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A DISCUSSION OF HILL'S METHOD OF SECULAR PERTURBATIONS AND ITS APPLICATION TO THE DETERMINATION OF THE ZERO-RANK EFFECTS IN NON-SINGULAR VECTORIAL ELEMENTS OF A PLANETARY MOTION

by Peter Musen

Goddard Space Flight Center
Greenbelt, Md. 20771

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16. Abstract Perturbations of zero-rank can cause large oscillations in both the shape and position of an orbit over a long time interval. This report presents formulas for the numerical integration of the zero-rank effects, using a modified Hill's theory and suitable vectorial elements. Scaler elements of our theory are the two components of Hamilton's vector in a moving ideal reference frame, and the three components of Gibbs' rotation vector in an inertial system. The integration step can be several hundred years for a planet or comet, and a few days for a near-Earth satellite, We re-discuss Hill's method, and apply vectorial analysis in a pseudo-euclidian space M_3 to obtain a symmetrical computational scheme in terms of traces of dyadics in M_3. The method is inapplicable for two orbits very close together. The numerical difficulty appears as a small divisor in Hill's method; as a slow convergence of a hypergeometric series in Halphen's method. Thus, in Hill's method the difficulty can be watched more directly than in Halphen's method. Numerical averaging can treat a large range of orbital eccentricities and inclinations; it can treat both the free and forced "Secular" oscillations, and together with their mutual cross-effects. At present, no analytical theory can do this fully.

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BASIC NOTATION

m = the mass of the disturbed body,

M = the mass of the Sun,

f = the gravitational constant,

$\mu = f(M + m)$,

$\vec{r}$ = the heliocentric position vector of the disturbed body,

$r = |\vec{r}|$,

$\vec{r}^{\circ}$ = the unit vector along $\vec{r}$,

$\vec{n}^{\circ}$ = the unit vector normal to $\vec{r}$ and lying in the orbital plane of the disturbed body,

a = the semi-major axis of the orbit of the disturbed body,

e = the eccentricity of the orbit of the disturbed body,

g = the mean anomaly of the disturbed body,

ϵ = the eccentric anomaly of the disturbed body,

$p = a(1 - e^2)$,

$\vec{P}_1$ = the unit vector directed from the Sun toward the perihelion of the disturbed body,

$\vec{P}_2$ = the unit vector normal to $\vec{P}_1$ and lying in the orbital plane of the disturbed body,

$\vec{s} = \frac{e}{\sqrt{1 - e^2}} \vec{P}_1$,

λ = the true orbital longitude of the disturbed body, reckoned from the departure point of the ideal system of coordinates,

χ = the true orbital longitude of the perihelion of the disturbed body in the ideal system of coordinates reckoned from the departure point,

σ = the angular distance of the ascending node from the departure point,

$\vec{R}_1, \vec{R}_2, \vec{R}_3$ = the unit vectors along the axes of the ideal system of coordinates. $\vec{R}_1$ and $\vec{R}_2$ are in the osculating orbital plane of the disturbed body, $\vec{R}_3$ is normal to this plane. The intersection of $\vec{R}_1$ with the celestial sphere is the departure point.

$\vec{R}_3 = \vec{P}_1 \times \vec{P}_2$,

$\vec{S}_1, \vec{S}_2, \vec{S}_3$ = the initial values of $\vec{R}_1, \vec{R}_2, \vec{R}_3$, respectively.

$\vec{q}$ = the Gibbs' vector. This vector defines the rotation of the orbital plane of the disturbed body from its initial position to the position at the given time t,

m' = the mass of the disturbing body,

$\vec{r}'$ = the heliocentric position vector of the disturbing body,

a' = the semi-major axis of the orbit of the disturbing body,

e' = the eccentricity of the orbit of the disturbing body,

g' = the mean anomaly of the disturbing body,

ϵ' = the eccentric anomaly of the disturbing body,

$\vec{P}_1'$ = the unit vector directed from the Sun toward the perihelion of the disturbing body,

$\vec{P}_2'$ = the unit vector normal to $\vec{P}_1'$ and lying in the orbital plane of the disturbing body,

$\vec{A}_1' = a' \vec{P}_1'$,

$\vec{A}_2' = a' \sqrt{1 - e'^2}\, \vec{P}_2'$,

$\Delta = |\vec{r}' - \vec{r}|$.

A DISCUSSION OF HILL'S METHOD OF SECULAR PERTURBATIONS AND ITS APPLICATION TO THE DETERMINATION OF THE ZERO-RANK EFFECTS IN NON-SINGULAR VECTORIAL ELEMENTS OF A PLANETARY MOTION

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Peter Musen
Goddard Space Flight Center

INTRODUCTION

In the motion of planets and comets the purely "secular" part of the disturbing function produces perturbations in the elements with periods of many thousands of years. Poincaré (1905) classified them as of "zero-rank." They constitute a most essential part of those perturbations which regulate behavior of the orbit over an interval of, say, some millions of years. Knowledge of these perturbations of zero-rank is also valuable in cosmogony and paleoclimatology. Hirayama's discovery (1923) of families of minor planets represents one of the most beautiful cosmogonical applications of the theory of zero-rank secular perturbations in its linearized form. The paleo-climatological significance of the zero-rank effects in the motion of the Earth is explained in the work of Milankovitch (1948).

In the case of artificial satellites the time scale becomes contracted, and the periods of the perturbations become only a few years instead of thousands. Shute (1964) noticed large oscillations in the orbital eccentricity and inclination of satellites launched deep into cislunar space. Thus, knowledge of the zero-rank perturbations facilitates planning the launchings of satellites into elongated ellipses in cislunar space with lifetime prolonged or shortened as needed.

In earlier times, beginning with Lagrange, information about the secular behavior of the orbital elements was obtained on the basis of linearized differential equations and under the assumption that the orbital eccentricities and the inclinations of both the disturbed and disturbing bodies remain small. Since then many problems have arisen, particularly in connection with artifical satellites, for which this basic supposition is not valid and which require a more accurate treatment. The original approach is now of mainly historical interest: it permitted us to understand the basic features of the secular planetary perturbations of minor and major planets.

The Lagrangian theory cannot fully solve the problem of the existence of the mean motion of the node and of the perihelion. Important theoretical progress in solving this problem has been achieved by Bohl (1909, 1912), Jessen (1935), Weyl (1938, 1939) and Tornehave and Jessen (1945).

