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FREE-FALL PERIODIC ORBITS CONNECTING EARTH AND MARS

by

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OCTOBER 1969

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ABSTRACT

Free-fall periodic orbits which join Earth and Mars and which go back and forth between Earth and Mars forever are found through use of a patched conic analysis. Each of the periodic orbits found includes round trips from Earth to Mars and back to Earth along with series of direct returns at Earth. The periodic orbits are first established by computer solution in the case where the two planets are in circular coplanar orbits; then computer solution is attempted in the eccentric inclined case and has been found for several of the periodic orbit schemes attempted. In order to find these periodic orbits, a logical approach is developed to combine two round trips to Mars and two separate series of direct returns at Earth in an arrangement which is "symmetric" in time. The selection of the series of direct returns is facilitated by the formation of a list of all combinations of direct returns. The most efficient periodic orbit found requires four Earth-Mars synodic periods or 8.33 years on the average to make two round trips to Mars. Other periodic orbits found require five or more synodic periods to make the two visits to Mars.

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LIST OF SYMBOLS

Abbreviations

☉--Sun

⊕--Earth

♀--Venus

♂--Mars

E--Earth

M--Mars

a.u.--astronomical unit

EMOS--Earth Mean Orbital Speed unit which is equal to 2π a.u. per year or 29.77 km./sec. or 97,702 ft./sec.

RMS--root mean square

FR--full revolution return

HR--half revolution return

SiSR--symmetric return which traverses the Sun more than i times but which is shorter than the length indicated by linear analysis (about (i+0.45) revolutions or planetary periods)

LiSR--symmetric return which traverses the Sun more than i times and which is longer than the length indicated by linear analysis

IP--interplanetary trajectory

Coordinate Systems

R,G,Z--orthogonal coordinate system which is usually associated with a planet and which has the coordinate directions respectively pointing away from the Sun along the Sun-planet line (R), in the plane of the planet's orbit pointing approximately in the direction of the planet's motion (G), and vertical to the planet's orbital plane to complete the triad

P,T,Z--orthogonal coordinate system which is usually associated with a planet where the P-axis points approximately away from the Sun in the plane of the planet's orbit, the T-axis is aligned with the planet's velocity vector and points in the same direction, and the Z-axis is the vertical to the planet's orbital plane and is the same Z-axis as in the above triad. This coordinate system is exactly the same as the R,G,Z system for planets in circular orbits.

<u>Quantities</u>

a--semimajor axis

a_1--semimajor axis of the orbit of planet 1.

A_{IO}--the angle between the incoming hyperbolic excess velocity and C_O corresponding to a minimum T_O

A_{INT}--the size of one of the equal M-1 or M turns between points on the full revolution return locus where M is the number of full revolution returns

A_{OI}--the angle between C_I corresponding to a minimum T_I and the outgoing hyperbolic excess velocity

C_A--cone half angle

C_I--specifies the point on the full revolution return locus corresponding to T_I. Can be an angle around the locus

C_O--specifies the point on the full revolution return locus corresponding to T_O. Can be an angle around the locus

D--maximum date of interest as defined by Equation 4-1

e--eccentricity

E--eccentric anomoly

f--true anomoly. In particular, the true anomoly after a series of FR and one HR.

f_o--true anomoly before a series of FR and one HR

f^*--incorrect true anomoly corresponding to a point near the end of a series of FR and one HR. Equal to f if e=0

M--the number of full revolution returns in a row, either alone or before or after a half revolution return
Also, the mean anomoly. In particular, the mean anomoly after a series of FR and one HR

M_o--mean anomoly before a series of FR and one HR

M^*--incorrect mean anomoly corresponding to a point near the end of a series of FR and one HR. Equal to M if e=0

N--basic dimension of the numerical problem. Also, the dimension of the column matrices $\underline{v}$, $\underline{t}$, and $\underline{0}$

n--number of synodic periods. Also, the population of elememts

N_n^r--the number of combinations of elements taken r at a time from a population of n elements with replacement and without regard for order

$\underline{0}$--column matrix of all zeros

P_1--period of planet 1

q--convenient symbol for the quantity $(4 \sin 2\pi t - 6\pi t)$

r--convenient symbol for the quantity $(2 - 2 \cos 2\pi t)$
Also, the number of elements taken at a time

r_i--one of n non-negative integers which must be less than or equal to n

s--convenient symbol for the quantity $(\sin 2\pi t)$

T--time from opposition as defined by Equations (4-2) and (D-1)

t--independent variable of time measured in units of the planetary period.

$\underline{t}$--column matrix of encounter dates

t_i--the i^{th} element of this column matrix which is the i^{th} independent date of planetary encounter

$\underline{t}_{new}$--improved estimate of the desired $\underline{t}$

T_d--length of a series of direct returns

T_I--turn angle from the incoming hyperbolic excess velocity vector onto the locus which produces a full revolution return. Always starts out as the smallest such angle

T_O--turn angle from the locus which produces a full revolution return to the outgoing hyperbolic excess velocity vector. Always starts out as the smallest such angle

T_s--length of a synodic period

T_{cycle}--the length of time required for the periodic orbit to repeat or almost repeat

T --time spent on a series of direct returns at Earth

T --time spent on a series of direct returns at Venus

$\underline{v}$--column matrix of hyperbolic excess speed differences at the encountered planets

v_i--the i^{th} element of this column matrix of speed differences $(= v_{out_i} - v_{in_i})$

v_{in_i}--hyperbolic excess velocity of arrival at the i^{th} planetary encounter

v_{out_i}--hyperbolic excess speed of departure at the i^{th} planetary encounter

V_H--hyperbolic excess speed relative to the planet of encounter

V_P--speed of planet relative to the Sun

V_x--R component of hyperbolic excess speed measured in units of the planet's orbital speed (in EMOS for Earth)

V_y--G component of hyperbolic excess speed measured in units of the planet's orbital speed (in EMOS for Earth)

x--distance in a.u. in the R direction of a vehicle from a planet such as Earth in circular orbit around the Sun

y--distance in a.u. in the G direction of a vehicle from a planet such as Earth in circular orbit around the Sun

ϵ --step size variable

μ--gravitational constant for an orbit around the Sun

τ--time of perihelion passage

$(\dot{\ })$--indicates the first time derivative of ( )

$(\ddot{\ })$--indicates the second time derivative of ( )

CHAPTER 1

INTRODUCTION

1.1 What is Meant by the Term Periodic Orbit

What one means by the term "periodic orbit" is an interplanetary, free-fall trajectory which visits one or more planets and continues to visit these same planets repeatedly for an indefinite period of time. Each planetary encounter is assumed to involve a flyby maneuver by which the velocity vector of the vehicle relative to the encountered planet is rotated in space; that is, rotated by the effect of the gravitational mass of the encountered planet. The planets are to be visited repeatedly in a certain sequence which involves a certain order for the planetary encounters and certain specific types of trajectories between the planetary encounters. The periodicity of the periodic orbit comes from the fact that the certain sequence of encounters repeats indefinitely and from the fact that one has at least approximately repeating absolute or relative positions of the encountered planets.

Important qualities of these periodic orbits are that they require no propulsive thrust except for guidance once they are established and that they definitely require guidance in order to be maintained. On a periodic orbit during a flyby maneuver, no thrust is needed except for guidance, because each flyby maneuver in a periodic orbit must be designed as an unthrusted flyby which does not intersect the surface of the planet. This quality of a periodic orbit is an essential one. Interplanetary periodic orbits are expected to be unstable in the sense that any small deviation in position or velocity from the periodic orbit will result in an increasing departure from the periodic orbit and eventual breakdown of the periodic orbit scheme. Because of this instability, continuing guidance including velocity corrections is necessary to maintain a vehicle on or near the periodic orbit.

The types of trajectory legs which are used to make up the interplanetary elements of a periodic orbit may be broadly divided into two categories. These two categories are interplanetary trajectory legs and what are called direct return trajectory legs. A great deal of effort has been expended in the past in the examination of interplanetary trajectory legs, because these are basically the trajectory legs which get one somewhere in the solar system other than back to the place from which he started. The other class of trajectories, the direct return trajectories, only transport one back to the planet which he last left. Such direct returns are very useful, however, as a part of periodic orbits, because they make it possible for a vehicle to remain in the vicinity of one planet without going on to another one until a favorable opportunity presents itself when the planets are in the desired relative position. A series of direct returns can also help to turn the hyperbolic excess velocity vector in the desired direction for the transfer to the next planet by supplying several additional flybys of the planet so that the desired turn can be made in several smaller steps and hitting the planet during a flyby can thereby be avoided. Because interplanetary periodic orbits have not been considered for very long, direct returns have been neglected; and hence, the third chapter is devoted to direct returns..

A periodic orbit is the logical extension of having an increasing number of planetary flybys in a single trajectory with a series of planetary flybys. Various people[4,9,10,11] have found trajectories which depart from and return to Earth after having encountered either Mars or Venus or both. Minovitch[13] has found a mission involving six flybys at Earth, Mars, and Venus. VanderVeen[5] has found multiple flyby trajectories which depart from and return to Earth and intermediately encounter Venus, Mars, and Venus a second time. Then, in order to increase indefinitely the number of planetary encounters, one comes to the idea of a periodic orbit joining several planets.

Hollister[1] and Menning[2] discovered periodic orbits connecting Earth and Venus which repeat after 16 years. Their periodic orbits involve direct return trajectories at both Earth and Venus connected through flyby maneuvers at the planets with fairly quick transfers between the two planets. Obtaining periodic orbits which connect Earth and Mars can be expected to be a more difficult and more complicated task, because the small mass of Mars means that less change in velocity or rotation of the velocity vector is available from a flyby of that planet. In addition, the much larger eccentricity of the orbit of Mars over that of Venus might be expected to cause difficulty in numerically calculating the periodic orbit.

Broucke[16] has found periodic orbits in the Earth-Moon system. His problem, however, is considerably different from the problem of this thesis, as his mathematical model involved integration of three-body equations of motion. This thesis and the work of Hollister and Menning involve, however, only two-body equations of motion coupled with a patched conic approximation to handle the planetary encounters.

1.2 Possible Applications of Periodic Orbits

One might ask at this point, "Of what possible use are these periodic orbits?" Hollister[1] presents several possible applications. Probably the most obvious application for periodic orbits is that of a manned, reuseable interplanetary vehicle. A fairly large and comfortable vehicle could be used efficiently for interplanetary travel, because it would be reuseable and no additional thrust would be required (except for guidance) once the vehicle had been set upon the desired periodic orbit. Of course, additional vehicles would be necessary to shuttle personnel and material between the vehicle on the periodic orbit and the surface of the encountered planets in order to complete the interplanetary transportation system.

Hollister[1] suggests several more less obvious uses for a vehicle following a periodic orbit. He suggests that it could also serve as a communications link, a rescue station, or an interplanetary navigation beacon. In time, more uses will probably be discovered.

1.3 Approximations Used in This Thesis

There are two main assumptions or approximations on which the work of this thesis (and the work of Hollister[1] and Menning[2]) is based. The first approximation is that of patched conic trajectories in which the size of the sphere of influence of the flyby planet is neglected. The second approximation involves the periodicity of the solar system.

1.3.1 The Patched Conic Approximation

The patched conic approximation involves the use of only two-body conic trajectory segments in order to form a good approximation to the periodic orbit which is desired. For a trajectory which involves close approaches to several planets, one can divide the trajectory into segments which are Sun-centered and connect the planetary spheres of influence and into other segments which describe the trajectory inside the sphere of influence of a planet as a hyperbola relative to planet-centered, non-rotating coordinates. The Sun-centered segments are considered to be only under the influence of the Sun and are generally elliptical with respect to Sun-centered, non-rotating coordinates.

In addition, the approximation is made that the size of the planetary sphere of influence may be neglected. The radius of the sphere of influence of Earth is approximately 0.006 a.u. while the radius of the sphere of influence of Mars is approximately 0.004 a.u. The smallness of these planetary spheres of influence relative to the astronomical unit suggests perhaps that their finite radius may be neglected for purposes of approximately calculating trajectories involving a series of planetary flybys. A more detailed investigation[20] also indicates that this approximation is a reasonable one.

One would like to know the net effect of the patched conic approximation discussed here which neglects the size of the sphere of influence. The size of differences between quantities on the exact trajectory and quantities on the approximate trajectory are the measure of the effect in which one is interested. In particular, there is interest in possible changes in encounter dates, hyperbolic excess speeds, and minimum passing distances for multiple flyby trajectories. The changes in the end velocities and times for a simple trajectory between two planets are only a rough indication of the net changes in which one is interested.

Not much work has been done in obtaining accurate results for multiple flyby trajectories, although Bayliss[28] is working on a method to obtain these desired accurate results. An indication of the changes which can be expected when going from the patched conic approximation to an accurate integrated trajectory is given by the work of Sturms and Cutting[17] which compares the results of the approximate and the accurate calculations for three different Earth-Venus-Mercury trajectories. The differences at the encounter at Venus are less than one day for the time of closest approach, less than 0.03 venusian radius in the distance of closest approach, and less than one unit in the third significant digit of the hyperbolic excess speed. The conclusion is that the patched conic approximation is sufficiently accurate for the purposes of this investigation.

In short, for the purposes of this thesis, trajectory segments are to be calculated from planetary center to planetary center. From this calculation, hyperbolic excess velocities relative to the planet may be approximately determined. If the planetary flyby is to be accomplished, the hyperbolic excess speeds relative to the planet before and after the encounter must be equal. In addition, the angle between the incoming hyperbolic excess velocity vector and the outgoing hyperbolic excess velocity vector must be sufficiently small so that the flyby can be accomplished without hitting the planet; this requirement is checked only after the hyperbolic excess speeds have been approximately matched. In summary, this approximation means that one models a planet as a moving massless point in space from which an impulse of acceleration is available by passing through the point. The impulse is constrained such that the speed relative to the moving point must be the same immediately before and immediately after the encounter with the point.

1.3.2 The Periodicity of the Solar System

An important requirement for the existence of periodic orbits joining two or more planets is that the relative positions and the relative velocities of the encountered planets and the Sun must repeat after a certain length of time. In order for this to occur for planets in general elliptic, mutually inclined orbits, the absolute positions must repeat. Of course, for any real case, the absolute or even the relative positions of two or more planets and the Sun will never repeat exactly. If one approximates the elliptic orbits of two planets by circular orbits in the same plane and having the same periods as the elliptic orbits, then the relative positions and velocities will repeat exactly every synodic period. The synodic period in this circular coplanar case will be the same as the average synodic period for the elliptic, inclined case. An integer number of these synodic periods will be the length of a basic repeating cycle of a periodic orbit involving these two planets in the circular coplanar case. Conditions are similar with three planets instead of two, with the additional complexity that there must exist three numbers which differ by integers, which are associated individually with each of the three planets, and which, when multiplied by the period of the corresponding planet, each produce the same time. This period of time for the three planets is analogous to the synodic period for two planets in circular coplanar orbits and is the time required for the relative positions of the three planets to repeat.

The length of time for the repeating cycle of a periodic orbit will, in general, be an integer multiple of the length of time after which the relative positions and velocities of the Sun and the planets of concern repeat. In the circular coplanar case, the basic repeating cycle must be some integer multiple of the synodic period, or, in the case of three or more planets, some integer multiple of the basic time for the repeat of the relative positions. In the elliptic inclined case, the time for the basic repeating cycle must be some integer multiple of the time after which the absolute positions of the planets of concern and the Sun repeat. In fact, in the case of a periodic orbit involving two planets, the time for the elliptic inclined case to repeat is equal in synodic periods, to the product divided by the greatest common divisor of two integers. These two numbers are the number of synodic periods for the absolute planetary positions to repeat and the number of synodic periods involved in the basic circular coplanar scheme.

In reality, truly periodic orbits which join two or more planets do not exist, because the absolute positions of the Sun and the planets involved will never repeat exactly. However, in the case of two planets, one would expect that an indefinitely long series of unthrusted flybys would exist which are arranged in the same scheme as in the circular coplanar case in which the relative positions and velocities repeat exactly. One would also expect that the dates which are calculated for a series of flybys following the scheme of a periodic orbit and which have the initial and final speeds the same and which last the correct number of synodic periods will give a very good approximation to the dates of encounter for the continuous series of flybys. Hollister[1] and Menning[2], however, took care of this periodicity problem by modeling the planets' orbits as truly periodic.

In the case of periodic orbits involving three planets, one would expect that they would not exist at all, because the relative times of important planetary oppositions involving planets in different pairs would not continue to have the same timing relative to each other. One would expect, however, that a very long series of flybys could be found from an approximately periodic scheme involving three planets. One might be able to find an indefinitely long series of planetary flybys involving three planets by changing the basic scheme of the series of encounters when the relative alignment of oppositions strays too far from the desired spacing.

The planets of interest for this investigation are Earth, Mars, and Venus. To within an accuracy of a few degrees, the absolute positions of these three planets and the Sun repeat after 32 years. In that period of time Earth makes 32 revolutions of the Sun, Venus makes 52, and Mars makes 17. Hence, in the circular coplanar case, the relative positions and the relative velocities approximately repeat after 6.4 years. In this shorter period of time, Earth makes 6.4 revolutions of the Sun, Venus makes 10.4, and Mars makes 3.4. These figures imply certain relations also concerning average synodic periods. The relative positions and periods are indicated in Table 1-1 and in Figure 1-1.

	planet	Earth	Venus	Mars
1.	revolutions of Sun in 32 years	32	52	17
2.	synodic periods relative to Earth in 32 yr.	-	20	15
3.	revolutions of Sun in 6.4 years	6.4	10.4	3.4
4.	synodic periods relative to Earth in 6.4 yr.	-	4	3
5.	approximate period in years	1	8/13	32/17

6.	approximate synodic period in years	-	32/20=1.6	32/15=2.133
	... in days	-	584.4	779.2
7.	more exact synodic period in days7	-	583.92	779.94
8.	row 4 times row 6 in days	-	2337.6	2337.6
9.	row 4 times row 7 in days	-	2335.68	2339.82

Table 1-1. Approximate relative periods of Earth, Venus, and Mars.

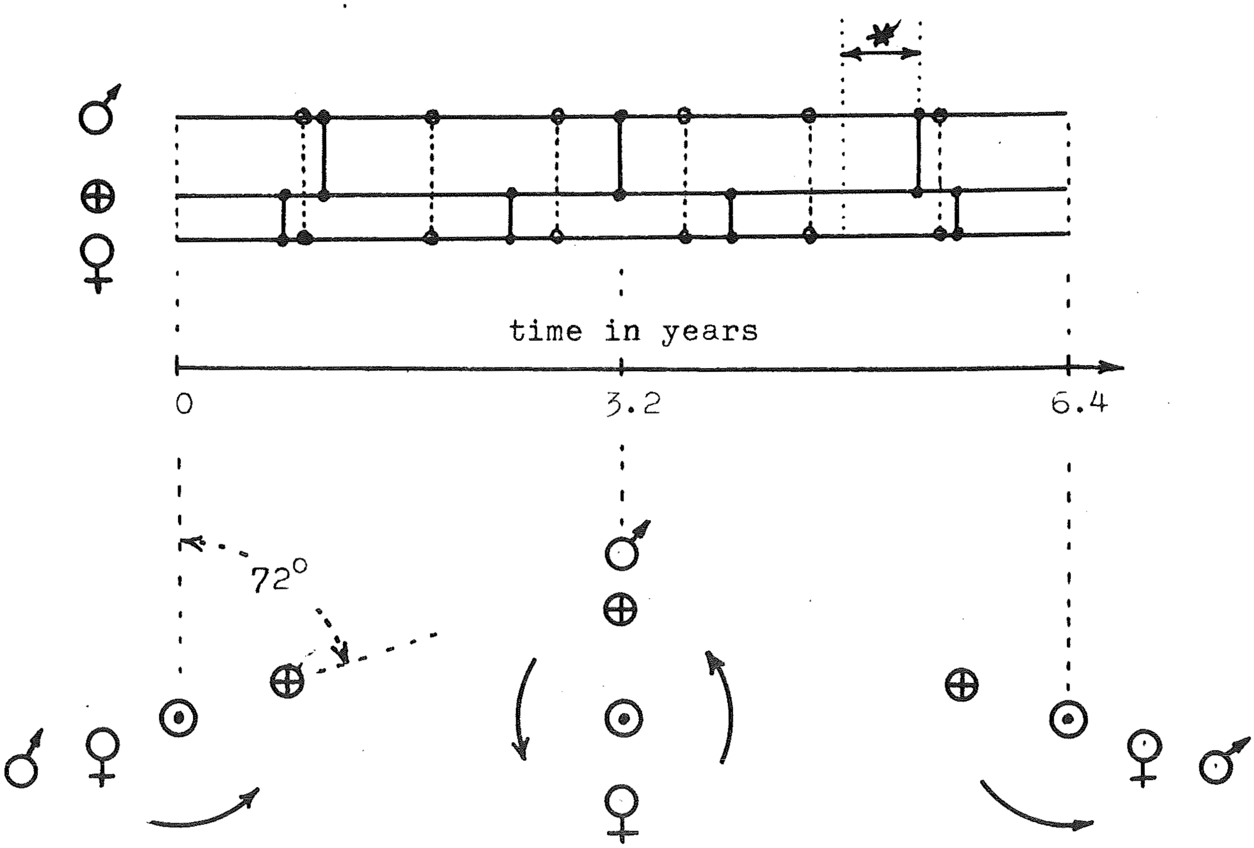


Figure 1-1. Spacing of the planetary alignments. (Vertical lines indicate oppositions between the planets indicated.)(Obtained from Reference 1.)

The alignments indicated by vertical lines in Figure 1-1 do not occur at exactly the same instants which the figure tends to demonstrate; the time of the opposition of Mars does not correspond exactly with the time of the superior conjunction of Venus as the middle sketch of Figure 1-1 indicates. And, the relative timing of an opposition of Mars and a superior conjunction of Venus does not remain the same. The times of opposition vary due to the eccentricities of the planets' orbits. Also, the relative timing tends to drift slightly with each passage of 6.4 years. The characteristics of this drift are indicated by the differences among the numbers in rows 8 and 9 of Table 1-1. The next time that an opposition of Mars will correspond closely with a superior conjunction of Venus will be some time around A.D. 1995, according to ephemerides which ignore the effect of planetary eccentricities. The amount of drift in the relative times, which is required before the above arrangement of alignments occurs again is indicated by the asterix in Figure 1-1--the minimum amount of drift before a superior conjunction of Venus again corresponds with an opposition of Mars. This drift is,

$$(1/3 - 1/4)\ 2337.6 \text{ days} = (779.2 \text{ days} - 584.4 \text{ days}) = 195 \text{ days} \qquad (1\text{-}1)$$

Every 6.4 years the drift that occurs is,

$$2339.82 \text{ days} - 2335.68 \text{ days} = 4.14 \text{ days} \qquad (1\text{-}2)$$

The total time for this drift to be accomplished is,

$$(195/4.14)\ 6.4 \text{ years} = 300 \text{ years} \qquad (1\text{-}3)$$

The following time that an opposition of Mars will correspond closely with a superior conjunction of Venus will be some time around A.D. 2295.

One might conclude from this discussion that opportunities for multiple series of three-planet flybys will occur over only a few years once every 300 years. Such is not necessarily the case. The drifts in the relative planetary alignments are quite slow; and for a period of about 40 years before and 40 years after the time of alignment according to the above calculation, the effects of the eccentricities of the planets' orbits can cause a greater change in the relative timing of alignments than can the drift. Therefore, for some time in the vicinity of each period of accurate alignment, real repeating flyby trajectories will be quite similar to those based on the assumption of a periodic solar system.

Remaining chapters discuss numerical techniques for determining periodic orbits, direct return orbit which can be linked together into series which last many different lengths of time, several possible approaches for finding both two- and three-planet periodic orbits, and a detailed presentation of some successful Earth-Mars periodic orbits.

CHAPTER 2

NUMERICAL TECHNIQUES

2.1 The Basic Numerical Problem

The basic numerical technique for obtaining the encounter dates for a periodic orbit is the iterative solution of a system of transcendental relations. One must first choose initial guesses for the N independent dates which determine the periodic orbit. These N dates then determine uniquely the positions of the encountered planets at the times of encounter. The trajectories between the encounter points are uniquely determined through the solution of Lambert's problem. The hyperbolic excess velocities are consequently determined by the velocities at the ends of the interplanetary trajectory legs. One then has a difference in hyperbolic excess speed before and after each planetary encounter; this results in N speed differences. The remaining problem is to change the N dates so that the N speed differences can be made to approach zero.

Only after the N speed differences are made very close to zero are the conditions during each flyby checked to see if the vehicle always misses the encountered planets.

The iteration procedure to make the N differences in hyperbolic excess speed zero can be summarized by some very simple expressions. The N independent dates can be indicated by a mathematical column matrix $\underline{t}$ and the N differences in hyperbolic excess speed can be characterized by a column matrix $\underline{v}$. Also, $\underline{v}$ is a function of $\underline{t}$. Then the desired relationship for the zeroing of the N speed differences can be summarized by the expression,

$$\underline{v}(\underline{t}) = \underline{0} \tag{2-1}$$

In order to know how to change $\underline{t}$ to make zero all of the differences in speed, one can form by numerical differencing an approximation to the matrix of all first derivatives which may be indicated by $\left[\frac{d\,\underline{v}}{d\,\underline{t}}\right]$. Then there are two basic iteration schemes which are used in the computer program to develop periodic orbits and which were used in the computer program of Menning[2]. The first is called a steepest descent iteration,

$$\underline{t}_{new} = \underline{t} - \epsilon \left[\frac{d\,\underline{v}(\underline{t})}{d\,\underline{t}}\right]^T \underline{v}(\underline{t}) \tag{2-2}$$

The second is known as a Newton-Raphson iteration,

$$\underline{t}_{new} = \underline{t} - \left[\frac{d\,\underline{v}(\underline{t})}{d\,\underline{t}}\right]^{-1} \underline{v}(\underline{t}) \tag{2-3}$$

The ϵ in Equation (2-2) is chosen separately for each step. It is chosen relatively arbitrarily but so that some measure of the convergence such as $\underline{v}(\underline{t})^T\underline{v}(\underline{t})$ actually decreases in value with each step. The step size is also reduced, if necessary, in the Newton-Raphson iteration in order to assure convergence.

The determination of the matrix $\left[\frac{d\,\underline{v}(\underline{t})}{d\,\underline{t}}\right]$ by numerical differencing methods requires much less computation in this case than it does in the general case. If one member of the column matrix $\underline{v}$ is indicated by v_i, then the functional dependence of each member of the speed difference vector on the vector of dates $\underline{t}$ may be indicated by,

$$\begin{aligned} v_i &= v_i(t_{i-1},\ t_i,\ t_{i+1}) \\ &= v_{out_i}(t_i,\ t_{i+1}) - v_{in_i}(t_{i-1},\ t_i) \end{aligned} \tag{2-4}$$

Each speed difference is a function of only three dates as opposed to the general case where one has a functional dependence on all N independent variables. The relative difficulty may be expressed by counting the number of trajectories which must be computed. N trajectories must be computed in either case in order to determine $\underline{v}(\underline{t})$. In the present case, only 2N additional trajectories must be computed in order to form $\left[\frac{d\,\underline{v}(\underline{t})}{d\,\underline{t}}\right]$ from one-sided numerical differencing. In the general case, however, one-sided numerical differencing would require the additional calculation of N^2 trajectories or equivalent functions.

This investigation could involve considerable experimentation with different iteration techniques. However, it was felt, after a review of the numerical techniques available, that it was much more important for the purpose of this investigation to concentrate on obtaining good initial guesses for possible periodic orbits. The numerical techniques used are primarily those of Menning[2], although several extensions of his work are carried out. Many of these techniques are demonstrated in the computer program of Appendix A.

A summary follows of what the computer program does to solve the numerical problem:

1. Read in data containing the starting encounter dates.
2. Solve Lambert's problem for the N or N+1 trajectories and form the column matrix $\underline{v}$ of speed differences $v_i = v_{out_i} - v_{in_i}$ at the planetary encounters. Each solution must go through the steps listed below under the subroutine called "Lambert."
3. If this is not the first time through, check that $\underline{v}^T\underline{v} = \sum_{i=1}^{N} v_i^2$ has been reduced in value. If it has not, halve the step size and do (2.) again.
4. If $\sum_{i=1}^{N} |v_i|$ is less than the requirement for convergence, skip down to number (8.).
5. Form $\left[\frac{d\,\underline{v}}{d\,\underline{t}}\right]$ by numerical differencing, solving Lambert's problem as needed (2N more times).
6. Take a Newton-Raphson step if $\sum_{i=1}^{N} |v_i|$ is less than some preset value. Otherwise take a steepest descent step.
7. Go up to (2.) and repeat steps ((2.)-(6.)) until convergence is attained at step (4.).
8. Compute the turn angle and passing distance at each planetary encounter and print the results.

<u>Lambert</u> (subroutine):

1. Begin with the encounter dates at each end of the

i^{th} trajectory.

2. Calculate the planetary positions and heliocentric velocities at each end of the trajectory to determine the space triangle for which one must solve Lambert's problem. Also determine the time of flight.

3. Iterate to determine the semimajor axis of the transfer trajectory.

4. For this transfer trajectory calculate the heliocentric velocities at the ends.

5. Compute v_{out_i} and $v_{in_{i+1}}$ by differencing the heliocentric vector velocities of the encountered planets and the trajectory ends.

6. Return these quantities to the main program.

2.2 Convergence Criteria for the "Ends" of a Periodic Orbit

A real periodic orbit has no ends; but as mentioned previously, there is no such thing as a truly periodic orbit joining two or more planets. When the approximation of a periodic solar system is dropped, the computer program needs some criterion for handling the ends of a multiple flyby trajectory in order to approximate a periodic orbit.

A computer program to find a periodic orbit does essentially what was discussed in Section 2.1; but some criterion must be chosen for deciding the first, last, or both hyperbolic excess speed differences. The computer program must essentially choose the dates for a series of unpowered flybys and must operate on some criterion for the speed

$$t_{N+1} = t_1 + T_{cycle} \tag{2-5}$$

This last requirement implies that the computer program is to perform its calculations for N+1 encounters and N-1 flybys and N trajectory legs. For encounters 2 through N, the speed difference at the planet is obvious; but an additional speed difference is necessary to define the numerical problem. There are two ways in which this speed difference has been defined for this investigation.

The first method was to supply an extra flyby, interplanetary trajectory, and planetary encounter. One would then have N+2 encounters, N flybys, and N+1 trajectory legs. The N flybys supply the N speed differences for the numerical problem, and the dimension of the problem is kept equal to N by the two requirements,

$$\begin{aligned} t_{N+1} &= t_1 + T_{cycle} \\ t_{N+2} &= t_2 + T_{cycle} \end{aligned} \tag{2-6}$$

These two requirements mean that the N+1st trajectory leg must be exactly equal in time to the first trajectory leg. This method was used by Menning[2] in his computer work. For the purposes of this discussion, let this method of obtaining the Nth speed difference be known as the "A" modification to determine the "end" speed differences for a periodic orbit.

The second way used to supply the Nth speed difference was simply to difference the speeds at the ends of the differences at the ends of the unpowered series of flybys in order for the series to approximate a periodic orbit.

A basic criterion for the "ends" of the periodic orbit is the length of time in which the basic cycle of the periodic orbit repeats or almost repeats. This length of time will be designated here by the symbol T_{cycle}. If there are to be N independent dates in the periodic orbit problem, and if the dates are to be designated by t_i, then the requirement for the scheme to repeat or almost repeat in the time T_{cycle} is indicated by the requirement,

multiple flyby trajectory. In other words, one will have N+1 encounters, N-1 flybys, and N trajectory legs. The N^{th} speed difference is obtained by differencing the hyperbolic excess departure speed at the date t_1 with the hyperbolic excess arrival speed at the date t_{N+1}. Let this method be known for now as the "B" modification to determine the "end" speed differences for the periodic orbit.

These two differences in calculation method have no effect if the solar system or the solar system model is exactly periodic in the sense that the relative positions and velocities of the planets of interest and the Sun repeat exactly after a certain length of time. However, they will make a difference for a more realistic model of the solar system.

One would expect, even with an accurate solar system model, that as the dimension of the problem is increased from N to 2N to 3N to 4N, *etc.*, that the encounter dates near the "middle" of this increasingly long series of flybys would converge toward the actual encounter dates corresponding to the indefinitely long series of flybys independent of which model one used for the "end" speed differences.

2.3 Models of the Solar System

There are several different mathematical models used for the solar system in the investigation. These models differ in the ephemerides which were used for the three planets of interest: Earth, Mars, and Venus. The different models used are probably best presented in outline form. The basic reason for using different models is to obtain convergence of the numerical problem while proceeding from simplified models of periodic orbits to more accurate aproximations. The different models of the solar system for the planets, Earth, Mars, and Venus, are outlined as follows:

I. Circular coplanar. Eccentricities and relative inclinations are set equal to zero.

A. Approximate values for semimajor axis a and period P in order to achieve periodicity.
$P_{♂} = 32/17$ year, $a_{♂} = (32/17)^{3/2}$ a.u.
$P_{♀} = 8/13$ year, $a_{♀} = (8/13)^{3/2}$ a.u.

1. Exactly symmetric. A date of martian opposition corresponds to a date of venusian superior conjunction.

2. Approximates the near future. The relation between the dates of martian opposition and venusian superior conjunction is determined by an approximation to reality at an arbitrary point in time which is taken to be in the near future.

B. Values for the semimajor axis and period correspond to the correct numerical values.

II. Eccentric inclined, exactly periodic. The eccentricities and relative inclinations are made equal to the correct numerical values at one time while the periods and semimajor axes are made equal to the values given in (I.A.) above. With these values, the absolute positions of Earth, Mars, and Venus repeat exactly after 32 years.

1. Corresponds to (I.A.1) above.

2. Corresponds to (I.A.2) above.

III. Eccentric inclined, constant element. All of the orbital elements of the three planets, Earth, Mars, and Venus, are set equal to the instantaneous mean elements obtained from Reference 7 or 8. With this model, one is also interested in how the speed differences at the "ends" of the periodic orbit are defined.

A. The "A" modification to determine the "end" speed differences is used. ($t_{N+1} = t_1 + T_{cycle}$ and $t_{N+2} = t_2 + T_{cycle}$.)

B. The "B" modification to determine the "end" speed differences is used. ($t_{N+1} = t_1 + T_{cycle}$ and the arrival speed at t_{N+1} is subtracted from the departure speed at t_1 to form the N^{th} speed difference.)

These last two differences in the numerical technique do not affect solar system Models I. or II., because both of these first two models are exactly periodic. Because Model III. most closely approximates the real solar system, its use can be expedted to give the most accurate approximation to the actual dates of the continuous, indefinitely long series of flybys which is based on the periodic orbit. That is, both versions of Model III. can be expected to give the most accurate estimates of the dates of any model used here. In all cases, the patched conic approximation of Section 1.2.1 above is used.

The distinction between (1.) and (2.) under Model I.A. and Model II. can be expected to make no basic difference if the periodic orbit only visits two planets. The options (1.) and (2.) are not applicable to Model I.B., because this model includes the effect of the drift in the relative timing in the oppositions of Mars and superior conjunctions of Venus; only the effects of eccentricity and relative inclination are neglected.

Some of the reasons for considering so many different solar system models are discussed in the following section.

2.4 Complexity of the Function Space

In the expression $\underline{v}(\underline{t})$, one seems to imply a simple functional relationship between the N encounter dates and the N speed differences. The actual relationship is neither simple nor neat, and the numerical solution will frequently not converge to the desired solution of obtaining the t_i's such that $\underline{v}(\underline{t}) = \underline{0}$.

In the space of the N t_i's and one additional dimension for the value of the function, one can either think of the N hypersurfaces $\underline{v}(\underline{t})$ or of the single hypersurface defined by $(\underline{v}^T(\underline{t})\underline{v}(\underline{t}))$. The function space is of dimension N+1, and the N+1st dimension is the function value of interest. If one thinks of the N hypersurfaces defined by $\underline{v}(\underline{t})$, then the solution desired is that point in the N dimensional space of all the t's such that all of the speed differences (the v_i's) are equal to zero. If one thinks of the single hypersurface determined by $\underline{v}^T\underline{v}$, then the solution which one desires is a minimum such that $\underline{v}^T\underline{v} = 0$. This minimum is an absolute minimum, since the expression $\underline{v}^T\underline{v}$ is a positive definite form. In general, for a very complex function, one would expect several local minima in addition to at least one absolute minimum. Both of these ways of thinking of the problem are equivalent, but they result in the visualization of different surfaces.

One can obtain some idea of the complexity of the function by looking at the published trajectory charts[4,8,9], remembering that one is now primarily interested in the speed <u>differences</u> at the N flybys given the N dates, and remembering that one wants to find N dates such that all of the speed differences are zero. The complexity in the appearance of the speed contours on the trajectory charts indicates the complexity of the function. The fact that the speed contours are very close together in some places and far apart in other places suggests also that one could have difficulty in selecting the step size to use in order to obtain the differences to be used as approximations to the matrix of first derivatives.

The complexity of the function is expected to be increased by increases in orbital eccentricities and mutual inclinations for the planets. Conversely, the circular coplanar case would be expected to result in a somewhat less complex function of encounter dates. The increased complexity with the eccentric inclined case is partially caused by the increase in the basic dimension N of the problem due to the lengthened time before the relative positions and velocities repeat. However, an even more basic cause of the complexity of the problem is due to the very large increases in the hyperbolic excess speeds for transfers near 180°. This can result in non-convergence of the numerical problem in the eccentric inclined case, in spite of starting with dates which would converge immediately in the circular coplanar case.

A method of dealing with this convergence difficulty is to increment the eccentricities and mutual inclinations of the planets' orbits in small steps while going from the circular coplanar case to the eccentric inclined case. One should start with the encounter dates for the circular coplanar case and use these dates in the computer program with very small values for the eccentricities and mutual inclinations. Then, with convergence in this case, one should use these new numbers for the encounter dates in the computer program with slightly larger values for the eccentricities and mutual inclinations. One should then continue in this manner until the correct values for the eccentricities and mutual inclinations have been reached. In other words, one should go from the circular coplanar case (model I.) to the eccentric inclined case (model II. or model III.) in several steps. The number of steps, the size of the steps, and whether the eccentricities and inclinations are incremented proportionally should be determined by the investigator by practical considerations. Three or four proportional steps in eccentricities and inclination were found adequate for the worst cases in this investigation. A more general principle, of which this technique may be considered a special case, is that one should not try to solve a numerical problem with an answer "too different" from the initial guess.

This method of incrementing the eccentricities and inclinations only is a method of dealing with convergence problems; it does not guarantee convergence. In fact, for the periodic orbit scheme which has been labeled M5-3 in Appendix E, a solution apparently does not exist in the eccentric inclined case, at least in the region of the circular coplanar solution. For this particular periodic orbit scheme, convergence was achieved quite readily in the circular coplanar case and in the case with the inclination and eccentricities at about one quarter of their actual values. However, with these parameters at about half of their actual values, no convergence was achieved. Then, the parameters were incremented in smaller amounts between these two values. In each case, the encounter dates with which each attempt was started were the the converged dates for the parameter values set to the values just below the ones tried. With the smaller increments in the parameter values, the problem did converge with the parameter values equal to about 0.4 of their correct values but did not converge on the second attempt with the parameter values equal to about 0.5 of the correct values. This behavior seems to indicate that a solution to the problem of the type labeled M5-3 simply does not exist for an inclination and eccentricities above certain values. This apparent lack of a solution is another indication of the complexity of the function under study.

2.5 Other Numerical Problems

There are several other numerical problems about which one might concern himself in the search for periodic orbits. These are all problems which have been recognized but which have not been pursued much further. Better handling of these problems would result in a computer program which would give better results.

As was mentioned above, one could explore the possibility of using different convergence techniques in the search for periodic orbits. One method, consisting of simply searching along the coordinates (the different encounter dates) successively in a cyclic fashion, may turn out to be a satisfactory one in terms of computer time, because a search involving a change in one date does not involve very much recalculation. In any case, different methods of numerical solution could be explored which make use of the unique properties of this particular problem.

One would desire a Lambert problem numerical solution which would give a continuous solution from the elliptic case, through the parabolic case and into the hyperbolic case particularly for periodic orbit attempts which involve Jupiter or planets even further out in the solar system. Battin[6,22,23] has done a great deal of work with this problem, and his latest method[23] probably offers the most promise. A better numerical solution to the space triangle or Lambert problem could also help to relieve some of the numerical difficulties associated with transfers near 180° which occur even in the circular coplanar case.

Another problem is that of obtaining better ephemerides for the planets. More accurate numbers for the planets could easily be included by using a published series expansion[7] for the instantaneous mean orbital elements. In addition, the angles could be measured from the instantaneous equinox instead of from the equinox of some arbitrary year such as 1960.

A final problem is that of the whole question of numerical accuracy in the program. It would be advantageous to be able to switch into a more accurate, double precision routine in order to check the numerical accuracy of the results.

In fact, there are so many things which might be done, that if the author were to do them he would begin by rewriting the program and dividing it into several subroutines in order to make it easier to change the program. In addition, several of the above mentioned improvements would be included.

