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SINGULAR EXTREMALOIDS IN OPTIMAL CONTROL THEORY AND THE CALCULUS OF VARIATIONS

by Terry A. Straeter

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NORTH CAROLINA STATE UNIVERSITY
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SINGULAR EXTREMALOIDS IN OPTIMAL CONTROL THEORY AND THE CALCULUS OF VARIATIONS.

By Terry A. Straeter

Research Assistant, North Carolina State University,
Raleigh, North Carolina

The relationships of the various definitions proposed by Kelley, Dunn, Haynes and Hermes of the concept of singularity of an extremaloid obtained from the application of Pontryagin's principle are demonstrated. Also discussed is how the various definitions are related to the definition of a singular extremaloid of a Lagrange problem for those instances where the control problem can be formulated as an equivalent Lagrange problem.

INTRODUCTION

Kelley, Haynes, Hermes and Dunn have proposed various definitions of the concept of singularity as applied to controls that are obtained from an application of Pontryagin's principle to optimal control problems.

It is the purpose of this paper to demonstrate how these definitions are related and, specifically, how they are related to the definition of a singular extremaloid of a Lagrange problem for those instances where the control problem can be formulated as a Lagrange problem.

Section 1 is devoted to a review of the classical problem of Lagrange. In section 2 we have shown that Kelley's ([4]) definition of a singular control is equivalent to the definition of a singular extremaloid of a Lagrange problem if the control region is open. Section 3 exhibits the equivalence of a singular extremal in the Haynes-Hermes sense ([8]) and a singular extremaloid of the corresponding Lagrange problem formed by a transformation of the type discussed by Park ([5]). The same objective is accomplished in section 4 by using slack variables and a formulation of the control problem given by Berkovitz ([6]). Section 5 discusses the relationship between Dunn's definition for a singular extremaloid and the Haynes-Hermes definition in the case of a linear problem. Also in this section the relationship of Dunn's definition and the classical definition for a Lagrange problem is discussed.

ANALYSIS

1. The Classical Problem of Lagrange and the Definition of Singularity in the Calculus of Variations.

The problem of Lagrange is that of finding in the class of piecewise smooth functions $y = (y_1(t), y_2(t), \dots, y_n(t))$ satisfying differential equations of the form

$$\phi_i(t,y,y') = 0, \qquad (i = 1, 2, \dots, m < n),$$

with some or all of the y_i's fixed initially and/or terminally, the one which minimizes the functional $J = \int_{t_1}^{t_2} f(t,y,y')dt$.

Let R denote the open region of the $(2n + 1)$ dimensional (t,y,y')-space in which the functions ϕ_i and f have continuous partial derivatives of at least second order. Suppose $y = y(t)$ is the solution to this problem, all its lineal elements lie in R, and $\frac{\partial\phi_i}{\partial y_j'}$ has rank m along $y = y(t)$. Then, every smooth portion of $y = y(t)$ satisfies the multiplier rule, ([1], [2]), i.e., there is associated with $y = y(t)$ a set of piecewise continuous functions $(\lambda_0, \dots, \lambda_m) \neq (0, 0, \dots, 0)$, so that the Mayer equations

$$h_{y_i} = \frac{d}{dt} h_{y_i'} \qquad i = 1, 2, \dots, n,$$

where $h = -\lambda_0 f + \sum_{i=1}^{m} \lambda_i\phi_i$, are satisfied by every smooth portion of $y = y(t)$. Any smooth portion of $y = y(t)$ which satisfies the above stated multiplier rule is called an <u>extremal arc</u>. $y = y(t)$ itself, when pieced together from extremal arcs is called an <u>extremaloid</u>, and when smooth, an <u>extremal</u>.

<u>Definition 1</u>: An extremaloid E is said to be "Calculus of Variations Regular" if the Jacobian

$$J = \frac{\partial(h_{y'}, \phi)}{(y',\lambda)} \neq 0$$

along E. An extremaloid E defined on an interval I is "Calculus of Variations Singular" if $\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)} = 0$ along E on some subinterval of I. It is well known that if an extremaloid is calculus of variations regular then it has no corners (i.e., is smooth).