Skripnichenko (1968) applied their ideas to determine the mean motions of the ascending nodes of Venus, Earth, and Mars, and the mean motions of the perihelia of Venus, Earth, and Uranus.

Hagihara (1928) and Kozai (1954) both developed analytical theories of the secular perturbations of higher orders and higher degrees. They used Baker's (1916) method of integrating the differential equations, and the classical expansion of the secular disturbing function in powers of the eccentricities and inclinations (which thus are assumed to be small). The second work by Kozai (1962) treats the secular perturbations of large eccentricities and inclinations under the assumption that the motion of the disturbed body is circular.

With the advent of electronic computers these restrictions became unnecessary. By the use of step-by-step numerical integration, it became possible to penetrate the problem more deeply and to obtain a series of interesting disclosures about the secular behavior of the orbits of asteroids and of satellites in cislunar space. Musen (1963) and Hamid* suggested the application of the Gauss (1818) method based on averaging the components of the disturbing force over the orbits of the disturbed and disturbing body. We have two basic modifications of Gauss' method. The first was developed by Hill (1882) and re-discussed by Calladreau (1885). The second method was developed by Halphen (1888) and has been re-discussed by Goriachev (1937) and Musen (1963).

Hamid* applied Hill's method to compute the secular effects in the motion of comets. Halphen's method was used by Shute (1964) to compute the secular lunar effects in the motions of artificial satellites in elongated orbits in cislunar space, and by Smith (1964) for the motions of Enke's comet and minor planets. Recently Musen and his associates have applied it to the investigation of Hirayama families of minor planets. The results will be published in subsequent papers. The author (1963) undertook the modernization of Halphen's method, presenting it in terms of dyadics and vectors. In the present article we discuss Hill's method in a similar manner. It is of interest to note that a careful re-examination of Hill's original scaler development reveals its intimate connection with the vector algebra of Minkowski's pseudo-euclidean three-dimensional space M_3. To clarify the geometrical aspects of Hill's theory we use in the present exposition the algebra of dyadics and vectors in M_3, and, when necessary, the algebra of dyadics and vectors in euclidean space E_3. We propose a new symmetrical computational scheme in terms of traces of dyadics in M_3. For the computation of elliptic integrals which appear in our exposition we propose Cody's (1965) highly accurate approximations by Chebyshev polynomials.

We have previously experienced difficulties in computing the secular effects for nearly circular orbits when the eccentricity became negative and the Laplacian vector abruptly changed direction. A similar situation arose with small orbital inclinations. We develop in this article the differential equations for the secular perturbations in vectorial elements which are nonsingular for small e and i. These elements are $\vec{s} = e/\sqrt{1-e^2}\,\vec{P}_1$ where $\vec{P}_1$ is the unit vector directed toward the perihelion, in place of e and π, and Gibbs'vector $\vec{q}$ (Gibbs, 1901) which defines the rotation of the orbital plane from its initial position to the position at a given time and which replaces the standard elements i, Ω, and σ. The introduction of Gibbs' vector (Musen, 1961) leads

*Private Communication, 1963.

to differential equations for the perturbations of the position of the orbital plane in which the problem of small divisors in form of the sine of the inclination is removed. The semi-major axis is not affected by the perturbations of zero rank; thus the computation of da/dt can serve as a check on the applicability of the method.

We consider the case of only one disturbing planet; generalization to the case of a planetary system is not too difficult. The presented theory is applicable not only to the planetary case but also, for a more limited interval of time, to cometary orbits. It is applicable also to the orbits of space probes not approaching the Moon too closely.

SOME BASIC RULES OF THE VECTOR ALGEBRA IN THE PSEUDO-EUCLIDEAN SPACE M_3

For the sake of completeness of the exposition, we state briefly and without proof those rules of the vector algebra in M_3 which differ from the analogous rules in E_3. We shall require these rules in our exposition of Hill's theory.

Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the basic unit vectors in M_3. The fundamental multiplication table peculiar to M_3 (in fact, the conditions of pseudo-orthogonality) is

$$\vec{e}_1 \cdot \vec{e}_1 = +1, \quad \vec{e}_2 \cdot \vec{e}_2 = +1, \quad \vec{e}_3 \cdot \vec{e}_3 = -1, \quad \vec{e}_i \cdot \vec{e}_j = 0 \; (i \neq j) \tag{1}$$

$$\vec{e}_1 \times \vec{e}_2 = +\vec{e}_3, \quad \vec{e}_2 \times \vec{e}_3 = -\vec{e}_1, \quad \vec{e}_3 \times \vec{e}_1 = -\vec{e}_2, \quad \vec{e}_i \times \vec{e}_i = 0. \tag{2}$$

The dot product of two vectors

$$\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3,$$

$$\vec{b} = b_1 \vec{e}_1 + b_2 \vec{e}_2 + b_3 \vec{e}_3,$$

in M_3, in agreement with (1), is given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 - a_3 b_3; \tag{3}$$

in particular

$$\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 - a_3^2 .$$

The cross product in M_3, in accordance with (2), can be expressed as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & -a_3 \\ b_1 & b_2 & -b_3 \end{vmatrix} .$$

The expansion formula for the double cross product in M_3 is

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b})\,\vec{c} - (\vec{a} \cdot \vec{c})\,\vec{b} \tag{4}$$

with the sign of the right side opposite to the sign in the analogous formula in E_3.

A linear vectorial transformation from the basic set $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ to a new basic set $(\vec{e}_1^{\,*}, \vec{e}_2^{\,*}, \vec{e}_3^{\,*})$ which leaves (3) invariant is called a pseudo-orthogonal transformation. The idemfactor (the unit matrix) in M_3 has the form

$$E = \vec{e}_1\,\vec{e}_1 + \vec{e}_2\,\vec{e}_2 - \vec{e}_3\,\vec{e}_3,$$

where the products of vectors are dyadic. From two representations of the idemfactor

$$\vec{e}_1\,\vec{e}_1 + \vec{e}_2\,\vec{e}_2 - \vec{e}_3\,\vec{e}_3 = \vec{e}_1^{\,*}\,\vec{e}_1^{\,*} + \vec{e}_2^{\,*}\,\vec{e}_2^{\,*} - \vec{e}_3^{\,*}\,\vec{e}_3^{\,*},$$

and making use of (1) and of the analogous conditions for $\vec{e}_1^{\,*}, \vec{e}_2^{\,*}, \vec{e}_3^{\,*}$, we obtain the decompositions

