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GAUSS QUADRATURE RULES INVOLVING SOME NONCLASSICAL WEIGHT FUNCTIONS

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SYMBOLS

b_n	parameter in recurrence relation (see eq. (15))
$e_k^{(n)}$	modified differences in Q-D scheme
E_N	error term of a quadrature formula (see eq. (2))
g_n	parameter in recurrence relation (see eq. (15))
k_N	coefficient in the error term (see eq. (3))
M_n	nth moment of weight function $w(t)$ (see eq. (11))
N	number of nodes in quadrature rule (see eq. (1))
$q_k^{(n)}$	modified quotients in Q-D scheme
S	significant figures
t_{jN}	$j = 1(1)N$, nodal points in quadrature formula (see eq. (1))
$w(t)$	weight function
W_{jN}	$j = 1(1)N$, weight coefficients in quadrature rule (see eq. (1))
$\Gamma(z)$	gamma function of argument z
$\phi_k(t)$	orthogonal polynomial of degree k (see eq. (15))

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SUMMARY

Gauss quadrature rules are given for numerically evaluating integrals on the interval [0, 1] for some weight functions of the form $w(t) = t^{\gamma}(1 - t^{\alpha})^{\beta}$, where α is a positive integer ≥ 2 and β, $\gamma > -1$. The nodes, weight coefficients, and the parameters in the three-term recurrence relation for the sequence of orthogonal polynomials associated with $w(t)$ are tabulated to 25 significant figures for N-point rules with N = 2, 4, 6, 8, 12, 16, and 24. Numerical tables to 5S are also given for a coefficient in the expression of the error term.

INTRODUCTION

A wide variety of problems requiring numerical approximations to specific definite integrals arise very often in practice, usually as an intermediate step in a more extensive computation. There are many methods for approximating such integrals, but among all quadrature formulas of the form

$$I = \int_a^b g(t)dt \equiv \int_a^b w(t)f(t)dt \simeq \sum_{j=1}^{N} W_{jN}f(t_{jN}) \tag{1}$$

only the weighted gaussian types have all their parameters chosen to maximize performance. For these formulas, the nodes t_{jN} and the coefficients W_{jN} are uniquely determined so that all approximations to the given integral are exact for arbitrary polynomials f of the highest possible degree, $2N - 1$. If the function f is other than a polynomial of degree $\leq 2N - 1$, the integration formulas (1) have an error term given (e.g., see ref. 1) by

$$E_N = \frac{k_N}{(2N)!} f^{(2N)}(\tau) , \qquad a < \tau < b \tag{2}$$

where

$$k_N = \int_a^b w(t)\phi_N^2(t)dt \tag{3}$$

with ϕ_N being the associated Nth-degree orthogonal polynomial.

The literature concerning weighted gaussian formulas is very large, both classical and recent (refs. 2, 3, and 4). Widespread use of high-speed computers, with emphasis on economizing the number of operations required in computations, has resulted in considerable renewed interest in the power of the gaussian technique; gauss quadrature rules are being tabulated and used with increasing frequency. Very accurate rules, for example, were recently tabulated in references 1 and 5 for cases associated with the classical orthogonal polynomials of Legendre, Laguerre, Chebyshev, Jacobi, and Hermite. Recent numerical tables to many significant figures have also been given for some nonclassical cases such as a weight function on [0, 1] having logarithmic singularities (ref. 6) and the weight function e^{-x^2} on the interval $[0, \infty]$ (ref. 7).

Weighted gaussian integration formulas are excellent for large classes of functions arising in practice. They are particularly useful when a complete integrand $g(t)$ is factorable into two functions $f(t)$ and $w(t)$, where f is well approximated by a polynomial, with the nonnegative weight function w having a singularity or a singular derivative at one or both ends of the interval. Since neither the error term (2) nor the sum in the approximations (1) directly involves the weight function w, once the quadrature rules are constructed and programmed on a computer, they can be repeatedly applied to integrals of the form (1) with different functions $f(t)$ and the same $w(t)$. There are several other advantages of gauss-type formulas: (a) since all the coefficients W_{jN} turn out to be positive, the rules have good roundoff properties; (b) the sequence of formulas (1) for $N = 1, 2, \ldots$ will always converge to the value of the Riemann integral I whenever this integral exists; (c) the rules have more favorable error terms and generally achieve higher accuracy with the use of fewer nodes than Newton-Cotes methods.

This report presents tabulated gaussian quadrature formulas for

$$I(\alpha, \beta, \gamma; f) = \int_0^1 w(\alpha, \beta, \gamma; t)f(t)dt \approx \sum_{j=1}^{N} W_{jN}f(t_{jN}) \tag{4}$$

where the weight function w is of the form

$$w(\alpha, \beta, \gamma; t) \equiv w(t) = t^{\gamma}(1 - t^{\alpha})^{\beta} \tag{5}$$

with α as a positive integer $\geqslant 2$; $\beta, \gamma > -1$; and w or its derivative singular at one or both ends of the interval. For the special case $\alpha = 1$, the approximate relation (4) becomes the classical Gauss-Jacobi formula

$$\int_0^1 t^{\gamma}(1 - t)^{\beta}f(t)dt \approx \sum_{j=1}^{N} W_{jN}f(t_{jN}) \tag{6}$$

Krylov, Lugin, and Janovic (ref. 8) recently gave these quadrature rules to eight significant figures for $\beta, \gamma = -0.9(0.1)3.0$, $\beta \leqslant \gamma$ and $N = 1(1)8$.

We shall consider various cases where the parameters (α, β, γ) are

$$\left.\begin{aligned} \alpha &= 2, 3, 4, 6, 8 \\ \beta &= -3/4, -2/3, -1/2, 1/2 \\ \gamma &= -1/2, 0, 1/2 \end{aligned}\right\} \tag{7}$$

One of these, for example, is the integral

$$I(3, -1/2, 0; f) = \int_0^1 \frac{f(t)}{\sqrt{1 - t^3}} dt \tag{8}$$

which, when $f(t)$ is simply a rational function, may be reduced by a substitution involving Jacobian elliptic functions to a combination of three standard tabulated elliptic integrals (ref. 9). A similar statement applies to integrals $I(\alpha, \beta, \gamma; f)$ with regard to the other values α, β, γ given in equations (7).

If f is not a rational function (e.g., e^{-ct}), none of the integrals $I(\alpha, \beta, \gamma)$ are expressible in finite terms of known tabulated functions for the chosen parameters (α, β, γ), and the only recourse is numerical approximation via some quadrature formula. Even when such integrals can be written as combinations of standard elliptic integrals etc., it is probably more efficient to use quadrature rules having the highest degree of algebraic precision to obtain answers. Integrals with our weight functions $w(t)$ occur sufficiently frequently in practice to warrant the construction of such rules.

Numerical tables for N-point gaussian rules $\{W_{jN}, t_{jN}\}$ with $N = 2, 4, 6, 8, 12, 16$, and 24 are given to 25 significant figures for 12 special cases of $I(\alpha, \beta, \gamma; f)$. These are for

$$(\alpha, \beta, \gamma) = \left\{\begin{matrix} (3, -1/2, 0) & (3, 1/2, 0) & (4, -1/2, 0) & (4, 1/2, 0) \\ (6, -1/2, 0) & (6, 1/2, 0) & (8, -1/2, 0) & (8, 1/2, 0) \\ (3, -1/2, -1/2) & (3, 1/2, 1/2) & (2, -3/4, 0) & (2, -2/3, 0) \end{matrix}\right\} \tag{9}$$

The coefficients b_n, g_n in the three-term recursion relation (15) for the sequence of monic orthogonal polynomials corresponding to $w(t)$ are also tabulated to 25S for $N = 1(1)26$. Since the error coefficients k_N in the remainder term (2) may be useful when a bound on the 2Nth derivative can be estimated, we give tabulations of them to 5S.

Little difficulty is entailed in the actual construction of gauss-type quadrature formulas relative to classical weight functions where the three-term recurrence relation for the associated orthogonal polynomials are known explicitly. This is not generally true for nonclassical $w(t)$ such as ours where the sequence of related unknown orthogonal polynomials must be generated using the moments of w.

GENERAL DISCUSSION OF CONSTRUCTION

For a given nonnegative weight function $w(t)$ on [0, 1], the problem of constructing the N-point gaussian quadrature

$$G_N(f) = \sum_{j=1}^{N} W_{jN} f(t_{jN}) = \int_0^1 w(t)f(t) - E_N(f) , \qquad (N = 1, 2, \ldots) \tag{10}$$

consists in determining the nodes t_{jN} and the weight coefficients W_{jN} ($2N$ parameters in all) so that the error $E_N(f) = 0$ whenever f is a polynomial of degree $n \leqslant 2N - 1$. There are essentially only two approaches to the determination of these unknown parameters, and they shall be discussed briefly.

Before proceeding, however, we wish to emphasize in the strongest possible terms the numerical difficulties inherent in the construction of the quadrature rule (10) from the moments

$$M_k = \int_0^1 t^k w(t)dt , \qquad k = 0(1)2N - 1 \tag{11}$$

Gautschi (ref. 4) has shown that an approximate upper bound to the relative, asymptotic condition number of this problem is

$$C_N \doteq \frac{(17 + 12\sqrt{2})^N}{64N^2} \tag{12}$$

This means that the computation is extremely ill-conditioned numerically, and that the number of significant figures lost in determining the rule (10) using the moments (11) is approximately

$$S_N = \log_{10} C_N = 1.53N - 2 \log_{10}(8N) \tag{13}$$

Thus, if one wants to have the nodes and weights of G_N accurate to k significant figures, then he must work with more than

$$k + 1.53N - 2 \log_{10}(8N)$$

significant figures. For example, to obtain a 10-point quadrature rule with nodes and weights accurate to 10 significant digits (i.e., $N = k = 10$), one must work with more than 22 significant digits at a certain stage in the calculation of the nodes and weights.

Algebraic Approach

If the nodes t_{jN} and the weight coefficients W_{jN} in equation (10) are chosen so that the formula is exact for all polynomials $f(t)$ of degree $n \leqslant 2N - 1$, then it is sufficient to choose the formula to be exact for the monomials t^k, $k = 0(1)2N - 1$, so that one has the system

$$\sum_{j=1}^{N} W_{jN}\, t_{jN}^{k} = M_k , \qquad k = 0(1)2N - 1 \tag{14}$$

of $2N$ nonlinear algebraic equations in $2N$ unknowns to solve, where M_k are given by equation (11). The drawback to this formulation of determining the quadrature rule from the moments is that any attempt at a direct solution requires the highly precise arithmetic indicated by equation (13) at every step of the calculations. For this reason, the algebraic approach is to be rejected except for very small N.

Approach Via Orthogonal Polynomials

Orthogonal polynomials play an important role in gaussian quadratures. In fact, it can easily be shown that if the nodal points t_{jN} in formula (10) are selected to be the roots of the Nth-degree polynomial $\phi_N(t)$ of the sequence of orthogonal polynomials $\{\phi_n(t)\}$ associated with the given weight function $w(t)$, then the formula is exact for any polynomial $f(t)$ of degree $\leqslant 2N - 1$; the converse of this is also true (e.g., see ref. 10).

The orthogonal system of polynomials associated with $w(t)$ in (a, b) is unique and any three consecutive polynomials satisfy the recurrence relation

$$\phi_n(t) = (t - b_n)\phi_{n-1}(t) - g_n\phi_{n-2}(t) , \qquad n = 1(1)N \tag{15}$$

with $\phi_{-1}(t) = 0$, $\phi_0(t) = 1$, and

$$\left.\begin{aligned} b_n &= \frac{\int_0^1 tw(t)\phi_{n-1}^2(t)dt}{\int_0^1 w(t)\phi_{n-1}^2(t)dt} \\ g_{n+1} &= \frac{\int_0^1 w(t)\phi_n{}^2(t)dt}{\int_0^1 w(t)\phi_{n-1}^2(t)dt} \end{aligned}\right\} \tag{16}$$

The usual approach in the construction of gaussian rules is based on the use of relation (15) and the moments to generate the sequence of polynomials $\{\phi_k(t)\}$. After these are calculated, one solves, in some way, for the roots t_{jN} of $\phi_N(t)$, and then finds the weight coefficients W_{jN} by other means. However, any procedure that relies on the moments is essentially equivalent to solving the basic algebraic system (14) and is highly unstable numerically.

The advantage of the approach via orthogonal polynomials, as we shall see, is that the ill-conditioned nature of the problem is encountered in the determination of the three-term recurrence relation from the moments, but at no point in the determination of the nodes and weights of the quadrature rule. Since the calculation of b_n, g_n, $n = 1(1)N$, needs less than $4N^2$ arithmetic operations, the amount of multiply precise arithmetic is small and manageable, thus making this approach very tractable in comparison to the algebraic approach.

NUMERICAL METHODS

The numerical construction of our gaussian rules proceeds in three steps: (1) Determine, to sufficient accuracy, the moments M_k for $k = 0(1)2N - 1$; (2) using the moments, calculate the parameters b_k, g_k, $k = 1(1)N$, in the three-term recurrence relation (15) for the first N orthogonal polynomials associated with the weight function $w(t)$ on $(0, 1)$; (3) determine the quadrature nodes t_{jN} and the weight coefficients W_{jN} for $j = 1(1)N$. The three steps will be discussed in that order.

Determination of the Moments

The moments for the weight function of interest are

$$M_n(\alpha, \beta, \gamma) \equiv M_n = \int_0^1 t^n w(t)dt = \int_0^1 t^{n+\gamma}(1 - t^\alpha)^\beta \, dt \tag{17}$$

for $n = 0(1)2N - 1$. If we let $z = t^\alpha$, equation (17) becomes

$$M_n = \frac{1}{\alpha}\int_0^1 z^{(n+\gamma+1)/(\alpha-1)}(1-z)^{(1+\beta)-1}\,dz \ , \qquad (\alpha > 0;\ \beta,\ \gamma > -1)$$

or

$$M_n = \frac{1}{\alpha}\,B\left(n + \frac{\gamma + 1}{\alpha},\ 1 + \beta\right) = \frac{\Gamma(n + \gamma + 1/\alpha)\Gamma(1 + \beta)}{\alpha\Gamma[1 + \beta + (n + \gamma + 1/\alpha)]} \tag{18}$$

where B and Γ are the beta and gamma functions, respectively. Using the well-known recurrence relation for the gamma function, we have the recursion relation

$$M_k = \frac{M_{k-\alpha}}{1 + [\alpha(1 + \beta)/(k + \gamma - \alpha + 1)]} \ , \qquad k = \alpha(1)2N - 1 \tag{19}$$

so that all the moments can be generated from $M_0, M_1, \ldots, M_{\alpha-1}$.

