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COMBINING SATELLITE ALTIMETRY AND SURFACE GRAVIMETRY IN GEODETIC DETERMINATIONS

by

Ronald Ging-wei Eng Young

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MEASUREMENT SYSTEMS LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOG

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ABSTRACT

The path of an earth satellite is smooth enough so that measurement of the altitude, the distance from the satellite to the earth's surface, can provide information about undulations in this surface. Since the mean surface of the ocean coincides approximately with the equipotential surface of gravity known as the geoid, satellite altimetry can provide information about the shape of the geoid.

This thesis studies the deterministic problem of combining satellite altimetry observations over ocean areas with surface gravimetry over land to determine the geoid and the gravity potential. By examining the existence and uniqueness of solutions to the equivalent mathematical problem, a mixed boundary value problem in potential theory for which a general solution method is not yet available, conditions for the validity of a Neumann series method of successive approximations are established using both analytical and numerical techniques. When altimetry data are weighted more heavily than gravimetry data, sufficient conditions are given for establishing, analytically, the validity of the method. When the altimetry and gravimetry data are weighted more evenly, a computer calculation demonstrates the validity of the method for a distribution of altimetry and gravimetry like that of the earth's ocean-land distribution. Numerical studies then illustrate the determination of spherical harmonic representations of the gravity field from altimetry and gravimetry data generated by standard sets of harmonic coefficients that agree closely with the standard sets.

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a	(2.13)	$\Delta g(q)$	(2.21)
$a^{(\ell)}$	(4.128)	H	(D.03)
$B = [B_{ij}]$	(4.67)	$I(p,q) = I = [\delta_{ij}]$	(2.43)
B	(E.01)	$\mathcal{I}(p,q)$	(2.49)
$B_C = [B_{C_{ij}}]$	(4.71)	$\mathcal{I}n$	(3.06)
b	(2.13)	i	(2.38)
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$C = [C_{ij}]$	(4.68)	J	(3.28)
$\bar{C}_{nm}$	(2.02)	J_2	(2.13)
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$\delta\bar{C}_{nm}$	(2.15)	$K^{(m)}(p,q)$	(4.06),(4.40)
$\delta\bar{C}^{(i)}_{nm}$	(5.01)	$K'(p,q)$	(E.05)
c_ℓ	(4.129)	$K_B(p,q)$	(A.04)
c_i	(E.10)	$K_N(p,q) = K_N$	(2.41),(2.44)
$D = [D_{ij}]$	(4.51)	$K_P(p,q)$	(2.37)
$E = [E_{ij}]$	(4.52)	$\delta K(p,q)$	(E.06)
E_m	(4.63)	$\mathcal{K}_N(p,q)$	(2.48)
e	(2.13)	$L = [L_{ij}]$	(4.105)
f	(2.13)	ℓ	(3.12)
$f(p)$	(2.33)	$\ell(p,q)$	(A.16)
$f(p) = f$	(4.91)	ln = natural logarithm	

M	(2.02)
$M(p,q) = M = [M_{ij}]$	(2.57),(4.41)
$M^{(k)}$, $M^{(-)}$, $M^{(k-)}$	(4.44)
$M^* = [M^*_{ij}]$	(4.109)
M^*_+	(4.119)
M^*_-	(4.120)
$M_S = [M_{S_{ij}}]$	(4.43),(4.49)
m	(2.02)
$N(\phi_p,\lambda_p) = N(p)$	(2.19)
n	(2.02)
n_i	(2.38)
n_M	(4.44)
n_p	(2.08)
n'_p	(2.14)
$P_n(\mu)$	(B.02)
$P^m_n(\mu)$	(B.01)
$\overline{P}^m_n(\sin\phi_p)$	(2.03)
p	(2.02)
q	(2.18)
R	(2.01)
R_λ	(4.17)
r_G	(2.06)
r_M, r_p	(2.02)
r_γ	(2.18)
$r_\sigma(K)$	(4.16)
S, S_0, S_i	(2.01)
$S = [S_{ij}]$	(4.42),(4.47)
S^{-1}_{ij}	(4.48)
$S^{(i)} = [S^{(i)}_{jk}]$	(4.54),(4.53)
S'_0, S'_1	(E.04)
$\overline{S}_{nm}$	(2.02)
$\delta\overline{S}_{nm}$	(2.15)
$\delta\overline{S}^{(i)}_{nm}$	(5.01)
$\delta S, \delta S^+, \delta S^-$	(E.04)
∂S	(2.26)
s	(4.45)
s_k	(3.12)
$\mathrm{sgn}(S_1)$	(D.04)
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$T(p)$	(2.15)
$T'(p)$, $T''(p)$	(3.01)
t	(3.12)
t	(4.52)
$U(p)$	(2.13)
$U(p)$	(A.01)
U_γ	(2.13)
$\cup$ = union of two sets	
u	(4.11)
u	(A.09)
$V(p)$	(2.02)

$V(p)$	(A.01)
$V_\ell(p)$	(A.03)
v	(4.11)
$v(p)$	(3.01)
$v^*(p)$	(3.23)
$v_\perp(p)$	(3.30)
$v_{n_i mj}$	(3.05)
$W(p)$	(2.01)
W_G	(2.07)
δW	(2.20)
w_i	(A.06)
x	(4.09)
x	(4.89)
x	(4.91)
$x_i(p)$	(2.38)
y	(4.91)
$y = [y_i]$	(4.131),(4.132)
$Z = [Z_{ij}]$	(3.35)
Z^*	(3.37)
z	(4.91)
$\alpha = [\alpha_i]$	(3.34),(3.30)
α^*	(3.37)
β	(2.50)
Γ	(4.60)
γ	(2.14)

Δ	(4.60)
∇^2 = Laplacian	
δ_{ij}	(2.03)
$\in$ = is member of (relation of element to containing class)	
$\zeta(p) = \zeta$	(2.39)
$\zeta^{(n)}(p)$	(4.03)
ζ^*	(4.108)
$\Lambda(p) = \Lambda = [\Lambda_{ij}]$	(2.34),(3.32)
$\Lambda^{1/2}$	(3.36)
λ	(4.16)
λ_p	(2.02)
$\lambda(M_S), \lambda_i(M_S)$	(4.65)
$\lambda^{(\ell)}$	(4.131)
μ	(A.08)
μ_j	(4.30)
ν	(2.53)
Ξ	(4.60)
ξ	(4.121)
ξ_+	(4.122)
ξ_-	(4.124)
$\pi = 3.14159265...$	
ρ	(2.05)
σ	(2.36)
$\sigma(K)$	(4.16)
σ_0	(4.32)

τ	(4.68)
Φ	(2.04)
ϕ_p	(2.02)
ψ_{pq}	(A.20)
$\Omega(p) = \Omega = [\Omega_{ij}]$	(4.26),(4.32)
$\Omega_{nmj}^{st\ell}$	(C.01)
ω	(2.04)
$\|\| \quad \|\|$	(4.12),(4.15)
$\|\| \quad \|\|_L$	(4.113)

CHAPTER 1

INTRODUCTION

1.1 General Discussion

The path of a satellite in earth orbit is smooth enough so that measurement of the altitude, the distance from the satellite to the earth's surface, can provide information about undulations in this surface. Since the mean surface of the ocean coincides approximately with the equipotential surface of gravity known as the geoid, satellite altimetry can provide information about the shape of the geoid. This thesis is devoted to a technique for combining satellite altimetry observations over the oceans with surface gravimetry over the land to improve the knowledge of the geoid and the gravity potential.

This introductory chapter provides some basic information on the two fields involved, which are satellite altimetry and geodesy, and the formulation of the problem which is solved here. In order to reach a mathematically tractable solution, only purely deterministic methods are employed. The statistical problems imposed by real, noisy, redundant data that are avoided here can be handled by a statistical combination of this solution with others.

1.2 Satellite Altimetry and Geodesy

Proposals (including, Frey, *et al.*, 1966, Godbey, 1965, Greenwood, *et al.*, 1967, and Raytheon Company, 1968) have been made to put an altimeter on board a satellite. The altimeter functions by measuring the time delay, interpretable as a distance measurement, between emission of a radar or laser pulse and reception of its reflection from a portion of the earth's surface. This observation can have both geodetic and oceanographic uses, but only geodetic applications are considered in the sequel.

Measurements for conventional satellite geodesy (Kaula, 1966a, Mueller, 1964) involve ground station tracking of the orbits of satellites. By comparing these orbits with orbits predicted using spherical harmonic representations of the gravitational potential (Gaposchkin, 1966) and employing statistical data fits to minimize the residuals, improved estimates of the harmonic coefficients are obtained (Gaposchkin, 1969, Kozai, 1969). Because the effects of higher harmonic variations of the gravitational field fall off rapidly with distance from the earth, short period (small fractions of the orbital period) orbital perturbations have small amplitudes. Only a few resonant higher harmonics can be determined conveniently by satellite observation (Gedeon, 1969, Greene, 1968, Wagner, 1968).

In gravimetric geodesy (Heiskanen and Moritz, 1967, Molodenskii, *et al.*, 1962), measurements of the gravity magnitude are made; these provide data sensative to the higher harmonics. Conversion of the data to a harmonic representation entails a solution of a boundary value problem in potential theory of the third kind with a boundary condition containing constant coefficients (Heiskanen and Moritz, 1967, p. 36), yielding the gravitational potential as a linear integral transform of gravity anomalies on the whole surface of the earth. There are large gaps in data coverage, especially over southern hemisphere oceans (Uotila, 1962). Current practice is to extrapolate to fill the gaps (Kaula, 1959, 1966b, Köhnlein, 1967, Potter and Frey, 1967, Rapp, 1968), obtain an approximate solution, and then combine this in a statistical data fit (Kaula, 1961, 1966c, Köhnlein, 1967, Rapp, 1968) with satellite and other determinations, such as estimates of geoidal sections from geometrical geodesy (Bomford, 1962).

Altimetry data can also provide higher harmonic detail if corrections for various effects are assumed made. These include the pulse form (Price, 1968), atmospheric propagation effects (Frey, *et al.*, 1966), surface reflection characteristics (Greenwood, *et al.*, 1967), altimeter design (Frey, *et al.*, 1966, Godbey, 1965, Raytheon Company, 1968), and data processing technique (Price, 1968). If the satellite's orbit is assumed known and appropriately chosen, altimetry then defines the figure of the earth, in an initial implementation, to an accuracy of one meter (Kaula, 1969). According to the best judgments of oceanographers (Greenwood, *et al.*, 1967), the ocean's surface, averaged for waves and sea state, coincides to within a few meters with the geoid, that equipotential surface of the gravity field that best coincides over oceans with mean sea level. Since the geoid is closer to masses causing anomalies in the gravity field than the satellite is, the geoid exhibits short wavelength undulations (see, for example, von Arx, 1966) with amplitudes large compared to short period perturbations of the altimetry satellite. Thus even if the satellite's orbit is not known, as previously assumed, the estimate obtained from conventional satellite geodesy can be used as a first approximation without seriously masking the short wavelength detail of the geoid. After the geoid information is used to improve the representation of the gravity field, higher approximations can proceed, if necessary. For consistency with satellite geodesy, the gravity field at the geoid is also represented here in terms of the spherical harmonics. Even if such a representation is not strictly valid for representing the geoid, the error, in practice, is small and can be taken into account (Madden, 1968).

To improve the geodetic parameters, Lundquist (1967) proposes to include the difference of measured altitudes and those calculated from a model gravity potential in a massive statistical data fit computer program (Gaposchkin, 1966) in the same manner as with conventional satellite observations. He points out that a naive approach requires an excessively large gravity field model in a determination that must handle large amounts of nonuniformly distributed data. Lundquist, *et al*. (1969) propose a transformation of the harmonic representation into a sum of functions primarily sensitive to the shape of particular areas of the geoid. Difficulties in choosing a particular transformation and set of functions are unresolved at this time.

The approach taken here attempts to avoid statistical assumptions as much as possible, and makes use of potential theory, as does that of gravimetric geodesy. If the geoid is specified over the whole surface of the earth, solution of a boundary value problem in potential theory of the first kind yields the gravitational potential as an integral transform of the surface data. Because altimetry provides such data only on oceans, the direct approach fails, since with only partial data, the problem is not well-posed (Hadamard, 1923). A statistical extrapolation approach encounters problems similar to those in implementing current gravimetric determinations. A combination of the potential theory approach to altimetry and that of gravimetry seems appropriate, since their data bases complement each other. Altimetry will be applicable only on oceans, and gravimetry is available primarily on land (geoidal section data, physically similar to altimetry, is available to a limited extent on land). This thesis assumes that exactly one of two types of data is available at each point of the earth's surface, idealized as, or reduced to, the geoid. At surface points of the first kind, designated oceans, the physical form of the geoid is specified by altimetry (or geoidal section) data. At points of the second kind, designated land, the magnitude of gravity on the geoid is specified by gravimetry. Because gravity is measured on the earth's physical surface rather than on the geoid, necessary reductions of gravity to the geoid (see, for example, Heiskanen and Moritz, 1967) are assumed made. The purpose of this thesis is to solve the physical and mathematical problem of combining the two types of boundary data to obtain the gravitational potential of the earth.

1.3 Synopsis

In chapter 2 the physical problem is translated into a precise mathematical problem with several equivalent formulations convenient for the later analysis. Chapter 3 discusses some of the conditions sufficient to render the problem uniquely solvable. In chapter 4 the problem formulated in chapter 2 is put into several alternative forms suitable for solution by a method of successive approximations. When altimetry data are weighted more heavily than gravimetry data, an approximation of the problem becomes simple enough that the validity of the method can be established analytically. When altimetry and gravimetry data are weighted more evenly, the validity of the method is established numerically, for a distribution of gravimetry and altimetry data resembling the earth's land-ocean distribution. Chapter 5 discusses the actual determination of harmonic coefficients from altimetry and gravimetry data. Because actual altimetry data are unavailable, all data for the test examples were generated using standard sets of harmonic coefficients, which could easily be compared with those obtained by the proposed method. Finally, chapter 6 discusses the contributions of this thesis to using satellite altimetry in geodetic determinations.

CHAPTER 2

PROBLEM FORMULATION

2.1 General Discussion

The physical problem of combining altimetry data, which will be applicable only on oceans, and gravimetry data, which are assumed available on land, to obtain the gravitational potential of the earth is, in this chapter, reduced to several mathematical formulations convenient for the later analysis. Altimetry data define, geometrically, the surface of the geoid, on which the gravity potential is constant. Alternatively, gravimetry yields gravity, the gradient of the gravity potential, on the geoid, whose position, at points where gravimetry is given, is not known; indeed its determination is a part of the problem. This free boundary problem is transformed into a more traditional boundary value problem by linearizing about a known reference surface, such as a standard ellipsoid of revolution.

In section 2.2 the physical problem is reduced to a boundary value problem in potential theory. In section 2.3 integral representations are introduced, and the problem is written in terms of dual integral equations. The dual integral equations are combined formally into a single compact equation in section 2.4.

2.2 Partial Differential Equation Formulation

Let S denote a closed surface approximating that of the earth. It is initially taken to be the geoid, next an ellipsoid, and finally, a sphere. Let S_0 denote that subset, associated with oceans, on which altimetry is available. Let S_1 denote that subset, associated with land, on which gravimetry is available. Assume that S_0 and S_1 are mutually exclusive and collectively exhaustive. Let R denote the infinite region external to S.

Now consider S to be the geoid. The gravity potential, W(p), is composed of the gravitational potential, V(p), and the centrifugal potential, $\Phi(p)$,

$$W(p) = V(p) + \Phi(p) \tag{2.01}$$

where

$$V(p) = \frac{GM}{r_p}\left[1 + \sum_{n=1}^{\infty}\left(\frac{r_M}{r_p}\right)^n \sum_{m=0}^{n} \bar{P}_n^m(\sin\phi_p)\left\{\bar{C}_{nm}\cos m\lambda_p + \bar{S}_{nm}\sin m\lambda_p\right\}\right] \tag{2.02}$$

G = gravitational constant = 6.67×10^{-11} m^3/kg/sec^2

M = mass of the earth

GM = 3.98603×10^{14} m^3/sec^2

r_p = radius of the point, p

r_M = a mean radius of the earth

$\bar{P}_n^m(\sin\phi_p)$ = normalized associated Legendre function

$$= \left[\frac{(2-\delta_{m0})(n-m)!(2n+1)}{(n+m)!}\right]^{1/2} \frac{\cos^m\phi_p}{2^n n!} \frac{d^{n+m}(\sin^2\phi_p - 1)^n}{d(\sin\phi_p)^{n+m}} \tag{2.03}$$

n = degree of spherical harmonic expansion

m = order of spherical harmonic expansion

ϕ_p = geocentric latitude of the point, p

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\bar{C}_{nm}$, $\bar{S}_{nm}$ = normalized spherical harmonic coefficients of V

λ_p = geocentric longitude of the point, p

$$\Phi(p) = \frac{1}{2}\omega^2 r_p^2 \cos^2\phi_p \tag{2.04}$$

and ω = angular velocity of the earth's rotation

$= 0.729,211,51 \times 10^{-4}$/sec

The gravity potential at a general point satisfies Poisson's equation

(Heiskanen and Moritz, 1967, p. 47)

$$\nabla^2 W(p) = 2\omega^2 - 4\pi G\rho \qquad p \in R \qquad (2.05)$$

where ρ = mass density.

In general, there are masses in R, since most land areas are above sea level. On S_0, the oceans, altimetry defines the geoid,

$$r_p = r_G(\phi_p, \lambda_p) \qquad p \in S_0 \qquad (2.06)$$

where

r_G = radius of the geoid.

The boundary value for the gravity potential is that constant for which the geoid is an equipotential of gravity,

$$W(p) = W_G \qquad p \in S \qquad (2.07)$$

where

W_G = the constant value of the gravity potential on the geoid. On S_1, the land, equation (2.07) also holds. There is a free boundary, since the position of the geoid remains an unknown to be determined. Gravimetry data are available on the earth's physical surface. For a mathematically tractable problem, these data can be subjected to one of several gravity reductions (Heiskanen and Moritz, 1967) to obtain the equivalent values on the geoid. In the process all masses can be removed from R in a manner that modifies the obtained geoid and gravity potential. Since this indirect effect can be taken into account using higher approximations (for example, Molodenskii, et al., 1962), it is assumed hereafter that there are no masses outside the boundary surface and that gravity, g(p), is known on the geoid,

$$g(p) = -\frac{\partial W(p)}{\partial n_p} \qquad p \in S_1 \qquad (2.08)$$

where

g(p) = gravity at the point, p

n_p = normal to the geoid into R at the point, p.

The result is a free boundary value problem,

$$\nabla^2 W(p) = 2\omega^2 \qquad p \in R \qquad (2.09)$$

with the mixed boundary conditions

$$
\begin{aligned}
&1) \quad W(p) = W_G && p \in S_0 \\
&\quad S_0: \; r_p = r_G(\phi_p, \lambda_p) && \\
&2) \quad W(p) = W_G && p \in S_1 \\
&\quad \frac{\partial W(p)}{\partial n_p} = - g(p) && p \in S_1 \\
&\quad S_1: \; \text{free} &&
\end{aligned}
\qquad (2.10)
$$

When the differential equation is written in terms of the gravitational potential, a harmonic function,

$$\nabla^2 V(p) = 0 \qquad p \in R \qquad (2.11)$$

the boundary conditions become

$$
\begin{aligned}
&1) \quad V(p) = W_G - \tfrac{1}{2}\omega^2 r_p^2 \cos^2\phi_p && p \in S_0 \\
&\quad S_0: \; r_p = r_G(\phi_p, \lambda_p) && \\
&2) \quad V(p) = W_G - \tfrac{1}{2}\omega^2 r_p^2 \cos^2\phi_p && p \in S_1 \\
&\quad \frac{\partial V(p)}{\partial n_p} = - g(p) - \tfrac{1}{2}\omega^2 \frac{\partial}{\partial n_p}(r_p^2 \cos^2\phi_p) && p \in S_1 \\
&\quad S_1: \; \text{free} &&
\end{aligned}
\qquad (2.12)
$$

Free boundary problems are occasionally encountered in fluid dynamics (see, for example, Garabedian, 1964, p. 558). A free boundary problem is avoided here by linearizing about a known surface approximating the geoid, but nonuniqueness is not avoided; see chapter 3.

Without loss of generality the relatively simple, level rotational ellipsoid is adopted as the reference boundary surface. It is an equipotential surface of a 'normal' gravity potential, U(p), (Heiskanen and Moritz, 1967, p. 73):

$$U(p) = \frac{GM}{r_p}\left[1 + \sum_{n=1}^{\infty}\left(\frac{a}{r_p}\right)^{2n}\bar{C}^{(U)}_{2n,0}\bar{P}^{0}_{2n}(\sin\phi_p)\right] + \Phi(p) = U_\gamma \tag{2.13}$$

where

a = semi-major axis of the ellipsoid = 6378160. m

$$\bar{C}^{(U)}_{2n,0} = \frac{(-1)^n 3e^{2n}}{(2n+1)(2n+3)}(1 - n + \frac{5nJ_2}{e^2})$$

e = first eccentricity = $(a^2 - b^2)^{1/2}/a$

b = semi-minor axis of the ellipsoid = a(1 - f)

f = flattening of the ellipsoid = 1/298.25

J_2 = earth's dynamical form constant = 0.001,0827

U_γ = the constant value of the normal gravity potential on the level rotational ellipsoid.

The gradient of this potential is the normal gravity

$$\gamma(p) = -\frac{\partial U(p)}{\partial n'_p} \tag{2.14}$$

where n'_p = normal to the ellipsoid into R at the point, p. The centrifugal terms in W and U are identical. Next introduce the anomalous potential,

$$\begin{aligned} T(p) &= W(p) - U(p) \\ &= \frac{GM}{r_p}\sum_{n=0}^{\infty}\left(\frac{r_M}{r_p}\right)^n \sum_{m=0}^{n}\bar{P}^m_n(\sin\phi_p)(\delta\bar{C}_{nm}\cos m\lambda_p + \\ &\qquad + \delta\bar{S}_{nm}\sin m\lambda_p) \end{aligned} \tag{2.15}$$

where

$\delta\bar{C}_{nm}$, $\delta\bar{S}_{nm}$ = harmonic coefficients of the anomalous potential.

If $r_M \approx a$ the various harmonic coefficients are related by

$$\bar{C}_{nm} = \delta\bar{C}_{nm} + \bar{C}_{nm}^{(U)} \qquad \bar{S}_{nm} = \delta\bar{S}_{nm} \tag{2.16}$$

Since T does not contain any centrifugal term

$$\nabla^2 T(p) = 0 \qquad p \in R \tag{2.17}$$

To every point, p, on the geoid corresponds a point, q, located at the base of the ellipsoid normal that intersects p. The definitions of S_0 and S_1 can now be transfered from the geoid to the ellipsoid.

The boundary condition for W along S_0 (geoid) is next converted to one for the anomalous potential along S (ellipsoid). By assumption, the radius of the oceanic geoid is known (see equation (2.06)). The radius, r_γ, of the level rotational ellipsoid may be obtained, using equation (2.13), in the form,

$$r_q = r_\gamma(\phi_q) \qquad q \in S \text{ (ellipsoid)} \tag{2.18}$$

The geoidal undulation, $N(\phi_p,\lambda_p)$, is defined as the distance measured from the ellipsoid to the geoid along the ellipsoid normal. The maximum excursion of N is on the order of 100 meters, which is small compared to the dimensions of the ellipsoid. The generalized Brun's formula (Heiskanen and Moritz, 1967, p. 100) defines the relation between the anomalous potential and the geoidal undulation,

$$T(p) = \gamma(q)N(q) + \delta W \tag{2.19}$$

where

$$\begin{aligned} \delta W &= W_G - U_\gamma = W(p) - U(q) \\ &= \text{difference of equipotential constants} \end{aligned} \tag{2.20}$$

This is the boundary condition on the anomalous potential, valid for $p \in S_0$.

The boundary condition on land is transformed, similarly. On S_1 (geoid), g(p) is known by assumption, and on S_1 (ellipsoid), $\gamma(q)$ is known by definition, so that the gravity anomaly, $\Delta g(q)$, is well defined

$$\Delta g(q) = g(p) - \gamma(q) \tag{2.21}$$

The generalized fundamental equation of physical geodesy (Heiskanen and Moritz, 1967, p. 101) holds,

$$\Delta g(q) = -\frac{\partial T(p)}{\partial n_p} + \frac{\partial \gamma(q)}{\partial n_q'}\,\frac{T(p)-\delta W}{\gamma(q)} \tag{2.22}$$

Since, as a result of the linearization, the measured data, N(q) and $\Delta g(q)$, are small quantities, we may identify p with q and n_p with n_q'. A boundary value problem for the anomalous potential may be formulated. For S the rotational ellipsoid and R its external volume,

$$\nabla^2 T(p) = 0 \qquad p \in R \tag{2.23}$$

The boundary conditions on the two parts of S are

1) $$T(p) = \gamma(p)N(p) + \delta W \qquad p \in S_0 \tag{2.24}$$

2) $$T(p) - \frac{\gamma(p)}{\frac{\partial \gamma(p)}{\partial n_p}}\,\frac{\partial T(p)}{\partial n_p} = \frac{\gamma(p)}{\frac{\partial \gamma(p)}{\partial n_p}}\,\Delta g(p) + \delta W \qquad p \in S_1 \tag{2.25}$$

This is called a mixed boundary value problem in potential theory, a problem of the third kind, or the Robin's problem, since a linear combination of the potential and its first derivative are specified on the boundary (Kellogg, 1953). Equation (2.24), if specified on all of S, can be identified with the well known boundary value problem of potential theory of the first kind, the Dirichlet problem. If equation (2.25) holds over the whole surface, the Stokes (1849) problem, in which the coefficient of the derivative term is variable, but continuous, is obtained as a special Robin's problem. In the present case the coefficient of $\frac{\partial T}{\partial n_p}$ is discontinuous on ∂S, the boundary between S_0 and S_1, since its value drops to zero on S_0.

As in analysis of the Stokes problem, the ellipsoid is next approximated by the sphere of radius r_M. This is justified by the previous linearization to small quantities as well as the entailing simplicity.

The normal derivative becomes a radial derivative,

$$\frac{\partial}{\partial n_p} = \frac{\partial}{\partial r_p} \tag{2.26}$$

The ratio

$$\gamma(p) \Big/ \frac{\partial \gamma(p)}{\partial n_p} = \gamma(p) \Big/ \frac{\partial \gamma(p)}{\partial r_p} \tag{2.27}$$

may be approximated by taking

$$U = \frac{GM}{r} \tag{2.28}$$

Thus

$$\gamma(p) \Big/ \frac{\partial \gamma(p)}{\partial r_p} = -\frac{r_p}{2} \tag{2.29}$$

Thus the spherical approximation of the equation of physical geodesy (equation (2.22)) is

$$T(p) + \frac{r_p}{2} \frac{\partial T(p)}{\partial r_p} = -\frac{r_p}{2} \Delta g(p) + \delta W \qquad p \in S_1 \tag{2.30}$$

We may state our partial differential equation formulation as

$$\nabla^2 T(p) = 0 \qquad p \in R \tag{2.31}$$

with the boundary conditions

$$\begin{aligned} &1) \quad T(p) = \gamma(p) N(p) + \delta W && p \in S_0 \\ &2) \quad T(p) + \frac{r_M}{2} \frac{\partial T(p)}{\partial r_p} = -\frac{r_M}{2} \Delta g(p) + \delta W && p \in S_1 \end{aligned} \tag{2.32}$$

Introducing

$$f(p) = \begin{cases} \gamma(p) N(p) & p \in S_0 \\ -\frac{r_M}{2} \Delta g(p) & p \in S_1 \end{cases} \tag{2.33}$$

and the land function

$$\Lambda(p) = \begin{cases} 0 & p \in S_0 \\ 1 & p \in S_1 \end{cases} \tag{2.34}$$

the boundary condition may be written compactly (in terms of discontinuous functions) as

$$T(p) + \Lambda(p) \frac{r_M}{2} \frac{\partial T(p)}{\partial r_p} = f(p) + \delta W \qquad p \in S \qquad (2.35)$$

2.3 Dual Integral Equation Formulation

To obtain the dual integral equation formulation we first state the integral representations of a potential function for two types of boundary conditions. For the Stokes problem similar techniques are employed by Moritz (1965).

If any harmonic function, $T(q)$, is prescribed, $q \in S$, the solution of the Dirichlet problem for the sphere can be written

$$T(p) = \frac{1}{4\pi} \iint_\sigma K_P(p, q) T(q) d\sigma_q \qquad p \in R \qquad (2.36)$$

where

σ = solid angle corresponding to the earth's surface

$K_P(p, q)$ = Poisson kernel (Kellogg, 1953, or appendix A)

$$= \sum_{i=1}^{\infty} \left(\frac{r_M}{r_p}\right)^{n_i+1} x_i(p) x_i(q) \qquad (2.37)$$

$x_i(p)$ = normalized spherical harmonic function

$$= \bar{P}^m_{n_i}(\sin \phi_p) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (m\lambda_p) \begin{matrix} j=0 \\ j=1 \end{matrix} \qquad (2.38)$$

$$i = (n_i + j)n_i + m + 1$$

$$0 \le m \le n_i < \infty \qquad \text{for } j = 0$$

$$0 < m \le n_i < \infty \qquad \text{for } j = 1$$

If $\frac{\partial T(q)}{\partial r_q}$ is prescribed, $q \in S$, there results the boundary value problem in potential theory of the second kind, the Neumann problem. An integral representation of the solution of this problem is derived in appendix A. It is convenient here to introduce a harmonic function, $\zeta(p)$,

$$\zeta(p) = \frac{r_p}{2} \frac{\partial T(p)}{\partial r_p} \tag{2.39}$$

If $\zeta(q)$ is prescribed, $q \in S$

$$T(p) = \frac{1}{4\pi} \iint_\sigma K_N(p, q)\zeta(q)d\sigma_q \tag{2.40}$$

where

$$K_N(p, q) = \text{modified Neumann kernel (appendix A)}$$

$$= - \sum_{i=1}^{\infty} \frac{2}{n_i+1}\left(\frac{r_M}{r_p}\right)^{n_i+1} x_i(p)x_i(q) \tag{2.41}$$

In addition

$$T(p) + \zeta(p) = T(p) + \frac{r_p}{2} \frac{\partial T(p)}{\partial r_p}$$

$$= \frac{1}{4\pi} \iint_\sigma \left[K_N(p, q) + K_P(p, q)\right] \zeta(q)d\sigma_q \tag{2.42}$$

We take ζ as the unknown independent variable. We allow p to lie on the boundary so that we may use equations (2.32) in the left hand sides of equations (2.40) and (2.42). In the limit as p is brought down to the surface, the Poisson kernel becomes a delta function, the kernel of the identity operator,

$$I(p, q) = K_P(p, q)\Big|_{r_p = r_M} = \sum_{i=1}^{\infty} x_i(p)x_i(q) \tag{2.43}$$

For the application of generalized functions, of which the delta function is a special case, to partial differential equations, see Shilov (1968). With $r_p = r_M$ in equation (2.37) the transform causes a function to be represented in a spherical harmonic series (for convergence, see Hobson, 1955, p. 344). The equivalent form of the Neumann kernel is obtained from equation (2.41) with $r_p = r_M$

$$K_N(p, q) = - \sum_{i=1}^{\infty} \frac{2}{n_i+1} x_i(p)x_i(q) \tag{2.44}$$

We thus obtain the dual integral equations

1) $$\gamma(p)N(p) + \delta W = \frac{1}{4\pi}\iint_{\sigma} K_N(p, q)\zeta(q)d\sigma_q \qquad p \in S_0 \qquad (2.45)$$

and

2) $$-\frac{r_M}{2}\Delta g(p) + \delta W = \zeta(p) + \frac{1}{4\pi}\iint_{\sigma} K_N(p, q)\zeta(q)d\sigma_q \qquad p \in S_1 \qquad (2.46)$$

The equation (2.45) is a Fredholm integral equation of the first kind. The equation (2.46) is a Fredholm integral equation of the second kind. Using the identity kernel it may alternatively be written as a singular integral equation of the first kind,

$$-\frac{r_M}{2}\Delta g(p) + \delta W = \frac{1}{4\pi}\iint_{\sigma}\left[I(p, q) + K_N(p, q)\right]\zeta(q)d\sigma_q \qquad p \in S_1 \quad (2.47)$$

Dual integral equations have not been actively studied until recently (see Sneddon, 1966, or Tranter, 1966), and much of the work principally involves one dimensional integrals. See also Mikhlin (1965) concerning multidimensional singular integral equations.

