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TRANSFORMATION OF A POTENTIAL FUNCTION UNDER COORDINATE ROTATIONS

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ABSTRACT

A potential function is a solution of Laplace's equation, $\nabla^2 U=0$. The operator ∇^2 is form-invariant under rotations of the coordinate system, so solutions of Laplace's equation in two coordinate systems connected by a rotation satisfy the same differential equation. Hence, they have the same series expansions, but with different values of the expansion coefficients. This paper derives the detailed connection between the new and old expansion coefficients, and it provides an example to illustrate the concepts and the use of the equations involved.

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TRANSFORMATION OF A POTENTIAL FUNCTION UNDER COORDINATE ROTATIONS

1. INTRODUCTION AND SUMMARY

A potential function is a solution of Laplace's equation, $\nabla^2 U=0$. For a gravitating body the solution is commonly written as the infinite series

$$U(r,\theta,\phi) = \frac{\mu}{r}\left\{1 + \sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell}\left(\frac{a}{r}\right)^{\ell} P_{\ell}^{m}(\cos\theta)\left[C_{\ell m}\cos m\phi + S_{\ell m}\sin m\phi\right]\right\}. \tag{1}$$

In this expression, (r,θ,ϕ) are the spherical polar coordinates of a point in space relative to a body-fixed rectangular coordinate frame which will be denoted by $\mathcal{O}$, μ is the body's constant of attraction, and a is a length usually taken as the body's mean radius. P_{ℓ}^{m} denotes the associated Legendre polynomial defined in Section 3.

Suppose the expansion coefficients $C_{\ell m}$ and $S_{\ell m}$ are known such that the series gives a precise representation of the potential for all points outside the body for which $r>a$. Then it is natural to ask the question: How is $U(r,\theta,\phi)$ transformed under a pure rotation of $\mathcal{O}$?

The purpose of this paper is to answer the question in full detail. In particular, let $\mathcal{R}(\alpha,\beta,\gamma)$ be a rotation operator which takes the original coordinate frame $\mathcal{O}$ into the new frame $\mathcal{O}'$, leaving the physical body fixed. $\alpha,\beta,$ and γ are the Euler angles of the rotation.* Then the physical point (r,θ,ϕ) assumes the new representation (r,θ',ϕ') after the rotation, and the potential expansion $U(r,\theta,\phi)$ is taken to $U'(r,\theta',\phi')$ by the rotation--that is

$$U(r,\theta,\phi) \xrightarrow[R(\alpha,\beta,\gamma)]{} U'(r,\theta',\phi'). \tag{2}$$

It will be shown that

$$U'(r,\theta',\phi') = \frac{\mu}{r}\left\{1 + \sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell}\left(\frac{a}{r}\right)^{\ell} P_{\ell}^{m}(\cos\theta')\left[D_{\ell m}\cos m\phi' + T_{\ell m}\sin m\phi'\right]\right\}, \tag{3}$$

and the dependence of $D_{\ell m}$ and $T_{\ell m}$ on $C_{\ell m}$, $S_{\ell m}$, α, β, and γ will be presented. These relations are given in (30) and (31), and an example illustrating their use is given in Appendix C. It may be noted in advance that for each ℓ, $D_{\ell m}$ and $T_{\ell m}$ are each linear combinations of all the $C_{\ell m}$ and $S_{\ell m}$ of the same ℓ only. The Euler angles appear in the coefficients of the linear combinations.

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*The rotation convention assumed in the paper is that $R(\alpha,\beta,\gamma)$ means the following sequence of rotations:

1) Rotate through α about the z-axis,

2) Rotate through β about the new y-axis,

3) Rotate through γ about the new z-axis.

A positive rotation of the (positive) x-axis toward the (positive) y-axis is understood to be such that a right-handed screw would advance along the (positive) z-axis. In other words, positive rotations are consistent with the right-hand rule.

The paper is largely devoted to arriving at (30) and (31). The approach is to generalize the form of (1) in terms of spherical harmonics (Section 3), to examine the rotational properties of the spherical harmonics (Section 4), and finally to use these properties to obtain the desired expressions for $D_{\ell m}$ and $T_{\ell m}$ (Section 5). Section 2 is merely a quick justification of (3).

2. FORM-INVARIANCE OF THE POTENTIAL

The infinite series expansion (1) is a form of the most general power series expansion of the solution of Laplace's equation,

$$\nabla^2 U=0 \tag{4}$$

The series can be made to reproduce a body's physical potential by appropriate selection of the constants $C_{\ell m}$ and $S_{\ell m}$. Since the Laplacian operator ∇^2 is an operator scalar product, then under a rotation $R(\alpha,\beta,\gamma)$ its form is preserved:

$$\nabla^2 \xrightarrow[R(\alpha,\beta,\gamma)]{} \nabla'^2. \tag{5}$$

This may be verified by performing the transformation explicitly. As a result of this property, a rotation takes (4) into

$$\nabla'^2 U'=0 \tag{6}$$

Because of the formal equivalence of (4) and (6), the equations have the same solutions. This is the justification for (3).