Calculation of Parameters in the Three-Term Recurrence Relation

There are several methods of calculating the three-term recurrence relation from the moments in the general case (refs. 1 and 11). However, for several reasons, mainly because of the volume of arithmetic involved, we have chosen a more specialized approach. We use the quotient-difference algorithm (ref. 12) to obtain the three-term recurrence relation parameters from the moments. This method will always work when the interval of integration does not have the origin as an interior point. The procedure is briefly as follows: Define

$$e_o^{(j)} = 0 \ , \qquad q_1^{(j)} = \frac{M_{j+1}}{M_j} \ , \qquad j = 0(1)2N - 2 \tag{20}$$

Generate the rest of the table

$$\begin{array}{cccccc}
q_1^{(0)} & & & & & \\
 & e_1^{(0)} & & & & \\
q_1^{(1)} & & q_2^{(0)} & \cdot & & \\
 & e_1^{(1)} & & & \cdot & \\
q_1^{(2)} & & \cdot & & & e_{N-1}^{(0)} \\
 & \cdot & & & & & q_N^{(0)} \\
\cdot & & \cdot & & & e_{N-1}^{(1)} \\
 & \cdot & & & \cdot & \\
\cdot & & \cdot & \cdot & & \\
 & \cdot & & & & \\
\cdot & & q_2^{(2N-4)} & \cdot & & \\
 & e_1^{(2N-3)} & & & & \\
q_1^{(2N-2)} & & & & &
\end{array}$$

using the rhombus rules

$$\left.\begin{aligned} e_k^{(n)} &= e_{k-1}^{(n+1)} + \left[q_k^{(n+1)} - q_k^{(n)}\right], \qquad n = 0(1)2(N-k)-1 \\ q_{k+1}^{(n)} &= \frac{q_k^{(n+1)} e_k^{(n+1)}}{e_k^{(n)}}, \qquad n = 0(1)2(N-k) \end{aligned}\right\} \tag{21}$$

with $k = 1(1)N - 1$. The parameters for relation (15) are then expressed by

$$\left.\begin{aligned} b_n &= q_n^{(0)} + e_{n-1}^{(0)}, \qquad n = 1(1)N \\ g_n &= q_{n-1}^{(0)} e_{n-1}^{(0)}, \qquad n = 2(1)N \end{aligned}\right\} \tag{22}$$

with $g_1 = 0$.

Calculation of the Nodes and Weight Coefficients

The three-term recurrence relation (15) for the first N orthogonal polynomials can be written (ref. 13) as the maxtrix equation

$$t \quad \Phi(t) = A \quad \Phi(t) \quad + \quad B \quad \phi_N(t) \tag{23}$$

where

$$\Phi(t) = \begin{bmatrix} \phi_0(t) \\ \phi_1(t) \\ \phi_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \phi_{N-2}(t) \\ \phi_{N-1}(t) \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{24}$$

and

$$
A = \begin{bmatrix} b_1 & 1 & 0 & 0 & & & \\ g_2 & b_2 & 1 & 0 & & \bigcirc & \\ 0 & g_3 & b_3 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & \bigcirc & & & & b_{N-1} & 1 \\ & & & & & g_N & b_N \end{bmatrix} \tag{25}
$$

From equation (23) it is immediately seen that the roots of $\phi_N(t)$, which are the nodes of the quadrature rule, are also the eigenvalues of the matrix A. Applying a diagonal similarity transformation to the matrix (25) yields the symmetric Jacobi matrix

$$
J_N = \begin{bmatrix} b_1 & \sqrt{g_2} & 0 & & & \\ \sqrt{g_2} & b_2 & \sqrt{g_3} & & \bigcirc & \\ 0 & \sqrt{g_3} & b_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \bigcirc & & \ddots & b_{N-1} & \sqrt{g_N} \\ & & & & \sqrt{g_N} & b_N \end{bmatrix} \tag{26}
$$

which has the same eigenvalues as A. In reference 13, it is further shown that if $Q_{1,j}$ is the first component of the orthonormal eigenvector of J_N associated with the eigenvalue t_{jN}, then

$$
W_{jN} = Q_{1,j}^2 M_0 \tag{27}
$$

Hence, both the nodes and the weight coefficients of the quadrature rule (10) can be calculated from the eigensystem of the symmetric tridiagonal matrix (26). The version of the very fast numerically stable QR algorithm (ref. 14) given in reference 12 to accomplish this computation was used.

Calculation of the Error Coefficient

The coefficient of the error term (2) can be obtained from the g_n. From equations (16) and equation (3) it immediately follows that

$$k_N = M_0 \prod_{i=1}^{N} g_{i+1} \qquad (28)$$

These constants are useful in obtaining error bounds if one has an estimate of $\max\limits_{0<\tau<1} |f^{(2N)}(\tau)|$.

Details of the Calculations

From the formulation of the calculation of quadrature rule (1) from the three-term recurrence relation parameters (16) as the determination of the eigensystem of the symmetric tridiagonal matrix (26) with no multiple eigenvalues, it can be shown (ref. 15) that once the three-term recurrence relation is known, no error is incurred except for round-off error. Therefore, all of the ill-conditioning of the calculation of the quadrature rule from the moments must occur in the determination of the recurrence relation. With this fact in mind along with the aim of obtaining a 24-point quadrature rule with nodes and weights accurate to 25 significant figures, the following calculations were carried out:

(a) The moments M_n, $n = 0(1)51$, were calculated from the recurrence relation (19) in 70-digit floating-point arithmetic. The first α moments were calculated using appropriate values of the gamma function accurate to 65 significant figures. These values were obtained by the method given in reference 16.

(b) The coefficients b_n, g_n, $n = 1(1)26$, were obtained from the moments via the quotient-difference algorithm. This computation was also done in 70-digit floating-point arithmetic. Using Gautschi's estimate of the condition number (which is remarkably accurate) we find that b_{25}, g_{25} had

$$k = 65 - 1.53(25) + 2 \log_{10}(200) \simeq 30$$

significant figures.

(c) The eigenvalues and the first components of the orthonormal eigenvectors of the matrix J_N (eq. (26)) were calculated, using a version of the algorithm *Gaussquadrule* given in reference 12, for $N = 2(2)8(4)16, 24$. These computations were done using 30-digit floating-point arithmetic.

In order to check these quadrature rules, the identities

$$\sum_{j=1}^{N} w_{jN} = M_0 , \qquad \sum_{j=1}^{N} b_j = \sum_{j=1}^{N} t_{jN} \qquad (29)$$

were used. It is our experience that, for quadrature rules on a finite interval, these are sufficient to determine the computed accuracy of the nodes and weight coefficients. In no instance was the disagreement more than a relative error of 10^{-28}.

RESULTS

In tables 1 through 12, the nodes and weight coefficients of the N-point quadrature rules for $N = 2(2)8(4)16,24$ are given for the 12 previously enumerated weight functions. Each of the nodes and weight coefficients is rounded to 25S. In table 13 the error constants k_N, for the same values of N, are given to 5 significant figures.

Tables 14 through 25 are tabulations of M_0 and of b_n, g_n, $n = 1(1)26$ of the recurrence relation for the monic orthogonal polynomials associated with the enumerated weight function $w(t)$. These tabulations are also rounded to 25 significant figures.

Appendix A is a listing of a double precision version of the algorithm used to obtain the quadrature rules. This program can be used to determine practical accuracy quadrature rules from the three-term recurrence relation parameters. In particular, it can be employed to find quadrature formulas not tabulated in this report.

The parameters associated with each table are summarized by:

Table	α	β	γ
1, 14	3	-1/2	0
2, 15	3	1/2	0
3, 16	4	-1/2	0
4, 17	4	1/2	0
5, 18	6	-1/2	0
6, 19	6	1/2	0
7, 20	8	-1/2	0
8, 21	8	1/2	0
9, 22	3	-1/2	-1/2
10, 23	3	1/2	1/2
11, 24	2	-3/4	0
12, 25	2	-2/3	0

In the tables, a ± b denotes $a \cdot 10^{\pm b}$; for instance, 0.17 - 6 means $0.17 \cdot 10^{-6}$.

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Moffett Field, Calif. 94035, January 19, 1970

APPENDIX A

QR ALGORITHM PROGRAM OF GOLUB AND WELSCH

```
C     THE ROUTINE IS ACCESSED BY THE FORTRAN IV STATEMENT
C
C         CALL GAUSSQ ( M, B, G, ZM, T, W )
C
C     WHERE
C
C         M IS THE NUMBER OF NODES IN THE QUADRATURE RULE TO BE CALCULATED
C
C         B,G ARE DOUBLE PRECISION ARRAYS OF LENGTH AT LEAST M WHICH
C              CONTAIN THE PARAMETERS OF THE THREE TERM RECURRENCE
C              RELATION FOR THE FIRST M MONIC ORTHOGONAL POLYNOMIALS
C
C                   P (X) = (X - B )P    (X) - G P    (X) , J= 1(1) M.
C                    J              J  J-1        J J-2
C
C         ZM IS THE ZERO MOMENT OF THE WEIGHT FUNCTION (IN DOUBLE PRECISION).
C
C     ALL OF THE ABOVE PARAMETERS ARE INPUTS. THE OUTPUTS ARE THE DOUBLE
C         PRECISION ARRAYS OF LENGHT M
C
C         T, W  WHICH CONTAIN THE NODES AND WEIGHTS OF THE DESIRED
C              M-POINT GAUSS QUADRATURE RULE.
C
C     THE CONTENTS OF B AND G ARE DESTROYED BY THE ROUTINE.
C
C
      SUBROUTINE GAUSSQ ( M, B, G, ZM, T, W )
C
      DOUBLE PRECISION B(1), G(1), T(1), W(1), ZM
C
```

```
C                 INTERNAL VARIABLES
C
      DOUBLE PRECISION AA, AJ, BR, BZERO, B2, CJ, CT, DEPS, DET, EIGMAX,
     * F, LAM, LAM1, LAM2, NORM, Q, RHO, RR, ST, WJ
C
C          DEPS IS THE LARGEST POSITIVE NUMBER SUCH THAT 1+DEPS = 1
C                 TO MACHINE DOUBLE PRECISION
C
      DEPS = 0.5D0 ** 54
      MN = M - 1

C          INITIALIZE W, SYMMETRIZE MATRIX, AND DETERMINE MAXIMUM ROW
C                 NORM
C
      DO 100 J = 1, MN
          G(J) = DSQRT(G(J+1))
          W(J+1) = 0.0D0
  100 CONTINUE
      W(1) = 1.0D0
      NORM = DABS(B(1)) + G(1)
      DO 200 J = 2, MN
          NORM = DMAX1(G(J) + DABS(B(J)) + G(J-1), NORM )
  200 CONTINUE
      NORM = DMAX1( G(MN) + DABS(B(M)), NORM )
      DEPS = NORM*DEPS
      LAM = NORM
      LAM1 = NORM
      LAM2 = NORM
      RHO = NORM
      MN = M
      BZERO = 0.0D0
C
```

```
C          LOOK FOR CONVERGENCE OF LOWER DIAGONAL ELEMENT
C
  101 IF ( MN .EQ. 0 ) RETURN
          M1 = MN - 1
          K = M1
          I = K
      IF ( MN .EQ. 1 ) GO TO 110
      IF ( DABS (G(M1)) .GT. DEPS ) GO TO 120
  110 T(MN) = B(MN)
      W(MN) = ZM*W(MN)**2
      RHO = DMIN1(LAM1, LAM2)
      MN = M1
      GO TO 101
C
C               SMALL OFF-DIAGONAL ELEMENT MEANS MATRIX CAN BE SPLIT
C
  120 K = I
      I = I-1
      IF ( I .EQ. 0) GO TO 130
      IF ( DABS(G(I)) .GT. DEPS ) GO TO 120
C
C          FIND EIGENVALUES OF LOWER 2 X 2 MATRIX AND SELECT ACCELERATING
C               SHIFT

  130 B2 = G(M1)**2
      DET = DSQRT ((B(MN) - B(M1))**2 + 4.0D0*B2 )
      AA = B(MN) + B(M1)
      LAM2 = AA + DSIGN(DET,AA)
      LAM1 = ( B(MN)*B(M1) - B2 ) / LAM2
      EIGMAX =  DMAX1 (LAM1,LAM2)
      IF ( DABS(EIGMAX-RHO) .GT. 0.125D0*DABS(EIGMAX) ) GO TO 135
          LAM = EIGMAX
  135 RHO = EIGMAX
C
```

```
C           TRANSFORM BLOCK FROM K TO M
C
      CJ = G(K)
      IF ( K .EQ. 1 ) BZERO = B(K) - LAM
      IF ( K .GT. 1 ) G(K-1) = B(K) - LAM
C
      DO 140 J = K, M1
          JM = J - 1
          IF ( JM .GT. 0 ) BR = G(JM)
          IF ( JM .EQ. 0 ) BR = BZERO
          RR = DSQRT ( BR**2 + CJ**2 )
          ST = CJ / RR
          CT = BR / RR
          AJ = B(J)
          BR = RR
          CJ = G(J+1) * ST
          G(J+1) = -CT * G(J+1)
          F = CT * AJ + ST * G(J)
          Q = G(J) * CT  + ST * B(J+1)
          B(J) = F * CT + Q * ST
          G(J) = F * ST - Q * CT
          WJ = W(J)
          B(J+1) = AJ + B(J+1) - B(J)
          W(J) = WJ * CT + W(J+1) * ST
          W(J+1) = WJ * ST - W(J+1) * CT
C
          IF ( JM .GT. 0 ) G(JM) = BR
          IF ( JM .EQ. 0 ) BZERO = BR
  140 CONTINUE
C
      IF ( K .EQ. 1 ) BZERO = 0.0D0
      IF ( K .GT. 1 ) G(K-1) = 0.0D0
C
      GO TO 101
      END
```