2.4 Integral Operator Formulation

For convenience and compactness we introduce the integral operator notation. For any integrable function, $x(q)$,

$$\mathcal{K}_N(p, q)x(q) = \frac{1}{4\pi}\iint_{\sigma} K_N(p, q)x(q)d\sigma_q \qquad (2.48)$$

$$x(p) = \mathcal{I}(p, q)x(q) = \frac{1}{4\pi}\iint_{\sigma} I(p, q)x(q)d\sigma_q \qquad (2.49)$$

These operators are infinite-dimensional, since the representations of their kernels in terms of the normalized spherical harmonics (see equations (2.43) and (2.44)) each consist of an infinite number of terms. For practical work the series must be truncated, so that finite-dimensional operators result. For simplicity we write $K_N(p, q)$ and $I(p,q)$ for both the operator and the kernel. Write

$$\zeta(p) = \beta T(p) + \zeta(p) - \beta T(p) \qquad (2.50)$$

where β is a scalar free parameter weighting the influence of altimetry data relative to gravimetry data. For $p \in S_0$, replace the right-hand T of equation (2.50) with equation (2.32), replace the left-hand T by equation (2.40) using the notation of equation (2.48), and represent ζ using the notation of equation (2.49):

$$\zeta(p) = [\beta K_N(p, q) + I(p, q)]\zeta(q) - \beta\gamma(p)N(p) - \beta\delta W \qquad p \in S_0 \quad (2.51)$$

For $p \in S_1$, set $\beta = 1$ and give the right-hand T the same representation as that given the left-hand T of equation (2.51). Noting equation (2.39), the remaining terms of equation (2.50) are just the left-hand-side of equation (2.32) part 2), so that

$$\zeta(p) = -\frac{r_M}{2}\Delta g(p) + \delta W - K_N(p, q)\zeta(q) \qquad p \in S_1 \quad (2.52)$$

Equations (2.51) and (2.52) constitute a version of the dual integral equations in operator notation. They are next combined into the form of a single equation. We define the inhomogeneous term

$$\nu(p) = \begin{cases} -\beta\gamma(p)N(p) - \beta\delta W & p \in S_0 \\ -\frac{r_M}{2}\Delta g(p) + \delta W & p \in S_1 \end{cases} \quad (2.53)$$

The effect of the parameter, β, on the relative weighting of the two types of data is explicit in equation (2.53). We define the operator

$$K(p, q) = \begin{cases} \beta K_N(p, q) + I(p, q) & p \in S_0 \\ -K_N(p, q) & p \in S_1 \end{cases} \quad (2.54)$$

We have

$$\zeta(p) - K(p, q)\zeta(q) = \nu(p) \quad (2.55)$$

This operator equation is of the form of an inhomogeneous Fredholm integral equation of the second kind. It is unconventional in the sense that the operator, $K(p, q)$, has a kernel that is discontinuous as a function of the parameter point, p, along the irregular boundary, ∂S, between oceans, S_0, and land, S_1. The inhomogeneous term is similarly discontinuous. In addition, the appearance of the identity operator in part of the kernel is definitely nonclassical. The problem may also be cast in the form of an integral equation of the first kind,

$$M(p, q)\zeta(q) = \nu(p) \tag{2.56}$$

where

$$M(p, q) = \begin{cases} -\beta K_N(p, q) & p \in S_0 \\ I(p, q) + K_N(p, q) & p \in S_1 \end{cases} \tag{2.57}$$

This operator is similarly unconventional. Integral equations of the first kind are generally more difficult to solve; equation (2.56) is used primarily as a starting point to manipulate the problem into a problem involving an integral equation of the second kind. Equation (2.55) is the simplest form; others are developed in chapter 4.

Chapter 3

UNIQUENESS THEORY

3.1 Physical Considerations

Engineers usually do not concern themselves with mathematical questions such as uniqueness and existence; they prefer to rely on physical reasoning to guarantee these properties of the solution of their problems. However, these tools can be used as important checks on the validity of the analytical model of the physical problem, which arises because approximations must be made to physical reality in order to deal with the problem in a tractable manner and yet get useful results. A proper mathematical model should have enough restrictions so that there are not multiple solutions, but not so many that none exist.

We shall assume that the solution for the anomalous potential may be approximated by a function, T(p), defined outside of the earth's surface, appropriately approximated, which is:

1) finite
2) single-valued
3) regular at distances far from the earth (vanishes at least as fast as 1/r)
4) continuously differentiable

For compatibility the boundary data must be suitably restricted. As an approximation, altimetry should yield continuous geoidal undulations on oceans, S_0. Similarly, gravity anomalies should be extracted from gravimetry as a continuous function on land, S_1. At the boundary, ∂S, between ocean and land there are no further restrictions relating the physical data across the boundary. Some conditions sufficient for the full problem, in which all of the spherical harmonic coefficients are retained, to be unique are presented in section 3.2.

3.2 Uniqueness Results

To obtain conditions sufficient for the problem to be unique, we start with the partial differential equation formulation of section 2.2,

$$\nabla^2 T(p) = 0 \qquad p \in R \qquad (2.31)$$

with the unified boundary condition,

$$T(p) + \Lambda(p)\frac{r_M}{2}\frac{\partial T(p)}{\partial r_p} = f(p) + \delta W \qquad p \in S \qquad (2.35)$$

To examine uniqueness, suppose the contrary, that there exist at least two harmonic functions, $T'(p)$ and $T''(p)$, each satisfying the boundary condition. The difference,

$$v(p) = T'(p) - T''(p) \qquad (3.01)$$

satisfies

$$\nabla^2 v(p) = 0 \qquad p \in R \qquad (3.02)$$

with the boundary condition,

$$v(p) + \Lambda(p)\frac{r_M}{2}\frac{\partial v(p)}{\partial r_p} = 0 \qquad p \in S \qquad (3.03)$$

Since the boundary is a sphere, it is natural to expand v(p) in a series of spherical harmonics. Conditions under which various coefficients vanish indicate conditions for the uniqueness of T(p). We expand v(p) in solid spherical harmonics

$$v(p) = \sum_{i=1}^{\infty}\left(\frac{r_M}{r_p}\right)^{n_i+1} v_{n_i mj}\, x_i(p) \qquad (3.04)$$

where

$$v_{n_i mj} = \frac{1}{4\pi}\iint_{\sigma} v(p)\, x_i(p)\, d\sigma_p \qquad (3.05)$$

and $x_i(p)$, i, n_i, m, and j are defined in equation (2.38). According to Hobson (1955, p. 344) the assumptions imposed on T(p) (see section 3.1) and therefore v(p) assure the validity of the series representation.

We form the integral

$$In = \frac{-1}{4\pi} \iint_{\sigma} v(p)\left[v(p) + \frac{r_M}{2}\frac{\partial v(p)}{\partial r_p}\right] d\sigma_p \qquad (3.06)$$

Decomposing equation (3.03),

$$v(p) = 0 \qquad p \in S_0 \qquad (3.07)$$

$$v(p) + \frac{r_M}{2}\frac{\partial v(p)}{\partial r_p} = 0 \qquad p \in S_1 \qquad (3.08)$$

Since

$$S_0 \cup S_1 = S \qquad (3.09)$$

there results

$$In = 0 \qquad (3.10)$$

We insert the harmonic series, noting that

$$\left.\frac{\partial v(p)}{\partial r_p}\right|_{r_p = r_M} = -\sum_{i=1}^{\infty} \frac{n_i+1}{r_M} v_{n_i mj} x_i(p) \qquad (3.11)$$

There results

$$In = 0 = \frac{-1}{4\pi} \iint_{\sigma} \sum_{k=1}^{\infty} v_{s_k t\ell} x_k(p) \sum_{i=1}^{\infty}\left(1 - \frac{n_i+1}{2}\right) v_{n_i mj} x_i(p)\, d\sigma_p \qquad (3.12)$$

Using the orthonormality property

$$\frac{1}{4\pi} \iint_{\sigma} x_k(p) x_i(p)\, d\sigma_p = \delta_{ki} \qquad (3.13)$$

we obtain

$$In = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{n_i-1}{2} v_{s_k t\ell} v_{n_i mj} \delta_{ki} \qquad (3.14)$$

$$= 0$$

or

$$v_{000}^2 = \sum_{n_i=2}^{\infty} (n_i - 1) \sum_{m=0}^{n_i} \left(v_{n_i m0}^2 + v_{n_i m1}^2\right) \qquad (3.15)$$

Both the left hand side and the right hand side of the equation consist of nonnegative terms. If one side vanishes, then so must the other. If

$$v_{000} = 0 \tag{3.16}$$

then

$$V_{n_i mj} = 0 \tag{3.17}$$

for all n_i, m, j such that

$$\begin{aligned} 2 &\le n_i < \infty \\ 0 &\le m \le n_i \\ j &= 0,\ 1 \end{aligned} \tag{3.18}$$

By definition (see equation (3.05)),

$$v_{000} = \frac{1}{4\pi} \iint_{\sigma} \left[T'(p) - T''(p) \right] d\sigma_p \tag{3.19}$$

Thus if both solutions for the anomalous potential have the same average value over the surface of the earth,

$$V_{000} = 0 \tag{3.20}$$

This is equivalent to the requirement that the mass of the earth (in the constant, GM) and the difference of geoid and ellipsoid equipotential constants, δW, must be prescribed. Further, the constant, δW, behaves as a zeroth harmonic of the inner potential in the boundary conditions (2.35), violating requirement 3 of section 3.1. Hence choose

$$\delta W = 0$$

We still have to examine the differences of first degree harmonic coefficients, v_{100}, v_{110}, v_{111}, which are not involved in equation (2.15). By assumption (see equation (3.16)), these are the only remaining possible nonzero terms. Hence

$$v(p) = v_{100} x_2(p) + v_{110} x_3(p) + v_{111} x_4(p) \tag{3.21}$$

But in view of the boundary condition (see equation (3.03))

$$v(p) = \Lambda(p)v(p) \qquad (3.22)$$

The three first harmonic terms may be interpreted as the three orthogonal components of a translation of the center of the coordinate system (Heiskanen and Moritz, 1967, p. 62). Aside from a trivial translation, v(p) is zero only on the locus of points common to both the original reference sphere and its translation resulting from nonzero first harmonics.

Thus the first harmonic coefficients must vanish if the oceans cover a finite area, since

$$v(p) = 0 \qquad p \in S_0 \qquad (3.07)$$

Thus we have proved that, when both altimetry and gravimetry data are specified in the boundary condition, if a solution is assumed to exist, any other solution with the same zeroth harmonic is identical. The question of existence of solutions is handled in the next chapter; useful results are obtained only for solutions in which the potential is assumed to be the sum of a finite number of spherical harmonics. An analytical proof yields not only existence, but also uniqueness, for the finite approximation. An alternative numerical approach (which of course requires a finite approximation) demonstrates that for an altimetry-gravimetry distribution resembling the ocean-land distribution of the earth, a unique solution can be obtained.

Before turning to the finite-dimensional problem, a few more remarks will be made concerning the infinite-dimensional case. As a result of the linearity of solutions of equations (3.02) and (3.03), if a nontrivial solution exists, it may be expressed in the form,

$$v(p) = v_{000}\, v^*(p) \qquad (3.23)$$

where $v^*(p)$ is a unique function for a particular choice of $\Lambda(p)$.

The uniqueness analysis just discussed makes no use of the detailed form of the boundary between land and ocean (other than to eliminate the trivial boundary). The detail of the discontinuity is difficult to handle analytically, but J. E. Potter (personal communication) has extended the uniqueness proof by deriving criteria sufficient for the problem to be unique. These results are now obtained using the present notation.

Rewrite equation (3.14) in the form

$$In = \left[\sum_{i=1}^{4} \frac{1}{2} v_{n_i mj}^2 + \sum_{i=5}^{\infty} \frac{n_i - 1}{2} v_{n_i mj}^2\right] - \sum_{i=1}^{4} \frac{2-n_i}{2} v_{n_i mj}^2 \tag{3.24}$$

In the previous analysis it was shown that if $v_{000} = 0$, equation (3.14) is positive definite, so that only a trivial choice of coefficients satisfies equation (3.10). To show that equation (3.24) is positive definite, it is sufficient to show that a less positive function is positive definite. Hence replace $(n_i - 1)/2$ by $1/2$ in the second summation, yielding

$$In \geq \sum_{i=1}^{\infty} \frac{1}{2} v_{n_i mj}^2 - \sum_{i=1}^{4} \frac{2-n_i}{2} v_{n_i mj}^2 \tag{3.25}$$

It is easily seen that

$$\frac{1}{4\pi} \iint_{\sigma} [v(p)]^2 d\sigma_p = \sum_{i=1}^{\infty} v_{n_i mj}^2 \tag{3.26}$$

Substitute this into equation (3.25) and use also equations (3.05) and (3.22):

$$In \geq \frac{1}{4\pi} \iint_{\sigma} [v(p)]^2 d\sigma_p \left[\frac{1}{2} - J\right] \tag{3.27}$$

where

$$J = \frac{\sum_{i=1}^{4} \frac{1}{4\pi} \iint_{\sigma} v(p)\Lambda(p) x_i(p) d\sigma_p \frac{2-n_i}{2} \frac{1}{4\pi} \iint_{\sigma} \Lambda(q) x_i(q) v(q) d\sigma_q}{\frac{1}{4\pi} \iint_{\sigma} [v(p)]^2 d\sigma_p} \tag{3.28}$$

If

$$\max_{v(p)} \quad J < \frac{1}{2} \tag{3.29}$$

then equation (3.24) is positive definite. All trial functions in the maximization of equation (3.28) may be represented in the form

$$v(p) = \sum_{i=1}^{4} \alpha_i \Lambda(p) x_i(p) + v_\perp(p) \tag{3.30}$$

where $v_\perp(p)$ satisfies

$$\frac{1}{4\pi} \iint_\sigma v_\perp(p) \Lambda(p) x_i(p) d\sigma_p = 0 \qquad 1 \le i \le 4 \tag{3.31}$$

$v_\perp(p)$ does not contribute to the numerator of equation (3.28), so that it may be taken to be zero for the maximization. Insert equation (3.30) into equation (3.28) and define

$$\Lambda_{ij} = \frac{1}{4\pi} \iint_\sigma \Lambda(p) x_i(p) x_j(p) d\sigma_p = \Lambda_{ji} \tag{3.32}$$

There results

$$J = \frac{\sum_{i,j,k=1}^{4} \alpha_i \Lambda_{ij} \frac{2-n_j}{2} \Lambda_{jk} \alpha_k}{\sum_{i,j=1}^{4} \alpha_i \Lambda_{ij} \alpha_j} \tag{3.33}$$

or, in matrix notation,

$$J = \frac{\alpha^T \Lambda Z \Lambda \alpha}{\alpha^T \Lambda \alpha} \tag{3.34}$$

where

$$Z = [Z_{ij}] = [\delta_{ij} \frac{2-n_j}{2}] \tag{3.35}$$

Introducing the square root matrix, $\Lambda^{1/2}$,

$$\Lambda^{1/2} \Lambda^{1/2} = \Lambda \tag{3.36}$$

equation (3.34) may be written

$$J = \frac{\alpha^{*T} Z^* \alpha^*}{\alpha^{*T} \alpha^*} \tag{3.37}$$

where

$$\alpha^* = \Lambda^{1/2}\alpha$$

$$Z^* = \Lambda^{1/2} Z \Lambda^{1/2}$$

An eigenvalue problem is obtained by applying the calculus of variations to equation (3.37),

$$Z^* \alpha^* = \lambda \alpha^* \tag{3.38}$$

so that

$$\max J = \max |\lambda| \tag{3.39}$$

To avoid the necessity of actually obtaining the square root matrix, apply a similarity transform, which leaves the eigenvalues invariant,

$$\Lambda^{1/2} Z^* \Lambda^{-1/2} = \Lambda Z \tag{3.40}$$

If the magnitude of the largest eigenvalue of ΛZ is less than 1/2, so that relation (3.29) is satisfied, equation (3.24) is positive, and hence the problem is unique. The criterion thus developed depends on the zeroth and first harmonics of the land function, so that to verify uniqueness, a gravimetry-altimetry distribution must be chosen. The uniqueness verification process is pessimistic, because of the crudeness of approximation in the relation (3.25).

Uniqueness can be verified for the infinite-dimensional problem for an altimetry-gravimetry distribution considered later in the thesis (figure 1). The land coefficients may be obtained from the ocean coefficients, Ω_{ij}, obtained using the computer program given in appendix C.1,

$$\Lambda_{ij} = \delta_{ij} - \Omega_{ij} \tag{3.41}$$

Substituting the obtained values in equation (3.40),

$$\Lambda Z = \begin{bmatrix} 0.30 & 0.06 & 0.05 & 0.03 \\ 0.12 & 0.18 & 0.02 & 0.02 \\ 0.11 & 0.02 & 0.12 & 0.00 \\ 0.06 & 0.02 & 0.00 & 0.15 \end{bmatrix} \tag{3.42}$$

Since the maximum of the row sums bounds the eigenvalues (Todd, 1962, p. 284), the eigenvalues of this matrix are all less than 1/2, so that the problem of this thesis is unique for a gravimetry-altimetry distribution resembling the land-ocean distribution of the earth.

CHAPTER 4

EXISTENCE THEORY

4.1 General Discussion

In this chapter we discuss a method for solving the problem formulated in chapter 2 and study the conditions under which it will yield a solution. The problem is formulated as a Neumann series, which is valid when the operator is suitably "small". Next a nonsymmetric matrix approximation to the kernel of the operator is obtained. The matrix is then transformed into a form in which the matrix becomes symmetric under certain conditions. When these are made to hold, necessary and sufficient conditions for a solution to the finite problem are given. These conditions on the symmetric case are not satisfied when the full, infinite-dimensional operator is considered. A particularly simple version of the symmetric case is discussed in section 4.5. For the nonsymmetric form of the operator, analytic results are lacking, but for a finite approximation, numerical studies show that the problem can be solved for an altimetry-gravimetry distribution like that of the earth's ocean-land distribution.

4.2 Neumann Series Representation

To obtain a solution, we put the problem (equation (2.56)) in the classical form of a Fredholm integral equation of the second kind (see equation(2.55)),

$$[I(p,q) - K(p,q)]\zeta(q) = \nu(p) \qquad (4.01)$$

Unfortunately, the kernel and inhomogeneous terms contain discontinuities, and the kernel includes, in part, the identity operator. These considerations will be examined in later sections. Bitsadze (1968), Collatz (1960), and Courant and Hilbert (1953-1962) are representative of the mathematical methods to be considered for a solution. Here, an iterative solution and terminology used in investigating its validity are described. Rewrite the equation in the form

$$\zeta(p) = K(p,q)\zeta(q) + \nu(p) \tag{4.02}$$

If the operator, $K(p,q)$, is in some sense "small" compared to the identity operator, we try an iterative procedure,

$$\zeta^{(n+1)}(p) = K(p,q)\zeta^{(n)}(q) + \nu(p) \tag{4.03}$$

A convenient initial choice is

$$\zeta^{(0)}(p) = 0 \tag{4.04}$$

If the process converges it yields a solution to the equation. An alternative expression for the process is the Neumann series

$$\zeta(p) = \nu(p)+K(p,q)\nu(q)+K^{(2)}(p,q)\nu(q)+K^{(3)}(p,q)\nu(q)+\ldots \tag{4.05}$$

where the n^{th} iterated kernel is given by

$$K^{(n)}(p,q) = K(p,t)K^{(n-1)}(t,q) \tag{4.06}$$

and

$$K^{(0)}(p,q) = I(p,q) \qquad K^{(1)}(p,q) = K(p,q) \tag{4.07}$$

Another version is

$$\zeta(p) = \left[\sum_{n=0}^{\infty} K^{(n)}(p,q)\right]\nu(q) \tag{4.08}$$

The quantity in brackets is also known as the Neumann series and is in some sense the inverse of the operator

$$M(p,q) = [I(p,q) - K(p,q)]$$

Note the analogy with the well known series expansion

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n \tag{4.09}$$

which is valid for

$$|x| < 1 \tag{4.10}$$

We have a problem in functional analysis, since we are not considering a function, but a functional (or operator) on a class of functions. In order to establish the convergence of the Neumann series an analogous inequality must be established for the operator K. First, an operation analogous to taking the absolute value of a complex number must be defined. The admissible functions are functions defined on a sphere. Such functions constitute a linear vector space on which an inner product is defined:

$$(u,v) = \frac{1}{4\pi}\iint_{\sigma} u(p)v(p)d\sigma_p = (v,u) \tag{4.11}$$

Analogous to absolute value of a number or the length of a vector is the norm of a function,

$$||u|| = [(u,u)]^{1/2} \tag{4.12}$$

A complete set of basis vectors spanning this space is the set of normalized spherical harmonics, $x_i(p)$, defined in equation (2.38). Equation (3.13) can now be written

$$(x_i(p),x_j(p)) = \delta_{ij} \tag{4.13}$$

and

$$||x_i(p)|| = 1 \qquad\qquad 1 \leq i < \infty \tag{4.14}$$

The norm of an operator is defined in terms of the norm of a function by

$$||K(p,q)|| = \sup_x \left\{ ||K(p,q)x(q)||; ||x||=1 \right\} \quad (4.15)$$

It is the least upper bound on the norm of the function, K(p,q)x(q), when all possible x(q) of unit norm are considered. The norm of the operator corresponds to the absolute value operator of equation (4.10).

Corresponding to the radius of convergence of equation (4.09) is the spectral radius of the operator, K, $r_\sigma(K)$

$$r_\sigma(K) = \sup_{\lambda \in \sigma(K)} |\lambda| \quad (4.16)$$

$r_\sigma(K)$ is the least upper bound of the absolute value of the spectrum, $\sigma(K)$, of the operator, K, which for a finite-dimensional operator is a finite set of numbers, λ, its eigenvalues, for which the operator,

$$[\lambda I(p,q) - K(p,q]$$

fails to have an inverse. For infinite-dimensional operators, matters are more complicated; not only can there be an infinite number of eigenvalues, but other types of points can lie in the spectrum. These are too difficult to describe here; see Taylor (1958). The Neumann series (equation (4.08)) is a formal expansion of the resolvent operator,

$$R_\lambda = [\lambda I - K]^{-1} \quad (4.17)$$

with $\lambda = 1$. The kernel of the resolvent operator differs from the resolvent kernel used in classical integral equation terminology (Hildebrand, 1953, p. 430) in that the latter kernel does not contain the initial delta function corresponding to the identity operator. To establish the validity of the convergence of equation (4.08) the applicable theory of functional analysis (Taylor, 1958, p. 262) is quoted.

If K is an operator on a complete complex linear vector space, the resolvent is given by

$$R_\lambda = \sum_{n=1}^{\infty} \lambda^{-n} K^{(n-1)} \tag{4.18}$$

if

$$|\lambda| > r_\sigma(K) \tag{4.19}$$

This series also represents R_λ if the series converges and

$$|\lambda| = r_\sigma(K) \tag{4.20}$$

The series diverges if

$$|\lambda| < r_\sigma(K) \tag{4.21}$$

An alternative formulation in terms of norms of iterated kernels is

$$r_\sigma(K) = \lim_{n\to\infty} ||K^{(n)}||^{1/n} \tag{4.22}$$

For our problem with $\lambda = 1$, we require

$$r_\sigma(K) < 1 \tag{4.23}$$

This holds if some iterated kernel, K^n, is a contraction operator (Vulich, 1963),

$$||K^{(n)}|| < 1 \tag{4.24}$$

The resolvent operator then yields a unique solution (Chu and Diaz, (1965)).

Koch (1967) considers a similar iterative approach for the case when only gravimetry is prescribed.

4.3 Matrix Representation of the Operator

For the subsequent work, a matrix representation is needed for the kernel of the operator, M(p,q), defined in equation (2.57), or equivalently, $K(p,q)$, defined in equation (2.54), which is the kernel of equation (2.55), whose solution, if it converges, is the Neumann series of equation (4.08). Because the kernel is needed to verify convergence, its representation is obtained in this section. Alternative Neumann series formulations are developed later in the chapter; their matrix representations can be obtained directly from that of $M(p,q)$, which is related to $K(p,q)$ by

$$M(p,q) = I(p,q) - K(p,q) \tag{4.25}$$

Hence we need only find a suitable representation for the kernel,

$$K(p,q) = \begin{cases} I(p,q) + \beta K_N(p,q) & p \in S_0 \\ -K_N(p,q) & p \in S_1 \end{cases} \tag{2.54}$$

Since the boundary surface, S, is a sphere, the normalized spherical harmonics (see equation (2.38)), $x_i(p)$, are a suitable set of basis vectors for representing the kernel. From section 2.3, the kernel of the identity operator is

$$I(p,q) = \sum_{i=1}^{\infty} x_i(p)x_i(q) \tag{2.43}$$

Similarly the kernel of the modified Neumann operator is

$$K_N(p,q) = -\sum_{i=1}^{\infty} \frac{2}{n_i+1} x_i(p)x_i(q) \tag{2.44}$$

where n_i is defined in equation (2.38). To find a single representation for the kernel valid both on S_0 and S_1, define, in conjunction with the land function of equation (2.34), the ocean function,

$$\Omega(p) = \begin{cases} 1 & p \in S_0 \\ 0 & p \in S_1 \end{cases} \tag{4.26}$$

It is related to the land function by

$$\Omega(p) = 1 - \Lambda(p) \tag{4.27}$$

Then

$$K(p,q) = \Omega(p)[I(p,q)+\beta K_N(p,q)]-\Lambda(p)K_N(p,q) \tag{4.28}$$

We may also write

$$K(p,q) = -K_N(p,q)+\Omega(p)[I(p,q)+(1+\beta)K_N(p,q)] \tag{4.29}$$

or

$$K(p,q) = \sum_{j=1}^{\infty}\left[x_j(p)\frac{2}{n_j+1}x_j(q)+\Omega(p)x_j(p)(1-2\mu_j)x_j(q)\right] \tag{4.30}$$

where

$$\mu_j = \frac{1+\beta}{n_j+1}$$

It is also desirable to have an expression in which the arguments appear only in the form of spherical harmonics. We thus expand the function, $[\Omega(p)x_j(p)]$, in terms of spherical harmonics. With use of Parseval's identity, the representation

$$[\Omega(p)x_j(p)] = \sum_{i=1}^{\infty}\Omega_{ji}x_i(p) \tag{4.31}$$

follows, where the coefficients Ω_{ji} are given by

$$\begin{aligned}\Omega_{ji} &= ([\Omega(p)x_j(p)],x_i(p)) \\ &= \frac{1}{4\pi}\iint_{\sigma}[\Omega(p)x_j(p)]x_i(p)d\sigma_p \\ &= \frac{1}{4\pi}\iint_{\sigma_0}x_j(p)x_i(p)d\sigma_p\end{aligned} \tag{4.32}$$

and σ_0 is the solid angle corresponding to the area S_0. A listing of a computer program that calculates these coefficients may be found in appendix C.1.

Note that

$$\Omega_{ij} = \Omega_{ji} \tag{4.33}$$

Since equation (4.13) holds,

$$|\Omega_{ij}| \leq 1 \tag{4.34}$$

and

$$0 \leq \Omega_{ii} \leq 1 \tag{4.35}$$

For the nontrivial mixed data problem the strict inequalities holds. Application of the Cauchy - Buniakovskii - Schwarz inequality (Hardy, et al., 1934) yields the further restriction,

$$|\Omega_{ij}| \leq [\Omega_{ii}\Omega_{jj}]^{1/2} \tag{4.36}$$

Substituting into the representation for the kernel we obtain

$$K(p,q) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[\delta_{ij} \frac{2}{n_j+1} + \Omega_{ij}(1-2\mu_j) \right] x_i(p) x_j(q) \tag{4.37}$$

As a short hand notation we suppress writing the spherical harmonic basis vectors and express K(p,q) as an infinite matrix (Cooke, 1950) of spherical harmonic coefficients,

$$K(p,q) = [K_{ij}] \tag{4.38}$$

where

$$K_{ij} = \delta_{ij} \frac{2}{n_j+1} + \Omega_{ij}(1-2\mu_j) \tag{4.39}$$

Similarly, a vector is represented as a column of its spherical harmonic coefficients, and a product is the inner product of equation (4.11).

Note that iterated kernels may be formed by successive premultiplications of the matrix.

$$K^{(n)}(p,q) = [K_{ik}K_{k\ell}\cdots K_{nm}K_{mj}] \tag{4.40}$$

Here, and in the remainder of the chapter, we employ the summation convention when matrix or vector products are indicated. Analysis of infinite-dimensional matrices is difficult; for example, the associative law may not necessarily be valid for products of infinite-dimensional matrices, although it is true for diagonal matrices, such as the matrix representations of the identity and Neumann operators. The representation of equation (4.38) contains off-diagonal terms, which are intimately associated with the discontinuity of the kernel and the fact that the spherical harmonics are not orthogonal over the oceans (see equation (4.32)). In addition, the discontinuity is with respect to only one of the two variables, so that the matrix is nonsymmetric. In the practical case, of course, the matrices must be truncated. The associative law is then strictly valid for a given approximation, but the kernel is smoothed, and the discontinuity is lost. Truncation has the effect of confining the spectrum of the solution, eliminating the complications mentioned in section 4.2. Determination of the spectrum is simple for a diagonal matrix, since the eigenvalues are just the diagonal terms. For an arbitrary, nonsymmetric, finite matrix, it is difficult enough just to determine the largest modulus of these, the spectral radius.