3. SPHERICAL HARMONIC EXPANSION OF THE POTENTIAL

It remains to relate the $D_{\ell m}$ and $T_{\ell m}$ to $C_{\ell m}$, $S_{\ell m}$, α, β, and γ. To do this it will be convenient to write the solution of Laplace's equation in a more general form, namely

$$U(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{a}{r}\right)^{\ell+1} A_{\ell m} \, Y_{\ell m}(\theta,\phi), \tag{7}$$

where U is for the moment allowed to be complex. The $Y_{\ell m}(\theta,\phi)$ are spherical harmonics defined for $-\ell \leq m \leq \ell$ by

$$Y_{\ell m}(\theta,\phi) = N_{\ell m} \, P_\ell^{\,m}(\cos\,\theta) \, e^{im\phi} \tag{8}$$

$$N_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \,. \tag{9}$$

The $P_\ell^{\,m}(x)$ are the associated Legendre polynomials, defined for $-\ell \leq m \leq \ell$ by*

$$P_\ell^{\,m}(x) = \frac{(1-x^2)^{m/2}}{2^\ell \ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell \qquad (x^2 \leq 1). \tag{10}$$

The spherical harmonics form a complete set of orthonormal functions on the unit sphere:**

$$\int_0^\pi \sin\,\theta\, d\theta \int_0^{2\pi} d\phi \; Y^+_{\ell m}(\theta,\phi) \; Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'} \, \delta_{mm'}. \tag{11}$$

They have the property

$$Y_{\ell,-m}(\theta,\phi) = (-)^m \, Y^+_{\ell m}(\theta,\phi). \tag{12}$$

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*This definition is consistent with the one recommended by the International Astronomical Union. It differs by a sign convention from the definition given by some authors. See Jahnke and Emde for a list of the first several associated Legendre polynomials.

**In the following, N^+ means the complex conjugate of N.

If the value of U is known everywhere on the sphere r=a, then the $A_{\ell m}$ may be computed as

$$A_{\ell m} = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \; U(a,\theta,\phi) \; Y^{+}_{\ell m}(\theta,\phi). \tag{13}$$

Assuming now that U is real, as a physical potential must be, it follows that

$$A_{\ell,-m} = (-)^m A^{+}_{\ell m}. \tag{14}$$

For a real potential, it can be shown that (7) is equivalent to (1). The conversion formulas relating the expansion coefficients are

$$\begin{array}{l|l} C_{\ell m} = \frac{a}{\mu} N_{\ell m} z_m \, \mathrm{Re}(A_{\ell m}) & \mathrm{Re}(A_{\ell m}) = \frac{\mu}{aN_{\ell m} z_m} C_{\ell m} \\ \hline S_{\ell m} = -\frac{a}{\mu} N_{\ell m} z_m \, \mathrm{Im}(A_{\ell m}) & \mathrm{Im}(A_{\ell m}) = -\frac{\mu}{aN_{\ell m} z_m} S_{\ell m} \end{array} \tag{15}$$

where

$$z_m = \begin{cases} 2 \text{ if } m \neq 0 \\ 1 \text{ if } m = 0. \end{cases} \tag{16}$$

Note that the coefficient C_{00} must be set to unity in order to make the potential behave like μ/r when r is very large. With (15) in hand we may now proceed with a discussion of (7), whose rotational properties are much easier to describe than those of (1).

4. EFFECT OF ROTATING THE COORDINATE FRAME

Consider what happens to the potential function $U(r,\theta,\phi) \equiv U(r,\Omega)$ when it is operated on by the rotation operator $O_{R(\alpha,\beta,\gamma)}$, where α, β, and γ are the Euler angles of a rotation denoted by R. Denoting the function obtained as

$$U'(r,\Omega) = O_{R(\alpha,\beta,\gamma)}\, U(r,\Omega), \tag{17}$$

operate on (7) to get

$$U'(r,\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{a}{r}\right)^{\ell+1} A_{\ell m}\, O_R\, Y_{\ell m}(\Omega). \tag{18}$$

Since the $Y_{\ell m}$'s form a complete set of functions, we may expand

$$O_{R(\alpha,\beta,\gamma)} Y_{\ell m}(\Omega) = \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} D^{\ell'\ell}_{m'm}(\alpha,\beta,\gamma)\, Y_{\ell' m'}(\Omega), \tag{19}$$

where the $D^{\ell'\ell}_{m'm}(\alpha,\beta,\gamma)$ are expansion coefficients which are functions of the Euler angles of the rotation. The sum over ℓ' in (19) is actually fictitious, as the following argument will show.

The spherical harmonics are solutions of the eigenvalue problem [Morse and Feshbach, p. 1464]

$$L^2\, Y_{\ell m}(\Omega) = \ell(\ell+1)\, Y_{\ell m}(\Omega) \qquad (\ell \geq 0). \tag{20}$$

In Appendix A it is shown that the operator L^2 commutes with the rotation operator O_R. Using this fact, apply (20) to (19):

$$
\begin{aligned}
L^2 O_R Y_{\ell m}(\Omega) &= O_R L^2 Y_{\ell m}(\Omega) = \ell(\ell+1)\; O_R Y_{\ell m}(\Omega) \\
&= \ell(\ell+1) \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} D^{\ell'\ell}_{m'm}\, Y_{\ell' m'}(\Omega).
\end{aligned} \tag{21}
$$

On the other hand,

$$
\begin{aligned}
L^2 O_R Y_{\ell m}(\Omega) &= \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} D^{\ell'\ell}_{m'm}\, L^2 Y_{\ell' m'}(\Omega) \\
&= \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} D^{\ell'\ell}_{m'm}\, \ell'(\ell'+1)\; Y_{\ell' m'}(\Omega).
\end{aligned} \tag{22}
$$