CHAPTER 3

DIRECT RETURN TRAJECTORIES

3.1 Introduction to Direct Return Trajectories

Of the two main classes of trajectory legs used to make up periodic orbits or attempts at periodic orbits, interplanetary trajectory legs and direct return trajectory legs, the class which is the subject of this chapter has only recently been considered to any large extent. This class of direct return trajectories is a very important addition which makes possible the existence of periodic orbits. As the names imply, interplanetary trajectories go between two different planets, and direct return trajectory legs return to the planet from which they departed last.

Beyond this, the class of direct return trajectory legs could further be divided into those which go around the Sun the same number of times as does the launch and arrival planet and those which go around the Sun a different number of times than the planet of encounter. Obviously, the vehicle can only go around the Sun a number of times which differs by an integer from the number of times which the planet traverses the Sun; this is necessary so that the vehicle will return to the launch planet. An important characteristic which distinguishes these two classes of direct return trajectories is that the hyperbolic excess speeds relative to the planet can approach zero in the case where the vehicle traverses the Sun the same number of times as does the planet, while there may be a minimum hyperbolic excess speed in the case of a direct return trajectory which does not traverse the sun the same number of times as does the planet.

Further, each of these classes of direct return trajectories can be divided into three more classes. Under the classification system of Ross[4] two of these classes would be called symmetric and nonsymmetric direct return orbits. The third class might be called half revolution direct returns.

The symmetric returns are symmetric in the sense that the line of apsides of the ellipse which the vehicle follows is the line of symmetry for the encounters with the planet--when the planet is in circular orbit around the Sun. A symmetric return is also symmetric in the sense that it looks the same in backward time if one views the orbits from the opposite side of the orbital plane. A symmetric return is also characterized by the fact that the point of departure does not necessarily correspond to the point of arrival. Hence the plane of the vehicle's orbit and the plane of the planet's orbit must coincide.

The nonsymmetric return is characterized by the facts that the points of arrival and departure coincide, that the plane of the vehicle's orbit and the plane of the planet's orbit need not coincide, and that both the planet and the vehicle traverse the sun an integer number of times between departure and arrival. In order to launch a vehicle on a certain type of nonsymmetric return, one must merely be certain that the vehicle leaves the vicinity of the planet with a certain heliocentric speed. A nonsymmetric return which returns after exactly one planetary period will be called a full revolution return trajectory and will be abbreviated by "FR".

A half revolution return traverses the Sun an integer number of times plus exactly one-half revolution. It has some of the characteristics of each of the other two types. It looks like a symmetric return in terms of the symmetrical arrangement around the line of apsides and in terms of its symmetry in time. However, the points of arrival and departure and the Sun are colinear so that the plane of the vehicle's orbit does not, in general, correspond to the plane of the planet's orbit.

Most of the rest of this chapter will not concern itself further with trajectories which do not traverse the Sun the same number of times as does the planet. These trajectories are not expected to be as useful as those which traverse the sun the same number of times as does the planet, because in general, they require more time before returning to the launch planet. However, in certain instances, such as in a search for periodic orbits which connect Earth with a planet differing appreciably in semimajor axis and period from those of Earth, they could prove quite useful as a part of a periodic orbit which otherwise would not exist. The examination of direct return trajectories which do not traverse the Sun the same number of times as does the encountered planet is an area for further study.

3.2 Full Revolution Return Trajectories

3.2.1 General Characteristics of Full Revolution Return Trajectories

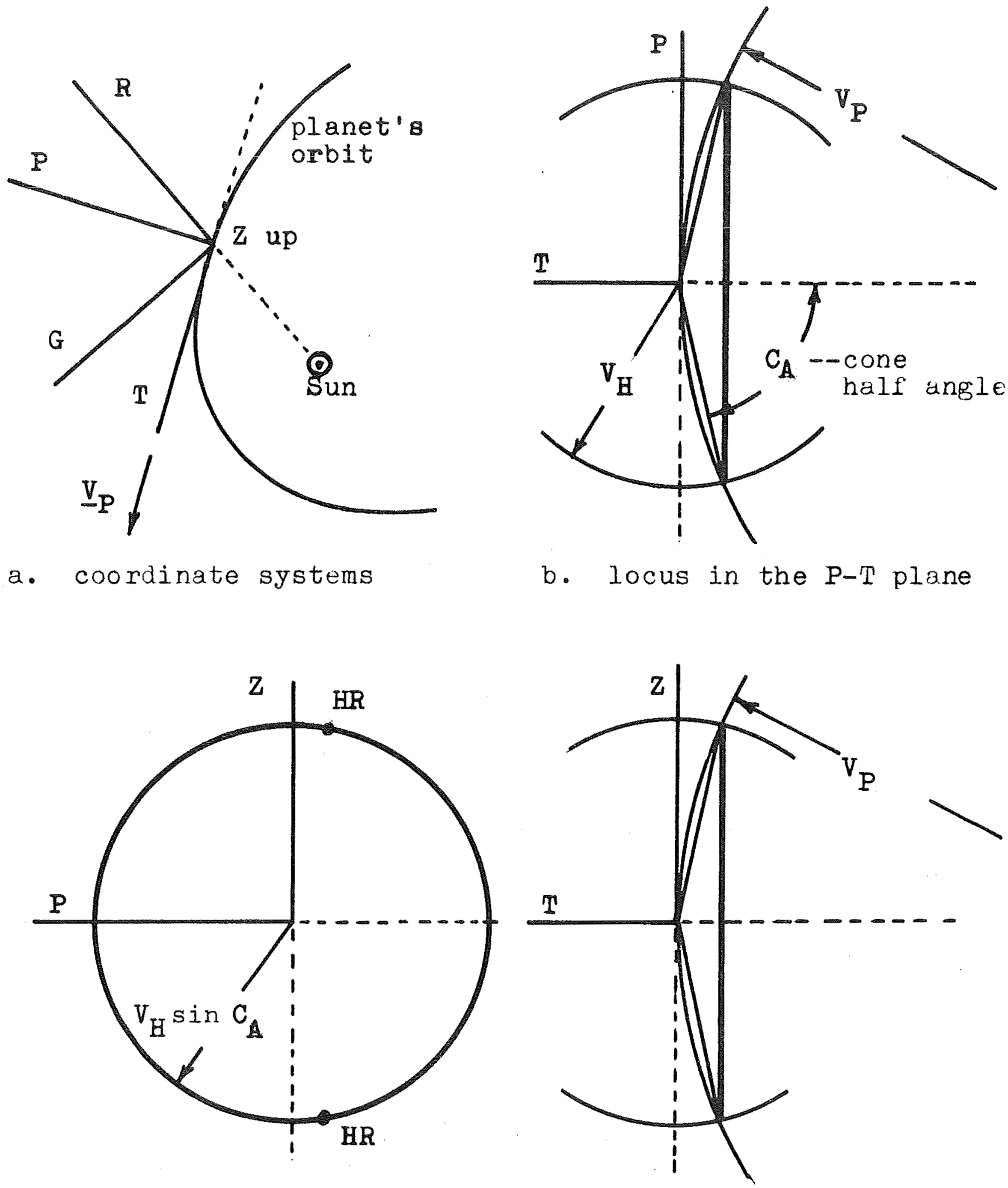
A full revolution return trajectory is one which returns to the planet of departure after one period of the planet. A full revolution return will frequently be abbreviated as "FR". In order to accomplish this, the vehicle's orbit must have the same period around the Sun as does the planet. This is accomplished, in turn, by the vehicle's having the same semimajor axis, the same energy per unit mass relative to the Sun, and the same heliocentric speed at the arrival and departure points as does the planet of arrival and departure. At a given encounter point, there are a double infinity of such full revolution return trajectories.

The locus of the tip of the hyperbolic excess velocity vector (relative to the planet), which will put a vehicle on such a full revolution return, is the surface of a sphere which passes through zero and is symmetric about the direction of the planet's heliocentric velocity vector at that point. Three views of this locus are shown in Figure 3-1. In addition, if one has a vehicle approaching the planet with some given hyperbolic excess velocity, and if one desires to put the vehicle on a full revolution return trajectory immediately after the planetary encounter, then the hyperbolic excess speed for the vehicle on the full revolution return is fixed. Then the locus which this fixed length hyperbolic excess velocity vector must reach is a small circle on the spherical surface mentioned above. The fact that this more restricted locus is a small circle of the sphere of the less restricted locus is best demonstrated in Figure 3-1 c. In Figure 3-1, V_P is the speed of the planet at the point in its orbit where the encounters take place, V_H is the hyperbolic excess speed at the planet, and the cone half angle C_A is determined by the formula,

$$C_A = \text{Arcos} \left(\frac{V_H}{2 V_P}\right) \qquad (3\text{-}1)$$

The orthogonal coordinate directions P,T,Z indicated in Figure 3-1 are based on the Z direction as the perpendicular to the planet's orbital plane and on the T direction parallel to the direction of the planet's heliocentric velocity vector at the given point.

One should also note that only one of the encounter dates for a series of full revolution return trajectories is an independent date; all of the remaining encounter



a. coordinate systems

b. locus in the P-T plane

c. locus in the P-Z plane

d. locus in the T-Z plane

Figure 3-1. Locus of points of the tip of the velocity vector for a full revolution return and a half revolution return.

dates can be determined uniquely from one of the encounter dates (to within the accuracy of the patched conic approximation mentioned in the first chapter). This interdependence of the encounter dates is a result of the fact that the length of the full revolution return is independent of the speed relative to the encountered planet. Hence, for the purposes of numerical calculation as in the computer program of Appendix A, only the first date of a series of full revolution returns need be specified; the remaining dates are determined by adding an integer number of planetary periods to the date of the initial encounter.

3.2.2. Turn Angle Selection for a Series of Full Revolution Returns

Because the turn angles are not specified completely; that is, because one has one degree of freedom in selecting the direct return trajectory given the hyperbolic excess speed, there is a problem of selecting the angles for a series of full revolution return trajectories. A criterion for making a selection is to maximize the minimum radius of closest approach which is equivalent to minimizing the maximum turn angle at the planet for the series of full revolution return trajectories. Basically, the turn angles are chosen by picking a number of hyperbolic excess velocity vectors equal to the number of full revolution returns, all lying on the locus which will produce a full revolution return. This selection may also be thought of as choosing points on the small circle of the sphere shown in Figure 3-1;

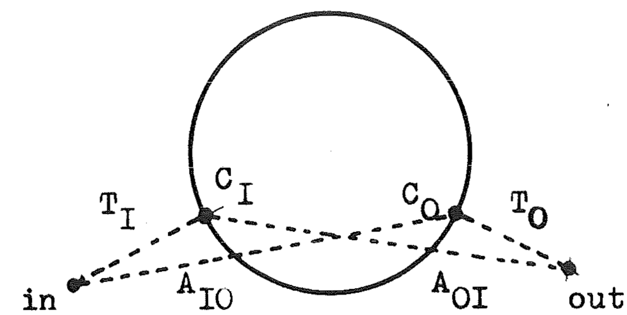
that is, choosing points on the locus which has been specified by V_H and V_P.

Such a criterion of minimizing the maximum turn angle is not necessary; but it will assure that the vehicle will miss the planet if it is possible to do so; and it will assure the investigator that the series of full revolution returns cannot work, due to the vehicle's hitting the planet one or more times, if such a series of full revolution return trajectories is indeed impossible.

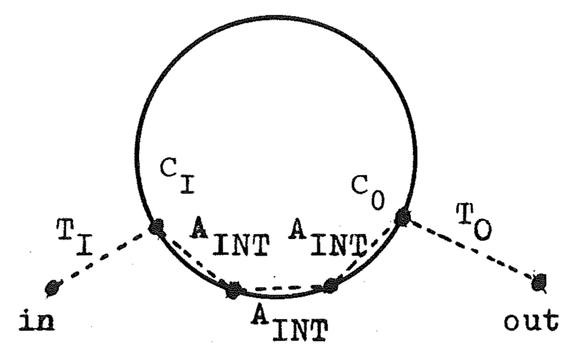
In order to better visualize the turn angle selection, it is perhaps more enlightening to distort the sphere in velocity space of radius V_H onto a plane. This sphere is the locus of the tips of all hyperbolic excess velocity vectors at the planet which have a hyperbolic excess speed of V_H. The locus on this sphere of all hyperbolic excess velocity vectors of length V_H, which will produce a full revolution return trajectory, is a small circle on the sphere. In fact, the latter locus is the intersection of the sphere of radius V_H mentioned here and the sphere of radius V_P mentioned above as the locus of the tips of all full revolution return hyperbolic excess velocity vectors. The distortion of this smaller sphere onto a plane will mean that different hyperbolic excess velocity vectors can be represented by points in this plane, that angles can be represented by the distances between points in this plane, and that the locus of full revolution return vectors can be represented by a circle in this plane. The angles will not be accurately represented in this case, but the basic layout will be more clearly visualized since the problem is basically two-dimensional. As the length of the hyperbolic excess velocity vector has already been determined by the speed relative to the planet at the ends of interplanetary transfers, only the angles can be varied. The planar representation of the problem is shown in Figure 3-2.

In order to calculate the turn angles so that one can pick the set which minimizes the maximum one, it is convenient to express vectors, which produce a full revolution return, in terms of V_H, the cone half angle, and an angle to express the position around the full revolution return locus. Then, incoming and outgoing hyperbolic excess velocity vectors, along with the full revolution return vectors, should be expressed in the P, T, Z coordinate frame so that the taking of scalar products can be used to obtain angles between the vectors. The calculation of the desired angles will not be shown here, however, since Menning[2] gives an adequate presentation of the necessary calculations.

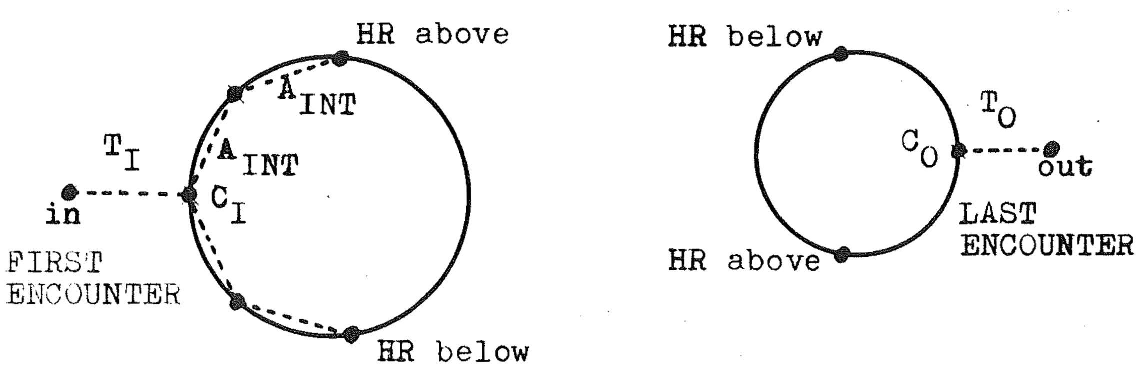
There are a few important angles and points which are common to turn angle selection for a single full revolution return trajectory, for two or more full revolution return trajectories, and for a half revolution return trajectory connected with full revolution return trajectories. These are represented in Figure 3-2 by the points labeled C_I and



a. single FR turn angle



b. two or more FR turn angles



c. HR turn angles

Figure 3-2. Turn Angle selection

C_O and by the turn angles labeled T_I and T_O. These are respectively the points on the full revolution return locus which require the minimum turns from and to the inbound and outbound hyperbolic velocity vectors and the turns associated with these points on the full revolution return locus. The inbound and outbound velocities in this case are, of course, the hyperbolic excess velocity vectors which are associated with the interplanetary trajectory legs immediately before and after the series of full revolution return trajectories.

3.2.2.1 Turn Angle Selection for One Full Revolution Return

The logic for the selection of the turn angles for the case of a single full revolution return is relatively straightforward. Referring to Figure 3-2a and the angles indicated in that figure, the logic for selecting the single full revolution return trajectory which minimizes the maximum turn angle can be summarized by the following statements:

1. If T_I is greater than A_{OI}, then C_I is the point for the FR and the angles are T_I and A_{OI}.

2. If T_O is greater than A_{IO}, then C_O is the point for the FR and the angles are A_{IO} and T_O.

3. Otherwise, the point for the FR is some point between C_I and C_O (going the short way around) such that the two turn angles are equal.

3.2.2.2 Turn Angle Selection for Two or More Full Revolution Returns

The logic for the selection of the turn angles for the case of two or more full revolution returns is more complex than for the case of one full revolution return. Again, the criterion for the selection is to minimize the maximum turn angle. In the general case of M full revolution returns, there will be M+1 turn angles; but only three of these angles will be different. There will be the turn T_I onto the full revolution return locus, the turn T_O off of the full revolution return locus, and the M-1 turns equal to A_{INT} between points on the full revolution return locus. In order to follow the logic, the reader should refer to Figure 3-2b. The points C_I and C_O and the corresponding turns T_I and T_O should be thought of as initially corresponding to the minimum turns onto and off of the full revolution return locus, but these points and turns should be thought of as altered as necessary. Then the logic for the selection of the turn angles for the case of two or more full revolution returns can be summarized as follows:

1. If T_I is greater than A_{INT} and T_O is greater than A_{INT}, then C_I and C_O and the M-2 intermediate points are the desired points on the full revolution return locus, and the initial turn angles are the desired ones.

2. If A_{INT} is greater than T_I and T_O is greater

than A_{INT}, then C_I should be moved toward C_O (the short way around) until T_I and A_{INT} are made equal.

3. If T_I is greater than A_{INT} and A_{INT} is greater than T_O, then C_O should be moved toward C_I (the short way around) until T_O and A_{INT} are made equal.

4. If A_{INT} is greater than T_I and is also greater than T_O, then C_I and C_O should be moved toward each other until the three angles (T_I, A_{INT}, and T_O) are all made equal.

The calculations used to carry out this logic and to calculate the desired angles are given by Menning[2] and are contained in the computer program of Appendix A.

3.3 Half Revolution Return Trajectories

3.1 General Characteristics of Half Revolution Return Trajectories

Half revolution return trajectories are, in a way, a special case of a full revolution return trajectory. A half revolution return will frequently be abbreviated as

"HR". The hyperbolic excess velocity vector necessary to produce a half revolution return trajectory is a special case of the locus of hyperbolic excess velocity vectors which produce a full revolution return trajectory. This special case is determined from the more general locus by the restriction that the R component of the hyperbolic excess velocity must be equal to zero. Given that the hyperbolic excess speed is V_H, the points for the velocity giving a half revolution return trajectory are shown by the two points labeled "HR" in Figure 3-1. On a half revolution return, the eccentricity of the vehicle's orbit around the Sun, as well as the semimajor axis of its orbit, must match those of the planet's orbit.

As in the case of a full revolution return trajectory, only one of the encounter dates is an independent variable. The initial date of encounter on a half revolution return decides the second date; however, the second date is not exactly one half planetary period later because of the eccentricity of the planet's and hence of the vehicle's orbit. The second encounter of the vehicle with the planet occurs after a heliocentric transfer angle of 180° which may be after a length of time which is slightly greater or slightly less than one half of a planetary period. Series expressions for the true anomoly before and after a half revolution return and a series of full revolution returns, are derived in Appendix B as a power series in the eccentricity e. That a series of full revolution returns is connected with a half revolution return does not alter the fact that for such a series of several FR and one HR, only one of the encounter dates is an independent variable; all of the remaining encounter dates can be calculated from one of the encounter dates, the first encounter date, for example.

Another quirk introduced with a half revolution return is that the speed relative to the planet at the encounter after a half revolution return is not the same as the speed at the encounter just before the half revolution return. This too is due to the eccentricity of the planet's and of the vehicle's orbit around the Sun. In each case, for a given half revolution return, the speeds before and after are proportional to the G component of the planet's heliocentric velocity at each encounter point. A consequence of this difference in hyperbolic excess speed is that for a periodic orbit, the arrival speed at a planet, after a transfer from another planet but before a series of one HR and several FR, must be different from the departure speed from that planet.

3.3.2 Turn Angle Selection for a Half Revolution Return and Several Full Revolution Returns

The problem of turn angle selection for one half revolution return and a series of several full revolution returns is considerably more difficult than the problem for just a series of full revolution returns. First of all, one should note that in this case the criterion of minimizing the maximum turn angle is <u>not</u> exactly equivalent to maximizing the minimum radius of closest approach, because the eccentricity of the orbits causes the speeds to be different before and after the half revolution return. The logic for carrying out either one of these criteria would be considerably more difficult than for the equivalent problem of a series of full revolution returns because of the number of choices involved.

There are quite a few choices as to how one might arrange a single half revolution return with a series of full revolution returns. First of all, the half revolution return with a given initial hyperbolic excess speed can be accomplished in two ways; the half revolution return trajectory can take place either above or below the plane of the encountered planet's orbit. These two choices are indicated by the two points labeled "HR" in Figure 3-1c and by the two pairs of points labeled respectively "HR above" and "HR below" in the two parts of Figure 3-2c. Secondly, one must decide how many of the full revolution returns are to be placed on each side of the half revolution return; one must decide how many FR should go before the HR and how many FR should go after the HR. Finally, one must decide how to pick the turn angles once the above two choices have been made; the positions of the full revolution return velocity vectors on the full revolution return locus must be decided.

Because of the complexity of the logic required to make all of these decisions in a truly optimum way, it was decided not to have the computer make all of them automatically. The decisions as to whether the half revolution return trajectory goes above or below the planet's orbit and as to how many full revolution returns go on each side of the half revolution return, are included by the investigator in the data for the computer program. Then, if the vehicle hits the planet in one or two cases, the investigator can run the problem again after having made a few reasonable changes in this information.

Once the decisions have been made as to how many FR are to occur on each side of the HR and whether the half revolution return trajectory will be above or below the planet's orbit, the minimization of the maximum turn angle is relatively straightforward. This logic is quite similar to that for a series of full revolution returns alone. The scheme of the problem is illustrated in the two sketches in Figure 3-2c. The two sketches correspond to the situations before and after the half revolution return. In Figure 3-2c, if the half revolution return is to be above the plane of the planet's orbit, then the points of the full revolution return locus which produce this desired half revolution return are labeled "HR above." Correspondingly, the points labeled "HR below" are connected with a half revolution return which lies under the plane of the planet's orbit.

In either case (either before or after the half revolution return), if there is to be no full revolution return before (or after) the half revolution return, the desired turn angle is simply the turn angle from the incoming velocity vector (to the outgoing velocity vector) to (from) the velocity vector which produces the desired half revolution return.

For the situation where there are to be one or more full revolution returns before (or after) the half revolution return, the problem of minimizing the maximum turn angle has similarities to the same problem for a series of two or more full revolution return trajectories. The logic will be explained here for one or more FR occuring before the HR, and the first sketch in Figure 3-2c will be referred to. The logic for the FR occuring after the HR is exactly analogous and can be demonstrated on the second sketch; because the logic is essentially the same, it will not be mentioned explicitly. As in the case of two or more full revolution returns, consider the minimum turn from the inbound velocity vector onto the locus which will produce a full revolution return and consider that point to be called C_I. The turn in is to be called T_I. Then, if there are M full revolution returns before the half revolution return, there will be M equal turn angles labeled A_{INT} to get to the velocity vector which produces a half revolution return. As before, consider

T_I to be initially the minimum turn onto the full revolution return locus; but consider the possibility of moving C_I so as to increase T_I and reduce A_{INT}.

The actual logic really only involves one test and one possible adjustment. If one initially has T_I greater than A_{INT}, then one already has the desired turn angles. Otherwise, one should move C_I so as to increase T_I and decrease A_{INT} so as to make equal these two angles. The same simple logic works for the FR's and departing velocity vector after the HR.

3.4 Symmetric Return Trajectories

Symmetric return trajectories return to the planet of departure after varying periods of time. Because the times vary and the transfer angles vary and are, in general, not some multiple of 180^o, the two points of encounter and the Sun are not colinear. Hence, the plane of the vehicle's orbit must coincide with the plane of the planet's orbit. Different symbols are used in this thesis to stand for symmetric return trajectories, because there are different types of symmetric return trajectories; but all of them end with the two letters "SR". The symmetric return trajectories differ considerably from the full revolution return or the half revolution return in that the length of the symmetric return is varied continuously in order to produce a continuous variation in the speed of the vehicle relative to the planet. For symmetric returns, the encounter dates at the ends of the trajectory are independent variables and must be chosen separately.

3.4.1 The Linear Case

One way to search for direct return trajectories which go around the sun the same number of times as the planet of launch and arrival is to consider equations of motion linearized about a point in circular orbit around the Sun. Hollister[12] uses these linearized equations as an initial step in mission planning; he presents their derivation in Appendix A of his doctoral thesis. Letting x be measured in the R direction from the planet and y be measured in the G direction from the planet--both in units of the radius of the planet's orbit (a.u. for the Earth)--one obtains the differential equations of motion in the plane of the planet's orbit which Hollister[12] presented as Equation (A.9),

$$\ddot{x} = 4\pi\dot{y} + 12\pi^2 x \qquad \ddot{y} = -4\pi\dot{x} \tag{3-2}$$

Hollister gives the solution of these equations as,

$$x = sV_x + rV_y \qquad y = -rV_x + qV_y \tag{3-3}$$

for zero initial conditions on x and y where,

$$\begin{aligned} q &= q(t) = 4\sin 2\pi t - 6\pi t \\ r &= r(t) = 2 - 2\cos 2\pi t \\ s &= s(t) = \sin 2\pi t \end{aligned} \tag{3-4}$$

and where V_x and V_y are the components of the hyperbolic excess speed measured in units of the planet's speed around the Sun (Earth Mean Orbital Speed units (EMOS) in the case of Earth). Time t is measured in units of the planet's period (years).

Then, in order to find a direct return trajectory in the R-G plane, one desires a V_x, a V_y, and a time t such that x and y are simultaneously zero. Hollister[3] in his Appendix D shows that this requirement is,

$$\frac{V_x}{V_y} = -\frac{r}{s} = \frac{q}{r} \qquad (3\text{-}5)$$

which reduces to,

$$4(1 - \cos 2\pi t) = 3\pi t \sin 2\pi t \qquad (3\text{-}6)$$

This equation has solutions whenever t is an integer. These solutions with t an integer correspond to full revolution return trajectories which lie in the plane of the planet's orbit. Half revolution return trajectories do not show up as solutions of this equation, because half revolution returns occur only perpendicular to the plane of the planet's orbit in the neglected Z direction of this simple model. The remaining solutions at non-integer times correspond to the linear solutions for symmetric return trajectories. The first ten of these non-integer solutions are presented in Table 3-1. In this table, the departure angle is measured from the G axis toward the R axis; the departure angle is equal to the inverse tangent of V_x/V_y. Both solutions from the inverse tangent are presented. After the given time in planetary periods, a vehicle would return with an arrival angle which is the negative of the departure angle. This reversal of the R component of velocity (V_x) between departure and arrival is another characteristic which is a result of the symmetry of symmetric returns.

i	Time in planetary periods	Departure angles in degrees: limit of SiSR trajectories	limit of LiSR trajectories
1	1.4067	98.578	-81.422
2	2.4453	94.960	-85.040
3	3.4612	93.509	-86.491
4	4.4699	92.718	-87.282
5	5.4754	92.219	-87.781
6	6.4792	91.876	-88.124
7	7.4820	91.625	-88.375
8	8.4841	91.433	-88.567
9	9.4858	91.282	-88.718
10	10.4871	91.159	-88.841

Table 3-1. Number of planetary periods and departure angles for symmetric returns obtained from the simple model of Hollister[1].

3.4.2 Symmetric Returns at a Planet in Circular Orbit Around the Sun

The values for the times and the angles obtained from the analysis of the linear case give the results for a more accurate analysis in the limit as the speed relative to the planet approaches zero. In the more general case, there will be symmetric returns which last longer and symmetric returns which last a shorter time than the time indicated by the linear analysis.

Now, we come to the point of explaining the symbols which this thesis uses in some places to indicate the different types of symmetric return trajectories. The first character in each symbol is either an S or an L standing respectively for symmetric returns which are shorter than or longer than the one indicated by the linear analysis. The second character in each symbol is a positive integer which indicates the number of revolutions of the Sun completed by the vehicle and the planet during the symmetric return. The last two symbols in each case are "SR" to indicate that a symmetric return is being indicated.

For the problem of a planet in circular orbit around the Sun, the length of time which the symmetric return trajectory is to last determines the speed relative to the planet and the angles of departure and arrival. Figures 3-3 through 3-7 give plots of departure and arrival speed and departure angle relative to a planet in circular orbit around the Sun with a semimajor axis of one astronomical unit. The lengths of time for the symmetric returns are given in days as well as in planetary periods or years. The speed relative to the planet is given in Earth Mean Orbital Speed units. The plots also give the departure angle for the hyperbolic excess speed vector. The angle is measured

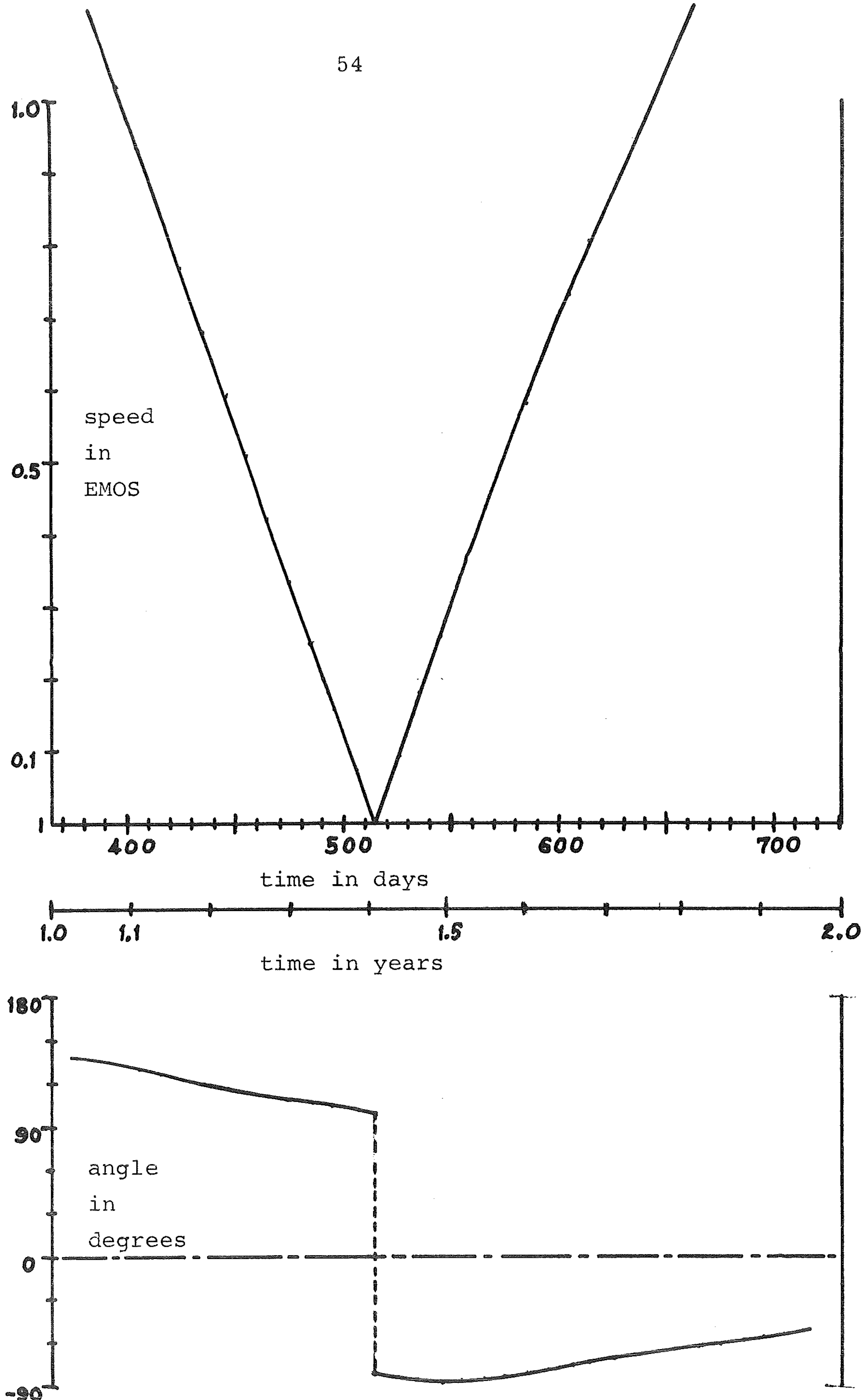


Figure 3-3. Speed and departure angle at Earth for $S1S_R$ and L1SR

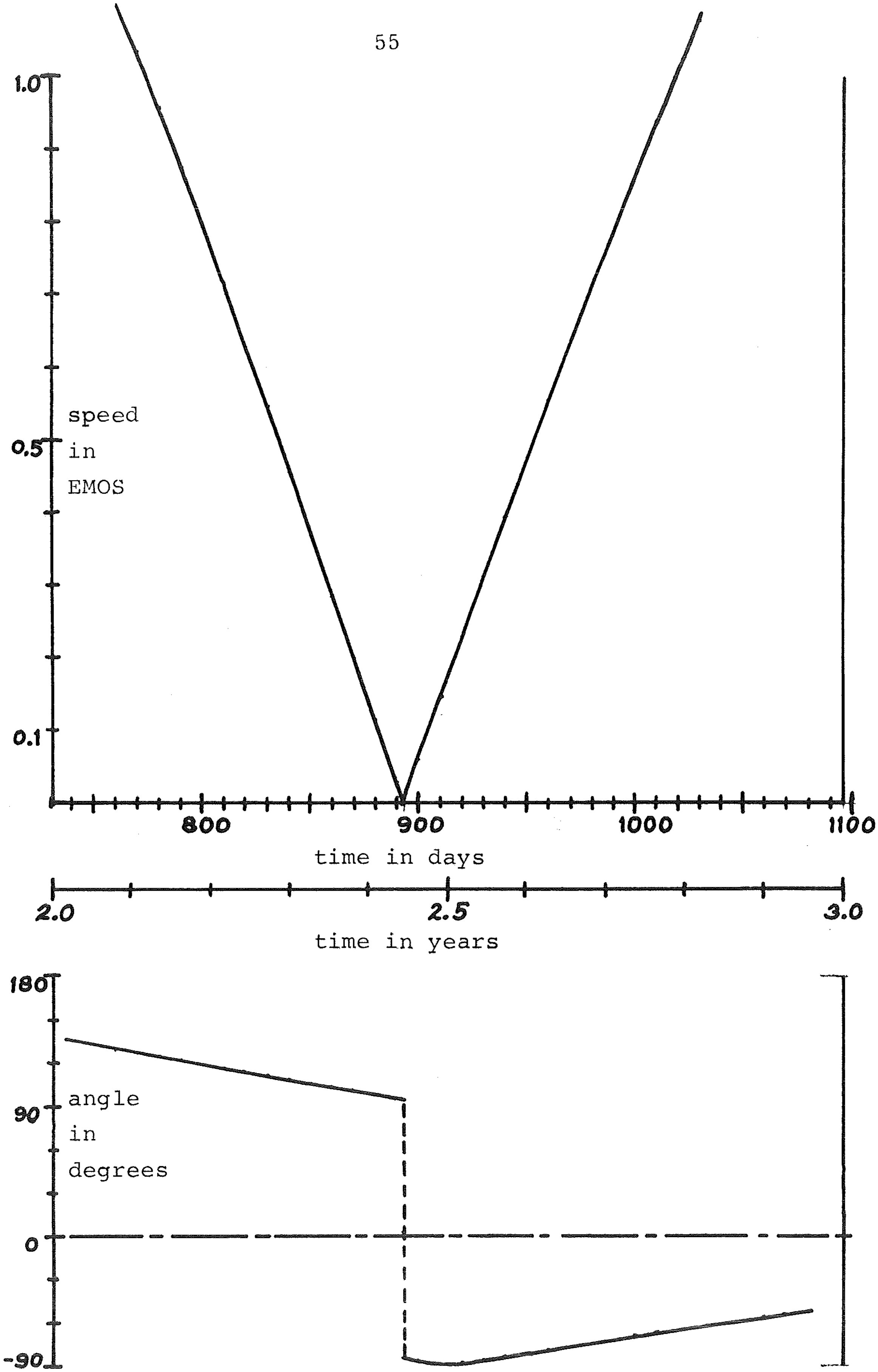


Figure 3-4. Speed and departure angle at Earth for S2SR and L2SR

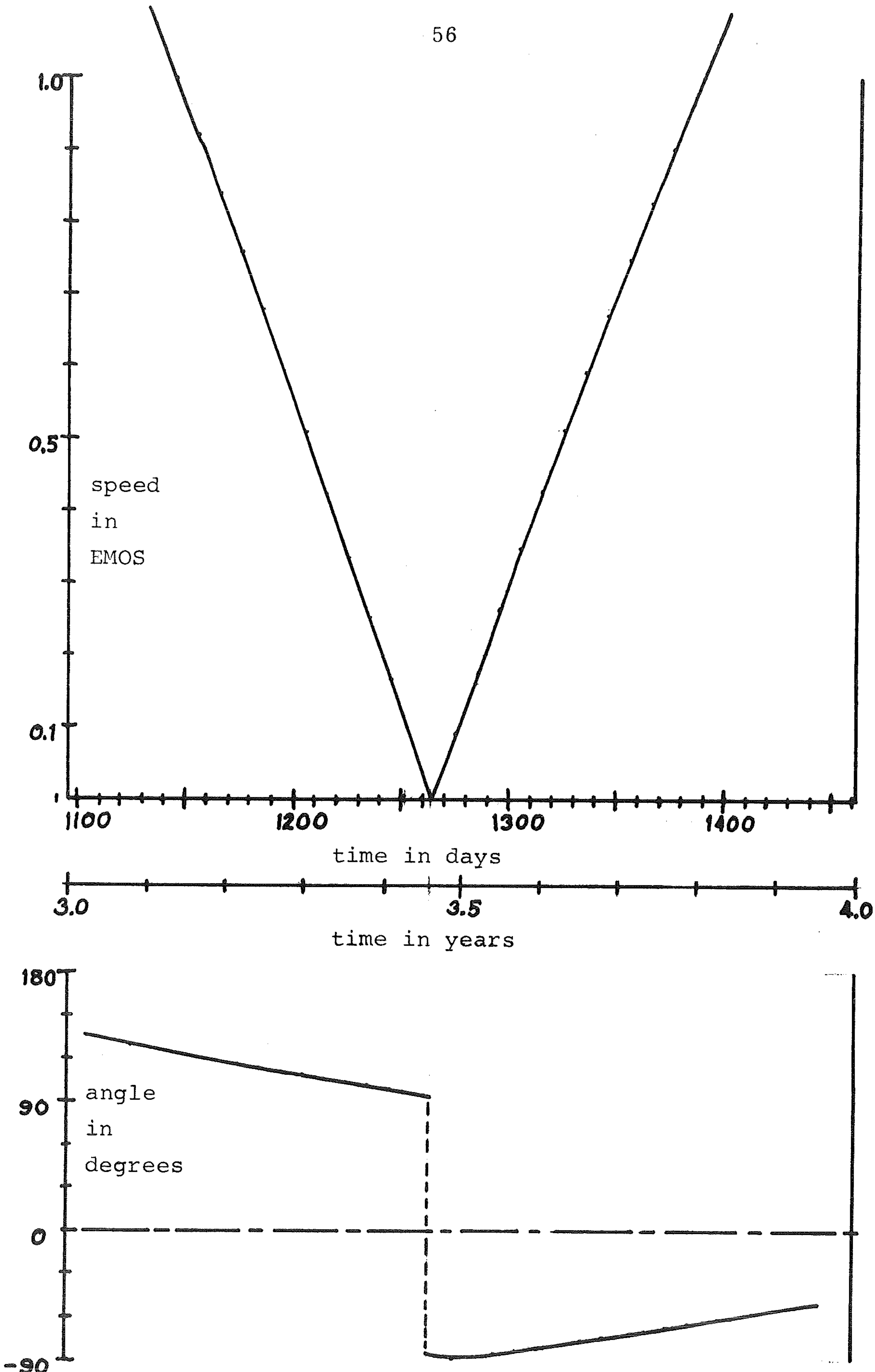


Figure 3-5. Speed and departure angle at Earth for S3SR and L3SR.