4.4 Analytical Criteria for Convergence

From the formulation of the problem of this thesis given in equation (2.55), an iterative solution (see equation (4.08)), has been developed. The iteration converges only when the spectral radius of the kernel satisfies inequality (4.23). Unfortunately, the kernel is nonsymmetric (see equation (4.39)), so that analytical conditions necessary or sufficient for the Neumann series solution to be valid are difficult to obtain. In section 4.6, numerical procedures establish that a truncated form of the kernel with special choices of the ocean function, Ω, and weight parameter, β, has a spectral radius that satisfies inequality (4.23). To obtain insight into the problem, an analytical study is also desirable.

To this end, we start with the problem in the form of equation (2.56). M(p, q) can also be written in matrix form, using equations (4.25) and (4.39); its elements are:

$$M_{ij} = \delta_{ij}\left(1 - \frac{2}{n_j+1}\right) + \Omega_{ij}(2\mu_j - 1) \tag{4.41}$$

where

$$\mu_j = \frac{1+\beta}{n_j+1}$$

and n_j is defined in equation (2.38). Because symmetric matrices are more convenient to handle analytically, a symmetrizing transformation is sought. A similarity transformation, $S = [S_{ij}]$, leaves the eigenvalues, and hence, the spectral radius, invariant (see, for example, Hildebrand, 1952), so that the spectra, $\sigma(M)$ and $\sigma(SMS^{-1})$, are identical (of course, S must be nonsingular). Hence an alternative formulation for equation (2.56) is

$$(SMS^{-1})(S\zeta) = S\nu \tag{4.42}$$

The solution of equation (2.56) can then be reduced to the inversion of

$$M_S = SMS^{-1} \tag{4.43}$$

using, for example, an appropriate Neumann series formulation.

Require that

$$\beta \neq \frac{n_j-1}{2} \tag{4.44}$$

so that $2\mu_j \neq 1$

For all n_j, $o \le n_j \le n_M$

where n_M = maximum degree of harmonic approximation. When this is violated, so that

$$\beta = \frac{n_k - 1}{2}$$

for some n_k, $0 \le n_k \le n_M$, the matrix is 'decomposable' (Todd, 1962, p. 285); the spherical harmonic basis vectors can be reordered so that the n_k th harmonic terms come first, yielding

$$M = \left[\begin{array}{c|c} M^{(k)} & M^{(k-)} \\ \hline 0 & M^{(-)} \end{array}\right]$$

Except when $\beta = 0$ (the problem is then clearly improperly posed) the $(2n_k + 1) \times (2n_k + 1)$ diagonal matrix, $M^{(k)}$, is clearly invertable. The standard partitioning technique for matrix inverses (see, for example, Todd, 1962, p. 238) thus yields

$$M^{-1} = \left[\begin{array}{c|c} M^{(k)-1} & -M^{(k)-1} \quad M^{(k-)} \quad M^{(-)-1} \\ \hline 0 & M^{(-)-1} \end{array}\right]$$

so that to study the validity of the inverse, M^{-1}, one need only consider the matrix, $M^{(-)}$, in which the rows and columns corresponding to $\beta = (n_k - 1)/2$ are removed. Define s so that

$$2\mu_j > 1 \qquad \text{or } \beta > \frac{n_j - 1}{2} \qquad j \le s \tag{4.45}$$

and

$$2\mu_j < 1 \qquad \text{or } \beta < \frac{n_j - 1}{2} \qquad j > s \tag{4.46}$$

The following similarity transform leaves diagonal terms invariant:

$$S_{ij} = \delta_{ij}(|2\mu_j - 1|)^{1/2} \tag{4.47}$$

Its inverse is

$$S_{ij}^{-1} = \delta_{ij}(|2\mu_j - 1|)^{-1/2} \tag{4.48}$$

Partition M_S,

$$M_S = \left[\begin{array}{c|c} A & -E^T \\ \hline E & D \end{array}\right] \tag{4.49}$$

where

$$A = [A_{ij}] = \left[\delta_{ij}(1 - \frac{2}{n_j+1}) + \Omega_{ij}(2\mu_i - 1)^{1/2}(2\mu_j - 1)^{1/2}\right] \tag{4.50}$$

$$1 \le i,\ j \le s$$

$$D = [D_{ij}] = \left[\delta_{ij}(1 - \frac{2}{n_j+1}) - \Omega_{ij}(1 - 2\mu_i)^{1/2}(1 - 2\mu_j)^{1/2}\right] \tag{4.51}$$

$$s < i,\ j \le t$$

$$E = [E_{ij}] = \left[- \Omega_{ij}(1 - 2\mu_i)^{1/2}(2\mu_j - 1)^{1/2}\right] \tag{4.52}$$

$$1 \le j \le s < i \le t$$

where $t = (n_M + 1)^2$. Hence M_S is the sum of a symmetric part,

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right],$$

and a skew-symmetric part,

$$\left[\begin{array}{c|c} 0 & -E^T \\ \hline E & 0 \end{array}\right].$$

It is always possible to obtain complete symmetry by a similarity transformation (Gantmacher, 1959, p. 13), but the symmetric matrix is complex in the case considered here. Let

$$S_{jk}^{(i)} = \begin{cases} (i)^{1/2}\delta_{jk} & j \le s \\ (i)^{-1/2}\delta_{jk} & j > s \end{cases} \tag{4.53}$$

Then

$$S^{(i)-1} M_S S^{(i)} = \left[\begin{array}{c|c} A & iE \\ \hline iE & D \end{array}\right] \tag{4.54}$$

This form will not be used here.

If $$\beta < -\frac{1}{2} \tag{4.55}$$

$$M_S = D \tag{4.56}$$

If

$$\beta > \frac{n_M - 1}{2} \tag{4.57}$$

$$M_S = A \tag{4.58}$$

These cases, in which the operator is symmetrized, will be considered subsequently.

In general, M is nonsymmetric. Even here there may be a solution involving only symmetric inverses (if the inverses exist). Using the standard partitioning technique for the inverse of a finite matrix (see, for example, Todd, 1962, p. 238),

$$M_S^{-1} = \left[\begin{array}{c|c} \Xi & -\Gamma^T \\ \hline \Gamma & \Delta \end{array}\right] \tag{4.59}$$

where

$$\begin{aligned} \Delta &= [D + BA^{-1} B^T]^{-1} \\ \Xi &= A^{-1} - A^{-1} B^T \Delta B A^{-1} \\ \Gamma &= - \Delta B A^{-1} \end{aligned} \tag{4.60}$$

If A is singular, but D is not, the obvious modification may be made. To obtain

$$\zeta = S^{-1} M_S^{-1} S \nu \tag{4.61}$$

the existence of the inverses is not necessary, but only sufficient for equation (4.59) to be valid. A practical verification of the existence of the inverses for matrices of useful size must rely on numerical procedures. The application of numerical techniques is considered later, but not using this form. The similarity transformation does not simplify the analysis when

$$-\frac{1}{2} < \beta < \frac{n_M - 1}{2} \tag{4.62}$$

so that in the numerical study, in which a typical value of $\beta = 1$ was chosen, the original K matrix (equation (4.39)) was examined. In section 4.6 a numerical determination of the spectral radius shows that a Neumann series solution is valid for a particular land-ocean geometry resembling the earth. Sufficiency having been established for the particular combination of β, Ω, and n_M, chapter 5 describes a numerical simulation illustrating the determination of the harmonic coefficients using this method.

To explore possible solution methods for which additional analytic tools are available the cases in which β satisfies the inequalities (4.55) and (4.57) are next examined in detail. M_S is now symmetric (see equations (4.56) and (4.58)), and its eigenvalues are all real. The minimax and maximin theorems (Courant and Hilbert, 1953, or Householder, 1964) are applicable:

$$\max_{E_m} \quad \min_{x \in E_m} \lambda(M_S) = \lambda_m(M_S) \tag{4.63}$$

and

$$\min_{E_m} \quad \max_{x \in E_m} \lambda(M_S) = \lambda_{t-m+1}(M_S) \tag{4.64}$$

where

$$\lambda(M_S) = \text{Rayleigh quotient} = \frac{x^T M_S x}{x^T x}$$

$$\lambda_1(M_S) \geq \lambda_2(M_S) \geq \dots \geq \lambda_t(M_S) \tag{4.65}$$

and E_m is a subspace of the entire space, E_t, for which the set of spherical harmonics, $x_i(p)$, $1 \leq i \leq t$, is a basis. Considering

$$m = t$$

any Rayleigh quotient of a real symmetric matrix lies on the closed interval between the largest and smallest eigenvalues. For example, let x have only the i^{th} component nonzero. Hence

$$\lambda_1(M_S) \geq M_{S_{ii}} \geq \lambda_t(M_S) \tag{4.66}$$

for all i, $1 \leq i \leq t$. Hence lower bounds on the maximum magnitude of the eigenvalues (spectral radius) may be obtained.

For convenience, introduce a parameter, τ, which can be chosen to facilitate convergence of the Neumann series. Let

$$\begin{aligned} M_S &= B + C^2 \\ &= C[I + C^{-1}BC^{-1}]C \\ &= C[I + B_C]C \end{aligned} \tag{4.67}$$

where

$$C = [C_{ij}] = [\delta_{ij}(1 + \tau)^{1/2}] \qquad \tau > -1 \tag{4.68}$$

When inequality (4.55) holds,

$$\begin{aligned} B = [B_{ij}] &= D - (1 + \tau)\ I = \\ &= \left[-\left(\tau + \frac{2}{n_j+1}\right)\delta_{ij} - \Omega_{ij}(1 - 2\mu_i)^{1/2}(1 - 2\mu_j)^{1/2} \right] \\ &\qquad \beta < -\frac{1}{2} \end{aligned} \tag{4.69}$$

When inequality (4.57) holds,

$$\begin{aligned} B = [B_{ij}] &= A - (1 + \tau)I \\ &= \left[-\left(\tau + \frac{2}{n_j+1}\right)\delta_{ij} + \Omega_{ij}(2\mu_i - 1)^{1/2}(2\mu_j - 1)^{1/2} \right] \\ &\qquad \beta > \frac{n_M-1}{2} \end{aligned} \tag{4.70}$$

In both cases

$$B_C = C^{-1}BC^{-1} = [B_{C_{ij}}]$$
$$= \frac{1}{1+\tau} B \tag{4.71}$$

And the diagonal terms are of the same form,

$$B_{C_{jj}} = \frac{1}{1+\tau} \left\{ -(\tau + \frac{2}{n_j+1}) + \Omega_{jj}(2\mu_j - 1) \right\} \tag{4.72}$$

To establish the validity of the Neumann series representation,

$$M_S^{-1} = C^{-1}\left[\sum_{n=0}^{\infty} (-1)^n B_C^{(n)}\right] C^{-1} \tag{4.73}$$

It must be shown that

$$r_\sigma(B_C) < 1 \tag{4.74}$$

(see section 4.2).

We now develop inequalities that must necessarily hold in order to invert M_S using the Neumann series of equation (4.73) when β satisfies one of the inequalities (4.55) and (4.57) and τ satisfies condition (4.68). Applying the inequality (4.66), it is necessary for the representation (4.73) to be valid that

$$|B_{C_{jj}}| < 1 \qquad \text{for all } j,\ 1 \leq j \leq t \tag{4.75}$$

Let inequality (4.55) hold, and set $j = 1$, so that $n_j = 0$ and $\mu_j = 1 + \beta$

$$B_{C_{11}} = \frac{1}{1+\tau} \left[-\tau - 2 - \Omega_{11}(-1 - 2\beta) \right]$$
$$= -1 - \frac{1}{1+\tau} - \Omega_{11}\left(\frac{-1-2\beta}{1+\tau}\right) \qquad \beta < -\frac{1}{2} \tag{4.76}$$

In view of the inequalities (4.35), (4.55), and (4.68),

$$B_{C_{11}} < -1 \qquad \beta < -\frac{1}{2} \tag{4.77}$$

Hence the Neumann series is not valid (a valid Neumann series may be obtained for suitable values of τ and β for which $\tau < -1$ and $\beta < -\frac{1}{2}$. The derivation is not given here, since it is similar to the one given shortly, see also section 4.5).

When inequality (4.57) holds, a necessary condition for the validity of the Neumann series is

$$\frac{1}{1+\tau}\left|\Omega_{jj}(2\mu_j - 1) - \tau - \frac{2}{n_j+1}\right| < 1 \qquad \beta > \frac{n_M-1}{2} \quad 1 \le j \le t \tag{4.78}$$

For example, let $n_M = 2 \qquad \beta = 1 \qquad \tau = 0$ (4.79)

For j = 1 inequality (4.78) becomes

$$|3\,\Omega_{11} - 2| < 1 \tag{4.80}$$

so that

$$\frac{1}{3} < \Omega_{11} < 1 \tag{4.81}$$

must hold for equation (4.73) to be valid. Similarly, for $2 \le j \le 4$, $n_j = 1$

$$|\Omega_{jj} - 1| < 1 \tag{4.82}$$

so that

$$\Omega_{jj} > 0 \tag{4.83}$$

is required. For $n_j = 2$, no useful result is obtained. When equations (4.79) hold, numerical studies, described in section 4.6, indicate which of several choices of the ocean function allow the spectral radius of the operator to be small enough so that equation (4.73) is valid. The smallest satisfactory amount of ocean is greater than the 1/3 requirement of inequality (4.81). (The original matrix was used, but the eigenvalues, and hence the necessary conditions are the same). It should be noted that even if convergence is not valid here for a particular land-ocean geometry, this does not rule out a solution in a different form.

In general, inequality (4.78) becomes

$$\frac{\frac{2}{n_j+1} - 1}{2\mu_j - 1} < \Omega_{jj} < \frac{2\tau+1+\frac{2}{n_j+1}}{2\mu_j-1} \qquad 2\mu_j > 1 \qquad \tau > -1 \tag{4.84}$$

The lower bound is independent of τ,

$$\Omega_{jj} > \frac{1-n_j}{1-n_j+2\beta} \qquad \beta > \frac{n_M-1}{2} \tag{4.85}$$

The only useful restrictions are

$$\Omega_{11} > \frac{1}{1+2\beta} \tag{4.86}$$

and

$$\Omega_{jj} > 0 \qquad 2 \leq j \leq 4 \tag{4.87}$$

If ocean areas are small, β can be chosen sufficiently large so that inequality (4.86) is satisfied. The upper bound in condition (4.84) is lowered by this action, but since τ is still available as a free parameter, it is plausible to assert that a combination of β satisfying inequality (4.57) and τ satisfying condition (4.68) can be so chosen that the necessary condition (4.75) for the representation (4.73) to be valid is satisfied as long as oceans cover a finite area.

It turns out that not only is this so, but β and τ can be chosen to assure convergence of this formulation: J. E. Potter (personal communication) has outlined a proof specifying values of β and τ that are sufficient for establishing equation (4.73). Potter's proof is now only sketched, since a similar, but simpler, proof under the same assumptions is provided for the formulation of the next section. Rewrite equation (4.71) in the form

$$B_C = -\frac{\tau}{1+\tau}I + \frac{1}{1+\tau}K_N + \frac{1}{1+\tau}S\Omega S \tag{4.88}$$

A matrix, B_C, is negative (positive) definite if the Rayleigh quotient,

$$\frac{x^T B_C x}{x^T x}$$

is less (greater) than zero for all nontrivial vectors, x. Its eigenvalues are hence all negative (positive). I is positive definite, and

$$|| \; I \; || = 1 \tag{4.89}$$

K_N is negative definite, and

$$|| \; K_N \; || = 2 \tag{4.90}$$

The infinite dimensional matrix, Ω, is only positive semidefinite, with eigenvalues of magnitude 0 and 1. The eigenvalues of the finite matrix are bounded by these, so that

$$|| \; \Omega \; || \leq 1 \tag{4.91}$$

An absolute inequality holds on the lower bound, since, as is now shown, the finite approximation is positive definite. If Ω is only positive semidefinite, there is at least one nontrivial function, f, such that

$$f^T \Omega f = 0$$

Hence Ω depends at most on only $(n_M + 1)^2 - 1$ independent basis vectors, which can be formed by the Gram-Schmidt orthogonalization process (see, for example, Garabedian, 1964), using f as the first component. Application of equation (4.32) yields

$$\Omega_{11} = 0 = \frac{1}{4\pi} \iint_{\sigma} \Omega(p) \, [f(p)]^2 d\sigma_p$$

$$= \frac{1}{4\pi} \iint_{\sigma_0} [f(p)]^2 d\sigma_p$$

Hence f(p) must be identically zero on oceans, but nontrivial on land. But f(p) is at most a polynomial (Hobson, 1955, p. 120) of degree n_M in (x, y, z), and z may be eliminated, since f(p) is confined to the surface of a sphere. On any interval, a polynomial of degree n_M can have at most n_M roots (see, for example, Cheney, 1966, p. 74). Considering y as a parameter, the locus of roots of the polynomial in x may be obtained; this is just a series of lines, of which at most n_M intersect any line,

$$y = \text{constant.}$$

The process may be repeated with the roles of x and y reversed. Only on the union of root loci does

$$f(p) = 0$$

but this does not constitute a finite area, so that Ω is positive definite. It is not difficult to extend this proof to show that the strict inequality holds in equation (4.91), but such a result is not needed in the sequel.

The finite matrix, S, is positive definite, and for

$$\beta > \frac{n_M - 1}{2} \qquad (4.57)$$

its norm is

$$|| S || = (1 + 2\beta)^{1/2} \qquad (4.92)$$

Hence τ may be chosen sufficiently large so that B_C is negative definite. Choose, for example,

$$\tau > 1 + 2\beta \geq || S ||^2 || \Omega || \geq || S\Omega S || \qquad (4.93)$$

(for the operator manipulations, see for example, Halmos, 1951). Hence

$$B_C < 0 \qquad (4.94)$$

Now take β so large that the operator

$$K_N + S\Omega S > 0 \qquad (4.95)$$

The positive definiteness of Ω insures the existence of the lower norm,

$$|| \Omega ||_L = \frac{1}{||\Omega^{-1}||}$$

This is a lower bound on the magnitude of the eigenvalue closest to the origin. For S,

$$|| S ||_L = \left[\frac{1+2\beta-n_M}{n_M+1}\right]^{1/2} \tag{4.96}$$

and

$$|| S ||_L^2 || \Omega ||_L \leq || S\Omega S ||_L \tag{4.97}$$

Bounds for the eigenvalues of the composite matrix may be formed by an appropriate translation of bounds of the individual matrices (see, for example, Householder, 1964, chapter 3). Since SΩS is positive definite and K_N is negative definite the inequality (4.95) holds if

$$|| S\Omega S ||_L > || K_N || \tag{4.98}$$

Hence require

$$\beta > \frac{n_M+1}{||\Omega||_L} + \frac{n_M-1}{2} \tag{4.99}$$

so that the relation (4.93) becomes

$$\tau > 2\,\frac{n_M+1}{||\Omega||_L} + n_M \tag{4.100}$$

These values assure the unique solvability of the finite approximation of equation (2.56). It should be cautioned that β must be increased greatly as n_M is increased, so that it is an open question whether the approximation to SΩS is thereby improved; successive solutions may not agree.

The requirements on β and τ are pessimistic; for better convergence smaller values might be tried. The conditions necessary for this formulation to converge than serve as lower bounds on the permissible values of β and τ. The relation (4.85) becomes

$$\beta > \frac{1-n_j}{2}\left[\frac{1}{\Omega_{jj}} - 1\right] \qquad 1 \le j \le 4 \quad (4.101)$$

The right-hand inequality in relation (4.84) may be written

$$\beta < \frac{n_j-1}{2} + \frac{n_j+1}{\Omega_{jj}} \tau + \frac{n_j+3}{2\Omega_{jj}} \qquad 1 \le j \le t \quad (4.102)$$

or

$$\tau > \frac{(1+2\beta-n_j)\Omega_{jj}-n_j-3}{2(n_j+1)} \qquad 1 \le j \le t \quad (4.103)$$

4.5 A Symmetric Reformulation

In this section the nonsymmetric matrix is factored into the product of a symmetric and a diagonal matrix; the eigenvalues of the matrix and its factors are not simply related. When

$$-\frac{1}{2} < \beta < \frac{n_M-1}{2} \qquad (4.62)$$

so that the similarity transformation of the last section does not symmetrize the matrix, the two matrix factors are indefinite; no further analysis is considered here. For the finite matrix approximation when

$$\beta < -\frac{1}{2} \qquad (4.55)$$

or

$$\frac{n_M-1}{2} < \beta \qquad (4.57)$$

conditions sufficient for the unique solvability of equation (2.56) are established. The results appear to be better than those obtained under these conditions in section 4.4; in effect, C in equation (4.67) is taken to be a diagonal matrix of variable elements rather than a scalar times the identity matrix. For simplicity, the results are obtained directly from equation (2.56), which may be written in the form,

$$M\zeta = \nu \qquad (4.104)$$

where

$$M = [M_{ij}] = [\delta_{ij}(1 - \frac{2}{n_j+1}) + \Omega_{ij}(2\mu_j - 1)] \qquad 1 \le i,\ j \le t \quad (4.41)$$

$$\mu_j = \frac{1+\beta}{n_j+1}$$

and n_j is defined in equation (2.38). ν is derived from measured altimetry and gravimetry data (see equation (2.53)), and ζ is the unknown function related to the gravitational potential (see equation (2.39)) to be determined. Define the matrix

$$L = [L_{ij}] = \delta_{ij}(2\mu_j - 1) \qquad (4.105)$$

It is required here that

$$\beta \ne \frac{n_j-1}{2} \qquad (4.44)$$

so that $2\mu_j \ne 1$ for all n_j, $0 \le n_j \le n_M$ where n_M = maximum degree of harmonic approximation. As discussed in section 4.4, restriction (4.44) can be relaxed. Comparing with equation (4.47)

$$L = S^2 \qquad \text{if } s \ge t \qquad (4.106)$$

write

$$\begin{aligned} \nu &= ML^{-1}L\zeta \\ &= M^*\zeta^* \end{aligned} \qquad (4.107)$$

where

$$\zeta^* = L\zeta \qquad (4.108)$$

and

$$\begin{aligned} M^* &= ML^{-1} = [M^*_{ij}] \\ &= \left[\delta_{ij}\,\frac{n_j-1}{1+2\beta-n_j} + \Omega_{ij}\right] \end{aligned} \qquad (4.109)$$

If the symmetric matrix, M^*, can be inverted, equation (4.104) may be solved,

$$\zeta = L^{-1}\zeta^{*} = L^{-1}M^{*-1}\nu \tag{4.110}$$

A sufficient condition for M^{*-1} to exist, and therefore for equation (4.104) to be uniquely solvable, is that M^{*} be positive definite, so that all of its eigenvalues exceed zero. M^{*} is composed of a diagonal matrix, whose eigenvalues are just

$$\lambda_j = \frac{n_j - 1}{1 + 2\beta - n_j} \tag{4.111}$$

and the ocean function, Ω, whose finite-dimensional approximation is positive definite (see section 4.4), so that

$$\begin{aligned} 0 < || \ \Omega \ ||_L = \min_j \ (\lambda_j(\Omega)) \leq \\ \leq \max_j \ (\lambda_j(\Omega)) = || \ \Omega \ || \leq 1 \end{aligned} \tag{4.112}$$

The norm, $|| \ \Omega \ ||$, is defined as in equation (4.15). The lower norm is

$$|| \ \Omega \ ||_L = \frac{1}{||\Omega^{-1}||} \tag{4.113}$$

In the infinite-dimensional case, the upper bound on the spectrum of Ω is unity and the lower bound is zero.

Bounds on the eigenvalues of M^{*} may be formed by taking the algebraic sums (see, for example, Householder, 1964, chapter 3; actually the strict inequalities hold, since the matrices are symmetric).

$$\min_j(\lambda_j(M^{*})) \geq \min_j(\lambda_j(\Omega)) + \min_j\left(\frac{n_j - 1}{1 + 2\beta - n_j}\right) \tag{4.114}$$

$$\max_j(\lambda_j(M^{*})) \leq \max_j(\lambda_j(\Omega)) + \max_j\left(\frac{n_j - 1}{1 + 2\beta - n_j}\right) \tag{4.115}$$

Hence M^{*} is positive definite if

$$|| \ \Omega \ ||_L + \min_j\left(\frac{n_j - 1}{1 + 2\beta - n_j}\right) > 0 \tag{4.116}$$

This can hold if

$$\beta > \frac{1}{2}\left(\frac{1}{||\Omega||_L} - 1\right) \tag{4.117}$$

or

$$-\beta > \frac{n_M-1}{2}\left(\frac{1}{||\Omega||_L} - 1\right) \tag{4.118}$$

For the remaining values of β positive definiteness cannot be guaranteed (But note that L and M^* could become indefinite in a manner in which M remains definite). The problem, although possibly not in this form, can still be solved, see the next section.

If they are compared to the sufficient condition requirement (4.99) of section 4.4, the inequalities (4.117) and (4.118) can be seen to require values of β of smaller magnitude; the inequality (4.117) is the best in this respect. Applying also relation (4.115), bounds on the spectrum of M^* are obtained. When inequality (4.117) holds,

$$0 < ||\ \Omega\ ||_L - \frac{1}{1+2\beta} \leq \lambda_j(M_+^*) \leq ||\ \Omega\ || + \frac{n_M-1}{1+2\beta-n_M} \tag{4.119}$$

When inequality (4.118) holds.

$$0 < ||\ \Omega\ ||_L - \frac{n_M-1}{n_M-1-2\beta} \leq \lambda_j(M_-^*) \leq ||\ \Omega\ || + \frac{1}{-1-2\beta} \tag{4.120}$$

Since the eigenvalues are real, bounded, and positive (as long as $||\ \Omega\ ||_L > 0$), a convergent Neumann series,

$$\begin{aligned} M^{*-1} &= [\xi I - (\xi I - M^*)]^{-1} \\ &= \sum_{n=1}^{\infty} \xi^{-n}(\xi I - M^*)^{(n-1)} \end{aligned} \tag{4.121}$$

can always be found by choosing ξ (which corresponds to $(1 + \tau)$ of section 4.4) sufficiently large. To minimize the spectral radius, choose, when inequality (4.117) holds,

$$\xi = \xi_+ = ||\ \Omega\ ||_L - \frac{1}{1+2\beta} + \frac{1}{2}\ ||\ \Omega\ || + \frac{1}{2}\ \frac{n_M-1}{1+2\beta-n_M} \tag{4.122}$$

so that

$$r_\sigma(M_+^*) = \frac{1}{2}\left[|| \ \Omega \ ||_L - \frac{1}{1+2\beta} + || \ \Omega \ || + \frac{n_M - 1}{1+2\beta - n_M} \right] \qquad (4.123)$$

and when inequality (4.118) holds,

$$\xi = \xi_- = || \ \Omega \ ||_L - \frac{n_M - 1}{n_M - 1 - 2\beta} + \frac{1}{2} \ || \ \Omega \ || + \frac{1}{2} \ \frac{1}{-1-2\beta} \qquad (4.124)$$

so that

$$r_\sigma(M_-^*) = \frac{1}{2}\left[|| \ \Omega \ ||_L - \frac{n_M - 1}{n_M - 1 - 2\beta} + || \ \Omega \ || + \frac{1}{-1-2\beta} \right] \qquad (4.125)$$

Convergence can be improved if β is chosen consistent with the previous constraints so that $r_\sigma(M^*)/\xi$ is minimized. Hence when

$$|| \ \Omega \ ||_L > 0 \qquad (4.126)$$

sufficient conditions are obtained for equation (4.104) to be uniquely solvable (see the end of section 4.2). These results are consistent with those of chapter 3; altimetry must cover a finite area, since otherwise the ocean function vanishes. As the degree of harmonic approximation is increased, $|| \ \Omega \ ||_L$ approaches zero, and β becomes very large. In equation (4.110), the operator, L^{-1}, is 'small', but M^{*-1} is 'large'. It is an open question whether successive solutions will approach a limit as n_M is increased. Trouble could occur if the configuration of the numerical approximation approaches conditions that give rise to a nonunique solution in the infinite-dimensional case. Development of solution methods to handle such occurences, possibly requiring consistency conditions on the measured data, must be left for the future (for a sufficient condition independent of the form of the ocean function, see appendix D).

4.6 Numerical Criteria for Convergence

It is not practical to attempt to determine analytically the spectral radius of the operator in its nonsymmetric formulation. A numerical study would determine most feasibly whether the Neumann series then forms the basis for a solution. This has the drawback that a determination can be made only for a particular choice of the land-ocean configuration. The drawback is not as restrictive as it sounds, since for finite matrix representations, the eigenvalues, and therefore the spectral radius, are continuous functions of the land-ocean configuration (Ostrowski, 1960). For the full operator we can show that the norm varies continuously with perturbations of the boundary between land and oceans; see appendix E. The norm is related to the spectral radius (see equation (4.22)), but the continuity of the spectral radius for the infinite-dimensional operator is an open question.

The spectral radius was determined numerically for the land-ocean configuration shown in figure 1. For simplicity, the land and ocean were chosen to coincide with multiples of five degrees of latitude and longitude. The kernel of equation (2.55) was approximated by truncating the infinite matrix to include only terms up to a given degree, ranging up to twelfth. To illustrate a typical situation when

$$-\frac{1}{2} < \beta < \frac{n_M-1}{2} \tag{4.62}$$

β was set to unity. Since there are $2n + 1$ harmonics of n^{th} degree, at a given degree of approximation there are

$$\sum_{m=0}^{n} (2m + 1) = (n + 1)^2 \tag{4.127}$$

spherical harmonic terms. Consequently K is approximated by an $(n + 1)^2$ by $(n + 1)^2$ nonsymmetric matrix. The eigenvalue of largest absolute magnitude then yields the spectral radius. If the matrix has a complete set of eigenvalues and eigenvectors and the eigenvalue of largest absolute magnitude is real, then the most practical method for determining the spectral radius is the well known iterative procedure, the power method (Bodewig, 1959, Wilkinson, 1965, p. 570).