Because of the orthogonality property (11), the coefficients of a series expansion in spherical harmonics are unique. Thus the coefficients of the series in (21) and (22) must be identical, and only the $\ell'=\ell$ term may be present. The conclusion is that (19) may be written as

$$
O_{R(\alpha,\beta,\gamma)} Y_{\ell m}(\Omega) = \sum_{m'=-\ell}^{\ell} D^{\ell}_{m'm}(\alpha,\beta,\gamma)\; Y_{\ell m'}(\Omega). \tag{23}
$$

This important result says that under rotations the functions

$$
Y_{\ell,\ell} \qquad Y_{\ell,\ell-1} \quad \cdots \quad Y_{\ell,0} \quad \cdots \quad Y_{\ell,-\ell+1} \qquad Y_{\ell,-\ell}
$$

are mixed among themselves exclusively. Substituting (23) into (18), we finally obtain the effect of applying the rotation operator O_R to the potential U. It is

$$U'(r,\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left|\frac{a}{r}\right|^{\ell+1} A_{\ell m} \sum_{m'=-\ell}^{\ell} D^{\ell}_{m'm}(\alpha,\beta,\gamma)\, Y_{\ell m'}(\Omega)$$
$$= \sum_{\ell=0}^{\infty} \sum_{m'=-\ell}^{\ell} \left|\frac{a}{r}\right|^{\ell+1} Y_{\ell m'}(\Omega)\left[\sum_{m=-\ell}^{\ell} A_{\ell m} D^{\ell}_{m'm}(\alpha,\beta,\gamma)\right]. \tag{24}$$

After relabeling the m- and m'-sums and defining

$$B_{\ell m} = \sum_{m'=-\ell}^{\ell} A_{\ell m'} D^{\ell}_{mm'}(\alpha,\beta,\gamma) \tag{25}$$

we obtain

$$U'(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left|\frac{a}{r}\right|^{\ell+1} B_{\ell m}\, Y_{\ell m}(\theta,\phi). \tag{26}$$

If the angles at which U' is evaluated are now chosen to be θ' and ϕ' (angular coordinates of a point (θ,ϕ) expressed in the new frame O') then $U'(r,\theta',\phi')$ will have the same value as $U(r,\theta,\phi)$. This requirement is discussed in Appendix A.

The result (26) reproduces the conclusion of Section 2, that the effect of applying a coordinate rotation to a solution of Laplace's equation is to produce an expansion which has the old form but new coefficients. The new information is in (25), which shows that for each ℓ the new coefficients are linear combinations of the old coefficients of the same ℓ. The quantities $D^{\ell}_{mm'}(\alpha,\beta,\gamma)$ are presented in Appendix B.

5. FINAL FORM OF THE ROTATED EXPANSION COEFFICIENTS

With the help of (15), (25) may be rendered into a form applicable to the potential expansion of (3). The procedure is as follows:

1. Use (15) to express $D_{\ell m}$ in terms of $B_{\ell m}$.

2. Use (25) to eliminate $B_{\ell m}$ in terms of the old coefficients $A_{\ell m}$.

3. Convert the resulting sum to a sum over only non-negative values of m'.

4. Use (15) to replace $\mathrm{Re}(A_{\ell m})$ and $\mathrm{Im}(A_{\ell m})$ in favor of $C_{\ell m}$ and $S_{\ell m}$.

5. Repeat the above steps, but begin with $T_{\ell m}$ in step (1).

Following this outline we obtain

$$D_{\ell m} = z_m N_{\ell m}\left[\frac{C_{\ell 0}}{N_{\ell 0}} \mathrm{Re}(D^{\ell}_{m0}) + \sum_{m'=1}^{\ell} \frac{1}{2N_{\ell m'}} \left\{ C_{\ell m'} \left[\mathrm{Re}(D^{\ell}_{mm'}) + (-)^{m'} \mathrm{Re}(D^{\ell}_{m,-m'})\right] + S_{\ell m'}\left[\mathrm{Im}(D^{\ell}_{mm'}) - (-)^{m'} \mathrm{Im}(D^{\ell}_{m,-m'})\right]\right\}\right] \tag{27}$$

and

$$T_{\ell m} = z_m N_{\ell m} \left[- \frac{C_{\ell 0}}{N_{\ell 0}} \mathrm{Im}(D^{\ell}_{m0}) + \sum_{m'=1}^{\ell} \frac{1}{2N_{\ell m'}} \left\{-C_{\ell m'} \left[\mathrm{Im}(D^{\ell}_{mm'}) + (-)^{m'} \mathrm{Im}(D^{\ell}_{m,-m'})\right] + S_{\ell m'}\left[\mathrm{Re}(D^{\ell}_{mm'}) - (-)^{m'} \mathrm{Re}(D^{\ell}_{m,-m'})\right]\right\}\right]. \tag{28}$$