$$\vec{e}_i^{\,*} = \alpha_{i1}\,\vec{e}_1 + \alpha_{i2}\,\vec{e}_2 - \alpha_{i3}\,\vec{e}_3, \tag{5}$$

$$\vec{e}_i = \alpha_{1i}\,\vec{e}_1^{\,*} + \alpha_{2i}\,\vec{e}_2^{\,*} - \alpha_{3i}\,\vec{e}_3^{\,*}, \tag{6}$$

where $\alpha_{ij} = \vec{e}_i^{\,*} \cdot \vec{e}_j$. From (5) and (6) we have the formulas for the transformation of coordinates

$$\mathrm{col}\,(\xi_1^*, \xi_2^*, -\xi_3^*) = (\alpha_{ij}) \cdot \mathrm{col}\,(\xi_1, \xi_2, \xi_3), \tag{7}$$

$$\mathrm{col}\,(\xi_1, \xi_2, -\xi_3) = (\alpha_{ji}) \cdot \mathrm{col}\,(\xi_1^*, \xi_2^*, \xi_3^*), \tag{8}$$
$$(i, j = 1, 2, 3)$$

where i is the row index and j is the column index. The multiplication of matrices and column vectors in (7) and (8) is performed in the standard manner. The conditions of pseudo-orthogonality are

$$\begin{aligned} \alpha_{i1}\,\alpha_{j1} + \alpha_{i2}\,\alpha_{j2} - \alpha_{i3}\,\alpha_{j3} &= \delta_{ij}, \\ \alpha_{1i}\,\alpha_{1j} + \alpha_{2i}\,\alpha_{2j} - \alpha_{3i}\,\alpha_{3j} &= \delta_{ij}, \end{aligned} \tag{9}$$

where Kronecker deltas in M_3 are defined as

$$\delta_{ij} = \begin{cases} +1 & i = j = 1, 2 \\ 0 & i \neq j \\ -1 & i = j = 3 \,. \end{cases}$$

Hill, in his exposition, resorts to (9). We do not use these relations in the present article.

The normal form of the dyadic

$$\Phi = \vec{a}_1 \vec{b}_1 + \vec{a}_2 \vec{b}_2 + \vec{a}_3 \vec{b}_3 = \sum_{i,j} c_{ij} \vec{e}_i \vec{e}_j \quad (i, j = 1, 2, 3)$$

in M_3 space is

$$\Phi = -\lambda_1 \vec{e}_1^* \vec{e}_1^* - \lambda_2 \vec{e}_2^* \vec{e}_2^* + \lambda_3 \vec{e}_3^* \vec{e}_3^* ,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the characteristic equation. In agreement with Callandreau we write this equation in the form

$$\varphi(\lambda) = - \begin{vmatrix} c_{11} + \lambda & c_{12} & c_{13} \\ c_{21} & c_{22} + \lambda & c_{23} \\ c_{31} & c_{32} & c_{33} - \lambda \end{vmatrix} = 0, \tag{10}$$

which when expanded becomes

$$\varphi(\lambda) = \lambda^3 + f_1 \lambda^2 + f_2 \lambda + f_3 = 0,$$

where f_1, f_2, f_3 are the scaler invariants of Φ. In M_3 they have the form

$$f_1 = \vec{a}_1 \cdot \vec{b}_1 + \vec{a}_2 \cdot \vec{b}_2 + \vec{a}_3 \cdot \vec{b}_3 = c_{11} + c_{22} - c_{33},$$

$$f_2 = -(\vec{a}_1 \times \vec{a}_2) \cdot (\vec{b}_1 \times \vec{b}_2) - (\vec{a}_2 \times \vec{a}_3) \cdot (\vec{b}_2 \times \vec{b}_3) - (\vec{a}_3 \times \vec{a}_1) \cdot (\vec{b}_3 \times \vec{b}_1)$$

$$= + \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} - \begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix} - \begin{vmatrix} c_{33} & c_{31} \\ c_{13} & c_{11} \end{vmatrix},$$

$$f_3 = -(\vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3)(\vec{b}_1 \cdot \vec{b}_2 \times \vec{b}_3) = - \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}.$$

Cayley's identity

$$\Phi^3 - f_1 \Phi^2 + f_2 \Phi - f_3 E = 0$$

remains valid in M_3, as does the identity

$$(\Phi + \lambda E)^{-1} = \frac{E\lambda^2 - (\Phi - E f_1)\lambda + (\Phi^2 - f_1 \Phi + E f_2)}{\lambda^3 + f_1 \lambda^2 + f_2 \lambda + f_3}, \tag{11}$$

provided that λ is not a root of the characteristic equation of Φ.

COMPONENTS OF THE SECULAR DISTURBING FORCE

To achieve compactness and symmetry of formulas, we use notations for the vectorial elements of the disturbed and the disturbing body which differ from the standard notations. Substituting

$$\vec{r}' = \vec{A}_1' (\cos \epsilon' - e') + \vec{A}_2' \sin \epsilon' \tag{12}$$

into

$$\vec{\Delta} = \vec{r}' - \vec{r},$$

we obtain

$$\vec{\Delta} = \vec{A}_1' \cos \epsilon' + \vec{A}_2' \sin \epsilon' - \vec{\rho},$$

where $\vec{\rho} = \vec{r} + e' \vec{A}_1'$ is the position vector of the disturbed body relative to the center of the orbit of the disturbing body. We assume e' remains greater than zero. Decomposing $\vec{\rho}$ along the axes $\vec{P}_1'$, $\vec{P}_2'$, $\vec{P}_3'$, we obtain

$$\vec{\rho} = x_1 \vec{P}_1' + x_2 \vec{P}_2' + x_3 \vec{P}_3',$$

where

$$x_1 = \vec{r} \cdot \vec{P}_1' + e' a', \quad x_2 = \vec{r} \cdot \vec{P}_2', \quad x_3 = \vec{r} \cdot \vec{P}_3',$$

and (Hansen, 1957)

$$\Delta^2 = a'^2 (\gamma_0 + \gamma_2 \cos^2 \epsilon' - 2 \gamma_1 \cos \epsilon' - 2 \beta_0 \sin \epsilon'),$$

where

$$\gamma_2 = e'^2, \ \gamma_1 = \frac{x_1}{a'}, \ \gamma_0 = 1 - e'^2 + \frac{\rho^2}{a'^2}, \ \beta_0 = \frac{x_2 \sqrt{1 - e'^2}}{a'}. \tag{13}$$

In the actual computations and programming, the system of formulas

$$\gamma_2 = e'^2, \ \gamma_1 = e' + \alpha \frac{\vec{r}}{a} \cdot \vec{P}_1', \ \gamma_0 = 1 - 2e'^2 + \alpha^2 \frac{r^2}{a^2} + 2e'\gamma_1, \ \beta_0 = \alpha \frac{\vec{r}}{a} \cdot \vec{P}_2' \sqrt{1 - e'^2} \tag{14}$$

where

$$\alpha = \frac{a}{a'} ,$$

is preferable to (13). The expressions

$$\frac{r}{a} = 1 - e \cos \epsilon, \quad \frac{\vec{r}}{a} = \vec{P}_1 (\cos \epsilon - e) + \vec{P}_2 \sqrt{1 - e^2} \sin \epsilon$$

are to be substituted into (14).