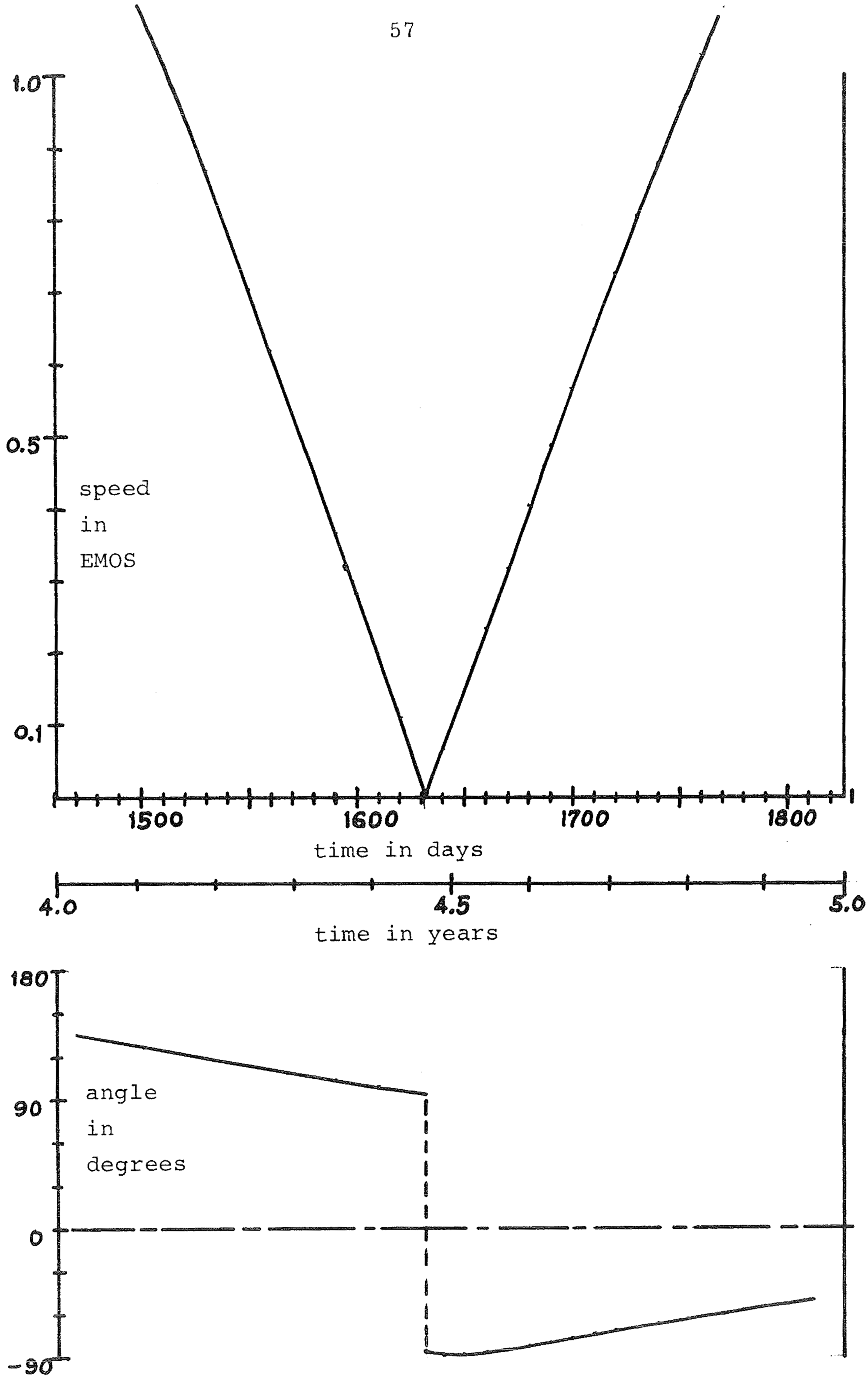


Figure 3-6. Speed and departure angle at Earth for S4SR and L4SR

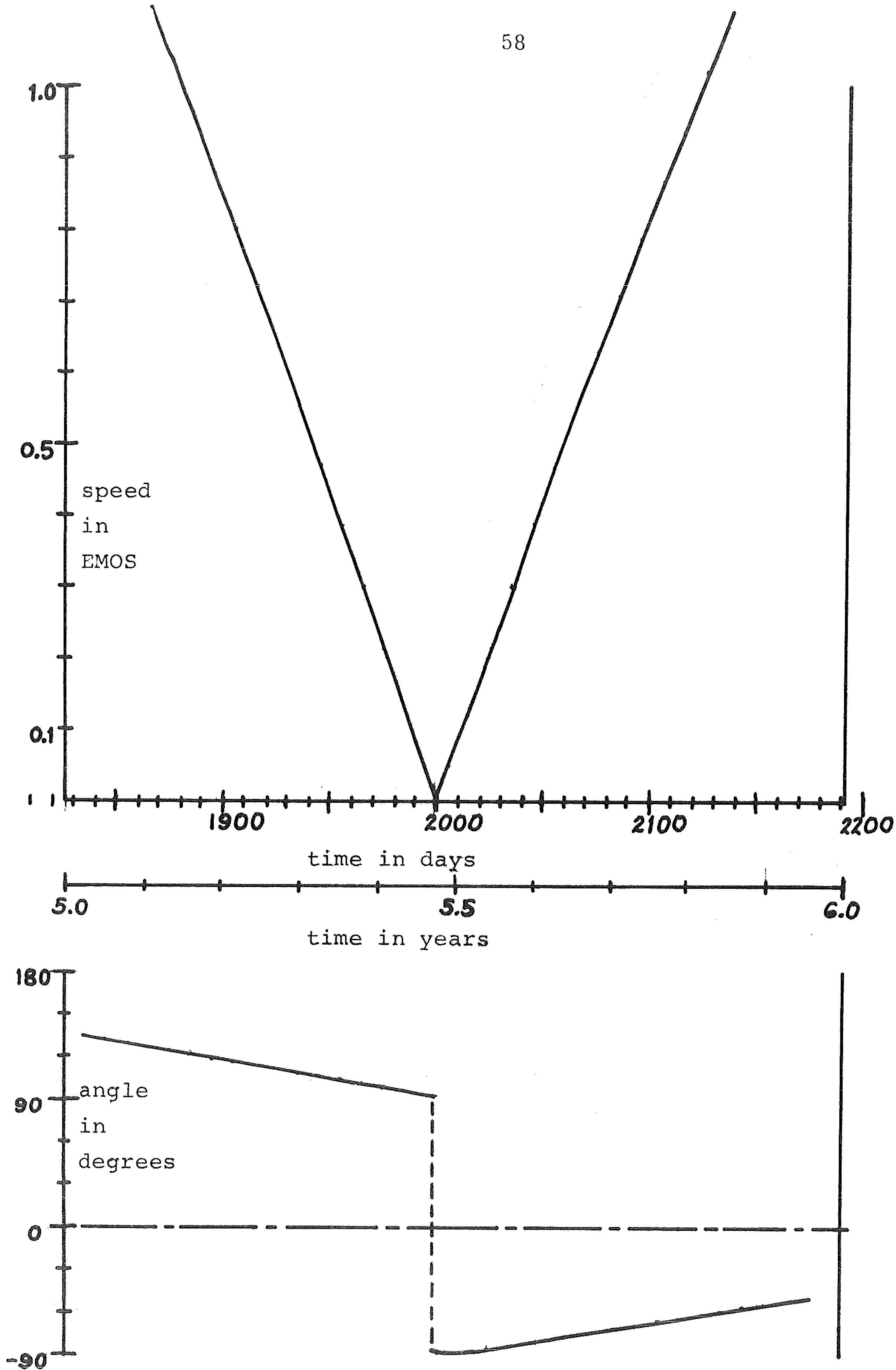


Figure 3-7. Speed and departure angle at Earth for S5SR and L5SR

positive from the G axis toward the R axis. The arrival angle for the velocity vector relative to the planet at the end of the symmetric return trajectory is the negative of the departure angle. The plots were made using the patched conic approximations mentioned in Chapter 1 of this thesis. The plots are essentially made for symmetric returns at Earth; but they can be easily scaled for symmetric returns at any other planet in almost circular orbit by going from years to planetary periods for the time measurement, and by going from EMOS to units of the planet's average orbital speed.

The longer symmetric returns here can be expected to be of less use than the shorter ones for the same reason that direct returns which do not go around the sun the same number of times as the planet can be expected to be of limited usefulness. Both types of returns spend a great deal of time away from the Earth.

3.4.3 Selection of the Semimajor Axis for a Symmetric Return Trajectory

The computer program must compute the speeds relative to a planet in an elliptic orbit for a vehicle on a symmetric return from and to the planet. For inputs, the computer needs the dates specifying the times of encounter immediately before and immediately after the symmetric return trajectory. These times of encounter specify the positions of encounter and the length of time for the symmetric return trajectory. The computer must then solve Lambert's problem for the resulting space triangle in order to find the trajectory followed by the vehicle. This point in the discussion brings us to an important characteristic of Lambert's problem.

If the vehicle is to make more than one full revolution around the Sun, then the solution to the space triangle problem is not unique. If the space triangle problem is to be solved with the number of full revolutions of the Sun specified (greater than or equal to one), then the specification of the time of flight will, in general, result in the possibility of two different values for the semimajor axis of the transfer ellipse. There will be two possible solutions. One of these solutions will correspond to the trajectory of the planet, and the other one will correspond to the trajectory of the vehicle. Menning[2] discusses how the logic of finding the vehicle trajectory is carried out.

3.5 Series of Direct Return Trajectories

Now consider the problem of how to combine series of types of direct return trajectories. One would like to know what lengths of time one can stay in the vicinity of a planet by use of a series of free-fall flybys before embarking on a trajectory leg which would take one to another planet. A list of wait times in the vicinity of one planet which are available for use as part of a periodic orbit is a desirable collection of information. One would also like to know how to achieve these different wait times and what restrictions, if any, are to be placed upon the use of the different schemes.

One must start with a logical plan of attack to achieve different waiting times. The author began by essentially obtaining all of the different possible combinations of direct return trajectories. Appendix C calculates the number of combinations which exist in general.

First, the number of types taken one at a time were obtained, then the number of combinations taken two at a time were determined, and the different combinations continued to be taken until all of the different combinations taken six at a time were obtained. The number of different types of direct return trajectories was restricted to the HR, FR, S1SR, and L1SR types. The other types of symmetric return trajectories were not considered in the initial determination of all of the possible combinations because of the additional complexity involved and because one L2SR, for example, takes very close to the same time to complete as does one FR and one L1SR; the L2SR could be considered a modification of (FR)(L1SR).

Once all of the different combinations of HR, FR, S1SR, and L1SR were determined, one could then begin eliminating and adding combinations. First of all, one could eliminate all of the combinations which involve two or more half revolution returns, because these combinations could more easily be replaced by a combination in which each pair of half revolution returns is replaced by one full revolution return. One full revolution return takes the same length of time as two half revolution returns and can, in general, be accomplished more easily in terms of one less encounter and a greater minimum passing distance. Then those combinations are added which can be created by substituting, for instance, one S3SR for (2FR)(S1SR). Thirdly, all of those combinations are eliminated which require unreasonable turn angles such as those greater than 90°. For instance, a combination such as (2S1SR) would be eliminated, because one could not expect the vehicle to make the turn between the two symmetric returns without hitting the planet.

Finally, lengths of time are associated with each direct return trajectory. The lengths of time taken for the symmetric returns are arbitrarily taken to be those at which the speed relative to the planet at encounter is 0.3 EMOS (or planetary orbital speed units in the case of a planet other than Earth). Then all of the different remaining combinations are placed in order of increasing time up to some arbitrary limit such as three average synodic periods of Earth and Mars. These results are presented in the table of Appendix D.

Appendix D also associates these different combinations and wait times with time differences between the ends of such a combination of direct returns and the ends of an interval. This interval is made up of an integer number of synodic periods, and the series of direct returns is centered in it. The time differences and the intervals may be seen easily in Figure 3-8. These time differences are useful in associating interplanetary trajectories with series of direct returns in order to develop periodic orbit schemes. They give dates relative to the date of planetary alignment. Appendix D finally gives a letter code of restrictions on the incoming and outgoing velocity vectors at the end of the interval formed by the series of direct return trajectories.

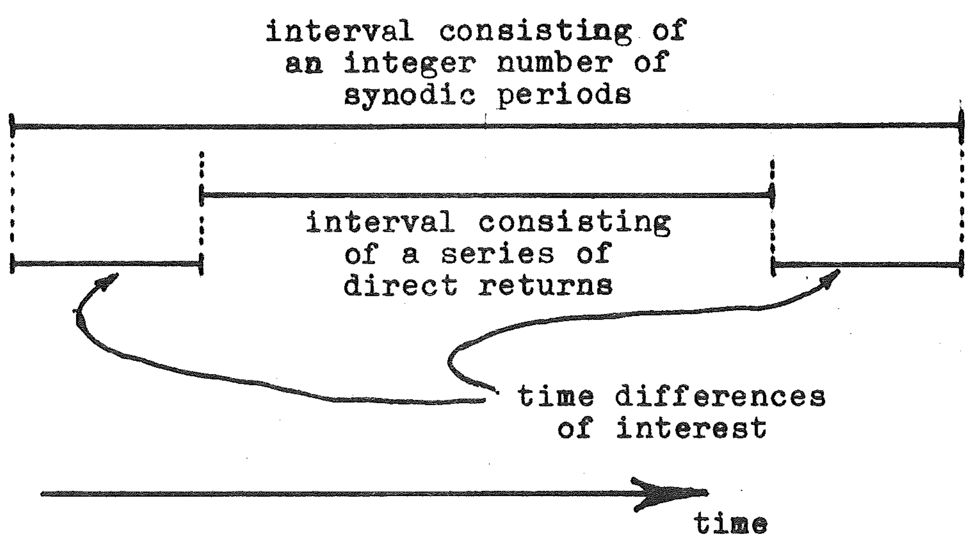


Figure 3-8. Times associated with series of direct returns.

3.6 Direct Return Trajectories Which Traverse the Sun a Different Number of Times Than the Planet

Direct return trajectories which go around the Sun a different number of times than does the planet which the vehicle encounters immediately before and after the direct return are an unexplored area. One would expect to divide these direct returns into categories analogous to the categories of the direct returns mentioned earlier. One would, in general, expect direct return trajectories which traverse the Sun (1.) an integer number of times, (2.) an integer number of times plus exactly 180°, and (3.) noninteger numbers of times. These direct returns would differ, however, from those covered more thoroughly in that the hyperbolic excess speed at encounter associated with them may not approach zero.

A much more complete coverage of the area of direct return trajectories should cover these other types of direct returns and should include them in the number of combinations of direct returns considered.

CHAPTER 4

POSSIBLE APPROACHES TO OBTAIN PERIODIC ORBITS CONNECTING EARTH AND MARS

The problem remaining, to find periodic orbits connecting Earth and Mars, consists of finding two or more interplanetary trajectories which can be patched together with a series of direct return orbits where needed through flyby maneuvers at the encountered planets to form a periodic orbit. Naturally, the vehicle on the periodic orbit must miss the encountered planet during the flyby maneuver. One must also work within the constraints of the basic periodicity of the solar system as mentioned in Chapter 1 and discussed by Hollister[1]. Finally, one would like to find periodic orbits connecting Earth and Mars which involve reasonably small velocities relative to Earth and Mars, which have reasonably short Earth-Mars and Mars-Earth interplanetary transfers, and which make the round trip between Earth and Mars reasonably frequently.

For each prospective periodic orbit tried, an attempt was made initially for the circular coplanar case where the basic repetition time for the periodic orbit is either some multiple of the synodic period or some multiple of the period in which the relative positions of the planets involved repeat themselves. If the scheme did not work in the circular coplanar case, it was assumed that it would not work in the elliptic inclined case with a basic repetition period of some multiple of 32 years.

In all cases, symmetry was used to improve the chances for finding a periodic orbit. Symmetry in this case means primarily the existence of "reciprocal" trajectories as discussed by Ross[4]. An interplanetary trajectory or an interplanetary round trip trajectory always has a "reciprocal" trajectory in the case of circular coplanar planetary orbits. The two reciprocal trajectories have the properties that the speeds at the encountered planets are the same for the two trajectories and the dates of the planetary encounters on one trajectory are the negative of the dates of the planetary encounters on the other trajectory when the dates are measured from the time of planetary alignment (conjunction or opposition). Symmetry means that if an interplanetary trajectory or an interplanetary round trip trajectory is used in an attempt at a periodic orbit, then the "reciprocal" trajectory is also used. Symmetry also means that if the basic period of repetition for a prospective periodic orbit in the circular coplanar case involves a certain number of synodic periods, then this prospective periodic orbit can be arranged within this period so that the dates of encounter for the periodic orbit are symmetrically arranged (positive and negative) about a point in time within the period. This concept of symmetry may easily be seen in Figures 4-1, 4-2, and 5-1 through 5-4.

Symmetry improves the chances of finding a periodic orbit, because it insures that the hyperbolic excess speeds can be made equal at both ends of a series of direct returns. This equality is insured by the fact that most of the series of direct returns used within periodic orbit schemes join corresponding points on two "reciprocal" trajectories. Remember that each member of a pair of corresponding points on two reciprocal interplanetary trajectories has a hyperbolic excess speed which is equal to that at the other point. Bear in mind that reciprocal trajectories exist exactly only in the circular coplanar case and that the relationships discussed here hold only approximately for more accurate solar system models. If the two ends of a given series of direct returns do not correspond to ends of two reciprocal trajectories, then one cannot guarentee that the hyperbolic excess speeds will be the same at each end of the series of direct returns without putting an additional restriction on the remaining dates of encounter. Such an additional restriction could very well result in making it impossible to match the hyperbolic excess speeds at each planetary encounter. In each of the periodic orbits attempted, there was symmetry in the sense discussed here.

All approaches used in this investigation are included in this chapter, but only one of them resulted in a workable periodic orbit. An additional approach may reasonably be expected to produce successufl results. The unsuccessful approaches are included in order to demonstrate what approaches are possible and what approaches might eventually be expected to lead to successful results in other problems.

4.1 Approach Involving Direct Return Orbits at Both Earth and Mars

The scheme used by Hollister[1] to obtain his Earth-Venus periodic orbits and the scheme used here was to make the interplanetary legs of the proposed periodic orbit fairly low energy transfers (fairly close to Hohman transfers) and to connect these interplanetary transfers with suitable series of direct return orbits at each of the two planets. Although this method worked for the Earth-Venus case, not much hope was held for the success of this method in the Earth-Mars case because of the much lower mass of Mars, 0.108 Earth mass, relative to the mass of Venus, 0.815 Earth mass. The lower mass of Mars means that much less velocity change is available from a flyby maneuver, and it was expected that no scheme involving direct returns at Mars would work even in the circular coplanar case. Success was not expected even in the circular coplanar case, because the Earth-Mars Hohman velocity relative to Mars is 0.09 EMOS and the approximate turn angle needed to get onto a full revolution return trajectory is 90°; at this speed, Mars can supply a turn of only about 80°

without the vehicle's intersecting the surface of the planet.

Use is made of the symmetry properties of trajectories; for this type of periodic orbit in the circular coplanar case, the Earth-Mars trajectory is the reciprocal of the Mars-Earth trajectory.

In attempting to obtain a periodic orbit of this type, one can calculate the dates for the circular coplanar case without the help of an electronic computer. One simply obtains approximate dates for a very low energy interplanetary trajectory between the two planets of interest (and at the same time the dates of the symmetric interplanetary trajectory)--both relative to a date of opposition. One then determines a series of direct returns at each planet which can patch together the interplanetary trajectories and give dates relative to opposition which are close to those determined for the low energy trajectory.

In the case of the Earth-Mars periodic orbit, two trajectories were found which worked, surprisingly, in the circular coplanar case. The reason that a turn onto the full revolution return locus at Mars is possible in this case is that, in going from the Hohman trajectory to the actual interplanetary trajectory, the increase in speed at Mars was small while the decrease in the required turn angle was more substantial. The two speeds at Mars were 0.11 and 0.15 EMOS while the two minimum turn angles onto the full revolution return locus were about 65° and 40° respectively. Each of these had a repeating cycle of 12.8 years (6 synodic periods), made one round trip between

Earth and Mars in this period of time, and involved 4 full revolution returns at Mars (lasting a total of 7.5 years). The large number of full revolution returns (and long waiting time) at Mars was allowed deliberately in order to permit the velocity vector relative to Mars to be turned as needed during a large number (5) of flybys. In the circular coplanar case, one of the periodic orbits involved 4 full revolution returns at Earth and missed Mars by 0.03 planetary radii; the other periodic orbit involved 2 full revolution returns and one short symmetric return at Earth and missed Mars by 0.007 planetary radii. In both cases the miss distances at Earth were more substantial.

When convergence was obtained for the elliptic inclined case with a repeating period of 64 years (30 synodic periods), however, both trajectories hit Mars. The prime reason for the increased difficulty with the elliptic inclined case is that the resulting inclination of the transfer trajectories between Earth and Mars results in a large out-of-plane velocity component relative to the encountered planets and hence much higher speeds relative to the encountered planets. The resulting higher speeds relative to Mars mean that the flyby maneuvers are simply no longer possible.

It is not expected that any periodic orbits of this type connecting Earth and Mars exist.

4.2 An Attempt to Obtain Directly an Earth-Venus-Mars Periodic Orbit

An attempt was made to put together Hollister's[1] periodic Orbit I connecting Earth and Venus and an Earth-Venus-Mars-Venus-Earth continuous flyby trajectory to form a periodic orbit connecting Earth, Mars, and Venus. Hollister's[1] periodic Orbit I has a basic repeating period in the circular coplanar case of 3.2 years, one year of which is spent in the vicinity of Earth. Hence, the time between successive Earth encounters on each side of the trip to Venus is 2.2 years in the circular coplanar case. VanderVeen[5] found several Earth-Venus-Mars-Venus-Earth flyby trajectories including some which are basically symmetric and last about two years. They are symmetric basically in the fact that two of the interplanetary trajectories are close to being the reciprocal trajectories to the other two. VanderVeen calls this type of trajectory a 5/5 type. Apparently, the reason for the existence of this type of low energy flyby trajectory has to do with the fact that a Mars-Venus Hohman transfer takes 216 days while Venus' period is 225 days. Note also that the relative positions of Earth, Venus, and Mars repeat every 6.4 years in the circular coplanar case. Also, once in every 6.4 year period, an opposition of Mars corresponds roughly in time to a superior conjunction of Venus. It is approximately at this time that VanderVeen's multiple flyby trajectory encounters Mars.

Therefore, it was thought to be reasonable to attempt to obtain an Earth-Earth-Venus-Venus-Venus-Earth-Earth-Venus-Mars-Venus-Earth (repeat) periodic orbit. In the circular coplanar case, it would repeat every 6.4 years. However, it was thought that it could be determined directly in the elliptic inclined case simply by putting down Hollister's Orbit I for 32 years and putting in an encounter at Mars instead of an encounter at Venus at the appropriate 5 times in 32 years. The basic arrangement of trajectories and encounters attempted is that of Figure 4-1. This approach was tried with the computer program, but convergence was not achieved.

Several other ideas similar in nature to this approach were tried, but without success. Instead of trying to send the vehicle to Mars 5 times in 32 years, an attempt was made on successive computer runs to send the vehicle to Mars 1, 2, 3, 4, and finally 5 times in 32 years. With successive submittals of attempts to the computer program, convergence was eventually obtained to send the vehicle to Mars 3 times in 32 years. Convergence could not be obtained to send the vehicle to Mars 4 or 5 times in 32 years. Unfortunately, even with the converged solution to send the vehicle to Mars 3 times in 32 years, the trajectory frequently intersected Venus in preparation to go to Mars or in coming back from Mars.

In addition, several attempts were made in the circular

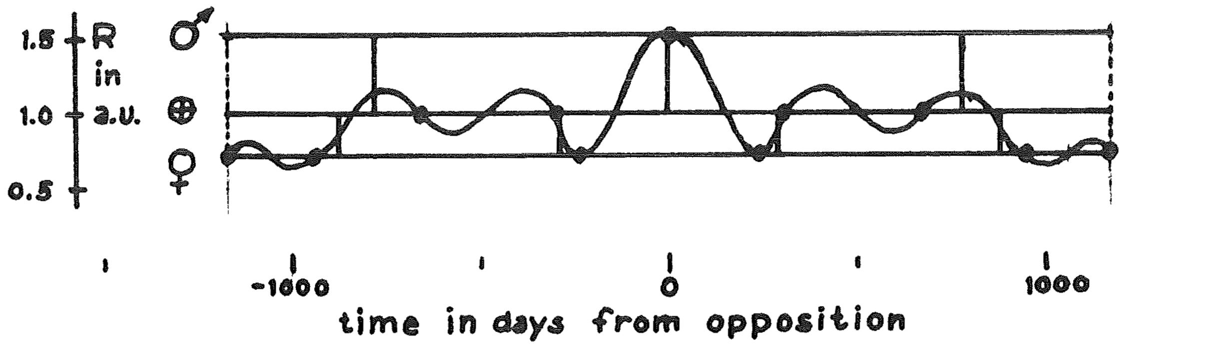


Figure 4-1. Distance from the Sun as a function of time for a periodic orbit scheme to Mars which involves direct returns at both Earth and Mars.

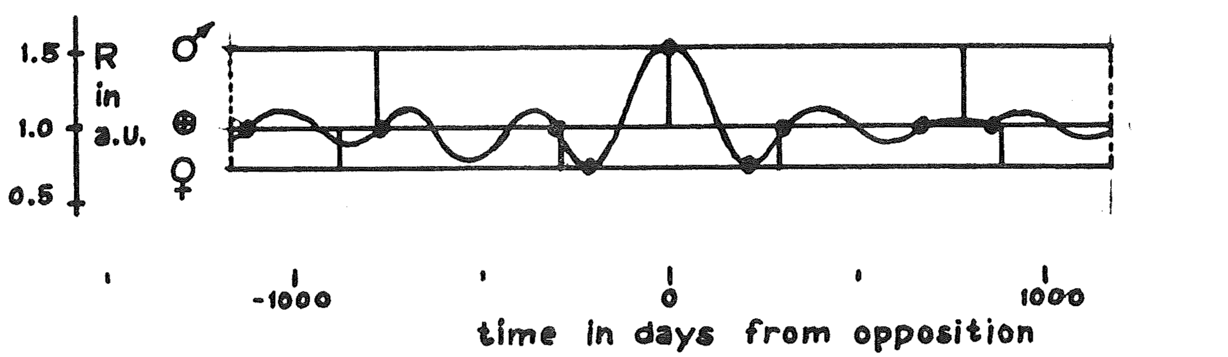


Figure 4-2. Distance from the Sun as a function of time for a periodic orbit scheme which encounters Earth, Mars, and Venus but has direct returns only at Earth.

coplanar case with a basic repeating cycle of 6.4 years. These attempts also were without success. Attempts were made in the circular coplanar case based on Hollister's periodic orbits I, II, and III; none of the attempts converged. The solar system model used was Model I.A.2. An attempt was also made with this solar system model to patch together into a periodic orbit a trajectory similar to that of VanderVeen with a wait at Earth of about 4.4 years ((3FR)(SISR), (HR)(4FR), or something similar). The arrangement of this last type of attempt is shown in Figure 4-2. Convergence also failed here.

4.3 Reduction of the Earth-Venus-Mars Periodic Orbit Problem to a Two-Dimensional Problem

In order to obtain convergence for the problem of an Earth-Venus-Mars periodic orbit based on the trajectories of VanderVeen[5] and Hollister[1] as shown in Figure 4-1, a much better initial guess for the encounter dates must be obtained.

In order to do this, the problem is first reduced to a two-dimensional problem which is solved to give a better initial guess. The first step in doing this is the selection of a solar system model which is circular, coplanar, exactly periodic and has an opposition of Mars which corresponds exactly to a superior conjunction of Venus; in other words, one is to use solar system Model I.A.1. With this model, the basic repeating period for the relative positions of the three planets, Earth, Mars, and Venus, is 6.4 years.

If the date of opposition of Mars and superior conjunction of Venus is placed in the center of this 6.4 year period as in Figure 1-1, then the remaining times of conjunction of Venus and opposition of Mars are arranged in a symmetrical pattern around this date. One can then argue by symmetry that the date of the encounter at Mars must correspond with the date of opposition and superior conjunction and that any series of direct returns at Venus must be centered around the date (3.2 years from the above mentioned one) of both Mars' and Venus' being exactly on the opposite side of the Sun from the Earth. Then, for a periodic orbit of the type considered here and in the previous section and shown in Figure 4-1, there remain four independent dates of encounter to be chosen--two at Venus and two at Earth. However, from the condition of symmetry, these dates should be symmetrically arranged about the date of opposition and superior conjunction. Hence, there should remain only two independent dates--one at Venus and one at Earth--left to solve the problem.

For the case mentioned at the end of Section 4.2 and shown in Figure 4-2, where the periodic orbit only goes to Venus on the way to and from Mars, there is only one independent variable to determine. This variable is the date relative to the time of opposition and superior conjunction which describes the two dates at Venus.

In order to solve the two-dimensional problem, plots were made. The axes of these plots are the date at Venus and the date at Earth, each of which is measured relative to the date of opposition and superior conjunction, which corresponds with the date of encounter with Mars. A point on the plot corresponds to two symmetrically arranged dates of encounter at Earth and two symmetrically arranged dates of encounter at Venus; the date of encounter at Mars is always assumed to correspond to the date of opposition and superior conjunction. The plots give locii of pairs of dates such that the difference in hyperbolic excess speed before and after the planetary encounters at Earth and Venus are zero. Each locus appears as a line on the plot, and a possible periodic orbit solution is indicated by the intersection of two locii.

Figure 4-3 shows the region of possible interest for these plots as a large triangular region. The borders of the region are very strict constraints on when the encounters can occur. The boundary on the left is determined by the fact that a vehicle going from Mars to Venus cannot get to Venus before it has been to Mars. The diagonal boundary is decided by the fact that trajectories which get to Earth before they get to Venus are not being considered here; hence, the date at Earth must always be a larger number than the date at Venus. The upper boundary on the region is determined by the length of time taken up by direct returns at Venus and at Earth and by the length of the 6.4 year interval. In fact, a formula for the position of the upper boundary can be written in the form,

$$D = 1169 \text{ days} - \tfrac{1}{2} T_{♀} - T_{⊕} \qquad (4\text{-}1)$$

where $T_{♀}$ is the length of time spent on direct returns in the vicinity of Venus and where $T_{⊕}$ is the length of time spent on direct returns in the vicinity of Earth on each of the two occasions which the vehicle spends in the vicinity of Earth. The smaller region inside of the triangle of Figure 4-3 is the region to which the plotting of locii was arbitrarily limited and is the region of Figures 4-4 and 4-5.

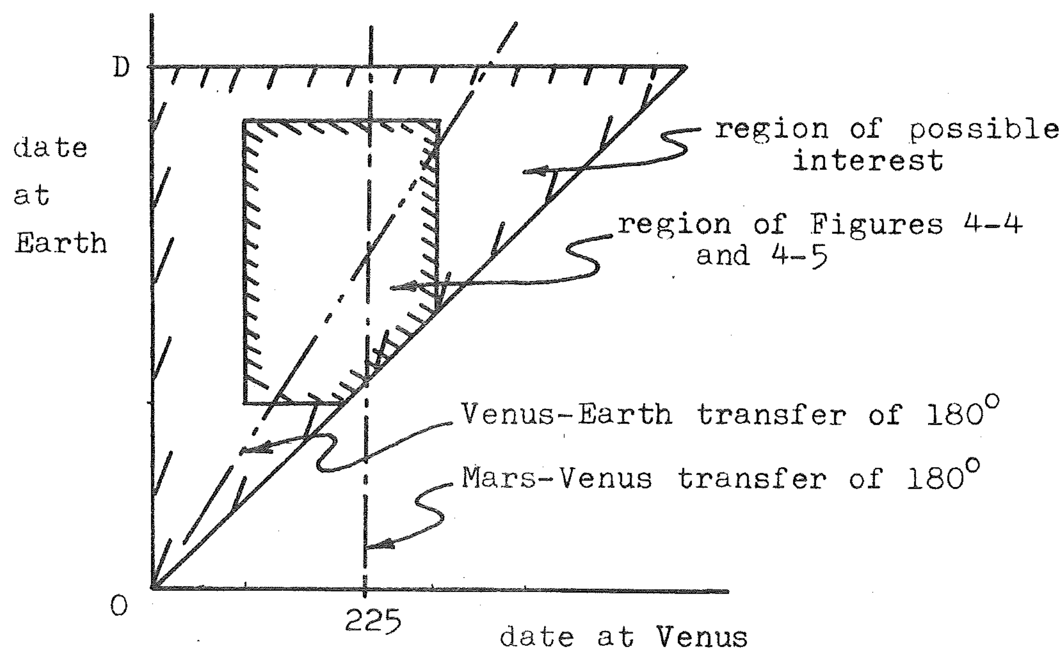


Figure 4-3. Regions of possible trajectories and investigation for periodic orbit attempts involving symmetric Earth-Venus-Mars-Venus-Earth flyby trajectories.

Figure 4-4 shows the locus of points such that the hyperbolic excess speed difference at Venus is equal to zero. This particular locus is needed in all cases. It will remain

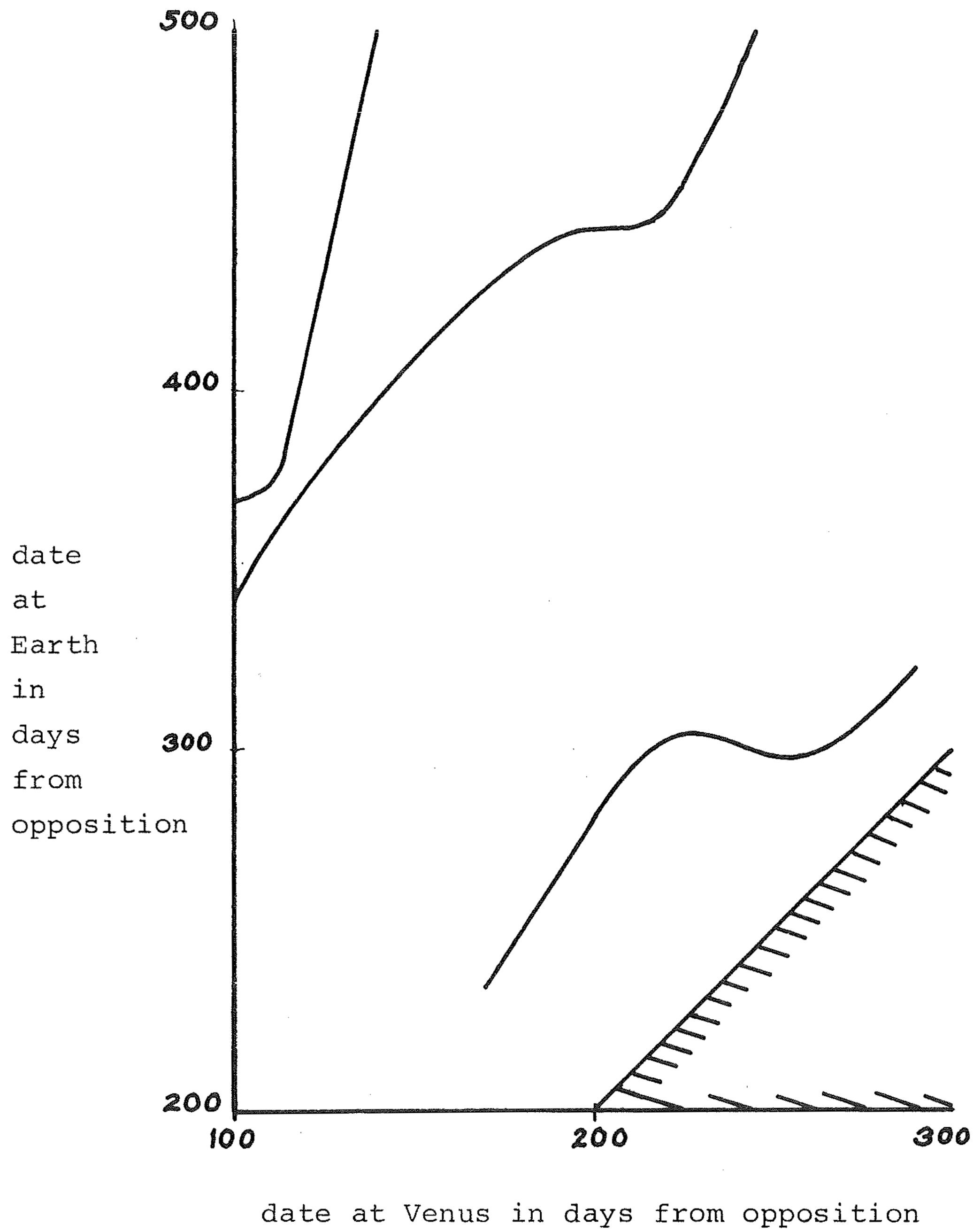


Figure 4-4. Locus of Points such that the hyperbolic excess speed difference at Venus is equal to zero.

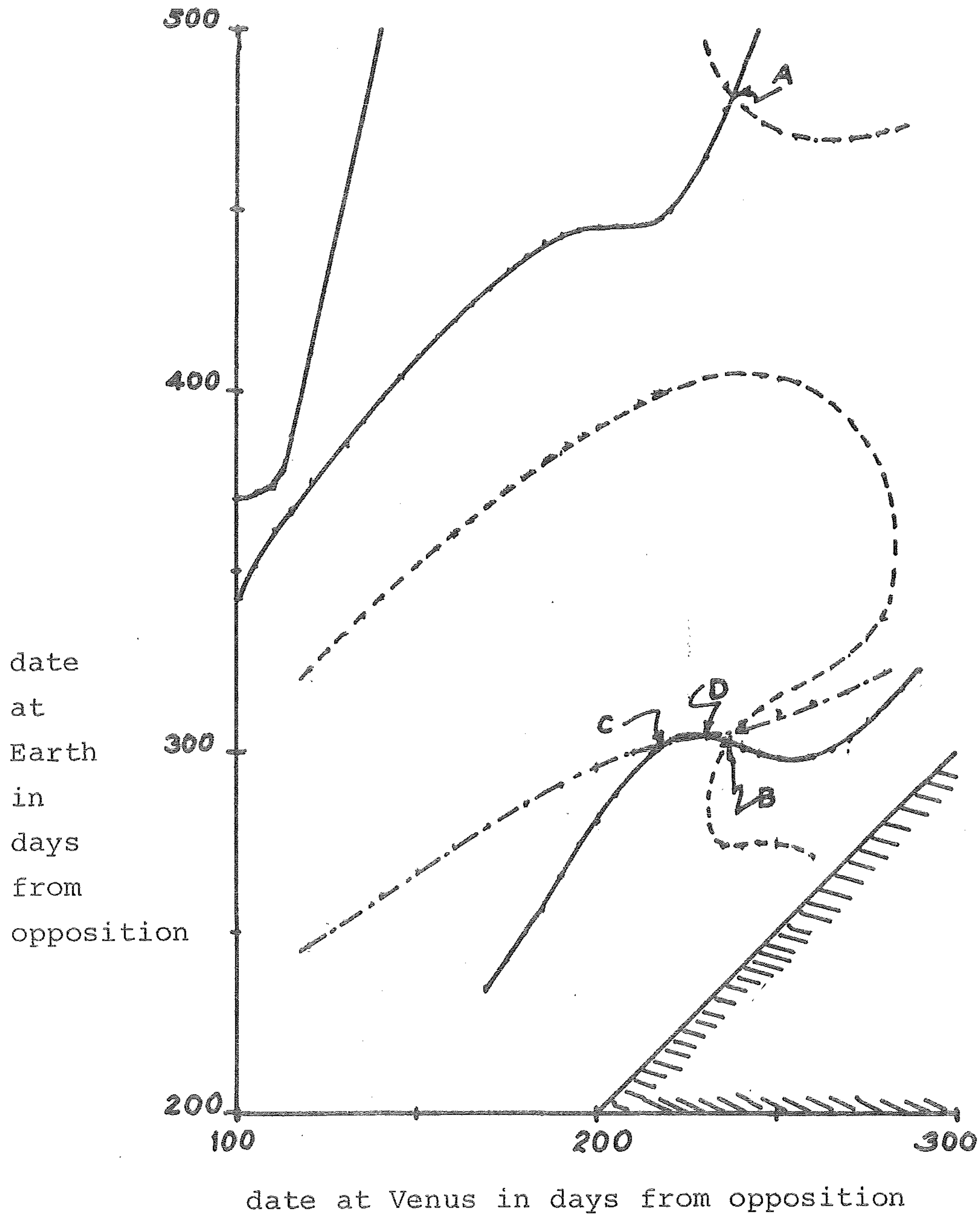


Figure 4-5. The locus of Figure 4-4 plus two different locii such that the hyperbolic excess speed difference at Earth is equal to zero. (------) with two full revolution returns at Venus. (-•-•-•-•-) with three full revolution returns at Venus.

the same regardless of how much time the vehicle spends in the vicinity of Earth and in the vicinity of Venus. It is the only locus of interest if the vehicle does not go to Venus except in going to and from Mars as in Figure 4-2. One more locus is needed if the vehicle is to go to Venus a second time as in Figure 4-1 in order to obtain a solution at an intersection. Note that Figure 4-4 says nothing about whether the required turns are possible at Earth, at Venus, or at Mars. Actually, it is impossible to say from the information on Figure 4-4 whether the turns are possible at Earth and the second time at Venus, unless one specifies what direct returns are to be used at Earth and at Venus. One could, however, tell whether the required turns were possible at Mars and at Venus on the way to and from Mars; but this information has not been determined for this plot. One can comment from experience, however, that the turns are usually impossible, except for small regions around dates at Earth of 300 and 445 days and dates at Venus not too far separated from about 220 days.

Figure 4-5 shows both the locus for equal hyperbolic excess speeds at Venus and for equal hyperbolic excess speeds at Earth--for two different schemes of direct returns at Venus. In both schemes, the two series of direct returns at Earth in each 6.4 years each consists of one full revolution return. The locus for no difference in hyperbolic excess speed at Venus is shown as a solid line (———).