To complete the discussion, expressions for the $D^{\ell}_{mm'}(\alpha,\beta,\gamma)$ are required. Since the computation of these quantities is not direct, they are dealt with in Appendix B. From (B1),

$$\begin{aligned} D^{\ell}_{mm'}(\alpha,\beta,\gamma) &= e^{im\gamma} d^{\ell}_{mm'}(\beta)\, e^{im'\alpha} \\ &= d^{\ell}_{mm'}(\beta)\left[\cos(m\gamma+m'\alpha) + i \sin(m\gamma+m'\alpha)\right], \end{aligned} \tag{29}$$

where $d^{\ell}_{mm'}(\beta)$ is a real matrix which reduces to the identity matrix $\delta_{mm'}$ as β goes to zero. Substituting (29) into (27) and (28) gives

$$D_{\ell m} = z_m N_{\ell m}\left[\frac{C_{\ell 0}}{N_{\ell 0}}\, d^{\ell}_{m0}\cos(m\gamma) + \sum_{m'=1}^{\ell} \frac{1}{2N_{\ell m'}}\left\{C_{\ell m'}\left[d^{\ell}_{mm'}\cos(m\gamma+m'\alpha) + (-)^{m'} d^{\ell}_{m,-m'}\cos(m\gamma-m'\alpha)\right] + S_{\ell m'}\left[d^{\ell}_{mm'}\sin(m\gamma+m'\alpha) - (-)^{m'} d^{\ell}_{m,-m'}\sin(m\gamma-m'\alpha)\right]\right\}\right] \tag{30}$$

$$T_{\ell m} = z_m N_{\ell m}\left[-\frac{C_{\ell 0}}{N_{\ell 0}}\, d^{\ell}_{m0}\sin(m\gamma) + \sum_{m'=1}^{\ell} \frac{1}{2N_{\ell m'}}\left\{-C_{\ell m'}\left[d^{\ell}_{mm'}\sin(m\gamma+m'\alpha) + (-)^{m'} d^{\ell}_{m,-m'}\sin(m\gamma-m'\alpha)\right] + S_{\ell m'}\left[d^{\ell}_{mm'}\cos(m\gamma+m'\alpha) - (-)^{m'} d^{\ell}_{m,-m'}\cos(m\gamma-m'\alpha)\right]\right\}\right] \tag{31}$$

Appendix B shows how to compute the matrices $d^{\ell}_{mm'}(\beta)$, and it displays them for several values of ℓ. $N_{\ell m}$ is defined in (9), and z_m is defined in (16).

Equations (30) and (31) exhibit in detail the dependence of the expansion coefficients appearing in (3) on the coefficients appearing in (1). An example illustrating the use of these equations is given in Appendix C.

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APPENDIX A

COMMUTATION PROPERTY OF L^2 AND O_R

In the text it was stated that the spherical harmonics satisfy an eigenvalue problem of the form

$$L^2 \, Y_{\ell m}(\Omega) = \ell(\ell+1) \, Y_{\ell m}(\Omega). \tag{A1}$$

In this equation, $L^2 = \overline{L} \cdot \overline{L}$ and*

$$\overline{L} = \overline{r} \times \overline{\nabla}. \tag{A2}$$

We will show that the rotation operator $O_{R(\alpha,\beta,\gamma)}$ commutes with the operator L^2. Since any finite rotation of the coordinate system can be generated as a sequence of infinitesimal rotations, it is sufficient to show that the operator $O_{R(\delta\overline{\phi})}$ for infinitesimal rotations commutes with L^2. In this notation, the vector $\delta\overline{\phi}$ gives the axis and the magnitude of the infinitesimal rotation.

To begin the proof, we need to know the effect of an operator O_R on an arbitrary scalar function $\psi(\overline{r})$. A scalar function assigns numbers to the points of Euclidian space. Since this assignment must be independent of the coordinate system, then O_R acting on ψ produces a new functional form which, when evaluated at some point, must equal the value of the old form evaluated at the same physical point. If the

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*Morse and Feshbach, p. 1463.

new function and point are denoted by

$$\psi' = O_R \psi \tag{A3}$$

and

$$\bar{r}' = R\bar{r}, \tag{A4}$$

then this statement has the mathematical expression

$$O_R \psi(\bar{r}') = \psi(\bar{r}) \tag{A5}$$

or, equivalently,

$$O_R \psi(\bar{r}) = \psi(R^{-1}\bar{r}). \tag{A6}$$

Now the effect of $O_{R(\delta\bar{\phi})}$ on the representation of a point $\bar{r}$ is*

$$\bar{r} \xrightarrow[O_{R(\delta\bar{\phi})}]{} \bar{r}' = \bar{r} + \delta\bar{r} \tag{A7}$$

where

$$\delta\bar{r} = \bar{r} \times \delta\bar{\phi}. \tag{A8}$$

With this information we can use (A6) and a Taylor series expansion to get

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*Goldstein, § 4.7.

$$O_{R(\delta\overline{\phi})}\ \psi(\overline{r}) = \psi(\overline{r}-\delta\overline{r})$$

$$= \psi(\overline{r}) - \delta\overline{r}\cdot\overline{\nabla}\ \psi(\overline{r}). \tag{A9}$$

Because the rotation is infinitesimal, it is acceptable to have dropped all but the linear terms in the expansion. Putting (A8) into (A9) results in

$$O_{R(\delta\overline{\phi})}\ \psi(\overline{r}) = (1 + \delta\overline{\phi}\cdot\overline{r}\ \mathrm{x}\ \overline{\nabla})\ \psi(\overline{r}). \tag{A10}$$

Substituting (A2) into this, the following expression is obtained for the operator for infinitesimal rotations of the coordinate system:

$$O_{R(\delta\overline{\phi})} = (1 + \delta\overline{\phi}\cdot\overline{L}). \tag{A11}$$

It is clear from (A11) that $O_{R(\delta\overline{\phi})}$ commutes with both $\overline{L}$ and L^2, so the demonstration is complete.

