------------------------------------------------------------------------------------
2                                    2    2            3   2                                 3            2    3
   + 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO   SO   + 1/16 A3 ZO   SO   + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO   + 1/16 A3 ZO   SO   +
-----------------------------------------------------------------------------------------------------------------------
             4       3                                                 2          3
1/32 A3 ZO SO   ) ) ZO   SO + 1/4 ( 1/8 ( ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C   ) ZO SO + 1/2 ( 1/2 A1 + 1/2 A2 C
-----------------------------------------------------------------------------------------------------------------------
           2      2                               3                4                                          2
 + 3/8 A3 C   ) ZO   SO + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO   SO + 1/32 A3 ZO   SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C   )
-----------------------------------------------------------------------------------------------------------------------
      2                                    2    2            3   2                                  3            2
ZO SO   + 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO   SO   + 1/16 A3 ZO   SO   + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO   + 1/16 A3 ZO   SO
-----------------------------------------------------------------------------------------------------------------------
3                  4                                                     2                            2          3
   + 1/32 A3 ZO SO   ) A3 + (  - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO   ) ( A0 + 1/2 C A1 + 1/4 A2 C   + 1/8 A3 C
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) • ( 1/4 A2 + 3/8 A3 C ) ( - ( AO + 1/2 C A1 + 1/4 A2 C^2 + 1/8 A3 C^3 ) SO - 1/2 ( 1/2 A1 • 1/2 A2 C • 3/8 A3 C^2 )
SC^2 - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO^3 - 1/32 A3 SO^4 ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ( - 1/2 ( 1/2 A1 • 1/2 A2
C + 3/8 A3 C^2 ) SO - 1/4 ( 1/2 A2 + 3/4 A3 C ) SO^2 - 1/16 A3 SO^3 ) ) ZO^4 SO + 1/5 ( - 1/32 ( AO • 1/2 C A1 + 1/4
A2 C^2 + 1/8 A3 C^3 ) A3 SC + 1/8 ( - ( AC + 1/2 C A1 + 1/4 A2 C^2 • 1/8 A3 C^3 ) SO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8
A3 C^2 ) SO^2 - 1/3 ( 1/4 A2 + 3/8 A3 C ) SC^3 - 1/32 A3 SO^4 ) A3 • ( 1/4 A2 • 3/8 A3 C ) ( - 1/2 ( 1/2 A1 + 1/2 A2 C
+ 3/8 A3 C^2 ) SC - 1/4 ( 1/2 A2 + 3/4 A3 C ) SO^2 - 1/16 A3 SO^3 ) • ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ( - 1/3 ( 1/
4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO^2 ) ) ZC^5 SO + 1/6 ( - 1/32 ( 1/2 A1 • 1/2 A2 C + 3/8 A3 C^2 ) A3 SO + 1/8 ( - 1/2
( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) SO - 1/4 ( 1/2 A2 + 3/4 A3 C ) SO^2 - 1/16 A3 SO^3 ) A3 • ( 1/4 A2 • 3/8 A3 C ) (
- 1/3 ( 1/4 A2 • 3/8 A3 C ) SO - 1/16 A3 SO^2 ) ) ZO^6 SO + 1/7 ( - 1/32 ( 1/4 A2 • 3/8 A3 C ) A3 SO • 1/8 ( - 1/3
( 1/4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO^2 ) A3 ) ZO^7 SO + 1/2 ( ( - ( AO + 1/2 C A1 • 1/4 A2 C^2 + 1/8 A3 C^3 ) ZO - 1/
2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ZO^2 - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO^3 - 1/32 A3 ZO^4 ) ( AO • 1/2 C A1 + 1/4 A2 C^2
+ 1/8 A3 C^3 ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ( ( AO • 1/2 C A1 • 1/4 A2 C^2 • 1/8 A3 C^3 ) ZO SO • 1/2 ( 1/2 A1