We have three classes of extremaloids: 1. Regular, where $J \neq 0$ along the entire extremal. 2. Extremaloids with corners where $J = 0$ only at isolated points and 3. Singular, where $J = 0$ on some subinterval (t_1,t_2).

For future reference

$$\frac{\partial(h_{y'},\phi)}{(y',\lambda)} = \begin{vmatrix} h_{y'y'} & \phi^T_{y'} \\ \phi_{y'} & 0 \end{vmatrix}$$

2. The Problem of Optimal Control and the Pontryagin Principle.

The usual type problem in the theory of optimal control is to minimize $J[u] = \int_{t_0}^{t_f} f_0(x,u,t)dt$ subject to constraining differential equations $x_i' = f_i(x,u,t)$ with some or all components of x specified initially and/or terminally. Here, the $f_i : E^k \times E^r \times E \rightarrow E$, $i = 0, 1, \ldots, k$ are assumed to have continuous second order partial derivatives and $x = (x_1, x_2, \ldots, x_k) \in X$ where X is an open subset of E^k and $u = (u_1, u_2, \ldots, u_r) \subset U$ where U is a given subset of E^r. The necessary condition that a sectionally continuous function $u = u(t)$ with values in U render J a minimum is given by the Pontryagin principle ([2], [3]), namely, that there exist sectionally smooth functions $(\lambda_0,\lambda(t)) \in E^{k+1}$ with $(\lambda_0,\lambda(t)) \neq 0$ and where λ_0 is a constant such that if

$$H(x,u,\lambda,t) = \lambda_0 f_0 + \sum_{i=1}^{k} \lambda_i f_i$$

then (1) $H(x,\tilde{u},\lambda,t) \leq H(x,u,\lambda,t)$ for all $\tilde{u} \in U$

(2) $\lambda' = - H_x$

(3) $\lambda_0 \leq 0$

We omit transversality conditions as they are not involved in any of the following discussion concerning singular arcs.

It is clear that if the set U given above is open then the preceeding problem can be considered as a Lagrange problem with $\phi_i \equiv x_i' - f_i(x,u,t)$. It has been shown that in this case the maximum principle implies the multiplier rule and Weierstrass' necessary condition for the Lagrange problem [2] and [3].

For this problem we have

$$h = -\lambda_0 f\ (x,u,t) + \sum_{i=1}^{k} \lambda_i (x_i' - f_i(x,u,t))$$

where we now have $h = h(x,u,t,\lambda)$ instead of $h = h(y,y',t,\lambda)$. Then

$$h_{y'y'} = \begin{vmatrix} h_{x'x'} & h_{x'u} \\ h_{ux'} & h_{uu} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & h_{uu} \end{vmatrix}$$

where we let $(y_1', \ldots, y_n') = (x_1', \ldots, x_k', u_1, \ldots, u_r)$ and

$$\phi_{y'} = (I_{k \times k} \mid -\frac{\partial f}{\partial u}).$$

Hence

$$\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)} = \begin{vmatrix} 0 & 0 & I \\ 0 & h_{uu} & -\frac{\partial f}{\partial u} \\ I & -\frac{\partial f}{\partial u} & 0 \end{vmatrix}$$

which is different from zero iff $\det |h_{uu}| \neq 0$. Since $|\det |h_{uu}|| = |\det |H_{uu}||$ we call a solution of the maximum principle regular in the classical sense if $\det |H_{uu}| \neq 0$ ([4]).

3. The concept of a singular arc in the case of the linear optimal control problem with a right parallelepiped as control region.