APPENDIX B

THE ROTATION MATRIX $D^{\ell}_{mm'}(\alpha,\beta,\gamma)$

The matrix $D^{\ell}_{mm'}(\alpha,\beta,\gamma)$ is a $(2\ell+1) \times (2\ell+1)$ matrix representation of operators in the group of complex, two-dimensional, unitary and unimodular transformations. This group is named SU(2). The Euler angles α,β,γ are involved in these representations because the elements of SU(2) are in one-to-one correspondence with the three-dimensional rotation group one customarily deals with.* The group SU(2) becomes involved in (23) in the transformation of the spherical harmonics $Y_{\ell m}(\theta,\phi)$ because these functions from a $(2\ell+1)$-dimensional basis, for each ℓ.

For each ℓ, $D^{\ell}_{mm'}(\alpha,\beta,\gamma)$ is said to form a representation of the group SU(2). Finding these representations is a reasonably straightforward process which has been documented in Wigner, Chapter 9 and Appendix A. The result, in a form consistent with the conventions of this paper, is

$$D^{\ell}_{mm'}(\alpha,\beta,\gamma) = e^{im\gamma}\, d^{\ell}_{mm'}(\beta)\, e^{im'\alpha} \tag{B1}$$

where

$$d^{\ell}_{mm'}(\beta) = \sum_{\kappa} (-)^{m-m'+\kappa} \frac{\sqrt{(\ell+m')!(\ell-m')!(\ell+m)!(\ell-m)!}}{(\ell-m-\kappa)!(\ell+m'-\kappa)!(\kappa+m-m')!\kappa!} \times \left(\cos\frac{\beta}{2}\right)^{2\ell-(m-m'+2\kappa)} \left(\sin\frac{\beta}{2}\right)^{m-m'+2\kappa} . \tag{B2}$$

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*This is shown in Wigner, pages 157-161, and in Goldstein, pages 109-118.

The sum runs over those values of κ for which the factorial arguments are non-negative. Since $d^{\ell}_{mm'}(\beta)$ is real, we have

$$\mathrm{Re}\left[D^{\ell}_{mm'}(\alpha,\beta,\gamma)\right] = d^{\ell}_{mm'}(\beta)\ \cos(m\gamma+m'\alpha) \tag{B3}$$

and

$$\mathrm{Im}\left[D^{\ell}_{mm'}(\alpha,\beta,\gamma)\right] = d^{\ell}_{mm'}(\beta)\ \sin(m\gamma+m'\alpha). \tag{B4}$$

It may be observed that $D^{\ell}_{mm'}(0,\beta,0) = d^{\ell}_{mm'}(\beta)$, so that $d^{\ell}_{mm'}(\beta)$ must give the effect of a rotation of the coordinate system about its y-axis.

Although the matrix $d^{\ell}_{mm'}(\beta)$ has dimension $2\ell+1$, only $(\ell+1)^2$ of the $(2\ell+1)^2$ elements are independent. This follows from the symmetry properties

$$d^{\ell}_{m'm}(\beta) = (-)^{m-m'}\ d^{\ell}_{mm'}(\beta) \tag{B5}$$

and

$$d^{\ell}_{m'm}(\beta) = d^{\ell}_{-m,-m'}(\beta). \tag{B6}$$

Thus, using the case $\ell=2$ as an example, one only needs to compute the upper triangular matrix indicated with dots in Figure B1. This triangle may be referred to as ΔAOB. Once the functions in ΔAOB have been computed,

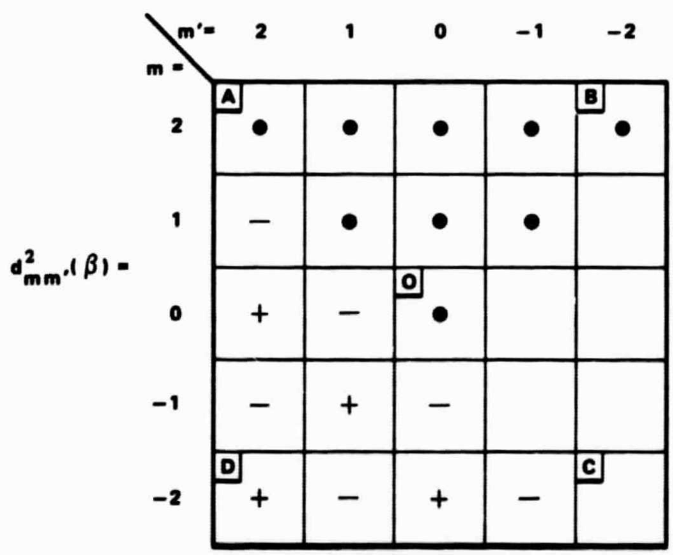

FIGURE B1

they are placed in ΔCOB by the mapping induced by folding the figure along line $\overline{BOD}$. The elements in ΔACB are now mapped into ΔACD by folding along the line $\overline{AOC}$ and then multiplying by the signs shown in the figure. This mapping technique applies for all ℓ-values.