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TABLE 1.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{1 - T^3}$

T(I)	W(I)
N=2	
.23565026281786422648902 39+0	.56524283044148194608307 84+0
.87123433373136355822175 30+0	.83693927488397231509194 07+0
N=4	
.73377297386531816794646 50-1	.18402040956528594163430 37+0
.35017788181504371582347 97+0	.35527807293575269797451 90+0
.71057400872514091971013 83+0	.42401681470006153166418 73+0
.96421300062083795395278 09+0	.43886680812435408990200 90+0
N=6	
.35044570480243434184495 11-1	.88933140503948397956917 33-1
.17604291365206955909015 30+0	.18831531510234235529376 22+0
.39645152902729397273250 90+0	.25230278759905485983142 25+0
.64492945894475697406607 49+0	.28293120635892598204547 65+0
.85950591621651806242494 22+0	.29360290700149660203737 40+0
.98363701218258463746220 39+0	.29609674875968606401006 66+0
N=8	
.20419815058272573418109 98-1	.52059698933706115450225 07-1
.10461363005751560329577 83+0	.11456393253240571068823 27+0
.24437101694411646173157 62+0	.16307133056536627257726 73+0
.42104261482291079789857 86+0	.19476109744324074425188 30+0
.61022394667953089094776 68+0	.21203689876043589933546 71+0
.78464791405151481927682 09+0	.21980795543267095930109 43+0
.91826947065445858455385 58+0	.22258466615448636726138 457+0
.99067674633707533364017 26+0	.22329653011276488695700 40+0
N=12	
.93947554688271318575531 92-2	.24036327587966588523728 09-1
.48858803409374463484714 56-1	.54506762005738996900455 46-1
.11728627579572479180164 08+0	.81708402023724617344357 98-1
.21045455700591805643925 85+0	.10420967149224569390304 77+0
.32256488533374446990803 34+0	.12136666031716294197284 60+0
.44654288669312182813076 71+0	.13338255363962156561864 95+0
.57441201181072120770668 24+0	.14109573163734765729537 43+0
.69777557792449843143178 50+0	.14561248807803629094172 37+0
.80838775163204343313809 99+0	.14799495428729082438472 65+0
.89875406533278769784459 11+0	.14909709023860875425908 27+0
.96269359596157152299709 11+0	.14952335475045204809192 63+0
.99580645490692094218712 08+0	.14964810926725828193910 09+0
N=16	
.53750720282923202695170 46-2	.13769924647408063900008 54-1
.28109152502829734570792 76-1	.31575086287206446717794 14-1
.68154245848006573375853 98-1	.48284389261644528307176 01-1
.12408745846495187499008 41+0	.63315992419950048899192 826-1
.19390982134406966642957 13+0	.76249170758420719551564 97-1
.27511709793081807350668 95+0	.86862635566278154791512 71-1
.36477600392075480614302 90+0	.95155228376619427646375 29-1
.45961462462815613287308 55+0	.10131837050513501615063 04+0
.55613076407532476499336 02+0	.10567088580574445375813 44+0
.65071800748340215216288 46+0	.10858443252404973284880 26+0
.73980454725254970409301 33+0	.11042288448264559371827 69+0
.81999677711987176267986 82+0	.11150514090121461596916 36+0
.88821894869917816746028 66+0	.11208918916109191859130 52+0
.94184102494176745231047 17+0	.11237024756310290427971 72+0
.97878829287086490473955 20+0	.11248584590685935535113 41+0
.99762776733873874246762 32+0	.11252274937853284060149 49+0
N=24	
.24292788284754403094547 99-2	.62293317623894397659501 89-2
.12756123996299512580685 46-1	.14401952424181842156490 20-1
.31157824046557190278248 90-1	.22352039244509806598195 72-1
.57338234540554902585137 48-1	.29940738722691304125013 44-1
.90870961948402929932407 61-1	.37050223195288019950856 45-1
.13120956402970029839674 39+0	.43580349888351161616058 99-1
.17769631966997719164896 67+0	.49454638133782575712744 65-1
.22957206464873914322775 70+0	.54625461054027239857998 79-1
.28598725046141150780298 94+0	.59076561642426095142240 46-1
.34601433182281046645494 61+0	.62822089888685527971104 75-1
.40866159987020319014138 41+0	.65902365992279091211048 66-1
.47288850686967240605151 73+0	.68377387395384668726049 58-1
.53762238991437009721996 42+0	.70319457239783929056475 93-1
.60177634641763440564678 14+0	.71806196345861395688179 85-1
.66426788882677220752832 89+0	.72914785560871007954676 87-1
.72403793239961124698606 62+0	.73717799592673275510237 91-1
.78006964764068439221003 42+0	.74280602341588518168752 89-1
.83140672470737030202285 29+0	.74660041057667233796207 75-1
.87717063483576543981706 83+0	.74904089147542687888181 52-1
.91657652113770722827868 39+0	.75052097733538567056957 03-1
.94894740058950539859183 91+0	.75135375203022049026176 49-1
.97372640743061790083325 03+0	.75177886554047290641108 06-1
.99048685479922798731770 24+0	.75196930550443548267654 52-1
.99893993666602098104087 88+0	.75203704654417985286660 37-1

TABLE 2.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{1 - T^3}$

T(I)	W(I)
N=2	
.1945179134435048844396394+0	.4567144640653860539471959+0
.7299837021604849105558436+0	.3845947991298865027578155+0
N=4	
.6620016630784749589898308-1	.1656934398410866831255331+0
.3137372655784858805014977+0	.3042133775864010686647449+0
.6373331862398621857745485+0	.2706455105812883441206441+0
.9000244758525026819403301+0	.1007569351864964607940894+0
N=6	
.3265016120507537780840628-1	.8281429551661712180297748-1
.1636297885906563589890199+0	.1736068736400336354135599+0
.3671361940733304348636674+0	.2195129152302947223176756+0
.5973299786650457034259124+0	.2012759089066132717405893+0
.8052299694850214662385373+0	.1256668806154776117658401+0
.9488328821876800658801781+0	.3843238928623619366436911-1
N=8	
.1934656167936207757272972-1	.4931331824303393679841235-1
.9901872922325172116918954-1	.1081739676403133092758345+0
.2308475404630563977330189+0	.1514504502617981717881725+0
.3969220717426023399055997+0	.1704456012399666570353590+0
.5752674877528983025006790+0	.1591003848655220277664058+0
.7431621970600495053513068+0	.1193867060920149889416669+0
.8799190591321317956729686+0	.6506658850952775093988879-1
.9690652760533958250658543+0	.1837224634309571415927157-1
N=12	
.9056647579583088582247288-2	.2316990845081379098999374-1
.4708750867032975062168546-1	.5250538335013947873397337-1
.1129686428956082166100345+0	.7849546315493191270086718-1
.2025246303277493996637521+0	.9919309759061723038438132-1
.3100860545698904280931601+0	.1126672241820038586382610+0
.4289235681712044714730553+0	.1171490517914960002416213+0
.5517319234523368407303446+0	.1115679434129505966942145+0
.6711040673758606280142086+0	.9626752957735582105026241-1
.7799612460795478509942482+0	.7350748512900176678957520-1
.8719333282123326248981670+0	.4740671496296363118275478-1
.9416890927288078833690682+0	.2322466711095466042813672-1
.9852104725820194456779623+0	.6154794482043808870969930-2
N=16	
.5227959788538227454999262-2	.1339271583690751503400923-1
.2733674290961945408013557-1	.3070222991546851968857281-1
.6626614245331592251922190-1	.4691262115799225324525275-1
.1206046761762865407015243+0	.6137203812169870621885229-1
.1883705498556211470321585+0	.7344644623057134954268726-1
.2670982014513314959182705+0	.8248849202599251187661479-1
.3539414243416655439082627+0	.8786501279812335100577276-1
.4457901255296638736811900+0	.8904561872582474309630673-1
.5393912277383994409714323+0	.8574122265102774163103146-1
.6314665071756231142510369+0	.7805283886633371764297321-1
.7188237864420323069501037+0	.6657915788811138590288314-1
.7984601932870751515752630+0	.5243989677407524212082705-1
.8676567213358183517432077+0	.3719407307099849880921719-1
.9240629032055792327917945+0	.2265907115084945477547454-1
.9657698685874632028037220+0	.1066029973847135256221360-1
.9913698122829605273071696+0	.2757528242826213552322657-2
N=24	
.2384317521299020051288686-2	.6113993099844821960953168-2
.1251962220725596642954834-1	.1413431694865500847045019-1
.3057820623969246018614188-1	.2193310987954520235986209-1
.5626592911546632136424419-1	.2936859537300242183987772-1
.8915875232221724633622034-1	.3631078252170591508543291-1
.1287129684673059356510456+0	.4263011007850173362339521-1
.1742749124369632060974863+0	.4819465411682937572833047-1
.2250927045659366332346948+0	.5286913461665508335494163-1
.2803299289816693446678810+0	.5651699559834750141296166-1
.3390808126799902552188817+0	.5900667658236846051012453-1
.4003863727203384011501618+0	.6022223810226121171817134-1
.4632510237168698591039599+0	.6007737654356975829878755-1
.5266592377341232458088721+0	.5853092795436701176791383-1
.5895919693795355817916679+0	.5560144386174354917542182-1
.6510426573166384947335978+0	.5137839027009853252979793-1
.7100326708217821117099211+0	.4602791102028447878210266-1
.7656260892059519084117792+0	.3979177969508672738982021-1
.8169436967071892108896303+0	.3297899923722074418537579-1
.8631760606964236057497395+0	.2595037140575991551149700-1
.9035955475035686094451520+0	.1909715090081975739149705-1
.9375671235654964891378251+0	.1281554778112044601631509-1
.9645577917305635933491144+0	.7479294539053714968893715-2
.9841445229716109399633630+0	.3412718594797478110511767-2
.9960205610306135219204399+0	.8667444736337065125761007-3

TABLE 3.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{1 - T^4}$

T(I)	W(I)
N=2	
.23166915830506519842906636+0	.55165965904078904014638885+0
.86597626745396221388502256+0	.75936911810527086508680313+0
N=4	
.72774080550079916215229089-1	.18238817961178258375534000+0
.34687084656442691252728824+0	.34682850575052888902031820+0
.70611449011049440132783650+0	.39394169108487287545501790+0
.96334341339230054107408770+0	.38787040069887555700174370+0
N=6	
.34851731255571427612816200-1	.88430130757550137290469150-1
.17496828863159156336319290+0	.18661093802854850831322200+0
.39390740835307363670847690+0	.24590210333373562155355780+0
.64173798762495339922265060+0	.26595937524183546334436510+0
.85756341641245792806721660+0	.26454798711011954469066520+0
.98336305662210139060684880+0	.25957824267427063004014050+0
N=8	
.20534961722577847485838070-1	.51840361254143340562012250-1
.10415140396872194908282520+0	.11395876706796743704726250+0
.24319271281627815433958810+0	.16130080105879790440101860+0
.41900841425072126700435200+0	.18955471350539545892850900+0
.60778936650666349188591230+0	.20060615073336420569875660+0
.78268280237399380051773650+0	.20103154825944809934910390+0
.91733447349086272016177740+0	.19769150465605740000596210+0
.99055827268493149898903230+0	.19504493061088605923979480+0
N=12	
.93684661854109666244221730-2	.23968691272382885012752360-1
.48718503695400448202894670-1	.54341768518336156585540560-1
.11693179532938423531002400+0	.81381738307175954341832120-1
.20977857775608904521334650+0	.10345320452580614224689100+0
.32149316482510730874389770+0	.11954930631584032664223000+0
.44510571769539379101040210+0	.12957227266713729086216480+0
.57278181954679911045329690+0	.13439496101054442126588450+0
.69622031868631454830147860+0	.13551907138125916653086670+0
.80716358191816294810328720+0	.13454864626941837527698650+0
.89800216324165957513885420+0	.13280693099518511166296610+0
.96238898578700237480255640+0	.13120669921500972557809270+0
.99577061042143819366143740+0	.13028548666796434922621250+0

T(I)	W(I)
N=16	
.53637215993827816402848200-2	.13740758886040776352974470-1
.28048969244457983103990350-1	.31505901657290669700482610-1
.68004142765920827987799300-1	.48165554396449471042387790-1
.12380246854338128910818350+0	.63104313998064829762819220-1
.19344333195280787256282500+0	.75820723875873911022215630-1
.27443191701681700917444770+0	.85967991652785743212097310-1
.36386273140906103730664400+0	.93420610896183897156904610-1
.45850772872580845615806800+0	.98311091267703920277064440-1
.55491133079763608996910610+0	.10100504772374763648038310+0
.64949926439991945556629620+0	.10200429976330594568390690+0
.73870554319867493955308820+0	.10185914381941080226562260+0
.81911414548967166108007510+0	.10106203056937903660530310+0
.88760536787997828361662100+0	.10001210285709846339524620+0
.94149489563070878362323970+0	.99002458173708882746500270-1
.97865529593362197538905890+0	.98231109653812504317864790-1
.99761250862348404881505860+0	.97817577775208415222646730-1
N=24	
.24258353590290008829104570-2	.62204901564541083533046150-2
.12737936571510423340494500-1	.14381249329020158348457550-1
.31112884677002026212551180-1	.22318810385663840954129820-1
.57254014916643815276151390-1	.29882277484237554312955010-1
.90734122549981327629675740-1	.36977990662485982048345280-1
.13100598498376778960902520+0	.43463583519248146334638330-1
.17741185220112600531138550+0	.49252507274193729685322370-1
.22919422661579626049094150+0	.54270463177872781249238840-1
.28550771342603381687765940+0	.58472359945731150983300800-1
.34543163620801417064207800+0	.61847931220288142516059640-1
.40798330196332057721712440+0	.64423969187749210152765600-1
.47213203571568539655739610+0	.66261878668544661238909230-1
.53681424065687846893921950+0	.67450692417935489623135600-1
.60094969432541361070661420+0	.68097095859592810390266650-1
.66345908615940631942611640+0	.68314711217992313378914950-1
.72328253622405585001653800+0	.68214686337659213549746910-1
.77939861550861444297556300+0	.67898822462455879423841530-1
.83084327298833673736557380+0	.67455548133914925893770630-1
.87672806363974459338189450+0	.66958363544488594077851530-1
.91625713079719884509986960+0	.66466055341532527079283780-1
.94874248662979887250384300+0	.66023948839272045004402370-1
.97361722844483926992051960+0	.65665592444252465975309330-1
.99044641261024346923633900+0	.65414445869266732421771740-1
.99893537900304575756006070+0	.65285308666207447226697470-1