Suppose that the system of constraining equations for the optimal control problem of section 2 is of the form $x' = A(x,t) + B(x,t)u$ where x and u are k and r vectors respectively, $A(x,t)$ is a n vector valued function, and $B(x,t)$ is a $k \times r$ matrix valued function satisfying suitable conditions so that $A(x,t) + B(x,t)u$ satisfies the hypotheses of section 2. Further let U be of the form $a_i \leq u_i \leq b_i$ (i.e., a right parallelepiped). Suppose further that $f_0(x,u,t) = a(x,t) + b_0(x,t) \cdot u$ where $a(x,t)$ is a real valued function and $b_0(x,t)$ is a vector valued function which satisfy the differentiability requirements set forth in section 2. Under these conditions the Hamiltonian H is linear in u and the optimal control (if it exists) is necessarily of the form

$$u_i = \frac{1}{2} [\mathrm{sgn}(s_i(x,\lambda,t)) - 1][b_i - a_i] + b_i$$

when $s_i(x,\lambda,t) \neq 0$, where $s_i(x,\lambda,t) = \lambda B_i + \lambda_0 b_{0i}$ where B_i is the i^{th} column of the B matrix, ($s_i(x,\lambda,t)$ is called the switching function), and

$$\mathrm{sgn}(\alpha) = \begin{cases} 1 & \alpha > 0 \\ -1 & \alpha < 0 \\ \text{undefined for } \alpha = 0. \end{cases}$$

In 1963, Haynes and Hermes published a precise definition of singular arcs for this class of problems ([8]). Since we are interested in piecewise continuous controls the definition stated below for a singular control is modified accordingly.

Definition 2: (Haynes-Hermes) Let $\Gamma(\ell)$ denote the set $\Gamma(\ell) = \{(\lambda,x,t) | s_\ell(\lambda,x,t) = 0\}$ and let $\Gamma = \bigcup_{\ell=1}^{r} \Gamma(\ell)$. Then an extremaloid $(\lambda(t), x(t))$ given on an interval I is said to be "singular" if the set $\beta = \{t \mid t \in I \text{ and } (\lambda(t),x(t),t) \in \Gamma\}$ contains an open interval.

Suppose we let $\frac{(a_i + b_i)}{2} + \frac{(b_i - a_i)}{2} \sin y'_{i+k} = u_i$ for $i = 1, 2, \ldots, r$ and let $y_i = x_i$, $i = 1, 2, \ldots, k$. Then the problem has been transformed into a classical problem of Lagrange [5] and the multiplier rule yields as a necessary condition that

$$y'_{i+k} = \frac{\Pi}{2}\,\mathrm{sgn}(s_i(x,\lambda,t)) \quad \text{whenever} \quad s_i \neq 0 \tag{3.1}$$

where $s_i(x,\lambda,t)$ is as given above.

Let us consider the Jacobian $\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)}$ for this Lagrange problem. Here we have

$$h_{y'y'} = \begin{pmatrix} 0 & 0 \\ & \\ 0 & h_{y'_{i+k}y'_{j+k}} \end{pmatrix}$$

$$\phi_{y'} = \left(I_{k\times k} \;\middle|\; -\frac{\partial(a+A)}{\partial y'_{i+k}}\right) \equiv (I \mid F)$$

so

$$\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)} = \begin{pmatrix} 0 & 0 & I \\ 0 & (h_{y'_{i+k},y'_{j+k}}) & F^T \\ I & F & 0 \end{pmatrix}.$$

This determinant is equal to det $\left[h_{y'_{i+k}},\ h_{y'_{j+k}}\right]$ and $h_{y'_{i+k},y'_{j+k}} = -\delta_{ij}\, s_i(x,\lambda,t)\left(\frac{(b_i - a_i)}{2}\right)\sin y'_{i+k}$. That is $(h_{y'_{i+k},\ y'_{j+k}})$ is a diagonal matrix with elements $-\, s_i(x,\lambda,t)\sin y'_{i+k}\left[\frac{b_i - a_i}{2}\right]$. Hence

$$\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)} = (-1)^r \prod_{i=1}^{r} s_i(x,\lambda,t)\sin y'_{i+k}\left[\frac{b_i - a_i}{2}\right] \tag{3.2}$$