+ 1/2 A2 C + 3/8 A3 C^2 ) ZO^2 SO • 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO^3 SO • 1/32 A3 ZO^4 SO + 1/2 ( 1/2 A1 + 1/2 A2 C • 3/
8 A3 C^2 ) ZO SO^2 + 1/4 ( 1/2 A2 • 3/4 A3 C ) ZO^2 SO^2 + 1/16 A3 ZO^3 SO^2 + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO^3 + 1/16
A3 ZC^2 SC^3 + 1/32 A3 ZO SO^4 ) ) ZO SO^2 + 1/4 ( ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ( - ( AO • 1/2 C A1 + 1/4 A2 C^2
+ 1/8 A3 C^3 ) SO - 1/2 ( 1/2 A1 • 1/2 A2 C + 3/8 A3 C^2 ) SO^2 - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO^3 - 1/32 A3 SO^4 ) • (
1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ( - ( AC + 1/2 C A1 + 1/4 A2 C^2 + 1/8 A3 C^3 ) ZO - 1/2 ( 1/2 A1 • 1/2 A2 C + 3/8 A3
C^2 ) ZO^2 - 1/3 ( 1/4 A2 • 3/8 A3 C ) ZO^3 - 1/32 A3 ZO^4 ) • ( 1/2 A2 • 3/4 A3 C ) ( ( AO • 1/2 C A1 • 1/4 A2 C^2 +
1/8 A3 C^3 ) ZC SC + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ZO^2 SO • 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO^3 SO + 1/32 A3 ZO^4
SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ZO SO^2 + 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO^2 SO^2 + 1/16 A3 ZO^3 SO^2 + 1/3 ( 1
/4 A2 + 3/8 A3 C ) ZC SO^3 + 1/16 A3 ZC^2 SO^3 + 1/32 A3 ZO SO^4 ) + ( AO + 1/2 C A1 + 1/4 A2 C^2 • 1/8 A3 C^3 )^2 ) ZO
^2 SO^2 + 1/6 ( 3/8 ( ( AC + 1/2 C A1 + 1/4 A2 C^2 • 1/8 A3 C^3 ) ZO SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 )^2 ZO
SO + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO^3 SO + 1/32 A3 ZO^4 SO • 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ZO SO^2 + 1/4 ( 1/2
A2 + 3/4 A3 C ) ZC^2 SO^2 • 1/16 A3 ZO^3 SO^2 • 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO^3 + 1/16 A3 ZO^2 SO^3 • 1/32 A3 ZO SO^4
) A3 + 3/2 ( 1/2 A1 + 1/2 A2 C • 3/8 A3 C^2 ) ( AO • 1/2 C A1 + 1/4 A2 C^2 + 1/8 A3 C^3 ) + ( 1/2 A2 + 3/4 A3 C ) (
- ( AO + 1/2 C A1 + 1/4 A2 C^2 • 1/8 A3 C^3 ) SO - 1/2 ( 1/2 A1 • 1/2 A2 C + 3/8 A3 C^2 ) SO^2 - 1/3 ( 1/4 A2 + 3/8 A3
C ) SO^3 - 1/32 A3 SO^4 ) + ( 1/2 A1 • 1/2 A2 C + 3/8 A3 C^2 ) ( - 1/2 ( 1/2 A1 + 1/2 A2 C • 3/8 A3 C^2 ) SO - 1/4 (
1/2 A2 + 3/4 A3 C ) SO^2 - 1/16 A3 SO^3 ) • ( 1/4 A2 • 3/8 A3 C ) ( - ( AO • 1/2 C A1 + 1/4 A2 C^2 • 1/8 A3 C^3 ) ZO
- 1/2 ( 1/2 A1 + 1/2 A2 C • 3/8 A3 C^2 ) ZO^2 - 1/3 ( 1/4 A2 • 3/8 A3 C ) ZO^3 - 1/32 A3 ZO^4 ) ) ZO^3 SO^2 + 1/8 ( 3/
8 ( - ( AO + 1/2 C A1 + 1/4 A2 C^2 + 1/8 A3 C^3 ) SO - 1/2 ( 1/2 A1 • 1/2 A2 C + 3/8 A3 C^2 ) SO^2 - 1/3 ( 1/4 A2 + 3/
8 A3 C ) SC^3 - 1/32 A3 SO^4 ) A3 • 1/8 ( - ( AO + 1/2 C A1 + 1/4 A2 C^2 + 1/8 A3 C^3 ) ZO - 1/2 ( 1/2 A1 • 1/2 A2 C
+ 3/8 A3 C^2 ) ZO^2 - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO^3 - 1/32 A3 ZO^4 ) A3 + 4/3 ( 1/4 A2 + 3/8 A3 C ) ( AO + 1/2 C A1
+ 1/4 A2 C^2 + 1/8 A3 C^3 ) + ( 1/2 A2 + 3/4 A3 C ) ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) SO - 1/4 ( 1/2 A2 • 3/
4 A3 C ) SO^2 - 1/16 A3 SO^3 ) • ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO^2 )
+ 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 )^2 ) ZC^4 SO^2 + 1/10 ( - 1/32 ( 1/2 A1 + 1/2 A2 C • 3/8 A3 C^2 ) A3 SO • 5/
32 ( AO + 1/2 C A1 + 1/4 A2 C^2 • 1/8 A3 C^3 ) A3 + 3/8 ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C^2 ) SO - 1/4 ( 1/2 A2
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                2                 3                                                                                  2
+ 3/4 A3 C ) SO  - 1/16 A3 SO  ) A3 + ( 1/2 A2 + 3/4 A3 C ) ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO  ) + 5/6