The iteration is started by choosing an arbitrary real vector of dimension, $(n + 1)^2$, $a^{(0)}$. At the ℓ^{th} stage premultiply by the real, truncated K matrix to obtain a new vector

$$b^{(\ell)} = Ka^{(\ell)} \tag{4.128}$$

The components of $a^{(\ell+1)}$ are taken as a scalar multiple, c_ℓ, of the components of $b^{(\ell)}$

$$a^{(\ell+1)} = c_\ell b^{(\ell)} \tag{4.129}$$

A convenient choice is

$$c_\ell = 1/ \max_j b_j^{(\ell)} \tag{4.130}$$

Hence the largest component of $a^{(\ell+1)}$ is unity. An estimate of the largest eigenvalue is given by

$$\lambda^{(\ell)} = (y, b^{(\ell)}),/ (y, a^{(\ell)}) \tag{4.131}$$

If

$$y_i = \delta_{ij} \tag{4.132}$$

where j corresponds to the largest component of $a^{(\ell)}$ and $b^{(\ell)}$,

$$\lambda^{(\ell)} = 1/c_\ell \tag{4.133}$$

This estimate converges linearly to the eigenvalue of maximum modulus (The iteration must be modified if several large eigenvalues are close or identical in magnitude and possibly complex). A listing of a computer program that can be used to calculate the spectral radius of a finite matrix may be found in appendix C.2. Results of this process are shown in figure 2. The estimate of the eigenvalue plotted, the Rayleigh quotient, uses

$$y = b^{(\ell)} \tag{4.134}$$

This choice accelerates convergence of the eigenvalue when K is a symmetric matrix (Ralston, 1965). In the present case the successive values in the iteration vary more smoothly than when equation (4.133) is used. The iteration for the 4^{th} degree approximation does not converge, indicating possibly complex eigenvalues. The analysis of section 4.4 indicates that the interaction of a spherical harmonic of degree greater than $1 + 2\beta$ with one less than this value could result in complex eigenvalues, since the matrix cannot be transformed into a real symmetric matrix, all of whose eigenvalues are real. The 4^{th} degree approximation is the first one exposed to a condition of this type, since $\beta = 1$. The dominant eigenvalues are not complex for the higher approximations. The value of the spectral radius varies smoothly, as a function of degree, and appears to approach an asymptote that need not necessarily exceed unity. The iteration is slow, indicating close eigenvalues.

An example, in which the amount of available altimetry data is that obtainable by a single altimetry satellite, with its orbit inclination as a parameter, is next considered. Gravimetry is assumed available over oceans at high latitudes inaccessible to the satellite. Results are shown in figure 3 for the second degree approximation. Since $\beta = 1 > 1/2 = (n_M - 1)/2$, the matrix is symmetrizable; this is the example of equation (4.79). If the inclination does not exceed about 35 degrees, this formulation of the Neumann series will not yield a solution to the problem. There is an implied requirement that there be over 43 percent coverage by altimetry, an increase from the one third requirement of inequality (4.81) for $\beta = 1$. If the $zero^{th}$ harmonic is suppressed the spectral radius is less than unity even for the low inclination satellites. This result is consistent with the uniqueness analysis of chapter 3 and with the character of the indefiniteness of the matrix whose eigenvalues are given in equation (4.111).

CHAPTER 5

CONSTRUCTIVE SOLUTION

5.1 General Discussion

As an illustration of the application of the method to the non-symmetric kernel when $\beta = 1$, a computer simulation is described. A description of the Neumann series algorithm solving equation (2.55) is given in section 5.2. Section 5.3 describes the examples in which simulated altimetry and gravimetry data derived from standard sets of harmonic coefficients serve to define the "measurements" from which the Neumann series algorithm described in section 5.2 extracts estimates of the earth's gravitational field, as defined by the standard sets. For a fourth degree harmonic approximation, three altimetry-gravimetry distributions are considered: all altimetry, all gravimetry, and a distribution based on the actual ocean-land distribution. For the latter distribution, 14^{th} and 15^{th} degree harmonic approximations are also considered. The problems arising because of the slow rate of convergence and the large number of coefficients relative to cell size are discussed.

5.2 Description of the Algorithm

A reference level rotational ellipsoid is adopted and used as a basis for the reduction of altimetry to geoidal undulations on oceans and gravimetry to gravity anomalies on land. Its normal gravity potential, U(p), also forms the basis for representation of the actual gravity potential, W(p), in terms of the anomalous potential, T(p),

$$T(p) = W(p) - U(p)$$

$$= \frac{GM}{r_M} \sum_{n=0}^{n_M} \sum_{m=0}^{n} \bar{P}_n^m(\sin \phi_p)(\delta\bar{C}_{nm}^{(i)} \cos m\lambda_p + \delta\bar{S}_{nm}^{(i)} \sin m\lambda_p) \qquad (5.01)$$

where

n_M = Maximum degree of harmonic approximation

The $(n_M+1)^2$ coefficients, $\delta\bar{C}_{nm}^{(i)}$, $\delta\bar{S}_{nm}^{(i)}$, define the i^{th} approximation to the potential function.

The actual iteration is as follows,

1) At each surface point, p, determine if it is land or ocean

a) If $p \in S_0$

form

$$[\beta T(p)+\zeta(p)]^{(i)} = \frac{GM}{r_M}\sum_{n=0}^{n_M}\frac{2\beta-1-n}{2}\cdot$$

$$\cdot\sum_{m=0}^{n}\bar{P}_n^m(\sin\phi_p)[\delta\bar{C}_{nm}^{(i)}\cos m\lambda_p+\delta\bar{S}_{nm}^{(i)}\sin m\lambda_p] \quad (5.02)$$

and

$$\zeta^{(i)}(p) = [\beta T(p) + \zeta(p)]^{(i)} - \beta\gamma(p)N(p) \quad (5.03)$$

b) If $p \in S_1$

form

$$[T(p)]^{(i)} = \frac{GM}{r_M}\sum_{n=0}^{n_M}\sum_{m=0}^{n}\bar{P}_n^m(\sin\phi_p)[\delta\bar{C}_{nm}^{(i)}\cos m\lambda_p+\delta\bar{S}_{nm}^{(i)}\sin m\lambda_p] \quad (5.04)$$

and

$$\zeta^{(i)}(p) = -\frac{r_M}{2}\Delta g(p) - [T(p)]^{(i)} \quad (5.05)$$

2) Since $\zeta^{(i)}(p)$ is now defined for each point of S, obtain the spherical harmonic coefficients

$$\begin{Bmatrix} \delta\bar{C}_{nm}^{(i+1)} \\ \delta\bar{S}_{nm}^{(i+1)} \end{Bmatrix} = \frac{1}{4\pi} \iint_{\sigma} - \frac{2}{n+1} \zeta^{(i)}(p) \bar{P}_n^m(\sin\phi_p) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (m\lambda_p) \, d\sigma_p \tag{5.06}$$

3) If the $i+1^{st}$ and i^{th} sets of harmonic coefficients are in close enough agreement, stop. Otherwise, continue the iteration at step 1).

Simple initial coefficients are

$$\delta\bar{C}_{nm}^{(0)} = 0 \qquad \delta\bar{S}_{nm}^{(0)} = 0 \tag{5.07}$$

The iterative process is then just the Neumann series of equation (4.08). Section 4.6 shows that this algorithm converges. A better initial guess just decreases the number of iterations needed for convergence. To handle a practical problem, the use of a digital computer is essential. In particular, the surface integral is replaced by a finite sum of cells, here taken to be bounded by lines of latitude and longitude, with land geometry so chosen that no cell contains both land and ocean. The division of ocean from land is taken, as shown previously in figure 1, along multiples of five degrees of latitude and longitude. After setting $\zeta^{(i)}(p)$ in a cell as constant at a central value of p, the surface integral over the cell separates. The λ integral just involves a constant or a sinusoid. The ϕ integral is

$$\int_{\phi_1}^{\phi_2} \bar{P}_n^m(\sin\phi_p) \cos\phi_p \, d\phi_p \tag{5.08}$$

Appendix B derives the appropriate recursion relations from which the integral may be evaluated for all required values of degree and order. For numerical accuracy, especially that of the higher harmonics, the cell dimensions should be kept small, but this increases the time required for each iteration, so that a judicious choice of cell size must be made.

5.3 Numerical Examples

For numerical testing of the algorithm, a simulation is needed, because no actual altimetry data are available at this time; no altimetry satellite is yet operational. In addition, since the formulation of this problem has avoided real, noisy data, so should the examples, to be consistent with the assumptions of the analysis. Therefore, the altimetry data on oceans and the gravimetry data on land were simulated using the spherical harmonic series representations in which the harmonic coefficients were obtained from outside sources (Köhnlein, 1967, Rapp, 1968). To determine the accuracy of the harmonic coefficients obtained by the iteration from the altimetry and gravimetry data, a comparison need only be made with the standard coefficients used to define the data. The computer program to estimate the harmonic coefficients, written in Fortran IV for the IBM 360 is given in appendix C.3.

The Rapp (1968) coefficients, truncated at fourth degree, were used in the first example. The associated values of the mass of the earth and the reference radius of the earth were ignored in favor of the values previously given in this thesis. Table 1 displays these coefficients as well as the results of the algorithm of this thesis for three different ocean-land configurations:

1) The globe of figure 1
2) A globe with all altimetry (oceans)
3) A globe with all gravimetry (land)

Consistent with the existence and uniqueness analysis, the zeroth and first harmonics for the case with all gravimetry data diverge. All other coefficients for each of the cases differ from Rapp (1968) by less than one per cent (or about 10^{-8} when the original coefficient is zero). For these cases the cell size was 2 1/2 degrees of latitude by 2 1/2 degrees of longitude.

Cases with the spherical harmonics carried to 14^{th} degree (Rapp, 1968) and 15^{th} degree (Köhnlein, 1967) were also examined. In order to store the necessary number of coefficients to be estimated and keep computer time usage at reasonable levels it was necessary to increase the cell size to 5 degrees of latitude by 5 degrees of longitude. The results, which are shown in table 2, are not as impressive as the lower degree case, especially when the magnitudes of the coefficients are small. The discrepancy arises from numerical limitations. In addition, since the Neumann series algorithm has linear convergence, convergence is slow. An improvement of the numerical technique including accelerating the convergence (Shanks, 1955) might economically allow continued calculation to obtain better agreement.

The effect of varying the parameter, β, which was here chosen to be unity, in the range,

$$-\frac{1}{2} < \beta < \frac{n_M - 1}{2} \tag{4.62}$$

could also be explored. Numerical explorations could also determine whether the symmetrical formulations (β is then outside of the range of inequality (4.62)), in which the parameters, τ and ξ, are introduced, provide a more suitable solution.

Chapter 6

CONCLUSIONS AND SYNTHESIS

6.1 Conclusions

This thesis has shown that a Neumann series solution of successive approximations can be used to combine satellite altimetry data given on oceans with surface gravimetry data given on land to determine the parameters of the earth's gravitational field. The validity of truncated approximations to the infinite-dimensional problem is established by different methods, depending on how heavily altimetry data are weighted relative to gravimetry data. The surface integration of a point function on the globe is required at each iteration step in order to obtain its spherical harmonic representation. Convergence is linear and is slow for the small-magnitude higher harmonics.

The important points in the formulation of the problem, establishment of uniqueness criteria, conditions for convergence of the proposed iterative method, and numerical application of the method to test examples are tabulated below.

6.2 Summary of Contributions

The original contributions of this thesis to the field of geodesy, by which a method for combining satellite altimetry and surface gravimetry data is developed, are:

(1) Formulation of the problem of combining satellite altimetry and surface gravimetry data as a mixed boundary value problem in potential theory for which a general solution method is not yet available.

(2) Analytic proof that it is sufficient for the problem to be unique if the zeroth harmonic is prescribed and if altimetry covers a

finite area (This proof has been extended, so that if altimetry covers a sufficiently large area, such as that corresponding to the earth's oceans, the problem is unique).

(3) Formulation of the problem as a formal integral equation of the first kind, which combines, in a weighted sum, an integral equation of the first kind with an integral equation of the second kind.

(4) Expression of the nonsymmetric kernel of the formal integral equation in terms of an appropriate spherical harmonic expansion.

(5) Transformation of the kernel in several ways to obtain a formal integral equation of the second kind, for which a Neumann series of successive approximations provides a solution if the spectral radius of the kernel is sufficiently small.

(6) Determination of a transformation of the kernel that symmetrizes it when altimetry data are weighted much more heavily than gravimetry data, and the derivation of conditions sufficient for the problem to be uniquely determined by a Neumann series (Altimetry must cover finite area, and a finite approximation must be made).

(7) Computer calculations of the spectral radius of truncated approximations of the nonsymmetric kernel that results when altimetry and gravimetry data are evenly weighted, demonstrating that the spectral radius is less than one for these approximations and that the trend of the spectral radius with increasing degree of approximation indicates that higher approximations can be used.

(8) Demonstration by computer simulation that, when altimetry and gravimetry data are evenly weighted, the iterative method will recover the values of geodetic parameters used to generate simulated altimetry and gravimetry data (4^{th}, 14^{th}, and 15^{th} degree models).

The following results were independently obtained, their appearance in the literature is unknown:

a) Derivation of recursion relations for the indefinite integral of an associated Legendre function.

b) Independent derivation of the Bergman kernel function and the Neumann kernel function for a spherical boundary for the external potential, in terms of spherical harmonics and in closed form.

The form of the Neumann kernel function is known, but its derivation is not readily accessible.

This minor result was also obtained:

Proof that the norm varies continuously with changes in the land-ocean boundary, ∂S.

6.3 Synthesis

The proposed method permits altimetry and gravimetry data (in principle, also geoidal section data) to be combined in a single determination of the geodetic parameters without requiring the statistical assumptions that have been necessary previously when different types of measured data were combined.

Methods for the determination of the higher harmonic detail of the earth's gravitational field are well known (Stokes, 1849), but require, in principle, a single type of data of uniform accuracy to be available over the whole earth's surface. The lack of sufficient amounts of such data, even for practical applications, retarded progress for a long time. Satellite geodesy, using new techniques and allowing new measurements, has revitalized the field of geodesy. Conventional geodetic satellite observations determine well the lower harmonics, but are less effective, except for special cases of resonance, in determining the higher harmonics. The ability to combine data types, using the techniques developed here permits added flexibility for obtaining valid data of uniform accuracy over the whole globe. The addition of satellite altimetry along with compensating surface data then could serve to improve the determination of the higher harmonic detail of the earth's gravitational field.

Practical implementation of the method developed here requires further improvements, such as making the calculations, including the surface integrations, more accurate and efficient, to insure that the higher harmonics can be determined to sufficient accuracy to obtain information of interest. There are many techniques (Shanks, 1955) that can be employed to accelerate the linear convergence and thus make the algorithm more useful. A comparison could then be made to determine the best weighting of altimetry relative to gravimetry. In practice, the measured data are corrupted by noise in various amounts, so that the method should be modified to take into account statistical considerations, such as handling redundant measurements. Simultaneous geoidal undulation and gravity anomaly estimates present in certain areas might also be used, even though in standard analyses of potential theory the resulting problem is overconstrained (Lavrentiev, 1967). The technique of constructing a kernel by summing separate integral representations using weighting factors and characteristic functions might be extended to accomodate these generalizations. In addition there will still be some regions, although fewer than before, without any genuine measurements. The statistical extrapolations into these regions could possibly make use of both the available undulations and the available anomalies.

Extending the results to the infinite-dimensional operator might also prove to be an interesting mathematical problem. It should be noted that uniqueness of the infinite-dimensional operator is not fully established. It is conceivable that an attempt to apply the method to a problem with restricted altimetry (For example, only a low inclination altimetry satellite is available) might lead to numerical problems if the finite-dimensional approximation resembles a situation giving rise to nonuniqueness in the infinite-dimensional problem. The technique developed here might also be applicable to other problems that can be formulated as mixed boundary value problems in potential theory.

Appendix A

DERIVATION OF THE KERNEL FUNCTIONS

The Neumann kernel for the representation of the potential external to a sphere is here obtained from the Bergman kernel function (Bergman and Schiffer, 1953, p. 198), $K_B(p, q)$, a harmonic function, in a manner that also yields the solution of the Dirichlet problem, the Poisson kernel.

Define an inner product space of functions harmonic in R. Introduce the inner product (different from equation (4.11))

$$(U(p), V(p) = -\frac{1}{4\pi}\iint_{\sigma} U(p)\frac{\partial V(p)}{\partial r_p}d\sigma_p = (V(p), U(p)) \qquad (A.01)$$

The Bergman kernel function satisfies a reproducing property (Bergman and Schiffer, 1953, p. 201, see also Krarup, 1969),

$$V(p) = (K_B(p, q), V(q)) \qquad (A.02)$$

From this may be obtained integral representations of the potential for the Dirichlet and Neumann problems. In terms of a set of orthonormal functions, $V_m(p)$, spanning the space,

$$(V_\ell(p), V_m(p)) = \delta_{\ell m} \qquad (A.03)$$

the Bergman kernel function has the representation (Bergman and Schiffer, 1953, p. 202)

$$K_B(p, q) = \sum_{m=1}^{\infty} V_m(p)V_m(q) \qquad (A.04)$$

The normalized spherical harmonics (see equation (2.38)), $x_i(p)$, are orthogonal under this inner product as well as under equation (4.11), but do not satisfy the normalization required in equation (A.03). To determine the correct normalization set

$$V_i(p) = w_i\left(\frac{r_M}{r_p}\right)^{n_i+1} x_i(p) \qquad (A.05)$$

where n_i is defined in equation (2.38). The constant, w_i, is determined by substituting equation (A.05) into equation (A.03) and using equation (A.01). Thus

$$w_i = \left[\frac{r_M}{n_i+1}\right]^{1/2} \tag{A.06}$$

Consequently, equation (A.04) becomes

$$K_B(p,\ q) = \sum_{i=1}^{\infty} \frac{r_M^{2n_i+3}}{(r_p r_q)^{n_i+1}} \frac{x_i(p)x_i(q)}{n_i+1} \tag{A.07}$$

The addition theorem for the spherical harmonics may be written in the form,

$$(2n_i+1)P_{n_i}(\mu) = \sum_{i=n_i^2+1}^{(n_i+1)^2} x_i(p)x_i(q) \tag{A.08}$$

where μ = cosine of the angle between the radii to the points, p and q. Thus

$$K_B(p,\ q) = r_M \sum_{n=0}^{\infty} \frac{2n+1}{n+1} u^{n+1} P_n(\mu) \qquad u < 1 \tag{A.09}$$

where $u = \dfrac{r_M^2}{r_p r_q}$

This series may be summed to closed form using the identity,

$$(1 - 2\mu u + u^2)^{-1/2} = \sum_{n=0}^{\infty} u^n P_n(\mu) \qquad u < 1 \tag{A.10}$$

Integration of equation (A.10) with respect to u between the limits of 0 and u yields

$$\ln\left[\frac{u-\mu+(1-2\mu u+u^2)^{1/2}}{1-\mu}\right] = \sum_{n=0}^{\infty} \frac{u^{n+1}}{n+1} P_n(\mu) \tag{A.11}$$

Noting that

$$\frac{2n+1}{n+1} = 2 - \frac{1}{n+1} \tag{A.12}$$

equation (A.09) becomes

$$K_B(p,\ q) = r_M\left[2u(1 - 2\mu u + u^2)^{-1/2} - \ln\frac{u-\mu+(1-2\mu u+u^2)^{1/2}}{1-\mu}\right] \tag{A.13}$$

Comparing equations (2.36), (A.01), and (A.02), it is apparent that

$$K_P(p,\ q) = -\left.\frac{\partial K_B(p,q)}{\partial r_q}\right|_{r_q=r_M} \tag{A.14}$$

After substitution and simplification

$$K_P(p,\ q) = \frac{r_M(r_p^2-r_M^2)}{[\ell(p,q)]^3} \tag{A.15}$$

where

$$\ell(p,\ q) = (r_p^2 + r_M^2 - 2r_p r_M \mu)^{1/2} \tag{A.16}$$

The result is just the Poisson kernel, the well-known integral representation for the spherical Dirichlet problem. Using equations (A.07) and (A.14), the well-known spherical harmonic series representation can be obtained in the form

$$K_P(p,\ q) = \sum_{i=1}^{\infty}\left(\frac{r_M}{r_p}\right)^{n_i+1} x_i(p)\,x_i(q) \tag{A.17}$$

Comparing equations (2.40), (A.01), and (A.02), it is apparent that when $\frac{\partial T(p)}{\partial r_p}$ is prescribed, $p \in S$, the integral representation for the Neumann problem is

$$T(p) = \frac{1}{4\pi}\iint_{\sigma}\left[-\left.K_B(p,\ q)\right|_{r_q=r_M}\right]\frac{\partial T(q)}{\partial r_q}\,d\sigma_q \tag{A.18}$$

The standard Neumann kernel is the term in brackets,

$$\left[-\left.K_B(p,\ q)\right|_{r_q=r_M}\right] = r_M \ln \frac{r_M - r_p\mu + \ell(p,q)}{r_p - r_p\mu} - \frac{2r_M^2}{\ell(p,q)} \tag{A.19}$$

In the limit, when p also lies on S, we have

$$\left[-\left.K_B(p,\ q)\right|_{r_p,r_q=r_M}\right] = r_M \ln\left(1 + \csc\frac{\psi_{pq}}{2}\right) - r_M \csc\frac{\psi_{pq}}{2} \tag{A.20}$$

where $\psi_{pq} = \cos^{-1}\mu$

This last result is given without proof by MacMillan (1958, p. 406, prob. 15) and Prasad (1930, p. 45, prob. 9).

For our purposes it is desirable to define a modified Neumann kernel

$$K_N(p, q) = \frac{2}{r_M}\left[- K_B(p, q)\Big|_{r_q=r_M}\right] \tag{A.21}$$

By defining

$$\zeta(p) = \frac{r_p}{2}\frac{\partial T(p)}{\partial r_p} \tag{A.22}$$

we obtain

$$T(p) = \frac{1}{4\pi}\iint_{\sigma} K_N(p, q)\zeta(q)d\sigma_q \tag{A.23}$$

The spherical harmonic representation of the modified Neumann kernel may be obtained by substituting equation (A.07) into equation (A.21),

$$K_N(p, q) = -\sum_{i=1}^{\infty}\frac{2}{n_i+1}\left(\frac{r_M}{r_p}\right)^{n_i+1} x_i(p)x_i(q) \tag{A.24}$$

Appendix B

INDEFINITE INTEGRAL OF THE ASSOCIATED LEGENDRE FUNCTION

The associated Legendre function is

$$P_n^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m} \tag{B.01}$$

where the Legendre polynomial is

$$P_n(\mu) = P_n^0(\mu) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n \tag{B.02}$$

Differentiation of the associated Legendre function with respect to μ and multiplication by $(1 - \mu^2)^{1/2}$ results in the well-known recursion relation

$$(1 - \mu^2)^{1/2} P_n^{m\prime}(\mu) = \frac{m\mu}{(1-\mu^2)^{1/2}} P_n^m(\mu) + P_n^{m+1}(\mu) \tag{B.03}$$

Integrate the left hand side by parts,

$$\int_{\mu_1}^{\mu_2} (1 - \mu^2)^{1/2} \frac{dP_n^m(\mu)}{d\mu} d\mu = (1 - \mu^2)^{1/2} P_n^m(\mu) \Big|_{\mu_1}^{\mu_2} + \\ + \int_{\mu_1}^{\mu_2} \mu (1 - \mu^2)^{-1/2} P_n^m(\mu) d\mu \tag{B.04}$$

This may be combined with a formal integration of the right hand side of the recursion (B.03)

$$\int_{\mu_1}^{\mu_2} P_n^{m+1}(\mu) d\mu = (1 - \mu^2)^{1/2} P_n^m(\mu) \Big|_{\mu_1}^{\mu_2} + \\ + (m + 1) \int_{\mu_1}^{\mu_2} \mu (1 - \mu^2)^{-1/2} P_n^m(\mu) d\mu \tag{B.05}$$

Solution of the last integral requires the well-known recursion relation for varying order

$$P_n^{m+2}(\mu) - \frac{2(m+1)\mu}{(1-\mu^2)^{1/2}} P_n^{m+1}(\mu) + + (n - m)(n + m + 1)P_n^m(\mu) = 0 \tag{B.06}$$

This is obtained by differentiating Legendre's differential equation

$$(1 - \mu^2)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + n(n + 1)y = 0 \tag{B.07}$$

m times and noting tha $y = P_n(\mu)$ is a solution. After redefining m, equation (B.06) becomes

$$\frac{\mu P_n^m(\mu)}{(1-\mu^2)^{1/2}} = \frac{1}{2m}\left[P_n^{m+1}(\mu) + (n + m)(n - m + 1)P_n^{m-1}(\mu)\right] \quad m \neq 0 \tag{B.08}$$

After substituting equation (B.08) into equation (B.05) and solving for the low order term, there results,

$$\int_{\mu_1}^{\mu_2} P_n^{m-1}(\mu)d\mu = [(m + 1)(n + m)(n - m + 1)]^{-1} \cdot \left[(m - 1)\int_{\mu_1}^{\mu_2} P_n^{m+1}(\mu)d\mu - 2m(1 - \mu^2)^{1/2}P_n^m(\mu)\Big|_{\mu_1}^{\mu_2}\right] \tag{B.09}$$

This recursion, relating, for constant degree, an associated Legendre function and its integral at adjacent orders, is valid for

$$0 < m \leq n \tag{B.10}$$

There are two special cases, m = 1, and m = n. For m = 1

$$\int_{\mu_1}^{\mu_2} P_n^0(\mu)d\mu = \frac{-1}{n(n+1)}(1 - \mu^2)^{1/2}P_n^1(\mu)\Big|_{\mu_1}^{\mu_2} \tag{B.11}$$

It is not related by the recursion to integrals of higher order and thus is isolated. The known alternate form depending only on Legendre polynomials is

$$\int_{\mu_1}^{\mu_2} P_n(\mu)\,d\mu = \frac{1}{2n+1}\Big[P_{n+1}(\mu) - P_{n-1}(\mu)\Big]\Big|_{\mu_1}^{\mu_2} \qquad (B.12)$$

For $m = n$

$$\int_{\mu_1}^{\mu_2} P_n^{n-1}(\mu)\,d\mu = \frac{-1}{n+1}(1 - \mu^2)^{1/2} P_n^n(\mu)\Big|_{\mu_1}^{\mu_2} \qquad (B.13)$$

Using this as a starting value $\int_{\mu_1}^{\mu_2} P_n^m(\mu)\,d\mu$ may be obtained for alternate orders. To obtain the remainder a value is needed for

$$\int_{\mu_1}^{\mu_2} P_n^n(\mu)\,d\mu = \frac{(2n)!}{2^n n!}\int_{\mu_1}^{\mu_2}(1 - \mu^2)^{n/2}\,d\mu \qquad (B.14)$$

Using integral formula #146 in Burington (1957)

$$\int_{\mu_1}^{\mu_2} P_n^n(\mu)\,d\mu = \frac{1}{n+1}\Big[\mu P_n^n(\mu)\Big|_{\mu_1}^{\mu_2} + n(2n - 1)(2n - 3)\int_{\mu_1}^{\mu_2} P_{n-2}^{n-2}(\mu)\,d\mu\Big] \qquad (B.15)$$

Thus knowledge of the Legendre functions and the initial conditions,

$$\int_{\mu_1}^{\mu_2} P_0^0(\mu) = \mu_2 - \mu_1 \qquad (B.16)$$

and

$$\int_{\mu_1}^{\mu_2} P_1^1(\mu)\,d\mu = \tfrac{1}{2}[\mu(1 - \mu^2)^{1/2} + \sin^{-1}\mu]\Big|_{\mu_1}^{\mu_2} \qquad (B.17)$$

suffice, in principle, to obtain integrals

$$\int_{\mu_1}^{\mu_2} P_n^m(\mu)\,d\mu \qquad (B.18)$$

for all integer, n and m, $0 \le m \le n < \infty$

The recursion in equation (B.15) is, however, unstable near the poles. A direct evaluation of equation (B.14) with $\mu = \sin\phi$ using #2.512, 2. and 3., of Gradshteyn and Ryzhik (1965) was actually used in the computer program (see appendix C), where the algorithm is written in terms of the normalized spherical harmonics.

Appendix C

COMPUTER PROGRAMS

C.1 The Calculation of the Ocean Functions

A listing of a computer program that calculates the ocean coefficients, Ω_{ik}, is given below. Sample values (Ω_{ii}, Ω_{i1}, $\Omega_{i,85}$, $\Omega_{i,169}$) are given in table 3 for the land-ocean configuration of figure 1, along with previously published (Lee and Kaula, 1967, Munk and MacDonald, 1960) values, up to eighth degree, of Ω_{i1}. For ease of comparison, the linear subscripts were transformed to degree and order subscripts,

$$\Omega_{nmj}^{st\ell} = \Omega_{ik} \tag{C.01}$$

where the subscripts are related as in equation (2.38). The comparison with the published values is not favorable, but the choice of geometry here is relatively crude and intended to be a distribution typical of altimetry and gravimetry, rather than of ocean and land. The Ω_{ii}'s do not deviate from $\Omega_{11} = \Omega_{000}^{000}$ by more than 20 percent. Actually, it can be shown that, for all n,

$$\Omega_{000}^{000} = \frac{1}{2n+1} \sum_{\substack{m=0 \\ j=0,1}}^{n} \Omega_{nmj}^{nmj} \tag{C.02}$$

The coefficients, Ω_{ik}, $i \neq k$, generally are an order of magnitude smaller.