The locus for no difference in hyperbolic excess speed at Earth for the scheme which involves two full revolution returns at Venus is shown as a dotted line (-------). The locus for equal hyperbolic excess speeds at Earth for the scheme which involves three full revolution returns at Venus is shown as a line which alternates dots and dashes (-·—·—·-). The intersections of the dotted lines with the solid line are solutions which are to be investigated for the possible existence of periodic orbits. The intersections are solutions; but they must be investigated to see that they will work to the extent of missing the encountered planets and to the extent that they will converge for more accurate models of the solar system than the one used here, solar system Model I.A.1.

Intersections A and B in Figure 4-5 indicate solutions to the scheme which involves one FR at each Earth visit and 2FR at the visit to Venus which occurs half way through the 6.4 year cycle of the scheme. The solution corresponding to intersection A results in the vehicle's intersecting the surface of Venus both immediately before and immediately after the encounter with Mars. The solution corresponding to intersection B results in the vehicle's intersecting the surface of Venus at each encounter associated with the two full revolution returns at that planet.

Intersections C and D in Figure 4-5 indicate solutions to the scheme which involves one FR at each Earth visit and

3FR at the visit to Venus which occurs half way through the 6.4 year cycle of the scheme. These two intersections are very close together. With a more accurate circular coplanar model for the solar system such as Model I.A.2 or I.B in which the opposition of Mars does not exactly correspond to the time of the superior conjunction of Venus, one could expect a number of different possible solutions, perhaps none through four in number. Apparently, the number of solutions in this case is zero, because the circular coplanar computer solution always failed, regardless of the starting point for the dates, which were always in the vicinity of the dates indicated by the intersections C and D.

This behavior for solutions obtained in this manner is fairly typical. Many other schemes of this type were tried, and none of them worked; some of them did not even supply intersections of locii. All of the schemes were of the type Earth-Venus-Mars-Venus-Earth-(Earth)-Venus-(Venus)-...-Venus-(Earth)-Earth-repeat. Fourteen different schemes were tried. Nine involved a wait at Earth each time of one full revolution return, four involved a wait at Earth of one symmetric return, and one involved two full revolution returns at Earth. They all involved different waits at Venus. Whenever a symmetric return was involved, for the purposes of obtaining a plot, it was assumed to be a constant 1.4 planetary periods. One could then make runs with S1SR and L1SR and attempt to obtain convergence. This approach gave line intersections in eleven of the fourteen cases. A great many of the subsequent computer runs did not converge to a solution, and all of the solutions which did converge resulted in the trajectory's intersecting the surface of one or more planets.

If the vehicle is not to go to Venus a second time and the time other than the Earth-Venus-Mars-Venus-Earth trajectory is to be spent only in direct returns at Earth, then only the locus of Figure 4-4 is of interest. In this case, one selects different combinations of flybys from Appendix D and reads the fifth (or next to last) column in the main table of Appendix D in order to obtain the "date at Earth" which is to decide the ordinate on Figure 4-4. From Figure 4-4, then read off the date at Venus (relative to the date of opposition). Approximate dates for a circular coplanar computer attempt are obtained in this manner. There were thirteen such attempts possible. Not all were attempted on the computer, but a sufficient number were attempted to determine that the others would not work. A few did not converge, but most hit one or more planets. Only one scheme seemed to be a possibility. It involved a scheme which one can symbolically represent as (FR)(HR)(2FR)(S1SR). It is this scheme which is shown in Figure 4-2. With this arrangement, the vehicle only grazes Earth on the side of the half revolution return associated with the single full revolution return. Unfortunately, when an eccentric inclined computer run was attempted, the problem never converged even with the smallest values for the eccentricities and inclinations used in the computer program; apparently, a solution does not exist with this scheme.

The fact that no periodic orbits or long series of flybys were found with the schemes of this section does not prove that there are no working schemes of these types; but the existence of working schemes of this type seems unlikely.

4.4 The Possibility of Obtaining a Mars-Venus Periodic Orbit Which Visits Earth Occasionally

Because a Mars-Venus Hohman transfer takes about 216 days while the period of Venus is 225 days, a fairly similar number, it was thought possible to more easily obtain a Mars-Venus periodic orbit than to obtain a Mars-Earth periodic orbit. If such a Mars-Venus periodic orbit could be found, one could then examine it to see when a vehicle following it would be brought fairly close to Earth; when the vehicle was in the vicinity of Earth, an attempt could instead be made to have the vehicle encounter Earth and perform a flyby maneuver.

The approach to obtaining such a periodic orbit was to have the trajectory go from Venus to Mars to Venus in a low energy fashion (taking around 400 or 450 days for the round trip and making about one revolution of the Sun) while Venus makes about two revolutions of the Sun. Then find appropriate direct return trajectories in the vicinity of Venus until the next opportunity for a transfer to Mars presents itself. The opportunity for the next transfer is indicated by the fact that the synodic period of Venus relative to Mars is about 333 days.

Attempts were made to find a periodic orbit connecting Venus and Mars involving many different schemes of direct return orbits. In each case, investigation was limited to the circular coplanar problem. Attempts were made with repeating periods of up to four Mars-Venus synodic periods. In all cases, the attempted method either did not converge, or the trajectory intersected the surface of Venus. Schemes involving full revolution returns at Venus did not work, primarily because the velocity vector for the interplanetary transfer is primarily in the direction of Venus' motion around the Sun while the velocity vector relative to Venus for a full revolution return at Venus is primarily perpendicular to the direction of Venus' motion. Venus could not supply the necessary change in direction of the velocity vector of a vehicle following the proposed periodic orbit scheme.

More promising direct return orbits appeared to be those which travel around the Sun a different number of times than does Venus; the direct return orbits attempted here are similar to symmetric return orbits in that they do not travel around the Sun an integral number of times and in that they must lie in the same plane as that of the departure and arrival planet--Venus in this case. This investigation attempted to include several different direct return orbits of this type into several different periodic orbits connecting Mars and Venus. In each case, the attempt either failed to converge; or the resulting trajectory intersected the surface of Venus. The failure is partially due to the lack of knowledge concerning direct returns which traverse the Sun a different number of times than the planet.

This investigation did not succeed in finding any periodic orbits connecting Mars and Venus under the circular coplanar approximation. Hence, the investigation never proceeded to the point of trying to distort a Mars-Venus periodic orbit to visit Earth or to the point of going from the circular coplanar case to the eccentric inclined case. In conclusion, it can be said that this investigation did not prove that no periodic orbits of the type considered connect Mars and Venus; this investigation simply failed to find any.

4.5 Use of Earth-Mars-Earth Round Trip Trajectories Plus Suitable Waiting Times in the Vicinity of Earth

The scheme which was used in this section finally led to success in the search for free-fall periodic orbits connecting Earth and Mars. It involves a systematic combination of Earth-Mars-Earth flyby trajectories and a suitable series of direct return trajectories in the vicinity of

Earth.

Ross[4] investigated Earth-Mars-Earth (and Earth-Venus-Earth) flyby trajectories. He created trajectory maps for the circular coplanar case as well as for the eccentric inclined case. His results were reprinted in the NASA handbooks[8]. The trajectory maps are reprinted here as Figures 4-6 through 4-10.

Since the determination of exactly what these trajectory plots mean is not entirely clear, they will be explained here. The first important fact to note is that all dates are measured with respect to the date of opposition of Mars. Figures 4-6 and 4-7 give a broad range of dates and describe round trips to Mars which are not perturbed by the flyby of Mars. In other words, the flyby distance at Mars is large. Each round trip trajectory is described by two points. Both of these points are on the smooth lines which are not the contours of speed at Earth. Each point corresponds to a two-planet interplanetary trajectory--one, from Earth to Mars; and one, from Mars to Earth. The first point is to be chosen by looking only at the encounter dates which appear upright when the chart is in its normal position. The first point corresponding to the Earth-Mars trajectory can be chosen by considerations of encounter dates at the planets and/or the departure speed from Earth. With the choice of the first point made, the second point is determined uniquely. The chart must be turned over and attention paid only to those

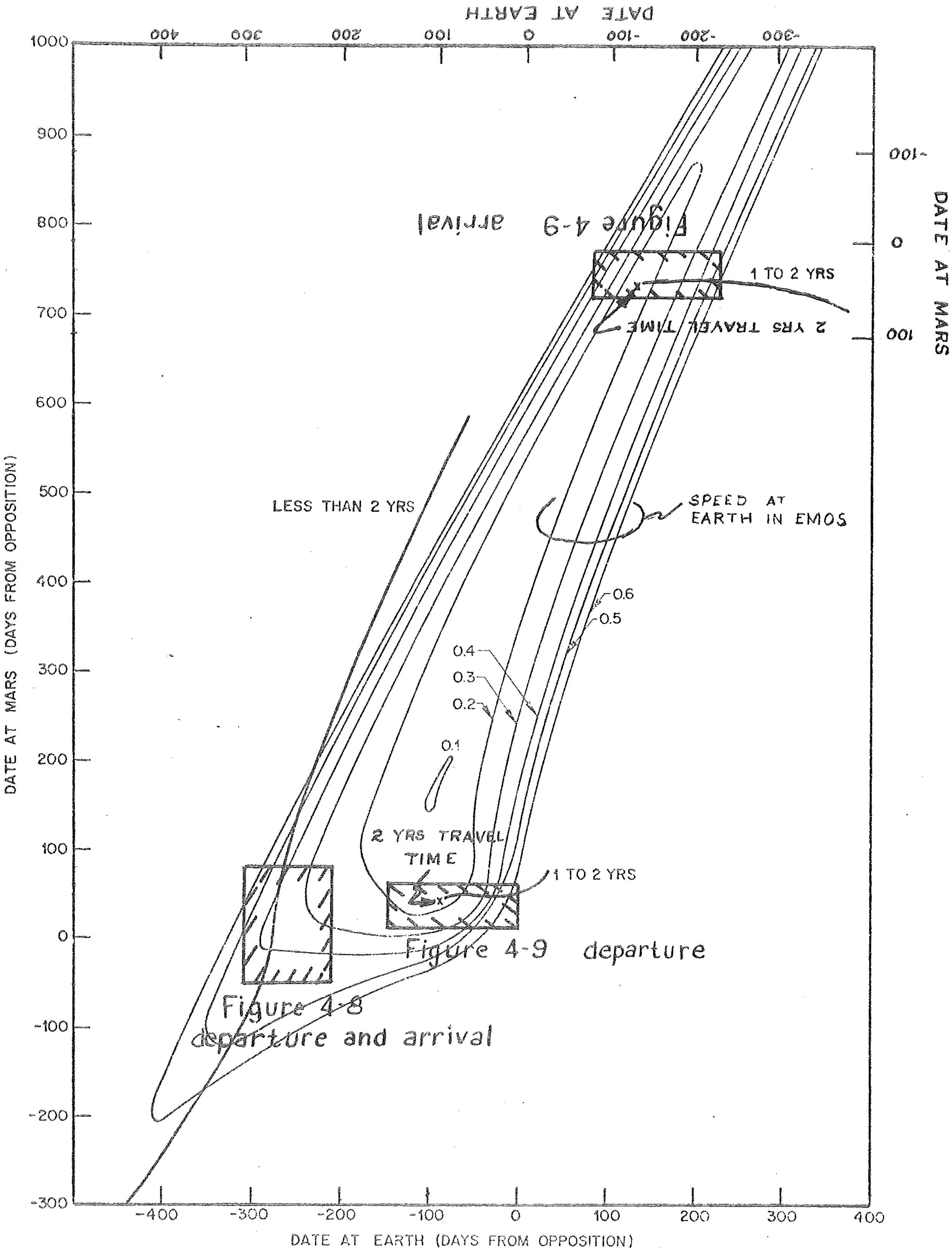


Fig. 4-6 Symmetric Round Trips Past Mars, Overlaid on Constant Speed Curves

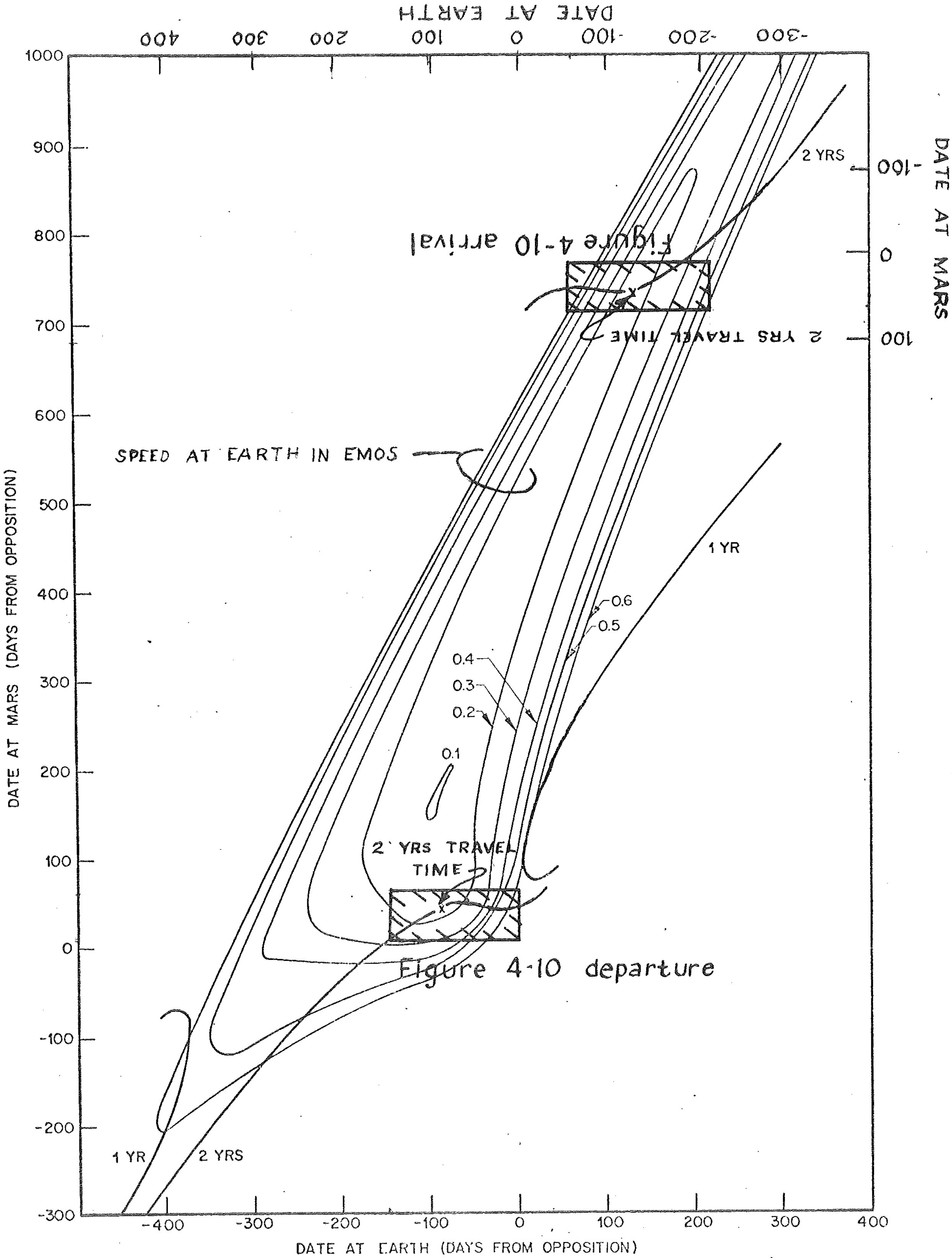


Fig. 4-7 Nonsymmetric Round Trips Past Mars, Overlaid on Constant Speed Curves

encounter dates which then present themselves as upright. From the date at Mars which has been chosen and which also describes the date at Mars for the second half of the trip, one determines the corresponding point on the same or another of the smooth curves. The procedure is the same for both of these charts.

Figures 4-8 through 4-10 are very similar to Figures 4-6 and 4-7 with only two important differences. Figures 4-8 through 4-10 include a smaller range of dates, but they include the effect of different passing distances at Mars. The contours on these charts correspond to the speed at Earth measured in EMOS and the minimum planetocentric passing distance at Mars measured in units of that planet's radius. The passing distance curves are the ones marked 1, 2, 3, 5, and ∞. Points which describe legs of a round-trip trajectory may now be within large regions of these figures. The contours of periplanet distance now perform the same function as the smooth lines in Figures 4-6 and 4-7. One chooses the first half of an Earth-Mars round trip by considerations of dates at the planets, the departure speed from Earth, and/or the periplanet distance at Mars. One then has a point which corresponds to some date at Mars and some passing distance at Mars. One then turns over the chart as before in order to find the point corresponding to the second half of the round trip. This second point is determined by the intersection of the date at Mars with the periplanet distance at Mars.

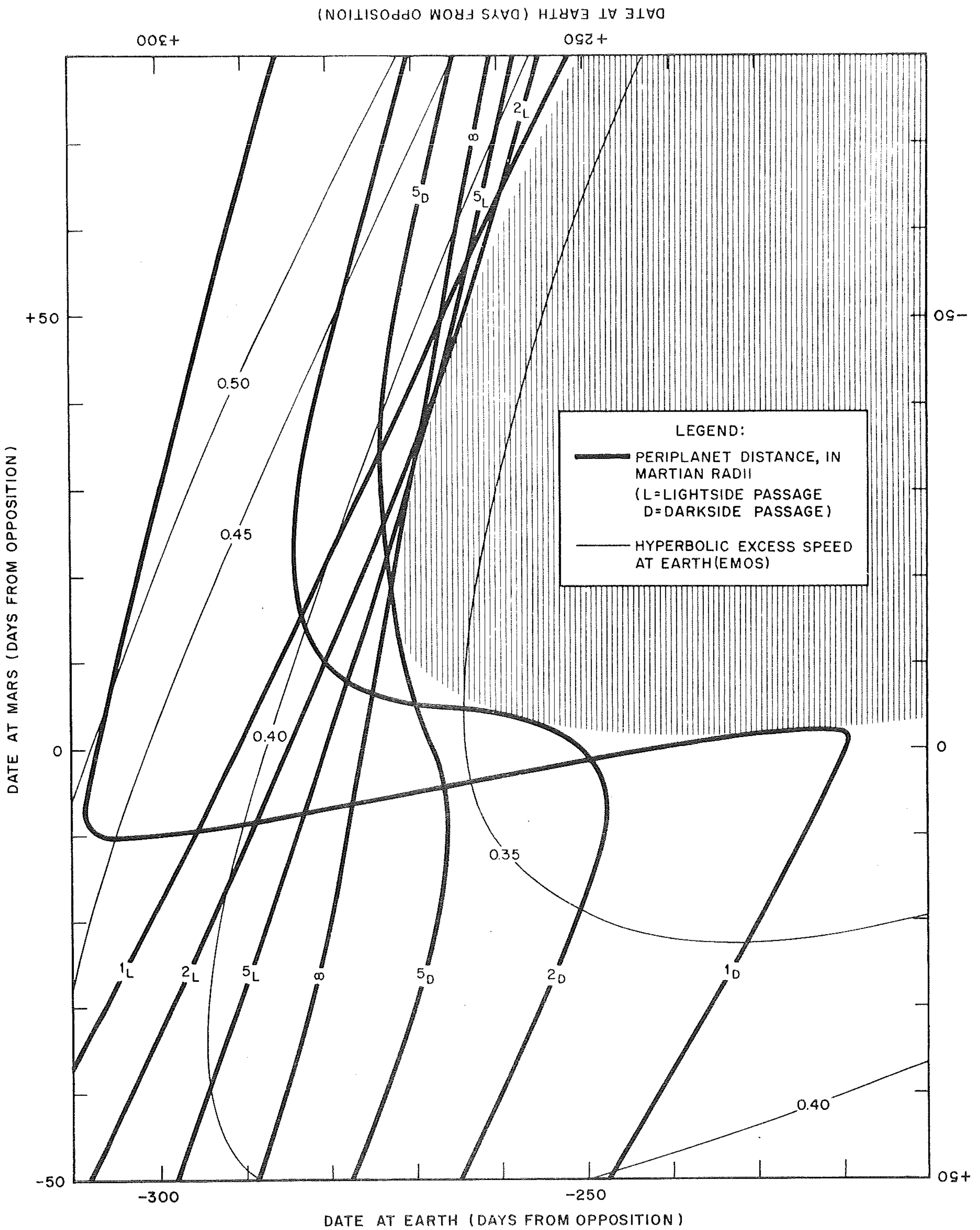


Fig. 4-8. Symmetric High-Energy Trips Past Mars, Modified by Close Approaches at Various Distances

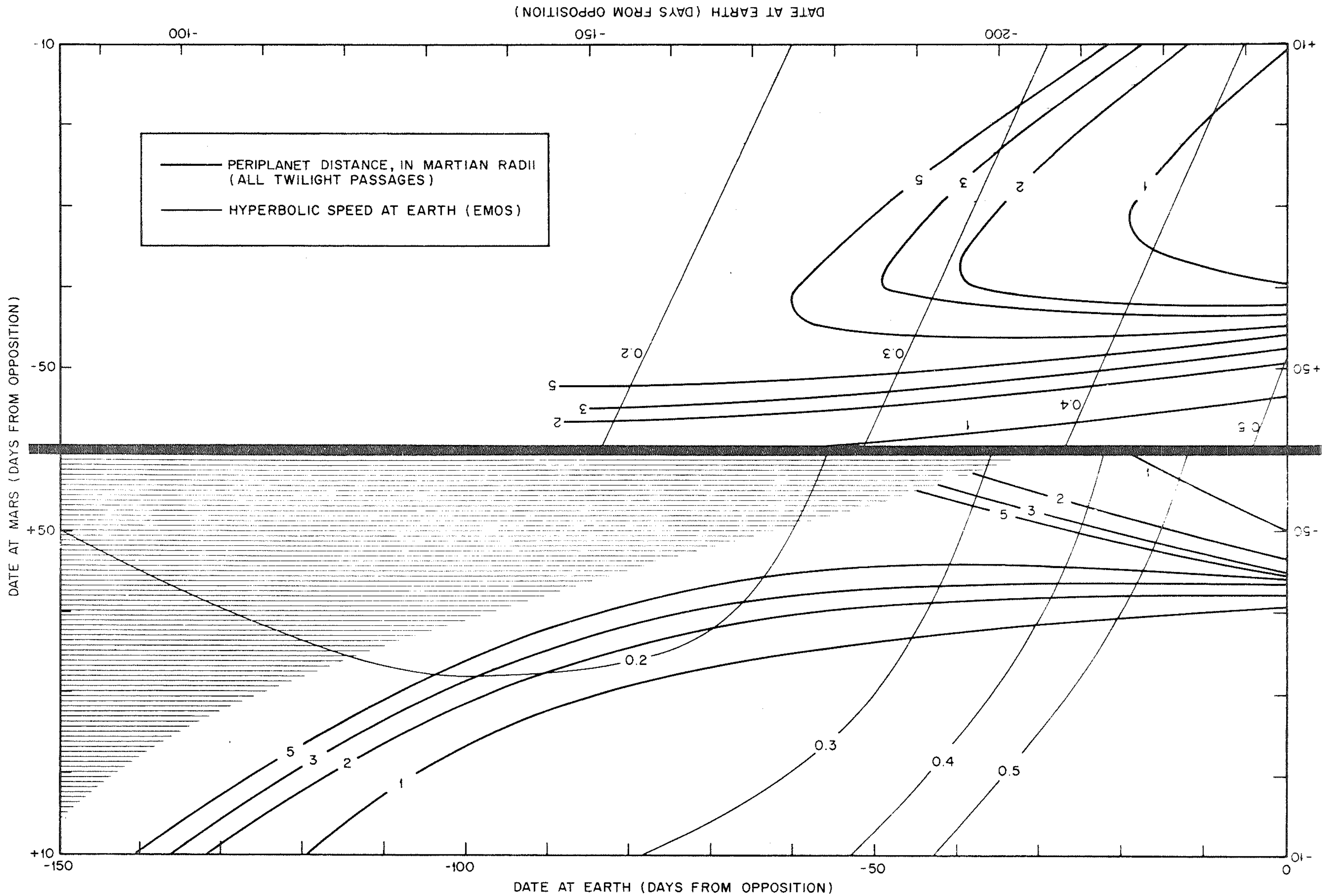


Fig. 4-9. Symmetric Low-Energy Trips Past Mars, Modified by Close Approaches at Various Distances

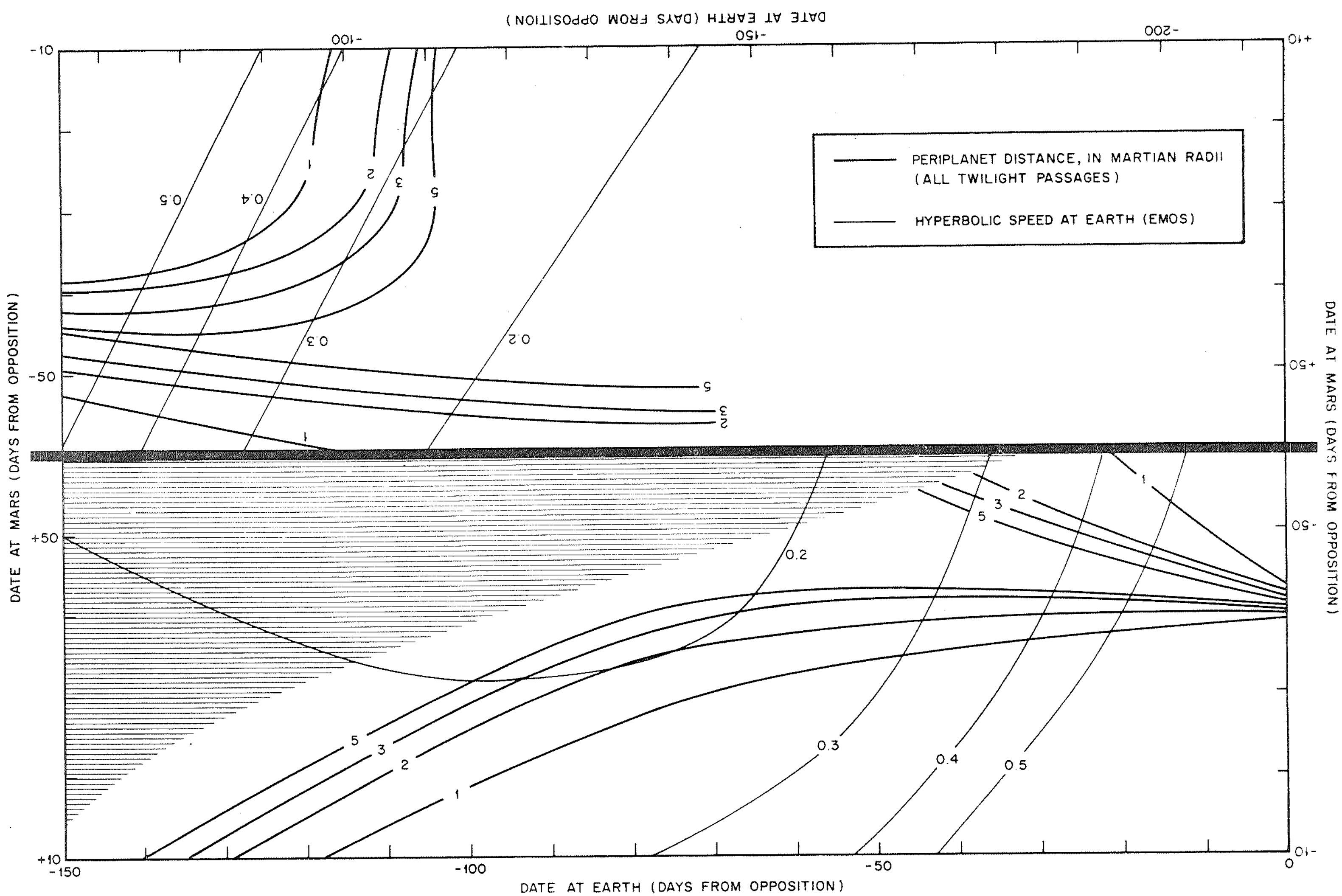


Fig. 4-10. Nonsymmetric Low-Energy Trips Past Mars, Modified by Close Approaches at Various Distances

As before, one only looks at the dates which are upright when one is selecting a round-trip trajectory. As before, one has two points, each of which corresponds to one leg of the round-trip trajectory.

A few more comments should be added here about these charts. Note that the areas of Figures 4-8, 4-9, and 4-10 are indicated on Figures 4-6 and 4-7. Also note that the arrival regions of Figures 4-9 and 4-10 are almost the same; there is a large amount of overlap. The important thing to see then is that the contours of periplanet distance in these two figures are continuous from the one chart to the other. Finally, note that the departure regions for these two figures are exactly the same.

The existence of "reciprocal" trajectories is also important for this scheme. The existence of these "reciprocal" trajectories is discussed at the beginning of this chapter. Briefly, if one thinks of an interplanetary trajectory arranged in some manner in time around an opposition of the two planets in circular coplanar orbits, then there also exists a "reciprocal" trajectory, for which the dates of encounter are the negatives of the dates of the original trajectory when all dates are measured from the date of opposition. The existence of these reciprocal trajectories for planets in circular coplanar orbits also implies that each round-trip flyby trajectory in such a system also has a reciprocal round-trip flyby trajectory, since each leg of the flyby trajectory is a simple interplanetary trajectory. For each of the Earth-Mars-Earth round-trip trajectories indicated in the Figures 4-6 through 4-10, there exists a reciprocal round-trip trajectory for which the dates of encounter relative to an opposition of Mars are reversed in sign and for which the speeds at each planet, corresponding to the same numbers for dates (but having different signs), are the same.

The basic scheme used in attempting to create these periodic orbits involves both the Earth-Mars-Earth round trip trajectory and its reciprocal. That is to say, if a certain round trip trajectory is used in a periodic orbit attempt, then the reciprocal trajectory is also used in the basic circular coplanar scheme. The only exception to this rule is if a round trip trajectory used in a periodic orbit attempt is its own reciprocal; the round trip trajectory will be its own reciprocal if the date of encounter with Mars corresponds with the date of opposition and if the dates of Earth encounter are placed symmetrically about the date of martian opposition. In the general case, the two round trip trajectories are spaced by two different series of direct return trajectories at Earth so that the length of the basic scheme will be equal to an integral number of synodic periods and so that the desired pair of reciprocal round trip trajectories can be properly placed in time with respect to different oppositions of Mars. This basic scheme is made clearer in the next chapter in Figures 5-1 through 5-4.

This method of attempting to obtain an Earth-Mars periodic orbit therefore requires the selection of two series of direct return trajectories at Earth. Each of these two series of direct return trajectories is chosen so that when the series of direct returns is centered in a time interval which consists of an integral number of synodic periods, the times of the ends of the series of direct returns and the times of the oppositions differ by amounts which correspond to times which are desirable for creating an Earth-Mars-Earth round trip. Desirable times are indicated by the departure and arrival dates of Figures 4-8 through 4-10.

Appendix D was created with the problem in mind of selecting appropriate combinations of direct returns to use in periodic orbit attempts. The table of this appendix contains direct returns which last up to a maximum of about 6.4 years. Note that from a given length of time spent in the vicinity of Earth on a series of direct returns, one can easily calculate a corresponding date relative to an opposition for use in selecting an Earth-Mars-Earth round trip trajectory to be used in a periodic orbit attempt. Let T_s be the average synodic period of Earth and Mars; let n be the number of synodic periods between the oppositions around which the Earth-Mars-Earth round trip trajectories are to be centered; let T_d be the waiting time in the vicinity of Earth due to a series of direct return orbits; then the date relative to the opposition, T, on which we must leave Earth on a round trip trajectory, can be expressed simply as,

$$T = \frac{1}{2} (n T_s - T_d) \qquad (4\text{-}2)$$

The resulting numbers are columns three, four, and five of the table of Appendix D. The number of columns with this information is limited to three, because oppositions of interest are arbitrarily limited to those which are three synodic periods apart or less. In other words, successive round trip trajectories of interest in a periodic orbit attempt are not arranged around oppositions which are separated by more than three synodic periods. This explanation of how the two series of direct returns are selected is also presented in Appendix D, but it has been included here for additional clarity.

The next task in obtaining periodic orbits is to examine carefully Figures 4-8, 4-9, and 4-10 as well as the numbers in Appendix D in order to see how many round trips can reasonably be patched together with series of direct return trajectories at Earth. The Figures 4-8 through 4-10 give no information about the direction of the hyperbolic excess velocity vector; hence, because of the lack of information, no series of direct return trajectories can be eliminated because the direction is wrong, although some can be eliminated because the speed at the end of the round trip trajectory is too large. A close look at the dates and the charts will reveal quickly that a very large number of periodic orbits of the class considered seem to be reasonable to try--a number on the order of 10^3. In order to reduce the number of periodic orbit candidates which have to be examined, it seems reasonable to try several of the many possible candidates and then to make judgements on the candidates with encounter dates not very far different from the ones of the scheme tried. In this manner a large number of possible candidates can be eliminated without having to set up and run every one of them on the computer.

In this manner, eighteen reasonable candidates were obtained which worked or almost worked in the circular coplanar case (solar system Model I.). None of these candidates involve round trip trajectories from Figure 4-8. These eighteen reasonable candidates are presented in Appendix E along with comments as to how well they worked in the eccentric inclined case (solar system Models II. or III.). Ten of these work in at least one version with the best system model.

These eighteen circular coplanar possibilities may not be all of the reasonable circular coplanar possibilities, but they are all that this investigation has found. In fact, there may be some other scheme or schemes of the types considered which lead to a working periodic orbit; but the author considers this possibility unlikely. Many of the possibilities listed in Appendix E do not work in the eccentric inclined case (solar system Models II. or III.).

Further discussion of how these periodic orbits work and what they look like is presented in the following chapter. What happens in the eccentric inclined case is also discussed. An understanding of how these periodic orbits work and what they look like is important if one is to consider their application to real missions.

4.6 Periodic Orbits Which Involve Earth, Mars, and Venus

Now that a method has been found to obtain candidates for Earth-Mars periodic orbits, a better method to obtain candidates for Earth-Mars-Venus periodic orbits is seen clearly. As discussed in the first chapter, a continuous set of flybys involving these three planets may not be found; but one might reasonably expect to find a very long series of flybys involving these three planets.

The key to finding the three planet periodic orbits is the same as that described in the preceding section of matching up round trip trajectories which leave and return to Earth with suitable series of direct returns at Earth.

The important things to start with are the information of Appendix D for different series of direct returns in the vicinity of Earth and information contained in Earth-Venus-Mars-Earth, Earth-Mars-Venus-Earth, and Earth-Venus-Mars-Venus-Earth round trip trajectory charts. Unfortunately, these round trip charts involving three planets apparently do not exist. Therefore, the investigator must start by creating a set of these charts for his own use. These charts should plot the date of departure from Earth versus the date of arrival back at Earth and contain contours of hyperbolic excess speed at the different Earth encounters. The same chart or other charts should include the dates of encounter at the intervening planets, information about the direction of the hyperbolic excess velocity vectors at Earth, and a clear indication of the region where the vehicle will not hit any of the intermediately encountered planets. The solar system model which should be used for the creation of these charts is the circular coplanar, exactly symmetric and periodic model (solar system Model I.A.1) so that reciprocals of most of the trajectories exist; and hence, fewer charts are required; and the charts are an approximation for a longer period of time. The charts must then be examined along with the information of Appendix D in order to obtain candidates for the periodic orbits. Other trajectory charts and round trip charts which end at Venus could also be created so that one could also consider periodic orbit candidates which involve direct returns at Venus as well as at Earth.

This investigation has not been carried out, but it is the next logical step.

CHAPTER 5

DETAILED DEMONSTRATION OF THE MORE PROMISING PERIODIC ORBITS

5.1 Earth-Mars Periodic Orbits in the Circular Coplanar Case

An understanding of the character of the periodic orbits found should be important to the reader. This character is best understood by making the solar system model as simple as possible for the initial examination. Understanding the circular coplanar case should be quite useful to the mastery of more exact cases, because one knows from experience that the periodic orbits do not change too much when going from the less accurate to the more accurate models.

Solar system Model I.B. is to be used here. That is, the orbits of Earth and Mars are to be circular and coplanar but are to have accurate values for semimajor axis and period.

The labeling system used for periodic orbits is explained in this paragraph. Each label for the periodic orbits considered here begins with the letter M to indicate that the vehicle is to go to Mars as well as to Earth. The second element of each label is a number such as 4, 5, or 6 which stands for the number of synodic periods which are required for the completion of the basic circular coplanar scheme. Each of the periodic orbits found makes two round trips to Mars in the time required for the basic circular coplanar scheme. Next in each label comes a hyphen followed by another integer. This integer is basically arbitrary and is used to differentiate among the periodic orbits within each group. The last element of each label is a lower case letter which is used to differentiate among the different variations of each periodic orbit scheme. The difference among versions of each scheme depends upon which Earth-Mars opposition begins the scheme. The number of different versions will be equal to the number of synodic periods which are required for the completion of the basic circular coplanar scheme. Further discussion of this fact can be found in Section 5.3. However, the difference among the variations is not significant in the circular coplanar case; and hence, the lower case letter at the end of the label is dropped when referring to the circular coplanar case or to the basic scheme in general

Only the more promising Earth-Mars periodic orbits are discussed in this chapter. "Promising" periodic orbits are those for which the probability is high for the existence of the actual, indefinitely long series of multiple flybys. These periodic orbits also deliver more round trips to Mars in a given length of time than the less "promising" periodic orbits. All of the circular coplanar periodic orbits found in the investigation of this thesis are presented in Appendix E, but only a small number of them are very "promising." The most promising is periodic orbit M4-1. Other periodic orbits which are also to be considered in some detail here are M5-1 and M5-2. Those periodic orbits numbered M6-... or greater can be considered unnecessarily long for the number of round trips to Mars which are achieved.

Better understanding can be obtained by referring to the figures of this chapter as one reads the chapter. Figures 5-1 through 5-4 give the distance from the Sun as a function of the time for periodic orbits M4-1, M5-1, M5-2, and M6-1, respectively. These figures present the circular coplanar case. The vertical lines in these figures indicate oppositions of Earth and Mars. The large dots indicate points and times of flyby encounter with the planets. Figures 5-5 and 5-6 give respectively the path of periodic orbit M4-1 in the circular coplanar case in a Sun-centered inertial frame and in a Sun-centered coordinate frame which rotates with the Earth-Sun line. In Figure 5-5, the direct returns are not shown in order not to confuse the picture. The Arabic numbers indicate consecutive planetary encounters beginning with the encounter immediately following the

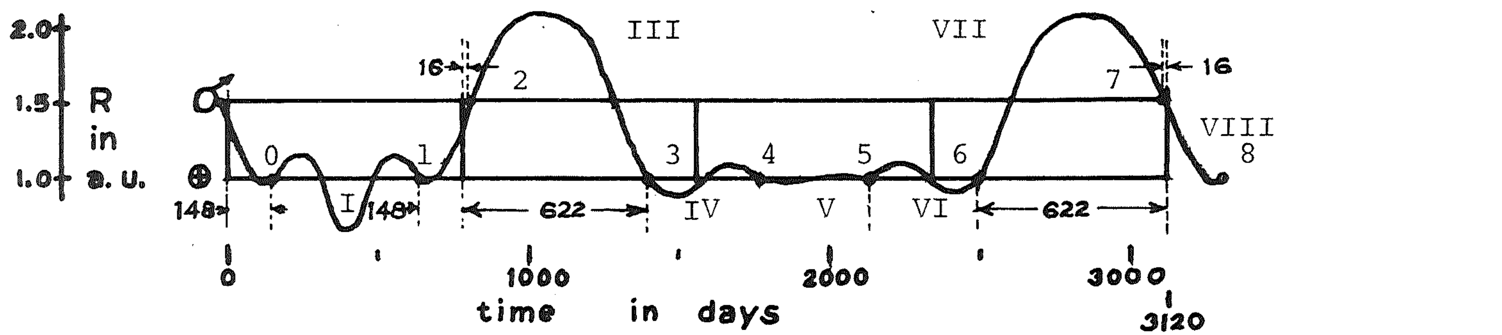


Figure 5-1. Distance from the Sun as a function of time for periodic orbit M4-1.

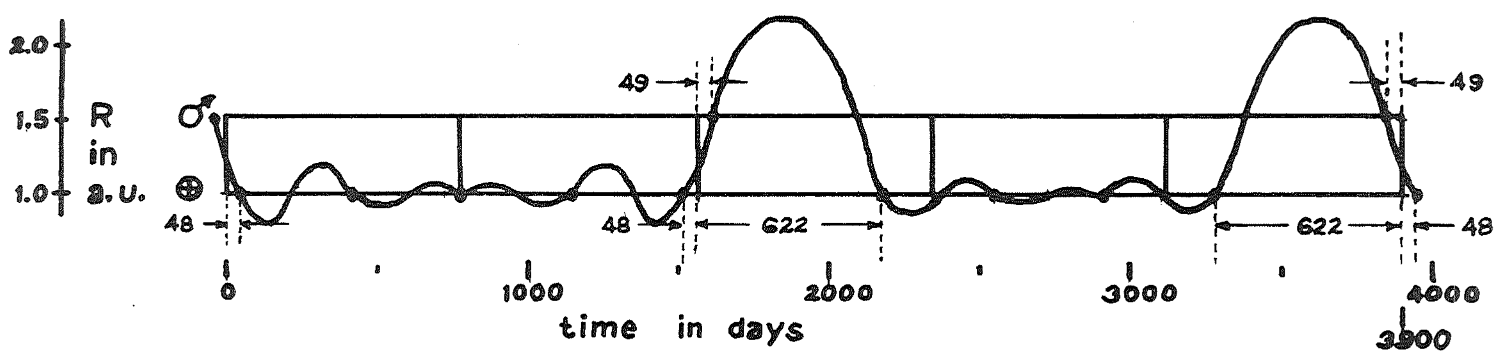


Figure 5-2. Distance from the Sun as a function of time for periodic orbit M5-1.