In the ensuing exposition it will be convenient to express Δ^2 in terms of the position vector

$$\vec{\xi} = \xi_1 \vec{e}_1 + \xi_2 \vec{e}_2 + \xi_3 \vec{e}_3 \tag{15}$$

of the point

$$\xi_1 = \tau \cos \epsilon', \quad \xi_2 = \tau \sin \epsilon', \quad \xi_3 = \tau \tag{16}$$

lying on the surface

$$\vec{\xi} \cdot \vec{\xi} = \xi_1^2 + \xi_2^2 - \xi_3^2 = 0$$

in the space M_3 ; here τ is a factor to be discussed later.

Introducing the M_3 dyadic

$$\Phi = \gamma_2 \vec{e}_1 \vec{e}_1 + \gamma_1 (\vec{e}_1 \vec{e}_3 + \vec{e}_3 \vec{e}_1) + \beta_0 (\vec{e}_2 \vec{e}_3 + \vec{e}_3 \vec{e}_2) + \gamma_0 \vec{e}_3 \vec{e}_3 \tag{17}$$

and taking (1), (15), and (16) into account, we obtain

$$\tau^2 \left(\frac{\Delta}{a'}\right)^2 = \vec{\xi} \cdot \Phi \cdot \vec{\xi} . \tag{19}$$

By means of a pseudo-orthogonal transformation we can reduce (17) to the normal form

$$\Phi = G_1 \vec{e}_1^{\,*} \vec{e}_1^{\,*} - G_2 \vec{e}_2^{\,*} \vec{e}_2^{\,*} + G_3 \vec{e}_3^{\,*} \vec{e}_3^{\,*}, \tag{19}$$

where $-G_1$, $+G_2$ and $+G_3$ are the roots of the characteristic equation

$$\varphi(\lambda) = \begin{vmatrix} \lambda + \gamma_2 & 0 & -\gamma_1 \\ 0 & \lambda & -\beta_0 \\ \gamma_1 & \beta_0 & \lambda - \gamma_0 \end{vmatrix} = 0. \tag{20}$$

To help the reader compare the original expositions of Hill and Callandreau with the present exposition, we are using Hill's notations for the characteristic roots. In the expanded form we have

$$\varphi(\lambda) = [\lambda(\lambda - \gamma_0) + \beta_0^2](\lambda + \gamma_2) + \gamma_1^2 \lambda = 0, \tag{21}$$

or

$$\varphi(\lambda) = \lambda^3 + f_1 \lambda^2 + f_2 \lambda + f_3 = 0, \tag{22}$$

where

$$f_1 = \gamma_2 - \gamma_0, \quad f_2 = \gamma_1^2 + \beta_0^2 - \gamma_0 \gamma_2, \quad f_3 = \beta_0^2 \gamma_2$$

are the scalar invariants of Φ.

Hill has shown that all roots of (22) are real. We repeat his proof here, with slight modifications. From (21) we obtain

$$\varphi(-\gamma_2) = -\gamma_1^2 \gamma_2, \quad \varphi(0) = +\beta_0^2 \gamma_2, \quad \varphi(+\gamma_0) = \beta_0^2 (\gamma_0 + \gamma_2) + \gamma_1^2 \gamma_0,$$

and, taking (13) into account,

$$\varphi(1 - e'^2) = -\frac{x_3^2 (1 - e'^2)}{a'^2}.$$

Thus

$$\varphi(-\gamma_2) < 0, \quad \varphi(0) > 0, \quad \varphi(1 - e'^2) < 0, \quad \varphi(+\gamma_0) > 0.$$

Consequently, the characteristic equation has three real roots, $-G_1$, $+G_2$, $+G_3$, located in the intervals

$$(-\gamma_2, 0), \quad (0, 1 - e'^2) \text{ and } (1 - e'^2, \gamma_0).$$

The root $-G_1$ can be easily obtained by the method of iteration. From (22) we have

$$G_1 = \frac{f_3}{G_1^2 - f_1 G_1 + f_2}$$

and, because of the smallness of γ_2, we obtain G_1 in the case of a minor planet, or a satellite in cislunar space, after only a few iterations. After G_1 is computed, the remaining roots $+G_2$ and $+G_3$ can be determined from the reduced equation

$$\lambda^2 + (f_1 - G_1)\lambda + (G_1^2 - f_1 G_1 + f_2) = 0.$$

In fact, (22) appears in Hansen's work (1857) in connection with the numerical expansion of the disturbing function, where the use of the iteration procedure is also suggested.

In our exposition we shall need only the roots of (22), so we can dispense with the determination of the matrix of the pseudo-orthogonal transformation, as well as with the actual determination of the factor τ. We select τ such that the new coordinate ξ_3^* of the point (16) will be equal to unity. Then, taking $\vec{\xi}\cdot\vec{\xi} = 0$ into account, we can set

$$\xi_1^* = \cos T, \quad \xi_2^* = \sin T, \quad \xi_3^* = +1. \tag{23}$$

By substituting

$$\vec{\xi} = \vec{e}_1^* \cos T + \vec{e}_2^* \sin T + e_3^* \tag{24}$$

and (19) into (18) we obtain

$$\tau^2 \left(\frac{\Delta}{a'}\right)^2 = G_1 \cos^2 T - G_2 \sin^2 T + G_3.$$

Differentiating the identity

$$\vec{e}_1^* \cos T + \vec{e}_2^* \sin T + \vec{e}_3^* \equiv \tau (\vec{e}_1 \cos \epsilon' + \vec{e}_2 \sin \epsilon' + \vec{e}_3), \tag{24'}$$

and taking (2) into account, we deduce

$$\vec{\xi} \times \vec{e}_3^* \, dT = \frac{\vec{\xi}}{\tau} d\tau + \vec{\xi} \times \vec{e}_3 \, d\epsilon'$$

or

$$\vec{\xi} \times (\vec{\xi} \times \vec{e}_3^*) \, dT = \vec{\xi} \times (\vec{\xi} \times \vec{e}_3) \, d\epsilon',$$

and, after the application of (4),

$$dT = \tau \, d\epsilon'. \tag{25}$$

The disturbing force averaged over the orbit of the disturbing body is

$$\vec{F}_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\vec{r}' - \vec{r}}{\Delta^3} dg'.$$