Appendix C.1

```
C
C       MAIN PROGRAM CALCULATES THE OCEAN COEFFICIENTS FOR SPECIFIED
C            LAND-OCEAN GEOMETRY.
C            CALLS OCLAUD, DCOS, DSIN, NLEGND, OCLA(, EXIT, ERRSET).
        IMPLICIT REAL * 8 ( A-H , O-Z )
        REAL * 8 THETA ( 9 ) , X ( 9 ) , CX ( 9 ) , PM ( 20 , 20 , 9 )
        REAL * 8                  OMEGA ( 13 , 13 , 13 , 13 )
        REAL * 8 COSMPL (25 , 72 ) , SINMPL (25 , 72 )
        REAL * 8 FMT ( 8 ) , OM ( 4 )
        REAL*8     FOURPI / 12.566370614359l7 /
        REAL*8                      PIHALF / 1.570796326794897 /
        LOGICAL * 4 MAP / T /
        INTEGER * 4 OCLA
        INTEGER * 4 MAXDEG / 12/ , NOCELL / 18 / , NSIMP / 4 /
        INTEGER * 4 I1 ( 4 ) , I2 ( 4 ) , NP1MIN / 1 / , NP1MAX / 13 /
        NAMELIST / CNSTNT / MAXDEG , NOCELL , NP1MIN , NP1MAX , MAP,NSIMP
        I1 ( 1 ) = 0
        I2 ( 1 ) = 0
        I1 ( 3 ) = 1
        I2 ( 3 ) = 1
        CALL ERRSET ( 217 , 1 , -1 , 1 )
      1 CONTINUE
        READ  ( 5 , CNSTNT , END = 99999 )
        WRITE ( 6 , CNSTNT )
        NSP  = NSIMP     + NSIMP
        CELL = PIHALF / NOCELL
        CELLH = CELL / 2D0
        CELLN = CELLH/ NSIMP
        CONST =          CELLN/(FOURPI * 3D0 )
        LATMAX=NOCELL
        LONGMX=4*NOCELL
        MXDEGP=MAXDEG+1
        MDDP = MAXDEG + MXDEGP
        CALL  OCLAUD ( MAP )
        DO 50 MP2 = 1 , MXDEGP
        DO 50 NP2 = 1 , MXDEGP
        DO 50 MP1 = 1 , MXDEGP
        DO 50 NP1 = 1 , MXDEGP
     50 OMEGA ( NP1 , MP1 , NP2 , MP2 ) = 0D0
        DO 200 LONGNO = 1 , LONGMX
        ALONG = ( LONGNO - .5D0 ) * CELL
        COSMPL ( 1 , LONGNO ) = 1D0
        SINMPL ( 1 , LONGNO ) = 0D0
        DO 100 M1 = 2 , MDDP
        FACTOR = ( M1 - 1 ) * ALONG
        COSMPL ( M1 , LONGNO ) = DCOS ( FACTOR )
        SINMPL ( M1 , LONGNO ) = DSIN ( FACTOR )
    100 CONTINUE
    200 CONTINUE
        THETA ( 1 ) = 0D0
        I = 1
        CALL NLEGND ( MAXDEG , THETA ( I ) , PM( 1 , 1 , I ) , X(I),CX(I))
        IH = NSP + 1
        L = 1
```

```
      DO 1000 NTHETA = 1 , LATMAX
      I = IH
      IH = L
      L = I
      THETA ( L ) = NTHETA * CELL
      CALL NLEGND ( MAXDEG , THETA ( I ) , PM( 1 , 1 , I ) , X(I),CX(I))
      THETA ( 2 ) = THETA ( IH ) + CELLN
      DO 205 I = 2 , NSP
      CALL NLEGND ( MAXDEG , THETA ( I ) , PM( 1 , 1 , I ) , X(I),CX(I))
      IF ( I .LT. NSP ) THETA ( I + 1 ) = THETA ( I ) + CELLN
  205 CONTINUE
      ALAT = PIHALF - THETA ( NSIMP + 1 )
      DO 900 LONGNO = 1 , LONGMX
      ALONG = ( LONGNO - .5D0 ) * CELL
      ION = OCLA (  ALAT , ALONG )
      IOS = OCLA ( -ALAT , ALONG )
      IF ( ION .EQ. 1 .AND. IOS .EQ. 1 ) GO TO 900
      ID1 = -1
      DO 800 NP1 = NP1MIN , NP1MAX
      ID1 = -ID1
      IO1 = -ID1
      DO 700 MP1 = 1 , NP1
      IO1 = -IO1
      M1 = MP1 - 1
      MMP1 = M1 + MP1
      IF ( M1 .NE. 0 ) TDM1 = 2D0 / M1
      ID2 = -1
      DO 600 NP2 = 1 , NP1
      ID2 = -ID2
      IO2 = -ID2
      MP2M = NP2
      IF ( NP2 .EQ. NP1 ) MP2M = MP1
      DO 500 MP2 = 1 , MP2M
      IO2 = -IO2
      M2 = MP2 - 1
      NCOEF = 1 - ION + ( 1 - IOS ) * IO1 * IO2
      IF ( NCOEF .EQ. 0 ) GO TO 500
      F = 0D0
      T = 0D0
      DO 208 I = 2 , NSP , 2
      F  = PM ( NP1 , MP1 , I ) * PM ( NP2 , MP2 , I ) * CX ( I ) + F
      IF ( I .LT. NSP ) T = T
     1 +   PM ( NP1 , MP1 ,I+1) * PM ( NP2 , MP2 ,I+1) * CX (I+1)
  208 CONTINUE
      FACTPC = 4D0 * F + 2D0 * T
     1 +   PM ( NP1 , MP1 , L ) * PM ( NP2 , MP2 , L ) * CX ( L )
     2 +   PM ( NP1 , MP1 , IH) * PM ( NP2 , MP2 , IH) * CX ( IH)
      COEF = NCOEF * FACTPC * CONST
      IF ( M1 .GE. M2 ) GO TO  210
      IF ( M1 .GT. 0 ) GO TO 240
      FACTOR = ( 2D0 / M2 ) * SINMPL ( MP2 , 1 )
      CC = FACTOR * COSMPL ( MP2 , LONGNO )
      CS = FACTOR * SINMPL ( MP2 , LONGNO )
      GO TO 250
```

```
 210 IF ( M1 .NE. 0 ) GO TO 220
     CC = CELL
     GO TO 250
 220 IF ( M2 .NE. 0 ) GO TO 230
     FACTOR = TDM1           * SINMPL ( MP1 , 1 )
     CC = FACTOR * COSMPL ( MP1 , LONGNO )
     SC = FACTOR * SINMPL ( MP1 , LONGNO )
     GO TO 250
 230 IF ( M1 .NE. M2 ) GO TO 240
     FACTOR = SINMPL ( MMP1 , 1 ) * TDM1 / 4D0
     CS = FACTOR * SINMPL ( MMP1 , LONGNO )
     SC = CS
     CC = FACTOR * COSMPL ( MMP1 , LONGNO )
     SS = CELLH - CC
     CC = CC + CELLH
     GO TO 250
 240 MD = M1 - M2
     MS = M1 + M2
     MSP = MS + 1
     MDP = IABS ( MD ) + 1
     FACTP = SINMPL ( MSP , 1 ) / MS
     FACTM = SINMPL ( MDP , 1 ) / IABS ( MD )
     FACT1 = FACTM * COSMPL ( MDP , LONGNO )
     FACT2 = FACTP * COSMPL ( MSP , LONGNO )
     CC = FACT1 + FACT2
     SS = FACT1 - FACT2
     FACT2 = FACTP * SINMPL ( MSP , LONGNO )
     FACT1 = FACTM * SINMPL ( MDP , LONGNO )  * ISIGN ( 1 , MD )
     SC = FACT2 + FACT1
     CS = FACT2 - FACT1
 250 CONTINUE
     OMEGA ( NP1 , MP1 , NP2 , MP2 ) =
    1OMEGA ( NP1 , MP1 , NP2 , MP2 )+COEF * CC
     IF ( M1 .NE. 0 ) GO TO 300
     IF ( M2 .EQ. 0 ) GO TO 500
     GO TO 400
 300 OMEGA ( M1  , NP1 , NP2 , MP2 )  =
    1OMEGA ( M1  , NP1 , NP2 , MP2 )+COEF * SC
     IF ( M2 .EQ. 0 ) GO TO 500
     OMEGA ( M1  , NP1 , M2  , NP2 )  =
    1OMEGA ( M1  , NP1 , M2  , NP2 )+COEF * SS
     IF ( MP1 .EQ. MP2 .AND. NP1 .EQ. NP2 ) GO TO 500
 400 OMEGA ( NP1 , MP1 , M2  , NP2 ) =
    1OMEGA ( NP1 , MP1 , M2  , NP2 )+COEF * CS
 500 CONTINUE
 600 CONTINUE
 700 CONTINUE
 800 CONTINUE
 900 CONTINUE
1000 CONTINUE
1100 FORMAT(12X,8A8)
     READ ( 5 , 1100 ) FMT
     WRITE ( 6 , 1100 ) FMT
     WRITE ( 7 , 1100 ) FMT
```

```
1200 FORMAT ( '1 N1 M1 I N2 M2 I       OMEGA       N1 M1 I N2 M2 I       O
    1MEGA       N1 M1 I N2 M2 I       OMEGA       N1 M1 I N2 M2 I       OME
    1GA      ' )
     WRITE ( 6 , 1200 )
     DO 1800 NP1 = NP1MIN , NP1MAX
     N1 = NP1 - 1
     DO 1700 MP1 = 1 , NP1
     M1 = MP1 - 1
     DO 1600 NP2 = 1 , NP1
     N2 = NP2 - 1
     MP2M = NP2
     IF ( NP2 .EQ. NP1 ) MP2M = MP1
     DO 1500 MP2 = 1 , MP2M
     M2 = MP2 - 1
     I = 1
     OM ( 1 ) = OMEGA ( NP1 , MP1 , NP2 , MP2 )
     IF ( M1 .NE. 0 ) GO TO 1300
     IF ( M2 .EQ. 0 ) GO TO 1450
     GO TO 1400
1300 I = 2
     I1 ( 2 ) = 1
     I2 ( 2 ) = 0
     OM ( 2 ) = OMEGA ( M1 , NP1 , NP2 , MP2 )
     IF ( M2 .EQ. 0 ) GO TO 1450
     I = 3
     OM ( 3 ) = OMEGA ( M1 , NP1 , M2 , NP2 )
     IF ( MP1 .EQ. MP2 .AND. NP1 .EQ. NP2 ) GO TO 1450
1400 I = I + 1
     I1 ( I ) = 0
     I2 ( I ) = 1
     OM ( I ) = OMEGA ( NP1 , MP1 , M2 , NP2 )
1450 WRITE ( 7 , FMT ) ( I , N1 , M1 , I1 ( II ) , N2 , M2 , I2 ( II )
    1 , OM ( II ) , II = 1 , I )
     WRITE ( 6 , 1475 )(N1 , M1 , I1 ( II ) , N2 , M2 , I2 ( II ) ,
    1 OM ( II ) , II = 1 , I )
1475 FORMAT ( '0' , 4 ( 2I3 , I2 , 2I3 , I2 , F16.12 ) )
1500 CONTINUE
1600 CONTINUE
1700 CONTINUE
1800 CONTINUE
     I = -1
     WRITE ( 7 , FMT ) I
     GO TO  1
99999 CONTINUE
     CALL EXIT
     STOP
     END
```

Appendix C.1

```
C      OCLAUD READS-IN THE LAND OCEAN MAP(COLUMNS 1-72 OF 36 CARDS).
C      OCLA(IOCLA) SPECIFIES LAND OR OCEAN FOR A GIVEN LATITUDE AND
C           LONGITUDE.
       LOGICAL FUNCTION OCLAUD*4 ( MAP )
       LOGICAL *1 LOUT( 72 ) ,    QLAND / 'T' / , QOCEAN / ' ' /
       LOGICAL*1 LOCLA ( 36 , 72 )              , MAP*4
       INTEGER *4 OCLA , THENO
       REAL*8 LAT , LONG , THETA , PIHALF / 1.570796326794897 / , A (36)
   100 FORMAT ( 72L1 , A8 )
   200 FORMAT('1',  'LAND = T     OCEAN = _',T66,'0 DEG LONG',T83,'90',
      1  T28,'180',T46,'270',T100,'180',T120,'LABEL' )
   250 FORMAT ( T29 , '|' , 72A1 , '|' , T120 , A8 )
       IF ( MAP )
      1READ(5, 100 ) ( ( LOCLA ( I , J ) , J = 1 , 72),A(I),I=1 , 36 )
       WRITE(6,200 )
       DO 400 I = 1 , 36
       DO 350 J = 1 , 72
       IF ( LOCLA ( I , J ) ) GO TO 300
       LOUT ( J ) = QOCEAN
       GO TO 350
   300 LOUT ( J ) = QLAND
   350 CONTINUE
       WRITE ( 6 , 250 ) ( LOUT ( J ) , J = 1 , 72 ) , A ( I )
   400 CONTINUE
       OCLAUD = .FALSE.
       RETURN
       ENTRY IOCLA( LAT , LONG )
       ENTRY OCLA ( LAT , LONG )
C           - PI/2  <   LAT  <=  PI/2
C           0.  =<  LONG  <   2*PI ;      0.  =<  THETA  <   PI ;
       THETA = PIHALF - LAT
       ENTRY IOCLAT ( THETA , LONG )
       THENO =IDINT( 18.D0*THETA / PIHALF ) + 1
       LONGNO =IDINT(  18.D0*LONG / PIHALF ) - 35
       IF ( LONGNO .LE. 0 ) LONGNO = LONGNO + 72
       OCLA = 0
       IF(LOCLA ( THENO , LONGNO ))OCLA = 1
       RETURN
       END
```

Appendix C.1

```
C      NLEGND CALCULATES THE NORMALIZED ASSOCIATED LEGENDRE FUNCTIONS.
C           CALLS DSIN, DCOS, DSQRT.
       SUBROUTINE NLEGND(M,THETA,P , X , RT )
C      ****************************************************************
C      CALCULATE ASSOCIATED LEGENDRE FUNCTIONS
C      ****************************************************************
C______M       = MAXIMUM DEGREE OF THE LEGENDRE FUNCTIONS ( <= 19 ) .
C______THETA = COLATITUDE ( RADIANS ) . ________________________________
C______ P =             THE NORMALIZED LEGENDRE FUNCTIONS ( OUTPUT ) .
C______THE MATRICES IN THE CALLING PROGRAM ARE ASSUMED TO BE DIMENSIONED
C                                                             (20,20).
       IMPLICIT REAL*8 (A-H,O-Z )
       REAL * 8 ONE / .99999999999 /
       REAL * 8 R3 / 1.732050807568877 /
       DIMENSION P(20,20)
       N=M+1
       X=DCOS(THETA)
       RT = DSIN ( THETA )
       P(1,1)=1.DO
       IF(N.LE.1) GO TO 114
       IF ( DABS ( X ) .GT. 1D-11 .AND. DABS ( RT ) .LT. ONE ) GO TO 20
            X  = 0D0
            RT = DSIGN ( 1D0 , RT )
            GO TO 40
   20  IF ( DABS ( RT ) .GT. 1D-11 .AND. DABS ( X ) .LT. ONE ) GO TO 40
            RT = 0D0
            X  = DSIGN ( 1D0 , X )
   40  CONTINUE
       P(2,1)=X  * R3
       P(2,2)=RT * R3
       IF(N.EQ.2) GO TO 114
       IF ( N .GT. 20 ) N = 20
       DO 112 I=3,N
C             I AND J ARE ONE HIGHER THAN ACTUAL DEGREE AND ORDER.........
       P ( I , I ) =       RT            * P ( I - 1 , I - 1 )
      1  * DSQRT ( 1.DO + 1.DO / ( 2 * I - 2 ) )
       P ( I ,I-1) =       X             * P ( I - 1 , I - 1 )
      1      * DSQRT ( DFLOAT ( 2 * I - 1 ) )
       IMAX = I - 2
       DO 112 J = 1 , IMAX
  112  P ( I , J )=(        X            * P ( I - 1 ,    J    )      *
     1  DSQRT(((2*I-1) * (2*I-3)) / DFLOAT((I+J-2)*(I-J)))   )    -
     2 DSQRT ((( 2*I-1)*(I+J-3)*(I-J-1))/ DFLOAT((I+J-2)*(I-J)*(2*I-5)))
     3 *                    P ( I - 2 , J )
  114  CONTINUE
  102  RETURN
       END
```

Appendix C.2

THE CALCULATION OF THE NORMS OF THE OPERATOR

```
C
C     MAIN PROGRAM  CALCULATES THE SPECTRAL RADIUS OR NORM OF THE
C           KERNEL.
C           CALLS AKKYIM, AKKYI, AKZERO, AKKY, DMAX1(, EXIT, ERRSET).
      IMPLICIT REAL * 8 ( A-H , O-Z )
      REAL * 8             A (169 ) /9*1D0,16*1D-1,56*1D-2,88*1D-3/
      REAL * 8 B (169 )                          , DLIM / 5D-4 /
      INTEGER * 4 IOMEGA / 1 / , IRMIN / 1 / , IRMAX / 5 / , MXDEGP /13/
      INTEGER * 4 ITERM / 0 / , IB / 11/ , ISAMP / 84 / , IKSK / 1 /
      INTEGER * 4 ITMAX /10 / , KNORM / 0 / , KZERO / 1 / , I5 / 8 /
      NAMELIST / CNTRL / IRMIN , IRMAX , IOMEGA , MXDEGP , ITMAX
     1 , IKSK , KNORM , KZERO , I5    , ITERM , IB , ISAMP , DLIM , A
      CALL ERRSET ( 217 , 1 , -1 , 1 )
    1 CONTINUE
      READ ( 5 , CNTRL , END = 99999 )
      WRITE ( 6 , CNTRL )
      ALAM = 9999.D0
      IDEG = MXDEGP - 1
      IMAX = MXDEGP * MXDEGP
      CALL AKKYIM ( IMAX , ITERM , IB , IKSK )
      IF ( IOMEGA .NE. 1 ) GO TO 6000
      IOMEGA = 0
      CALL AKKYI ( MXDEGP , I5 , ISAMP )
 6000 IF ( KZERO .EQ. 0 ) CALL AKZERO
      IF ( IRMIN .GT. IRMAX ) GO TO 1
      DO 9000 IR = IRMIN , IRMAX
      IRM = IR - 1
      OALAM = ALAM
      OBF = - 10D0
      OLAM = -10D0
      DO 8000 IT = 1 , ITMAX
      CALL AKKY ( A , B , IRM )
      ALAMN = 0D0
      ALAMD = 0D0
      DO 7000 I = 1 , IMAX
      ALAMN = ALAMN + B ( I ) * B ( I )
      ALAMD = ALAMD + A ( I ) * B ( I )
 7000 CONTINUE
      ALAM = ALAMN / ALAMD
      DLAM = DABS ( ALAM - OLAM )
      OLAM = ALAM
      BF = DABS ( B ( 1 ) )
      IF ( IMAX .EQ. 1 ) GO TO 7090
      DO 7080 I = 2 , IMAX
 7080 BF = DMAX1 ( DABS ( B ( I ) ) , BF )
      DBF = DABS ( BF - OBF )
      OBF = BF
 7090 CONTINUE
      DO 7100 I = 1 , IMAX
      A ( I ) = B ( I ) / BF
 7100 CONTINUE
      WRITE ( 6 , 7050 ) IDEG , IR , IT , ALAM , DLAM , BF , DBF
 7050 FORMAT ( 'ODEG=',I3,', K IT=',I3,', L IT=',I3,', LAMBDA=',G24.16,
     1 ', D LAM=', G16.8,', BF=',G16.8,', DBF=',G16.8 )
```

```
      WRITE ( 6 , 7150 ) ( A ( I ) , I = 1 , IMAX )
 7150 FORMAT ( 'OA(I)=' , 9 ( G13.5 , ',' ) )
      IF ( DABS ( 1D0 - ALAM ) .LE. DLAM .OR.  DBF * 2000 .GT. BF )
     1 GO TO 8000
      IF ( 1D0 .GE. ALAM ) GO TO 7500
      IF ( DLAM .GE. ALAM* DLIM ) GO TO 8000
      IF ( OALAM .GE. ALAM .OR. KNORM .GT. IR ) GO TO 7300
      WRITE ( 6 , 7200 )
 7200 FORMAT ( 'OTHE NORM DIVERGES' )
 7250 FORMAT( 6X, 'MXDEGP=',I11,',IRMIN=',I11,',' /
     1         ' A=' , 3 ( G24.16 , ',' ) )
      WRITE ( 7 , 7250 ) MXDEGP , IR , ( A ( I ) , I = 1 , IMAX )
      GO TO 10000
 7300 CONTINUE
      WRITE ( 6 , 7400 )
 7400 FORMAT ( 'OTRY NEXT ITERATED KERNEL ' )
      WRITE ( 7 , 7250 ) MXDEGP , IR , ( A ( I ) , I = 1 , IMAX )
      GO TO 9000
 7500 IF ( DLAM .GE. ALAM* DLIM ) GO TO 8000
      WRITE ( 6 , 7700 )
 7700 FORMAT ( 'ONORM LESS THAN ONE' )
      WRITE ( 7 , 7250 ) MXDEGP , IR , ( A ( I ) , I = 1 , IMAX )
      IF ( KNORM .GT. IR ) GO TO 9000
      GO TO 10000
 8000 CONTINUE
      WRITE ( 6 , 8500 )
 8500 FORMAT ( 'OITERATION FOR LAMBDA EXCEEDED' )
      WRITE ( 7 , 7250 ) MXDEGP , IR , ( A ( I ) , I = 1 , IMAX )
 9000 CONTINUE
      WRITE ( 6 , 9500 )
 9500 FORMAT ( 'OMAXIMUM NUMBER OF ITERATIONS OF KERNEL EXCEEDED' )
10000 CONTINUE
      GO TO  1
99999 CONTINUE
      CALL EXIT
      STOP
      END
```

Appendix C.2

```
C      AKKY TRANSFORMS A INTO B BY MULTIPLYING BY A VERSION OF K,
C           INCLUDING: THE KERNEL(POWER METHOD), A SPECIFIED ITERATED
C           KERNEL, THE ADJOINT ONTO THE KERNEL, ITERATED ADJOINT ONTO
C           ITERATED KERNEL.
C      AKKYIM INITIALIZES THE DEGREE OF HARMONIC APPROXIMATION AND
C           OTHER CONTROL VARIABLES.
C      AKKYI READS-IN THE OCEAN COEFFICIENTS(OMEGA) ACCORDING TO
C           READ-IN FORMAT.
C      AKZERO CAUSES THE ZEROTH HARMONIC TO BE SUPPRESSED.
       SUBROUTINE AKKY ( A , BB , IRMM)
       IMPLICIT REAL * 8 ( A-H , O-Z )
       REAL * 8 A (169 ) , B (169 , 2 ),BB (169 ) , DF (169 ) , OF (169 )
       REAL * 8 OMEG (169 ,169 ) , FMT ( 8 )             , TERM ( 169 )
       IRM = IRMM
       IF ( IKSK .NE. 1 ) IRM = IRMM + 1
       DO 100 I = 1 , IMAX
   100 B ( I , 1 ) = A ( I )
       IF ( IZERO .EQ. 0 ) B ( 1 , 1 ) = 0D0
       IOLD = 1
       NEW = 2
       IF ( IRM .EQ. 0 ) GO TO 1000
       DO 900 ICNT = 1 , IRM
       DO 800 I = 1 , IMAX
       B ( I , NEW ) = 0D0
       DO 700 JC= 1 , IMAX
       J = JC
       IF ( IABS ( IB).GT. 1 ) J = IMXI - JC
       IF ( I .EQ. J ) GO TO  700
       B ( I , NEW ) = B ( I , NEW ) + OF ( J ) * OMEG ( I , J ) *
      1 B ( J , IOLD )
       IF ( I .EQ. ITERM ) TERM ( J ) = B ( I , NEW )
   700 CONTINUE
       B ( I , NEW ) = B ( I , IOLD ) * ( DF ( I ) + OF ( I ) *
      1 OMEG ( I , I ) ) + B ( I , NEW )
       IF ( I .EQ. ITERM ) TERM ( I ) = B ( I , NEW )
   800 CONTINUE
   850 FORMAT( '0('     ,I3,')=' , 5 ( G24.16 , ',' ) )
       IF ( ITERM .GT. 0 ) WRITE (6,850)ITERM,(TERM ( J ) , J = 1 , IMAX)
  6500 FORMAT ( '0B(I)=' , 9 ( G13.5 , ',' ) )
       IF ( IB .GT. 0 )
      1WRITE ( 6 , 6500 ) ( B ( I , NEW ) , I = 1 , IMAX )
       I = IOLD
       IOLD = NEW
       NEW = I
   900 CONTINUE
       IF ( IKSK .EQ. 2 ) GO TO 1950
       IF ( IKSK .NE. 1 ) GO TO 1500
  1000 DO 1300 I = 1 , IMAX
       B ( I , NEW ) = 0D0
       DO 1200 JC= 1 , IMAX
       J = JC
       IF ( IABS ( IB).GT. 1 ) J = IMXI - JC
       IF ( I .EQ. J ) GO TO 1200
       B ( I , NEW ) = B ( I , NEW ) + (1D0- DF ( I ) - DF ( J ) ) *
```

```
    1 OMEG ( I , J ) * B ( J , IOLD )
      IF ( I .EQ. ITERM ) TERM ( J ) = B ( I , NEW )
 1200 CONTINUE
      B ( I , NEW ) = B ( I , IOLD ) * ( DF ( I ) * DF ( I ) + ( 1D0 -
    1 DF ( I ) - DF ( I ) ) * OMEG ( I , I ) ) + B ( I , NEW )
      IF ( I .EQ. ITERM ) TERM ( I ) = B ( I , NEW )
 1300 CONTINUE
      IF ( ITERM .GT. 0 ) WRITE (6,850)ITERM,(TERM ( J ) , J = 1 , IMAX)
      IF ( IB .GT. 0 )
    1WRITE ( 6 , 6500 ) ( B ( I , NEW ) , I = 1 , IMAX )
      I = IOLD
      IOLD = NEW
      NEW = I
      IF ( IRM .EQ. 0 ) GO TO 2000
 1500 CONTINUE
      DO 1900 ICNT = 1 , IRM
      DO 1800 I = 1 , IMAX
      B ( I , NEW ) = 0D0
      DO 1700 JC= 1 , IMAX
      J = JC
      IF ( IABS ( IB).GT. 1 ) J = IMXI - JC
      IF ( I .EQ. J ) GO TO 1700
      B ( I , NEW ) = B ( I , NEW ) + OMEG ( I , J ) * B ( J , IOLD )
      IF ( I .EQ. ITERM ) TERM ( J ) = B ( I , NEW )
 1700 CONTINUE
      B ( I , NEW ) = B ( I , IOLD ) * ( DF ( I ) + DF ( I ) * OMEG ( I
    1 , I ) ) + B ( I , NEW ) * DF ( I )
      IF ( I .EQ. ITERM ) TERM ( I ) = B ( I , NEW )
 1800 CONTINUE
      IF ( ITERM .GT. 0 ) WRITE (6,850)ITERM,(TERM ( J ) , J = 1 , IMAX)
      IF ( IB .GT. 0 )
    1WRITE ( 6 , 6500 ) ( B ( I , NEW ) , I = 1 , IMAX )
      I = IOLD
      IOLD = NEW
      NEW = I
 1900 CONTINUE
 1950 CONTINUE
      IF ( IB .LE. 0 )
    1WRITE ( 6 , 6500 ) ( B ( I , IOLD) , I = 1 , IMAX )
 2000 DO 2200 I = 1 , IMAX
 2200 BB ( I ) = B ( I , IOLD )
      RETURN
      ENTRY AKKYIM ( IMAX , ITERM , IB , IKSK )
      IMXI = IMAX + 1
      RETURN
      ENTRY AKKYI (       MXDEGP , I5 , ISAMP )
      IZERO = 1
      I8 = 8
      IF ( I5 .EQ. 5 ) I8 = 5
      INC = -1
      IV = 0
      DO 3000 N1 = 1 , MXDEGP
      INC = INC + 2
      DFV = 2D0 / N1
```

```
      OFV = 1D0 - DFV - DFV
      DO 2900 ICNT = 1 , INC
      IV = IV + 1
      DF ( IV ) = DFV
      OF ( IV ) = OFV
2900 CONTINUE
3000 CONTINUE
4100 FORMAT ( I5 , 7X , 8A8 )
      READ (I8 , 4100 ) INPUT , FMT
      WRITE ( 6 , 4100 ) INPUT , FMT
4200 READ (I8 , FMT )    I , N1 , M1 , I1          , N2 , M2 , I2 , OM1
     1 ,                 ID, N3 , M3 , I3          , N4 , M4 , I4 , OM3
      IF ( ISAMP .LE. 0 )
     1WRITE( 6 , FMT )   I , N1 , M1 , I1          , N2 , M2 , I2 , OM1
     2 ,                 ID, N3 , M3 , I3          , N4 , M4 , I4 , OM3
4350 IF ( I .LE. 0 .OR. N1 .GE. MXDEGP ) GO TO 4500
      IV1 = ( N1 + I1          ) * N1 + M1 + 1
      IV2 = ( N2 + I2          ) * N2 + M2 + 1
      OMEG ( IV1 , IV2 ) = OM1
      OMEG ( IV2 , IV1 ) = OM1
      IF ( N3 .LE. 0 ) GO TO 4200
      IV1 = ( N3 + I3          ) * N3 + M3 + 1
      IV2 = ( N4 + I4          ) * N4 + M4 + 1
      OMEG ( IV1 , IV2 ) = OM3
      OMEG ( IV2 , IV1 ) = OM3
      GO TO 4200
4500 CONTINUE
      IF ( ISAMP .LE. 0 ) ISAMP = 1 + IABS ( ISAMP )
      DO 5000 IV = 1 , IMAX , ISAMP
      WRITE ( 6 , 4700 ) IV , ( OMEG ( II , IV ) , II = 1 , IMAX )
4700 FORMAT (   '0OMEG(I,'   , I3 , ')=' , 9 ( G12.4 , ',' ) /
     1                                    10 ( G12.4 , ',' ) )
5000 CONTINUE
5200 FORMAT ( '0 DF ( I ) = ' , 9 ( G12.4 , ',' ) )
      WRITE ( 6 , 5200 ) ( DF ( II ) , II = 1 , IMAX , ISAMP )
5400 FORMAT ( '0 OF ( I ) = ' , 9 ( G12.4 , ',' ) )
      WRITE ( 6 , 5400 ) ( OF ( II ) , II = 1 , IMAX , ISAMP )
      RETURN
      ENTRY AKZERO
      IZERO = 0
      DO 6000 I = 1 , IMAX
      OMEG ( 1 , I ) = 0D0
      OMEG ( I , 1 ) = 0D0
6000 CONTINUE
      RETURN
      END
```