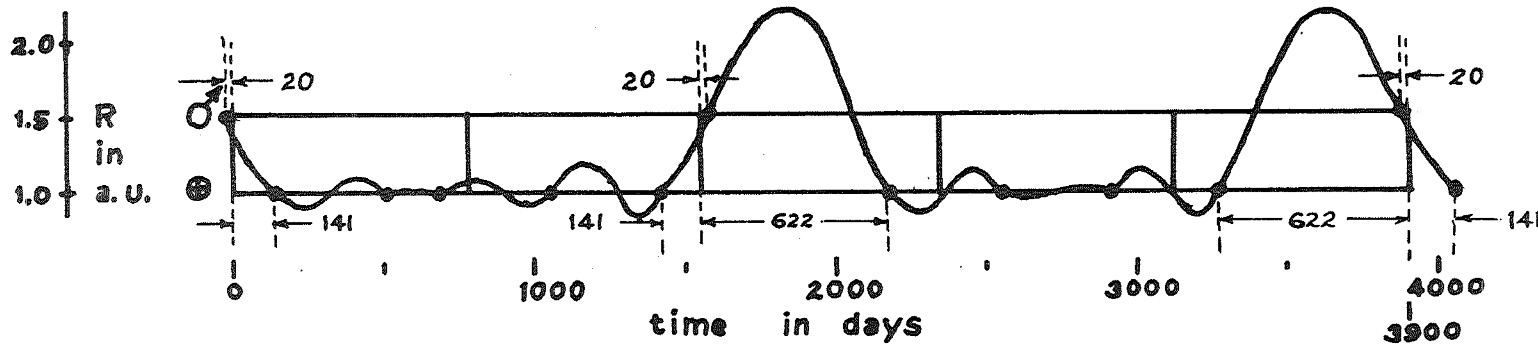


Figure 5-3. Distance from the Sun as a function of time for periodic orbit M5-2.

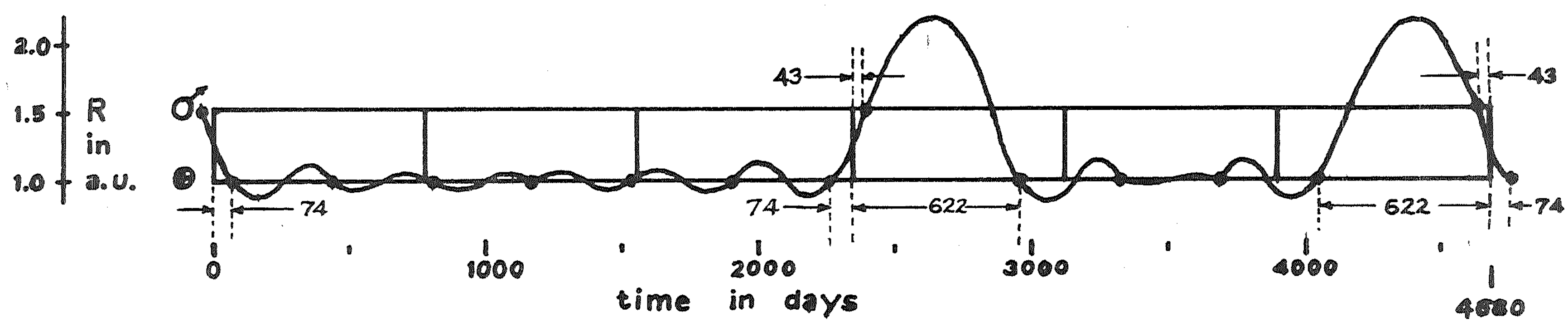


Figure 5-4. Distance from the Sun as a function of time for periodic orbit M6-1.

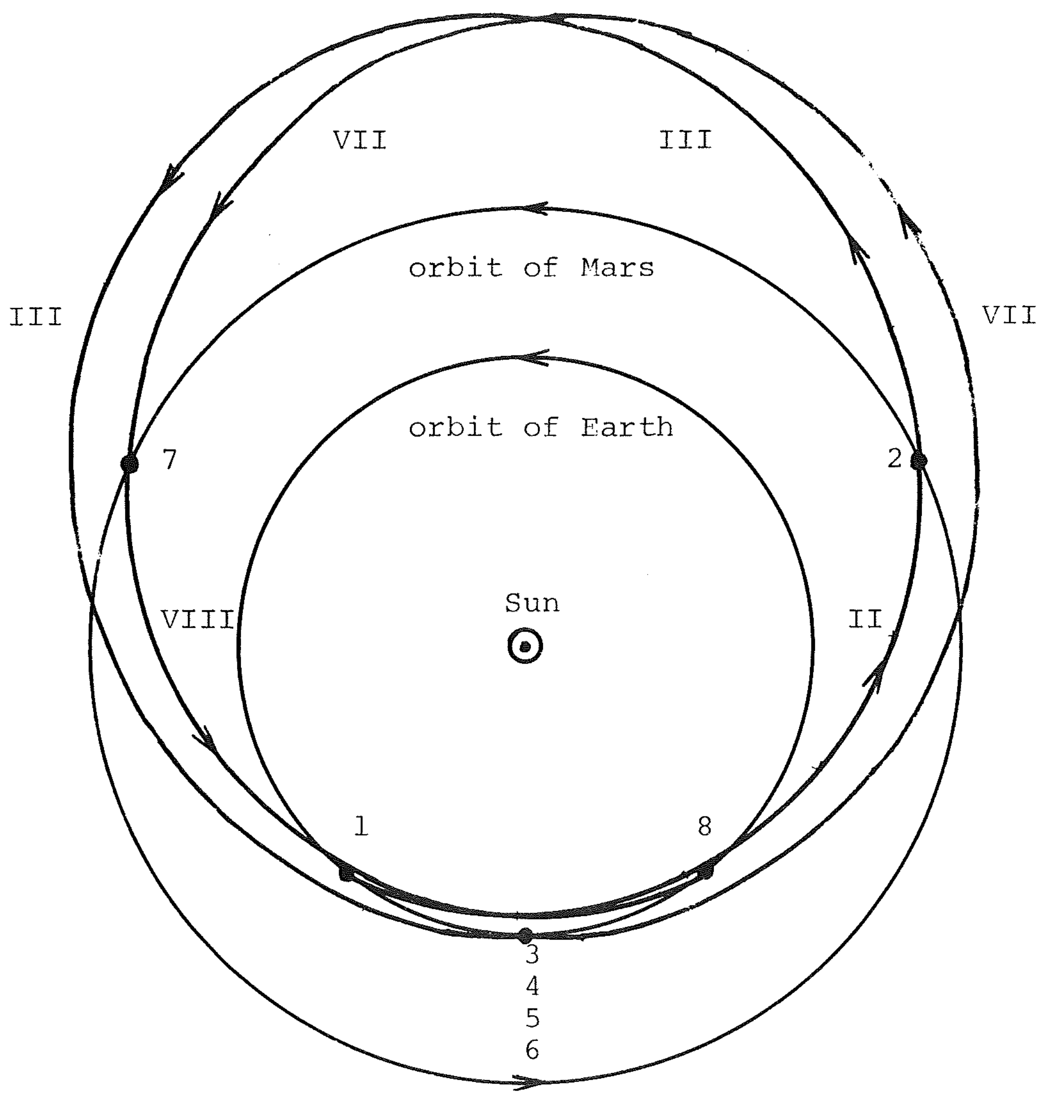


Figure 5-5. The interplanetary trajectory legs of periodic orbit M4-1 shown in a sun-centered, inertial coordinate frame.

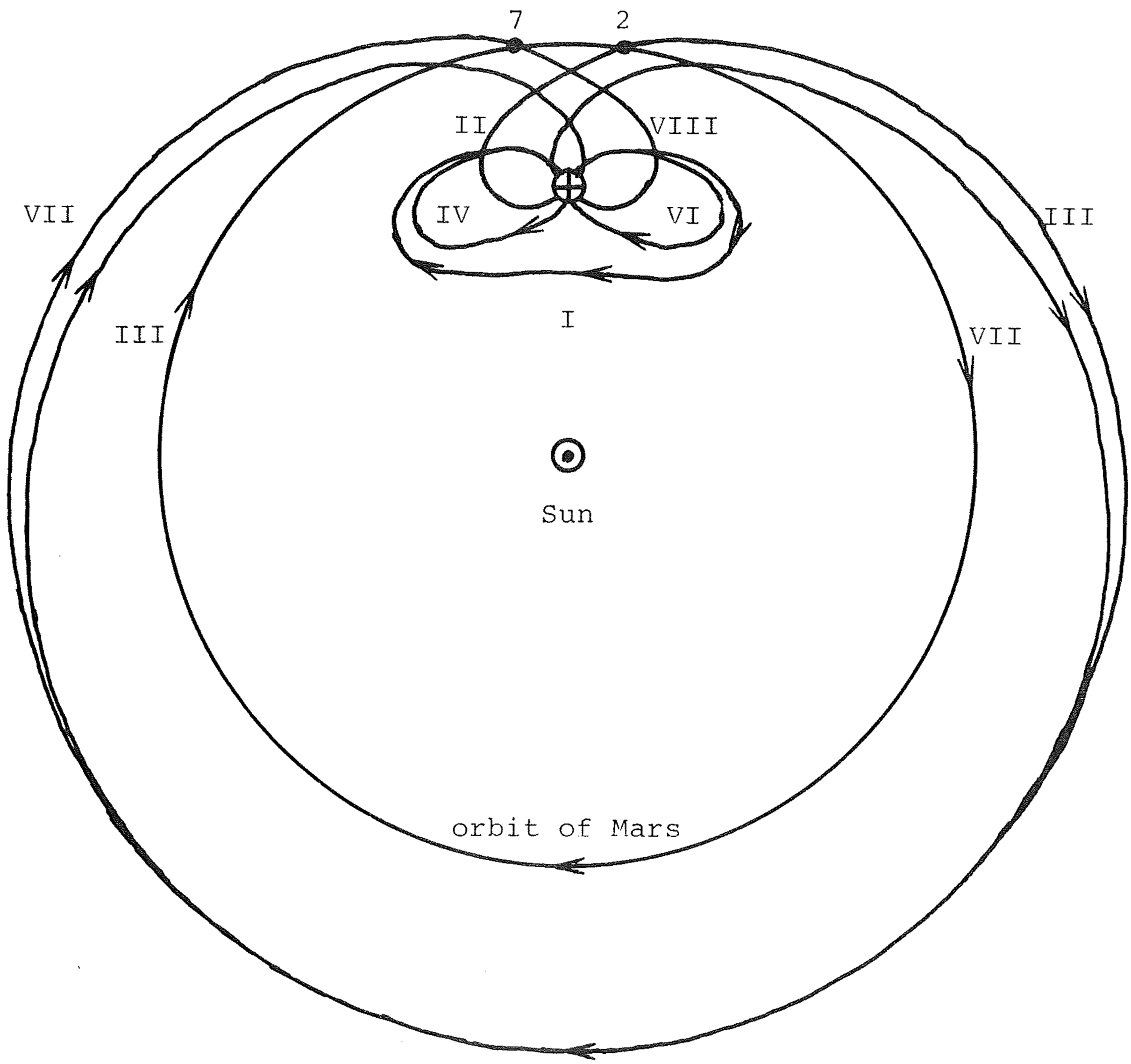


Figure 5-6. Periodic orbit M4-1 shown in the Sun-centered coordinate frame which rotates with the Earth in its orbit.

symmetric return trajectory at Earth (refer to Figure 5-1). The four numbers at Earth in the "middle" of the trajectory and in the middle of the figure indicate the four encounters associated with the three full revolution returns at Earth. The Roman numerals number the different trajectory legs in the periodic orbit; the symmetric return at Earth is the first trajectory leg according to this numbering system. The same numbering system is used in both Figure 5-5 and Figure 5-6. Almost all of the encounters and trajectory legs are shown in Figure 5-6, even though the resulting figure appears to be quite confusing, because some of the trajectory legs almost lie on top of each other. Trajectory leg number V is not shown in Figure 5-6, however, because it is confined to the near vicinity of a line perpendicular to the plane of the planetary orbits, which passes through the Earth. On this full revolution return, the vehicle travels about 0.18 a.u. above (or below) the plane of the ecliptic, then travels the same distance below (or above) the ecliptic, and finally returns to Earth. The common and differing characteristics are easily seen in Figures 5-1 through 5-4. The characteristics can also be seen from the listing of Appendix E, although they are not quite so obvious there.

One should note that the character of the round trip trajectory segments to and from Mars is always similar for the periodic orbits found. Since these round trip trajectory segments come from the round trip charts of Ross[4], this similarity implies that these trajectories are not too far apart on the charts considered. This is indeed the case. All of the round trip trajectory segments come from the central regions of Figures 4-9 and 4-10; these two figures, as the reader will remember, cover almost the same region of arrival and departure dates. None of the round trip trajectory segments come from the high speed symmetric trips of Figure 4-8. Each of the round trip trajectory segments which is used in a periodic orbit consists of two trajectory legs: one from Earth to Mars and the second from Mars to Earth. The first leg from Earth to Mars, in each case, involves a relatively short transfer time which always spans the time of opposition of Earth and Mars. The second leg from Mars to Earth always involves a relatively long transfer time which never includes the time of opposition of Earth and Mars. This long second leg always travels well outside the orbit of Mars.

In addition to this Earth-Mars-Earth round trip trajectory segment, each periodic orbit involves its reciprocal. For the circular coplanar case, the reciprocal is always exact. For the eccentric inclined case, the reciprocal is only approximate. A more detailed explanation of the term "reciprocal" can be found in Ross[4] or Chapter 4 of this thesis. This use of the reciprocal round trip trajectory can easily be seen in Figures 5-1 through 5-4. The reciprocal trajectory, of course, involves a long trajectory leg from Earth to Mars, followed by a short trajectory leg from Mars to Earth.

A final common characteristic of the circular coplanar periodic orbits found, all of which are listed in Appendix E, is that each one involves three full revolution returns between the two long trajectory segments between Earth and Mars. For all of the periodic orbits found, between the long transfer between Mars and Earth and the long transfer between Earth and Mars, there exist three full revolution returns (3FR) at Earth. The fact that the series of direct returns is the same in the same place in each of the periodic orbits found is due to the fact that no other series of direct returns was found at that point which resulted in reasonable Earth-Mars-Earth round trip trajectories and which missed Earth at all of the encounters associated with the series of direct returns.

If this series of three full revolution returns had not worked and if none of the other possible series of direct returns had worked, then there would be no periodic orbits of the type considered, except possibly ones involving a greater number of synodic periods to complete the cycle and/or perhaps ones involving series of direct returns which do not traverse the Sun the same number of times as does the planet of launch and return.

There are a few additional characteristics which are common to all of the periodic orbits found. Most of them are a result of the fact that the Earth-Mars-Earth round trip trajectory segments are similar. As was mentioned before, all of the periodic orbits spend a lot of time outside of the orbit of Mars. In the circular coplanar case, all of the periodic orbits reach distances greater than 2.0 a.u. from the Sun on two occasions in each basic repeating cycle. The periodic orbits found seldom spend much time or go very far inside of the orbit of Earth. The only times that the periodic orbits reach very far inside the orbit of Earth is on direct returns at Earth. One final observation is that the hyperbolic excess speeds at Mars are always much larger than the hyperbolic excess speeds at Earth.

The characteristic which makes all of the periodic orbits listed in Appendix E different is that each of the series of direct returns, which lies between the short transfers between Earth and Mars, is different.

5.2 Earth-Mars Periodic Orbits in the Eccentric Inclined Case

Not much more can be said about periodic orbits in the eccentric inclined case, because the basic character of the periodic orbits is well explained by examination of the periodic orbits in the circular coplanar case. However, several comments can be made. In general, when one goes from the circular coplanar case to the eccentric inclined case, the dates of actual opposition of Earth and Mars change; and the dates of encounter on a periodic orbit change by a number of days each. In addition, the planes of the interplanetary trajectory legs move out of the plane of the ecliptic; and hence, because of the out-of-plane components of hyperbolic excess velocity at the encountered planets, the speeds at the encountered planets increase; and the distances of closest approach during flybys frequently decrease, thereby making collision with an encountered planet more likely.

The one thing which makes the numerical problem much more difficult for the eccentric inclined case is the great increase in the numerical dimension of the problem. The basic repeating cycle is obtained by using the product divided by the greatest common divisor of the number of Earth-Mars synodic periods in the scheme of the basic circular coplanar case (4, 5, or 6) and the number of synodic periods in 32 years (15). The M5-... periodic orbits are the easiest, because they have a repeating cycle of 15 synodic periods or 32 years. Periodic orbit M4-1 is the most difficult, because it has a basic repeating cycle of 60 synodic periods or 128 years; the numerical dimension of the problem in terms of the number of independent dates and the dimension of the matrix which must be inverted is seventy-five.

These changes in the eccentric inclined case are probably best presented in the form of statistics for each periodic orbit. Statistics are presented in Table 5-1 for each of the periodic orbits M4-1, M5-1, and M5-2. The statistics are based on solar system Models I.B. and III.B. as the

	PERIODIC ORBIT		
	M4-1	M5-1	M5-2 1FR next to HR / 2FR next to HR
Encounters at Earth next to the short transfers to Mars:			
hyperbolic excess speed in EMOS:			
circular coplanar	0.257	0.249	0.245
average	0.260	0.276	0.244
highest	0.270	0.371	0.283
lowest	0.250	0.188	0.212
passing distance in Earth radii:			
circular coplanar	1.54	1.78	1.42/2.06
average	1.40	1.65	1.32/2.06
highest	1.63	2.52	1.73/2.41
lowest	1.21	1.07	1.02/1.81
turn angle in degrees:			
circular coplanar	48.3	46.0	54.1/42.7
average	51.0	45.7	57.9/43.6
highest	57.8	80.7	67.5/54.8
lowest	44.6	37.2	41.4/33.4
change in encounter date from the circular coplanar case in days:			
average	-0.8		
RMS	17.7		
average abs. value	15.2		
Encounters at Mars:			
hyperbolic excess speed in EMOS:			
circular coplanar	0.314	0.316	0.314
average	0.324	0.322	0.325
highest	0.405	0.390	0.399
lowest	0.245	0.252	0.246
passing distance in Mars radii:			
circular coplanar	3.77	7.30	4.79
average	3.04	2.59	5.57
highest	7.63	6.86	14.22
lowest	1.12	1.00	1.31
turn angle in degrees:			
circular coplanar	4.3	2.3	3.4
average	6.1	7.8	4.4
highest	11.2	11.1	13.6
lowest	2.9	3.3	1.4
change in encounter date from the circular coplanar case in days:			
average	-0.3		
RMS	28.2		
average abs. value	25.3		

(Table 5-1)

Encounters at Earth next to the long transfers to Mars:			
hyperbolic excess speed in EMOS:			
circular coplanar	0.181	0.211	0.183
average	0.195	0.216	0.195
highest	0.229	0.260	0.230
lowest	0.178	0.171	0.172
passing distance in Earth radii:			
circular coplanar	1.30	1.55	1.37
average	3.60	1.53	2.64
highest	38.7	1.76	23.53
lowest	1.16	1.31	1.11
turn angle in degrees:			
circular coplanar	77.4	60.7	74.8
average	61.1	60.8	62.3
highest	83.0	80.8	86.2
lowest	6.3	51.2	9.9
change in encounter date from the circular coplanar case in days:			
average	-0.8		
RMS	5.9		
average abs. value	4.7		

Table 5-1. Statistics of periodic orbits M4-1, M5-1, and M5-2.

circular coplanar and eccentric inclined cases respectively. The statistics are based on all of the five possible variations for both M5-1 and M5-2. Statistics are based only on the "a" variation of periodic orbit M4-1. Only the "a" variation of periodic orbit M4-1 was run on the computer due to the large dimension (75) of the numerical problem and the resulting large requirement for computer time. The passing distances and turn angles listed under periodic orbit M5-2 for the encounters at Earth next to the short transfers to Mars are in pairs to reflect the difference in having one full revolution return or two full revolution returns on one side of a half revolution return. These FR associated with the HR are not optimally arranged. The minimum passing distance could be increased in several instances by rearrangement of the FR and the HR. Complete cycles for these periodic orbits are presented in the listings of Appendix F.

5.3 An Interplanetary Transportation System Based on Periodic Orbits to Mars

There are several characteristics of the Earth-Mars periodic orbits found which relate in particular to possible applications.

The first thing that one should note here is that for the hypothetical periodic orbit Mn-i, there should be n separate periodic orbits of this type. That is to say, if the basic scheme of a periodic orbit requires n synodic periods to complete, then there will be, in general, n different, indefinitely long sequences of planetary encounters.

This fact can be easily understood by referring again to Figures 4-1 through 4-4. Each of these figures shows the basic scheme for a different periodic orbit along with the times of Earth-Mars opposition. One can imagine a very long strip of the type shown in these figures showing planetary distance from the Sun as a function of time and showing indications of the times of opposition. Then just imagine that at each of the opposition times, the basic scheme of one of the figures begins again and continues indefinitely in time. Only on beginning the $n+1^{st}$ pattern of the periodic orbits will the pattern coincide with the continuation of the first periodic orbit pattern.

A consequence of the fact that there are n separate periodic orbits of a given scheme of the type Mn-i is the availability of a short transfer between Earth and Mars and between Mars and Earth during every opposition period. A further consequence is that n vehicles would be needed to complete an interplanetary transportation system based on this hypothetical scheme. Since each oeriodic orbit pattern has one short transfer from Earth to Mars and one short transfer from Mars to Earth in the time of n synodic periods and since there are n separate such patterns covering different times of opposition, each time of opposition is covered by one short transfer from Earth to Mars and by one short transfer from Mars to Earth. In order to make all of these transfers available, n different vehicles would be necessary.

These n vehicles would complete a transportation system based on this hypothetical periodic orbit scheme. Hence, one can easily see in what way the periodic orbits with a smaller n are more efficient; fewer vehicles are required to complete a transportation system based on them. The n vehicles are visualized as completing the Earth-Mars transportation system by continuing in the n separate periodic orbit patterns. In addition, shuttle vehicles are required to transport material and personnel between the vehicles and the encountered planet during a flyby.

One would also like to make good use of those portions of the periodic orbits which are not the short transfers between Earth and Mars. The short transfers are best used for transportation. The long transfers between Earth and Mars are also available as transportation, although they are less efficient because of the very long transfer times. These long transfers could also be used for purposes of interplanetary research between the orbit of Earth and the inner reaches of the asteroid belt. The direct returns at Earth could also be used for research and for purposes of repair and maintenance of the vehicles.

Such a transportation system as described above offers the opportunity for a relatively efficient and comfortable system of vehicles traveling between Earth and Mars.

CHAPTER 6

SUMMARY AND CONCLUSIONS

6.1 Summary

The primary accomplishment of this thesis has been the discovery of continuous, ballistic, periodic orbits visiting both Earth and Mars. In order to obtain these periodic orbits, several important techniques were developed. This summary is a review of the successful technique.

First, it is important to remember that the patched conic approximation discussed in the Introduction is used throughout. Also remember that different approximations to the solar system and ephemerides of the planets have been used.

An important class of trajectory legs which are used to make up a part of a periodic orbit are the direct return orbits. There are several types of these direct return trajectories. There are half revolution return trajectories which return to the departure planet when it has completed one half revolution around the Sun. There are full revolution return trajectories which return to the departure planet when it has completed one full revolution around the Sun and when one planetary period has passed since the departure of the vehicle. In addition, there are symmetric return trajectories which come in different lengths. There are different symmetric returns which return to the departure planet in either slightly more than or slightly less than 1.407 planetary periods, in slightly more than or slightly less than 2.445 planetary periods, in slightly more than or slightly less than 3.461 planetary periods, *etc*. These trajectory segments vary in length with the speeds at their ends. The existence of all of these direct returns has been recognized by other workers in the field and several have been used[1,2] in periodic orbits. All of the above mentioned direct return trajectories travel around the Sun the same number of times as does the planet from which the vehicle was launched and to which the vehicle returns.

In order to be able to investigate all reasonable possibilities of series or combinations of direct returns to find suitable series for inclusion in periodic orbit schemes, it was found to be very helpful to create a list, which is given in Appendix D, of all possible combinations of direct returns up to some maximum length of time. This list is arranged in order of increasing length of time necessary to complete each series of direct returns. The list also includes some comments on possible restrictions on the incoming and outgoing hyperbolic excess velocity vectors at the ends of the series of direct returns so that the required flyby maneuvers can be performed at each encounter without intersecting the planet. The list will not give the exact length of time required for the given series of direct returns because of the possibility of the varying lengths of time for the symmetric returns due to different hyperbolic excess speeds required; nor will the list allow one to pick a series of direct returns of exactly the desired length of time; but it will help one to obtain a good starting point for computer solution.

Hollister[1] and Menning[2] found periodic orbits which join Earth and Venus. These periodic orbits involved direct returns at both Earth and Venus. Obtaining periodic orbits to Mars was expected to be and was found to be a more difficult problem, however, because the small mass of Mars makes the necessary flyby maneuvers at Mars impossible, because the calculated trajectories intersect the surface of the planet.

The approach used to obtain periodic orbits joining Earth and Mars was to combine two Earth-Mars-Earth round trips of Ross[4] with two separate series of direct returns at Earth in a "symmetric" manner. Use of the Earth-Mars-Earth round trips as segments of the periodic orbit schemes tried avoided the difficulty of making direct returns at Mars. All schemes were first attempted in the case of circular coplanar planetary orbits and were eliminated at that point if they did not work with this simplified solar system model. The two round trip trajectory segments which are used in each scheme are "reciprocal" to each other. Reciprocity of trajectories is explained in detail in Chapter 4 and by Ross[4], but briefly, a trajectory which is reciprocal to another one is that trajectory whose encounter dates are the negatives of the encounter dates for the original trajectory when all dates are measured relative to the date of opposition. The two reciprocal, round trip trajectory segments which are used are centered around different dates of opposition. The two separate series of direct returns are then used to connect the ends of the round trip trajectories. Two series of direct returns are needed for each scheme, because each scheme involves two different round trips as well as the two series of direct returns to create the basic repeating pattern of the periodic orbit. The basic repeating pattern in the circular coplanar case must, of course, last some integral number of synodic periods of Earth and Mars.

This basic approach to the attainment of periodic orbits can, of course, be extended to periodic orbits between planets other than Earth and Mars. It can also be extended to involve direct returns at two different planets and to include flybys of three or more different planets. The inclusion of more than two planets requires a basic repeating relative pattern for the three planets which is analogous to the synodic period for the case of two planets. The approach, however, is probably limited to the use of direct returns at fairly massive planets so that the flyby maneuvers can be performed without hitting the planet.

Computer solution is necessary in order to refine the circular coplanar estimate of the periodic orbit scheme and in order to improve the estimate of the actual encounter dates for more accurate models of the planets which are in eccentric, inclined orbits. The computer techniques used are basically those of Menning[2], although several extensions of his work were made in order to handle more types of trajectories. On the order of one hundred or so periodic orbit schemes were attempted in the circular coplanar case; of these, 18 missed all of the planets or at least did nothing more than graze a planet during an encounter or two. Convergence to a feasible solution was achieved for several.

One can then list in summary form the technical accomplishments of the thesis:

1. Several Earth-Mars periodic orbits are discovered.
2. A procedure is developed to obtain all possible series of direct return trajectories in the vicinity of one planet so that one can have available a list of all of the possible waiting times in the vicinity of one planet.

Just such a list is made for Earth for direct returns which go around the Sun the same number of times as does Earth and which require a total of no more than three synodic periods of Earth and Mars to accomplish. It is the table of Appendix D.

3. The concepts of symmetry and reciprocal trajectories are introduced and applied to periodic orbits and periodic orbit attempts.
With this concept, a procedure is developed to obtain good possibilities for periodic orbits between two planets, given that one has a family of round trips from one planet to the other and back and given that one has a list like the one mentioned above and given in Appendix D. The method is probably limited to having the direct returns at fairly massive planets.

4. Several extensions of the numerical work of Menning[2] were carried out, such as a scheme to minimize the maximum turn angle for any specified number of full revolution returns (FR), the handling of the encounter speeds, the encounter times, and the turn angles at the ends of a half revolution return (HR), and the handling of trajectories which travel more than one or two full revolutions around the Sun.

5. Suggestions were made as to how one might extend this work by including direct returns which do not go around the Sun the same number of times as does the planet of launch and arrival, their inclusion in the different combinations of direct returns, and their possible use to form periodic orbits at planets with widely differing semimajor axes.

6.2 Conclusions

The major conclusion of the thesis is that periodic orbits connecting Earth and Mars have been demonstrated to exist under the assumptions inherent in patched conic analysis. The shortest period found was four synodic periods. This would require a minimum of four spacecraft in order to have one going and one returning on fast transfers during each opposition period. The hyperbolic speeds at each planet are competitive with one-way transfers. Average speeds at Earth are 0.260 EMOS and 0.181 EMOS in different parts of the period. Average speeds at Mars are 0.324 EMOS. The passing distances at each of the planetary encounters are almost always satisfactory for missing the planet with reasonable guidance errors. The average minimum geocentric altitudes during the encounters at the Earth in the two different parts of the period are 1.40 and 3.60 Earth radii. The corresponding figure for Mars is 3.04 Mars radii. The closest the vehicle ever gets to the surface of a planet is at Mars where at one point a minimum planetocentric distance of 1.12 Mars radii is reached. These numbers are also characteristic of the other Earth-Mars periodic orbits found.

6.3 Recommendations for Further Study

The author has four recommendations for further study, and two of them have been made earlier in this thesis.

The first recommendation is that one look for periodic orbits or at least long series of interplanetary trajectories which are connected by planetary flybys of the three planets, Earth, Mars, and Venus. One should begin by making trajectory charts for all flyby trajectories of the types Earth-Venus-Mars-Venus-Earth, Earth-Venus-Mars-Earth, Earth-Venus-Mars-Venus, and Earth-Mars-Venus and their reciprocals. The solar system model which should be used for these charts should be the circular, coplanar, exactly symmetric and periodic model (Solar System Model I.A.1). With this solar system model, the trajectory charts can be applied approximately to a large period of time which is centered around the year 1995. The lists of series of direct returns (similar to Appendix D) should then be used to attempt to join various different trajectories with series of direct returns at Earth and/or Venus.

The second recommendation is that one extend the knowledge of direct returns to those direct return trajectories which do not travel around the Sun the same number of times as does the planet of launch and arrival. There should be many different types of such direct returns including ones which travel around the Sun more or fewer times than does the planet of launch and return. These should be separable into classes of direct returns which are analogous to the three different classes of direct returns which are discussed in Chapter 3. These additional types of direct returns should then also be included in a list of all possible combinations of direct returns which will be similar to Appendix D.

The third recommendation is that one extend the class of multiple flyby trajectories considered to include those which require small thrusted velocity changes. Small thrusted velocity changes or a small continuous thrust will be needed in any case for guidance. The addition of small required changes in velocity may make possible some convenient series of encounters which would not be otherwise possible. One would still want to keep the thrusted velocity changes as small as possible. This extension would probably be useful in the eccentric inclined case after one has found periodic orbits which converge in the circular coplanar case; the periodic orbit schemes which do not converge in the eccentric inclined case could be useful with the addition of a few small changes in velocity. In addition, this extension could be useful, particularly with low thrust, in creating trajectories which do not exist as free-fall trajectories even in the circular coplanar case.

The final recommendation is that the mathematical model of the different periodic orbits found be improved. The more accurate methods should consider a series of flybys determined by a periodic orbit; but the initial and final encounter dates should be specified by the researcher; and the length of the periodic orbit segment which one should investigate should be limited by practical considerations such as the amount of computer time which one desires to expend and the amount of the periodic orbit which one desires to investigate. Bayliss[28] is carrying out such an investigation which will apply to all multiple flyby trajectories.

APPENDIX A

COMPUTER PROGRAM

The following program is designed to look for periodic orbits which involve Earth, Mars, and Venus. It calculates the speed differences at the planetary encounters. It forms the matrix of derivatives of the speed differences with respect to the planetary encounter dates. Then it tries to make the speed differences approach zero, first by a steepest descent method and then by a Newton-Raphson type of iteration. Finally, the turn angles and planetary passage distances are checked.

Dates are in Julian Date minus 2440000.

The patched conic approximation is used throughout with the size of the sphere of influence neglected.

The ephemerides are based on the mean orbital elements of 1960.

The following are some of the more important parameters in the program along with explanations of how they work:

A--Semimajor axis of a planet.

PER--Period of a planet in days.

GFP--True longitude of perihelion for the planet.

E--Eccentricity of the planetary orbit.

TJP--Date of perihelion for the planet.

Subscripts for the above arrays and elements of the array NP refer to the planets according to the following list:

1--Earth

2--Venus

3--Mars

CYCLE--Basic repeating time of the scheme in days.

NDATE--The number of encounter dates read into the program which is one more than the basic dimension of the problem.

NP--Column matrix of planetary encounter dates (in the case of FR and/or HR, the initial date for each series).

ALONG--Column matrix of the number of FR associated with each planetary encounter up to and including 9. When ALONG(I) is greater than 9, the nonzero first digit indicates that an HR is to be added to the series of FR, the number of which is indicated by the second digit. The first digit will become LEHR(I).

LEHR--Column matrix which becomes at each location the first digit of ALONG in the same location. A nonzero LEHR(I) indicates that an HR will occur at the I^{th} planetary encounter. LEHR(I) becomes LEHRI for the I^{th} pass through the last part of the program. If LEHR(I) is 1 through 5, the half revolution return will be above the orbit plane of the planet encountered; and if it is 6 through 9, the HR will be below the orbit of the planet encountered. The scheme of how many FR are put before the HR and how many after is indicated by the following table. In each box, these numbers are indicated by $\frac{\text{number of FR before the HR}}{\text{number of FR after the HR}}$

first digit in ALONG (I) which becomes LEHR(I) \ second digit in ALONG (I)	1	2	3	4	5	6	7
1 or 6	$\frac{1}{0}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{4}{3}$
2 or 7		$\frac{0}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{3}{4}$
3 or 8		$\frac{2}{0}$	$\frac{3}{0}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{5}{2}$
4 or 9			$\frac{0}{3}$	$\frac{0}{4}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{2}{5}$
5				$\frac{4}{0}$	$\frac{5}{0}$	$\frac{5}{1}$	$\frac{6}{1}$

Table A-1. The arrangement of a series of FR before and after one HR according to LEHR(I).

CIR--Column matrix of the number of complete revolutions of the Sun made between one planetary encounter and the next up to and including 9, except that in the case of a symmetric return, it is in general, one less than the number of complete revolutions of the Sun before the next encounter. As with ALONG, when CIR(I) is greater than 9, the first digit indicates something different than does the second. The second digit means essentially what is explained immediately above.

CIRC--Becomes the second digit of CIR(I) in the subroutine.

NTEST--Becomes the first digit of CIR(I) in the subroutine. It decides which of the 2N+1 possible solutions of Lambert's problem is to be chosen, where N is the number of full revolutions of the Sun which may be made by the vehicle. There are two separate situations in which one determines what NTEST does: an interplanetary trajectory and a symmetric return trajectory.

In the case of an interplanetary trajectory, the meaning of NTEST (or the first digit of CIR(I)) is indicated by Figure A-1. N, the number of full revolutions to be made by the vehicle, is simply decided by the number CIRC. Figures A-1 and A-2 are sketches of semi-major axis versus time of flight as may be found in Chapter 3 of Battin[6].

In the case of a symmetric return trajectory where the departure planet is the same as the arrival planet, the situation is somewhat more complicated. If NTEST=0 and if the number of full revolutions of the Sun made by the planet is CIRC+1, then the subroutine will choose the branch of the curve corresponding to the semimajor axis which is not the semimajor axis of the planet as discussed in Chapter 3 and in Menning[2]. This choice results in a symmetric return which revolves around the Sun the same number of times as does

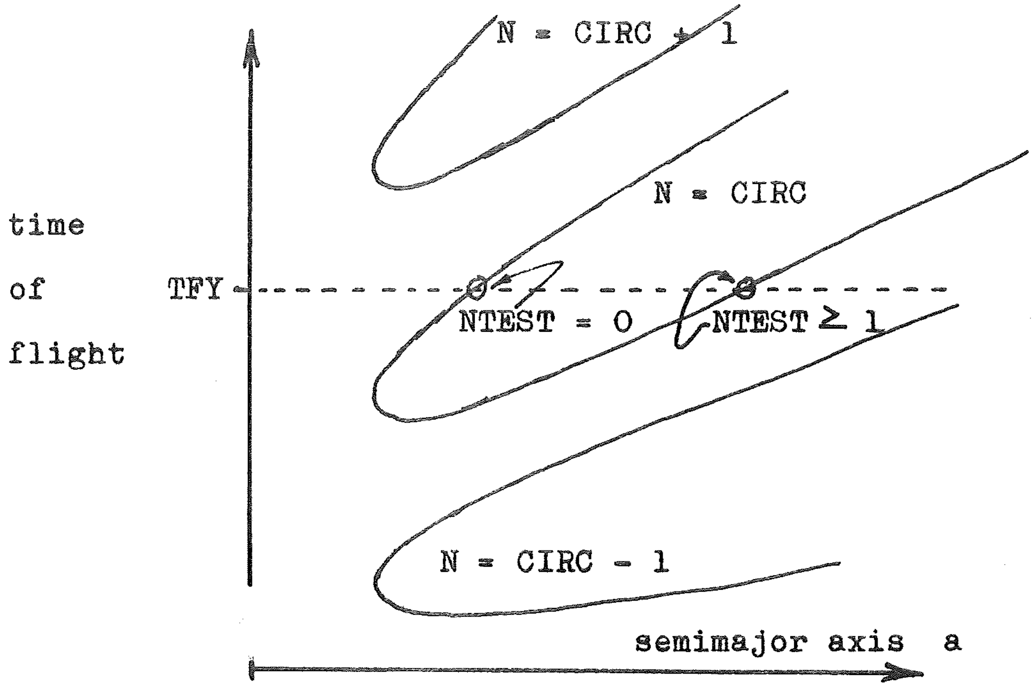


Figure A-1. Lambert solution selection for an interplanetary trajectory.

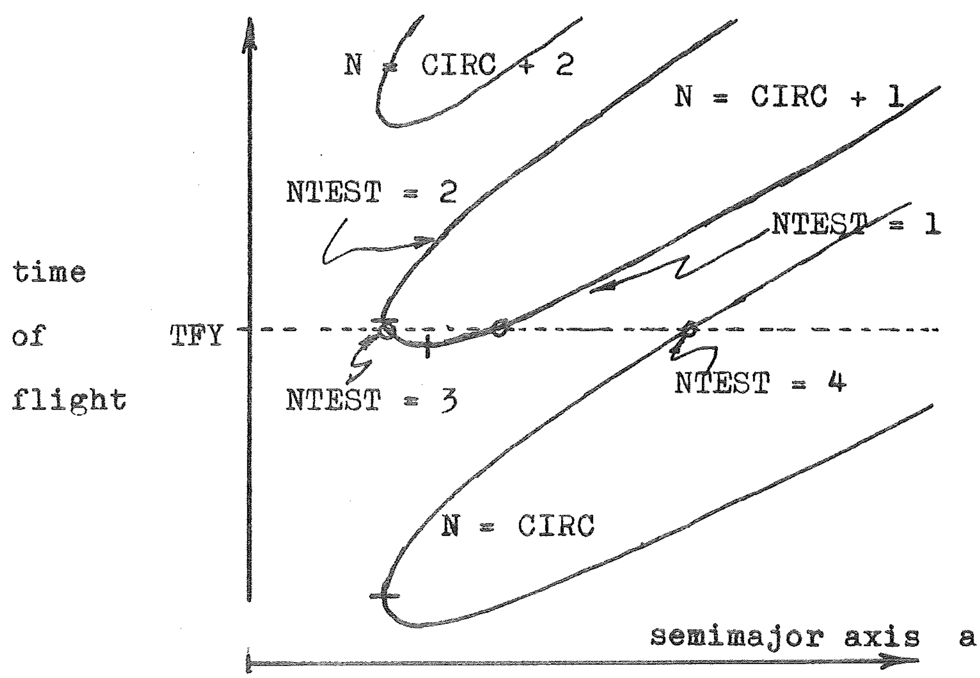


Figure A-2. Lambert solution selection for a symmetric return trajectory.

the planet. When NTEST is not equal to zero, the scheme is best demonstrated by Figure A-2.

TEST--Convergence criterion for SUM (below).