TABLE 4.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{1 - T^4}$

N=2

T(I)	W(I)
.1974237945187505962159729+0	.4655093301964751153818097+0
.7363260885489955514029448+0	.4085098545675648214398035+0

N=4

T(I)	W(I)
.6671843902813787620345253-1	.1670728285270703586380140+0
.3164908738500143725160001+0	.3101649259203136336343709+0
.6414708667330448965993946+0	.2859274812041764659755736+0
.9018508275330637907392523+0	.1108539491124794785736546+0

N=6

T(I)	W(I)
.3282449145482232564997399-1	.8326705559622805018908282-1
.1645854675152141536378311+0	.1750417775079680941773229+0
.3693795247848900463213310+0	.2242874446510954033681721+0
.6002817721648648647435764+0	.2116428118176476084722913+0
.8074115889905357861359224+0	.1368097432847228295498204+0
.9495446108446129332923910+0	.4297035190637795106492362-1

N=8

T(I)	W(I)
.1942520141529435294409544-1	.4951624946233404137005629-1
.9944388251406683185089363-1	.1087190987556033325146374+0
.2319197278984014769457291+0	.1529413132309200157670059+0
.3987718482898717747730782+0	.1744820864755212398896874+0
.5775420360065801577480006+0	.1667782489885567568343932+0
.7451560368255965248072726+0	.1287989459709585213439622+0
.8810939400075294263991941+0	.7206635345043414406078184-1
.9694085376125641692347153+0	.2071688842971188504108901-1

N=12

T(I)	W(I)
.9081625243256722896068787-2	.2323413985228197981026063-1
.4722051495678480760727549-1	.5266109949015637064046647-1
.1133032347146602956248536+0	.7879716147785233806827536-1
.2031593910790823337759724+0	.9986685735828600275083932-1
.3110893866019913446425329+0	.1142237104171352888480175+0
.4302725424186819869254738+0	.1202414402703369646769067+0
.5532808759994113664281759+0	.1165272169528736613666480+0
.6726227736866722132128896+0	.1026418157218967640555386+0
.7812206713912676424811427+0	.8004726801322293952637477-1
.8727876664757243775890914+0	.5261557786525529613676088-1
.9421199468616726476371718+0	.2616351966249146671631370-1
.9853263381762561168919870+0	.6999377682250864225211225-2

N=16

T(I)	W(I)
.5238880081916714983811427-2	.1342077082401426204394644-1
.2739459335892662149545982-1	.3076863205849485166371821-1
.6641016141608248718859137-1	.4702583283654015192377642-1
.1208773707265071554448609+0	.6157021407626646221132632-1
.1888155723592493269615684+0	.7383863308766138624417644-1
.2677503490542639184384173+0	.8328851286835142775829836-1
.3548102620377389126785167+0	.8937256235627862558583759-1
.4468458635329833140541219+0	.9154912596510011397785308-1
.5405624097281343646571991+0	.8937068074492539138415139-1
.6326524723605657412259872+0	.8265610463633997414370908-1
.7199169146267776159266629+0	.7169556718806446708912261-1
.7993695451731488025810808+0	.5740091361970563610952680-1
.8683262502340278641616361+0	.4132146848204729345341804-1
.9244809382923489334656035+0	.2548793510227340326662459-1
.9659695553623580566310523+0	.1210334840869214829096594-1
.9914218919211441955502528+0	.3148882509284341675161869-2

N=24

T(I)	W(I)
.2387672779195479726520233-2	.6122607710795912904555501-2
.1253733915569277575134309-1	.1415447715261713974120604-1
.3062196072467974986062373-1	.2196541888346776646552949-1
.5634786189897509967073675-1	.2941554037045358395872908-1
.8929172931319086671696595-1	.3638023754219567117567382-1
.1289105406841534559330915+0	.4274119693389322756040533-1
.1745506061260063189584691+0	.4838476058695132559350565-1
.2254584382056001839489014+0	.5319926192693544625110806-1
.2807937060964253575960866+0	.5707187329531747679969179-1
.3396442192018204771420740+0	.5988690618527008043672351-1
.4010426143953767085647880+0	.6152804689498367183115263-1
.4639841562177834398552379+0	.6188654673820973947939941-1
.5274449095439605545751563+0	.6087591803788615203899223-1
.5903995769064866311946681+0	.5845167940738630838284983-1
.6518384787695364557562333+0	.5463286255390870528611835-1
.7107833958612919628206611+0	.4952094764797752315424432-1
.7663021894154921377519060+0	.4331189815247003574843679-1
.8175222197319197284964807+0	.3629792369879171083324069-1
.8636426023138325744672855+0	.2885719917938712258731853-1
.9039453087652278701321110+0	.2143163925135783868440588-1
.9378050723789461462899838+0	.1449463859781906854130704-1
.9646980190430999535300406+0	.8512182708824765871410317-2
.9842089222618837570334485+0	.3901732853641618757238192-2
.9960369772784407389501067+0	.9936884534980447383697103-3

TABLE 5.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{1 - T^6}$

N=2

T(I)	W(I)
.2275715979585846041106961+0	.5396250766377089967178524+0
.8571031565242873380969259+0	.6747002473060818091921185+0

N=4

T(I)	W(I)
.7212564772417261791344756-1	.1807113803650086784537596+0
.3432544710951128088353072+0	.3403355734366210569910157+0
.6996280160268079844888851+0	.3645092051233290614438561+0
.9618133380174989453715734+0	.3287691650188320090213394+0

N=6

T(I)	W(I)
.3464064059329285167774036-1	.8788812226009906562400217-1
.1738371378998293513538924+0	.1851978002913559900953312+0
.3910263221669968170957644+0	.2413127540563494242581581+0
.6373357041279886583415256+0	.2504770241229544842489740+0
.8544097924098136674234587+0	.2327459354897027858484999+0
.9828826764458324881865124+0	.2167036877233290558350055+0

N=8

T(I)	W(I)
.2024142580063658862619342-1	.5160038594556015328914058-1
.1036566185863387993093046+0	.1133847276574747003465534+0
.2419436207017047237525148+0	.1600835945713080655364412+0
.4166573401290029870433783+0	.1859291321566303106017261+0
.6045203635673331582034694+0	.1906568817477191072031545+0
.7796628154694230593708342+0	.1815639112785202922248961+0
.9157667296761967301394336+0	.1693780894239286477538975+0
.9903514267781723841627602+0	.1617286011626485289541615+0

N=12

T(I)	W(I)
.9339328521134557282451370-2	.2389393891292663344950964-1
.4856504098397518720138830-1	.5416747728559574390458385-1
.1165528137833381415563990+0	.8109567799504302844644799-1
.2090631287133629190469690+0	.1029605150751151088775399+0
.3203219461689760718843399+0	.1184648909956516431709882+0
.4434076804369578485004445+0	.1269701877892378988511544+0
.5706512113330910886379781+0	.1288947829218173402447284+0
.6939798832691580721760148+0	.1258824054659614489728578+0
.8052557693724250451817117+0	.1203155242449796183664725+0
.8967627815081241706578995+0	.1144482159786554418766757+0
.9618689889121458455218392+0	.1098515170235062445182999+0
.9957083921729528527991001+0	.1073801902553006552307126+0

N=16

T(I)	W(I)
.5351113809427072608442839-2	.1370841062528893609173151-1
.2798258465146087668230399-1	.3143065893946315022316427-1
.6784091218342635634521003-1	.4804638144523592995817756-1
.1234979286140205703158452+0	.6293238958117820393657079-1
.1929491916472123105349557+0	.7555418103775377898406364-1
.2736960922637372392879847+0	.8547464622238238230948059-1
.3628398297576300897779148+0	.9240197342630746434482978-1
.4571847529093397375210405+0	.9626708794838849642406876-1
.5533389146000698023264119+0	.9730415879197964684931308-1
.6478052277124858827437712+0	.9606618044443370154183532-1
.7370731057314719776452986+0	.9332540711199260586974823-1
.8177301976215492070672670+0	.8989554381515707508335647-1
.8866027466593136263792395+0	.8647291336496365034536576-1
.9409124766791088735150425+0	.8356202379097204049118719-1
.9784272968154604247626860+0	.8148061961589674737035017-1
.9975861131075394273035640+0	.8040274778239699608672771-1

N=24

T(I)	W(I)
.2422008286060772050054837-2	.6210656995012059540275417-2
.1271775477525497549545132-1	.1435837824878390925263551-1
.3106331039803663470142352-1	.2228288200453923518582349-1
.5716196553340063670593680-1	.2984305144231008558284727-1
.9058629550459076955082901-1	.3691416615902560480010051-1
.1307885186320622863338079+0	.4338043472611628637824633-1
.1771099655810435804963677+0	.4913670808627855559868705-1
.2287922192647805848503809+0	.5409130446950203302082055-1
.2849899071662064658939877+0	.5817024568134060157939501-1
.3447847755082304653840692+0	.6132362996607207633886022-1
.4072004446635404844282340+0	.6353402683485402234838702-1
.4712171585332585916564732+0	.6482516855195177557517142-1
.5357860526301514757894338+0	.6526747148750469364294950-1
.5998429902290044412233525+0	.6497665266318900586911039-1
.6623226015787848020540519+0	.6410377976721584218808820-1
.7221734173631736578407328+0	.6281865117725849228626807-1
.7783747032882435679515398+0	.6129118953724405826466743-1
.8299549380037479744111974+0	.5967582627610940706470057-1
.8760112162319662517110063+0	.5810186850653308137829202-1
.9157285004212557257250008+0	.5667018278654222346787269-1
.9483976360616956069622984+0	.5545466539238632231591124-1
.9734312573741479128495390+0	.5450634773204470107835428-1
.9903769817821039656283918+0	.5385825842902753968113112-1
.9989275241325624881035366+0	.5352977602294919347137555-1

TABLE 6.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{1 - T^6}$

N=2

T(I)	W(I)
.2005997929169993432400113+0	.4740991886530253343040270+0
.7455726698730123736810423+0	.4366448043048177701284512+0

N=4

T(I)	W(I)
.6729205978379392283597651-1	.1685447083214997227959661+0
.3195667357275720163337610+0	.3150404130016933665797106+0
.6472033373752203309482565+0	.3022174823903069664484664+0
.9047467142100877677667082+0	.1249413892443430486083349+0

N=6

T(I)	W(I)
.3301885643412491154815001-1	.8376507044377595806290145-1
.1656148592788018079826461+0	.1762928849233464878475253+0
.3719440513073763263104491+0	.2278783213265854585495529+0
.6042534252879632473644538+0	.2218957826868740298687443+0
.8107505487088211779818326+0	.1511573069865178104108606+0
.9507043728336749152835526+0	.4975462659074335969289359-1

N=8

T(I)	W(I)
.1951306825743082864735983-1	.4974148298505323155883309-1
.9990659271881994989911413-1	.1092515608099633718402118+0
.2330751160369460966485501+0	.1540142321790286987612533+0
.4009198967214111018079766+0	.1774132739855488599103919+0
.5805491864258133907482330+0	.1739923449628008957699110+0
.7481094100782634165877557+0	.1400542623555034265461363+0
.8829642543544963105782317+0	.8193509348243441384817405-1
.9699753665725955758262449+0	.2434174219751020619756674-1

N=12

T(I)	W(I)
.9109558248461558923743778-2	.2330578289775986737083279-1
.4736745351176233563132086-1	.5282769168009784050521913-1
.1136651007748527051514341+0	.7906816327864060351218903-1
.2036391400794882179634748+0	.1008213020875723242808937+0
.3121950750777510842478183+0	.1151819105760995850049422+0
.4318684538958190654976739+0	.1224405332330382495733843+0
.5552889368442620718172042+0	.1208974487476286135419935+0
.6747709622272935615857870+0	.1094847784057625785392580+0
.7831302395106788384406737+0	.8826270854202248466544926-1
.8741483751766431951987678+0	.5996132602638893110073569-1
.9428281688576118996431721+0	.3064546184886796213198887-1
.9855201220048573167842806+0	.8346885633964064205591725-2

N=16

T(I)	W(I)
.5251091930692792494516502-2	.1345209969399729269995129-1
.2745886091369993856941566-1	.3084142723857861372194002-1
.6656802214227931184171355-1	.4714082576220562967593744-1
.1211713538501204050099609+0	.6173492695379547498300176-1
.1892912399212662306598531+0	.7408966568287124073737401-1
.2684561189947082689525087+0	.8374129080188018533748346-1
.3557879571662728275457087+0	.9028306597057529302556915-1
.4481079472745539636190390+0	.9332402812810482739838738-1
.5420647440967873729631360+0	.9244881495738857576422798-1
.6342828457076905365824792+0	.8726047682708929260931296-1
.7215137129715631261045821+0	.7758356627515352393935734-1
.8007647661971971503361267+0	.6379809737986536878468816-1
.8693926365805363634744017+0	.4713816666960182250578699-1
.9251648851214914186929425+0	.2974742325663529266786489-1
.9663021524482254446210369+0	.1437770696904860928213326-1
.9915095220901535354093542+0	.3782410891052061299462083-2