Taking (12) into account we obtain

$$\vec{F}_0 \cdot \vec{u}^0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{a' (N_1 \cos \epsilon' + N_2 \sin \epsilon' + N_3)}{\Delta^3} dg', \tag{26}$$

where

$$N_1 = \vec{P}_1' \cdot \vec{u}^0, \quad N_2 = \vec{P}_2' \cdot \vec{u}^0 \sqrt{1 - e'^2}, \quad N_3 = -\vec{P}_1' \cdot \vec{u}^0 e' - \alpha \frac{\vec{r}}{a} \cdot \vec{u}^0,$$

for the component of $\vec{F}_0$ in the direction of a unit vector $\vec{u}^0$. Then, taking (18) and

$$\epsilon' - e' \sin \epsilon' = g', \quad \frac{r'}{a'} = 1 - e' \cos \epsilon'$$

into account we have, from (26)

$$a'^2 \vec{F}_0 \cdot \vec{u}^0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\vec{\xi} \cdot \Pi \cdot \vec{\xi}}{(\vec{\xi} \cdot \Phi \cdot \vec{\xi})^{3/2}} \tau \, d\epsilon', \tag{27}$$

where

$$\Pi = -(N_1 \vec{e}_1 + N_2 \vec{e}_2 - N_3 \vec{e}_3)(\vec{e}_3 + e' \vec{e}_1) \tag{28}$$

is an M_3-dyadic. The trace of Π is

$$p_1 = -(N_3 + e' N_1).$$

From (7), (8) and (23) we obtain

$$-1 = \tau(\alpha_{31} \cos \epsilon' + \alpha_{32} \sin \epsilon' + \alpha_{33}),$$

$$-\tau = \alpha_{13} \cos T + \alpha_{23} \sin T + \alpha_{33}.$$

To avoid a contradiction between these last two relations we must conclude that τ cannot become zero or infinity, but oscillates between two fixed limits which, with the proper choice of the "direction cosines", can be assumed to be positive. From (25) it is evident that T is a monotonically increasing function of ϵ' and that ξ is a periodic function of T with period 2π. In addition, it is clear from (24') that when ϵ' covers a full period T also covers a full period.

When (25) is taken into account, (27) becomes

$$a'^2 \vec{F}_0 \cdot \vec{u}^0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\vec{\xi} \cdot \Pi \cdot \vec{\xi}}{(\vec{\xi} \cdot \Phi \cdot \vec{\xi})^{3/2}} \, dT, \tag{29}$$

where Φ is taken in its normal form (19), and correspondingly Π is taken in the form:

$$\Pi = \Gamma_1 \vec{e}_1^{\,*} \vec{e}_1^{\,*} + \Gamma_2 \vec{e}_2^{\,*} \vec{e}_2^{\,*} + \Gamma_3 \vec{e}_3^{\,*} \vec{e}_3^{\,*} + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} A_{ij} \vec{e}_i^{\,*} \vec{e}_j^{\,*}, \tag{30}$$

with $\vec{\xi}$ in the form (24). But, since

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\sin T \cos T \, dT}{(G_1 \cos^2 T - G_2 \sin^2 T + G_3)^{3/2}} = 0,$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos T \, dT}{(G_1 \cos^2 T - G_2 \sin^2 T + G_3)^{3/2}} = 0,$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\sin T \, dT}{(G_1 \cos^2 T - G_2 \sin^2 T + G_3)^{3/2}} = 0,$$

all terms having coefficients A_{ij} disappear from (29) and we have simply

$$a'^2 \vec{F}_0 \cdot \vec{u}^0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\Gamma_1 \cos^2 T + \Gamma_2 \sin^2 T + \Gamma_3}{(G_1 \cos^2 T - G_2 \sin^2 T + G_3)^{3/2}} dT,$$

or, taking into account the symmetry of the integrand,

$$a'^2 \vec{F}_0 \cdot \vec{u}^0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{\Gamma_1 \cos^2 T + \Gamma_2 \sin^2 T + \Gamma_3}{(G_1 \cos^2 T - G_2 \sin^2 T + G_3)^{3/2}} dT. \tag{31}$$

Consequently, we have to retain only the purely quadratic portion

$$\Gamma_1 \vec{e}_1^{\,*} \vec{e}_1^{\,*} + \Gamma_2 \vec{e}_2^{\,*} \vec{e}_2^{\,*} + \Gamma_3 \vec{e}_3^{\,*} \vec{e}_3^{\,*}$$

in the transformed Π. From

$$\Phi + \lambda E = (G_1 + \lambda) \vec{e}_1^{\,*} \vec{e}_1^{\,*} - (G_2 - \lambda) \vec{e}_2^{\,*} \vec{e}_2^{\,*} + (G_3 - \lambda) \vec{e}_3^{\,*} \vec{e}_3^{\,*}$$

we obtain

$$(\Phi + \lambda E)^{-1} = \frac{\vec{e}_1^{\,*} \vec{e}_1^{\,*}}{G_1 + \lambda} - \frac{\vec{e}_2^{\,*} \vec{e}_2^{\,*}}{G_2 - \lambda} + \frac{\vec{e}_3^{\,*} \vec{e}_3^{\,*}}{G_3 - \lambda},$$

provided that λ is not a root of (22). From this last equation and from (30) we have

$$\Pi \cdot (\Phi + \lambda E)^{-1} = \frac{\Gamma_1}{G_1 + \lambda} \vec{e}_1^{\,*} \vec{e}_1^{\,*} - \frac{\Gamma_2}{G_2 - \lambda} \vec{e}_2^{\,*} \vec{e}_2^{\,*} - \frac{\Gamma_3}{G_3 - \lambda} \vec{e}_3^{\,*} \vec{e}_3^{\,*} + \cdots$$

and, forming the trace of the last expression,

$$\{\Pi \cdot (\Phi + \lambda E)^{-1}\}_1 = \frac{\Gamma_1}{G_1 + \lambda} - \frac{\Gamma_2}{G_2 - \lambda} + \frac{\Gamma_3}{G_3 - \lambda}. \tag{32}$$