NDM1 = NDATE - 1

NDP1 = NDATE + 1

V--Column matrix which gives the speed differences at the planetary encounters.

$$FOX = \sum_{i=1}^{NDM1} (100\ V_i)^2$$

$$SUM = \sum_{i=1}^{NDM1} V_i$$

ARRAE--The formed matrix of derivatives $\left[\frac{d\ \underline{v}}{d\ \underline{t}}\right]$ (NDM1 x NDM1) where $\underline{v}$ is the column matrix of speed differences (V) and $\underline{t}$ is the column matrix of encounter dates (DATE). This storage area also contains the inverse of this matrix for a Newton-Raphson step.

MINV--A matrix inversion routine contained in the I.B.M. Scientific Subroutine Package (SSP).

STP--Step size for the numerical differencing used to form the matrix of derivatives $\left[\frac{d\ \underline{v}}{d\ \underline{t}}\right]$. Its units are days (STPI = 1/STP).

VX4RL,VX4GL,VX4ZL,VX4TL--Column matrices containing respectively the R, G, Z, and total velocity components relative to the planets at departure points from the planets.

VY4RA,VY4GA,VY4ZA,VY4TA--Column matrices containing respectively the R, G, Z, and total velocity components relative to the planets at arrival points at the planets.

VHYPA,VHYPL--Column matrices containing respectively the arrival and departure hyperbolic excess speeds.

VSPR,VSPG,VSPT--Column matrices containing respectively the R, G, and total heliocentric velocity components of the planets at the departure points.

VSPRA,VSPGA,VSPTA--Column matrices containing respectively the R, G, and total heliocentric velocity components of the planets at the arrival points.

GLON--Column matrix containing the true longitudes of the planets at the encounter points relative to the equinox of 1960. The units are degrees of arc.

DHICK--Column matrix containing the semimajor axes of the trajectory legs.

DHICK2--Column matrix containing the eccentricities of the trajectory legs.

There are many other parameters which might be included here; but the ones above are most of the important ones. Other parameters are explained by comments within the program and the main subroutine.

A listing of the computer program and the main subroutine follows and completes the appendix. The language used is Fortran IV.

```
      DIMENSION A(9), PER(9), E(9), TJP(9), GEP(9),
     1      NP(90), DATE(90), DATEL(90), ALONG(90), CIR(90),
     2      GLON(90), LEHR(90), DHICK(90), DHICK2(90),
     3      VX4RL(90),VX4GL(90),VX4ZL(90),VX4TL(90), VY4RA(90),
     4      VY4GA(90),VY4ZA(90),VY4TA(90), VHYPL(90), VHYPA(90),
     5      V(90), VC(90), RA(90), DELT(90),   VSPG(90),VSPT(90),
     6      VSPR(90), VSPTA(90),VSPGA(90),VSPRA(90),
     7      VLN(181),VAN(182),ARRAE(90,90),L(90),M(90),DUBB(8100)
      COMMON PI,RTD,YR, PER,A,E,TJP,GEP, NP,DATEL,CIR,LEHR,
     1      GLON,DHICK,DHICK2,  VX4RL,VX4GL,VX4ZL,VX4TL, VY4RA,
     2      VY4GA,VY4ZA,VY4TA,  VSPT,VSPG,VSPR, VSPTA,VSPGA,VSPRA,
     3      NERR
      PI = 3.1415926536
      RTD=180./PI
      ASCALE = 100.
      STP = 1.0
      STPI = 1./STP
      YR = 365.25636
      A(1) = 1.0
      A(2) = 0.723332
      A(3) = 1.523691
      E(1) = 0.016726
      E(2) = 0.006793
      E(3) = 0.093368
      PER(1) = YR
      PER(2) = 224.7008
      PER(3) = 686.9796
      GEP(1) = 102.25253
      GEP(2) = 131.00831
      GEP(3) = 335.33269
      TJP(1) = +2.124962 + (40.*PER(1) - 3065.)
      TJP(2) = -27.01776 + (65.*PER(2) - 3065.)
      TJP(3) = +146.08905 + (22.*PER(3) - 3065.)
 1    READ (5,701)  CYCLE
      READ(5,700) NDATE,(NP(I),I=1,NDATE)
      READ(5,701)(DATE(I),I=1,NDATE)
      READ (5,705)  (ALONG(I), I = 1,NDATE)
      READ (5,705)  (CIR(I), I=1,NDATE)
      WRITE (6,702) CYCLE
      WRITE(6,703) (DATE(I),I=1,NDATE)
      WRITE(6,704) (NP(I),I=1,NDATE)
      WRITE(6,706) (ALONG(I),I=1,NDATE)
      WRITE (6,706) (CIR(I), I=1,NDATE)
700   FORMAT (I2, 8X, (70I1) )
 701  FORMAT(6F12.6)
705   FORMAT (40F2.0)
702   FORMAT ('1   CYCLE =',  F13.3 , 'DAYS')
703   FORMAT (1H0, 10F13.5/ (1H , 10F13.5 ))
704   FORMAT((1H0, 65(I1,1X)))
706   FORMAT (1H0, 10F6.1, 2X, 10F6.1/ (1H ,10F6.1,2X,10F6.1))
      DO 719  I = 1,NDATE
```

```
      LEHR(I) = 0
  710 IF  (ALONG(I) .LT. 9.9999)   GO TO 719
      LEHR(I) = LEHR (I) + 1
      ALONG (I) = ALONG(I) - 10.0
      GO TO 710
  719 CONTINUE
      NOWPK=0
      ITER=2
      DAYS=5.0
      ILESS=0
      CNEUT=0.8
      STORE=100000.
      NDM1=NDATE-1
      NDP1=NDATE+1
      TEST = NDM1*2.E-4
      DATE(NDP1)=DATE(2)+CYCLE
      NP(NDP1)=NP(2)
C
C     CALCULATION OF FUNCTION VALUE
C
  4   DO 10   II=1,NDM1
      NPY=NP(II+1)
      NPX=NP(II)
      ALON = ALONG(II)*PER(NPX)
      IF (LEHR(II) .GT. 0)   ALON = ALON + 0.5*PER(NPX)
      DATED = DATE(II) + ALON
      DATEA=DATE(II+1)
      CALL LAMBRT(DATED,DATEA,II)
      IF (NERR.NE.1) GO TO 5
      IF (ITER.EQ.2) GO TO 815
      GO TO 9
    5 VHYPL(II) = VX4TL(II)
      IF (LEHR(II) .GT. 0 .AND. II .NE. 1)
     1      VHYPL(II) = VX4TL(II)*VSPGA(II)/VSPG(II)
   10 VHYPA(II+1)=VY4TA(II+1)
      VHYPA(1)=VHYPA(NDATE)
      DO 20   I=1,NDM1
   20 V(I)=VHYPL(I)-VHYPA(I)
      VHYPL(NDATE) = 0.
      V(NDATE) = 0.
      WRITE (6,777)
  777 FORMAT (1H0)
      WRITE (6,707) (DATE(I), VHYPA(I), VHYPL(I), V(I),
     1      DHICK(I), DHICK2(I), GLON(I), I=1,NDATE)
  707 FORMAT((1H , 7(F12.4, 5X)))
      SUM=0.0
      FOX=0.0
      DO 21   I=1,NDM1
      SUM=SUM + ABS(V(I))
   21 FOX=FOX + (100.*V(I))*(100.*V(I))
      WRITE(6,342) FOX,SUM
```

```
 342  FORMAT (8H   FOX = , F12.6, 5X, 6HSUM = , F12.6 / )
C
C     PROCEDURE TO REDUCE STEP SIZE IF FUNCTION VALUE
C     (SUM(V(I)**2) )   INCREASES.
C
      IF(FOX.LT.STORE) GO TO 892
    9 NOWRK = NOWRK + 1
      IF(NOWRK.EQ.10) GO TO 815
      ILESS=1
      DO 891 I=1,NDM1
      DELT(I)=DELT(I)/2.0
  891 DATE(I)=DATE(I) + DELT(I)
      DATE(NDATE)=DATE(1) + CYCLE
      GO TO 4
  892 STORE=FOX
      IF(ITER.EQ.2.AND.ILESS.GT.0) DAYS=0.5*DAYS
      NOWRK=0
      ILESS=0
      IF (SUM.GE. 0.300 .AND.  DAYS .GT. 0.30)  GO TO  175
      ITER=1
      DAYS=5.
  175 IF (SUM .LT. TEST)  GO TO 814
C
C     CALCULATION OF PARTIAL DERIVATIVES
C
      DO 40  II=1,NDM1
      NPX=NP(II)
      NPY=NP(II+1)
      ALON = ALONG(II)*PER(NPX)
      IF (LEHR(II) .GT. 0)  ALON = ALON + 0.5*PER(NPX)
      DATED = DATE(II) + ALON + STP
      DATEA=DATE(II+1)
      CALL LAMBRT(DATED,DATEA,II)
      IF(NERR.EQ.1) GO TO 815
      VLN(2*II) = VX4TL(II)
      IF (LEHR(II) .GT. 0 .AND. II .NE. 1)
     1     VLN(2*II) = VX4TL(II)*VSPGA(II)/VSPG(II)
      VAN(2*II + 1) = VY4TA(II + 1)
      DATED = DATE(II) + ALON
      DATEA=DATE(II+1) + STP
      CALL LAMBRT(DATED,DATEA,II)
      IF(NERR.EQ.1) GO TO 815
      VLN(2*II + 1) = VX4TL(II)
      IF (LEHR(II) .GT. 0 .AND. II .NE. 1)
     1     VLN(2*II + 1) = VX4TL(II)*VSPGA(II)/VSPG(II)
   40 VAN(2*II + 2) = VY4TA(II + 1)
      DO 30  I = 1,NDP1
      DO 30  J = 1,NDP1
   30 ARRAF(I,J)=0.0
      DO 50  KK=2,NDM1
      ARRAF(KK,KK) =(VLN(2*KK) - VAN(2*KK) - V(KK))*STPI
```

```
      ARRAE(KK-1, KK) =(VLN(2*KK - 1) - VHYPL(KK - 1))*STPI
   50 ARRAE(KK + 1, KK) =(-VAN(2*KK + 1) + VHYPA(KK + 1))*STPI
      ARRAE(1,1) = (VLN(2) - VAN(2*NDATE) - V(1))*STPI
      ARRAE(2,1) = (VHYPA(2) - VAN(3))*STPI
      ARRAE(NDM1,1) = (VLN(2*NDATE - 1) - VHYPL(NDM1))*STPI
      ARRAE(1,NDM1)=ARRAE(NDATE,NDM1)
      IF(ITER.EQ.1) GO TO 283
C
C     STEEPEST DESCENT STEP
C
      DO 70 I=1,NDM1
      DELT(I)=0.0
      DO 70 J=1,NDM1
   70 DELT(I) = DELT(I) + (100000.0*ARRAE(J,I))*V(J)
      D = 0.0
      GO TO 284
  283 CONTINUE
C
C     NEWTON RAPHSON STEP
C
      ID=0
      DO 48 I=1,NDM1
      DO 48 J=1,NDM1
      ID=ID+1
   48 DUBB(ID) = ARRAE(J,I)*ASCALE
      CALL MINV(DUBB,NDM1,D,L,M)
      ID=0
      DO 84 I=1,NDM1
      DO 84 J=1,NDM1
      ID=ID+1
   84 ARRAE(J,I) = DUBB(ID)*ASCALE
      DO 98 I=1,NDM1
      DELT(I)=0.0
      DO 98 J=1,NDM1
   98 DELT(I)=DELT(I) + CNEUT*ARRAE(I,J)*V(J)
  284 CONTINUE
C
C     LARGEST INCREMENT FOUND
C
      BIGDEL=ABS(DELT(1))
      DO 91 I=2,NDM1
   91 IF(ABS(DELT(I)).GT.ABS(BIGDEL)) BIGDEL=ABS(DELT(I))
C
C     THE INCREMENTS IN THE ENCOUNTER DATES ARE SCALED SO
C     THAT THE LARGEST IS EQUAL TO  (DAYS)  EXCEPT IN THE
C     NEWTON RAPHSON ITERATION WHERE THE LARGEST CHANGE IS
C     SIMPLY REQUIRED TO BE LESS THAN  (DAYS).
C
      IF(ITER.EQ.1.AND.BIGDEL.LT.DAYS) GO TO 92
      SCALE=DAYS/BIGDEL
      DO 93 I=1,NDM1
```

```
   93 DELT(I)=SCALE*DELT(I)
   92 WRITE (6,841)    D,(DELT(I), I=1,NDM1)
  841 FORMAT (1H0,  E13.6 / (1H , 10E13.6 )   )
      DO 80 I=1,NDM1
   80 DATE(I)=DATE(I)-DELT(I)
      DATE(NDATE)=DATE(1) + CYCLE
      DATE(NDP1)=DATE(2) + CYCLE
      GO TO 4
  814 CONTINUE
C
C     ***********************************************************
C
C     ALL OF THE REST OF THE MAIN PROGRAM CALCULATES THE TURN
C     ANGLES AT THE PLANETARY ENCOUNTERS.
C
C     VH IS THE EXCESS HYPERBOLIC VELOCITY.
C     RA IS THE MINIMUM ALLOWABLE RADIUS IN PLANETARY RADII.
C     VC IS THE CIRCULAR ORBIT SPEED IN EMOS AT THE PLANET'S
C          SURFACE.
C     TMAX IS THE MAXIMUM TURN POSSIBLE AT THE PLANET IN
C          QUESTION FOR THE VALUE OF THE ALLOWABLE RADIUS.
C     TMAXD IS TMAX IN DEGREES.
C     CO & COO ARE THE HALF CONE ANGLES WHICH PERMISSIBLE
C          FULL REVOLUTION VELOCITY VECTORS MAKE WITH PLANET
C          VELOCITY VECTOR.
C     TR & TRO ARE THE ANGLES BETWEEN THE PLANET VELOCITY VECTOR
C          AND THE G AXIS.
C     (AN  'O'  AT THE END OR NEXT TO THE END OF A VARIABLE
C          USUALLY MEANS THAT THAT VARIABLE HAS TO DO WITH
C          THE PLANETARY ENCOUNTER JUST BEFORE DEPARTURE FROM
C          A PLANET AT WHICH HAS BEEN  FR  AND/OR  HR.)
C
      RA(1) = 1.0
      RA(2) = 1.0
      RA(3) = 1.0
      VC(1)=0.266
      VC(2)=0.243
      VC(3)=0.121
      VY4RA(1) = VY4RA(NDATE)
      VY4GA(1) = VY4GA(NDATE)
      VY4ZA(1) = VY4ZA(NDATE)
      VY4TA(1) = VY4TA(NDATE)
      VSPTA(1) = VSPTA(NDATE)
      VSPRA(1) = VSPRA(NDATE)
      VSPGA(1) = VSPGA(NDATE)
      DO 440  I=1,NDM1
      N=NP(I)
      LHRM = 0
      LEHRI = LEHR(I)
      VH=VY4TA(I)
      VHO = VX4TL(I)
```

```
      TMAX=2.0*ARSIN(1./(1. + VH*VH/(VC(N)*VC(N))))
      TMAXD=TMAX*RTD
      ELEVI=ARSIN(VY4ZA(I)/VH)*RTD
      ELEVO=ARSIN(VX4ZL(I)/VHO)*RTD
      CO = ARCOS(VH/(2.0*VSPTA(I)))
      TR = ARSIN(-VSPRA(I)/VSPTA(I))
      COO = ARCOS(VHO/(2.0*VSPT(I)))
      TRO = ARSIN(-VSPR(I)/VSPT(I))
      ALG = ALONG(I)
      IF (ALG .GT. 0.0  .OR.  LEHRI .GT. 0)  GO TO 420
      WRITE(6,401) I,DATE(I),VH,TMAXD,RA(N)
  401 FORMAT(1H ,////,14H SIMPLE FLY-BY,2X,I2,2X,4(E15.4,2X))
      WRITE(6,292)
  292 FORMAT (' R, G, AND Z COMPONENTS OF ARRIVAL AND LAUNCH',
     1       ' VELOCITIES--' )
      WRITE(6,425) VY4RA(I),VY4GA(I),VY4ZA(I),VX4RL(I),
     1      VX4GL(I),VX4ZL(I)
C
C     TURN IS THE REQUIRED TURN AND RT IS THE RADIUS OF THE
C     POINT OF CLOSEST APPROACH FOR THE SIMPLE FLYBY.
C
      TURN=ARCOS((VY4RA(I)*VX4RL(I) + VY4GA(I)*VX4GL(I) +
     1      VY4ZA(I)*VX4ZL(I) )/(VH*VHO)  )
      TURND=TURN*RTD
      IF (ABS(TURN) .LT. 1.E-5)  TURN = 1.E-5
      RT=VC(N)*VC(N)*(1./SIN(TURN/2.) - 1.)/(VH*VH)
      WRITE(6,293)
  293 FORMAT(1H ,34HTURNS AND MINIMUM RADIAL DISTANCES)
      WRITE(6,425) TURND,RT
  425 FORMAT(1H ,6(E10.4,2X))
      WRITE(6,294)
  294 FORMAT(1H ,10HELEVATIONS)
      WRITE(6,425) ELEVI,ELEVO
      IF(TURN.LT.TMAX) GO TO 440
      WRITE(6,403)
  403 FORMAT(1H ,20HTHIS TURN IMPOSSIBLE)
      GO TO 440
C
C     CO  IS THE HALF-CONE ANGLE WHICH PERMISSIBLE FULL-REVO-
C          LUTION VELOCITY VECTORS MAKE WITH THE PLANET VELOCITY.
C     TR  IS THE ANGLE BETWEEN THE PLANET VELOCITY VECTOR AND
C          THE G AXIS.
C
 420  CONTINUE
      C1=SIN(CO)*VY4ZA(I)/VH
      C2=-SIN(CO)*(COS(TR)*VY4RA(I) + SIN(TR)*VY4GA(I))/VH
      C3=COS(CO)*(SIN(TR)*VY4RA(I) - COS(TR)*VY4GA(I))/VH
      C4 = -SIN(COO)*(COS(TRO)*VX4RL(I) + SIN(TRO)*VX4GL(I))/VHO
      C5 = COS(COO)*(SIN(TRO)*VX4RL(I) - COS(TRO)*VX4GL(I))/VHO
      C6=SIN(CO)*SIN(CO)
      C7=COS(CO)*COS(CO)
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      C8 = SIN(COO)*VX47L(I)/VHO
      THETAI=ARCOS(C1/SQRT(C1*C1 + C2*C2))
      THETAO=ARCOS(C8/SQRT(C8*C8 + C4*C4))
      IF(C2.LT.0.0) THETAI=2.0*PI - THETAI
      IF(C4.LT.0.0) THETAO=2.0*PI - THETAO
      THEID=THETAI*RTD
      THEOD=THETAO*RTD
C
C     TURNI  IS THE MINIMUM TURN REQUIRED TO TURN THE INBOUND
C          HYPERBOLIC VELOCITU VECTOR INTO A VELOCITY VECTOR
C          PRODUCING A FULL REVOLUTION RETURN (FR).
C     TURNO  IS THE CORRESPONDING ANGLE FOR THE OUTBOUND
C          VELOCITY VECTOR.
C
      TURNI=ARCOS(C1*COS(THETAI) + C2*SIN(THETAI) + C3)
      TURNO=ARCOS(C8*COS(THETAO) + C4*SIN(THETAO) + C5)
      TURNID=TURNI*RTD
      TURNOD=TURNO*RTD
      DIF=ABS(THETAO-THETAI)
      THETAS=THETAI
      IF(THETAO.LT.THETAI) THETAS=THETAO
      THETAL=THETAS + DIF
      AIOM=ARCOS(C1*COS(THETAO) + C2*SIN(THETAO) + C3)
      AOIM=ARCOS(C8*COS(THETAI) + C4*SIN(THETAI) + C5)
      AIOMD=AIOM*RTD
      AOIMD=AOIM*RTD
      IF (LEHRI .LE. 0)  GO TO 419
C     *    *    *    *    *    *    *    *    *    *    *    *    *    *
C
C     THE FOLLOWING SECTION DOES THE CALCULATION FOR AN  HR
C          AND A SERIES OF  FR.
C
      WRITE (6,520)  ALONG(I), DATE(I), DATEL(I)
  520 FORMAT (////, ' THIS SERIES OF ENCOUNTERS INVOLVES ONE'
     1      ,'  HR  AND A SERIES OF', F6.1, '  FR.   ARRIVE--',
     2      F15.4, 2X, 'LEAVE--', F15.4 )
      IZD = 1
      IF (LEHRI .LT. 6)  GO TO 490
      IZD = -1
      LEHRI = LEHRI - 5
  490 CONTINUE
      ALGDT = ALONG(I)/2.
      ALGTE = AMOD(ALGDT,1.)
      ALG = ALGDT + ALGTE + 1.
      ALGN = ALGDT - ALGTE + 1.
      IF (LEHRI .GT. 1) GO TO 531
      GO TO 529
  531 IF (LEHRI .GT. 2)  GO TO 522
      ALG = ALG - 1.
      ALGN = ALGN + 1.
      GO TO 529
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522   IF (LEHRI .GT. 3)  GO TO 523
      ALG = ALG + 1.
      ALGN = ALGN - 1.
      GO TO 529
523   IF (LEHRI .GT. 4) GO TO 524
      ALG = ALG - 2.
      ALGN = ALGN + 2.
      GO TO 529
524   CONTINUE
      ALG = ALG + 2.
      ALGN = ALGN - 2.
529   CONTINUE
      THETHR = ARSIN(TAN(TR)/TAN(CO) )
      IF (IZD .LT. 0)  THETHR = PI - THETHR
      THEHRL = ARSIN( TAN(TRO)/TAN(COO) )
      IF (IZD .GT. 0)  THEHRL = PI - THEHRL
      IMPOS = 0
      IF (ALG .GT. 1.00001)  GO TO 900
      TURN = ARCOS(C1*COS(THETHR) + C2*SIN(THETHR) + C3)
      TURND = TURN*RTD
      RT = VC(N)**2*(1./SIN(TURN/2.) - 1.)/VH**2
      ELEV = ARCOS(COS(THETHR)*SIN(CO) )*RTD
      IF (TURND .GE. TMAXD)  IMPOS = 1
      WRITE (6,532) VH, VY4RA(I), VY4GA(I), VY4ZA(I), TMAXD,
     1      TURND, RT, ELEVI, ELEV
 532 FORMAT ('0(ZERO  ER  BEFORE THE  HR)--SPEED* R,G,Z AR',
     1      'RIVAL VELOCITY* ALLOWABLE TURN* REQUIRED TURN + ',
     2      'RADIUS* ELEV IN* ELEV TO  HR*'/
     3      1H , F9.5, 3X, 3F9.5, 10X, F8.2, 5X, F8.2, ',',
     4      F6.3, 5X, 2(5X, F8.2) )
      ALG = ALGN
      IF (ALG .GT. 1.0001)  GO TO 950
 533 TURN = ARCOS(C8*COS(THEHRL) + C4*SIN(THEHRL) + C5)
      TURND = TURN*RTD
      RT = VC(N)**2*(1./SIN(TURN/2.) - 1.)/VHO**2
      ELEV = ARCOS(COS(THEHRL)*SIN(COO))*RTD
      TMAXOD = RTD*2.0*ARSIN(1./(1. + (VHO/VC(N))**2) )
      IF (TURND .GE. TMAXOD)  IMPOS = 1
      WRITE (6,534) VHO, VX4RL(I), VX4GL(I), VX4ZL(I), TMAXOD,
     1      TURND, RT, ELEV, ELEVO
 534 FORMAT ('0(ZERO  ER  AFTER THE  HR)--SPEED* R,G,Z DEP',
     1      'ARTURN VELOCITY* ALLOWABLE TURN* REQUIRED TURN + ',
     2      'RADIUS* ELEV FROM  HR* ELEV OUT*'/
     3       1H , F9.5, 3X, 3F9.5, 10X, F8.2, 5X, F8.2, ',',
     4      F6.3, 5X, 2(5X, F8.2)  )
      IF (IMPOS .LE. 0)  GO TO 440
      WRITE (6,403)
      GO TO 440
 900 ALGM1 = ALG - 1.
      DIF = ABS(THETHR - THETAI)
      DEL = DIF
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      IF (DIF .GT. PI)  DEL = DIF - 2.0*PI
      IF (THETAI .GT. THETHR)  DEL = -DEL
      DIF = DEL
      DEL = DEL/ALG
      TURNIO = ARCOS(C6*COS(DIF/ALGM1) + C7)
      IF (TURNIO .GT. (TURNT + 1.E-3) )  GO TO 901
      TURN1 = TURNT
      TURN2 = TURNIO
      THETA = THETAI
      GO TO 904
  901 CONTINUE
      DLT = 0.1/ALGM1
      DO  902  J = 1,30
      THET = THETHR - ALGM1*DEL
      F = C1*COS(THET) + C2*SIN(THET) + C3 - C6*COS(DEL) - C7
      G = ALGM1*(C1*SIN(THET) - C2*COS(THET)) + C6*SIN(DEL)
      IF (ABS(F) .LT. 0.0001)  GO TO 903
      CHAN = 0.9*F/G
      IF (CHAN .GT. DLT)  CHAN = DLT
      IF (CHAN .LT. -DLT)  CHAN = -DLT
  902 DEL = DEL - CHAN
      WRITE (6,436) F
      GO TO 440
  903 TURN1 = ARCOS(C1*COS(THET) + C2*SIN(THET) + C3)
      TURN2 = ARCOS(C6*COS(DEL) + C7)
      THETA = THET
  904 TURN1D = TURN1*RTD
      TURN2D = TURN2*RTD
      IF (TURN1D .GE. TMAXD .OR. TURN2D .GE. TMAXD)  IMPOS = 1
      RT1 = VC(N)**2*(1./SIN(TURN1/2.) - 1.)/VH**2
      RT2 = VC(N)**2*(1./SIN(TURN2/2.) - 1.)/VH**2
      ELEV1 = ARSIN(COS(THETA)*SIN(CO))*RTD
      ELEVHR = ARSIN(COS(THETHR)*SIN(CO))*RTD
      WRITE (6,905) ALGM1, VH, VY4RA(I), VY4GA(I), VY4ZA(I),
     1     TMAXD, TURN1D, RT1, TURN2D, RT2, ELEVI, ELEV1,
     2     ELEVHR
  905 FORMAT ('0(', F4.1, '  FR  BEFORE THE  HR)--SPEED* R,G',
     1     ',Z ARRIVAL* ALLOWABLE TURN* REQUIRED TURNS + ',
     2     'RADII* ELEV IN* NEXT ELEV* ELEV TO HR*'/
     3          1H , F9.5, 3X, 3F9.5, 10X, F8.2, 5X, 2(F8.2,
     4     ',', F6.3, '*'), 10X, 3F8.2  )
      ALG = ALGN
      IF (ALG .LE. 1.0001)  GO TO 533
  950 C6 = SIN(COO)**2
      C7 = COS(COO)**2
      ALGM1 = ALG - 1.
      DIF = ABS(THETAO - THEHRL)
      DEL = DIF
      IF (DIF .GT. PI)  DEL = DIF - 2.0*PI
      IF (THEHRL .GT. THETAO)  DEL = -DEL
      DIF = DEL
```

```
      DEL = DEL/ALG
      TURNIO = ARCOS(C6*COS(DIF/ALGM1) + C7)
      IF (TURNIO .GT. (TURNO + 1.E-3) )  GO TO 951
      TURN2 = TURNIO
      TURN3 = TURNO
      THETA = THETAO
      GO TO 954
 951  TURN1 = PI
      DLT = 0.1/ALGM1
      DO 952  J = 1,30
      THET = THEHRL+ ALGM1*DEL
      F = C8*COS(THET) + C4*SIN(THET) + C5 - C6*COS(DEL) - C7
      G = ALGM1* (C4*COS(THET) - C8*SIN(THET) ) + C6*SIN(DEL)
      IF (ABS(F)  .LT. 0.0001)  GO TO 953
      CHAN = 0.9*F/G
      IF (CHAN .GT. DLT)  CHAN = DLT
      IF (CHAN .LT. -DLT)  CHAN = -DLT
 952  DEL = DEL - CHAN
      WRITE (6,436) F
      GO TO 440
 953  TURN2 = ARCOS(C6*COS(DEL) + C7)
      TURN3 = ARCOS(C8*COS(THET) + C4*SIN(THET) + C5)
      THETA = THET
  954 TURN2D = TURN2*RTD
      TURN3D = TURN3*RTD
      TMAXOD = RTD*2.0*ARSIN(1./(1. + (VHO/VC(N))**2) )
      IF (TURN2D .GE. TMAXOD .OR. TURN3D .GE. TMAXOD)  IMPOS=1
      RT2 = VC(N)**2*(1./SIN(TURN2/2.) - 1.)/VHO**2
      RT3 = VC(N)**2*(1./SIN(TURN3/2.) - 1.)/VHO**2
      ELVHRL = ARSIN(COS(THEHRL)*SIN(COO))*RTD
      ELEV3 = ARSIN(COS(THETA)*SIN(COO))*RTD
      WRITE (6,955)  ALGM1, VHO, VX4RL(I), VX4GL(I), VX47L(I),
     1      TMAXOD, TURN2D, RT2, TURN3D, RT3, ELVHRL, ELEV3,
     2      ELEVO
  955 FORMAT  ('O(', F4.1, '  FR  AFTER THE HR)--SPEED* R,G,7',
     1      ' OUT* ALLOWABLE TURN* REQUIRED TURNS + RADII* ',
     2      'ELEV FROM  HR* NEXT TO LAST ELEV* ELEV OUT*'/
     3                 1H , F9.5, 3X, 3F9.5, 10X, F8.2, 5X,
     4      2(F8.2, ',', F6.3, '*'), 10X, 3F8.2  )
      IF (IMPOS .LE. 0)  GO TO 440
      WRITE (6,403)
      GO TO  440
C     *   *   *   *   *   *   *   *   *   *   *   *   *   *
C
C     THE REMAINDER OF THE MAIN PROGRAM CALCULATES THE TURN
C          ANGLES FOR ONE OR MORE  FR  WITHOUT AN  HR.
C
 419  IF (ALG.GE.2.0)  GO TO 430
      WRITE(6,421) I,DATE(I),VH,TMAXD,RA(N)
  421 FORMAT(1H ,////,25H SINGLE REVOLUTION RETURN,2X,I2,2X,
     1      4(F12.3, 2X) )
```

```
      WRITE(6,292)
      WRITE(6,425) VY4RA(I),VY4GA(I),VY4ZA(I),VX4RL(I),
     1      VX4GL(I), VX4ZL(I)
C
C     TURN1 IS THE FIRST TURN AND TURN2 IS THE SECOND TURN IN
C         A SINGLE FULL REVOLUTION RETURN.
C     RT1 AND RT2 ARE THE RADII OF CLOSEST APPROACH CORRES-
C         PONDING TO  TURN1  AND  TURN2  RESPECTIVELY.
C
      IF (LEHRI.LE. 0) GO TO 422
      IF (LHRM .EQ. 2)  GO TO 449
      IF (LHRM .GT. 2)  GO TO 440
      TURN1 = ARCOS(C1*COS(THETHR) + C2*SIN(THETHR) + C3)
      TURN2 = PI
      THETAM = THETHR
      LHRM = LHRM + 1
      GO TO 429
  449 TURN1 = PI
      TURN2 = ARCOS(C8*COS(THEHRL) + C4*SIN(THEHRL) + C5)
      THETAM = THEHRL
      LHRM = LHRM + 1
      GO TO 429
 422  IF (TURNO .LT. AIOM) GO TO 423
      TURN1=AIOM
      TURN2=TURNO
      THETAM=THETAO
      GO TO 429
  423 IF(TURNI.LT.AOIM) GO TO 424
      TURN1=TURNI
      TURN2=AOIM
      THETAM=THETAI
      GO TO 429
  424 IF(DIF.LE.PI) THETAM=THETAS + DIF/2.
      IF(DIF.GT.PI) THETAM=THETAS -(2.0*PI-DIF)/2.
      DO 396 J=1,30
      F=(C1-C8)*COS(THETAM) + (C2-C4)*SIN(THETAM) + C3 - C5
      G=(C8-C1)*SIN(THETAM) + (C2-C4)*COS(THETAM)
      IF(ABS(F).LT.0.0001) GO TO 521
      CHAN1=0.8*F/G
      IF(CHAN1.GT.0.10) CHAN1=0.10
      IF(CHAN1.LT.-0.10) CHAN1=-0.10
  396 THETAM=THETAM-CHAN1
      WRITE(6,436) F
      GO TO 815
  521 TURN1=ARCOS(C1*COS(THETAM) + C2*SIN(THETAM) + C3)
      TURN2=ARCOS(C8*COS(THETAM) + C4*SIN(THETAM) + C5)
  429 TURN1D=TURN1*RTD
      TURN2D=TURN2*RTD
      RT1=VC(N)*VC(N)*(1./SIN(TURN1/2.)-1.)/(VH*VH)
      RT2=VC(N)*VC(N)*(1./SIN(TURN2/2.)-1.)/(VH*VH)
      ELEV=ARSIN(COS(THETAM)*SIN(CO))*RTD
```

```
      WRITE(6,293)
      WRITE(6,425) TURN1D,RT1,TURN2D,RT2
      WRITE(6,294)
      WRITE(6,425) ELEVI,ELEV,ELEVO
      IF(TURN1.LT.TMAX.AND.TURN2.LT.TMAX) GO TO 440
      WRITE(6,403)
      GO TO 440
 430  WRITE (6,431)  ALG, I, DATE(I), VH, TMAXD, RA(N)
 431  FORMAT (//// '    MULTIPLE (', F3.1,') REVOLUTION RETURN',
     1      4X, I2, 2X, 4F10.3)
      WRITE(6,292)
      WRITE(6,425) VY4RA(I),VY4GA(I),VY4ZA(I),VX4RL(I),
     1      VX4GL(I),VX4ZL(I)
      ALGM1 = ALG - 1.0
C
C     TURN1,TURN2,TURN3,RT1,RT2,RT2, ARE THE TURNS AND CORRES-
C          PONDING RADII OF CLOSEST APPROACH FOR A MULTIPLE
C          (MORE THAN ONE  ER) RETURN.
C
 940  IFAKE = 0
      DEL  = DIF
      IF (DIF.GT.PI)  DEL  = DIF - 2.0*PI
      IF (THETAI.GT.THETAO)  DEL  = -DEL
      DIF = DEL
      DEL  = DEL /ALG
      TURNIO = ARCOS(C6*COS(DIF/ALGM1) + C7)
      IF(TURNIO.GT.TURNI.OR.TURNIO.GT.TURNO) GO TO 450
      TURN1=TURNI
      TURN2=TURNIO
      TURN3=TURNO
      THETA1=THETAI
      THETA2=THETAO
      GO TO 488
  450 IF(TURNI.LT.TURNO.OR.TURNI.LT.TURNIO) GO TO 434
  433 TURN1=TURNI
      DO 435 J=1,30
      THET = THETAI + ALGM1*DEL
      F = C8*COS(THET) + C4*SIN(THET) + C5 - C6*COS(DEL ) - C7
      G = ALGM1*(C4*COS(THET) - C8*SIN(THET)) + C6*SIN(DEL)
      IF(ABS(F).LT.0.0001) GO TO 437
      CHAN1=0.8*F/G
      IF(CHAN1.GT.0.10) CHAN1=0.10
      IF(CHAN1.LT.-0.10) CHAN1=-0.10
 435  DEL  = DEL  - CHAN1
      WRITE(6,436) F
  436 FORMAT(1H ,16HITERATION FAILED,5X,4HF = ,F12.6)
      GO TO 440
 437  TURN2 = ARCOS(C6*COS(DEL ) + C7)
      TURN3 = ARCOS(C8*COS(THET) + C4*SIN(THET) + C5)
      THETA1=THETAI
      THETA2 = THET
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      IF(IFAKE) 488,488,114
  434 IF(TURNO.LT.TURNI.OR.TURNO.LT.TURNIO) GO TO 442
  438 TURN3=TURNO
      DO 439 J=1,30
      THET = THETAO - ALGM1*DEL
      F = C1*COS(THET) + C2*SIN(THET) + C3 - C6*COS(DEL ) - C7
      G = ALGM1*(C1*SIN(THET) - C2*COS(THET)) + C6*SIN(DEL )
      IF(ABS(F).LT.0.0001) GO TO 486
      CHAN1=0.8*F/G
      IF(CHAN1.GT.0.10) CHAN1=0.10
      IF(CHAN1.LT.-0.10) CHAN1=-0.10
 439  DEL  = DEL  - CHAN1
      WRITE(6,436) F
      GO TO 440
 486  TURN1 = ARCOS(C1*COS(THET) + C2*SIN(THET) + C3)
      TURN2 = ARCOS(C6*COS(DEL ) + C7)
      THETA1 = THET
      THETA2=THETAO
      IF(IFAKE) 488,488,121
  442 IF(TURNI.LT.TURNO) GO TO 119
      IFAKE=1
      GO TO 433
  119 IFAKE=1
      GO TO 438
 114  IF (TURN2.LE.TURN1)  GO TO 488
      THETA1 = THETA1 + DEL /(2.0*ALG)
      GO TO 101
 121  IF (TURN2.LE.TURN3)  GO TO 488
      THETA2 = THETA2 - DEL /(2.0*ALG)
 101  DELTA = THETA2 - THETA1
 102  ADELTA = ABS(DELTA)
      SDELTA = DELTA/ADELTA
      IF (ADELTA.LE.PI) GO TO  109
      IF (ADELTA.LT.(2.0*PI))  GO TO 103
      DELTA = DELTA - 2.0*PI*SDELTA
      GO TO 102
 103  DELTA = DELTA - 2.0*PI*SDELTA
      THETA2 = THETA1 + DELTA
  109 DO 447 J=1,60
      F1 = C6*COS((THETA2 - THETA1)/ALGM1) + C7 - C8*
     1      COS(THETA2) - C4*SIN(THETA2) - C5
      F2 = C6*COS((THETA2-THETA1)/ALGM1) + C7 - C1*COS(THETA1)
     1      - C2*SIN(THETA1) - C3
      F = ABS(F1) + ABS(F2)
      IF(F.LT.0.0002) GO TO 497
      T11 = C6*SIN((THETA2-THETA1)/ALGM1)/ALGM1
      T22=-T11
      T12=T22 + C8*SIN(THETA2) - C4*COS(THETA2)
      T21=T11 + C1*SIN(THETA1) - C2*COS(THETA1)
      DETER=T11*T22 - T12*T21
      CHAN1=(0.8/DETER)*(T22*F1 - T12*F2)
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```
      CHAN2=(0.8/DETER)*(T11*F2-T21*F1)
      IF(CHAN1.GT.0.05) CHAN1=0.05
      IF(CHAN2.GT.0.05) CHAN2=0.05
      IF(CHAN1.LT.-0.05) CHAN1=-0.05
      IF(CHAN2.LT.-0.05) CHAN2=-0.05
      THETA1=THETA1-CHAN1
  447 THETA2=THETA2-CHAN2
      WRITE(6,436) F
      GO TO 440
  497 TURN1=ARCOS(C1*COS(THETA1) + C2*SIN(THETA1) + C3)
      TURN2 = ARCOS(C6*COS((THETA2-THETA1)/ALGM1) + C7)
      TURN3=ARCOS(C8*COS(THETA2) + C4*SIN(THETA2) + C5)
  488 TURN1D=TURN1*RTD
      TURN2D=TURN2*RTD
      TURN3D=TURN3*RTD
      RT1=VC(N)*VC(N)*(1./SIN(TURN1/2.) -1.)/(VH*VH)
      RT2=VC(N)*VC(N)*(1./SIN(TURN2/2.) -1.)/(VH*VH)
      RT3=VC(N)*VC(N)*(1./SIN(TURN3/2.) -1.)/(VH*VH)
      ELEV1=ARSIN(COS(THETA1)*SIN(CO))*RTD
      ELEV2=ARSIN(COS(THETA2)*SIN(CO))*RTD
      WRITE(6,293)
      WRITE(6,425) TURN1D,RT1,TURN2D,RT2,TURN3D,RT3
      WRITE(6,294)
      WRITE(6,425) ELEVI,ELEV1,ELEV2,ELEVO
      IF(TURN1.LT.TMAX.AND.TURN2.LT.TMAX.AND.TURN3.LT.TMAX)
     1      GO TO 440
      WRITE(6,403)
  440 CONTINUE
  815 GO TO 1
  819 STOP
      END
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      SUBROUTINE LAMBRT(TJL,TJA,JJ)
C     THIS SUBROUTINE CALCULATES THE PLANETARY POSITIONS
C     FROM THE ENCOUNTER DATES, SOLVES LAMBERT'S PROBLEM FOR
C     THE RESULTING SPACE TRIANGLE PROBLEM, AND ENDS UP BY
C     CALCULATING THE VELOCITIES AND SPEEDS AT THE ENCOUNTERS
C     AT EACH END OF A GIVEN TRAJECTORY.
C     IMPORTANT PARAMETERS HERE ARE--
C          GEN--LONGITUDE OF THE NODE
C          ANINC--RELATIVE INCLINATIONS OF THE PLANETARY ORBITS
C          ANOMX,ANOMY--MEAN ANOMOLIES OF ENCOUNTERED PLANETS
C          TJL, TJA--JULIAN DATE (MINUS 2440000) OF DEPARTURE
C             AND ARRIVAL  (THE DEPARTURE DATE MAY BE CHANGED
C             BY THE SUBROUTINE IN THE CASE OF A HALF REVOLUTION
C             BY THE SUBROUTINE IN THE CASE OF A HALF
C             REVOLUTION RETURN)
C     OTHER PARAMETERS OF IMPORTANCE WILL BE DESCRIBED IN
C     OTHER COMMENTS IN THE SUBROUTINE.
C
      DIMENSION PER(9), A(9), E(9), TJP(9), GEP(9),  NP(90),
     1     DATEL(90), LEHR(90), CIR(90), GLON(90), DHICK(90),
     2     DHICK2(90),  VX4RL(90),VX4GL(90),VX4ZL(90),VX4TL(90),
     3     VY4RA(90),VY4GA(90),VY4ZA(90),VY4TA(90),  VSPG(90),
     4     VSPT(90),VSPR(90), VSPTA(90),VSPGA(90),VSPRA(90)
      COMMON PI,RTD,YR, PER,A,E,TJP,GEP, NP,DATEL,CIR,LEHR,
     1     GLON,DHICK,DHICK2,  VX4RL,VX4GL,VX4ZL,VX4TL, VY4RA,
     2     VY4GA,VY4ZA,VY4TA,  VSPT,VSPG,VSPR, VSPTA,VSPGA,VSPRA,
     3     NERR
      NERR=0
      JJP1=JJ+1
      NPX=NP(JJ)
      NPY=NP(JJP1)
      CIRC = CIR(JJ)
      IF(NPX.EQ.1.AND.NPY.EQ.3) GO TO 42
      IF(NPX.EQ.3.AND.NPY.EQ.1) GO TO 43
      IF(NPX.EQ.1.AND.NPY.EQ.2) GO TO 40
      IF(NPX.EQ.2.AND.NPY.EQ.1) GO TO 41
      IF(NPX.EQ.2.AND.NPY.EQ.3) GO TO 44
      IF (NPX .EQ. 3 .AND. NPY .EQ. 2)  GO TO 45
      GO TO 46
   40 GEN = 76.31972
      ANINC = 1.9391
      GO TO 99
   41 GEN = 256.31972
      ANINC = 1.9391
      GO TO 99
   42 GEN = 49.24903
      ANINC = 1.84991
      GO TO 99
   43 GEN = 229.24903
      ANINC = 1.84991
      GO TO 99
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44   GEN = 282.013
     ANINC = 3.39423
     GO TO 99
45   GEN = 102.013
     ANINC = 3.39423
     GO TO 99
46   ANINC = 0.0
99   CONTINUE
C    *    *    *    *    *    *    *    *    *    *    *
C
C    THE FIRST PART OF THIS SUBROUTINE DETERMINES THE
C         POSITIONS OF THE ENCOUNTERED PLANETS.
C
     ANOMX = (TJL - TJP(NPX))*2.*PI/PER(NPX)
     EX=ANOMX + 2.*E(NPX)*SIN(ANOMX) +
    1      1.25*E(NPX)**2*SIN(2.*ANOMX)
     IF (LEHR(JJ) .LE. 0)  GO TO 49
     EX = EX - 4.0*E(NPX)*SIN(ANOMX)
     TJL = TJL - 2.0*E(NPX)*SIN(EX)*PER(NPX)/PI
  49 DATEL(JJ) = TJL
     TLA = TJA - TJL
     TEY = TLA/YR
     RX=(A(NPX)*(1.-E(NPX)*E(NPX)))/(1. + E(NPX)*COS(EX))
     GX=GEP(NPX)/RTD + EX
     GLON(JJ)=GX*RTD
     VSXTL=SQRT(2./RX-1./A(NPX))
     VSPT(JJ)=VSXTL
     GGX=ATAN(E(NPX)*SIN(EX)*RX/(A(NPX)-A(NPX)*E(NPX)*E(NPX)))
     VSXGL=VSXTL*COS(GGX)
     VSPG(JJ)=VSXGL
     VSXRL=VSXTL*SIN(GGX)
     VSPR(JJ)=VSXRL
     ANOMY = (TJA - TJP(NPY))*2.*PI/PER(NPY)
     EY=ANOMY + 2.*E(NPY)*SIN(ANOMY) +
    1      1.25*E(NPY)**2*SIN(2.*ANOMY)
     RY=(A(NPY)*(1. - E(NPY)*E(NPY)))/(1.+E(NPY)*COS(EY))
     GY=GEP(NPY)/RTD + EY
     GLON(JJ+1)=GY*RTD
     VSYTA=SQRT(2./RY-1./A(NPY))
     VSPTA(JJ+1) = VSYTA
     GGY=ATAN(E(NPY)*SIN(EY)*RY/(A(NPY)-A(NPY)*E(NPY)*E(NPY)))
     VSYGA=VSYTA*COS(GGY)
     VSPGA(JJ+1) = VSYGA
     VSYRA=VSYTA*SIN(GGY)
     VSPRA(JJ+1) = VSYRA
     SINTH=SIN(GY-GX)
     COSTH=COS(GY-GX)
     IF (ABS(SINTH) .LE.  1.E-4)  GO TO 1000
     C=SQRT(RX*RX+RY*RY-2.*RX*RY*COSTH)
     S=(RX+RY+C)/2.
     BEM=ARCOS((2.*C-S)/S)
```

```
C
C      B1  EQUALS  -1  IF THE TRANSFER ANGLE MODULO  360
C           DEGREES IS LESS THAN  180  DEGREES.  B1  EQUALS
C           +1  OTHERWISE.
C      TMM  EQUALS THE MINIMUM ENERGY TIME OF FLIGHT FOR ZERO
C           CIRCUITS.
C      TM  EQUALS THE MINIMUM ENERGY TIME OF FLIGHT FOR  "CIRC"
C           CIRCUITS.
C
      B1=+1.
      IF(SINTH) 27,27,26
   26 B1=-1.
  27  NTEST = 0
      IF (CIRC.LT.10.0)  GO TO 16
      CIRC = CIRC - 10.0
      NTEST = 1
      IF (CIRC.LT.10.0)  GO TO 16
      CIRC = CIRC - 10.0
      NTEST = 2
      IF (CIRC.LT.10.0)  GO TO 16
      CIRC = CIRC - 10.0
      NTEST = 3
      IF (CIRC.LT.10.0)  GO TO 16
      CIRC = CIRC - 10.0
      NTEST = 4
      IF (CIRC.LT.10.0)  GO TO 16
      CIRC = CIRC - 10.0
      NTEST = 5
  16  SQSB23 = SQRT((S**3)/8.)
      TMM = (SQSB23/2.)*(1. +B1*(BEM-SIN(BEM))/PI)
           TM = TMM + CIRC*SQSB23
      TM1 = SQSB23 + TM
C
C      TM1  EQUALS THE MINIMUM ENERGY TIME OF FLIGHT FOR
C           CIRC+1  CIRCUITS
C      SWITCH FOR SYMMETRIC TRANSFER OR INTERPLANETARY TRANSFER
C
      IF(NPX.EQ.NPY) GO TO 601
C      *    *    *    *    *    *    *    *    *    *    *
C
C      THIS PART OF THE SUBROUTINE SOLVES LAMBERT'S PROBLEM
C           FOR AN INTERPLANETARY TRANSFER.
C
      B3=-1.
      IF (TM .GE. TFY)  B3 = +1
   14 LIM=1
      AOLD=S/1.9
      IF (NTEST.EQ.0)  GO TO 17
      B3 = +1.
      AOLD = RX + RY
  17  AA = AOLD
```

```
      B2 = (1. - B3)/2. + CIRC
  306 COSAL=1.-S/AA
      IF(1.+COSAL) 310,307,307
  310 AA=(S+2.*AOLD)/4.
      GO TO 306
  307 COSBE=1.-(S-C)/AA
      AL=ARCOS(COSAL)
      BE=ARCOS(COSBE)
      SINAL=SIN(AL)
      SINBE=SIN(BE)
      PRR=SQRT(AA**3)
      TOFA=PRR*(B2+(B3*(AL-SINAL)+B1*(BE-SINBE))/(2.*PI))
      LIM=LIM+1
      IF (LIM - 30)   308,308,1001
  308 TEST=ABS(TOFA-TEY)-0.00010
      IF(SINAL.LT.0.0032.OR.SINBE.LT.0.0032) GO TO 610
      IF(TEST) 610,300,300
  300 AOLD=AA
      AA=AA*(1.-(TOFA-TEY)/(1.5*TOFA-PRR*(B3*(1.-COSAL)**2
     1      /SINAL + B1*(1. - COSBE)**2/SINBE)/(2.*PI) ) )
      IF(AA) 310,306,306
  601 CONTINUE
C
C     IF  (CIRC+1)  IS EQUAL TO THE NUMBER OF FULL CIRCUITS
C          OF THE PLANET, THEN  TEP  EQUALS THE PLANET TIME OF
C          FLIGHT FOR ZERO CIRCUITS.
C
      IF (NTEST.EQ.0)  GO TO 18
      NS = 1
      IF (NTEST.EQ.2)  GO TO 19
      B3 = +1.
      AA = RX + RY
      IF (NTEST.EQ.3)  AA=S/1.95
      IF (NTEST.LT.4)  GO TO 67
      B3 = -1.
      CIRC = CIRC - 1.
      AA = S/1.9
      GO TO 67
  18  TEP = TEY -SQRT(A(NPX)**3)*(CIRC + 1)
C     *    *    *    *    *    *    *    *    *    *    *
C
C     THIS PART OF THE SUBROUTINE SOLVES LAMBERT'S PROBLEM
C          FOR A SYMMETRIC RETURN.
C
      KONT=50
      IF (TEP.LT.TMM)  GO TO 65
      AA=1.25
      B3=+1.
      NS=1
      GO TO 67
   65 IF(TEY.LT.TM1) GO TO 64
```

```
 19   AA = S/1.9
      B3=-1.
      NS=1
      GO TO 67
  64  AA=1.25
      B3=+1.
      NS=0
 67   COSAL = 1. - S/AA
      COSBE = 1. - (S-C)/AA
      AL = ARCOS(COSAL)
      BE = ARCOS(COSBE)
      SINAL=SIN(AL)
      SINBE=SIN(BE)
      IF(SINAL.LT.0.0032.OR.SINBE.LT.0.0032) GO TO 615
      T4 = SQRT(AA**3)*( (3. + 2.*CIRC)*PI + B3*(AL - PI -
     1      SINAL) + B1*(BE - SINBE) )/(2.*PI)
      IF(ABS(TEY-T4)-0.00010) 98,98,7
    7 SLOPE= (1.5*T4/AA) - AA*((B1*(1.-COSBE)**2./SINBE) +
     1      (B3*(1.-COSAL)**2/SINAL) )/(2.*PI)
      AA=AA+(TEY-T4)/SLOPE
      IF(AA-S/2.) 89,89,15
   89 AA=(AA-(TEY-T4)/SLOPE + S/2.)/2.
   15 KONT=KONT-1
      IF(KONT.EQ.0) GO TO 1003
      GO TO 67
   98 IF(NS.EQ.1) GO TO 615
      IF(ABS(AA-A(NPX))-0.001) 9,9,615
    9 AA=S/1.99
      KONT=30
      NS = 1
      GO TO 67
  615 CONTINUE
 610  CONTINUE
      A4 = AA
C     *    *    *    *    *    *    *    *    *    *    *
C
C     THE REMAINING PART OF THIS SUBROUTINE DETERMINES THE
C          NECESSARY VELOCITIES CONNECTED WITH THE ENDS OF
C          THE TRAJECTORY.
C
      DHICK(JJ)=A4
      P4=4.*A4*(S-RX)*(S-RY)*SIN((AL-B1*B3*BE)/2.)*
     1      SIN((AL-B1*B3*BE)/2.)/C**2
      IF (P4 .LT. 1.E-4)  GO TO 1007
      E4=SQRT(1.-P4/A4)
      DHICK2(JJ)=E4
      VS4RL=SQRT(P4)*((1./RX-1./P4)*COSTH+(1./P4-1./RY))/SINTH
      VS4GL=SQRT(P4)/RX
      VS4RA=SQRT(P4)*((1./P4-1./RY)*COSTH+(1./RX-1./P4))/SINTH
      VS4GA=SQRT(P4)/RY
      VX4RL(JJ)=VS4RL-VSXRL
```

```
      VY4RA(JJ+1)=VS4RA-VSYRA
      IF(NPX.EQ.NPY) GO TO 605
      VX4ZL(JJ)=SIN(ANINC/RTD)*SIN(GY-GEN/RTD)*VS4GL/SINTH
      DOG1=VS4GL*VS4GL-VX4ZL(JJ)*VX4ZL(JJ)
      IF(DOG1) 1000,522,522
  522 VX4GL(JJ)=SQRT(DOG1)-VSXGL
      VY4ZA(JJ+1)=SIN(ANINC/RTD)*SIN(GX-GEN/RTD)*VS4GA/SINTH
      DOG2=VS4GA*VS4GA-VY4ZA(JJ+1)*VY4ZA(JJ+1)
      IF(DOG2) 1000,533,533
  533 VY4GA(JJ+1)=SQRT(DOG2)-VSYGA
      GO TO 712
  605 VX4ZL(JJ)=0.0
      VY4ZA(JJ+1)=0.0
      VX4GL(JJ)=VS4GL-VSXGL
      VY4GA(JJ+1)=VS4GA-VSYGA
  712 VX4TL(JJ)=SQRT(VX4RL(JJ)*VX4RL(JJ)+VX4GL(JJ)*VX4GL(JJ)
     1      + VX4ZL(JJ)*VX4ZL(JJ) )
      VY4TA(JJ+1)=SQRT(VY4RA(JJ+1)*VY4RA(JJ+1)+VY4GA(JJ+1)
     1      *VY4GA(JJ+1) + VY4ZA(JJ+1)*VY4ZA(JJ+1) )
      GLON(JJ) = AMOD( GLON(JJ), 360.)
      GLON(JJ + 1) = AMOD( GLON(JJ + 1), 360.)
      RETURN
 1000 WRITE(6,1010) TJL,TJA,GLON(JJ),GLON(JJ+1)
 1010 FORMAT(1H ,34HTHETA IS A MULTIPLE OF 180 DEGREES,10X,
     1      4(F12.6, 5X) )
      GO TO 106
 1001 WRITE(6,1002) TJL,TJA,GLON(JJ),GLON(JJ+1),AA,AOLD,
     1      RX, RY, NPX, NPY
 1002 FORMAT(1H ,29HINTERPLANETARY LAMBERT FAILED,10X,
     1      4(F12.6, 5X)/ 1H , 4(F12.6, 5X), I2, 5X, I2)
      GO TO 106
 1003 WRITE(6,1004) TJL,TJA,GLON(JJ),GLON(JJ+1),AA,RX,RY,NPX,NPY
 1004 FORMAT(1H ,24HSYMMETRIC LAMBERT FAILED,10X,4(F12.6,5X)/
     1      1H , 3(F12.6, 5X), I2, 5X, I2)
      GO TO 106
 1007 WRITE (6,1008) TJL, TJA, GLON(JJ), GLON(JJ+1),AA,AOLD,P4
 1008 FORMAT (' PARAMETER TOO SMALL  ', 4F12.4, 3F15.6)
  106 NERR=1
  900 CONTINUE
      RETURN
      END
```