N=24

T(I)	W(I)
.2391423526684103488920166-2	.6132231864223953988692883-2
.1255708973557172596019554-1	.1417685538270803940568687-1
.3067046207907868337713867-1	.2200055167170374120707319-1
.5643787845168766420527154-1	.2946362085805960501083262-1
.8943619453818105740777429-1	.3644242691921109527537358-1
.1291228594367632265857156+0	.4282180228837628052250396-1
.1748449750827521699847984+0	.4849598394426214627468618-1
.2258498316477631190904219+0	.5336919450244786806646235-1
.2812969907965889073113144+0	.5735469310089535585084064-1
.3402719281884675172097869+0	.6037155887021235743291026-1
.4018013633538108967836668+0	.6233936531848027886083251-1
.4648704320680175884990508+0	.6317272633679396716748727-1
.5284416187607439187605798+0	.6277982647584800529829880-1
.5914748548838299611226788+0	.6107002703801017445096946-1
.6529476481801802211372656+0	.5797403589363990242497086-1
.7118739053161202337489638+0	.5347580926268904485392416-1
.7673204016627160758446176+0	.4764997009237507205974737-1
.8184205453886782733735782+0	.4069458930128173643267888-1
.8643855687990872019822612+0	.3294847954929462393818544-1
.9045137230790078468546150+0	.2488491545665348761515852-1
.9381979368898297369056241+0	.1707883669170867390595913-1
.9649322830534519618639740+0	.1015052500631794811795453-1
.9843173853240356552901910+0	.4694008278756735393849997-2
.9960647578184413762566587+0	.1201958853848830877298654-2

TABLE 7.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{1 - T^8}$

T(I)	W(I)
N=2	
.22539606718959225549652 31+0	.53391677868811496596554 97+0
.84991618353275699863347 96+0	.62967579253015440933696 84+0
N=4	
.71762952639000869915295 86-1	.17978811313032777055617 49+0
.34133231158996667922573 16+0	.33778409782399450651551 74+0
.69525623993019712177202 50+0	.35129125634393168552754 57+0
.96046910205241626481984 90+0	.29472910392001541270328 01+0
N=6	
.34521394773905346757859 93-1	.87583590924913555232466 47-1
.17321777365984753988297 86+0	.18449834652011713750869 67+0
.38947471776300291382810 97+0	.23974941848616405641149 86+0
.63454124207168189551364 44+0	.24425276289401140224705 05+0
.85192949843591461026016 33+0	.21598923078505138212118 12+0
.98246066766474904742117 54+0	.19151922160801184178162 48+0
N=8	
.20188386826810265855866 13-1	.51464688357330522331224 36-1
.10338030728069329650408 06+0	.11307495457702752988337 05+0
.24126925226534023244953 30+0	.15957631268537630150167 61+0
.41537744549386626827531 84+0	.18480370882951339689265 14+0
.60250359811577138053081 42+0	.18694766917542079817001 22+0
.77745633965161345664539 04+0	.17215640723689182043842 87+0
.91447187235169044136555 63+0	.15353958612100969770050 78+0
.99017044169643517783279 57+0	.14202924423569930838464 71+0
N=12	
.93227537457263756758247 57-2	.23851466511637861162921 17-1
.48478239621508044599244 33-1	.54069760193367496363717 05-1
.11634139736098625244817 94+0	.80944248747037649509637 75-1
.20867315859578831040838 76+0	.10274963856187300943680 70+0
.31969318525928044761057 85+0	.11813410554363295741506 82+0
.44247029324719162661808 62+0	.12623993197428006311091 62+0
.56937168125787215989834 50+0	.12703438590613499380116 29+0
.69247351804090742405650 64+0	.12180867741161016433102 83+0
.80383056633833171945310 57+0	.11321859900266755434110 80+0
.89576093975378497544970 26+0	.10432626122473060375829 79+0
.96142729025266636921306 19+0	.97440063425404979163435 16-1
.99565428961534388137008 87+0	.93775432715892042908418 65-1

T(I)	W(I)
N=16	
.53439320294384032616913 41-2	.13689996099537138946103 40-1
.27944882672469469478292 90-1	.31388100480711852064308 14-1
.67748812863300229407032 97-1	.47980253447969253975905 63-1
.12332814908558312765237 78+0	.62843074308524713511999 31-1
.19267848184047408756792 12+0	.75439362100465092864692 72-1
.27329937453055954067721 07+0	.85318187274877390507000 88-1
.36228803747701789915945 42+0	.92139487888937684272803 34-1
.45644839112279211862088 30+0	.95721018371560371638119 01-1
.55240649809901684179942 45+0	.96127135497132576042483 59-1
.64671434736512797142805 40+0	.93777237248127886153227 39-1
.73592926852806447844351 20+0	.89471419365316750073827 24-1
.81668600605147288329618 48+0	.84238726751604396322334 71-1
.88580108014063817766396 32+0	.79070489072062365397668 73-1
.94042712154344368943378 88+0	.74715159820314580182508 82-1
.97823226215451055822803 94+0	.71629742931510428329926 81-1
.99756324813642435349002 24+0	.70043180559616895019608 41-1
N=24	
.24198177105678068452546 86-2	.62050503758064001671511 48-2
.12706258824243254565573 34-1	.14345371457262162838794 11-1
.31035141991776176812737 83-1	.22262563979695071274673 75-1
.57109878826201680211603 33-1	.29815556138274848207098 54-1
.90503182031535696749792 54-1	.36879615545539267781817 27-1
.13066736943919770623509 66+0	.43338793028205681301617 40-1
.17694373221097005227883 10+0	.49087289687354150633028 64-1
.22857353248444360767231 54+0	.54031285856791831842228 34-1
.28471070024201954086536 95+0	.58091053720073715244650 46-1
.34443608733020322159191 02+0	.61203979451154971465691 45-1
.40677298340389552060335 72+0	.63329667747362902248672 49-1
.47070332044154116100176 33+0	.64458303127513393876754 32-1
.53518365014025360918940 47+0	.64621994331087631197919 51-1
.59915987288385506471895 28+0	.63905711166230332341261 72-1
.66158021476110357577573 71+0	.62451444265649124710959 66-1
.72140709721282499410418 01+0	.60449710769563462166278 75-1
.77762962964428028169902 75+0	.58117966321928287350883 49-1
.82927854864131615463465 36+0	.55672585917290555522110 06-1
.87544431469243320134514 20+0	.53304242845877175748303 42-1
.91529753189371199605984 74+0	.51163632758369061630984 35-1
.94810985321337139758024 15+0	.49358666026646625039029 05-1
.97327342508478379803184 68+0	.47959834061339976125623 49-1
.99031741126444288811272 37+0	.47009164052035983394332 57-1
.99892074872264005433590 58+0	.46529088587216763192654 16-1

TABLE 8.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{1 - T^8}$

T(I)	W(I)
N=2	
.20237506853620430020080766+0	.47855439384035367997909999+0
.75203919231189057671249720+0	.45231966313426182026291460+0
N=4	
.67620206666347796475132020-1	.16937638234592249362993480+0
.32125341638696936013220800+0	.31711207446486106390900860+0
.65083108760234031552188550+0	.30997643572939584212523600+0
.90700430417964620326668090+0	.13440916443443610057783510+0
N=6	
.33130274756683413211836870-1	.84049241912621537900067510-1
.16618955185526476474129890+0	.17693428658412150756588570+0
.37335207251416716333349120+0	.22917732252233833280826820+0
.60674006232481809998238010+0	.22624407905014434105824480+0
.81322129085926817731508810+0	.15974830246026364746662880+0
.95163939756713760470686760+0	.54720824445126133442919460-1
N=8	
.19563442620784351027483080-1	.49870291239934826665378660-1
.10016831797789672738166190+0	.10954385184521788395184680+0
.23370932024210588677917850+0	.15448170450746208305497870+0
.40210481099615277898660530+0	.17836639800623067592588480+0
.58239261293999365595518200+0	.17681563378022329644898380+0
.75019449303152267646440000+0	.14610192859864812886940830+0
.88442255946384008882691850+0	.88586341143134000080304790-1
.97043963846849843694098550+0	.27107907853764605245228640-1
N=12	
.91255651222752296273436190-2	.23346793060849437628379850-1
.47451216544765819404683890-1	.52921893567150938552024480-1
.11386878666112545317225620+0	.79213592438264276398068700-1
.20421367799180641044895680+0	.10052173455667180701022870+0
.31279547835810122001015030+0	.11548647698568606368270390+0
.43275616443520372765749940+0	.12307795833875082357234360+0
.55649296295734388450211620+0	.12243937958652883699019040+0
.67619524800911709774833040+0	.11261794750259510442982170+0
.78451890003220642340131850+0	.92951876635873004189671460-1
.87520775177962961355750880+0	.64888550023362853216808440-1
.94340400989719101959015220+0	.33991866353820398896023900-1
.98568139913755117427390030+0	.94159879250619556757493330-2

T(I)	W(I)
N=16	
.52580868642331205592011530-2	.13470033815792554815274080-1
.27495570163505634909272810-1	.30882848025416027943649570-1
.66657639714374592003114900-1	.47205098265449244504572750-1
.12133638310114253839091800+0	.61821510442385301906087700-1
.18955391676786061515862210+0	.74200316169778479698885650-1
.26883997708364954144000370+0	.83889768426221972975291030-1
.35631964512891204206217550+0	.90524942785229997269701960-1
.44881400951241407493711130+0	.93811154095197228730955130-1
.54295545422796103081001000+0	.93466904475701313133073610-1
.63532540321715947675303920+0	.89156565929413158242270530-1
.72261647324988212814897820+0	.80538573206492680333506020-1
.80179484315705694325174100+0	.67571558750166297327220730-1
.87022191561236628802833970+0	.51024905945834703741821350-1
.92571690235307379130664630+0	.32866093602730777957024240-1
.96657721421955987586469600+0	.16149686721764941282663990-1
.99158298990157690948094780+0	.42940963170408203800141800-2
N=24	
.23935707288359344954653140-2	.61377399087502045052505390-2
.12568382668543978019431090-1	.14189630841480204633422540-1
.30698127894067902969467860-1	.22020500857205920787334630-1
.56489021646685794761121480-1	.29490601206326731883588200-1
.89517770346041052747289610-1	.36476299745698004058513350-1
.12924170166760067351547580+0	.42862566738487633072631970-1
.17500791726348224368041810+0	.48544228272421855492325120-1
.22606395836774593471716500+0	.53427450422566296711566130-1
.28156997553604497335192660+0	.57430729402416829074965600-1
.34061220494569642839386840+0	.60484677702872356810609990-1
.40221760084352416664556110+0	.62529504543797176381621510-1
.46536967920254845594627360+0	.63509177477443613791656150-1
.52902590414854208756289990+0	.63362805251533600150312980-1
.59213700849276561739526660+0	.62017250563258871338031730-1
.65366813065035923278568440+0	.59388940139138633037158670-1
.71262059201238133696735100+0	.55403774829827297890778720-1
.76805220149508365211143540+0	.50038787506307052832042240-1
.81909400338832105717076500+0	.43378564316918321455593280-1
.86496249613877186172819880+0	.35668900778894223888751820-1
.90496775057437292352722270+0	.27345967445913891124742420-1
.93851866980605303505477350+0	.19023944941562469809523810-1
.96512663974062798387073620+0	.11435541802234417640938890-1
.98440836686417253843713930+0	.53335253765987882430099120-2
.99608821030286633196162300+0	.13729469029611056276442950-2

TABLE 9.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{T(1 - T^3)}$

N=2

T(I)	W(I)
.13030212209544721459062O3+0	.1392129283523042851524713+1
.8352940721613854730867259+0	.1036521364364538760295228+1

N=4

T(I)	W(I)
.3567125592932102832876668-1	.7471880363857365358553503+0
.2939560710570540734322316+0	.6586793946745750508333283+0
.6763017806500842684811674+0	.5451246934004264188608391+0
.9593732452797612837682852+0	.4776585234268436062703239+0

N=6

T(I)	W(I)
.1630137488570383040237659-1	.5080762231947474424783142+0
.1407897126148294074386980+0	.4777325646804528407385396+0
.3597469018926891332442995+0	.4262817602419202144594560+0
.6185077970552631386327448+0	.3725351435282757707329870+0
.8477627723245529340650405+0	.3322997355697048849407952+0
.9821938708204399052016066+0	.3117252206724804584698497+0

N=8

T(I)	W(I)
.9291941826561905519983069-2	.3844316310299928436073746+0
.8166048171091533857020868-1	.3709115736153533562649943+0
.2162471487738233373723057+0	.3458423218533769927317627+0
.3940389663179270982833099+0	.3143703092123741543738179+0
.5891268784792185760052748+0	.2832855990820694622058603+0
.7717856561446507583624536+0	.2577958655217896138176260+0
.9130835154316830469580874+0	.2403670922524762517160444+0
.9900678408662408994433459+0	.2316462553101489371024615+0

N=12

T(I)	W(I)
.4182797654613511262828163-2	.2583471632093658573314330+0
.3723863509774276958814549-1	.2541602290311987910205067+0
.1012156291669530271748544+0	.2459837928299936620355372+0
.1919896451304628650608087+0	.2344101057468472728018143+0
.3036876578155392305757587+0	.2205448139203199875780012+0
.4289917621729014214568807+0	.2058278948127025026687837+0
.5595311070683353676119461+0	.1916574391246276673771197+0
.6863967833095710863294938+0	.1790897087931024703817512+0
.8007653032146166233036092+0	.1687569506336595171441316+0
.8945657275537707515962959+0	.1609502704813483415365915+0
.9611110218468115377298035+0	.1557550863409237765498344+0
.9956263595197316077650052+0	.1531671929634917653944373+0

N=16

T(I)	W(I)
.2367407232573675923254878-2	.1944735719826803951039245+0
.2117526341205507221766248-1	.1926732464149563768611977+0
.5809629413084753512523175-1	.1891104744182135918024864+0
.1117690621256004629254749+0	.1838979977217325404372531+0
.1802156480345581976511365+0	.1772677547308297307861596+0
.2609075346745081337067323+0	.1695840313372804638882497+0
.3508443714049945296521878+0	.1613066250140909026470397+0
.4466481420507367879554784+0	.1529186288545970986236469+0
.5446767586204551688184804+0	.1448532155305628098268206+0
.6411576419708340867533213+0	.1374474078079547923412617+0
.7323364610513124964112125+0	.1309301925738414808864438+0
.8146323700888199925506794+0	.1254359599956809701388411+0
.8847899667635598184153682+0	.1210294654771558349061820+0
.9400190156822257711800249+0	.1177316931336763656600726+0
.9781145826336963259847945+0	.1155411781812748548402822+0
.9975519086742717441995692+0	.1144492047130534030700801+0