Assuming that (B2) will only be applied to the upper triangular portion of $d^{\ell}_{mm'}(\beta)$, as just described, then Σ_{κ} ranges over integral values of κ such that

$$0 \leq \kappa \leq \ell - m. \tag{B7}$$

Hence each sum will have $\ell-m+1$ terms. This is shown as follows: The rule on Σ_{κ} is that κ is to assume all integral values for which the following conditions are true:

$$\kappa \geq 0 \qquad \text{(a)}$$

$$\kappa + m - m' \geq 0 \qquad \text{(b)}$$

$$\ell + m' - \kappa \geq 0 \qquad \text{(c)}$$

$$\ell - m - \kappa \geq 0 \qquad \text{(d)}$$

Condition (a) gives a definite lower limit on κ. Due to the assumption just stated, then $m \geq m'$, so (b) reduces to (a). In addition, $m \geq m'$ implies

$$\ell + m' - \kappa \geq \ell - m - \kappa \geq 0$$

in which the left sides of (c) and (d) appear. It is seen that (c) follows if (d) holds, so (a) and (d) are the only independent conditions in the set. Taken together, these amount to the condition given in (B7).

The preceding preliminaries simplify the computation of the matrices $d^{\ell}_{mm'}(\beta)$ from (B2). The matrices have been computed for ℓ = 1, 2, 3, 4 and are presented on the following pages. Notice the abbreviations

$$c = \cos(\tfrac{\beta}{2}) \tag{B8}$$

$$s = \sin(\tfrac{\beta}{2}) \tag{B9}$$

have been used. In order to preserve the symmetries of the matrices, no attempt has been made to simplify the trigonometric expressions which occur.

It was stated in the text that $d^{\ell}_{mm'}(\beta)$ tends to $\delta_{mm'}$ as β tends to zero. This point will now be demonstrated. In ΔAOB of Figure B1 consider those entries for which $m-m'>0$. According to (B7), $\kappa\geq 0$, so $m-m'+2\kappa>0$. Referring to (B2), it may be seen that each term of every such entry will contain the factor $\sin(\beta/2)$, and so every such term goes to zero with β. Due to the special symmetries of $d^{\ell}_{mm'}(\beta)$, then all the off-diagonal terms go to zero with β. Now consider the diagonal, for which $m=m'$. Except for the $\kappa=0$ terms, each term of every diagonal entry is proportional to $\sin(\beta/2)$ and tends to zero with β. Each of the $\kappa=0$ terms is equal to $\cos^{2\ell}(\beta/2)$, which tends to unity when β tends to zero.

$$
d^{1}_{m\,m'}(\beta) = \begin{array}{c|ccc} m \backslash m' & 1 & 0 & -1 \\ \hline 1 & c^2 & -\sqrt{2}\,cs & s^2 \\ 0 & \sqrt{2}\,cs & c^2 - s^2 & -\sqrt{2}\,cs \\ -1 & s^2 & \sqrt{2}\,cs & c^2 \end{array} \qquad \text{(B10)}
$$

$$
c \equiv \cos\left(\frac{\beta}{2}\right) \qquad s \equiv \sin\left(\frac{\beta}{2}\right)
$$

$$
d^{2}_{m\,m'}(\beta) = \begin{array}{c|ccccc}
m \backslash m' = & 2 & 1 & 0 & -1 & -2 \\
\hline
2 & C^4 & -2C^3S & \sqrt{6}C^2S^2 & -2CS^3 & S^4 \\
1 & 2C^3S & C^4 - 3C^2S^2 & \sqrt{6}\,(-C^3S + CS^3) & 3C^2S^2 - S^4 & -2CS^3 \\
0 & \sqrt{6}C^2S^2 & -\sqrt{6}\,(-C^3S + CS^3) & C^4 - 4S^2C^2 + S^4 & \sqrt{6}\,(-C^3S + CS^3) & \sqrt{6}C^2S^2 \\
-1 & 2CS^3 & 3C^2S^2 - S^4 & -\sqrt{6}\,(-C^3S + CS^3) & C^4 - 3C^2S^2 & -2C^3S \\
-2 & S^4 & 2CS^3 & \sqrt{6}C^2S^2 & 2C^3S & C^4
\end{array} \qquad \text{(B11)}
$$

$C \equiv \cos\left(\frac{\beta}{2}\right)$ $\qquad$ $S \equiv \sin\left(\frac{\beta}{2}\right)$

$d^{3}_{mm'}(\beta) =$ (B12)

m \ m' =	3	2	1	0	−1	−2	−3
3	c^6	$-\sqrt{6}c^5s^1$	$\sqrt{15}c^4s^2$	$-2\sqrt{5}c^3s^3$	$\sqrt{15}c^2s^4$	$-\sqrt{6}cs^5$	s^6
2	$\sqrt{6}c^5s$	$c^6-5c^4s^2$	$\sqrt{10}\,[-c^5s+2c^3s^3]$	$\sqrt{30}\,[c^4s^2 -c^2s^4]$	$\sqrt{10}\,[-2c^3s^3 +cs^5]$	$5c^2s^4-s^6$	$-\sqrt{6}cs^5$
1	$\sqrt{15}c^4s^2$	$-\sqrt{10}\,[-c^5s +2c^3s^3]$	$c^6-8c^4s^2 +6c^2s^4$	$2\sqrt{3}\,[-c^5s^1 +3c^3s^3-cs^5]$	$6c^4s^2 -8c^2s^4+s^6$	$\sqrt{10}\,[-2c^3s^3 +cs^5]$	$\sqrt{15}c^2s^4$
0	$2\sqrt{5}c^3s^3$	$\sqrt{30}\,[c^4s^2 -c^2s^4]$	$-2\sqrt{3}\,[-c^5s +3c^3s^3-cs^5]$	$c^6-9c^4s^2 +9c^2s^4-s^6$	$2\sqrt{3}\,[-c^5s +3c^3s^3-cs^5]$	$\sqrt{30}\,[c^4s^2 -c^2s^4]$	$-2\sqrt{5}c^3s^3$
−1	$\sqrt{15}c^2s^4$	$-\sqrt{10}\,[-2c^3s^3 +cs^5]$	$6c^4s^2 -8c^2s^4+s^6$	$-2\sqrt{3}\,[-c^5s +3c^3s^3-cs^5]$	$c^6-8c^4s^2 +6c^2s^4$	$\sqrt{10}\,[-c^5s+2c^3s^3]$	$\sqrt{15}c^4s^2$
−2	$\sqrt{6}cs^5$	$5c^2s^4-s^6$	$-\sqrt{10}\,[-2c^3s^3 +cs^5]$	$\sqrt{30}\,[c^4s^2 -c^2s^4]$	$-\sqrt{10}\,[-c^5s +2c^3s^3]$	$c^6-5c^4s^2$	$-\sqrt{6}c^5s$
−3	s^6	$\sqrt{6}cs^5$	$\sqrt{15}c^2s^4$	$2\sqrt{5}c^3s^3$	$\sqrt{15}c^4s^2$	$\sqrt{6}c^5s$	c^6