APPENDIX B

MEAN ANOMOLY CALCULATION FOR A HALF REVOLUTION RETURN

In order to facilitate calculations for a half revolution return, it is desired to find a series solution for the time or mean anomoly as a function of the true anomoly and the eccentricity.

Kepler's equation is given by

$$M = 2\pi\ (t - \tau)\sqrt{\frac{\mu}{a^3}} = E - e \sin E \tag{B-1}$$

where M is the mean anomoly, E is the eccentric anomoly, e is the eccentricity, a is the semimajor axis, μ is the constant of gravitation, τ is the time of perhelion passage, and t is the time.

Now, in order to create a power series for M in terms of true anomoly f and in powers of e, one needs to relate f and E such as by the expression from page 39 of Battin[6],

$$\tan \frac{f}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \tag{B-2}$$

Next, one needs to use this relation and Kepler's equation to obtain,

$$M\Big|_{e=0} = f \tag{B-3}$$

$$\left.\frac{\partial M}{\partial e}\right|_{e=0} = - 2 \sin f \tag{B-4}$$

$$\left.\frac{\partial^2 M}{\partial e^2}\right|_{e=0} = \frac{3}{2} \sin 2f \tag{B-5}$$

$$\left.\frac{\partial^3 M}{\partial e^3}\right|_{e=0} = - 2 \sin 3f \tag{B-6}$$

Then the general Taylor series for the mean anomoly as a function of eccentricity e and true anomoly f is given to the third power of e by,

$$M = f - 2e \sin f + \frac{3}{4} e^2 \sin 2f - \frac{1}{3} e^3 \sin 3f + \ldots \tag{B-7}$$

In the case of m FR's and one HR, we would like to know what happens to the time or mean anomoly at the end of the series of FR and HR as a function of the time at the beginning of the series. Hence, we let,

$$f = f_o + 2\pi m + \pi \tag{B-8}$$

where it is assumed that one already knows f_o as a function of the initial time. If one then substitutes this expression for f into Equation (B-7), one obtains an expression for M in terms of f_o, m, and e. In addition, the expression for M_o can be obtained by substituting f_o into Equation (B-7) in place of the f already there. The result is an expression for M_o in terms of f_o and e. This last expression for M_o can then be used to form an expression for M in terms of M_o, e, and sines of multiples of f_o as,

$$M = M_o + 2\pi m + \pi + 4 e \sin f_o + \frac{2}{3} e^3 \sin 3f_o + \ldots \tag{B-9}$$

One can also obtain another expression for the mean anomoly M after a series of FR and one HR through the use of an incorrect initial expression for M and for f. Battin[6] on page 55 gives a series expression for f in terms of M and e to fourth order in e as,

$$f = M + (2e - \frac{1}{4} e^3) \sin M + (\frac{5}{4} e^2 - \frac{11}{24} e^4) \sin 2M + \frac{13}{12} e^3 \sin 3M + \frac{103}{96} e^4 \sin 4M \qquad \text{(B-10)}$$

If one uses this expression to obtain an incorrect value for the true anomoly f* at the end of the series of FR and HR by using,

$$M^* = M_o + 2\pi m + \pi \qquad \text{(B-11)}$$

then one has,

$$\begin{aligned} f^* &= M_o + 2\pi m + \pi - (2e - \frac{1}{4} e^3) \sin M_o \\ &\quad + (\frac{5}{4} e^2 - \frac{11}{24} e^4) \sin 2M_o - \frac{13}{12} e^3 \sin 3M_o \\ &\quad + \frac{103}{96} e^4 \sin 4M_o \\ &= f_o + 2\pi m + \pi - 2\,(2e - \frac{1}{4} e^3) \sin M_o \\ &\quad - \frac{13}{6} e^3 \sin 3M_o \qquad \text{(B-12)} \end{aligned}$$

Hence, one can write the desired correct value for the true anomoly as,

$$\begin{aligned} f &= f_o + 2\pi m + \pi \\ &= f^* + (4e - \frac{1}{2} e^3) \sin M_o + \frac{13}{6} e^3 \sin 3M_o \\ &= f^* - (4e - \frac{1}{2} e^3) \sin M^* - \frac{13}{6} e^3 \sin 3M^* \qquad \text{(B-13)} \end{aligned}$$

One can also write from Equations (B-9) and (B-11),

$$M = M^* + 4e \sin f_o + \frac{2}{3} e^3 \sin 3f_o + \ldots$$

$$= M^* - 4e \sin f - \frac{2}{3} e^3 \sin 3f - \ldots \qquad \text{(B-14)}$$

The above relations are useful in calculating dates and true anomolies for half revolution returns which are associated with a series of full revolution returns.

APPENDIX C

THE NUMBER OF COMBINATIONS AVAILABLE

In order to know if all possible combinations of r of the n different types of direct return orbits have been counted, one would like to know the answer to the auxiliary problem of determining the number of possible combinations of r elements, each of which can be one of n possible things. This information is important as a check that one has all of the desired combinations in Appendix D. The list of combinations in Appendix D has been created without regard for the order of the elements in each combination although the order does matter in a periodic orbit. The auxiliary problem may be stated as follows:

> It is desired to determine the number of combinations or r elements from a population n <u>with</u> replacement and <u>without</u> regard for order.

In the specific case of the direct return orbits, n will be equal to four and r will be any number up to about 6.

An equivalent problem is determining the number of ways that one can choose n nonnegative integers, $r_1, r_2, \ldots r_n$, such that,

$$\sum_{i=1}^{n} r_i = r \tag{C-1}$$

In order to obtain the solution of this problem, one must first note that if the numbers, r_1, r_2, ... r_{i-1}, have already been chosen and if i is less than n, then the number r_i can be any integer between 0 and $(r - \sum_{j=1}^{i-1} r_j)$ inclusive.

Hence, there are $(r + 1 - \sum_{j=1}^{i-1} r_j)$ ways to choose the i^{th} number r_i. If, however, i is equal to n, then one must have,

$$r_n = r - \sum_{i=1}^{n-1} r_i \qquad (C-2)$$

in order to satisfy Equation (C-1); and there is only one way to choose r_n.

Hence, if the numbers, r_1, r_2, ... r_{n-2}, have already been chosen, then the number r_{n-1} can be chosen in $(r + 1 - \sum_{j=1}^{n-2} r_j)$ different ways. The choice of r_{n-1} then determines the choice of r_n. This choice of r_{n-1} will result in a different combination of the n r_i's for each different combination of r_1, r_2, ... r_{n-2}. One can then surmise that the desired number N_n^r of possible combinations meeting the above conditions is given by the sum of the number of all possible values for r_{n-1},

$$N_n^r = \sum_{r_1=0}^{r} \sum_{r_2=0}^{r-r_1} \cdots \sum_{r_{n-3}=0}^{r-\sum_{j=1}^{n-4} r_j} \sum_{r_{n-2}=0}^{r-\sum_{j=1}^{n-3} r_j} (r + 1 - \sum_{j=1}^{n-2} r_j)$$

$$= \sum_{r_1=0}^{r} \sum_{r_2=0}^{r-r_1} \cdots \sum_{r_{n-3}=0}^{r-\sum_{j=1}^{n-4} r_j} \sum_{k=1}^{r+1-\sum_{j=1}^{n-3} r_j} k \qquad (C-3)$$

Note that this expression for the value of N_n^r can be put into the following form:

$$N_n^r = \sum_{r_1=0}^{r} \sum_{r_2=0}^{r-r_1} \cdots \sum_{r_{n-3}=0}^{r-\sum_{j=1}^{n-4} r_j} \left[\left(r + 1 - \sum_{j=1}^{n-3} r_j\right) + \sum_{k=1}^{r-\sum_{j=1}^{n-3} r_j} k \right]$$

$$= \left[\sum_{r_1=0}^{r} \sum_{r_2=0}^{r-r_1} \cdots \sum_{k=1}^{r+1-\sum_{j=1}^{n-4} r_j} k \right]$$

$$+ \left[\sum_{r_1=0}^{r-1} \sum_{r_2=0}^{r-r_1-1} \cdots \sum_{k=1}^{r-\sum_{j=1}^{n-3} r_j} k \right]$$

$$= N_{n-1}^r + N_n^{r-1} \qquad \text{(C-4)}$$

Note that the first term in the final expression is the value of N_n^r for n one less than the present value and that the second term in the final expression is the value of N_n^r for r one less than the present value; a recursion relation for N_n^r has been developed. One can then set up a table for the value of N_n^r as a function of the values of n and r such that each number or value of N_n^r in the body of the table is equal to the sum of the number immediately to the left of it and the number immediately above it. Such a table is Table (C-1). From an examination of this table and by remembering how this table was formed, note that these numbers are those of Pascal's triangle; the numbers are the binomial coefficients. Then, the number of combinations of r elements taken from a population n <u>with</u> replacement and <u>without</u> regard for order can be given by,

$$N_n^r = \binom{r+n-1}{n-1} = \frac{(r+n-1)!}{(n-1)!r!} \qquad \text{(C-5)}$$

r \ n	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	1	3	6	10	15	21	28	36	
3	1	4	10	20	35	56	84		
4	1	5	15	35	70	126			
5	1	6	21	56	126	252			
6	1	7	28	84	210	462			
7	1	8	36	120	330	792			
8	1	9	45	165	495	1287			

Table C-1. The number of combinations N_n^r of r elements from a population n with replacement and without regard for order.

APPENDIX D

COMBINATIONS OF DIRECT RETURNS AT EARTH

This appendix lists all of the reasonable combinations of direct return trajectories which depart from and return to Earth and which traverse the Sun the same number of times as the Earth. The different combinations are listed in order of increasing time required for their completion. The first part discusses how this list was obtained.

The first step of the procedure is the writing down of all of the possible combinations of direct returns taken r at a time. r is successively set equal to 1, 2, 3, 4, 5, and 6. 6 is chosen as the upper limit so that one is sure that all of the desired combinations which last up to 3.2 years are included. The different direct returns which are taken to form these combinations are the ones indicated by the symbols HR, FR, S1SR, and L1SR. Symmetric returns of longer duration are not included, because the addition of more types of direct returns greatly increases the number of combinations possible and because symmetric returns of the type S2SR, for instance, can simply be considered a modification of the combination (FR)(S1SR), having a duration of only a few days longer. The number of combinations in each then known for the different combinations of direct returns.

Next, the different combinations of direct returns are placed in order of increasing duration so that the user of the list can easily look within a range of time for a combination which lasts lasts approximately some desired length of time. One can now eliminate from the list those combinations which last longer than the desired time--3.2 years in this case.

Finally, different combinations, which are based on combinations already in the list, are added. The reader will remember that, in the creation of the list up to this point, symmetric returns which last longer than two years have been excluded. Combinations which include long symmetric returns of this type are now added to the list. These additional combinations are formed by substituting symmetric returns which last more than two years for those combinations of direct returns which include one or more full revolution returns and a symmetric return. They supply series which require slightly more time to complete than the series from which they are formed. The added combinations are basically small variations on the original combinations. A reason for delaying the inclusion of combinations with long symmetric returns (such as S2SR or L3SR) is that the delay makes the earlier list formation easier. If the list contains a series of direct returns which includes a subcombination of the form, (2FR)(S1SR), then two additional series of direct returns can be formed which include the subcombinations of (FR)(S2SR) and (S3SR) respectively. The two additional series require respectively 15 and 20 days more to complete than does the original combination. In the process of adding these additional combinations, one must maintain the order of the list so that each series in the list requires a nominally greater length of time than the preceding series of direct returns. In creating these additional series of direct returns, one must also take care not to include impossible combinations such as (S2SR)(S3SR). Many series of direct returns were added to the list in this fashion.

A list of the different direct returns used and the times used for them follows:

direct return	duration in days
HR--half revolution return	182.6
FR--full revolution return	365.3
S1SR--symmetric return less than 1.41 yr.	478
L1SR--symmetric return greater than 1.41 yr.	550
S2SR--symmetric return less than 2.46 yr.	858
L2SR--sym. return greater than 2.46 yr.	930
S3SR--sym. return less than 3.46 yr.	1229
L3SR--sym. return greater than 3.46 yr.	1301
S4SR--sym. return less than 4.47 yr.	1598
L4SR--sym. return greater than 4.47 yr.	1670
S5SR--sym. return less than 5.48 yr.	1965
L5SR--sym. return greater than 5.48 yr.	2037

The third, fourth, and fifth columns of the table give times which are useful for matching a combination of direct returns with round trips to Mars and back. If one has a series of direct returns situated in the center of a time interval determined by one or more synodic periods, then the times which describe the distances between the ends of the interval of one or more synodic periods and the ends of the interval determined by the series of direct returns. will be equal. In other words, there are two intervals. The larger interval is a few synodic periods in length, and the ends of it are given by times of opposition. The smaller interval has a length determined by a series of direct returns. The smaller interval is centered within the larger interval so that the time differences between the beginnings of both intervals and between the ends of both intervals are equal. This time between the corresponding ends of the two intervals gives time from opposition, since the ends of the larger interval correspond to opposition times. These times from opposition are extremely useful for matching a series of direct returns with the ends of round trips to another planet. If n is the number of synodic periods, T_s is the length of a synodic period, and T_d is the length of time spent on the series of direct returns, then the time T from opposition is given by,

$$T = \tfrac{1}{2} (n \, T_s - T_d) \qquad \text{(D-1)}$$

The purpose here is to go from Earth to Mars; hence, one takes $T_s = 779.2$ days and n equal to 1, 2, and 3, to obtain columns three, four, and five, respectively, in the table.

One must now understand the restrictions of the last column of the table. This column contains an attempt to declare any possible restrictions which may be placed on a round trip which is to be patched from a given series of direct returns. Restrictions are based on speed such that certain turn angles can be accomplished at Earth and on direction such that the hyperbolic excess velocity at the end of the round trip should be approximately in a certain direction. Beyond this, if there is no letter in the last column indicating a restriction, then no difficulty will arise if the speed is sufficiently slow (0.53 EMOS) so that about a 45° turn can be accomplished at Earth without hitting Earth. The restrictions corresponding to the letters used are given below. The R,G,Z coordinate system referred to for the hyperbolic excess velocity components is explained in Chapter 3, Figure 3-1a, and in the list of symbols. To allow as few restrictions as possible, the order of the individual direct returns in a series may be changed as needed. The interplanetary trajectory leg at one end of the series of direct returns is assumed to be the reciprocal of the leg at the other end. The list of restrictions follows:

<u>letter</u>	<u>corresponding restriction</u>
A	Arrival velocity vector at the beginning of the series of direct returns should have a positive R component (and the departure velocity vector should have a negative R component).
B	Arrival velocity vector at the beginning of the series of direct returns should have a negative R component (and the departure velocity vector should have a positive R component).

C Arrival (and departure) speed should be slow enough so that the vehicle can perform about a 60° turn at Earth without hitting the planet (0.27 EMOS).

D Arrival (and departure) speed should be slow enough so that the vehicle can perform about a 90° turn at Earth without hitting it (0.17 EMOS).

E Arrival (and departure) hyperbolic excess velocity should be mostly in either the positive or the negative G direction and small enough so that at least a 90° turn can be made at Earth without hitting it.

As is explained above, the following table is designed for series of direct returns at Earth and round trips to Mars and back. However, with a little bit of scaling, the table can be useful for direct returns at any other planet and for transfers between that planet and any other in the search for periodic orbits. The length of the series of direct returns must be scaled with the length of the year for the planet in question and columns three, four, and five must be recomputed using the new length for the series of direct returns and the appropriate value for the synodic period of the two planets in question. The restrictions would mean the same thing as far as the approximate required turns are concerned, but the speeds would be different.

direct returns in the series	time in days for the series	times from opposition 1 synodic period	2 synodic periods	3 synodic periods	re-strictions
(1)	(2)	(3)	(4)	(5)	(6)
(HR)	183	298	688	1077	D
(FR)	365	207	597	986	
(S1SR)	478	151	540	930	A
(HR)(FR)	548	116	505	895	D
(L1SR)	550	115	504	894	B
(HR)(S1SR)	661	59	449	838	D&A
(2FR)	731	24	414	804	
(HR)(L1SR)	733	23	413	802	D&B
(FR)(S1SR)	843		358	747	A
(S2SR)	858		350	740	A
(HR)(2FR)	913		323	712	
(FR)(L1SR)	915		322	711	B
(L2SR)	930		314	704	B
(HR)(FR)(S1SR)	1026		266	656	D
(HR)(S2SR)	1041		259	648	D
(S1SR)(L1SR)	1028		265	655	E
(3FR)	1096		231	621	
(HR)(FR)(L1SR)	1098		230	620	D
(HR)(L2SR)	1113		223	612	D&B
(HR)(2S1SR)	1139		210	599	D&A
(2FR)(S1SR)	1209		175	565	A
(HR)(S1SR)(L1SR)	1211		174	563	D
(FR)(S2SR)	1223		168	557	A
(S3SR)	1229		165	554	A
(HR)(3FR)	1278		140	530	
(2FR)(L1SR)	1281		139	529	BorD
(HR)(2L1SR)	1283		138	527	D&B

(FR)(L2SR)	1295	132	521	B
(L3SR)	1301	129	518	B
(FR)(2S1SR)	1321	119	508	D&A
(HR)(2FR)(S1SR)	1391	84	473	A
(FR)(S1SR)(L1SR)	1393	83	472	D
(HR)(FR)(S2SR)	1406	76	466	D&A
(HR)(S3SR)	1412	73	463	D&A
(4FR)	1461	49	438	
(HR)(2FR)(L1SR)	1463	48	437	B
(FR)(2L1SR)	1465	47	436	D&B
(HR)(FR)(L2SR)	1478	40	430	D&B
(HR)(L3SR)	1484	37	427	D&B
(HR)(FR)(2S1SR)	1504	27	417	D&B
(2S1SR)(L1SR)	1506	26	416	A
(HR)(S1SR)(S2SR)	1519	20	409	D&A
(3FR)(S1SR)	1574		382	
(HR)(FR)(S1SR)(L1SR)	1576		381	D
(S1SR)(2L1SR)	1578		380	B
(2FR)(S2SR)	1589		375	AorD
(HR)(S2SR)(L1SR)	1591		373	D
(HR)(S1SR)(L2SR)	1591		373	D
(FR)(S3SR)	1594		372	A
(S4SR)	1598		370	A
(HR)(4FR)	1644		347	
(3FR)(L1SR)	1646		346	
(HR)(FR)(2L1SR)	1648		345	D&B
(2FR)(L2SR)	1661		339	BorD
(HR)(L2SR)(L1SR)	1663		337	D&B
(FR)(L3SR)	1666		336	B
(L4SR)	1670		334	B
(2FR)(2S1SR)	1687		326	A&C
(HR)(2S1SR)(L1SR)	1689		324	D&A
(FR)(S1SR)(S2SR)	1701		318	D&A
(HR)(3FR)(S1SR)	1756		291	AorD
(2FR)(S1SR)(L1SR)	1759		290	

(HR)(S1SR)(2L1SR)	1761	288	D
(HR)(2FR)(S2SR)	1771	283	AorD
(FR)(S2SR)(L1SR)	1773	282	D
(FR)(S1SR)(L2SR)	1773	282	D
(HR)(FR)(S3SR)	1777	280	D
(HR)(S4SR)	1781	278	D&A
(5FR)	1826	256	
(HR)(3FR)(L1SR)	1828	255	
(2FR)(2L1SR)	1831	254	C
(HR)(2FR)(L2SR)	1843	247	BorD
(FR)(L1SR)(L2SR)	1845	246	D
(HR)(FR)(L3SR)	1849	244	D
(HR)(L4SR)	1853	242	D&B
(HR)(2FR)(2S1SR)	1869	234	AorD
(FR)(2S1SR)(L1SR)	1871	233	A
(HR)(FR)(S1SR)(S2SR)	1884	227	D&A
(S1SR)(S2SR)(L1SR)	1886	226	A
(2S1SR)(L2SR)	1886	226	A
(HR)(S1SR)(S3SR)	1890	224	D&A
(HR)(2S2SR)	1899	219	D&A
(4FR)(S1SR)	1939	199	AorC
(HR)(2FR)(S1SR)(L1SR)	1941	198	D
(3FR)(S2SR)	1954	192	AorD
(HR)(FR)(S2SR)(L1SR)	1956	191	D
(HR)(FR)(S1SR)(L2SR)	1956	191	D
(2FR)(S3SR)	1960	189	AorD
(HR)(S3SR)(L1SR)	1962	188	C
(HR)(S1SR)(L3SR)	1962	188	C
(FR)(S4SR)	1963	187	A
(S5SR)	1965	186	A
(HR)(S2SR)(L2SR)	1971	183	D
(HR)(FR)(3S1SR)	1982	178	D&A
(HR)(5FR)	2009	164	
(4FR)(L1SR)	2011	163	BorC
(HR)(2FR)(2L1SR)	2013	162	BorD

(3FR)(L2SR)	2026	156	BorD
(HR)(FR)(L1SR)(L2SR)	2028	155	D&B
(2FR)(L3SR)	2032	153	BorD
(HR)(L1SR)(L3SR)	2034	152	D&B
(FR)(L4SR)	2035	151	B
(L5SR)	2037	150	B
(HR)(2L2SR)	2043	147	D&B
(3FR)(2S1SR)	2052	143	AorD
(HR)(FR)(2S1SR)(L1SR)	2054	142	D
(2S1SR)(2L1SR)	2056	141	E
(2FR)(S1SR)(S2SR)	2067	136	A&C
(HR)(S1SR)(S2SR)(L1SR)	2069	134	D&A
(HR)(2S1SR)(L2SR)	2069	134	D&A
(FR)(S1SR)(S3SR)	2072	133	D&A
(FR)(2S2SR)	2081	128	D&A
(HR)(4FR)(S1SR)	2122	108	AorC
(3FR)(S1SR)(L1SR)	2124	107	
(HR)(FR)(S1SR)(2L1SR)	2126	106	D&B
(HR)(3FR)(S2SR)	2136	101	AorD
(2FR)(S2SR)(L1SR)	2139	100	
(2FR)(S1SR)(L2SR)	2139	100	
(HR)(S2SR)(2L1SR)	2141	98	D&B
(HR)(S1SR)(L1SR)(L2SR)	2141	98	D&B
(HR)(2FR)(S3SR)	2142	97	A
(FR)(S3SR)(L1SR)	2144	97	D
(FR)(S1SR)(L3SR)	2144	97	D
(HR)(FR)(S4SR)	2146	96	D
(HR)(S5SR)	2148	95	D&A
(FR)(S2SR)(L2SR)	2153	92	D
(2FR)(3S1SR)	2165	87	D
(HR)(3S1SR)(L1SR)	2167	85	D&A
(6FR)	2192	73	
(HR)(4FR)(L1SR)	2194	72	
(3FR)(2L1SR)	2196	71	C
(HR)(FR)(3L1SR)	2198	70	D&B

(HR)(3FR)(L2SR)	2208	65	BorD
(2FR)(L1SR)(L2SR)	2211	64	B&C
(HR)(2FR)(L3SR)	2214	62	BorD
(FR)(L1SR)(L3SR)	2216	61	D&B
(HR)(FR)(L4SR)	2218	60	D
(HR)(L5SR)	2220	59	D&B
(FR)(2L2SR)	2225	56	D&B
(HR)(3FR)(2S1SR)	2234	52	AorD
(2FR)(2S1SR)(L1SR)	2237	51	AorD
(HR)(2S1SR)(2L1SR)	2239	49	D
(HR)(2FR)(S1SR)(S2SR)	2249	44	AorD
(FR)(S1SR)(S2SR)(L1SR)	2251	43	A
(FR)(2S1SR)(L2SR)	2251	43	A
(HR)(FR)(S1SR)(S3SR)	2255	41	D&A
(S1SR)(S3SR)(L1SR)	2257	40	A
(2S1SR)(L3SR)	2257	40	A
(HR)(S1SR)(S4SR)	2259	39	D&A
(HR)(FR)(2S2SR	2264	37	D&A
(HR)(S2SR)(S3SR)	2270	34	D&A
(5FR)(S1SR)	2304	17	
(HR)(3FR)(S1SR)(L1SR)	2306	16	
(2FR)(S1SR)(2L1SR)	2309	15	BorD
(4FR)(S2SR)	2319	9	
(HR)(2FR)(S2SR)(L1SR)	2321	8	
(HR)(2FR)(S1SR)(L2SR	2321	8	
(FR)(S2SR)(2L1SR)	2323	7	B
(FR)(S1SR)(L1SR)(L2SR)	2323	7	B
(3FR)(S3SR)	2325	6	
(HR)(FR)(S3SR)(L2SR)	2327	5	D
(HR)(FR)(S1SR)(L3SR)	2327	5	D
(2FR)(S4SR)	2329	5	AorD
(S3SR)(2L1SR)	2329	4	B
(S1SR)(L1SR)(L3SR)	2329	4	B
(HR)(S4SR)(L1SR)	2331	3	D
(HR)(S1SR)(L4SR)	2331	3	D

(FR)(S5SR)	2330	4	A
(S6SR)	2332	3	A
(HR)(FR)(S2SR)(L2SR)	2336	1	D

group of direct returns taken r at a time must be equal to the corresponding number in Table C-1 in the column $n = 4$.

The next step is to eliminate all combinations which contain two or more half revolution returns (2HR). In these combinations of direct returns, each pair of half revolution returns may be replaced by one full revolution return without changing the length of time for the combination and yet allowing greater flexibility in the turn angles at the planet.

The third step is the elimination of combinations which are, in general, impossible for any finite hyperbolic excess speed. For instance, the combination of two short symmetric returns ((2S1SR) or (2L1SR) or (3S1SR) etc.) cannot be expected to work in any realistic case because the turn angle between the two symmetric returns is very close to 180°. Hence, such combinations are to be eliminated from the list.