Suppose an extremaloid of the linear control problem is singular in the Haynes-Hermes sense, then we have one of the s_i (i.e., switching function) is zero on some open interval. Notice that if one of the s_i is zero on an interval the Jacobian $\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)} = 0$ by (3.2). Moreover we see that $\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)}$ is zero whenever $s_i(x,\lambda,t) = 0$ for some i. And if no $s_i(x,\lambda,t) = 0$ then by using (3.1) we see $\sin(y'_{i+k}) = \pm 1 \neq 0$, which using (3.2) implies that $\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)} \neq 0$.

Theorem 1: $\frac{\partial(h_{y'},\phi)}{\partial(y',\lambda)} = 0$ if and only if $s_i(x,\lambda,t) = 0$ for some $i = 1, 2, \ldots, r$.

Corollary The concept of a singular extremaloid by Haynes-Hermes for the control problem linear in u with a right parallelepiped as control region and for the associated Lagrange problem are equivalent.

4. Slack Variables and the linear control problem with a closed parallelepiped as a control region.

The linear control problem discussed in section 3 can be transformed into an equivalent Lagrange problem by introducing slack variables. The necessary conditions for optimality in this case are given by Berkowitz ([6]). We shall use his notation and let $u_i = y_i'$ $i = 1, 2, \ldots, r$ and we define

$$R_i(y') = y_i' - a_i$$
$$R_{i+r}(y') = b_i - y_i'.$$

So the constraining differential equations become in terms of the slack variables ξ

$$\begin{aligned} x' - A(x,t) - B(x,t)y' &= 0 \\ \xi_i'^2 - R_i &= 0 \end{aligned} \tag{4.1}$$

for $i = 1, 2, \ldots, r, r+1, \ldots, 2r$. The h function is given by

$$h = -\lambda_0(a(x,t) + b_0(x,t) \cdot y') + \lambda \cdot (x' - A(x,t) - B(x,t)y') + \sum_{i=1}^{2r} \mu_i(\xi_i'^2 - R_i),$$

where the λ's are the multipliers associated with the first k constraining equations and the μ's those associated with the last 2r equations in (4.1).

Now the Jacobian which determines regularity is in this case

$$S = \frac{\partial(h_{x'},h_{y'},h_{\xi'},\ x' - A(x,t) - B(x,t)y',\ (\xi_i'^2 - R_i))}{\partial(x',y',\xi',\lambda,\mu)}$$

By taking the appropriate partials we see that

$$S = \begin{array}{c|c|c|c|c|c|c|c|}
 & k & r & r & r & k & r & r \\
\hline
k & 0 & 0 & 0 & 0 & I_k & 0 & 0 \\
\hline
r & 0 & 0 & 0 & 0 & -B & -I_r & +I_r \\
\hline
r & 0 & 0 & d(2\mu_i) & 0 & 0 & d(2\xi'_i) & 0 \\
\hline
r & 0 & 0 & 0 & d(2\mu_{i+r}) & 0 & 0 & d(2\xi'_{i+r}) \\
\hline
k & I_k & -B & 0 & 0 & 0 & 0 & 0 \\
\hline
r & 0 & -I_r & d(2\xi'_i) & 0 & 0 & 0 & 0 \\
\hline
r & 0 & +I_r & 0 & d(2\xi'_{i+r}) & 0 & 0 & 0 \\
\hline
\end{array}$$

where $d(\alpha_i)$ denotes a diagonal matrix with the α_i as diagonal entries.