                                                    2      5    2
( 1/4 A2 + 3/8 A3 C ) ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ) ZO   SO  + 1/12 ( - 1/32 ( 1/2 A2 + 3/4 A3 C ) A3 SO + 3/

                                                   2                                           2
8 ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO  ) A3 + 3/32 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) A3 + 1/3 ( 1/4 A2 + 3

          2      6     2                        2                                                7     2                2    8    2
/8 A3 C )  ) ZO   SO  + 1/14 ( - 3/256 A3  SO + 7/96 ( 1/4 A2 + 3/8 A3 C ) A3 ) ZO  SO  - 1/4096 A3  ZO  SO

                                               2                                         2                3
+ 1/3 ( ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  - 1/16 A3 ZO  ) ( A0 + 1/2 C

               2              3                                          2                                 2              3
A1 + 1/4 A2 C  + 1/8 A3 C  ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ( - ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) ZO

                                           2     2                                  3                 4
- 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO  - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO  - 1/32 A3 ZO  ) + ( 1/4 A2 + 3/8 A3 C

                              2             3                                               2     2
) ( ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) ZO SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO  SO + 1/3 ( 1/4 A2

                  3                 4                                        2        2
+ 3/8 A3 C ) ZO  SO + 1/32 A3 ZO  SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO SO  + 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO

2   2            3   2                                     3          2    3             4       3
  SO  + 1/16 A3 ZO  SO  + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO  + 1/16 A3 ZO  SO  + 1/32 A3 ZO SO  ) ) ZO SO  + 1/6

                                2             3                                               2    2
( 3/8 ( ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) ZO SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO  SO + 1/3 ( 1/4

                  3                 4                                        2        2
A2 + 3/8 A3 C ) ZO  SO + 1/32 A3 ZO  SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO SO  + 1/4 ( 1/2 A2 + 3/4 A3 C )

  2    2           3    2                                    3           2    3               4
ZO   SO  + 1/16 A3 ZO  SO  + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO SO  + 1/16 A3 ZO  SO  + 1/32 A3 ZO SO  ) A3 + 3/2 ( 1

                                2                                  2             3
/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) + ( 1/4 A2 + 3/8 A3 C ) ( - ( A0 + 1/2 C

               2              3                                           2     2                                 3
A1 + 1/4 A2 C  + 1/8 A3 C  ) SO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) SO  - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO  - 1/32

     4                                                          2             3
A3 SO  ) + ( 1/2 A2 + 3/4 A3 C ) ( - ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) ZO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8

     2     2                                3               4                                        2
A3 C  ) ZO  - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO  - 1/32 A3 ZO  ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ( - 1/2 ( 1/2 A1

                      2                                      2              3     2     3
+ 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  - 1/16 A3 ZO  ) ) ZO  SO  + 1/9 ( 3/8 ( - ( A0 + 1/2

                 2             3                                           2     2                                 3
C A1 + 1/4 A2 C  + 1/8 A3 C  ) SO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) SO  - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO  - 1

          4                                           2              3                                              2    2
/32 A3 SO  ) A3 + 3/8 ( - ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) ZO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO

                           3              4                                                               2
- 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO  - 1/32 A3 ZO  ) A3 + 5/4 ( 1/2 A2 + 3/4 A3 C ) ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3

3                                                           2                                       2
C  ) + ( 1/4 A2 + 3/8 A3 C ) ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) SO - 1/4 ( 1/2 A2 + 3/4 A3 C ) SO  - 1/16

     3                                                          2                                       2
A3 SO  ) + ( 1/4 A2 + 3/8 A3 C ) ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  - 1/