$c \equiv \cos\left(\frac{\beta}{2}\right)$ $\qquad$ $s \equiv \sin\left(\frac{\beta}{2}\right)$

$$d^{4}_{mm'}(\beta) =$$

$m \backslash m'$	4	3	2	1	0	−1	−2	−3	−4
4	c^8	$-2\sqrt{2}c^7s$	$2\sqrt{7}c^6s^2$	$-2\sqrt{14}c^5s^3$	$\sqrt{70}c^4s^4$	$-2\sqrt{14}c^3s^5$	$2\sqrt{7}c^2s^6$	$-2\sqrt{2}cs^7$	s^8
3	$2\sqrt{2}c^7s$	$c^8-7c^6s^2$	$\sqrt{14}[-c^7s+3c^5s^3]$	$\sqrt{7}[3c^6s^2-5c^4s^4]$	$2\sqrt{35}[-c^5s^3+c^3s^5]$	$\sqrt{7}[5c^4s^4-3c^2s^6]$	$\sqrt{14}[-3c^3s^5+cs^7]$	$7c^2s^6-s^8$	$-2\sqrt{2}cs^7$
2	$2\sqrt{7}c^6s^2$	$-\sqrt{14}[-c^7s+3c^5s^3]$	$c^8-12c^6s^2+15c^4s^4$	$\sqrt{2}[-3c^7s+15c^5s^3-10c^3s^5]$	$\sqrt{10}[3c^6s^2-8c^4s^4+3c^2s^6]$	$\sqrt{2}[-10c^5s^3+15c^3s^5-3cs^7]$	$15c^4s^4-12c^2s^6+s^8$	$\sqrt{14}[-3c^3s^5+cs^7]$	$2\sqrt{7}c^2s^6$
1	$2\sqrt{14}c^5s^3$	$\sqrt{7}[3c^6s^2-5c^4s^4]$	$-\sqrt{2}[-3c^7s+15c^5s^3-10c^3s^5]$	$c^8-15c^6s^2+30c^4s^4-10c^2s^6$	$\sqrt{5}[-2c^7s+12c^5s^3-12c^3s^5+2cs^7]$	$10c^6s^2-30c^4s^4+15c^2s^6-s^8$	$\sqrt{2}[-10c^5s^3+15c^3s^5-3cs^7]$	$\sqrt{7}[5c^4s^4-3c^2s^6]$	$-2\sqrt{14}[c^3s^5]$
0	$\sqrt{70}c^4s^4$	$-2\sqrt{35}[-c^5s^3+c^3s^5]$	$\sqrt{10}[3c^6s^2-8c^4s^4+3c^2s^6]$	$-\sqrt{5}[-2c^7s+12c^5s^3-12c^3s^5+2cs^7]$	$c^8-16c^6s^2+36c^4s^4-16c^2s^6+s^8$	$\sqrt{5}[-2c^7s+12c^5s^3-12c^3s^5+2cs^7]$	$\sqrt{10}[3c^6s^2-8c^4s^4+3c^2s^6]$	$2\sqrt{35}[-c^5s^3+c^3s^5]$	$\sqrt{70}c^4s^4$
−1	$2\sqrt{14}c^3s^5$	$\sqrt{7}[5c^4s^4-3c^2s^6]$	$-\sqrt{2}[-10c^5s^3+15c^3s^5-3cs^7]$	$10c^6s^2-30c^4s^4+15c^2s^6-s^8$	$-\sqrt{5}[-2c^7s+12c^5s^3-12c^3s^5+2cs^7]$	$c^8-15c^6s^2+30c^4s^4-10c^2s^6$	$\sqrt{2}[-3c^2s+15c^5s^3-10c^3s^5]$	$\sqrt{7}[3c^6s^2-5c^4s^4]$	$-2\sqrt{14}c^5s^3$
−2	$2\sqrt{7}c^2s^6$	$-\sqrt{14}[-3c^3s^5+cs^7]$	$15c^4s^4-12c^2s^6+s^8$	$-\sqrt{2}[-10c^5s^3+15c^3s^5-3cs^7]$	$\sqrt{10}[3c^6s^2-8c^4s^4+3c^2s^6]$	$-\sqrt{2}[-3c^2s+15c^5s^3-10c^3s^5]$	$c^8-12c^6s^2+15c^4s^4$	$\sqrt{14}[-c^7s^2+3c^5s^3]$	$2\sqrt{7}c^6s^2$
−3	$2\sqrt{2}cs^7$	$7c^2s^6-s^8$	$-\sqrt{14}[-3c^3s^5+cs^7]$	$\sqrt{7}[5c^4s^4-3c^2s^6]$	$-2\sqrt{35}[-c^5s^3+c^3s^5]$	$\sqrt{7}[3c^6s^2-sc^4s^4]$	$-\sqrt{14}[-c^7s+3c^5s^3]$	$c^8 \cdot 7c^6s^2$	$-2\sqrt{2}c^7s$
−4	s^8	$2\sqrt{2}cs^7$	$2\sqrt{7}c^2s^6$	$2\sqrt{14}c^3s^5$	$\sqrt{70}c^4s^4$	$2\sqrt{14}c^5s^3$	$2\sqrt{7}c^6s^2$	$2\sqrt{2}c^7s$	c^8