We next obtain this trace using the unreduced forms of Φ (17) and Π (28) and the identity (11); and then, by comparing the two expressions for the trace, we determine $\Gamma_1, \Gamma_2, \Gamma_3$. From (17) we have

$$\Phi^2 = (\gamma_2^2 - \gamma_1^2) \vec{e}_1 \vec{e}_1 - \gamma_1 \beta_0 (\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_1) + \gamma_1 (\gamma_2 - \gamma_0)(\vec{e}_1 \vec{e}_3 + \vec{e}_3 \vec{e}_1)$$

$$- \beta_0^2 \vec{e}_2 \vec{e}_2 - \beta_0 \gamma_0 (\vec{e}_2 \vec{e}_3 + \vec{e}_3 \vec{e}_2) + (\beta_0^2 + \gamma_1^2 - \gamma_0^2) \vec{e}_3 \vec{e}_3;$$

and, from this last equation, and (17) and (28), we obtain

$$\Pi \cdot \Phi = (N_1 \vec{e}_1 + N_2 \vec{e}_2 - N_3 \vec{e}_3)(s_{11} \vec{e}_1 + s_{12} \vec{e}_2 + s_{13} \vec{e}_3), \tag{33}$$

where

$$s_{11} = \gamma_1 - e'\gamma_2,\ s_{12} = \beta_0,\ s_{13} = \gamma_0 - e'\gamma_1;$$

and

$$\Pi \cdot \Phi^2 = (N_1 \vec{e}_1 + N_2 \vec{e}_2 - N_3 \vec{e}_3)(s_{21} \vec{e}_1 + s_{22} \vec{e}_2 + s_{23} \vec{e}_3), \tag{34}$$

where

$$s_{21} = \gamma_1 (\gamma_2 - \gamma_0) - e'(\gamma_2^2 - \gamma_1^2),\ s_{22} = -\beta_0 \gamma_0 + e' \gamma_1 \beta_0,\ s_{23} = (\beta_0^2 + \gamma_1^2 - \gamma_0^2) - e' \gamma_1 (\gamma_2 - \gamma_0).$$

The traces s_1 and s_2 of (33) and (34) are

$$s_1 = (\Pi \cdot \Phi)_1 = N_1 s_{11} + N_2 s_{12} + N_3 s_{13},$$

$$s_2 = (\Pi \cdot \Phi^2)_1 = N_1 s_{21} + N_2 s_{22} + N_3 s_{23}.$$

Multiplying Π by (11) and taking the traces of the left and right sides we deduce

$$\{\Pi \cdot (\Phi + \lambda E)^{-1}\}_1 = \frac{p_1 \lambda^2 - (s_1 - f_1 p_1)\lambda + (s_2 - s_1 f_1 + p_1 f_2)}{\lambda^3 + f_1 \lambda^2 + f_2 \lambda + f_3}. \tag{35}$$

Then by equating the right sides of (32) and (35) and making use of l'Hopital's rule we obtain:

$$\Gamma_1 = + \frac{p_1 G_1^2 + (s_1 - f_1 p_1) G_1 + (s_2 - s_1 f_1 + p_1 f_2)}{3 G_1^2 - 2 f_1 G_1 + f_2},$$

$$\Gamma_2 = + \frac{p_1 G_2^2 - (s_1 - f_1 p_1) G_2 + (s_2 - s_1 f_1 + p_1 f_2)}{3 G_2^2 + 2 f_1 G_2 + f_2},$$

$$\Gamma_3 = - \frac{p_1 G_3^2 - (s_1 - f_1 p_1) G_3 + (s_2 - s_1 f_1 + p_1 f_2)}{3 G_3^2 + 2 f_1 G_3 + f_2}.$$

With $\Gamma_1, \Gamma_2, \Gamma_3$ now determined, we can rewrite (31) in the form

$$a'^2 \vec{F}_0 \cdot \vec{u}^0 (G_1 + G_3)^{3/2} = \frac{2}{\pi} \int_0^{\pi/2} \frac{\Gamma_1 \cos^2 T + \Gamma_2 \sin^2 T + \Gamma_3}{(1 - k^2 \sin^2 T)^{3/2}} dT, \tag{36}$$

where

$$k^2 = \frac{G_1 + G_2}{G_1 + G_3}.$$

Making use of the formulas

$$\int_0^{\pi/2} \frac{dT}{(1 - k^2 \sin^2 T)^{3/2}} = \frac{E\ (k)}{k'^2},$$

$$\int_0^{\pi/2} \frac{\sin^2 T\, dT}{(1 - k^2 \sin^2 T)^{3/2}} = \frac{E\ (k) - k'^2 K\ (k)}{k^2\, k'^2},$$

$$\int_0^{\pi/2} \frac{\cos^2 T\, dT}{(1 - k^2 \sin^2 T)^{3/2}} = \frac{K(k) - E(k)}{k^2},$$

where $k'^2 = 1 - k^2$ and $K(k)$ and $E(k)$ are the normal elliptic integrals of the first and second kind respectively, we can put (36) into a form convenient for the numerical computations:

$$\vec{F}_0 \cdot \vec{u}^0 = \frac{2}{\pi\, a'^2\, (G_1 + G_3)^{3/2}} \left[(\Gamma_2 + \Gamma_3) \frac{E}{k'^2} + (\Gamma_1 - \Gamma_2) \frac{K - E}{k^2} \right].$$

For the numerical evaluation of elliptic integrals, we recommend their highly accurate approximations of Chebyshev polynomials as obtained by Cody (1965).