         3                                  2   2      3    3                                                2            3
16 A3 ZO  ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  )  ) ZO   SO  + 1/12 ( 7/16 ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C

                                              2                                       2              3
) A3 + 3/8 ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) SO - 1/4 ( 1/2 A2 + 3/4 A3 C ) SO  - 1/16 A3 SO  ) A3 + 1/8

                                          2                                      2                3
( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  - 1/16 A3 ZO  ) A3 + ( 1/4 A2 + 3/8 A3

                                                  2                                                           2
C ) ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO  ) + 5/6 ( 1/4 A2 + 3/8 A3 C ) ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  )

                                                               2     4    3
+ 3/4 ( 1/2 A2 + 3/4 A3 C ) ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ) ZO  SO  + 1/15 ( - 1/32 ( 1/4 A2 + 3/8 A3 C ) A3

                                                  2                                            2
SO + 3/8 ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO - 1/16 A3 SO  ) A3 + 11/32 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) A3 + 7/12 ( 1

                                              5    3                      2
/2 A2 + 3/4 A3 C ) ( 1/4 A2 + 3/8 A3 C ) ) ZO   SO  + 1/18 ( - 3/256 A3  SO + 3/16 ( 1/4 A2 + 3/8 A3 C ) A3 + 1/16

                                 6    3                 2    7    3                                          2             3
( 1/2 A2 + 3/4 A3 C ) A3 ) ZO   SO  + 5/5376 A3  ZO  SO  + 1/4 ( 1/8 ( ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  )

                                     2    2                                    3                 4
ZO SO + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO  SO + 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO  SO + 1/32 A3 ZO  SO + 1/2 ( 1/

                          2        2                                     2    2             3   2
2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO SO  + 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  SO  + 1/16 A3 ZO  SO  + 1/3 ( 1/4 A2 + 3/8

                3             2    3             4                                                2
A3 C ) ZO SO  + 1/16 A3 ZO  SO  + 1/32 A3 ZO SO  ) A3 + ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO  ) ( A0 + 1

                2             3                                                           2             3
/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) + ( 1/4 A2 + 3/8 A3 C ) ( - ( A0 + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) ZO - 1/2