N=24

T(I)	W(I)
.1058551256599176188873521-2	.1300962489144271767790083+0
.9500470636577049438343733-2	.1295533624421680571198363+0
.2624356686323587846405858-1	.1284714407624273593532948+0
.5100890468044952129995145-1	.1268609742479330084230030+0
.8338425637413211552596118-1	.1247436651916811822325191+0
.1228310224356064192750215+0	.1221570801740983595312989+0
.1686928256584271714469156+0	.1191575273508282376358938+0
.2202055297233880130405974+0	.1158198118986842047590616+0
.2765085968459051417058345+0	.1122335336422364381919616+0
.3366578470894900528553694+0	.1084967032067134824132217+0
.3996397422279969661928756+0	.1047082074580395150679562+0
.4643872630314025802534917+0	.1009607892131652646702278+0
.5297973079612462304461522+0	.9733577988777187218584927-1
.5947493715454998136829500+0	.9390012883373142026738219-1
.6581251167990389753950110+0	.9070562898232974993311197-1
.7188283674669352577185219+0	.8778983479768393051706401-1
.7758050151206948194015029+0	.8517803345876072346931614-1
.8280623494487953941100798+0	.8288568870556051525226063-1
.8746873595551466266667296+0	.8092082773725685626399568-1
.9148636052278770000476960+0	.7928680598301284077035461-1
.9478863111774369315858682+0	.7798321963200572370647345-1
.9731753902181926832464217+0	.7700842802692209837774873-1
.9902861523274998697271967+0	.7636020069613013542119557-1
.9989175058794761117551852+0	.7603662664391345947791117-1

TABLE 10.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{T(1 - T^3)}$

N=2

T(I)	W(I)
.26618581655995979649512 41+0	.24339216373783495583748 54+0
.76653615516379321955405 09+0	.28020661186046391723962 18+0

N=4

T(I)	W(I)
.10043631271162057413953 53+0	.61224037194592825174449 98-1
.35838748941039164157773 29+0	.17758002149781938574354 79+0
.66686204441274906806532 13+0	.20074560564251773684896 41+0
.90801169958928974492447 75+0	.84049111263368925310145 18-1

N=6

T(I)	W(I)
.51410049637323424712255 38-1	.22881267399823755886342 77-1
.19447904578578793151744 73+0	.79055374403344082536567 66-1
.39835517648889100770049 24+0	.13474336885321657888865 73+0
.62063126642700969482018 25+0	.14882322967944621604703 43+0
.81758288349056435555363 50+0	.10417754191795257252708 27+0
.95223914864556712495522 86+0	.33917993344515667191422 44-1

N=8

T(I)	W(I)
.31060743307199164696615 29-1	.10828660021945029782891 93-1
.12022010895639767578715 86+0	.39811534784624137480681 79-1
.25592194504092790563246 33+0	.77019253417570879550760 93-1
.42078351671328830425713 75+0	.10778445257158239155943 64+0
.59429718711959569988333 99+0	.11703953987967671667694 44+0
.75564871462903975229427 77+0	.97492368456761363520277 43-1
.88606894677171482932652 22+0	.56911877092483589009551 09-1
.97069640791075794198360 31+0	.16711089373654765496563 22-1

N=12

T(I)	W(I)
.14833519539001095612875 33-1	.35947648724582274435000 04-2
.58429169060280014955968 42-1	.13832113021318334823972 21-1
.12812243303408577621912 78+0	.29120997197411499453269 90-1
.21965312519032710533027 47+0	.46937470450036006113967 56-1
.32745804674184969731911 81+0	.63989408056344838724231 18-1
.44506815278834643268067 27+0	.76584625911246717298467 34-1
.56554834587160722120495 66+0	.81384364853788387279781 76-1
.68192278909617637699205 25+0	.76473775375907000316084 39-1
.78756019110361543714194 53+0	.62337013550878488446928 70-1
.87651537416625793402253 92+0	.42195465669738484626570 83-1
.94382624890213444792695 77+0	.21372839301012518257044 53-1
.98576017495526866108752 11+0	.57759373381583702932888 29-2

N=16

T(I)	W(I)
.86500147622604323567056 27-2	.16042452080955559277911 43-2
.34294480982810004865114 02-1	.62755728365515901136275 59-2
.76026376325800390825512 38-1	.13598233448459880707048 96-1
.13236783243334789782903 52+0	.22901915256573475190985 17-1
.20132438782629440397101 84+0	.33284213542364351700879 19-1
.28046335168074173945003 25+0	.43640461827882571535772 64-1
.36701186317970446684949 02+0	.52724845963910974293784 56-1
.45796629344661103948456 91+0	.59271690314543834412157 07-1
.55020429555667067060314 93+0	.62185563175360356380973 18-1
.64059349453074176697889 32+0	.60772269556195453846696 95-1
.72609396420079005794565 73+0	.54950575307498465697760 28-1
.80385343507228328965786 98+0	.45373752751235450069890 55-1
.87129449183491064724484 07+0	.33405897236707276921674 15-1
.92619260570597612727085 01+0	.20934508566925854589847 17-1
.96674339153807342108994 56+0	.10045813916731822253682 74-1
.99161729137193384410670 73+0	.26292216892619594345359 07-2

N=24

T(I)	W(I)
.39856441620131976700994 19-2	.50256411242608908807077 82-3
.15878147662431565905336 62-1	.19899690365580262411026 45-2
.35485174598117109620225 26-1	.44019850100855396112037 84-2
.62489384546261633806343 80-1	.76400918976467234296069 40-2
.96453407586453340914734 29-1	.11569572544715210796844 64-1
.13682701428511747198553 17+0	.16021536923035650146032 56-1
.18295651340661135404789 40+0	.20794880142558768288040 58-1
.23409624748701596469264 58+0	.25658945814517502049210 94-1
.28942187324482683938565 94+0	.30358332002384294312321 19-1
.34804497492048409191061 92+0	.34621480068157562143264 63-1
.40902850837432109173986 92+0	.38174238122168670709919 85-1
.47140261496814933088833 71+0	.40758559237039945453381 20-1
.53418044301714623108774 73+0	.42155164659654232846156 35-1
.59637372430228840647012 09+0	.42207745063035557973263 69-1
.65700793851048891028853 30+0	.40845411215442220661902 39-1
.71513694595811371777509 42+0	.38099832799253506183528 95-1
.76985698306197147867842 01+0	.34113867632450657072769 07-1
.82031990868417008926659 20+0	.29139408104744752282789 18-1
.86574557578851475359954 62+0	.23523498202667697892198 08-1
.90543319100185080225398 53+0	.17683297335043979745125 28-1
.93877151940458882972148 03+0	.12071965732676861974961 71-1
.96524779457470520888010 33+0	.71388124008803354648282 75-2
.98445520406942196353416 57+0	.32879084219191932797123 96-2
.99609883683388681640630 64+0	.83970911923589543087211 59-3

TABLE 11.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/(1 - T^2)^{3/4}$

T(I)	W(I)
N=2	
.2651104697351809990529214+0	.66919130283172978494696840+0
.93328991578435802970643400+0	.19528662514603900255178710+1
N=4	
.77578482444002621920837270-1	.19582611298213688147112290+0
.37092256668574864615304390+0	.40953253380491261879589790+0
.74342984922391992267456010+0	.62273169363369108031392330+0
.98276767632287712397424060+0	.13939672138713792298838960+1
N=6	
.36336435145560561879958080-1	.92379382974406466305823910-1
.18292224939661174618475239+0	.20042575657461027849663770+0
.41161355759679648438313137+0	.29004692106483496685159000+0
.66586195468113116572480990+0	.37965693043938415756850320+0
.87879624913350164572360700+0	.51945529864223372675466610+0
.99228650339093897686396160+0	.11400932645966502144876190+1
N=8	
.20975411123145326543531280-1	.53514924333774495486340050-1
.10759706825597412795812450+0	.11893329689581529359060200+0
.25151791038796482000091700+0	.17496504497011846112937130+0
.43288729033058504321243720+0	.22397184639192541862457910+0
.62540162547447988417285310+0	.27338547626164583629840130+0
.80016032481978710332123780+0	.33698787826051012588809840+0
.93035551575201293867189930+0	.45231024160773871106940630+0
.99564873139504534519992950+0	.98798884557059146837804110+0
N=12	
.95627935633959770121247040-2	.24471065085487920438431620-1
.49756723910748981761377560-1	.55637912043116580506654980-1
.11951151106849944795956570+0	.84156590865996126216177400-1
.21452138781998124566462960+0	.10955743650394456237261220+0
.32874473450640321522208450+0	.13232107249655039087125990+0
.45473648567644849664132930+0	.15363697223114951502774390+0
.58411885623567274157127450+0	.17526782868049812784208370+0
.70815853403112656747975460+0	.19972728414843376147119940+0
.81839234068001993091948190+0	.23116494727920237531519380+0
.90724248605534572925700800+0	.27849064818180266983937610+0
.96856765994287935969096550+0	.37050408362586991696738610+0
.99806145669488076135403790+0	.80712171315006786359672050+0
N=16	
.54467440441263105376372190-2	.13954627690976410838955150-1
.28490310765834251110361740-1	.32030632693471503308205960-1
.69100807354588366408833770-1	.49146878322904436280189750-1
.12585092609329211755378940+0	.64962500295090999622078750-1
.19670524226075528145619180+0	.79403285757797902470406350-1
.27908207286684014660766120+0	.92621525359134537328018980-1
.36993586391797630718646890+0	.10494952832695434670679350+0
.46586488698443568168896260+0	.11685975443437986916194610+0
.56323971723542339288121750+0	.12895532231441259618562690+0
.65834555143555595058514560+0	.14201266944955869796441150+0
.74753075731768617553595360+0	.15711420094716846597936390+0
.82735423970520748500644800+0	.17597319080610628456878650+0
.89472477511806154247369450+0	.20178330740569979404794780+0
.94702611474774873548991740+0	.24194957181956576669705790+0
.98222160619717713221427670+0	.32119156290100801257327620+0
.99890844757901095588298800+0	.69914899576789018673177430+0
N=24	
.24507607081204803967603270-2	.62845529467557349706650030-2
.12869834814090201104864970-1	.14533356815702051108163070-1
.31439216727617078332072070-1	.22574489148050090784926020-1
.57864384108950716977997800-1	.30297460737518360344221750-1
.91718538843130913204754730-1	.37632729847635755626881540-1
.13245089954272573089103900+0	.44545693452492064184040390-1
.17939410444655866694255140+0	.51035689353848627896993520-1
.23177361471929502590946040+0	.57132585615389521191306660-1
.28871934956181436479564920+0	.62892044106966783813224440-1
.34927945878956781323228950+0	.68390850561211083865534220-1
.41243598948132351410859260+0	.73723448684719269486171810-1
.47712211113443152877638960+0	.79000528277782774294870190-1
.54224051159154169458274000+0	.84350392748174670711519540-1
.60668255341357908990745080+0	.89924005150185797914774380-1
.66934777776262744817472690+0	.95905256632536434027069340-1
.72916335264526841759926870+0	.10252948592105769541800140+0
.78510307952457402879736470+0	.11011652648135262795244300+0
.83620559397711726533505110+0	.11913204059638848989038160+0
.88159142073096034353404160+0	.13030985938560433435336260+0
.92047856967456081438157270+0	.14492270034936439738142010+0
.95219638252517476846669290+0	.16547639614186421745831240+0
.97619732955354143628283860+0	.19793851005533217979146250+0
.99206609665377920880254310+0	.26243757251662646489131860+0
.99951442174846767541872940+0	.57097137876356038310777560+0

TABLE 12.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/(1 - T^2)^{2/3}$

T(I)	W(I)
N=2	
.2570976948190302772043214+0	.6419849115809653975193695+0
.9135412085939296756106419+0	.1461288246407215994243159+1
N=4	
.7653233938162432946381075-1	.1929856568699660324281308+0
.3656154327757110181960210+0	.3974989635025475272558533+0
.73401695919370998648872131+0	.5680256510921996992560396+0
.9769204026221799078755514+0	.9447628865234679328225050+0
N=6	
.3602196017706051441781718-1	.9155550156709031109846226-1
.1812596346320509902849203+0	.1978812519807219192116496+0
.4076001915726511517200313+0	.2823192190986969659268558+0
.6601500399582385666353307+0	.3364973738548132865082475+0
.8730226249610561619666100+0	.4495745095098818365869159+0
.9895629626953394871995762+0	.7254453019769770724303975+0
N=8	
.2084164908665650465654161-1	.5316820947072682296604277-1
.1068872083788446261851235+0	.1179943583985030242324827+0
.2498032088529165408252479+0	.1726834127852213762564197+0
.4299305744013840706186309+0	.2182490838586488582094152+0
.6213652195601669713899530+0	.2597310161548458330586683+0
.7957227345940077109705212+0	.3056748839714324063128131+0
.9266520688687292902739341+0	.3753166298403824934135317+0
.9940831687467691740027790+0	.6004555635084205773359551+0
N=12	
.9522796806409047438131568-2	.2436805964941454829532664-1
.4954493127894422114136726-1	.5538409304310284127512968-1
.1189901697882763172553211+0	.8366889948233679091726461-1
.2135645322795341638158066+0	.1085925812746155676184991+0
.3272625532237672327537250+0	.1308768263966818041547260+0
.4527048879609909767363908+0	.1498456496384564155117133+0
.5816047196924606463397596+0	.1682309488959117823945766+0
.7053328012456180694872836+0	.1871677878485580625475975+0
.8155267119849053544225609+0	.2090679118163116169261529+0
.9046891075507325292131218+0	.2385273341646520210799933+0
.9667247948525723015962074+0	.2886846337671386774780527+0
.9973512365128062061950650+0	.4593584320110012635634962+0
N=16	
.5429788413416968380732802-2	.1391103885278687716824364-1
.2840068715674590759239410-1	.3192635397215304558642838-1
.6887985272320408385162462-1	.4896517269825292984576983-1
.1254409556317280585382583+0	.6465266493602585425443367-1
.1960536143168552220171595+0	.7885671385605241947004000-1
.2781473417500304313838024+0	.9164837842042515296430830-1
.3686949392533592022938416+0	.1032553877123435937031784+0
.4643198422214430526570766+0	.1140165806279100370084840+0
.5614230178214642328267213+0	.1243557070742238827001678+0
.6563228104863825961320366+0	.1347876812758415779436306+0
.7454005204052867443898808+0	.1459745814246057647292905+0
.8252446671488822729014610+0	.1588761953193046161651780+0
.8927874042541401767806857+0	.1751391415493901301624227+0
.9454271736998061753928660+0	.1982896087629924391971207+0
.9811319351219213389887062+0	.2389929189352609985540311+0
.9985050085989204636036171+0	.3796250825706120723098011+0
N=24	
.2445710935107822797453948-2	.6271585350916622304767417-2
.1284318750927283794015242-1	.1450288127873926749839482-1
.3137358528490702374308480-1	.2252481411378850588273953-1
.5774229062126189544404887-1	.3022338182712332277208206-1
.9152265990831601977476978-1	.3752258634461899354661572-1
.1321645365241536910885975+0	.4437812760294266640101332-1
.1790018710891144679921088+0	.5077643327871736729338776-1
.2312622926318885481675080+0	.5673142517374345406468185-1
.2880788378684833370034769+0	.6227988654151781192498377-1
.3485037344855510034755054+0	.6747661370292565430889636-1
.4115239971384101771317705+0	.7239080182525497726749463-1
.4760785108021207416835926+0	.7710081763657254169110365-1
.5410762283149457451755300+0	.8169831034946114285342065-1
.6054150878535464422908734+0	.8628459338064329265368239-1
.6680012531107167367682701+0	.9097747280793932475322563-1
.7277682874068713229591989+0	.9591935775547927605544475-1
.7836958884338745148028214+0	.1012929537534091044138457+0
.8348278303068094393638829+0	.1073501607143330633277165+0
.8802887828053098284068161+0	.1144684964495337817743457+0
.9192997031149594162653681+0	.1232722899665470992654114+0
.9511915199940648974344861+0	.1349317518240564311497563+0
.9754168352947288898399360+0	.1520740783040970258306163+0
.9915591635768725210049186+0	.1828016156571662549162769+0
.9993333998143269862485147+0	.2900232223486544098126234+0