For the special cases of (26),

$$K_1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos \epsilon' - e'}{\Delta^3}\, dg',$$

$$K_2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin \epsilon'}{\Delta^3}\, dg',$$

$$K_3 = \frac{1}{2\pi} \int_0^{2\pi} \frac{dg'}{\Delta^3},$$

we obtain

$$K_1 = \frac{2}{\pi\, a'^2\, (G_1 + G_3)^{3/2}} \left[(\Gamma_{12} + \Gamma_{13}) \frac{E}{k'^2} + (\Gamma_{11} - \Gamma_{12}) \frac{K - E}{k^2} \right],$$

where

$$\Gamma_{11} = + \frac{(s_{21} - e' s_{23}) + (s_{11} - e' s_{13})(G_1 - f_1)}{3 G_1^2 - 2 f_1 G_1 + f_2},$$

$$\Gamma_{12} = + \frac{(s_{21} - e' s_{23}) - (s_{11} - e' s_{13})(G_2 + f_1)}{3 G_2^2 + 2 f_1 G_2 + f_2},$$

$$\Gamma_{13} = - \frac{(s_{21} - e' s_{23}) - (s_{11} - e' s_{13})(G_3 + f_1)}{3 G_3^2 + 2 f_1 G_3 + f_2};$$

and

$$K_2 = \frac{2}{\pi a'^2 (G_1 + G_3)^{3/2}} \left[(\Gamma_{22} + \Gamma_{23}) \frac{E}{k'^2} + (\Gamma_{21} - \Gamma_{22}) \frac{K - E}{k^2} \right],$$

where

$$\Gamma_{21} = + \frac{s_{22} + s_{12}(G_1 - f_1)}{3 G_1^2 - 2 f_1 G_1 + f_2},$$

$$\Gamma_{22} = + \frac{s_{22} - s_{12}(G_2 + f_1)}{3 G_2^2 + 2 f_1 G_2 + f_2},$$

$$\Gamma_{23} = - \frac{s_{22} - s_{12}(G_3 + f_1)}{3 G_3^2 + 2 f_1 G_3 + f_2};$$

and

$$K_3 = \frac{2}{\pi a'^2 (G_1 + G_3)^{3/2}} \left[(\Gamma_{32} + \Gamma_{33}) \frac{E}{k'^2} + (\Gamma_{31} - \Gamma_{32}) \frac{K - E}{k^2} \right],$$

where

$$\Gamma_{31} = - \frac{G_1^2 - (s_{13} + f_1) G_1 - (s_{23} - s_{13} f_1 - f_2)}{3 G_1^2 - 2 f_1 G_1 + f_2},$$

$$\Gamma_{32} = - \frac{G_2^2 + (s_{13} + f_1) G_2 - (s_{23} - s_{13} f_1 - f_2)}{3 G_2^2 + 2 f_1 G_2 + f_2} ,$$

$$\Gamma_{33} = + \frac{G_3^2 + (s_{13} + f_1) G_3 - (s_{23} - s_{13} f_1 - f_2)}{3 G_3^2 + 2 f_1 G_3 + f_2} .$$

EQUATIONS FOR SECULAR VARIATION OF ELEMENTS

To avoid the difficulty associated with division by a small eccentricity, when the elements π and e are used separately, we suggest instead the use of the vectorial element

$$\vec{s} = \frac{e}{\sqrt{1 - e^2}} \vec{P}_1 ,$$

which is intimately related to Hamilton's vector and to the element $(h/h_0)\, e\vec{P}_1$ of Hansen's lunar theory. The projections of $\vec{s}$ on the axes of the ideal reference frame (a frame rigidly connected with the osculating orbit plane of the disturbed body) are

$$s_1 = \frac{e}{\sqrt{1 - e^2}} \cos \chi , \quad s_2 = \frac{e}{\sqrt{1 - e^2}} \sin \chi ,$$

where χ is the true orbital longitude of perihelion of the disturbed body reckoned from the departure point of the ideal system.

The differential equation of motion of the disturbed body, as referred to the ideal system, can be written in the form

$$\frac{d^2 \vec{r}}{dt^2} + \frac{\mu \vec{r}}{r^3} = \frac{\mu\, m'}{M + m} (\vec{F}) , \tag{37}$$

where $(\vec{F})$ is the projection of the total disturbing acceleration

$$\vec{F} = \frac{\vec{r}' - \vec{r}}{\Delta^3} - \frac{\vec{r}'}{r'^3}$$

onto the osculating orbit plane. For our purpose it is convenient to write the Laplace quasi-integral of (37) in the form

$$\vec{R}_3 \times \frac{d\vec{r}}{dt} + \vec{r}^{\,0}\sqrt{\frac{\mu}{p}} + \sqrt{\frac{\mu}{a}}\,\vec{s} = 0. \tag{38}$$

The unit vector $\vec{R}_3$, normal to the orbital plane, is a constant in the ideal system. Differentiating (38), taking into account (37) and

$$\frac{d}{dt}\frac{1}{\sqrt{p}} = -\frac{m'\sqrt{\mu}}{M+m}\,\frac{\vec{F}\cdot\vec{R}_3\times\vec{r}}{p}, \quad \frac{d\vec{r}^{\,o}}{dt} = \frac{\vec{n}^{\,o}\sqrt{\mu p}}{r^2},$$

and considering the fact that a is not affected by the perturbations of zero-rank, we average the result over the orbits of both bodies and obtain for the perturbations of zero-rank in $\vec{s}$:

$$\frac{d\vec{s}}{dt} = \frac{m'\,n\,a^2}{M+m}\,\frac{1}{2\pi}\int_0^{2\pi} \Psi\cdot\vec{F}_0\,dg, \tag{39}$$

where

$$\Psi = \vec{R}_1\,\vec{R}_2 - \vec{R}_2\,\vec{R}_1 + \frac{1}{p}\,\vec{r}\,\vec{n}^{\,0},$$

and

$$\vec{F}_0 = \vec{A}_1'\,K_1 + \vec{A}_2'\,K_2 - \vec{r}\,K_3. \tag{40}$$

In the frame of the present theory the ideal system ($\vec{R}_1$, $\vec{R}_2$, $\vec{R}_3$) is affected only by the perturbations of zero-rank. The averaging process in (39) can be greatly facilitated by replacing the integration with respect to g by integration with respect to the true orbital longitude λ. Making use of the geometric relations

$$\lambda = v + \chi, \quad \frac{\partial v}{\partial g} = \frac{a^2}{r^2}\sqrt{1-e^2}, \quad \frac{\partial \chi}{\partial g} = 0,$$

where v is the true anomaly,

$$\vec{r} = r\,(\vec{R}_1\cos\lambda + \vec{R}_2\sin\lambda), \quad \vec{n}^{\,0} = -\vec{R}_1\sin\lambda + \vec{R}_2\cos\lambda, \tag{41}$$