(B13)

$c \equiv \cos\left(\frac{\beta}{2}\right)$ $\qquad$ $s \equiv \sin\left(\frac{\beta}{2}\right)$

APPENDIX C

EXAMPLE OF THE EFFECT OF ROTATING THE COORDINATE FRAME

Suppose there is a potential in the form of (1) for which all the coefficients are zero except C_{20}. Then

$$U(r,\theta,\phi) = \frac{\mu}{r}\left\{1 + \left(\frac{a}{r}\right)^2 C_{20}\, P_{20}(\cos\theta)\right\}. \tag{C1}$$

The variables in (C1) are understood to be measured in the coordinate frame $\mathcal{O}$.

Consider a rotation of $\mathcal{O}$ through angle β about its y-axis to a new frame $\mathcal{O}'$. Then the Euler angles of the rotation are

$$(\alpha,\beta,\gamma) = (0,\beta,0). \tag{C2}$$

What does the potential function look like in this new frame?

The answer to this is given by (3), (30), and (31). According to (3), the new function $U'(r,\theta',\phi')$ has the same form as the general expansion which led to (C1). However, the new coefficients, $D_{\ell m}$ and $T_{\ell m}$, are different. Referring to (31), we find that $S_{\ell m}=0$ and $\alpha=\gamma=0$ imply that $T_{\ell m}=0$, for all ℓ and m. From (30) it can be seen that all the $D_{\ell m}$ are zero also, except when $\ell=2$. For this case the non-zero coefficient C_{20} appears, and

$$D_{2m} = z_m\, N_{2m}\left[\frac{C_{20}}{N_{20}}\, d^2_{m0}(\beta)\right]. \tag{C3}$$

Writing this for $m=0,1,2$, then

$$\left.\begin{aligned} D_{20} &= C_{20}\, d^2_{00}(\beta) \\ D_{21} &= \frac{2}{\sqrt{6}}\, C_{20}\, d^2_{10}(\beta) \\ D_{22} &= \frac{C_{20}}{\sqrt{6}}\, d^2_{20}(\beta) \end{aligned}\right\} . \tag{C4}$$

The functions $d^2_{00}(\beta)$, $d^2_{10}(\beta)$, and $d^2_{20}(\beta)$ may be read off of the display in (B11). Doing so and simplifying the resulting trigonometric expressions gives

$$\left.\begin{aligned} D_{20} &= C_{20}\left[1 - \frac{3}{2}\sin^2(\beta)\right] \\ D_{21} &= \frac{-C_{20}}{2}\sin(2\beta) \\ D_{22} &= \frac{C_{20}}{4}\sin^2(\beta) \end{aligned}\right\} . \tag{C5}$$

With this information about the new coefficients $D_{\ell m}$ and $T_{\ell m}$, the expression (3) for the potential in the new frame may be written as

$$\begin{aligned} U'(r,\theta',\phi') = \frac{\mu}{r}\Bigg\{1+C_{20}\left(\frac{a}{r}\right)^2\Bigg[&\left(1-\frac{3}{2}\sin^2\beta\right) P_{20}(\cos\theta') \\ &- \frac{1}{2}\sin 2\beta\cos\phi'\, P_{21}(\cos\theta') \\ &+ \frac{1}{4}\sin^2\beta\cos 2\phi'\, P_{22}(\cos\theta')\Bigg]\Bigg\}, \end{aligned} \tag{C6}$$

where the variables are now measured in the new coordinate frame O'. Note that (C6) reduces to (C1) as the rotation angle β tends to zero.

It may be verified that <u>(C1) and (C6) have precisely the same values when evaluated at the same physical points</u>. This occurs because the two formulas are only different representations of the potential of the same physical object.

REFERENCES

Emde, P., and E. Jahnke, Tables of Functions with Formulae and Curves, New York: Dover, Fourth Edition, pp. 107-111.

Feshbach, H., and P. M. Morse, Methods of Theoretical Physics, New York: McGraw Hill, 1953, pp. 1463-4.

Goldstein, H., Classical Mechanics, Reading, Massachusetts: Addison-Wesley, 1959, pp. 109-18, 124-8.

Wigner, E. P., Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra, Translated by J. J. Griffen, New York: Academic, 1964, Chapter 9 and Appendix A.