Fourthly, times must be assigned to the different combinations of direct returns. The times for the half revolution return and the full revolution return are, quite obviously, one half year and one year respectively. The times for the symmetric returns are a problem, because the length of time for a symmetric return varies with the hyperbolic excess speed. The length of time for a symmetric return is arbitrarily assigned on the basis of assuming the hyperbolic excess speed to be equal to 0.3 Earth Mean Orbital Speed units (EMOS). With the times selected for the individual direct return trajectory segments, times are

APPENDIX E

CIRCULAR COPLANAR EARTH-MARS PERIODIC ORBITS

This appendix consists of a list of the basic arrangements for all of the Earth-Mars periodic orbits which work or almost work in the circular coplanar case. The solar system model taken for this purpose is a circular coplanar one with correct values for semimajor axis and period (solar system Model I.B.). One could, in a few cases, create slightly different periodic orbits by changing slightly the order of some of the direct return orbits; however, in these cases, the slightly different periodic orbit would not vary in encounter dates from the original one. There may be some other working circular coplanar periodic orbits which are basically different and which are not included in the list; but the author feels that it is unlikely that he missed any reasonable ones in the regions which were searched.

The numbering system for the periodic orbits is an attempt to be logical. The "M" in each case stands for the fact that the periodic orbit goes to Mars as well as to Earth. The second digit stands for the number of synodic periods of Earth and Mars required for the circular coplanar periodic orbit to repeat. In this number of synodic periods, in each case, a vehicle following the periodic orbit will make two round trips to Mars and back to Earth. The remaining number after the hyphen is essentially arbitrary and simply numbers the periodic orbits within the given group.

An explanation is necessary for the meaning of the numbers and symbols listed for each periodic orbit. The first column indicates the planet encountered. "E" stands for Earth and "M" stands for Mars. In each case, only a simple flyby of Mars occurs; but usually several direct return trajectories occur at Earth. In order to keep the list fairly short, only the encounters at Earth are indicated which occur immediately after and immediately before the encounter with Mars. The second column gives the dates of encounter rounded to the nearest day corresponding to those which would occur if the solar system did correspond to the circular coplanar Model I.B.; the dates are listed as Julian Date minus 2440000. The dates listed are close to those listed as the "a" version (M4-1a, M5-1a, M5-2a, etc.) of the periodic orbits listed in Appendix F. These dates, or these dates plus an integer number of synodic periods, are the basic starting points in the search for eccentric inclined periodic orbits (solar system Model III.). The third column in each case gives two types of information. It gives the scheme of direct return orbits for each series of direct returns at Earth, using the symbols of Appendix D. It also gives the dates (rounded to the nearest day) of the first

Earth-Mars-Earth round trip relative to the date of opposition. For the second Earth-Mars-Earth round trip contained in each periodic orbit, the dates relative to that opposition date will be the negatives of the dates given for the first round trip segment in each case. The fourth column gives the hyperbolic excess speed at each encounter in EMOS. Only three speeds need to be listed, because the speeds at both Mars encounters are the same by symmetry and because a series of direct return trajectories at a planet in a circular orbit will result in the same hyperbolic excess speed at each encounter at that planet. The fourth column gives the planetary passing distances in units of the radius of the planet encountered. In each case, all of the <u>different</u> planetary passing distances are listed.

The comments associated with each periodic orbit listing indicate the possibilities of the existence of an indefinitely long series of flybys according to the given scheme.

planet	date	direct returns & date	speed	passing distance
(1)	(2)	(3)	(4)	(5)
M4-1				
E	546	(S1SR)		
E	1030	-148	0.257	1.54
M	1194	+16	0.314	3.77
E	1800	+622	0.181	1.30
E	2531	(3FR)		
M	3502			
E	3666	The "a" version exists for the best approximations used.		
M5-1				
E	448	(4FR)		
E	1901	-48	0.249	1.78
M	2008	+49	0.316	7.3
E	2580	+622	0.211	1.55
E	3676	(3FR)		
M	4249			
E	4348	All versions exist for the best approximations used.		

M5-2

E	539	(FR)(HR)(2FR)		1.42
E	1818	-141	0.245	2.06
M	1978	+20	0.314	4.79
E	2580	+622	0.183	1.37
E	3676	(3FR)		
M	4278			
E	4439	All versions exist for the best approximations used.		

M5-3

E	574	(2FR)(S1SR)		2.21, ∞, 8.95
E	1783	-175	0.302	1.54
M	1958	-0	0.316	2.00
E	2580	+622	0.175	0.94
E	3676	(3FR)		
M	4298			
E	4473	The "a" version does not exist, at least in the region of the circular coplanar solution. Discussed in Chapter 2.		

M5-4

E	566	(FR)(S2SR)		2.19, 18.0
E	1791	-168	0.289	1.79
M	1963	+5	0.316	2.33
E	2580	+622	0.176	1.06
E	3676	(3FR)		
M	4293			
E	4466	Existence can neither be confirmed nor denied because of the lack of convergence near 180°.		

M5-5

E	563	(S3SR)		
E	1794	-164	0.284	1.89
M	1965	+7	0.315	2.49
E	2580	+622	0.177	1.10
E	3676	(3FR)		
M	4292			
E	4462	The "a" version, at least, does not exist, because it intersects Mars twice in 32 years.		

M5-6

E	451	(FR)(HR)(FR)(S1SR)		1.46, 2.03
E	1906	-52	0.237	2.70
M	2007	+49	0.316	11.0
E	2580	+622	0.210	1.55
E	3676	(3FR)		
M	4250			
E	4350	The "a" version does not exist, because it intersects Earth.		

M6-1

E	473	(6FR)		1.27, 4.90
E	2664	-74	0.180	1.27
M	2782	+43	0.315	14.
E	3360	+622	0.204	1.56
E	4456	(3FR)		
M	5034			
E	5152	All versions exist for the best approximations used.		

M6-2

E	534	(S1SR)(3FR)(S1SR)		1.52, 2.05
E	2603	-136	0.237	1.52
M	2760	+22	0.314	5.84
E	3360	+622	0.185	1.41
E	4456	(3FR)		
M	5056			
E	5214	The "a" version exists for the best approximations used.		

M6-3

E	479	(2FR)(HR)(2FR)(L1SR)		.998, 3.74, 6.55
E	2657	-81	0.175	1.22
M	2780	+41	0.315	12.5
E	3360	+622	0.202	1.56
E	4456	(3FR)		
M	5036			
E	5158	The "a" version exists for the best approximations used, although rearrangement of the direct returns next to the short transfer to Mars is occasionally necessary.		

M6-4

E	500	(FR)(HR)(FR)(S1SR)(2FR)		1.42, 3.32
E	2636	-102	0.186	24., ∞, 1.47
M	2774	+36	0.315	33.
E	3360	+622	0.196	1.54
E	4456	(3FR)		
M	5042			
E	5180	The "a" version exists for solar system Model II.		

M6-5

E	500	(2FR)(HR)(2FR)(S1SR)		1.47, 3.33, 5.83
E	2636	-102	0.186	1.13
M	2774	+36	0.315	33.
E	3360	+622	0.196	1.54
E	4456	(3FR)		
M	5042			
E	5180	The "a" version exists for solar system Model II.		

M6-6

E	499	(S2SR)(L1SR)(2FR)		1.24, 15.7
E	2637	-101	0.185	1.32
M	2774	+36	0.314	30.
E	3360	+622	0.196	1.54
E	4456	(3FR)		
M	5042			
E	5179	The "a" version exists for solar system Model II.		

M6-7

E	499	(S1SR)(L2SR)(2FR)		1.11, 15.2
E	2637	-101	0.185	1.32
M	2774	+36	0.314	30.
E	3360.	+622	0.196	1.54
E	4456.	(3FR)		
M	5042			
E	5179	The "a" version exists for solar system Model II. Convergence was obtained only by using most of the encounter dates from M6-6 above.		

M6-8

E	506	(FR)(S1SR)(L1SR)(2FR)		1.63, 22., 10.8
E	2630	-108	0.193	1.35
M	2772	+34	0.314	∞
E	3360	+622	0.194	1.53
E	4456	(3FR)		
M	5045			
E	5186	The "a" version apparently does not exist for solar system Model II; behavior was exhibited similar to that of periodic orbit M5-3.		

M6-9

E	506	(S1SR)(L1SR)(3FR)		1.25, 10.8
E	2630	-108	0.193	1.89, 1.89, 1.62
M	2772	+34	0.314	∞
E	3360	+622	0.194	1.53
E	4456	(3FR)		
M	5045			
E	5186	The "a" version apparently does not exist, because of the similarity to M6-8 above.		

M6-10

E	486	(L1SR)(3FR)(L1SR)		0.92, 3.71
E	2651	-87	0.175	0.92
M	2778	+40	0.315	13.5
E	3360	+622	0.201	1.55
E	4456	(3FR)		
M	5038			
E	5165	The "a" version does not exist, because it intersects Earth twice in 64 years.		

<u>M6-11</u>

E	564	(3FR)(HR)(2FR)		2.18, 2.52
E	2573	-166	0.286	1.90
M	2744	+6	0.315	2.42
E	3360	+622	0.176	1.09
E	4456	(3FR)		
M	5072			
E	5243			

APPENDIX F

ECCENTRIC INCLINED EARTH-MARS PERIODIC ORBITS

This appendix contains lists of encounter dates, hyperbolic excess speeds, and passing distances for eleven different periodic orbits. These periodic orbits are the ones known as M4-1a, all five versions of M5-1, and all five versions of M5-2. Each of these periodic orbits is listed for the length of time required for it to approximately repeat in the eccentric inclined case. The solar system model used for the creation of these lists is solar system Model III.B.

An explanation of the different columns of information is helpful. The first column lists the planet encountered at each encounter; all of the planetary encounters are listed. "E" stands for Earth and "M" stands for Mars. The second column in each list gives the Julian Date of each encounter rounded to the nearest day. Only the last five digits of the Julian Date are listed, and the first digit listed is separated from the other four by a hyphen. The third column gives the hyperbolic excess speed in EMOS at each planetary encounter. Columns four and five give respectively the turn angle in degrees and the planetocentric passing distance in units of the local planetary radii for each planetary encounter. Column six gives the type of trajectory which is to follow each encounter. The symbols are the same as those used earlier with the addition of "IP" to stand for an interplanetary trajectory.

The series of full revolution returns in each case is optimal in that the minimum passing distance for the series is maximized. However, the combinations of full revolution returns with a half revolution return in all of the versions of periodic orbit M5-2 are not optimally arranged. The minimum passing distance could be increased in several instances by reordering the FR and HR and/or by changing the side of the ecliptic plane on which the HR is chosen.

The lists of trajectories follow.

Periodic Orbit M4-1a

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	4-0538	0.249	48.8	1.63	S1SR
E	4-1023	0.249	49.1	1.60	IP
M	4-1164	0.349	7.5	1.71	IP
E	4-1806	0.178	83.0	1.14	FR
E	4-2172	0.178	10.9	21.3	FR
E	4-2537	0.178	10.9	21.3	FR
E	4-2902	0.178	82.5	1.16	IP
M	4-3495	0.353	6.1	2.10	IP
E	4-3672	0.254	55.1	1.28	S1SR
E	4-4156	0.253	56.2	1.24	IP
M	4-4340	0.321	6.1	2.51	IP
E	4-4912	0.189	67.6	1.57	FR
E	4-5277	0.189	62.8	1.81	FR
E	4-5643	0.189	62.8	1.81	FR
E	4-6008	0.189	68.3	1.54	IP
M	4-6635	0.282	10.6	1.80	IP
E	4-6784	0.270	45.2	1.55	S1SR
E	4-7266	0.271	44.7	1.58	IP
M	4-7414	0.295	9.8	1.81	IP
E	4-8046	0.188	72.2	1.39	FR
E	4-8411	0.188	58.8	2.07	FR
E	4-8776	0.188	58.8	2.07	FR
E	4-9142	0.188	71.2	1.43	IP
M	4-9718	0.326	5.9	2.55	IP
E	4-9900	0.251	57.4	1.22	S1SR
E	5-0385	0.250	53.5	1.38	IP
M	5-0563	0.348	5.0	2.66	IP
E	5-1149	0.178	77.3	1.35	FR
E	5-1514	0.178	6.3	38.7	FR
E	5-1879	0.178	6.3	38.7	FR
E	5-2244	0.178	78.4	1.30	IP
M	5-2896	0.347	11.2	1.12	IP
E	5-3044	0.269	44.8	1.59	S1SR
E	5-3527	0.270	46.8	1.48	IP
M	5-3683	0.252	5.3	4.73	IP
E	5-4287	0.211	60.3	1.58	FR
E	5-4652	0.211	60.3	1.58	FR
E	5-5017	0.211	60.3	1.58	FR
E	5-5382	0.211	60.3	1.58	IP
M	5-5945	0.301	4.6	3.82	IP

E	5-6130	0.252	57.8	1.19	S1SR
E	5-6614	0.252	50.8	1.48	IP
M	5-6785	0.376	3.6	3.17	IP
E	5-7399	0.179	76.1	1.37	FR
E	5-7764	0.179	75.1	1.41	FR
E	5-8130	0.179	75.1	1.41	FR
E	5-8495	0.179	76.7	1.35	IP
M	5-9141	0.392	6.2	1.66	IP
E	5-9293	0.266	45.3	1.60	S1SR
E	5-9775	0.265	51.1	1.32	IP
M	5-9946	0.247	5.0	5.29	IP
E	6-0524	0.226	58.2	1.47	FR
E	6-0889	0.226	58.2	1.47	FR
E	6-1254	0.226	58.2	1.47	FR
E	6-1619	0.226	58.2	1.47	IP
M	6-2178	0.278	3.7	5.64	IP
E	6-2362	0.257	56.0	1.21	S1SR
E	6-2846	0.256	48.8	1.54	IP
M	6-3009	0.398	2.9	3.53	IP
E	6-3639	0.189	69.9	1.47	FR
E	6-4004	0.189	69.9	1.47	FR
E	6-4369	0.189	69.9	1.47	FR
E	6-4735	0.189	69.9	1.47	IP
M	6-5374	0.405	3.5	2.82	IP
E	6-5532	0.265	47.3	1.50	S1SR
E	6-6015	0.266	55.5	1.15	IP
M	6-6196	0.261	5.8	4.01	IP
E	6-6759	0.229	57.2	1.47	FR
E	6-7124	0.229	57.2	1.47	FR
E	6-7489	0.229	57.2	1.47	FR
E	6-7854	0.229	57.2	1.47	IP
M	6-8420	0.256	3.4	7.36	IP
E	6-8598	0.261	52.4	1.32	S1SR
E	6-9082	0.260	47.3	1.57	IP
M	6-9237	0.404	4.8	2.07	IP
E	6-9878	0.188	70.1	1.48	FR
E	7-0243	0.188	70.1	1.48	FR
E	7-0608	0.188	70.1	1.48	FR
E	7-0973	0.188	70.1	1.48	IP
M	7-1600	0.393	4.4	2.35	IP
E	7-1766	0.262	50.2	1.40	S1SR
E	7-2249	0.262	57.6	1.11	IP
M	7-2434	0.283	5.9	3.38	IP
E	7-2993	0.222	56.6	1.60	FR
E	7-3358	0.222	56.6	1.60	FR
E	7-3724	0.222	56.6	1.60	FR
E	7-4089	0.222	56.6	1.60	IP
M	7-4673	0.245	3.6	7.63	IP

E	7-4839	0.265	48.4	1.45	S1SR
E	7-5322	0.265	45.8	1.58	IP
M	7-5472	0.382	8.0	1.33	IP
E	7-6118	0.180	79.4	1.24	FR
E	7-6483	0.180	52.8	2.73	FR
E	7-6848	0.180	52.8	2.73	FR
E	7-7213	0.180	78.8	1.26	IP
M	7-7822	0.369	5.9	1.98	IP
E	7-7996	0.257	53.4	1.32	S1SR
E	7-8479	0.256	57.2	1.17	IP
M	7-8664	0.308	6.1	2.75	IP
E	7-9229	0.203	60.9	1.67	FR
E	7-9595	0.203	60.9	1.67	FR
E	7-9960	0.203	60.9	1.67	FR
E	8-0325	0.203	60.9	1.67	IP
M	8-0936	0.258	7.4	3.18	IP
E	8-1089	0.270	45.9	1.52	S1SR
E	8-1572	0.270	44.6	1.59	IP
M	8-1719	0.330	10.7	1.31	IP
E	8-2361	0.180	81.2	1.18	FR
E	8-2727	0.180	32.7	5.59	FR
E	8-3092	0.180	32.7	5.59	FR
E	8-3457	0.180	79.8	1.23	IP
M	8-4044	0.341	6.2	2.19	IP
E	8-4224	0.252	56.3	1.25	S1SR
E	8-4708	0.251	55.2	1.30	IP
M	8-4890	0.333	5.7	2.52	IP
E	8-5468	0.183	72.2	1.47	FR
E	8-5833	0.183	58.2	2.23	FR
E	8-6198	0.183	58.2	2.23	FR
E	8-6564	0.183	72.5	1.46	IP
M	8-7194	0.301	7.8	2.20	IP
E	8-7334	0.249			

Periodic Orbit M5-1a

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	4-0447	0.188	79.2	1.14	FR
E	4-0813	0.188	56.1	2.26	FR
E	4-1178	0.188	56.1	2.26	FR
E	4-1543	0.188	56.1	2.26	FR
E	4-1908	0.188	80.7	1.09	IP
M	4-2018	0.252	3.9	6.57	IP
E	4-2592	0.239	53.0	1.53	FR
E	4-2957	0.239	53.0	1.53	FR
E	4-3322	0.239	53.0	1.53	FR
E	4-3688	0.239	53.0	1.53	IP
M	4-4227	0.328	9.4	1.53	IP
E	4-4339	0.333	37.6	1.34	FR
E	4-4704	0.333	37.6	1.34	FR
E	4-5069	0.333	37.6	1.34	FR
E	4-5435	0.333	37.6	1.34	FR
E	4-5800	0.333	37.6	1.34	IP
M	4-5881	0.390	9.6	1.06	IP
E	4-6480	0.171	80.8	1.31	FR
E	4-7245	0.171	80.8	1.31	FR
E	4-7610	0.171	80.8	1.31	FR
E	4-7975	0.171	80.8	1.31	IP
M	4-8176	0.389	9.0	1.13	IP
E	4-8256	0.326	38.2	1.37	FR
E	4-8621	0.326	38.2	1.37	FR
E	4-8986	0.326	38.2	1.37	FR
E	4-9351	0.326	38.2	1.37	FR
E	4-9717	0.326	38.2	1.37	IP
M	4-9830	0.322	9.2	1.63	IP
E	5-0368	0.240	51.8	1.59	FR
E	5-0734	0.240	51.8	1.59	FR
E	5-1099	0.240	51.8	1.59	FR
E	5-1464	0.240	51.8	1.59	IP
M	5-2035	0.249	4.1	6.41	IP
E	5-2147	0.188			

Periodic Orbit M5-1b

planet	date	speed	turn angle	passing distance	trajectory type
(1)	(2)	(3)	(4)	(5)	(6)
E	4-1233	0.208	55.2	1.90	FR
E	4-1598	0.208	48.2	2.37	FR
E	4-1963	0.208	48.2	2.37	FR
E	4-2329	0.208	48.2	2.37	FR
E	4-2694	0.208	62.6	1.52	IP
M	4-2821	0.261	9.8	2.32	IP
E	4-3364	0.260	54.0	1.26	FR
E	4-3730	0.260	54.0	1.26	FR
E	4-4095	0.260	54.0	1.26	FR
E	4-4460	0.260	54.0	1.26	IP
M	4-4995	0.286	9.2	2.04	IP
E	4-5120	0.248	46.1	1.78	FR
E	4-5485	0.248	44.7	1.87	FR
E	4-5850	0.248	44.7	1.87	FR
E	4-6215	0.248	44.7	1.87	FR
E	4-6581	0.248	44.7	1.87	IP
M	4-6656	0.350	5.7	2.27	IP
E	4-7267	0.185	69.5	1.56	FR
E	4-7632	0.185	67.5	1.66	FR
E	4-7998	0.185	67.5	1.66	FR
E	4-8363	0.185	69.0	1.58	IP
M	4-8937	0.377	10.3	1.04	IP
E	4-9029	0.370	37.9	1.07	FR
E	4-9394	0.370	37.9	1.07	FR
E	4-9759	0.370	37.9	1.07	FR
E	5-0125	0.370	37.9	1.07	FR
E	5-0490	0.370	37.9	1.07	IP
M	5-0589	0.362	9.7	1.22	IP
E	5-1148	0.198	60.7	1.76	FR
E	5-1513	0.198	60.7	1.76	FR
E	5-1879	0.198	60.7	1.76	FR
E	5-2244	0.198	60.7	1.76	IP
M	5-2851	0.308	4.0	4.32	IP
E	5-2932	0.208			

Periodic Orbit M5-1c

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	4-0235	0.174	75.8	1.46	FR
E	4-0600	0.174	75.4	1.48	FR
E	4-0966	0.174	75.4	1.48	FR
E	4-1331	0.174	75.4	1.48	IP
M	4-1942	0.370	6.8	1.69	IP
E	4-2016	0.274	40.6	1.77	FR
E	4-2382	0.274	40.6	1.77	FR
E	4-2747	0.274	40.6	1.77	FR
E	4-3112	0.274	40.6	1.77	FR
E	4-3477	0.274	40.6	1.77	IP
M	4-3599	0.296	10.5	1.65	IP
E	4-4135	0.255	51.2	1.43	FR
E	4-4500	0.255	51.2	1.43	FR
E	4-4865	0.255	51.2	1.43	FR
E	4-5231	0.255	51.2	1.43	IP
M	4-4780	0.252	5.4	4.68	IP
E	4-5907	0.192	75.1	1.22	FR
E	4-6272	0.192	51.1	2.52	FR
E	4-6637	0.192	51.1	2.52	FR
E	4-7002	0.192	51.1	2.52	FR
E	4-7368	0.192	72.4	1.33	IP
M	4-7462	0.268	3.3	6.86	IP
E	4-8052	0.225	54.8	1.64	FR
E	4-8418	0.225	54.8	1.64	FR
E	4-8783	0.225	54.8	1.64	FR
E	4-9148	0.225	54.8	1.64	IP
M	4-9694	0.343	9.5	1.37	IP
E	4-9801	0.355	37.2	1.20	FR
E	5-0166	0.355	37.2	1.20	FR
E	5-0531	0.355	37.2	1.20	FR
E	5-0896	0.355	37.2	1.20	FR
E	5-1262	0.355	37.2	1.20	IP
M	5-1347	0.389	10.1	1.00	IP
E	5-1934	0.174			

Periodic Orbit M5-1d

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	4-1025	0.176	76.9	1.39	FR
E	4-1390	0.176	69.5	1.73	FR
E	4-1756	0.176	69.5	1.73	FR
E	4-2121	0.176	75.5	1.45	IP
M	4-2710	0.390	9.8	1.03	IP
E	4-2794	0.351	37.9	1.20	FR
E	4-3159	0.351	37.9	1.20	FR
E	4-3525	0.351	37.9	1.20	FR
E	4-3890	0.351	37.9	1.20	FR
E	4-4255	0.351	37.9	1.20	IP
M	4-4363	0.338	8.8	1.53	IP
E	4-4907	0.226	53.7	1.69	FR
E	4-5272	0.226	53.7	1.69	FR
E	4-5637	0.226	53.7	1.69	FR
E	4-6003	0.226	53.7	1.69	IP
M	4-6589	0.261	4.2	5.57	IP
E	4-6688	0.190	73.2	1.32	FR
E	4-7053	0.190	45.2	3.13	FR
E	4-7418	0.190	45.2	3.13	FR
E	4-7784	0.190	45.2	3.13	FR
E	4-8149	0.190	76.4	1.20	IP
M	4-8274	0.252	7.1	3.50	IP
E	4-8827	0.254	53.0	1.36	FR
E	4-9192	0.254	53.0	1.36	FR
E	4-9558	0.254	53.0	1.36	FR
E	4-9923	0.254	53.0	1.36	IP
M	5-0457	0.302	9.4	1.79	IP
E	5-0578	0.284	39.9	1.69	FR
E	5-0943	0.284	39.9	1.69	FR
E	5-1308	0.284	39.9	1.69	FR
E	5-1673	0.284	39.9	1.69	FR
E	5-2039	0.284	39.9	1.69	IP
M	5-2114	0.374	7.6	1.48	IP
E	5-2724	0.176			

Periodic Orbit M5-1e

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	3-9662	0.214	59.9	1.55	FR
E	4-0027	0.214	50.7	2.07	FR
E	4-0392	0.214	50.7	2.07	FR
E	4-0757	0.214	50.7	2.07	FR
E	4-1123	0.214	53.4	1.90	IP
M	4-1202	0.318	3.8	4.27	IP
E	4-1810	0.198	62.2	1.69	FR
E	4-2176	0.198	62.2	1.69	FR
E	4-2541	0.198	62.2	1.69	FR
E	4-2906	0.198	62.2	1.69	IP
M	4-3468	0.365	10.1	1.14	IP
E	4-3566	0.371	37.4	1.09	FR
E	4-3931	0.371	37.4	1.09	FR
E	4-4296	0.371	37.4	1.09	FR
E	4-4662	0.371	37.4	1.09	FR
E	4-5027	0.371	37.4	1.09	IP
M	4-5121	0.374	10.2	1.07	IP
E	4-5691	0.184	67.9	1.65	FR
E	4-6056	0.184	67.9	1.65	FR
E	4-6422	0.184	67.9	1.65	FR
E	4-6787	0.184	67.9	1.65	IP
M	4-7399	0.342	5.1	2.67	IP
E	4-7474	0.239	45.1	1.99	FR
E	4-7839	0.239	45.1	1.99	FR
E	4-8205	0.239	45.1	1.99	FR
E	4-8570	0.239	45.1	1.99	FR
E	4-8935	0.239	47.6	1.84	IP
M	4-9061	0.281	11.1	1.72	IP
E	4-9598	0.260	52.0	1.34	FR
E	4-9963	0.260	52.0	1.34	FR
E	5-0329	0.260	52.0	1.34	FR
E	5-0694	0.260	52.0	1.34	IP
M	5-1234	0.264	7.3	3.07	IP
E	5-1360	0.215			

Periodic Orbit M5-2a

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	4-0539	0.281	45.7	1.42	FR
E	4-0904	0.281	45.7	1.42	HR
E	4-1083	0.280	35.3	2.07	FR
E	4-1449	0.280	35.3	2.07	FR
E	4-1814	0.280	35.3	2.07	IP
M	4-1968	0.266	8.3	2.67	IP
E	4-2586	0.202	63.3	1.56	FR
E	4-2951	0.202	63.2	1.57	FR
E	4-3316	0.202	63.2	1.57	FR
E	4-3681	0.202	63.2	1.57	IP
M	4-4245	0.312	1.8	9.38	IP
E	4-4421	0.223	67.2	1.16	FR
E	4-4786	0.223	67.2	1.16	HR
E	4-4969	0.230	42.6	2.34	FR
E	4-5335	0.230	42.6	2.34	FR
E	4-5700	0.230	46.0	2.08	IP
M	4-5853	0.398	2.4	4.33	IP
E	4-6479	0.179	76.5	1.36	FR
E	4-6844	0.179	76.5	1.36	FR
E	4-7209	0.179	76.5	1.36	FR
E	4-7575	0.179	76.5	1.36	IP
M	4-8203	0.399	1.9	5.33	IP
E	4-8352	0.226	57.4	1.50	FR
E	4-8717	0.226	57.4	1.50	HR
E	4-8903	0.226	51.3	1.81	FR
E	4-9269	0.226	51.3	1.81	FR
E	4-9634	0.226	51.3	1.81	IP
M	4-9812	0.307	2.1	8.37	IP
E	5-0374	0.204	60.6	1.67	FR
E	5-0739	0.204	60.6	1.67	FR
E	5-1104	0.204	60.6	1.67	FR
E	5-1469	0.204	60.6	1.67	IP
M	5-2082	0.258	9.0	2.59	IP
E	5-2238	0.281			

Periodic Orbit M5-2b

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	4-1340	0.262	43.8	1.73	FR
E	4-1705	0.262	43.8	1.73	HR
E	4-1889	0.262	44.7	1.69	FR
E	4-2255	0.262	44.7	1.69	FR
E	4-2620	0.262	44.7	1.69	IP
M	4-2793	0.250	4.2	6.10	IP
E	4-3364	0.230	58.0	1.42	FR
E	4-3729	0.230	58.0	1.42	FR
E	4-4094	0.230	58.0	1.42	FR
E	4-4459	0.230	58.0	1.42	IP
M	4-5017	0.272	1.8	12.6	IP
E	4-5196	0.245	62.9	1.08	FR
E	4-5562	0.245	62.9	1.08	HR
E	4-5742	0.252	37.7	2.34	FR
E	4-6107	0.252	37.7	2.34	FR
E	4-6472	0.252	37.7	2.34	IP
M	4-6618	0.380	6.5	1.67	IP
E	4-7258	0.176	83.8	1.13	FR
E	4-7623	0.176	37.8	4.76	FR
E	4-7988	0.176	37.8	4.76	FR
E	4-8353	0.176	81.7	1.21	IP
M	4-8957	0.372	2.4	4.93	IP
E	4-9118	0.213	54.8	1.83	FR
E	4-9483	0.213	52.2	1.99	FR
E	4-9848	0.213	52.2	1.99	HR
E	5-0034	0.217	63.1	1.37	FR
E	5-0399	0.217	63.1	1.37	IP
M	5-0569	0.350	2.4	5.66	IP
E	5-1154	0.174	80.6	1.27	FR
E	5-1519	0.174	28.3	7.21	FR
E	5-1884	0.174	28.3	7.21	FR
E	5-2249	0.174	83.3	1.18	IP
M	5-2894	0.343	10.0	1.31	IP
E	5-3039	0.262			

Periodic Orbit M5-2c

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	4-0240	0.176	78.8	1.31	FR
E	4-0605	0.176	78.7	1.31	FR
E	4-0970	0.176	78.7	1.31	FR
E	4-1336	0.176	80.4	1.25	IP
M	4-1974	0.392	3.9	2.71	IP
E	4-2119	0.240	51.4	1.61	FR
E	4-2484	0.240	51.4	1.61	HR
E	4-2671	0.237	49.9	1.72	FR
E	4-3036	0.237	49.9	1.72	FR
E	4-3401	0.237	49.9	1.72	IP
M	4-3580	0.282	1.9	11.1	IP
E	4-4137	0.222	56.5	1.60	FR
E	4-4503	0.222	56.5	1.60	FR
E	4-4868	0.222	56.5	1.60	FR
E	4-5233	0.222	56.5	1.60	IP
M	4-5818	0.246	4.6	5.83	IP
E	4-5988	0.277	52.6	1.16	FR
E	4-6353	0.277	52.6	1.16	HR
E	4-6532	0.276	33.4	2.31	FR
E	4-6897	0.276	33.4	2.31	FR
E	4-7263	0.276	33.4	2.31	IP
M	4-7411	0.297	10.6	1.63	IP
E	4-8044	0.189	71.8	1.40	FR
E	4-8409	0.189	59.9	1.99	FR
E	4-8775	0.189	59.9	1.99	FR
E	4-9140	0.189	69.9	1.48	IP
M	4-9712	0.328	2.2	6.91	IP
E	4-9885	0.217	67.5	1.21	FR
E	5-0250	0.217	67.5	1.21	HR
E	5-0434	0.224	45.0	2.28	FR
E	5-0799	0.224	45.0	2.28	FR
E	5-1164	0.224	48.5	2.03	IP
M	5-1322	0.389	1.9	5.89	IP
E	5-1939	0.176			

Periodic Orbit M5-2d

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	4-1019	0.179	78.1	1.29	FR
E	4-1385	0.179	69.9	1.65	FR
E	4-1750	0.179	69.9	1.65	FR
E	4-2115	0.179	77.8	1.31	IP
M	4-2734	0.392	1.9	5.58	IP
E	4-2887	0.219	60.8	1.44	FR
E	4-3252	0.219	60.8	1.44	HR
E	4-3438	0.221	51.7	1.88	FR
E	4-3804	0.221	51.7	1.88	FR
E	4-4169	0.221	51.7	1.88	IP
M	4-4345	0.323	2.5	6.31	IP
E	4-4914	0.189	67.5	1.59	FR
E	4-5280	0.189	62.5	1.84	FR
E	4-5645	0.189	62.5	1.84	FR
E	4-6010	0.189	69.1	1.52	IP
M	4-6641	0.287	13.6	1.32	IP
E	4-6793	0.283	41.4	1.61	FR
E	4-7158	0.283	41.4	1.61	HR
E	4-7340	0.274	40.9	1.76	FR
E	4-7705	0.274	40.9	1.76	FR
E	4-8070	0.274	40.9	1.76	IP
M	4-8237	0.246	5.3	4.96	IP
E	4-8825	0.222	58.1	1.52	FR
E	4-9190	0.222	58.1	1.52	FR
E	4-9555	0.222	58.1	1.52	FR
E	4-9921	0.222	58.1	1.52	IP
M	5-0477	0.287	1.4	14.2	IP
E	5-0657	0.235	65.1	1.10	FR
E	5-1022	0.235	65.1	1.10	HR
E	5-1203	0.243	39.4	2.36	FR
E	5-1568	0.243	39.4	2.36	FR
E	5-1933	0.243	41.1	2.22	IP
M	5-2081	0.396	4.5	2.27	IP
E	5-2718	0.179			

Periodic Orbit M5-2e

planet (1)	date (2)	speed (3)	turn angle (4)	passing distance (5)	trajectory type (6)
E	3-9740	0.261	57.8	1.11	FR
E	4-0105	0.261	57.8	1.11	HR
E	4-0284	0.261	35.1	2.42	FR
E	4-0649	0.261	35.1	2.42	FR
E	4-1015	0.261	35.1	2.41	IP
M	4-1160	0.354	9.0	1.39	IP
E	4-1806	0.179	81.9	1.16	FR
E	4-2172	0.179	9.9	23.5	FR
E	4-2537	0.179	9.9	23.5	FR
E	4-2902	0.179	79.0	1.27	IP
M	4-3487	0.355	2.2	5.92	IP
E	4-3653	0.212	66.5	1.29	FR
E	4-4018	0.212	66.5	1.29	HR
E	4-4203	0.218	49.1	2.10	FR
E	4-4569	0.218	49.1	2.10	FR
E	4-4934	0.218	51.1	1.97	IP
M	4-5099	0.367	2.2	5.62	IP
E	4-5701	0.172	84.2	1.17	FR
E	4-6066	0.172	35.6	5.41	FR
E	4-6432	0.172	35.6	5.41	FR
E	4-6797	0.172	86.2	1.11	IP
M	4-7436	0.374	6.5	1.76	IP
E	4-7581	0.251	37.7	2.36	FR
E	4-7946	0.251	37.7	2.36	FR
E	4-8312	0.251	37.7	2.36	HR
E	4-8497	0.246	64.8	1.02	FR
E	4-8863	0.246	64.8	1.02	IP
M	4-9041	0.268	2.2	10.4	IP
E	4-9599	0.230	56.3	1.50	FR
E	4-9964	0.230	56.3	1.50	FR
E	5-0330	0.230	56.3	1.50	FR
E	5-0695	0.230	56.3	1.50	IP
M	5-1264	0.252	3.0	8.42	IP
E	5-1439	0.261			

REFERENCES

1. Hollister, W. M., "Periodic Orbits for Interplanetary Flight," M.I.T. Experimental Astronomy Laboratory, No. RE-36, Cambridge, January 1968, (also, Journal of Spacecraft and Rockets, Vol. 6, No. 4, April 1969, pp. 366-369).

2. Menning, M. D., "Free-Fall Periodic Orbits Connecting Earth and Venus," S.M. Thesis in the Department of Aeronautics and Astronautics, M.I.T., Cambridge, July 1968.

3. Hickman, D. E., "Guidance Requirements for Periodic Orbits," S.M. Thesis in the Department of Aeronautics and Astronautics, M.I.T., Cambridge, August 1968.

4. Ross, S., "A Systematic Approach to the Study of Non-stop Interplanetary Round Trips," Interplanetary Missions Conference (9th Annual AAS Meeting), Los Angeles, January 1963.

5. VanderVeen, A. A., "Triple-Planet Ballistic Flybys of Mars and Venus," AAS/AIAA Astrodynamics Specialist Conference, Jackson, Wyoming, (AAS Paper No. 68-114), September 1968, (also, Journal of Spacraft and Rockets, Vol. 6, No. 4, April 1969, pp. 383-389).

6. Battin, R. H., Astronautical Guidance, New York, McGraw-Hill Book Company, 1964.

7. Nautical Almanac Offices of the United Kingdom and the United States of America, Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, London, Her Majesty's Stationery Office, 1961.

8. National Aeronautics and Space Administration, Planetary Flight Handbook (Space Flight Handbooks, Vol. 3), Office of Scientific and Technical Information, Washington, D.C., 1963.

9. Lockheed Missiles and Space Company, Final Report: A Study of Interplanetary Transportation Systems, No. 3-17-62-1, Sunnyvale, June 1962.

10. Crocco, G. A., "One Year Exploration Trip Earth-Mars-Venus-Earth," Proceedings of the VIIth International Astronomical Congress, Rome, September 1956, pp. 201-252.

11. Battin, R. H., "The Determination of Round-Trip Reconnaissance Trajectories," M.I.T. Instrumentation Laboratory Report No. R-219, Cambridge, January 1959, (also, Journal of the Aero-Space Sciences, Vol. 26, No. 9, September 1959).

12. Hollister, W. M., "The Mission for a Manned Expedition to Mars," Sc.D. Thesis in Instrumentation, M.I.T., May 1963.

13. Minovitch, M. A., "The Determination and Characteristics of Ballistic Interplanetary Trajectories Under the Influence of Multiple Planetary Attractions," Jet Propulsion Laboratory, California Institute of Technology, Technical Report No. 32-464, Pasadena, 1963.

14. Hollister, W. M., "Mars Transfer Via Venus," AIAA/ION Astrodynamics Guidance and Control Conference, (AIAA Paper No. 64-647), Los Angeles, August, 1964.

15. Sohn, R. L., "Venus Swingby Mode for Manned Mars Missions," Journal of Spacecraft and Rockets, Vol. 1, No. 5, September-October 1964, pp. 565-567.

16. Broucke, R., "Periodic Moon Orbiter Trajectories in a Restricted Three-Body Approximation," Space Programs Summary No. 37-30, Volume IV, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, December 1964.

17. Sturms, F. M., Jr., and Cutting, E., "Trajectory Analysis of a 1970 Mission to Mercury Via a Close Encounter with Venus," AIAA 2nd Aerospace Sciences Meeting, (AIAA Paper No. 65-90) New York, January, 1965.

18. Gillespie, R. W. and Ross, S., "The Venus Swingby Mission Mode and Its Role in the Manned Exploration of Mars," AIAA 3rd Aerospace Sciences Meeting, (AIAA Paper No. 66-37), New York, January 1966.

19. Hollister, W. M. and Prussing, J. E., "Optimum Transfer to Mars Via Venus," Astronautica Acta, Vol. 12, No. 2, March-April 1966, pp. 169-179.

20. Breakwell, J. V. and Perko, L. M., "Matched Asymptotic Expansions, Patched Conics and the Computation of Interplanetary Trajectories," AIAA/ION Astrodynamics Specialist Conference, (AIAA Paper No. 65-689) Monterey, September 1965.

21. Breakwell, J. V., "Trajectories Launched Normal to the Ecliptic," Proceedings of the XIV International Astronautical Congress, Vol. IV, Paris, 1963, pp. 125-141.

22. Battin, R. H., "Universal Formulae for Conic Trajectory Calculations," Report R-382, Instrumentation Laboratory, M.I.T., Cambridge, Mass., September 1962.

23. Battin, R. H., "A New Solution for Lambert's Problem," Report R-618, Instrumentation Laboratory, M.I.T., Cambridge, 1968.

24. Barnes, J. G. P., "An Algorithm for Solving Nonlinear Equations Based on the Secant Method," The Computer Journal, Vol. 8, No. 1, April 1965, p. 66.

25. Powell, M. J. D., "An Efficient Method for Finding the Minimum of a Function of Several Variables Without Calculating Derivatives," The Computer Journal, Vol. 7, No. 2, July 1964, p. 155.

26. Fletcher, R. and Reeves, C. M.,"Function Minimization by Conjugate Gradients," The Computer Journal, Vol. 7, No. 2, July 1964, pp. 149-154.

27. Fletcher R. and Powell, M. J. D., "A Rapidly Convergent Descent Method for Minimization," The Computer Journal, Vol. 6, No. 2, July 1963, p. 163.

28. Bayliss, S. S., private communication relating to expected M.I.T. Sc.D. thesis, "Precision Targeting for Multiple Interplanetary Flyby Trajectories," January 1970.

BIOGRAPHICAL SKETCH

Charles Sherman Rall was born on 28 December 1943 in Ogden, Utah. He graduated from Peabody High School in Pittsburgh, Pennsylvania in 1961. He received the degrees of Bachelor of Science in 1965 and Master of Science in 1966, both from The Massachusetts Institute of Technology. As an undergraduate he was elected to Sigma Gamma Tau and Tau Beta Pi and, as a graduate student, to Sigma Xi. Since 1965 he has been a National Science Foundation Fellow in the Department of Aeronautics and Astronautics at M.I.T.

New Technology Appendix

After a diligent review of the work performed under this contract, no new innovation, discovery, improvement or invention was made.