TABLE 13.- ERROR CONSTANTS k_N

Weight function	N						
	2	4	6	8	12	16	24
1	0.88323-2	0.35333-4	0.13925-6	0.54651-9	0.83793-14	0.12817-18	0.29917-28
2	.38568-2	.15381-4	.60574-7	.23766-9	.36431-14	.55722-19	.13006-28
3	.83723-2	.33428-4	.13195-6	.51784-9	.79396-14	.12145-18	.28347-28
4	.40962-2	.16231-4	.63926-7	.25081-9	.38488-14	.58808-19	.13726-28
5	.78509-2	.31380-4	.12366-6	.48528-9	.74402-14	.11381-18	.26564-28
6	.43400-2	.17319-4	.68214-7	.26764-9	.41029-14	.62755-19	.14647-28
7	.75454-2	.30175-4	.11891-6	.46663-9	.71543-14	.10943-18	.25543-28
8	.45125-2	.18010-4	.70939-7	.27834-9	.42669-14	.65265-19	.15233-28
9	.18611-1	.72707-4	.28401-6	.11094-8	.16925-13	.25831-18	.60142-28
10	.20227-2	.79014-5	.30865-7	.12057-9	.18397-14	.28071-19	.65359-29
11	.13386-1	.51613-4	.20089-6	.78342-9	.11935-13	.18198-18	.42341-28
12	.11944-1	.46749-4	.18286-6	.71488-9	.10918-13	.16668-18	.38828-28

TABLE 14.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 1

[Zero moment = 0.1402182105325454261175019+1]

J	B(J)	G(J)
1	0.6150198678196335788169563+0	0.0000000000000000000000000+0
2	0.4918847287293942058938206+0	0.9719998072864766978734869-1
3	0.4942420946542773527667192+0	0.6480442960076585929694835-1
4	0.4972154973440492668035494+0	0.6347162017942392756940588-1
5	0.4983553817384675085928096+0	0.6302665582117424823634536-1
6	0.4989138302174447250865230+0	0.6283096286916349924533615-1
7	0.4992291580192525787958213+0	0.6272727656339084492637858-1
8	0.4994245960626758460064597+0	0.6266571647154172186701221-1
9	0.4995540782953546455435342+0	0.6262618415588194270540389-1
10	0.4996442788137826402990677+0	0.6259929023366112311948653-1
11	0.4997096265296906928777585+0	0.6258016628848901163648707-1
12	0.4997584830310309334443183+0	0.6256608214529109093540000-1
13	0.4997959664415942918933535+0	0.6255540992892579903320774-1
14	0.4998253524094227033312671+0	0.6254712991341269109622364-1
15	0.4998488165401011149931981+0	0.6254057684446898891785706-1
16	0.4998678497832047459205261+0	0.6253530168863810152861730-1
17	0.4998835016646389778878341+0	0.6253099244959812457538696-1
18	0.4998965283447508966728075+0	0.6252742686957877446515471-1
19	0.4999074858084827709476154+0	0.6252444314072017553104359-1
20	0.4999167902564709870214112+0	0.6252192119340475533176225-1
21	0.4999247582582129859329286+0	0.6251977041042140630985484-1
22	0.4999316340639115500779560+0	0.6251792136294655017448687-1
23	0.4999376085632916995228546+0	0.6251632013092715978563307-1
24	0.4999428326887550657745987+0	0.6251492432368882826861675-1
25	0.4999474270486818254884930+0	0.6251370024294201042101103-1
26	0.4999514889558611242126238+0	0.6251262082814268729665467-1

TABLE 15.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 2

[Zero moment = 0.8413092631952725567050114+0]

J	B(J)	G(J)
1	0.4392999055855954134406831+0	0.0000000000000000000000000+0
2	0.4852017100183943815546000+0	0.7115415880902764271108506-1
3	0.4952340721964686178118784+0	0.6442813666042976091605580-1
4	0.4975594061782398313079980+0	0.6332966745806871531576270-1
5	0.4985136962886463803089016+0	0.6296973592602236653531604-1
6	0.4990001839394647825814603+0	0.6280211458494534797182123-1
7	0.4992813856065603058674026+0	0.6271063547685535342098320-1
8	0.4994585632933782518982225+0	0.6265525449907410750921175-1
9	0.4995773992440699269037173+0	0.6261918246049314627814951-1
10	0.4996609771701953955817794+0	0.6259437591540021127428069-1
11	0.49972199307331511660646697+0	0.6257658535884642161731121-1
12	0.4997678923911061752064208+0	0.6256339261223142347617901-1
13	0.4998032923765095789488353+0	0.6255333880575099221001711-1
14	0.4998311673467357528099563+0	0.6254550122226665985960952-1
15	0.4998535090524275498334804+0	0.6253927301393207138839993-1
16	0.4998716911400500233692867+0	0.6253424175026022321354261-1
17	0.4998866858891195975515897+0	0.6253011917047419224366962-1
18	0.4998991971771692743253767+0	0.6252669886828711531437449-1
19	0.4999097447421078882977161+0	0.6252382989725239067226655-1
20	0.4999187191160587306248568+0	0.6252139979967882585789658-1
21	0.4999264183447872010118851+0	0.6251932340168467508461636-1
22	0.4999330730828293368672846+0	0.6251753523559262072777206-1
23	0.4999388640880470893800002+0	0.6251598431194931680803091-1
24	0.4999439346373893106544550+0	0.6251463043927422864958548-1
25	0.4999483994818230691187482+0	0.6251344159158641393491053-1
26	0.4999523514007295296044060+0	0.6251239199535311867247599-1

TABLE 16.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 3

[Zero moment = 0.1311028777146059905232420+1]

J	B(J)	G(J)
1	0.5990701173677961037199612+0	0.0000000000000000000000000+0
2	0.4985753083912313065941279+0	0.9806157552139862646437001-1
3	0.4942065962236433906526132+0	0.6512306687296883760940316-1
4	0.4972508186534630970177334+0	0.6345674644453224600076581-1
5	0.4983683347062163400625880+0	0.6302166220438324487745683-1
6	0.4989207115762312343521780+0	0.6282864671434263341294804-1
7	0.4992332802030389296191316+0	0.6272595745632631642853379-1
8	0.4994272607319624542687284+0	0.6266489347407277283216556-1
9	0.4995558999123650428058766+0	0.6262563614141393223819131-1
10	0.4996455788146774711040877+0	0.6259890703886055597170153-1
11	0.4997105865356024598802937+0	0.6257988789342640280374204-1
12	0.4997592119907668423085779+0	0.6256587355905226469287555-1
13	0.4997965329291087615448246+0	0.6255524963080244789721383-1
14	0.4998258013316745562848209+0	0.6254700407730020482077893-1
15	0.4998491783012137528924902+0	0.6254047625922939490837249-1
16	0.4998681455611728783667530+0	0.6253522002612800804597432-1
17	0.4998837465770987898195050+0	0.6253092524613064958977411-1
18	0.4998967334164776476489372+0	0.6252737090370467398344471-1
19	0.4999076592322160168824781+0	0.6252439604041707708777642-1
20	0.4999169382228966046379858+0	0.6252188118087462402414351-1
21	0.4999248855154766978370628+0	0.6251973613204479311178697-1
22	0.4999317443028827493188126+0	0.6251789177340569642615230-1
23	0.4999377046879372664677897+0	0.6251629441264089274169945-1
24	0.4999429170091685794121872+0	0.6251490182977750387040971-1
25	0.4999475014210552731992583+0	0.6251368045618503435808750-1
26	0.4999515548854324930980912+0	0.6251260333100228893355157-1

TABLE 17.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 4

[Zero moment = 0.8740191847640399368216132+0]

J	B(J)	G(J)
1	0.4493025880258470777899709+0	0.0000000000000000000000000+0
2	0.4844472950418990698289468+0	0.7229513301995411333776938-1
3	0.4952496639498663135967710+0	0.6439933981136391471174203-1
4	0.4975314601266484748424108+0	0.6333844246251481963630555-1
5	0.4985023989292226020748584+0	0.6297404143551978993652384-1
6	0.4989940496814565716480681+0	0.6280415357384917522671346-1
7	0.4992776422578280706692689+0	0.6271182345569966339525942-1
8	0.4994561125571365342526840+0	0.6265600692090025239883507-1
9	0.4995757074945356397630047+0	0.6261968906479339953848994-1
10	0.4996597605308907849517661+0	0.6259473322475693390116272-1
11	0.4997210866633170393234334+0	0.6257684674954987965057991-1
12	0.4997672023488136769714523+0	0.6256358957201063222902333-1
13	0.4998027537947649678678176+0	0.6255349088761947173211291-1
14	0.4998307389515027002553390+0	0.6254562108885519244057762-1
15	0.4998531627246158081290975+0	0.6253936915837527579414533-1
16	0.4998714071878286934799331+0	0.6253432004126267878853528-1
17	0.4998864501898053314500696+0	0.6253018376844604379888347-1
18	0.4998989993891209937789153+0	0.6252675278900052877356511-1
19	0.4999095771521773948082559+0	0.6252387536989785377342982-1
20	0.4999185758769712546284759+0	0.6252143850096430134909688-1
21	0.4999262949586761333477443+0	0.6251935661164539603196196-1
22	0.4999329660441485916410501+0	0.6251756394597657275221424-1
23	0.4999387706320396402101213+0	0.6251600930023114625219419-1
24	0.4999438525599051594134650+0	0.6251465232200624994089398-1
25	0.4999483270082625276689169+0	0.6251346086276909425429012-1
26	0.4999522870893819907785304+0	0.6251240905454199895408462-1

TABLE 18.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 5

[Zero moment = 0.12143253239437908059099 71+1]

J	B(J)	G(J)
1	0.57735026918962576450914 88+0	0.00000000000000000000000000+0
2	0.50732448529324617769847 32+0	0.97851593204965089158919 17-1
3	0.49487951056762227333886 12+0	0.66071513310352115903727 86-1
4	0.49726720781309814106273 00+0	0.63425956648239527355011 40-1
5	0.49838250394211850721846 19+0	0.63017735977560869897835 38-1
6	0.49892829683804297025121 89+0	0.62826060669976448943688 88-1
7	0.49923784825461929189203 71+0	0.62724500394244149814868 37-1
8	0.49943021881043530463695 73+0	0.62663980396645002888051 71-1
9	0.49955792382108561279055 67+0	0.62625027491047948350220 06-1
10	0.49964702387344254960812 92+0	0.62598481177804346822020 12-1
11	0.49971165401219119543722 61+0	0.62579578380228603503183 75-1
12	0.49976002276011401200273 51+0	0.62565641591599461392229 56-1
13	0.49979716312278292802812 98+0	0.62555071323757373053872 82-1
14	0.49982630082758983184874 49+0	0.62546864077176255970715 78-1
15	0.49984958087965889276419 45+0	0.62540364333581425331887 12-1
16	0.49986847475758002172937 38+0	0.62535129143227000834857 44-1
17	0.49988401919534515613750 46+0	0.62530850444898083050109 64-1
18	0.49989696171320754256164 61+0	0.62527308603048188678019 60-1
19	0.49990785231701578996161 00+0	0.62524343603065682422350 83-1
20	0.49991710298021980476390 34+0	0.62521836629937293716256 28-1
21	0.49992502722625797000923 54+0	0.62519697962109213683028 59-1
22	0.49993186707263302570246 82+0	0.62517858821789478865738 21-1
23	0.49993781174722044830704 74+0	0.62516265769863111388864 51-1
24	0.49994301092815090553880 34+0	0.62514876776169245191946 94-1
25	0.49994758426510112102070 25+0	0.62513658416268439777968 21-1
26	0.49995162832955638525130 43+0	0.62512583840166390710162 47-1

TABLE 19.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 6

[Zero moment = 0.91074399295784310443247 81+0]