We expand S with respect to the first k rows and the first k columns and obtain

$$S = \begin{vmatrix}
0 & 0 & 0 & -I_r & I_r \\
0 & d(2\mu_i) & 0 & d(2\xi'_i) & 0 \\
0 & 0 & d(2\mu_{i+r}) & 0 & d(2\xi'_{i+r}) \\
-I_r & d(2\xi'_i) & 0 & 0 & 0 \\
I_r & 0 & d(2\xi'_{i+r}) & 0 & 0
\end{vmatrix}$$

Next, add the $4r + j^{th}$ column to the $3r + j^{th}$ column and do the same for the corresponding rows, $j = 1, 2, \ldots, r$; and we have

$$S = \begin{vmatrix}
0 & 0 & 0 & 0 & I_r \\
0 & d(2\mu_i) & 0 & d(2\xi'_i) & 0 \\
0 & 0 & d(2\mu_{i+r}) & d(2\xi'_{i+r}) & d(2\xi'_{i+r}) \\
0 & d(2\xi'_i) & d(2\xi'_{i+r}) & 0 & 0 \\
I_r & 0 & d(2\xi'_{i+r}) & 0 & 0
\end{vmatrix}$$

Expanding with respect to the 1st r rows and then the 1st r columns we obtain

$$S = \begin{vmatrix} d(2\mu_i) & 0 & d(2\xi_i') \\ 0 & d(2\mu_{i+r}) & d(2\xi_{i+r}') \\ d(2\xi_i') & d(2\xi_{i+r}') & 0 \end{vmatrix} \tag{4.2}$$

From the multiplier rule, it is necessary that

$$y_i' = \frac{1}{2}(\mathrm{sgn}\,(s_i) - 1)(b_i - a_i) + b_i \tag{4.3}$$

when $s_i \neq 0$ where s_i are the switching functions as defined in section 2. Berkowitz showed that $h_{y'} = 0$; that is, $-(\lambda_0 b + \lambda B) + \mu R_{y'} = 0$ since $R_{y'} = (I_r \mid - I_r)$ and $-(\lambda_0 b_i + \lambda \circ B_i) = - s_i$, we can say that for $i = 1, 2, \ldots, r$

$$- s_i + \mu_i - \mu_{i+r} = 0 \tag{4.4}$$

Now suppose $s_i \neq 0$ for $i = 1, 2, \ldots, r$. We can assume without loss of generality that $s_i > 0$ for $i = 1, 2, \ldots, r$. So then (4.3) implies that $y_i' = b_i$ and $R_{i+r} = 0$ and $R_i = b_i - a_i \neq 0$. Berkowitz has shown that it is necessary that $\mu_i R_i = 0$. So, $\mu_i = 0$ and (4.4) imply that $\mu_{i+r} = - s_i < 0$. And since $\xi_i'^2 - R_i = 0$ we have $\xi_i' = \sqrt{b_i - a_i} \neq 0$, and $\xi_{i+r}' = 0$. So (4.2) tells us that

$$S = \begin{vmatrix} 0 & 0 & d(2\xi_i') \\ 0 & d(-2s_i) & 0 \\ d(2\xi_i') & 0 & 0 \end{vmatrix} \neq 0.$$

Conversely assume $s_j = 0$ for some j. Since by definition both R_j and R_{j+r} cannot be zero and, $\mu_j R_j$ and $\mu_{j+r} R_{j+r}$ must be zero, we have either μ_{j+r} or $\mu_j = 0$. But (4.4) with $s_j = 0$ implies $\mu_{j+r} = \mu_j = 0$. Then in the determinant (4.2) the j^{th} row and the $(j + r)^{th}$ row are linearly dependent. Hence $S = 0$.

Theorem 2: A control is singular in the sense of Haynes-Hermes if and only if the corresponding extremaloid in the Berkowitz formulation of the control problem as a Lagrange problem is singular in the calculus of variations sense.

5. Dunn Definition of a singular arc.

Recently Dunn published a classification of controls that are obtained from Pontryagin's Principle ([7]). We shall state here the restriction of his definition to the case of piecewise continuous controls. In order to simplify the expressions we make the following slight change of notation: let $x = (x_0, x_1, \ldots, x_n)$ where $x_0' = f_0(x,u,t)$.