Appendix C.3

THE CALCULATION OF THE HARMONIC COEFFICIENTS

(Listings of subroutines NLEGND and OCLAUD may be found in appendix C.1)

```
C
C     MAIN PROGRAM ESTIMATES THE HARMONIC COEFFICIENTS FROM ANOMALIES
C          AND UNDULATIONS GENERATED FROM A REFERENCE SET OF HARMONIC
C          COEFFICIENTS.
C          CALLS DSIN, DSQRT, NLEGND, SNPXDX, CSPCH, OCLA, CSRDR,OCLAUD,
C                       CSTBL, DCOS, DATAN(, EXIT, ERRSET).
      IMPLICIT REAL*8 (A-H,O-Z )
      REAL  * 8 SMCT( 15 ) , SINMLT ( 15 ,288 ) , COSMLT( 15 , 288 )
      REAL * 8 P ( 20 , 20 ) , PT ( 136 , 72 ) , SPT ( 136 , 72 )
      REAL*8 DCS(20,20)             , CBN(10) , QPCD / '%DIFF  6' /
      REAL*8 CSB ( 20 , 20 ) , QBLANK / '        6' /,QDIFF/'DIFF   6'/
      REAL *8 DCRDT(20,20,2),  QSTART / 'START  6' /
      REAL*8 QDCRDT / 'DCRDT  6' / , QRDTCH / 'RDTCH    ' /
      REAL * 8 FMT ( 8 )    ,    QDELTA / 'DELTA  6' /
      REAL*8    FOURPI / 12.56637061435917 /
      REAL*8                       PIHALF / 1.570796326794897 /
      LOGICAL * 4 MAP / T / , OCLAUD
      INTEGER * 4 OCLA
      INTEGER*4 MEAN /0/        , LATPRT /18/ , JMXDEG / 0 / , JNCELL /0/
      INTEGER*4 MAXDEG/19/,NOCELL/18/ , ITERST / 1 / , IOCLAI / -1 /
C_____________________________________________________________________________
      CALL ERRSET ( 212 ,-1 , -1 , 1 )
      CALL ERRSET ( 217 , 1 , -1 , 1 )
C_____________________________________________________________________________
   19 FORMAT('1',20X,'COMBINING SATELLITE ALTIMETRY AND SURFACE GRAVIMET
     1RY IN GEODETIC DETERMINATIONS.  BY  RONALD GING-WEI ENG YOUNG.')
C     ******************************************************************
  100 WRITE ( 6 , 19 )
      NAMELIST /CNSTNT/ MAXDEG , NOCELL , MEAN      , A , F
     1, OMEGA , STDMU  , ITERMX , IOCLAI    , LATPRT,ITERST, MAP
C_____MAXDEG = MAXIMUM DEGREE OF THE SPHERICAL HARMONIC FUNCTIONS
C_____      DEFINING THE EARTH'S GRAVITY FIELD.
C     NOCELL = NUMBER OF CELLS IN EACH 90 DEGREES OF LATITUDE AND
C          LONGITUDE.
C_____MEAN = 1  REQUESTS MEAN VALUES RATHER THAN POINT VALUES OF GRAVITY
C          DATA.
C_____A = EQUATORIAL RADIUS OF THE MEAN EARTH ELLIPSOID.
C     F=FLATTENING =(A-B)/A
C     OMEGA= ANGULAR VELOCITY OF REVOLUTION
C_____STDMU = GAUSSIAN CONSTANT TIMES THE MASS OF THE STANDARD EARTH.
C_____ITERMX = NUMBER OF ITERATIONS THAT THE SPHERICAL INTEGRATIONS ARE
C          DONE.
C     IOCLAI = 0 FORCES OCEANS.  IOCLAI .GT. 0 FORCES LAND.  OTHERWISE
C          AS INPUT.
C_____WHENEVER LATNO >= LATPRT DCRDT IS PRINTED. _______________________
C_____ITERST = VALUE OF THE NEXT ITERATION IN A SERIES OF ITERATIONS.
C     MAP = T REQUESTS A READ-IN OF THE LAND-OCEAN CONFIGURATION.
C         = F SUPPRESSES A READ-IN.
C
      READ ( 5 , CNSTNT , END = 8500 )
C
      WRITE ( 6 , CNSTNT )
  265 FORMAT('
     1                         ','            ITERATION NUMBER IS ',I3)
```

```
C
      READ ( 5 , 265 )
C
      MXDEGP=MAXDEG+1
      AREA = FOURPI
      B=A*(1.D0 - F )
      RT=DSQRT((A-B)*(A+B))
      RTA2 = ( RT / A ) ** 2
      EM = ( OMEGA * A ) ** 2 * B / STDMU
C_____E2 = E' = SECOND ECCENTRICITY.___________________________________
      E2 = RT / B
      ATANE2 = DATAN ( E2 )
       QUZERO = ( .5D0 + 1.5D0 / E2 ** 2 ) * ATANE2 - 1.5D0 / E2
      STDJ2 = RTA2             * ( .3333333333333333 - EM*E2/(QUZERO
     1  * 22.5D0 ) )
      ARTJ = 5.D0 * STDJ2 / RTA2
CBN(I) ARE COEFFICIENTS FOR THE NORMAL ELLIPSOID ( I = (DEG+2)/2 )  ....
      CBN ( 1 ) =   1.D0
      DO 150 I = 1 , 9
      CBN ( I + 1 ) =   ( ( ARTJ - 1.D0)* I +1.D0)* 3.D0*( - RTA 2 )**I
     1  / ( ( 2 * I + 1 ) * ( 2 * I + 3 ) * DSQRT ( DFLOAT ( 4 * I +1)))
  150 CONTINUE
 3000 FORMAT ( I5 , 7X , 8A8 )
C
      READ ( 5 , 3000 ) I , ( FMT ( J ) , J = 1 , 8 )
C
      IF ( I .GT. 0 ) GO TO 3100
      IF ( I .LT. 0 ) GO TO 151
C     IF ( I .EQ. 0 )
      DO  44 J = 1 , MXDEGP
      DO  44 I = 1 , MXDEGP
   44 CSB ( I , J ) = 0D0
      DO  550 I = 1 , MXDEGP , 2
  550  CSB (   I     , 1 )= CBN ( I / 2 + 1 )
C
 3100 CALL CSRDR ( CSB , FMT )
C
      CALL CSTBL ( CSB ,MXDEGP,QSTART )
      WRITE ( 6 , 265 ) ITERST
C(N,M) = CSB (N+1,M+1) , S(N,M) = CSB (M,N+1) ........................
      DO 1550 I = 1 , MXDEGP , 2
 1550  CSB (   I     , 1 )=-CBN ( I / 2 + 1 ) + CSB ( I , 1 )
  151 CONTINUE
      IF ( JNCELL .NE. NOCELL ) GO TO 157
      IF ( MAXDEG .GT. JMXDEG ) GO TO 193
      GO TO 197
  157 JNCELL = NOCELL
C    IF SUCCESSIVE VALUES OF NOCELL ARE THE SAME, THE GEOID
C           READ-IN IS SKIPPED.
      GEOP    =    STDMU* ATANE2    /     RT +OMEGA**2*A**2   / 3.D0
      EARAD = ( A ** 2 * B ) ** ( .3333333333333333 )
      GRAVM = STDMU / ( EARAD * EARAD )
      CELL = PIHALF / NOCELL
      CELLH = CELL / 2D0
```

```
      ACELL = ( CELL + CELL ) * DSIN ( CELLH ) * FOURPI
      LATMAX=NOCELL
      LONGMX=4*NOCELL
C
      MAP = OCLAUD ( MAP )
C
      NAMELIST /PARAMS/ CBN , CELL , B , EARAD , GRAVM , RT , GEOP
       WRITE ( 6 , PARAMS )
C     *****************************************************************
 193  JMXDEG = MAXDEG
C     CALCULATE SINES AND COSINES FOR THE LONGITUDE TERMS.
      DO 3810 IORD = 1 , MAXDEG
      ORDH = IORD / 2D0
      SMCT ( IORD ) = DSIN ( CELL * ORDH ) / ORDH
 3810 CONTINUE
      LONGMS = 2 * LONGMX
      LONGNO = 0
      DO 3830 LONGNH = 1 , LONGMS , 2
      LONGNO = LONGNO + 1
      J = 0
      DO 3820 IORD = 1 , MAXDEG
      J = J + LONGNH
      I = J
  101 IF ( I .LT. LONGMX ) GO TO 102
      I = I - LONGMS
      GO TO 101
  102 FACTOR = I * CELLH
C_____SINMLT(1,LATNO)=SIN(THETA)=COS(ALAT)______________________________
      SINMLT( IORD , LONGNO ) = DSIN ( FACTOR )
      COSMLT( IORD , LONGNO ) = DCOS ( FACTOR )
 3820 CONTINUE
 3830 CONTINUE
C     IF NOCELL IS CHANGED OR MAXDEG IS INCREASED, THE LEGENDRE
C          FUNCTIONS ARE RECOMPUTED.
      THETAH = 0D0
      CALL NLEGND ( MAXDEG , THETAH , PT     ( 1 , 1 )     , XH ,CXH )
      IHIGH = 1
C_____IF DIMENSIONS OF PT(I,J) ARE CHANGED CHECK THAT I*ILOW >= 400.
      ILOW = 10
      DO 3900 I = 1 , LATMAX
      THETAL = CELL * I
C     CALCULATE ASSOCIATED LEGENDRE FUNCTIONS
      CALL NLEGND ( MAXDEG , THETAL , PT        ( 1,ILOW),XL,CXL)
      CALL SNPXDX ( MAXDEG , P ( 1 , 1 ) , PT ( 1 , IHIGH ) , PT ( 1 ,
     1                            ILOW ) , THETAH,XH,CXH,THETAL,XL,CXL)
      J = IHIGH
      IHIGH = ILOW
      ILOW = J
      THETAH = THETAL
      XH = XL
      CXH = CXL
      NN = 0
      DO 3890 N1 = 1 , MXDEGP
      DO 3890 M1 = 1 , N1
```

```
      NN = NN + 1
 3890 SPT ( NN       , I ) = P ( N1 , M1 )
 3900 CONTINUE
      IF ( MEAN .GT. 0 ) GO TO 197
C     CALCULATE ASSOCIATED LEGENDRE FUNCTIONS
      DO 210 I = 1 , LATMAX
      THETAL= ( I - .5D0 ) * CELL
      CALL NLEGND ( MAXDEG , THETAL,  P       ( 1 , 1 )     , XL ,CXL )
      NN = 0
      DO 205 N1 = 1 , MXDEGP
      DO 205 M1 = 1 , N 1
      NN = NN + 1
  205 PT ( NN       , I ) = P ( N1 , M1 )
  210 CONTINUE
C     ******************************************************************
  197 CONTINUE
C          INITIALIZE THE ARRAYS FOR THE SPHERICAL INTEGRATIONS ........
C
      READ ( 5 , 3000 ) I , ( FMT ( J ) , J = 1 , 8 )
C
      IF ( I .GT. 0 ) GO TO 3300
      IF ( I .EQ. 0 ) GO TO 3250
      IF ( I .EQ. -1 ) GO TO 3400
      DO 3200 J = 1 , MXDEGP
      DO 3200 I = 1 , MXDEGP
 3200 DCS ( I , J ) = CSB ( I , J )
      GO TO 3350
 3250 DO  121 J = 1 , MXDEGP
      DO  121 I = 1 , MXDEGP
  121 DCS ( I , J ) = 0.D0
C
 3300 CALL CSRDR ( DCS , FMT )
C
 3350 CONTINUE
      CALL CSTBL ( DCS , MXDEGP , QDELTA )
      WRITE ( 6 , 265 ) ITERST
      DO 179 IDEGP = 1 , MXDEGP
      FACTOR = - IDEGP / 2D0
      DCRDT ( IDEGP , 1 , 1 )  =   FACTOR * DCS ( IDEGP , 1 )
      IF ( IDEGP  .EQ. 1 ) GO TO 179
      DO 173 IORDP = 2 , IDEGP
      IORD = IORDP - 1
      DCRDT ( IDEGP , IORDP , 1 ) = FACTOR * DCS ( IDEGP , IORDP )
      DCRDT ( IORD  , IDEGP , 1 ) = FACTOR * DCS ( IORD  , IDEGP )
  173 CONTINUE
  179 CONTINUE
      NOLD = 1
      NEW = 2
 3400 CONTINUE
      ITERND = ITERST + ITERMX - 1
C     ******************************************************************
      DO 8300 LOOPVR = ITERST , ITERND
C_____ITERATION LOOP STARTS HERE. ________________________________
      DO 141  J = 1 , MXDEGP
```

```
      DO 141  I = 1 , MXDEGP
  141 DCROT ( I , J , NEW ) = 0DO
C     *************************************************************
      DO 1140 LATNO=1,LATMAX
C__________ POLES TO EQUATOR _________________________________________
      IF ( MEAN .NE. 0 ) AREA = ACELL * SINMLT ( 1 , LATNO )
      ALAT = PIHALF   - ( LATNO - .5DO ) * CELL
      DO  145 J = 1 , MXDEGP
      DO  145 I = 1 , MXDEGP
  145 P ( I , J ) = 0DO
C     *************************************************************
      DO 1120 LONGNO = 1 , LONGMX
C__________ 0 DEGREES TO 360 DEGREES LONGITUDE EASTWARD . ____________
      ALONG = ( LONGNO - .5DO ) * CELL
      IF ( IOCLAI ) 2035 , 2025 , 2030
 2025 IOCLAN = 0
      IOCLAS = 0
      GO TO 273
 2030 IOCLAN = 1
      IOCLAS = 1
      GO TO 273
 2035 CONTINUE
      IOCLAN = OCLA ( ALAT , ALONG )
      IOCLAS = OCLA (-ALAT , ALONG )
  273 CONTINUE
      RDTN = 0.D0
      RDTS = 0.D0
      IMD = - 1
      NN = 0
C     *************************************************************
      DO 780 IDEGP = 1 , MXDEGP
      NN = NN + 1
      IMD = - IMD
      IDEG = IDEGP - 1
      FACTOR = ( 1 - IDEG ) / 2DO
      IF ( IOCLAN .EQ. 1 ) GO TO 306
      CN = DCS ( IDEGP , 1 ) * FACTOR
                       CN=CN-CSB(IDEGP,1)
      GO TO 315
  306 CN = - DCS ( IDEGP , 1 )
                       CN=CN+CSB(IDEGP,1)    * FACTOR
  315 IF ( IOCLAS .NE. IOCLAN ) GO TO 191
      CS = CN
      GO TO 316
  191 IF ( IOCLAS .EQ. 1 ) GO TO 313
      CS = DCS ( IDEGP , 1 ) * FACTOR
                       CS=CS-CSB(IDEGP,1)
      GO TO 316
  313 CS = - DCS ( IDEGP , 1 )
                       CS=CS+CSB(IDEGP,1)    * FACTOR
  316 IF ( MEAN .EQ. 0 ) GO TO 200
      PNM = SPT ( NN , LATNO )  * CELL
      GO TO 201
  200 PNM = PT ( NN , LATNO )
```

```
  201 CONTINUE
      RDTN = RDTN + CN * PNM
C_____IMD =               ( - 1   ) **IDEG       __________________________
      IF ( IMD .LT. 0 ) CS   = -CS
      RDTS = RDTS + CS * PNM
      IF ( IDEG .EQ. 0 ) GO TO 780
      IMO = IMD
C     ****************************************************************
      DO 760 IORD = 1 , IDEG
      NN = NN + 1
      IMO = - IMO
      IORDP = IORD + 1
      IF ( MEAN .EQ. 0 ) GO TO 211
      PNM = SPT ( NN , LATNO )  * SMCT ( IORD )
      GO TO 212
  211 PNM = PT ( NN , LATNO )
  212 CONTINUE
      COSINE = COSMLT( IORD , LONGNO )
        SINE = SINMLT( IORD , LONGNO )
      IF ( IOCLAN .EQ. 1 ) GO TO 307
      CN = DCS ( IDEGP , IORDP )* FACTOR
      CN = CN - CSB ( IDEGP , IORDP )
      SN = DCS ( IORD , IDEGP ) * FACTOR
      SN = SN - CSB ( IORD   , IDEGP )
      GO TO 401
  307 CN = - DCS ( IDEGP , IORDP )
      CN = CN + CSB ( IDEGP , IORDP )        * FACTOR
      SN = - DCS ( IORD , IDEGP )
      SN = SN + CSB ( IORD   , IDEGP )       * FACTOR
  401 IF ( IOCLAS .NE. IOCLAN ) GO TO 228
      CS = CN
      SS = SN
      GO TO 402
  228 IF ( IOCLAS .EQ. 1 ) GO TO 314
      CS = DCS ( IDEGP , IORDP )* FACTOR
      CS = CS - CSB ( IDEGP , IORDP )
      SS = DCS ( IORD , IDEGP ) * FACTOR
      SS = SS - CSB ( IORD   , IDEGP )
      GO TO 402
  314 CS = - DCS ( IDEGP , IORDP )
      CS = CS + CSB ( IDEGP , IORDP )        * FACTOR
      SS = - DCS ( IORD , IDEGP )
      SS = SS + CSB ( IORD   , IDEGP )       * FACTOR
  402 RDTN = RDTN +     PNM                   * ( CN * COSINE+SN*SINE)
      IF ( IMO .LT. 0 ) PNM = -PNM
C_____IMO =                ( - 1 ) ** (IDEG -IORD )      _________________
      RDTS =RDTS +      PNM             *(CS*COSINE+SS*SINE)
  760 CONTINUE
C     ****************************************************************
  780 CONTINUE
C     ****************************************************************
      CN = ( RDTN + RDTS ) / AREA
      CS = ( RDTN - RDTS ) / AREA
      IMD = - 1
```

Appendix C.3

```
C      ************************************************************
       DO 1100 IDEGP = 1 , MXDEGP
       IMD = - IMD
       IDEG = IDEGP - 1
C______IMD =              ( - 1   ) **IDEG      ____________________________
       IF ( IMD .LT. 0 ) GO TO 254
       FACTOR = CN
       GO TO 255
   254 FACTOR = CS
   255 CONTINUE
       P ( IDEGP , 1 ) = P ( IDEGP , 1 ) + FACTOR
       IF (IDEG .EQ.  0   ) GO TO 1100
       IMO = IMD
C      ************************************************************
       DO 1070 IORD= 1 ,IDEG
       IMO = - IMO
       IORDP = IORD + 1
C______IMO =              ( - 1 ) ** (IDEG -IORD )      _________________
       IF ( IMO .LT. 0 ) GO TO 262
       FACTOR = CN
       GO TO 263
   262 FACTOR = CS
   263 CONTINUE
       P     (IDEGP,IORDP     )= P    (IDEGP,IORDP     )+COSMLT(IORD,LONGNO)
      1 * FACTOR
       P     (IORD,IDEGP      )=  P   (IORD,IDEGP      )+SINMLT(IORD,LONGNO)
      1 * FACTOR
  1070 CONTINUE
C      ************************************************************
  1100 CONTINUE
C      ************************************************************
  1120 CONTINUE
C      ************************************************************
       IF ( LATPRT .LE. LATNO )
      1CALL CSTBL ( P , MXDEGP , QDIFF )
       NN = 0
       DO 326 IDEGP = 1 , MXDEGP
       NN = NN + 1
       DCRDT ( IDEGP , 1 , NEW ) = DCRDT ( IDEGP , 1 , NEW ) + P ( IDEGP
      1  , 1 ) * SPT ( NN , LATNO )
       IF ( IDEGP .EQ. 1 ) GO TO 326
       DO 320 IORDP = 2 , IDEGP
       IORD = IORDP - 1
       NN = NN + 1
       DCRDT(IDEGP,IORDP,NEW)=DCRDT(IDEGP,IORDP,NEW)+
      1  P  (IDEGP,IORDP ) * SPT ( NN , LATNO )
       DCRDT(IORD,IDEGP,NEW)=DCRDT(IORD,IDEGP,NEW)+
      1  P  (IORD,IDEGP  ) * SPT ( NN , LATNO )
   320 CONTINUE
   326 CONTINUE
       IF ( LATPRT .LE. LATNO )
      1CALL CSTBL (  DCRDT(1,1,NEW) , MXDEGP , QDCRDT )
  1131 FORMAT(' ITERATION=',I3,',  ZONE =',I3,',  DCRDT =',4(G23.16,','))
       WRITE ( 6 , 1131 ) LOOPVR , LATNO , ( DCRDT(J,1,NEW),J=1,4)
```

```
 1140 CONTINUE
C      ******************************************************************
      DO  286   IDEGP = 1 , MXDEGP
  286 DCRDT ( IDEGP , 1 , NEW ) = DCRDT ( IDEGP , 1 , NEW ) * CELL
      DO  288   IORD = 1 , MAXDEG
      IORDP = IORD + 1
      FACTOR = SMCT ( IORD )
      DO  288   IDEGP = IORD P , MXDEGP
      DCRDT(IDEGP,IORDP,NEW)=DCRDT(IDEGP,IORDP,NEW)    * FACTOR
      DCRDT(IORD,IDEGP,NEW)=DCRDT(IORD,IDEGP,NEW)      * FACTOR
  288 CONTINUE
      CALL CSTBL (  DCRDT(1,1,NEW) , MXDEGP , QDCRDT )
      WRITE ( 6 , 265 ) LOOPVR
      DO 1195 IDEGP = 1 , MXDEGP
      I = IDEGP
      DO 1185 J = 1 , MXDEGP
 1185 DCRDT ( I , J , NOLD ) = DCRDT ( I , J , NEW ) - DCRDT (I,J,NOLD)
      FACTOR = - 2D0 / IDEGP
      DCS ( IDEGP , 1 ) = FACTOR * DCRDT ( IDEGP , 1 , NEW )
      IF ( IDEGP .EQ. 1 ) GO TO 1195
      DO 1175 IORDP = 2 , IDEGP
      IORD = IORDP - 1
      DCS ( IDEGP , IORDP ) = FACTOR * DCRDT ( IDEGP , IORDP , NEW )
 1175 DCS ( IORD , IDEGP ) = FACTOR * DCRDT ( IORD , IDEGP , NEW )
 1195 CONTINUE
      CALL CSTBL ( DCRDT ( I , 1 , NOLD ) , MXDEGP , QRDTCH )
      WRITE ( 6 , 265 ) LOOPVR
      CALL CSTBL ( DCS , MXDEGP , QDELTA )
      WRITE ( 6 , 265 ) LOOPVR
      I = NOLD
      NOLD = NEW
      NEW = I
       DO 1500 J = 1 , MXDEGP
       DO 1500 I = 1 , MXDEGP
 1500 DCRDT(I,J,NEW)= DCS ( I , J ) - CSB ( I , J )
C(N,M) = CSB (N+1,M+1) , S(N,M) = CSB (M,N+1) ..........................
      CALL CSTBL ( DCRDT ( 1 , 1 , NEW ) , MXDEGP , QDIFF )
      WRITE ( 6 , 265 ) LOOPVR
C          DIFF IS CALCULATED MINUS INPUT COEFFICIENT ..................
       DO 1600 I = 1 , MXDEGP
       DO 1600 J = 1 , MXDEGP
      IF(CSB(I,J).EQ.0.D0)      GO TO 1590
      DCRDT(I,J,NEW)=DCRDT(I,J,NEW)*100.D0         /DABS(CSB(I,J))
 1590 CONTINUE
 1600 CONTINUE
      CALL CSTBL ( DCRDT ( 1 , 1 , NEW ) , MXDEGP , QPCD  )
      WRITE ( 6 , 265 ) LOOPVR
C         %DIFF IS CALCULATED MINUS INPUT COEFFICIENT AS PERCENTAGE OF
C                INPUT.
      DO 1200 I = 1 , MXDEGP , 2
 1200  DCS (  I     , 1 ) =CBN ( I / 2 + 1 ) + DCS ( I , 1 )
      CALL CSTBL ( DCS , MXDEGP , QBLANK )
      WRITE ( 6 , 265 ) LOOPVR
      DO 178  I = 1 , MXDEGP , 2
```

```
       DCS (  I     , 1 )=-CBN ( I / 2 + 1 ) + DCS ( I , 1 )
  178 CONTINUE
 8300 CONTINUE
C     *****************************************************************
      ITERST = ITERND + 1
C
      READ ( 5 , 3000 ) I , ( FMT ( J ) , J = 1 , 8 )
C
      IF ( I .EQ. -9999 ) GO TO 197
      IF ( I .LE. 0 )GO TO 100
      WRITE ( 7 , 3000 ) I , ( FMT ( J ) , J = 1 , 8 )
      CALL CSPCH ( DCS , MXDEGP , FMT )
      IF ( I .EQ. 99999 ) GO TO 3400
      IF ( I .EQ.  9999 ) GO TO 197
      GO TO 100
C     *****************************************************************
 8500 CONTINUE
      CALL EXIT
      STOP
      END
```

Appendix C.3

```
C     CSRDR READS-IN THE NORMALIZED SPHERICAL HARMONIC COEEFFICIENTS.
C          THE READ-IN FORMAT STATEMENT MUST SPECIFY VARIABLES IN THE
C               ORDER: DEGREE, ORDER, C, S, ETC., ENDING WITH DEG=-1,
C               FOLLOWED BY, DEG1, DEG2, C1,C2, ETC., ENDING WITH DEG1=
C               -1.  THE ZONALS MAY BE READ-IN IN EITHER THE FIRST OR
C               THE SECOND GROUP.
      SUBROUTINE CSRDR ( CSB , F66 )
      REAL * 8  CSB ( 20 , 20 ) , F66 ( 8 )
444   READ(5,F66)I,J,C1,S1,K,L,C2,S2
C     READ IN THE NORMALIZED C(I,J)
      IF(I.LT.0)GO TO 888
       CSB ( I + 1 , J + 1 ) = C1
      IF ( J .LE. 0 ) GO TO 27
       CSB ( J      , I + 1 ) = S1
   27 CONTINUE
       CSB ( K + 1 , L + 1 ) = C2
      IF ( L .LE. 0)GO TO 444
       CSB ( L      , K + 1 ) = S2
      GO TO 444
888   READ(5,F66)I,J,C1,S1,K,L,C2,S2
C     READ IN THE NORMALIZED ZONAL COEFFS
      IF(I.LT.0)GO TO 10099
       CSB ( I + 1 , 1      ) = C1
       CSB ( J + 1 , 1      ) = S1
      IF(K.LT.0)GO TO 10099
       CSB ( K + 1 , 1      ) = C2
       CSB ( L + 1 , 1      ) = S2
      GO TO 888
10099 CONTINUE
      RETURN
      END
```

Appendix C.3

```
C       SNPXDX CALCULATES THE INTEGRAL OF THE NORMALIZED LEGENDRE
C            FUNCTIONS.
C            CALLS DSQRT, SNP2L, FAC.
        SUBROUTINE SNPXDX( MAXDEG , SP , PH , PL , THETAH , XH , CXH ,
      1  THETAL , XL , CXL )
C______MAXDEG = MAXIMUM DEGREE OF THE LEGENDRE FUNCTIONS ( <= 19 ) .
C______SP = INTEGRAL OF THE NORMALIZED LEGENDRE FUNCTIONS ( OUTPUT ) .
C       PH = NORMALIZED LEGENDRE FUNCTIONS AT NORTHERN BOUNDARY OF
C            INTEGRATION.
C       PS = NORMALIZED LEGENDRE FUNCTIONS AT SOUTHERN BOUNDARY OF
C            INTEGRATION.
C______THE MATRICES IN THE CALLING PROGRAM ARE ASSUMED TO BE DIMENSIONED
C                                                              (20,20).
        IMPLICIT REAL*8 (A-H,O-Z )
        REAL * 8 R3 / 1.732050807568877 /
        REAL * 8 SP ( 20 , 20 ) , PH ( 20 , 20 ) , PL ( 20 , 20 )
        MXDEGP = MAXDEG + 1
        SP ( 1 , 1 ) = XH - XL
        IF(MAXDEG .LE. 0 ) RETURN
        TASIN = ( THETAL - THETAH ) / 2D0
        SP ( 2 , 2 ) =(( XH * CXH - XL * CXL  ) / 2D0 + TASIN ) * R3
        SP ( 2 , 1 ) =(( CXL ** 2 - CXH ** 2 ) / 2.D0 ) * R3
        IF ( MAXDEG .EQ. 1 ) RETURN
        IF ( MXDEGP .GT. 20 ) MXDEGP = 20
        LMAX = MAXDEG / 2
        DO 1000 L = 1 , LMAX
        L1 = L + 1
        L21 = L + L1
        SP2L1 = 0D0
        DO 900 J1 = 1 , L1
        J = J1 - 1
        J21 = J + J1
        SP2L1 = SP2L1 + ( XH * CXH ** J21 - XL * CXL ** J21 ) * ( 4 ** J
      1 * J1 ) * FAC ( J ) ** 2 / FAC ( J21 + 1 )
    900 CONTINUE
        SP ( L21 , L21 ) =SNP2L ( L , XH , CXH , XL , CXL )
        SP2L1 = SP2L1 + TASIN
        SP ( L21 + 1 , L21 + 1 ) = SP2L1 * ( L21 + 1 ) * DSQRT (2*(L21 +
      1 L21 + 1 ) * FAC ( L21 + L21 ) ) / ( 4 ** L21 * FAC ( L1 ) ** 2 )
   1000 CONTINUE
        DO 5000 IDEGP = 3 , MXDEGP
        IDEG = IDEGP - 1
        IDEGM = IDEGP - 2
        DEGP = IDEGP
        DEG = IDEG
        SP (IDEGP,1) = (CXL*PL(IDEGP,2) - CXH*PH(IDEGP,2))/
      1  DSQRT ( 2.D0 * DEG * DEGP )
        SP(IDEGP,IDEG ) =((CXL*PL(IDEGP,IDEGP)-CXH*PH(IDEGP,IDEGP))/DEGP )
      1  * DSQRT ( 2.D0 * DEG )
        IF ( IDEGP .NE. 3 ) GO TO 3000
        GO TO 5000
   3000 CONTINUE
        DO 4000 I = 2 , IDEGM
        IORD = IDEGP - I
```

```
      SP ( IDEGP , IORD ) = ( ( IORD - 1 ) * SP ( IDEGP , IORD + 2 ) *
     1  DSQRT (DFLOAT ( ( IDEG - IORD ) * ( IDEGP + IORD ) ) ) -
     2  ( 2 * IORD ) * ( CXH * PH ( IDEGP , IORD + 1 ) - CXL * PL (
     3  IDEGP , IORD + 1 ) ) ) / ( ( IORD+1)*DSQRT(
     4  DFLOAT(( IDEGP - IORD ) * ( IDEG + IORD ) ) ) )
 4000 CONTINUE
 5000 CONTINUE
      RETURN
      END
```

Appendix C.3

```
C     FAC CALCULATES THE FACTORIAL FUNCTION.
C     FACINC CALCULATES THE RATIO OF FACTORIAL FUNCTIONS.
      REAL FUNCTION FAC * 8 ( NH )
      IMPLICIT REAL * 8 ( A - H , O - Z )
      NL = 2
      INC = 1
      ENTRY FACINC ( NH , NL , INC )
      T = NH
      D = INC
      S = NL - .5D0
      FACINC = 1D0
   10 IF ( S .GT. T ) RETURN
      FAC = FAC * T
      T = T - D
      GO TO 10
      END
```

Appendix C.3

```
C      SNP2L CALCULATES THE INTEGRAL OF ALTERNATE SECTORAL HARMONICS.
C           CALLS FACINC, DSQRT, FAC.
       REAL FUNCTION SNP2L * 8 ( L , XH , CXH , XL , CXL )
       IMPLICIT REAL * 8 ( A-H , O-Z )
       L2 = L + L
       L2M1 = L2 - 1
       SNP2L = 0D0
       L1 = L + 1
C      DO 100 K1 = 1 , L
       DO 100 JK1 = 1 , L
       K1 = L1 - JK1
       L2K2 = L2 - K1 - K1
            IF ( L2K2 .GT. 0 ) GO TO 50
            FACTOR = 1D0
            GO TO 75
50          FACTOR = CXH ** L2K2
75          CONTINUE
       SNP2L = SNP2L + 2 ** K1 * FACINC ( L , L - K1 + 1 , 1D0 ) * ( XH *
      1  FACTOR      - XL * CXL ** L2K2 ) / FACINC ( L2M1 , L2K2+1,2)
  100  CONTINUE
       SNP2L = XH * CXH ** L2 - XL * CXL ** L2 + SNP2L
       SNP2L = SNP2L * DSQRT ( ( 8 * L + 2 ) * FAC ( 4 * L ) ) / ( 4 ** L
      1  * FAC ( L2 + 1 ) )
       RETURN
       END
```

```
C      CSPCH PUNCHES OUT THE HARMONIC COEFFICIENTS ACCORDING TO A
C           READ-IN FORMAT STATEMENT.
C      CSTBL PRINTS OUT THE HARMONIC COEFFICIENTS ACCORDING TO A STANDARD
C           FORMAT.
       SUBROUTINE CSPCH (DCS , MXDEGP , FMT )
       INTEGER*4 DEG , DEGP                , ORD , ORDP
       INTEGER*4 II ( 4 ) , JJ ( 4 )
         REAL*8 DCS ( 20 , 20 )      , FMT ( 8 ) , T
       REAL*8 CC ( 4 ) , SS ( 4 )
       REAL * 8 TT
       INTEGER * 2 IT , IB / '  ' / , I1 / ' 1' /
       LOGICAL * 1 LL ( 8 ) , L
       EQUIVALENCE ( TT , LL(1) ) , ( IT , LL(7) ) , ( L , LL(8) )
       IOT = 7
       GO TO 1000
       ENTRY       CSTBL (DCS , MXDEGP , T )
       IOT = 6
       TT = T
       IF ( IT .EQ. IB ) IT = I1
 1200  FORMAT(A1,' ', 4(' N  M  ',A5,'''C''BAR  ',A5,'''S''BAR '))
       WRITE ( 6 , 1200 )L,T , T , T , T , T , T , T, T
 1000  IF ( MXDEGP .GT. 20 ) MXDEGP = 20
       I = 0
       DO 1400 DEGP = 1 , MXDEGP
       DEG = DEGP - 1
       DO 1400 ORDP = 1 , DEGP
       ORD = ORDP - 1
       I = I + 1
       II ( I ) = DEG
       JJ ( I ) = ORD
       CC ( I ) = DCS ( DEGP , ORDP )
       IF ( ORD .GT. 0 ) GO TO 1240
       SS ( I ) = 0.D0
       GO TO 1270
 1240  CONTINUE
       SS ( I ) = DCS ( ORD , DEGP )
 1270  CONTINUE
       IF ( IOT .EQ. 6 ) GO TO 1285
       IF ( I .LT. 4                          ) GO TO 1400
       WRITE ( 7 ,  FMT ) ( II ( I ) , JJ ( I ) , CC ( I ) , SS ( I ) , I
      1  = 1 , 4 )
       GO TO 1350
 1285  CONTINUE
       IF ( I .LT. 4.AND. ORDP .LT. MXDEGP ) GO TO 1400
 1300  FORMAT ( ' ' , 4(2I3 , 2 G13.5 ) )
       WRITE ( 6 , 1300 ) ( II ( J ) , JJ ( J ) , CC ( J ) , SS ( J ) , J
      1  = 1 , I )
 1350  CONTINUE
       I = 0
 1400  CONTINUE
       IF ( IOT .EQ. 6 ) RETURN
       IF ( I .EQ. 0 ) GO TO  9000
       IF ( I .EQ. 2 ) GO TO 7000
       IF ( I .EQ. 3 ) GO TO 8000
```

Appendix C.3

```
      II(2)=II(1)
      JJ(2)=JJ(1)
      CC(2)=CC(1)
      SS(2)=SS(1)
7000  WRITE ( 7 ,  FMT ) ( II ( I ) , JJ ( I ) , CC ( I ) , SS ( I ) , I
     1 = 1 , 2 )
9000  I = -1
      WRITE ( 7 , FMT ) I
      WRITE ( 7 , FMT ) I
      RETURN
8000  II(4)=II(3)
      JJ(4)=JJ(3)
      CC(4)=CC(3)
      SS(4)=SS(3)
      WRITE ( 7 ,  FMT ) ( II ( I ) , JJ ( I ) , CC ( I ) , SS ( I ) , I
     1 = 1 , 4 )
      GO TO 9000
      END
```

Appendix D

CONVERGENCE OF AN ALTERNATIVE SYMMETRIC FORMULATION

J. E. Potter and S. J. Madden (personal communication) suggest a formulation for which a sufficient condition for existence and uniqueness is obtained. Write equation (2.55) with

$$\beta = 1 \tag{D.01}$$

in the form

$$[I + H]\zeta = 2\nu \tag{D.02}$$

where

$$H = \text{sgn }(S_1)(I + 2K_N) \tag{D.03}$$

and $\text{sgn }(S_1) = \Lambda(p) - \Omega(p)$ (D.04)

An upper bound for $|| H ||$ may be obtained by applying the Cauchy-Buniakovskii-Schwarz inequality and noting that

$$|| \text{ sgn }(S_1) || = 1 \tag{D.05}$$

Hence

$$|| H || \leq \max_{\lambda_j \in \sigma(I+2K_N)} | \lambda_j | \tag{D.06}$$

where

$$\lambda_j = 1 - \frac{4}{n_j+1} \tag{D.07}$$

When the zeroth and first harmonics are suppressed and the series is truncated the bound is less than one. Hence a Neumann series for this problem converges.