                          2     2                               3               4
( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO  - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO  - 1/32 A3 ZO  ) + ( 1/2 A1 + 1/2 A2 C + 3/8
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     2                                         2                                   2                3          4
A3 C  ) ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  - 1/16 A3 ZO  ) ) ZO SO  + 1
                                     2              3                                            2     2
/8 ( 1/8 ( - ( AO + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) SO - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) SO  - 1/3 ( 1/4
                      3               4                                           2            3
A2 + 3/8 A3 C ) SC  - 1/32 A3 SO  ) A3 + 3/8 ( - ( AO + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) ZO - 1/2 ( 1/2 A1 + 1/2
                 2     2                                3               4
 A2 C + 3/8 A3 C  ) ZO  - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO  - 1/32 A3 ZO  ) A3 + 4/3 ( 1/4 A2 + 3/8 A3 C ) ( AO + 1/2 C
                 2           3                                                                2
 A1 + 1/4 A2 C  + 1/8 A3 C  ) + ( 1/2 A2 + 3/4 A3 C ) ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2
                 2             3                                      2
 + 3/4 A3 C ) ZO  - 1/16 A3 ZO  ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO
2                                            2  2     2    4                                           2          3
 ) + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  )  ) ZO  SO  + 1/12 ( 7/16 ( AO + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  )
                                             2                                  2               3
A3 + 1/8 ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) SO - 1/4 ( 1/2 A2 + 3/4 A3 C ) SO  - 1/16 A3 SO  ) A3 + 3/8 (
                                      2                                     2              3
 - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  - 1/16 A3 ZO  ) A3 + ( 1/4 A2 + 3/8 A3 C
                                              2                                                             2
 ) ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO  ) + 5/6 ( 1/4 A2 + 3/8 A3 C ) ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) +
                                                  2       3     4
3/4 ( 1/2 A2 + 3/4 A3 C ) ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ) ZO  SO  + 1/16 ( 1/8 ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) SO
             2                                                2                                             2
 - 1/16 A3 SO  ) A3 + 1/8 ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO  ) A3 + 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C
                                 2                          2     4     4                       2
 ) A3 + 2/3 ( 1/4 A2 + 3/8 A3 C )  + 1/4 ( 1/2 A2 + 3/4 A3 C )  ) ZO  SO  + 1/20 ( - 1/256 A3  SO + 25/96 ( 1/4
                                                     5     4                2     6     4
A2 + 3/8 A3 C ) A3 + 5/32 ( 1/2 A2 + 3/4 A3 C ) A3 ) ZO  SO  + 11/6144 A3  ZO  SO  + 1/5 ( - 1/32 ( AO + 1/2 C A1
            2     2                                3               4
 + 3/8 A3 C  ) ZO  - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO  - 1/32 A3 ZO  ) A3 + ( 1/4 A2 + 3/8 A3 C ) ( - 1/2 ( 1/2 A1 + 1
                  2                                   2              3                                    2
/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  - 1/16 A3 ZO  ) + ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ( - 1
                                               2        5                                                2
/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO  ) ) ZO SO  + 1/10 ( - 1/32 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) A3 ZO + 5/32
                          2           3                                                  2
 ( AO + 1/2 C A1 + 1/4 A2 C  + 1/8 A3 C  ) A3 + 3/8 ( - 1/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3
                2             3                                                                               2
/4 A3 C ) ZO  - 1/16 A3 ZO  ) A3 + ( 1/2 A2 + 3/4 A3 C ) ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO  ) + 5/6 ( 1
                                                  2     2     5
/4 A2 + 3/8 A3 C ) ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ) ZO  SO  + 1/15 ( - 1/32 ( 1/4 A2 + 3/8 A3 C ) A3 ZO + 3/8 (
                                           2                                              2
 - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO  ) A3 + 11/32 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) A3 + 7/12 ( 1/2 A2 + 3/4
                                3     5                2
 A3 C ) ( 1/4 A2 + 3/8 A3 C ) ) ZO  SO  + 1/20 ( - 1/256 A3  ZO + 25/96 ( 1/4 A2 + 3/8 A3 C ) A3 + 5/32 ( 1/2 A2 +
                  4     5              2     5     5                                                2
3/4 A3 C ) A3 ) ZO  SO  + 7/3200 A3  ZO  SO  + 1/6 ( - 1/32 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) A3 ZO + 1/8 ( - 1
                                  2                                   2              3
/2 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) ZO - 1/4 ( 1/2 A2 + 3/4 A3 C ) ZO  - 1/16 A3 ZO  ) A3 + ( 1/4 A2 + 3/8 A3 C )
                                            2        5
( - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO  ) ) ZO SO  + 1/12 ( - 1/32 ( 1/2 A2 + 3/4 A3 C ) A3 ZO + 3/8 ( - 1/
                                         2                                                2
3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO  ) A3 + 3/32 ( 1/2 A1 + 1/2 A2 C + 3/8 A3 C  ) A3 + 1/3 ( 1/4 A2 + 3/8 A3 C )
2     2     6                      2                                                                        3     6
 ) ZO  SO  + 1/18 ( - 3/256 A3  ZO + 3/16 ( 1/4 A2 + 3/8 A3 C ) A3 + 1/16 ( 1/2 A2 + 3/4 A3 C ) A3 ) ZO  SO  +
           2     4     6
11/6144 A3  ZO  SO  + 1/7 ( - 1/32 ( 1/4 A2 + 3/8 A3 C ) A3 ZO + 1/8 ( - 1/3 ( 1/4 A2 + 3/8 A3 C ) ZO - 1/16 A3 ZO
2                 7                    2                                            2     7              2     3     7
 ) A3 ) ZO SO  + 1/14 ( - 3/256 A3  ZO + 7/96 ( 1/4 A2 + 3/8 A3 C ) A3 ) ZO  SO  + 5/5376 A3  ZO  SO  - 1/
        2     2     8
4096 A3  ZO  SO
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(30)

CONCLUSION

The availability of the large-scale digital computer, together with the development of symbolic programming languages such as Formac, permits the use of the mathematical solution to problems in compressible fluid flow theory.

It has been shown that the memory capacity of the computer can accommodate many thousands of symbolic terms arising in the formula for the potential function of a compressible fluid flow. Furthermore, the possibility of numerically evaluating these many thousands of terms has been demonstrated.

The results presented in this report suggest that the computational techniques employed here permit the solution to problems in compressible fluid flow theory which could not be accomplished by other means.

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