J	B(J)	G(J)
1	0.46168021535170061160731 90+0	0.00000000000000000000000000+0
2	0.48429224743831110531373 46+0	0.74123284358865614994835 00-1
3	0.49513034586909235988145 42+0	0.64289624956837301225733 80-1
4	0.49750603843756996108219 43+0	0.63362723050606834434341 78-1
5	0.49849014537579043843058 00+0	0.62977443241086388308024 08-1
6	0.49898712137829791015580 20+0	0.62806503514308054830731 68-1
7	0.49927342494413310963101 03+0	0.62713158831125057439784 24-1
8	0.49945335237088117464137 16+0	0.62656854849366320396307 14-1
9	0.49957380383624814313487 57+0	0.62620259370557270585923 52-1
10	0.49965839265326314875614 69+0	0.62595135118324139693692 33-1
11	0.49972007096245247816219 21+0	0.62577140520887456189038 00-1
12	0.49976642759174141008867 05+0	0.62563810779338341688822 23-1
13	0.49980214941650935158910 21+0	0.62553661592836412676403 80-1
14	0.49983025844026160587916 17+0	0.62545755567327144907806 11-1
15	0.49985277441677142386944 41+0	0.62539476977356885581679 79-1
16	0.49987108892421445465371 65+0	0.62534407806957315121648 20-1
17	0.49988618608766151586010 96+0	0.62530256161122644219921 80-1
18	0.49989877782407546029955 38+0	0.62526813199363444905585 01-1
19	0.49990938945842152748190 10+0	0.62523926303036893926256 71-1
20	0.49991841548775016172927 42+0	0.62521481840054665296843 03-1
21	0.49992615682455072045245 16+0	0.62519393794401442697029 56-1
22	0.49993284623099349211903 31+0	0.62517596085350725893540 35-1
23	0.49993866603801167829134 50+0	0.62516037268617026879551 44-1
24	0.49994376071278587993461 90+0	0.62514676811032484279788 76-1
25	0.49994824591811282380462 02+0	0.62513482426447220625837 08-1
26	0.49995221513990677601158 39+0	0.62512428140860837236638 51-1

TABLE 20.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 7

[Zero moment = 0.11635925712182693753025 18+1]

J	B(J)	G(J)
1	0.56335387900140439567340 88+0	0.00000000000000000000000 00+0
2	0.51119583717209448584565 940+0	0.96845969387166064301155 48-1
3	0.49625404673071696960474 99+0	0.66957340493676726819436 45-1
4	0.49725430875851471199814 88+0	0.63467144933885798943236 79-1
5	0.49839213921247103990606 51+0	0.63012138260767852298107 68-1
6	0.49893254894504937802496 49+0	0.62824804237797974647370 85-1
7	0.49924044660006275102497 53+0	0.62723657047411175222989 01-1
8	0.49943190271305641477045 80+0	0.62663461797919091807704 79-1
9	0.49955907593296958286739 82+0	0.62624680995773023120914 88-1
10	0.49964784675963164950967 92+0	0.62598238711117059933756 20-1
11	0.49971226203892259461609 46+0	0.62579402106340346848105 36-1
12	0.49976048467564324034857 43+0	0.62565509448978016164028 07-1
13	0.49979752223350512131954 74+0	0.62554969727168892504778 78-1
14	0.49982658551314198710217 71+0	0.62546784291835231060121 87-1
15	0.49984981036481219799007 67+0	0.62540300536753760513348 07-1
16	0.49986866243948693639494 09+0	0.62535077332643662816793 06-1
17	0.49988417464120915492145 87+0	0.62530807796269686002445 40-1
18	0.49989709190254623710167 82+0	0.62527273077312721981394 51-1
19	0.49990796243812218073340 44+0	0.62524313698309861081003 39-1
20	0.49991719695467873040970 69+0	0.62521811220151772509293 86-1
21	0.49992510806266695407519 17+0	0.62519676189706466981178 00-1
22	0.49993193711024427679461 24+0	0.62517840024335039246419 96-1
23	0.49993787282697783073672 76+0	0.62516249429118253523981 75-1
24	0.49994306451483324664063 50+0	0.62514862482005896028342 10-1
25	0.49994763153591525065465 07+0	0.62513645840695319145866 66-1
26	0.49995167023931209996809 97+0	0.62512572718353702141732 50-1

TABLE 21.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 8

[Zero moment = 0.93087405697461550024201 45+0]

J	B(J)	G(J)
1	0.46946156583450366306117 40+0	0.00000000000000000000000 00+0
2	0.48495269501359121385939 96+0	0.75472668470712446415518 46-1
3	0.49487717972208515512340 72+0	0.64230434286440209963939 76-1
4	0.49751757426712366335192 54+0	0.63368847536188492308061 22-1
5	0.49848032285205289291480 41+0	0.62980768343674510932524 75-1
6	0.49898331218798263498025 23+0	0.62807706571374568744761 97-1
7	0.49927098312527078610531 30+0	0.62713947246284079299255 83-1
8	0.49945176273818459296613 90+0	0.62657341769435887562168 90-1
9	0.49957270871420814479204 63+0	0.62620587782617768376572 33-1
10	0.49965760630866299804998 10+0	0.62595366225126329614483 88-1
11	0.49971948744481371032236 88+0	0.62577309342462126543609 75-1
12	0.49976598271783761786740 56+0	0.62563937829381134601595 07-1
13	0.49980180251949190150142 99+0	0.62553759592505719053424 61-1
14	0.49982998273512828789357 33+0	0.62545832740521834081594 35-1
15	0.49985255168118345464774 17+0	0.62539538831727981745348 47-1
16	0.49987090641260788566785 62+0	0.62534458143282897317007 86-1
17	0.49988603466882952445407 84+0	0.62530297670760253716957 91-1
18	0.49989865081771810792906 09+0	0.62526847831267160803793 48-1
19	0.49990928188622573632741 15+0	0.62523955496571682820156 60-1
20	0.49991832357843414793882 49+0	0.62521506676890052538069 93-1
21	0.49992607767902451715214 07+0	0.62519415100099049942946 68-1
22	0.49993277759106960717220 88+0	0.62517614498821801149209 24-1
23	0.49993860612362533986026 70+0	0.62516053290558937310915 56-1
24	0.49994370810547961387774 97+0	0.62514690838312036424912 98-1
25	0.49994819947634745082536 13+0	0.62513494776904218125856 91-1
26	0.49995217393663853864761 69+0	0.62512439071464913669386 26-1

TABLE 22.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 9

[Zero moment = 0.2428650647887581611819942+1]

J	B(J)	G(J)
1	0.4311849265382984224922525+0	0.0000000000000000000000000+0
2	0.5344112677185342651850937+0	0.1215894930360789821822357+0
3	0.4997081663357153245114400+0	0.6302335181643522600332703-1
4	0.4999979923236526420216647+0	0.6250797750313006342035336-1
5	0.5000007535559464804948454+0	0.6249996102798235805729757-1
6	0.5000000013213312142794695+0	0.6249999662216613204448769-1
7	0.4999999999586801854786577+0	0.6250000000704277852836404-1
8	0.4999999999988624260515453+0	0.6250000000232057766949852-1
9	0.5000000000000277664504764+0	0.6250000000000495634934922-1
10	0.5000000000000010935031495+0	0.6249999999999810836830959-1
11	0.4999999999999999800980423+0	0.6249999999999998709267165-1
12	0.4999999999999999988887928+0	0.6250000000000000167223660-1
13	0.5000000000000000000140255+0	0.6250000000000000001995067-1
14	0.5000000000000000000011640+0	0.6249999999999999999845983-1
15	0.4999999999999999999999913+0	0.6249999999999999999997285-1
16	0.4999999999999999999999988+0	0.6250000000000000000000144-1
17	0.5000000000000000000000000+0	0.6250000000000000000000004-1
18	0.5000000000000000000000000+0	0.6250000000000000000000000-1
19	0.5000000000000000000000000+0	0.6250000000000000000000000-1
20	0.5000000000000000000000000+0	0.6250000000000000000000000-1
21	0.5000000000000000000000000+0	0.6250000000000000000000000-1
22	0.5000000000000000000000000+0	0.6250000000000000000000000-1
23	0.5000000000000000000000000+0	0.6250000000000000000000000-1
24	0.5000000000000000000000000+0	0.6250000000000000000000000-1
25	0.5000000000000000000000000+0	0.6250000000000000000000000-1
26	0.5000000000000000000000000+0	0.6250000000000000000000000-1

TABLE 23.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 10

[Zero moment = 0.5235987755982988730771072+0]

J	B(J)	G(J)
1	0.5348805958594948912557218+0	0.0000000000000000000000000+0
2	0.4998413758642581247934532+0	0.6178132826217020632362457-1
3	0.4999755312512212612701437+0	0.6252901993772103984020583-1
4	0.5000000431490767513877482+0	0.6250068874693278612242355-1
5	0.5000000242857029786729298+0	0.6249997553471393707650941-1
6	0.5000000000653895318792445+0	0.6249999918983040738478722-1
7	0.4999999999739740331698369+0	0.6250000002234182735731698-1
8	0.4999999999998037014320393+0	0.6250000000009618865540936-1
9	0.5000000000000285940367426+0	0.6249999999997963060999185-1
10	0.5000000000000003647314906+0	0.6249999999999876211407871-1
11	0.4999999999999999684488097+0	0.6250000000000001758908590-1
12	0.4999999999999999994158492+0	0.6250000000000000153639510-1
13	0.5000000000000000000345507+0	0.6249999999999999998677223-1
14	0.5000000000000000000008687+0	0.6249999999999999999810552-1
15	0.4999999999999999999999628+0	0.6250000000000000000000645-1
16	0.4999999999999999999999988+0	0.6250000000000000000000231-1
17	0.5000000000000000000000000+0	0.6250000000000000000000000-1
18	0.5000000000000000000000000+0	0.6250000000000000000000000-1
19	0.5000000000000000000000000+0	0.6250000000000000000000000-1
20	0.5000000000000000000000000+0	0.6250000000000000000000000-1
21	0.5000000000000000000000000+0	0.6250000000000000000000000-1
22	0.5000000000000000000000000+0	0.6250000000000000000000000-1
23	0.5000000000000000000000000+0	0.6250000000000000000000000-1
24	0.5000000000000000000000000+0	0.6250000000000000000000000-1
25	0.5000000000000000000000000+0	0.6250000000000000000000000-1
26	0.5000000000000000000000000+0	0.6250000000000000000000000-1

TABLE 24.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 11

[Zero moment = 0.2622057554292119810464840+1]

J	B(J)	G(J)
1	0.7627597635018131880623260+0	0.0000000000000000000000000+0
2	0.4356406220177258406970294+0	0.8486420984932468062146180-1
3	0.4836622253812257957504204+0	0.6015784166190404885710378-1
4	0.4926359637757834902129061+0	0.6193678264971025932719100-1
5	0.4958160622403241212889954+0	0.6225222585037623492812274-1
6	0.4973025753683422198594969+0	0.6236205886612335061715978-1
7	0.4981164296636529946910406+0	0.6241260383439648456735817-1
8	0.4986102355901269385187147+0	0.6243986224336240279938661-1
9	0.4989323521912857650120867+0	0.6245617352484519608591106-1
10	0.4991541052638607539612490+0	0.6246668234920447603665392-1
11	0.4993132670282797887930142+0	0.6247383767893202305716068-1
12	0.4994313681629815576863545+0	0.6247892354516655206672295-1
13	0.4995214144758561080843496+0	0.6248266485212445221838946-1
14	0.4995916401095205920752087+0	0.6248549552240358527784499-1
15	0.4996474647387156655116376+0	0.6248768795508785713376816-1
16	0.4996925741142250124147184+0	0.6248942003341689749557342-1
17	0.4997295457223993006853085+0	0.6249081184928942431655294-1
18	0.4997602259978321357255499+0	0.6249194681785359645619747-1
19	0.4997859657879120033126338+0	0.6249288432176976983229232-1
20	0.4998077718433315107473383+0	0.6249366755340057480597428-1
21	0.4998264069016278370299617+0	0.6249432852952401654116095-1
22	0.4998424573786985573892202+0	0.6249489138525996160567481-1
23	0.4998563801202517857375893+0	0.6249537458977838778488676-1
24	0.49986853553030452627082922+0	0.6249579246870493527987046-1
25	0.4998792099848722599216561+0	0.6249615627029216380232851-1
26	0.4998886352223224496996331+0	0.6249647492504550799883447-1

TABLE 25.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 12

[Zero moment = 0.2103273157988181391762529+1]

J	B(J)	G(J)
1	0.7131741278126598550076291+0	0.0000000000000000000000000+0
2	0.4574649756003002978073342+0	0.9138266341865190505355904-1
3	0.4880003168683295789202167+0	0.6214157546585639457435741-1
4	0.4944457136921354886874163+0	0.6256266089751222097592868-1
5	0.4968059789822665620923860+0	0.6256299807035465916956908-1
6	0.4979263010407050897093034+0	0.6254803891898191389681332-1
7	0.4985452766986458567019637+0	0.6253641066555803461296587-1
8	0.4989231367944321599964818+0	0.6252821015902222334908550-1
9	0.4991707046055156317255353+0	0.6252238768292636176628102-1
10	0.4993417030237524201212730+0	0.6251815423805393280826464-1
11	0.4994647555958287820432594+0	0.6251499729997178538489978-1
12	0.4995562544624473435839419+0	0.6251258760325053868405053-1
13	0.4996261382321980163249037+0	0.6251070987590363060099149-1
14	0.4996807182051692635412360+0	0.6250921991884649904453897-1
15	0.4997241588988035784313898+0	0.6250801871899012795326180-1
16	0.4997592984903970030365944+0	0.6250703668503265348254934-1
17	0.4997881253624322011219338+0	0.6250622384898620495766689-1
18	0.4998120661803502762728449+0	0.6250554363219312863216430-1
19	0.4998321661305429612525275+0	0.6250496878593744372225568-1
20	0.4998492051120634935611432+0	0.6250447868911064312231361-1
21	0.4998637746348907397280967+0	0.6250405751193489093328913-1
22	0.4998763299112804176981203+0	0.6250369294431442926087396-1
23	0.4998872258830428922387086+0	0.6250337529972380990634016-1
24	0.4998967426074243971984465+0	0.6250309687354327723032198-1
25	0.4999051034483297284078400+0	0.6250285147678126826373071-1
26	0.4999124883125537396555600+0	0.6250263409262308183342118-1

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