Appendix E

CONTINUITY OF THE NORM

In this appendix it is shown that the norm of the infinite-dimensional operator, K(p, q), and hence, those of the equivalent operators, such as M(p, q), varies continuously as the altimetry-gravimetry boundary is deformed.

If A and B are operators on a normed linear space, the triangle inequality holds (Halmos, 1951, p. 35)

$$|| A + B || \leq || A || + || B || \tag{E.01}$$

Similarly

$$|| A || = || (A + B) - B || \leq || A + B || + || - B || \tag{E.02}$$

or

$$\Big| || A + B || - || A || \Big| \leq || B || \tag{E.03}$$

We identify A with the operator,

$$K(p, q) = \begin{cases} I(p,q)+\beta K_N(p,q) & p \in S_0 \\ -K_N(p,q) & p \in S_1 \end{cases} \tag{2.54}$$

We identify A + B with the same operator but applied to a sphere where the boundary, ∂S, between S_0 and S_1 is perturbed slightly to obtain new surfaces S_0' and S_1'. Let

$$\begin{aligned} S_0 + \delta S &= S_0'. \\ S_1 - \delta S &= S_1' \end{aligned} \tag{E.04}$$

δS consists of "positive" areas, δS^+, that are in S_0' but not in S_0 and "negative" areas, δS^-, that are in S_0, but not in S_0'. We designate this new operator,

$$K'(p, q) = [K(p, q) + \delta K(p, q)]$$

$$= \begin{cases} I(p, q) + \beta K_N(p, q) & p \in S_0 + \delta S \\ - K_N(p, q) & p \in S_1 - \delta S \end{cases} \qquad (E.05)$$

We thus identify B with the perturbation operator, $\delta K(p, q)$,

$$\delta K(p, q) = \begin{cases} I(p, q) + (1 + \beta)K_N(p, q) & p \in \delta S \\ 0 & p \in S - \delta S \end{cases} \qquad (E.06)$$

We may correct for the positive and negative areas by including a signum function multiplying the operator, or alternatively,

$$\delta K = \begin{cases} I(p, q) + (1 + \beta)K_N(p, q) & p \in \delta S^+ \\ - I(p, q) - (1 + \beta)K_N(p, q) & p \in \delta S^- \\ 0 & p \in S - \delta S^+ - \delta S^- \end{cases} \qquad (E.07)$$

We wish to show that $|| K ||$ varies continuously with changes in ∂S, i.e., for small changes, δS, $|| K + \delta K ||$ is near $|| K ||$. Since

$$\Big| || K + \delta K || - || K || \Big| \leq || \delta K || \qquad (E.08)$$

We have to show that $|| \delta K ||$ is as small as desired when δS is sufficiently small.

As in equation (4.15), the norm is defined by

$$|| \delta K || = \sup_x \{ || \delta Kx || ; || x || = 1 \} \qquad (E.09)$$

Let

$$x(p) = \sum_{i=1}^{\infty} c_i x_i(p) \qquad (E.10)$$

The c_i's are any set of coefficients satisfying

$$\sum_{i=1}^{\infty} c_i^2 = 1 \qquad (E.11)$$

Define

$$\operatorname{sgn}(\delta S^{+}) = \begin{cases} 1 & p \in \delta S^{+} \\ -1 & p \in \delta S^{-} \\ 0 & p \in S - \delta S^{+} - \delta S^{-} \end{cases} \tag{E.12}$$

We have

$$\delta K(p, q)x(q) = \operatorname{sgn}(\delta S^{+})[I(p, q) + (1 + \beta)K_N(p, q)]x(q) \tag{E.13}$$

or, in the notation of equation (4.30),

$$\delta K(p, q)x(q) = \operatorname{sgn}(\delta S^{+})\frac{1}{4\pi}\iint_{\sigma}\sum_{i=1}^{\infty}(1 - 2\mu_i)x_i(p)x_i(q)\sum_{j=1}^{\infty}c_j x_j(q)\,d\sigma_q \tag{E.14}$$

Using the orthonormality of the spherical harmonics, $x_i(p)$,

$$\delta K(p, q)x(q) = \operatorname{sgn}(\delta S^{+})\sum_{i=1}^{\infty}(1 - 2\mu_i)c_i x_i(p) \tag{E.15}$$

Since for any c_i's such that equation (E.11) holds,

$$\sum_{i=1}^{\infty}c_i x_i(p) = x(p) \tag{E.16}$$

is bounded and convergent, and (see equation (2.38))

$$n_i \geq 0$$

by the Weierstrass M test so is

$$\sum_{i=1}^{\infty}(1 - 2\mu_i)c_i x_i(p) = x(p) - 2(1 + \beta)\sum_{i=1}^{\infty}\frac{c_i x_i(p)}{n_i+1} \tag{E.17}$$

We have

$$\|\ \delta K(p, q)x(q)\ \| =$$

$$= \left\{\frac{1}{4\pi}\iint_{\sigma}\left[\operatorname{sgn}(\delta S^{+})\sum_{i=1}^{\infty}(1 - 2\mu_i)c_i x_i(p)\right]^2 d\sigma_p\right\}^{1/2} \tag{E.18}$$

$$= \left\{\frac{1}{4\pi}\iint_{\delta S}\left[\sum_{i=1}^{\infty}(1 - 2\mu_i)c_i x_i(p)\right]^2 d\sigma_p\right\}^{1/2}$$

Here we use the fact that

$$[\mathrm{sgn}(\delta S^{+})]^{2} = \begin{cases} 1 & p \in \delta S = \delta S^{+} \cup \delta S^{-} \\ 0 & p \in S - \delta S \end{cases} \tag{E.19}$$

Thus if δS is sufficiently small in area, $|| \delta K(p, q)x(q) ||$ will be as small as desired for any $x(q)$, $|| x(q) || = 1$. Thus $|| \delta K ||$ can be as small as desired, and the continuity of the norm is established.

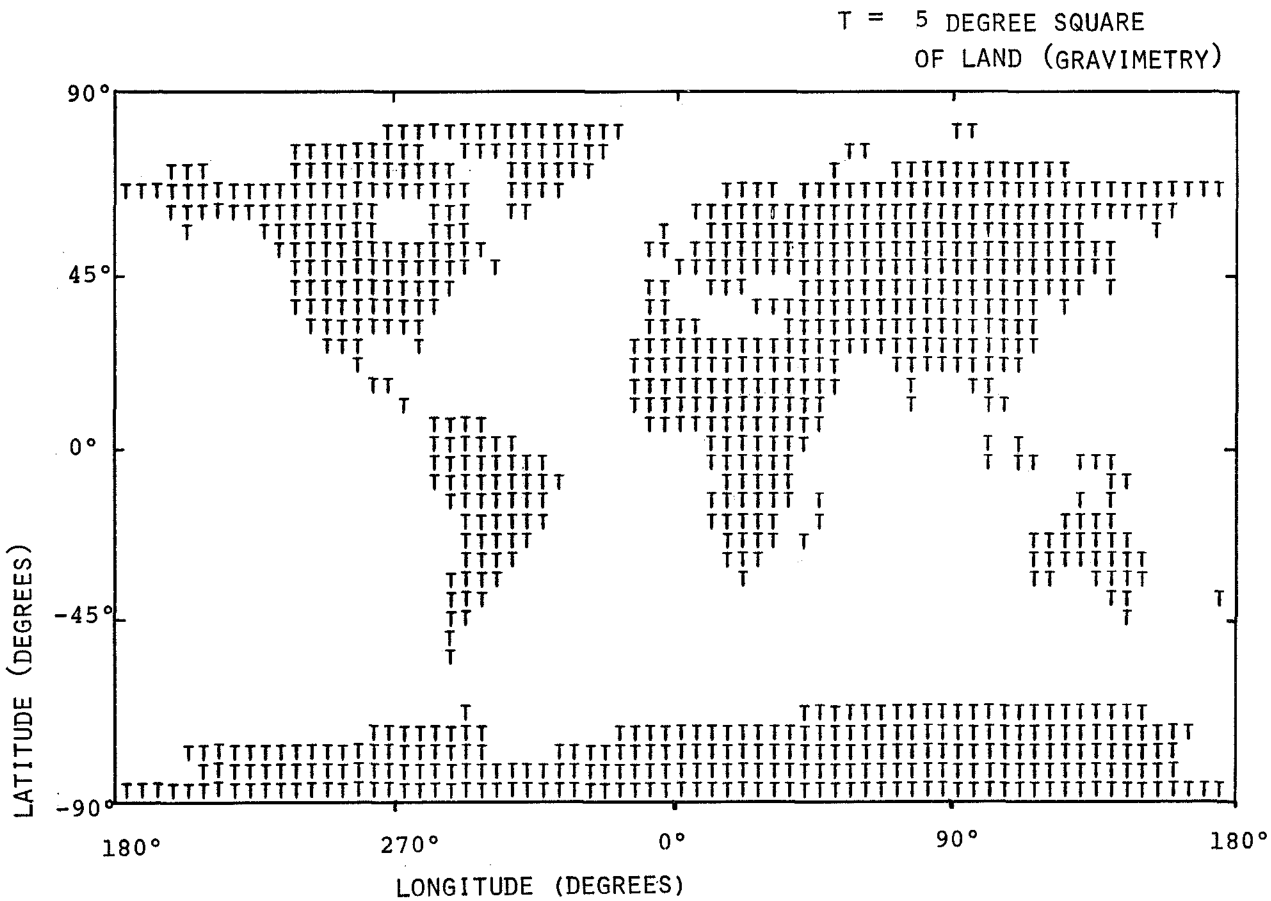

FIGURE 1 LAND AND OCEAN DISTRIBUTION

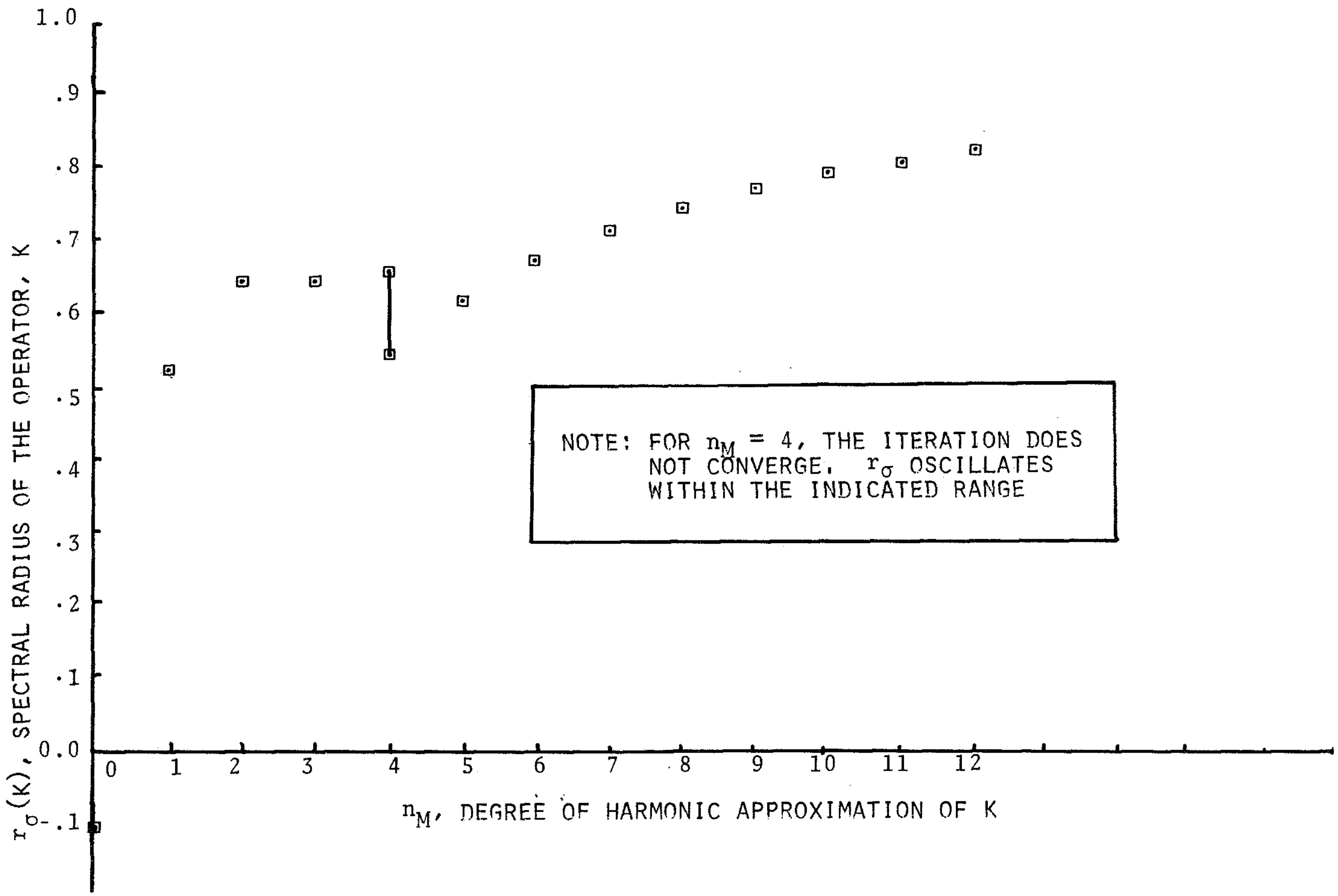

FIGURE 2. SPECTRAL RADIUS OF THE OPERATOR VS. DEGREE OF HARMONIC APPROXIMATION

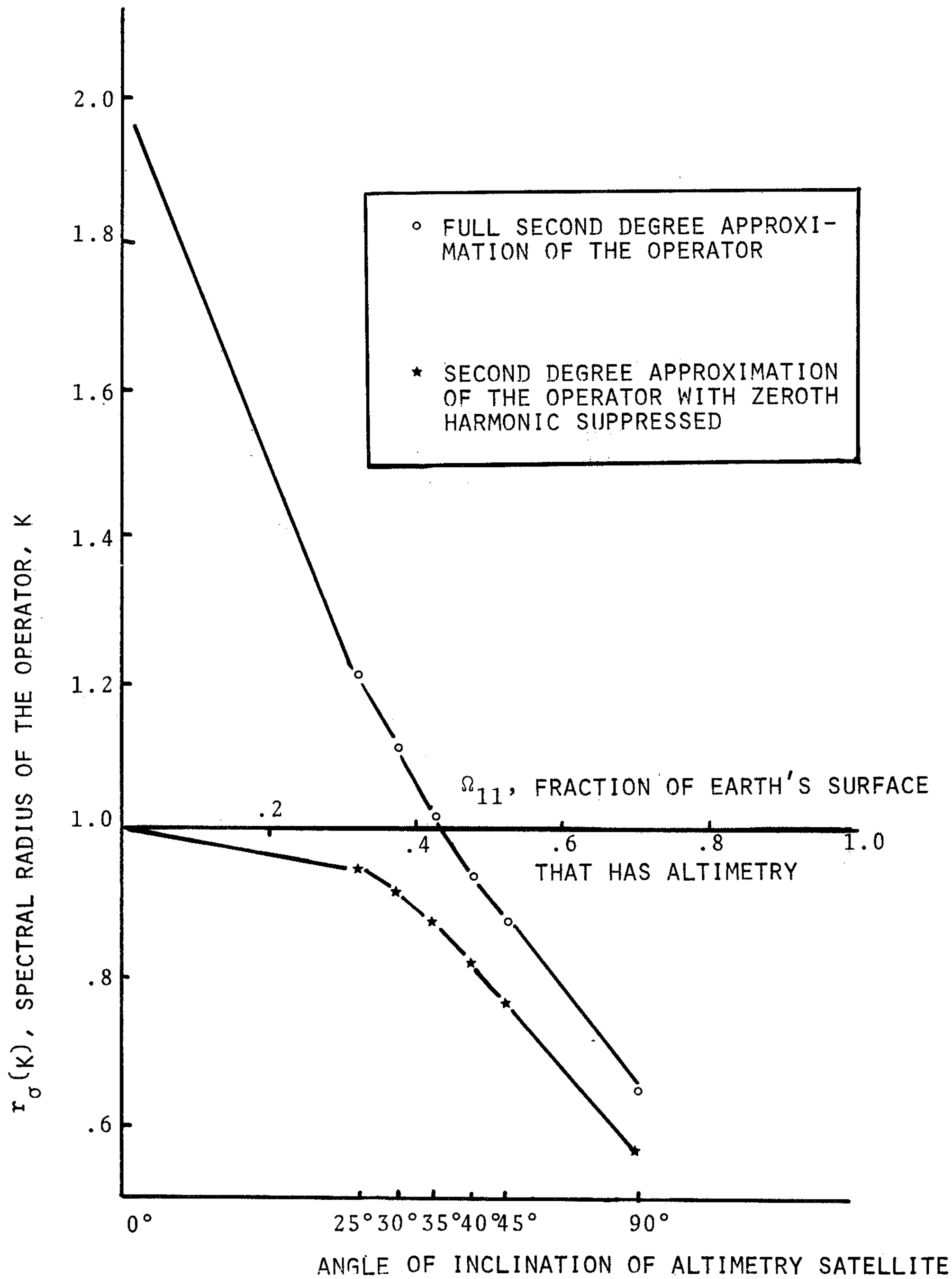

FIGURE 3. SPECTRAL RADIUS OF THE OPERATOR VS. FRACTION OF EARTH'S SURFACE THAT HAS ALTIMETRY

Table 1

HARMONIC COEFFICIENTS-FOURTH DEGREE MODEL

Normalized spherical harmonic coefficients [× 1000000] n m	The Rapp (1968) model	The globe of figure 1	A globe with all altimetry (oceans)	A globe with all gravimetry (land)
C 0 0	1000000.0000	1000000.0005	999999.9999	999787.7328
C 1 0	0.0	0.0014	0.0006	0.0123
C 1 1	0.0	0.0010	0.0005	0.0096
S 1 1	0.0	0.0002	0.0001	0.0013
C 2 0	-484.1778	-484.1780	-484.1779	-484.1782
C 2 1	0.0	-0.0002	-0.0004	-0.0007
S 2 1	0.0	0.0001	-0.0003	-0.0006
C 2 2	2.3509	2.3494	2.3495	2.3474
S 2 2	-1.3251	-1.3237	-1.3240	-1.3225
C 3 0	0.8906	0.8892	0.8893	0.8893
C 3 1	1.7134	1.7105	1.7107	1.7107
S 3 1	0.2334	0.2331	0.2330	0.2330
C 3 2	0.6717	0.6706	0.6707	0.6707
S 3 2	-0.5572	-0.5561	-0.5564	-0.5564
C 3 3	0.7172	0.7160	0.7159	0.7159
S 3 3	1.3390	1.3367	1.3366	1.3367
C 4 0	0.5606	0.5611	0.5614	0.5611
C 4 1	-0.5108	-0.5093	-0.5089	-0.5097
S 4 1	-0.4094	-0.4083	-0.4079	-0.4085
C 4 2	0.2528	0.2520	0.2522	0.2524
S 4 2	0.4842	0.4828	0.4824	0.4831
C 4 3	0.8946	0.8921	0.8915	0.8927
S 4 3	-0.2114	-0.2106	-0.2107	-0.2110
C 4 4	0.1467	0.1464	0.1461	0.1463
S 4 4	0.3338	0.3329	0.3325	0.3330

Table 2

HARMONIC COEFFICIENTS-14^{th} AND 15^{th} DEGREE MODELS

Normalized spherical harmonic coefficients [× 1000000]	n	m	Modified † Rapp (1968) model	The globe of figure 1 (Rapp)	The Köhnlein (1967) model	The globe of figure 1 (Köhnlein)
C	0	0	1000000.0000	1000000.0052	1000000.0000	1000000.0069
C	1	0	0.0	0.0086	0.0	0.0150
C	1	1	0.0	0.0125	0.0	0.0167
S	1	1	0.0	-0.0057	0.0	0.0003
C	2	0	-484.1741†	-484.1808	-484.1741	-484.1784
C	2	1	0.0	-0.0045	0.0	-0.0011
S	2	1	0.0	-0.0068	0.0	-0.0029
C	2	2	2.3509	2.3454	2.3800	2.3774
S	2	2	-1.3251	-1.3258	-1.3500	-1.3474
C	3	0	0.8906	0.8923	0.9695	0.9732
C	3	1	1.7134	1.7072	1.7100	1.7039
S	3	1	0.2334	0.2320	0.2300	0.2330
C	3	2	0.6717	0.6694	0.8400	0.8438
S	3	2	-0.5572	-0.5481	-0.5100	-0.5015
C	3	3	0.7172	0.7065	0.6600	0.6506
S	3	3	1.3390	1.3488	1.4300	1.4384
C	4	0	0.5606	0.5611	0.5360	0.5369
C	4	1	-0.5108	-0.5202	-0.4700	-0.4738
S	4	1	-0.4094	-0.4050	-0.3900	-0.3920
C	4	2	0.2528	0.2544	0.3500	0.3495
S	4	2	0.4842	0.4844	0.4800	0.4777
C	4	3	0.8946	0.8848	0.9200	0.9123
S	4	3	-0.2114	-0.2066	-0.2400	-0.2343
C	4	4	0.1467	0.1450	0.0400	0.0379
S	4	4	0.3338	0.3381	0.3000	0.3061
C	5	0	0.0286	0.0297	0.0525	0.0535
C	5	1	-0.0847	-0.0778	-0.0600	-0.0515
S	5	1	-0.0202	-0.0229	-0.0500	-0.0472
C	5	2	0.3703	0.3732	0.5300	0.5297
S	5	2	-0.1789	-0.1819	-0.2100	-0.2066
C	5	3	-0.1887	-0.1804	-0.4000	-0.3893
S	5	3	0.0204	0.0230	0.0700	0.0735
C	5	4	0.1552†	0.1566	-0.2000	-0.1946
S	5	4	0.1024	0.0917	0.0200	0.0115
C	5	5	0.0078	0.0144	0.1800	0.1795
S	5	5	-0.5450	-0.5312	-0.5600	-0.5483
C	6	0	-0.0782	-0.0774	-0.1503	-0.1453
C	6	1	-0.0893	-0.0952	-0.0800	-0.0786
S	6	1	-0.0198	-0.0208	0.0100	0.0030
C	6	2	-0.0065	-0.0067	0.0100	0.0108
S	6	2	-0.1998	-0.1903	-0.2700	-0.2617
C	6	3	-0.0616	-0.0516	-0.0400	-0.0351
S	6	3	0.0815	0.0754	0.0300	0.0281
C	6	4	-0.0461	-0.0421	-0.0800	-0.0815
S	6	4	-0.3647	-0.3588	-0.4800	-0.4723
C	6	5	-0.2671	-0.2579	-0.2600	-0.2524
S	6	5	-0.4441	-0.4357	-0.4600	-0.4479
C	6	6	0.0215	0.0173	-0.0200	-0.0177

Table 2

Normalized spherical harmonic coefficients [× 1000000]	n	m	Modified † Rapp model	The Globe of figure 1 (Rapp)	The Köhnlein (1967) model	The globe of figure 1 (Köhnlein)
S	6	6	-0.1916	-0.1879	-0.1600	-0.1582
C	7	0	0.0458	0.0464	0.1082	0.1061
C	7	1	0.0927	0.0924	0.1700	0.1661
S	7	1	0.0644	0.0631	0.1100	0.1116
C	7	2	0.2424	0.2406	0.3200	0.3177
S	7	2	0.1026	0.0972	0.1600	0.1556
C	7	3	0.1615	0.1472	0.1800	0.1697
S	7	3	0.0042	0.0048	0.0	-0.0009
C	7	4	-0.2278	-0.2191	-0.1600	-0.1559
S	7	4	-0.0911	-0.0848	-0.0400	-0.0357
C	7	5	0.0618	0.0582	0.0700	0.0679
S	7	5	0.0535	0.0484	-0.0100	-0.0113
C	7	6	-0.1381	-0.1361	-0.2300	-0.2236
S	7	6	0.1187	0.1133	0.1000	0.0990
C	7	7	0.0426	0.0348	0.0700	0.0614
S	7	7	-0.0737	-0.0787	0.0600	0.0486
C	8	0	0.0243	0.0233	0.0310	0.0363
C	8	1	-0.0372	-0.0395	-0.0100	-0.0073
S	8	1	0.0070	0.0065	-0.0100	-0.0121
C	8	2	0.0442	0.0415	0.0400	0.0357
S	8	2	0.1552	0.1472	0.0400	0.0364
C	8	3	0.0357	0.0332	-0.0300	-0.0282
S	8	3	0.0806	0.0754	0.0	0.0004
C	8	4	-0.0386	-0.0362	-0.1700	-0.1634
S	8	4	0.0625	0.0590	-0.0200	-0.0243
C	8	5	-0.0497	-0.0527	-0.0900	-0.0871
S	8	5	0.0618	0.0550	0.0900	0.0802
C	8	6	-0.1373	-0.1297	-0.0100	-0.0121
S	8	6	0.2520	0.2298	0.3000	0.2812
C	8	7	0.0358	0.0327	0.0200	0.0197
S	8	7	0.0286	0.0258	0.0400	0.0387
C	8	8	-0.0764	-0.0732	-0.1800	-0.1648
S	8	8	-0.0605	-0.0591	0.0300	0.0239
C	9	0	0.0179	0.0199	0.0050	0.0088
C	9	1	0.1367	0.1300	0.1100	0.1052
S	9	1	-0.0926	-0.0857	0.0	0.0056
C	9	2	0.0061	0.0105	0.0300	0.0349
S	9	2	-0.0387	-0.0411	0.0500	0.0447
C	9	3	-0.0844	-0.0776	-0.0300	-0.0302
S	9	3	-0.0119	-0.0088	-0.0100	-0.0104
C	9	4	0.0397	0.0332	0.0700	0.0591
S	9	4	-0.0139	-0.0136	0.0200	0.0183
C	9	5	-0.0579	-0.0541	-0.0400	-0.0370
S	9	5	0.0116	0.0072	0.0400	0.0387
C	9	6	-0.0091	-0.0106	0.0400	0.0349
S	9	6	0.0511	0.0452	0.0100	0.0109
C	9	7	0.0429	0.0388	0.0400	0.0361
S	9	7	0.0238	0.0194	-0.0200	-0.0203
C	9	8	0.2402	0.2240	0.1300	0.1248
S	9	8	0.0078	0.0047	0.0	-0.0013
C	9	9	0.0045	0.0072	0.0800	0.0739
S	9	9	-0.0401	-0.0346	0.0400	0.0416