<u>Definition 3</u>: A pair of functions (x,λ) given on an interval I is said to be an <u>extremaloid of the maximum principle on I</u> if and only if (a) the trajectory $x = x(t)$ is generated by an admissible control $u = u(t)$ (i.e., piecewise continuous with range in U) (b) $\lambda = \lambda(t)$ satisfy $\lambda' = -H_x(x,\lambda,u)$. (c) (u,x,λ) satisfy the maximum principle on I.

Let C denote the class of functions $c : E^k \times E^k \times E$ into E^r satisfying

$$H(x,\lambda,t,c(x,\lambda,t)) = \sup_{u \in U} H(x,\lambda,t,u) \tag{5.1}$$

identically on $E^k \times E^k \times E^1$ and let $\mathcal{D}$ denote the corresponding class of systems of differential equations

$$\begin{aligned} \lambda' &= -H_x(x,\lambda,t,c(x,\lambda,t)) \\ x' &= H_\lambda(x,\lambda,t,c(x,\lambda,t)) \end{aligned} \tag{5.2}$$

If N is a neighborhood in the (x,λ,t) space, let $C(N)$ denote the class of functions $c : N \to E$ satisfying (5.1) and let $\mathcal{D}(N)$ denote the corresponding class of ordinary differential equations defined on N.

<u>Definition 4</u>: A point $p : (x,\lambda,t) \in E^k \times E^k \times E^1$ is said to be a <u>singular point</u> if every neighborhood N of p contains a point q (possibly p itself) at which two or more members of the class $\mathcal{D}(N)$ are distinct. Conversely, if there is some neighborhood N^* of $\{p\}$ such that $\mathcal{D}(N^*)$ consists of exactly one member, then

p is said to be a regular point. The set Q of all singular points is called the singular set. The set R of all regular points is called the regular set.

As an immediate consequence of the above definition we have

<u>Theorem 3</u>: If $\mathcal{D}$ is non-empty then

(a) $Q \cap R = \emptyset$

(b) $Q \cup R = E^k \times E^k \times E$

(c) R is open

(d) Q is closed

(This is theorem 2 in Dunn's paper.)

<u>Definition 5</u>: An extremal (x,λ) defined on some interval is said to be regular if and only if it lies entirely in the regular set R.

<u>Definition 6</u>: An extremaloid (x,λ) on I is said to be singular if and only if the set $\alpha = \{t \mid t \in I, (x(t), (t,t)) \in Q\}$ contains an open subinterval.

So we have three categories of extremaloids

(1) Regular, those for which $\alpha = \emptyset$.

(2) Extremaloids where $\alpha \neq \emptyset$ and α contains a finite number of points.

(3) Singular extremaloids where $\alpha \neq \emptyset$ and $(t', t'') \subseteq \alpha$ for some $t' < t''$.

The question naturally arises how Dunn's scheme of classifying extremaloids is related to the other methods. First we consider the case of the linear optimization problem with a closed right parallelepiped as control region. There, the Hamiltonian is linear in u. This is the problem defined in section 2 and discussed as a Lagrange problem in sections 3 and 4.

<u>Theorem 4</u>: For the linear control problem of section 2, we have $C\Gamma \subset R$ where Γ is defined in definition 2 of section 3 and where R is Dunn's regular set.

Proof: $C\Gamma = \{(\lambda,x,t) \mid s_\ell(\lambda,x,t) \neq 0,\ \ell = 1, 2, \ldots, r\}$. If $(x,\lambda,t) \in C\Gamma$, then, (3.1) defines a unique system of canonical equations and $(x,\lambda,t) \in R$.

Corollary: $Q \subset \Gamma$ where Q is Dunn's singular set.

Proof: Since $C\Gamma \subset R$, we have $CR \subset \Gamma$ and the result follows from $Q = CR$.

Remark: The corollary states that if an extremaloid is singular in Dunn's sense, then it is also singular in the Haynes-Hermes sense.