Table 2

Normalized spherical harmonic coefficients [× 1000000] n m			Modified † Rapp model	The globe of figure 1 (Rapp)	The Köhnlein (1967) model	The globe of figure 1 (Köhnlein)
C	10	0	-0.0339	-0.0379	0.0738	0.0633
C	10	1	0.0553	0.0531	0.1000	0.0833
S	10	1	-0.0412	-0.0439	-0.0700	-0.0695
C	10	2	-0.0352	-0.0407	-0.0800	-0.0745
S	10	2	-0.0760	-0.0640	-0.0600	-0.0520
C	10	3	-0.0110	-0.0105	-0.0800	-0.0734
S	10	3	-0.1295	-0.1234	-0.0500	-0.0534
C	10	4	-0.0053	-0.0091	-0.0600	-0.0629
S	10	4	-0.0616	-0.0559	-0.0800	-0.0715
C	10	5	-0.0044	-0.0023	0.0200	0.0181
S	10	5	0.0087	-0.0004	-0.0200	-0.0148
C	10	6	-0.0536	-0.0507	-0.0400	-0.0349
S	10	6	-0.3760†	-0.3427	-0.0100	-0.0046
C	10	7	0.0857	0.0793	0.0400	0.0354
S	10	7	-0.0040	-0.0095	-0.0500	-0.0499
C	10	8	0.0328	0.0295	0.0400	0.0293
S	10	8	-0.1242	-0.1071	-0.0500	-0.0464
C	10	9	0.1027	0.0980	0.0500	0.0461
S	10	9	0.0002	0.0040	-0.0400	-0.0383
C	10	10	0.0709	0.0719	0.0300	0.0339
S	10	10	-0.0739	-0.0592	-0.0200	-0.0111
C	11	0	-0.1022	-0.0905	-0.0367	-0.0338
C	11	1	0.0328	0.0349	-0.0300	-0.0229
S	11	1	0.0147	0.0075	0.0200	0.0133
C	11	2	0.0276	0.0274	0.0500	0.0430
S	11	2	-0.0326	-0.0283	-0.0500	-0.0387
C	11	3	-0.0139	-0.0191	0.0100	0.0032
S	11	3	-0.0416	-0.0360	-0.0800	-0.0699
C	11	4	-0.0173	-0.0191	-0.0300	-0.0318
S	11	4	-0.0595	-0.0674	0.0	-0.0036
C	11	5	0.0196	0.0173	0.0300	0.0291
S	11	5	-0.0744	-0.0708	0.0200	0.0166
C	11	6	-0.0454	-0.0431	-0.0300	-0.0276
S	11	6	-0.0004	0.0012	-0.0200	-0.0135
C	11	7	0.0051	0.0099	0.0300	0.0316
S	11	7	-0.0922	-0.0907	-0.0300	-0.0311
C	11	8	0.0460	0.0415	0.0400	0.0364
S	11	8	0.0142	0.0142	-0.0200	-0.0185
C	11	9	0.0258	0.0210	0.0300	0.0222
S	11	9	-0.0017	0.0041	0.0100	0.0125
C	11	10	-0.0220	-0.0154	-0.0300	-0.0254
S	11	10	-0.0171	-0.0162	-0.0100	-0.0134
C	11	11	0.0737	0.0659	0.1000	0.0874
S	11	11	0.0172	0.0143	0.0600	0.0507
C	12	0	-0.0589†	-0.0543	-0.0106	-0.0062
C	12	1	-0.0445	-0.0384	-0.0900	-0.0761
S	12	1	-0.0602	-0.0559	-0.0700	-0.0608
C	12	2	-0.0184	-0.0187	-0.0600	-0.0523
S	12	2	0.0742	0.0675	0.0200	0.0108
C	12	3	0.0740	0.0683	0.0300	0.0285
S	12	3	-0.0052	-0.0027	0.0200	0.0160
C	12	4	-0.0205	-0.0134	-0.0500	-0.0459

Table 2

Normalized spherical harmonic coefficients [× 1000000] n m	Modified † Rapp model	The globe of figure 1 (Rapp)	The Köhnlein (1967) model	The globe of figure 1 (Köhnlein)
S 12 4	-0.0068	-0.0103	0.0100	0.0066
C 12 5	0.0408	0.0434	0.0200	0.0201
S 12 5	-0.0855	-0.0848	0.0100	0.0077
C 12 6	0.0070	0.0017	-0.0100	-0.0123
S 12 6	0.0304	0.0217	0.0100	0.0152
C 12 7	-0.0484	-0.0349	-0.0400	-0.0314
S 12 7	0.0392	0.0319	-0.0200	-0.0215
C 12 8	0.0263	0.0197	0.0	0.0013
S 12 8	0.0499	0.0439	0.0100	0.0064
C 12 9	-0.0231	-0.0163	-0.0100	-0.0052
S 12 9	0.0582	0.0550	0.0200	0.0251
C 12 10	-0.0061	-0.0083	-0.0100	-0.0083
S 12 10	0.0128	0.0086	0.0	0.0051
C 12 11	-0.0253	-0.0180	-0.0500	-0.0423
S 12 11	0.0071	0.0073	-0.0200	-0.0125
C 12 12	0.0295	0.0249	-0.0100	-0.0122
S 12 12	-0.0375	-0.0323	-0.0100	-0.0081
C 13 0	0.0590	0.0484	0.0281	0.0249
C 13 1	-0.0031	-0.0038	0.0	-0.0027
S 13 1	-0.0259	-0.0217	0.0400	0.0354
C 13 2	0.0001	-0.0005	-0.0300	-0.0247
S 13 2	0.0046	0.0034	0.0100	0.0030
C 13 3	0.0164	0.0176	0.0	0.0027
S 13 3	0.0748	0.0670	0.0300	0.0293
C 13 4	0.0081	0.0078	-0.0100	-0.0109
S 13 4	-0.0439	-0.0358	-0.0200	-0.0143
C 13 5	0.0650	0.0617	0.0300	0.0297
S 13 5	-0.0570	-0.0584	-0.0200	-0.0195
C 13 6	-0.0417	-0.0366	-0.0300	-0.0229
S 13 6	0.0441	0.0386	0.0500	0.0382
C 13 7	0.0055	0.0039	-0.0200	-0.0164
S 13 7	0.0219	0.0201	0.0	-0.0002
C 13 8	-0.0587	-0.0491	-0.0200	-0.0211
S 13 8	0.0041	0.0058	-0.0100	-0.0102
C 13 9	-0.0059	-0.0006	0.0200	0.0215
S 13 9	0.0604	0.0520	0.0500	0.0459
C 13 10	0.0084	0.0067	0.0400	0.0350
S 13 10	-0.0745	-0.0656	-0.0200	-0.0183
C 13 11	-0.0595	-0.0488	-0.0200	-0.0101
S 13 11	-0.0026	-0.0039	0.0100	0.0071
C 13 12	0.0054	0.0013	-0.0200	-0.0204
S 13 12	0.0653	0.0545	0.0600	0.0511
C 13 13	-0.0105	-0.0102	-0.0700	-0.0608
S 13 13	0.0375	0.0285	0.0	-0.0022
C 14 0	-0.0068	-0.0064	0.0323	0.0268
C 14 1	0.0162	0.0131	-0.0100	-0.0107
S 14 1	0.0014	0.0001	0.0200	0.0132
C 14 2	-0.0729	-0.0663	-0.0100	-0.0143
S 14 2	-0.0023	-0.0019	-0.0400	-0.0336
C 14 3	0.0230	0.0229	0.0600	0.0521
S 14 3	0.0132	0.0109	-0.0300	-0.0225
C 14 4	0.0319	0.0244	0.0	-0.0043

Table 2

Normalized spherical harmonic coefficients [× 1000000] n m	Modified † Rapp model	The globe of figure 1 (Rapp)	The Köhnlein (1967) model	The globe of figure 1 (Köhnlein)
S 14 4	-0.0044	-0.0039	0.0	-0.0005
C 14 5	0.0972	0.0897	0.0500	0.0492
S 14 5	-0.0887	-0.0800	-0.0300	-0.0296
C 14 6	0.0263	0.0237	0.0100	0.0097
S 14 6	-0.0552	-0.0509	-0.0300	-0.0266
C 14 7	0.0787	0.0650	0.0300	0.0227
S 14 7	0.0343	0.0311	0.0200	0.0204
C 14 8	-0.0154	-0.0135	-0.0300	-0.0264
S 14 8	-0.0252	-0.0190	-0.0300	-0.0262
C 14 9	0.0386	0.0316	0.0300	0.0273
S 14 9	0.0885	0.0760	0.0700	0.0609
C 14 10	0.0707	0.0610	0.0400	0.0360
S 14 10	-0.0666	-0.0585	0.0100	0.0071
C 14 11	0.0303	0.0250	0.0400	0.0336
S 14 11	-0.0071	-0.0063	0.0100	0.0108
C 14 12	-0.0128	-0.0071	0.0500	0.0365
S 14 12	-0.0013	-0.0004	-0.0300	-0.0221
C 14 13	0.0105	0.0068	0.0100	0.0052
S 14 13	0.0233	0.0211	0.0400	0.0312
C 14 14	-0.0392	-0.0306	-0.0400	-0.0354
S 14 14	-0.0122	-0.0048	0.0200	0.0171
C 15 0			0.0117	0.0091
C 15 1			0.0100	0.0124
S 15 1			-0.0100	-0.0053
C 15 2			-0.0200	-0.0150
S 15 2			-0.0300	-0.0259
C 15 3			0.0200	0.0193
S 15 3			0.0300	0.0232
C 15 4			0.0	0.0006
S 15 4			0.0100	0.0064
C 15 5			0.0300	0.0291
S 15 5			-0.0200	-0.0177
C 15 6			0.0300	0.0234
S 15 6			-0.0500	-0.0437
C 15 7			0.0300	0.0259
S 15 7			0.0400	0.0358
C 15 8			-0.0600	-0.0516
S 15 8			0.0	-0.0021
C 15 9			0.0	0.0008
S 15 9			0.0400	0.0369
C 15 10			0.0200	0.0195
S 15 10			0.0100	0.0090
C 15 11			0.0100	0.0074
S 15 11			0.0100	0.0096
C 15 12			-0.0700	-0.0546
S 15 12			0.0500	0.0414
C 15 13			-0.0500	-0.0405
S 15 13			-0.0300	-0.0244
C 15 14			0.0100	0.0083
S 15 14			-0.0300	-0.0233
C 15 15			-0.0200	-0.0173
S 15 15			-0.0100	-0.0049

† The four indicated (†) coefficients deviate from those of Rapp (1968).

Table 3

THE OCEAN COEFFICIENTS

$\Omega_{ik} = \Omega^{st\ell}_{nmj}$

		Ω_{ii}		Ω_{il}		Lee and Kaula (1967) Ω_{il}		Munk and MacDonald (1960) Ω_{il}	
n	m	Ω^{nm0}_{nm0}	Ω^{nm1}_{nm1}	Ω^{000}_{nm0}	Ω^{000}_{nm1}	Ω^{000}_{nm0}	Ω^{000}_{nm1}	Ω^{000}_{nm0}	Ω^{000}_{nm1}
0	0	0.702		0.702		0.709		0.714	
1	0	0.638		-0.124		-0.051		-0.123	
1	1	0.762	0.706	-0.106	-0.062	-0.144	-0.079	-0.108	-0.055
2	0	0.627		-0.071		-0.040		-0.058	
2	1	0.778	0.620	-0.045	-0.056	-0.053	-0.068	-0.039	-0.061
2	2	0.762	0.724	0.036	-0.004	0.051	0.002	0.077	-0.005
3	0	0.616		0.044		0.036		0.044	
3	1	0.754	0.618	0.043	-0.038	0.035	-0.046	0.046	-0.039
3	2	0.710	0.723	0.065	-0.095	0.074	-0.109	0.125	-0.179
3	3	0.745	0.750	-0.010	-0.088	-0.011	-0.122	-0.017	-0.252
4	0	0.635		-0.034		-0.016		-0.026	
4	1	0.710	0.600	0.038	0.033	0.035	0.016	0.041	0.025
4	2	0.716	0.713	0.093	-0.026	0.097	-0.040	0.175	-0.043
4	3	0.711	0.734	-0.047	0.006	-0.060	-0.001	-0.144	0.007
4	4	0.758	0.743	0.022	-0.096	0.033	-0.153	-0.069	-0.406
5	0	0.648		0.102		0.056		0.101	
5	1	0.717	0.562	-0.008	0.014	0.001	0.008	-0.008	0.018
5	2	0.705	0.684	0.050	0.024	0.060	0.020	0.097	0.052
5	3	0.711	0.738	-0.030	-0.012	-0.039	-0.012	-0.107	-0.036
5	4	0.726	0.726	-0.086	0.028	-0.118	0.027	-0.363	0.106
5	5	0.733	0.772	-0.002	-0.048	-0.002	-0.074	0.000	-0.257
6	0	0.649		-0.030		-0.007		-0.033	
6	1	0.709	0.571	0.009	0.029	0.007	0.017	0.009	0.020
6	2	0.686	0.658	0.020	-0.003	0.027	0.001	0.033	-0.006
6	3	0.709	0.725	-0.001	-0.029	-0.003	-0.032	0.002	-0.075
6	4	0.743	0.712	-0.028	0.028	-0.050	-0.036	-0.110	0.091
6	5	0.717	0.740	0.027	0.025	0.028	0.027	0.110	0.115
6	6	0.774	0.735	-0.003	-0.013	-0.009	-0.024	-0.012	-0.078
7	0	0.661		0.046		0.025		0.051	
7	1	0.706	0.557	-0.004	-0.029	0.002	-0.025	-0.006	-0.035
7	2	0.675	0.658	-0.028	-0.011	-0.017	-0.006	-0.049	-0.002
7	3	0.685	0.706	0.007	-0.018	0.010	-0.018	0.043	-0.032
7	4	0.730	0.725	0.028	-0.008	0.026	-0.003	0.109	-0.026
7	5	0.704	0.751	-0.006	0.024	-0.005	0.032	-0.021	0.119
7	6	0.736	0.726	0.002	0.034	0.007	0.039	0.031	0.163
7	7	0.759	0.754	0.000	0.037	0.013	0.049	0.043	0.224
8	0	0.648		0.007		0.012		0.010	
8	1	0.716	0.578	0.004	0.033	0.001	0.009	-0.002	0.027
8	2	0.655	0.635	-0.006	-0.004	-0.012	0.004	-0.020	0.024
8	3	0.687	0.699	-0.014	-0.016	-0.011	-0.015	-0.046	-0.027
8	4	0.723	0.694	0.003	-0.017	0.002	-0.014	0.000	-0.052
8	5	0.718	0.749	0.016	0.003	0.024	0.006	0.116	-0.002
8	6	0.723	0.731	-0.014	-0.009	-0.014	-0.013	-0.073	-0.115
8	7	0.731	0.736	-0.036	-0.015	-0.047	-0.023	-0.243	-0.122
8	8	0.759	0.757	0.008	0.015	-0.025	0.037	-0.136	0.175

Table 3

		Ω_{ii}		Ω_{i1}		$\Omega_{i,85}$		$\Omega_{i,169}$	
n	m	Ω^{000}_{nm0}	Ω^{nm1}_{nm1}	Ω^{nm0}_{nm0}	Ω^{nm1}_{nm1}	Ω^{930}_{nm0}	Ω^{930}_{nm1}	$\Omega^{12,12,1}_{nm0}$	$\Omega^{12,12,1}_{nm1}$
9	0	0.638		0.026		0.010		-0.004	
9	1	0.723	0.587	0.005	0.024	-0.036	0.023	-0.008	0.002
9	2	0.642	0.636	0.003	-0.007	0.018	-0.019	0.003	-0.002
9	3	0.673	0.677	-0.016	-0.001	0.673	-0.010	-0.008	0.002
9	4	0.721	0.695	0.007	-0.019	-0.001	-0.013	0.003	-0.003
9	5	0.707	0.726	-0.003	0.003	-0.007	-0.015	0.006	0.007
9	6	0.740	0.733	-0.008	-0.027	0.033	-0.006	-0.018	0.000
9	7	0.731	0.720	-0.017	0.012	0.024	-0.038	0.022	-0.007
9	8	0.736	0.736	-0.020	-0.006	0.000	-0.002	0.011	-0.062
9	9	0.756	0.762	0.016	0.013	0.003	-0.008	-0.076	0.009
10	0	0.640		0.015		-0.006		0.001	
10	1	0.713	0.591	0.004	0.017	-0.016	-0.041	0.003	-0.006
10	2	0.647	0.645	0.040	0.021	0.039	0.013	-0.003	-0.013
10	3	0.654	0.658	0.019	0.028	-0.122	-0.006	0.004	0.006
10	4	0.711	0.689	0.014	-0.009	-0.040	-0.035	0.001	-0.016
10	5	0.700	0.729	-0.012	0.001	0.017	-0.007	-0.001	0.002
10	6	0.733	0.712	-0.010	0.010	-0.003	0.037	0.008	0.003
10	7	0.742	0.733	-0.006	-0.006	0.021	-0.020	-0.004	0.005
10	8	0.724	0.726	-0.006	0.007	0.002	-0.028	0.033	-0.019
10	9	0.741	0.736	-0.036	0.001	0.014	0.022	0.006	-0.034
10	10	0.761	0.759	-0.020	0.008	-0.007	0.010	0.011	-0.007
11	0	0.645		0.002		-0.029		0.006	
11	1	0.711	0.582	-0.034	0.017	-0.023	-0.034	0.008	-0.005
11	2	0.653	0.650	0.001	0.009	0.029	-0.057	-0.001	0.004
11	3	0.647	0.651	0.009	0.006	-0.070	-0.002	0.008	-0.001
11	4	0.697	0.668	0.003	0.001	-0.018	-0.028	-0.005	-0.002
11	5	0.703	0.724	0.001	-0.009	0.052	-0.010	-0.001	-0.005
11	6	0.724	0.711	-0.010	-0.007	0.002	0.000	0.008	-0.004
11	7	0.728	0.728	0.011	0.004	0.008	0.035	-0.009	0.007
11	8	0.735	0.740	-0.008	0.002	-0.016	-0.013	0.001	0.030
11	9	0.720	0.729	0.003	-0.000	0.014	-0.006	0.011	-0.019
11	10	0.747	0.734	-0.020	-0.006	-0.008	-0.001	-0.057	-0.015
11	11	0.757	0.765	0.003	-0.005	-0.020	0.011	-0.021	-0.117
12	0	0.650		0.007		-0.017		-0.001	
12	1	0.706	0.577	-0.014	-0.000	-0.025	0.036	-0.002	0.003
12	2	0.658	0.649	0.005	0.003	-0.016	-0.019	0.002	0.008
12	3	0.648	0.648	0.004	0.009	0.026	-0.020	-0.004	-0.006
12	4	0.679	0.657	0.011	-0.002	0.032	-0.038	-0.000	0.015
12	5	0.691	0.716	0.003	-0.021	0.046	-0.056	0.001	-0.007
12	6	0.727	0.711	0.003	-0.011	-0.003	-0.031	-0.007	-0.001
12	7	0.721	0.719	0.014	0.004	0.024	-0.006	0.004	-0.009
12	8	0.732	0.731	0.021	-0.018	-0.006	0.015	-0.013	0.009
12	9	0.734	0.739	-0.010	0.002	0.001	0.008	0.005	0.019
12	10	0.731	0.719	-0.001	-0.005	0.002	0.016	-0.011	0.004
12	11	0.737	0.750	-0.004	-0.008	-0.009	-0.006	-0.026	-0.032
12	12	0.758	0.766	0.019	0.019	-0.002	-0.008	0.004	0.766

REFERENCES

Bergman, S., and M. Schiffer, Kernel Functions and Elliptic Differential Equations in Mathematical Physics, Academic Press, New York, 1953.

Bitsadze, A.V., Boundary Value Problems for Second Order Elliptic Equations, North-Holland Publishing Co., Amsterdam, 1968.

Bodewig, E., Matrix Calculus, 2nd Ed., North Holland Publishing Co., Amsterdam, 1959.

Bomford, G., Geodesy, 2nd Edition, Oxford University Press, London, 1962.

Burington, R.S., Handbook of Mathematical Tables and Formulas, 3rd Edition, Handbook Publishers, Sandusky, Ohio, 1957.

Cheney, E.W., Introduction to Approximation Theory, McGraw-Hill Book Co., New York, 1966.

Chu, S.C., and J.B. Diaz, Remarks on a generalization of Banach's principle of contraction mappings, J. Math. Anal. Appl.,11, 440-446, 1965.

Collatz, L., The Numerical Treatment of Differential Equations, 3rd Ed., Springer-Verlag, Berlin, 1960,

Cooke, R.G., Infinite Matrices and Sequence Spaces, Macmillan and Co., London, 1950.

Courant, R., and D. Hilbert, Methods of Mathematical Physics, 2 Vol., Interscience Publishers, New York, 1953-62.

Frey, E.J., J.V. Harrington, and W.S. von Arx, A study of satellite altimetry for geophysical and oceanographic measurement, in Proc. XVIth International Astronautical Congress, Athens, 1965, Meteorological and Communication Satellites, Gauthier-Villars, Paris, 53-72, 1966.

Gantmacher, F.R., Applications of the Theory of Matrices, Interscience Publishers, New York, 1959.

Gaposchkin, E.M., Orbit determination, in, C.A. Lundquist, and G. Veis, Geodetic parameters for a 1966 Smithsonian Institution standard earth, Smithsonian Astrophys. Obs. Spec. Rept. 200, Vol. 1, 77-184, 1966.

Gaposchkin, E.M., Improved values for the tesseral harmonics of the geopotential and station coordinates, presented at the 12th meeting of COSPAR, Prague, 1969.

Garabedian, P.R., Partial Differential Equations, John Wiley and Sons, New York, 1964.

Gedeon, G.S., Tesseral resonance effects on satellite orbits, Celestial Mechanics, 1, 167-189

Godbey, T.W., Oceanographic satellite radar altimeter and wind sea sensor, in, G.C. Ewing, Ed., Oceanography from Space, Woods Hole Oceanographic Inst. Rept. 65-10, 21-26, 1965.

Gradshteyn, I.S., and Ryzhik, I.M., Table of Integrals, Series, and Products, Academic Press, New York, 1965.

Greene, R.H., Maneuverable geodetic satellites in resonant orbits, J. Spacecraft, 5, 497-502, 1968.

Greenwood, J.A., A. Nathan, G. Neumann, W.J. Pierson, F.C. Jackson, and T.E. Pease, Radar altimetry from a spacecraft and its potential applications to geodesy and oceanography, Geophys. Sci. Lab. Rept. TR-67-3, New York University, 1967.

Hadamard, J., Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press, New Haven, Conn. 1923.

Halmos, P.R., Introduction to Hilbert Space and the Theory of Spectral Multiplicity, Chelsea Publishing Co., New York, 1951.

Hardy, G.H., J. E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, London, 1934.

Heiskanen, W.A., and H. Moritz, Physical Geodesy, W. H. Freeman and Co., San Francisco, 1967.

Hildebrand, F.B., Methods of Applied Mathematics, Prentice-Hall, Englewood Cliffs, N.J., 1952.

Hobson, E.W., The Theory of Spherical and Ellipsoidal Harmonics, Chelsea Publishing Co., New York, 1955.

Householder, A.S., The Theory of Matrices in Numerical Analysis, Blaisdell Publishing Co., New York, 1964.

Kaula, W.M., Statistical and harmonic analysis of gravity, J. Geophys. Res., 64, 2401-2421, 1959.

Kaula, W.M., A geoid and world geodetic system based on a combination of gravimetric, astro-geodetic, and satellite data, NASA Tech. Note D-702, 1961.

Kaula, W.M., Theory of Satellite Geodesy, Blaisdell Publishing Co., Waltham, Mass., 1966a.

Kaula, W.M., Global harmonic and statistical analysis of gravimetry, in, H. Orlin, Ed., Gravity Anomalies: Unsurveyed Areas, American Geophysical Union, Washington, 58-67, 1966b.

Kaula, W.M., Tests and combination of satellite determinations of the gravity field with gravimetry, J. Geophys. Res. 71, 5303-5314, 1966c.

Kaula, W.M., Chairman, Report of a study at Williamstown, Mass., to the National Aeronautics and Space Administration, The Terrestrial Environment, Solid-Earth and Ocean Physics, Application of Space and Astronomic Techniques, NASA-Electronics Research Center and M.I.T.-Meas. Syst. Lab., Cambridge, Mass., 1969.

Kellogg, O.D., Foundations of Potential Theory, Dover Publications New York, 1953.

Koch, K., Successive approximation of solutions of Molodensky's basic integral equation, Dept. Geodetic Sci. Rept. 85, Ohio State University, 1967.

Köhnlein, W., The earth's gravitational field as derived from a combination of satellite data with gravity anomalies, in, C.A. Lundquist, Ed., Geodetic satellite results during 1967, Smithsonian Astrophys. Obs. Spec. Rept. 264, 1967.

Kozai, Y., Revised zonal harmonics in the geopotential, Smithsonian Astrophys. Obs. Spec. Rept. 295, 1969.

Krarup, T., A Contribution to the Mathematical Foundation of Physical Geodesy, Geodaetisk Institute, Meddelelse No. 44, Copenhagen, 1969.

Lavrentiev, M.M., Some Improperly Posed Problems of Mathematical Physics, Springer-Verlag, Berlin, 1967.

Lee, W.H.K., and W. M. Kaula, A spherical harmonic analysis of the earth's topography, J. Geophys. Res., 72, 753-758, 1967.

Lundquist, C.,A., Satellite altimetry and orbit determination, Smithsonian Astrophys. Obs. Spec. Rept. 248, 1967.

Lundquist, C.A., G. E. O. Giacaglia, K. Hebb, and S. G. Mair, Possible geopotential improvement from satellite altimetry, Smithsonian Astrophys. Obs. Spec. Rept. 294, 1969.

MacMillan, W.D., Theoretical Mechanics, The Theory of the Potential, reprinted, Dover Publications, New York, 1958.

Madden, S.J., The geoid in spheroidal coordinates (Abstract), in Electronics Research Center National Aeronautics and Space Administration, Guidance Theory and Trajectory Analysis Seminar Abstracts, Cambridge, Mass. 17, 1968.

Mikhlin, S.G., Multidimensional Singular Integrals and Integral Equations, Pergamon Press, Oxford, 1965.

Molodenskii, M.S., V. F. Eremeev, and M.I. Yurkina, Methods for Study of the External Gravitational Field and Figure of the Earth, Israel Program for Scientific Translations, Jerusalem 1962.

Moritz, H., Green's functions in physical geodesy and the computation of the external gravity field and geodetic boundary value problem, Inst. Geod. Phot. Cart. Rept. 49, Ohio State University, 1965.

Mueller, I.I., Introduction to Satellite Geodesy, F. Ungar Publishing Co., New York, 1964.

Munk, W. H., and G. J. F. MacDonald, The Rotation of the Earth, Cambridge University Press, New York, 1960.

Ostrowski, A.M., Solution of Equations and Systems of Equations, Academic Press, New York, 1960.

Potter, J.E., and E. J. Frey, Rotation invariant probability distributions on the surface of a sphere, with applications to geodesy, Experimental Astronomy Laboratory Rept. RE-27, Massachusetts Institute of Technology, 1967.

Prasad, G., A Treatise on Spherical Harmonics and the Functions of Bessel and Lamé, 2 Vol., Benares Math. Soc., Benares, 1930-32.

Price, C.F., Signal processing in a satellite radar altimeter, Experimental Astronomy Lab. Rept. RE-48, Massachusetts Institute of Technology, 1968.

Ralston, A., A First Course in Numerical Analysis, McGraw-Hill Book Co., New York, 1965.

Rapp, R. H., Gravitational potential of the earth determined from a combination of satellite, observed, and model anomalies, J. Geophys. Res., 73, 6555-6562, 1968.

Raytheon Company, Space Geodesy Altimetry Study, Final Report, October 1968, NASA Contract NASW-1709, Space and Information Systems Division, Sudbury, Mass., 1968. (Also NASA Rept. CR 1298).

Shanks, D., Non-linear transformations of divergent and slowly convergent sequences, J. Math. Phys., 34, 1-42, 1955

Shilov, G. E., Generalized Functions and Partial Differential Equations, Gordon and Breach, New York, 1968.

Sneddon, I.N., Mixed Boundary Value Problems in Potential Theory, Interscience Publishers, New York, 1966.

Stokes, G.G., On the variation of gravity on the surface of the earth, Trans. Cambridge, Phil. Soc., 8, 672-695, 1849.

Taylor, A.E., Introduction to Functional Analysis, John Wiley and Sons, New York, 1958.

Todd, J., Ed., Survey of Numerical Analysis, McGraw-Hill Book Co., New York, 1962.

Tranter, C.J., Integral Transforms in Mathematical Physics, Methuen and Co., London, 1966.

Uotila, U.A., Harmonic analysis of World-wide gravity material, Ann. Acad. Scient. Fennicae, A. III. 67, Helsinki, 1962.

von Arx, W.S., Level surface profiles across the Puerto Rico Trench, Science, 154, 1651-1654, 1966.

Vulikh, B.Z., Introduction to Functional Analysis for Scientists and Technologists, Addison-Wesley Publishing Co., Reading Mass., 1963.

Wagner, C.A., Determination of low-order resonant gravity harmonics from the drift of two Russian 12-hour satellites, J. Geophys. Res., 73, 4661-4674, 1968.

Wilkinson, J.H., The Algebraic Eigenvalue Problem, Oxford University Press, London, 1965.

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Ronald Ging-wei Eng Young was born on 26 August 1941 in New York, New York. He was educated in the public schools of Newark, New Jersey, graduating from West Side High School in June, 1959. He received his undergraduate training with the Class of 1963 at the Massachusetts Institute of Technology on an M.I.T. Alumni Fund National Scholarship. For the industrial practice of the Cooperative Course in aeronautics and astronautics he was associated with the propulsion staff of the Martin Company, Baltimore, Maryland. He held a summer position, as an undergraduate, with the aerodynamics staff of Sikorsky Aircraft, Stratford, Connecticut. Participating also in the Honors Course of the Department of Aeronautics and Astronautics, he received, simultaneously, the S.B. and S.M. degrees in June, 1965. The results of his Masters thesis, involving the wind tunnel testing of simulated hailstones, were published in the Journal of Atmospheric Sciences. As a graduate research assistant in the Department of Aeronautics and Astronautics, he has been associated with the Aeroelastics and Structures Research Laboratory, the Instrumentation Laboratory, and the Measurement Systems Laboratory, previously known as the Experimental Astronomy Laboratory; his experience has included analysing elastic structural dynamic response, studying infrared thermistor detectors, and developing an interactive computer program for interplanetary mission analysis that was used to explore opportunities for double planet flybys (swingbys) of Mars and Venus. He also served briefly as a consultant in interplanetary mission analysis to the Raytheon Company.

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