It is shown by Dunn that the converse of theorem 4 is not true. This is seen by having $B(x,t) \overset{\Delta}{=} 0$ in the vector equation $x' = A(x,t) + B(x,t)u$ on some neighborhood N of the (x,t) space. Then it would be true that $s_i(\lambda,x,t) = \lambda \cdot B_i(x,t) = 0$ on $E^k \times N_1$ so $E^k \times N \subset \Gamma$. But the canonical equations would be uniquely given by

$$x' = A(x,t)$$

$$\lambda' = -\left(\frac{\partial A(x,t)}{\partial x}\right)^T \lambda$$

on N, hence $E^k \times N \subset R$, so the converse of theorem 4 does not hold in this case. However notice that for this example the system $x' = A(x,t) + B(t,x)u$ is no longer underdetermined on N.

If we consider linear optimization problems which are underdetermined with respect to each u_i for $i = 1, 2, \ldots, r$ for every (x,t), then while the converse of Theorem (4) is not generally true we do have the following theorem:

Theorem 5: If in the linear optimization problem we have $\lambda_0 \neq 0$ for all (x,t), and the columns of the matrix $\overline{B}(x,t) = \begin{pmatrix} b_0^T(x,t) \\ B(x,t) \end{pmatrix}$ are all not zero, then $\Gamma = Q$.

Proof: Suppose $(x,\lambda,t) \in \Gamma$. This implies that for some i, $i = 1,2,\ldots,r$, $s_i(x,\lambda,t) = 0$. Since $\sum_{j=0}^{n} b_{ij}(x,t)\,\lambda_j = s_i(x,\lambda,t) = 0$, the Hamiltonian is independent of u_i

hence any value of u_i satisfies the Pontryagin principle. But by hypothesis $(b_{i1}, b_{i2}, \ldots, b_{in}) \neq 0$ for all (x,t). So if $b_{ij} \neq 0$ for some $j = 0, 1, 2, \ldots, r$ the system $x' = -H_\lambda$ is not unique because the j^{th} equation has an undetermined term u_j in it. So the canonical equations are not uniquely determined at (λ,x,t), hence $(\lambda,x,t) \in Q$.

Corollary 1: $R = C\Gamma$.

Corollary 2: With the hypothesis of theorem (5), if an extremaloid of the linear optimization problem is singular in the Haynes-Hermes sense, then it is singular in the Dunn sense.

For the general control problem, Dunn ([7]) has shown that under certain conditions an extremal which is regular in the calculus of variations sense is regular in the Dunn sense. His theorem is

Theorem 6: Every extremal $\{x,\lambda\}$ which is regular in the calculus of variations sense is regular in the Dunn sense, if it satisfies a neighborhood form of the Weierstrass condition, i.e., the function $c = c(\tilde{x},\tilde{\lambda},t)$ which satisfies $H(\tilde{x},\tilde{\lambda},t,c(\tilde{x},\tilde{\lambda},t)) = \sup_{u \in U} H(\tilde{x},\tilde{\lambda},t,u)$ is such that $c \in C(N)$ for some neighborhood N of $(\tilde{x},\tilde{\lambda})$ and has continuous first partials on N.

Clearly the converse of theorem 6 need not be true. This follows since an extremal which is non-singular in the Dunn sense must satisfy the Pontryagin principle, that is $H(x,\lambda,u,t)$ is a maximum with respect to u. However the matrix H_{uu}, if it exists at all, is by necessity only negative <u>semi</u>-definite at the maximum. Hence H_{uu} is possibly singular at the maximum.

CONCLUDING REMARKS

We have shown for the linear optimization problem the equivalence of a singular extremal in the Haynes-Hermes sense and a singular extremaloid of the equivalent Lagrange problem formed either by a transformation of the type discussed by Park or the introduction of slack variables. Also we have shown that the Dunn definition of singularity implies the Haynes-Hermes definition, and the converse is true if the system is underdetermined with respect to each control variable for all (x,t). For the nonlinear case, if an extremal of a Lagrange problem is nonsingular and it satisfies a neighborhood form of the Weierstrass condition, then it is regular in the Dunn sense. But the converse is, in general, not true.

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