$$\frac{r}{a} = (1 + s^2)^{-1/2} \left(\sqrt{1 + s^2} + s_1 \cos\lambda + s_2 \sin\lambda\right)^{-1}, \tag{42}$$

we obtain, instead of (39),

$$\frac{d\vec{s}}{dt} = \frac{m' n a^2}{M + m} \frac{1}{2\pi} \int_0^{2\pi} (\vec{W}_1 K_1 + \vec{W}_2 K_2 + \vec{W}_3 K_3)\, d\lambda,$$

where we set

$$\vec{W}_j = \left(\frac{r}{a}\right)^2 \Psi \cdot \vec{A}_j' \sqrt{1 + s^2}, \quad (j = 1, 2)$$

$$\vec{W}_3 = -\left(\frac{r}{a}\right)^2 \Psi \cdot \vec{r} \sqrt{1 + s^2}.$$

In scalar form, noting that in the ideal system $\vec{R}_i$ $(i = 1, 2)$ are constant, we have

$$\frac{d s_i}{dt} = \frac{m' n a^2}{M + n} \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{3} S_{ij} K_j \, d\lambda, \quad (i = 1, 2) \tag{43}$$

where

$$S_{1j} = \vec{A}_j' \cdot \left[\vec{R}_2 + \frac{r}{a}(1 + s^2)\vec{n}^0 \cos\lambda\right] \left(\frac{r}{a}\right)^2 \sqrt{1 + s^2}, \quad (j = 1, 2)$$

$$S_{2j} = \vec{A}_j' \cdot \left[-\vec{R}_1 + \frac{r}{a}(1 + s^2)\vec{n}^0 \sin\lambda\right] \left(\frac{r}{a}\right)^2 \sqrt{1 + s^2}, \quad (j = 1, 2)$$

$$S_{13} = -a\left(\frac{r}{a}\right)^3 \sqrt{1 + s^2} \sin\lambda,$$

$$S_{23} = +a\left(\frac{r}{a}\right)^3 \sqrt{1 + s^2} \cos\lambda.$$

We determine the perturbations of zero-rank in the position of the orbital plane by means of Gibbs' vector

$$\vec{q} = \sum_{i=1}^{3} q_i \vec{S}_i ,$$

decomposed along the $\vec{S}_i$ $(i = 1, 2, 3)$; S_i are the initial values of the unit vectors of the ideal reference frame. The matrix of rotation Λ of the ideal frame from its initial position to its position at the time t, as expressed in terms of $\vec{q}$, has the form (Gibbs, 1901)

$$\Lambda = I + \frac{2}{1 + q^2} [\vec{q} \times I + \vec{q} \times (\vec{q} \times I)] = \frac{1 - q^2}{1 + q^2} I + \frac{2}{1 + q^2} (\vec{q} \times I + \vec{q}\,\vec{q}) .$$

In addition, we shall use the dyadic

$$\Xi = \frac{1}{2} (1 + q^2) (I + \Lambda) = I + \vec{q} \times I + \vec{q}\,\vec{q} ,$$

in the matrix form

$$\Xi = \sum_{i,j} \xi_{ij} \vec{S}_i \vec{S}_j , \quad (i, j = 1, 2, 3) ,$$

where

$$\xi_{11} = 1 + q_1^2 , \quad \xi_{12} = q_1 q_2 - q_3 , \quad \xi_{13} = q_1 q_3 + q_2 ,$$

$$\xi_{21} = q_1 q_2 + q_3 , \quad \xi_{22} = 1 + q_2^2 , \quad \xi_{23} = q_2 q_3 - q_1 ,$$

$$\xi_{31} = q_1 q_3 - q_2 , \quad \xi_{32} = q_2 q_3 + q_1 , \quad \xi_{33} = 1 + q_3^2 .$$

The unit vectors $\vec{R}_i$ $(i = 1, 2, 3)$ are given by the formula

$$\vec{R}_i = \vec{S}_i + \frac{2}{1 + q^2} [\vec{q} \times \vec{S}_i + \vec{q} \times (\vec{q} \times \vec{S}_i)] = \frac{1 - q^2}{1 + q^2} \vec{S}_i + \frac{2}{1 + q^2} (\vec{q} \times \vec{S}_i + \vec{q}\,\vec{q} \cdot \vec{S}_i) .$$

If we set

$$A'_{ji} = \vec{A}'_j \cdot \vec{R}_i ,$$

it now follows from (44) that

$$A'_{ji} = \frac{1 - q^2}{1 + q^2} \vec{A}'_j \cdot \vec{S}_i + \frac{2}{1 + q^2} (\vec{q} \cdot \vec{S}_i \times \vec{A}'_j + q_i \, \vec{q} \cdot \vec{A}_j). \tag{45}$$

After these A'_{ji} are determined using (45), we compute S_{ij} (i, j = 1, 2) by means of the formulas

$$S_{1j} = \left[A'_{j2} + \frac{r}{a}(1 + s^2)(-A'_{j1} \sin\lambda + A'_{j2} \cos\lambda) \cos\lambda\right] \left(\frac{r}{a}\right)^2 \sqrt{1 + s^2} ,$$

$$S_{2j} = \left[-A'_{j1} + \frac{r}{a}(1 + s^2)(-A'_{j1} \sin\lambda + A'_{j2} \cos\lambda) \sin\lambda\right] \left(\frac{r}{a}\right)^2 \sqrt{1 + s^2} .$$

These expressions are to be substituted into equations (43).

We have previously established (Musen, 1961) the following differential equation for the perturbations in Gibbs' vector:

$$\frac{d\vec{q}}{dt} = \frac{1}{2} \frac{m' n a^2}{M + m} \frac{1}{\sqrt{1 - e^2}} \frac{r}{a} (\vec{F} \cdot \vec{R}_3) \,\Xi \cdot (\vec{S}_1 \cos\lambda + \vec{S}_2 \sin\lambda).$$

Taking the average over the orbits of both bodies and replacing the integration with respect to g by integration with respect to the true orbital longitude, we obtain, for the zero-rank perturbations in $\vec{q}$

$$\frac{d\vec{q}}{dt} = \frac{1}{2} \frac{m' n a^2}{M + m} (1 + s^2) \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^3 (A'_{13} K_1 + A'_{23} K_2) \,\Xi \cdot (\vec{S}_1 \cos\lambda + \vec{S}_2 \sin\lambda) \, d\lambda. \tag{46}$$

By projecting (46) on $\vec{S}_i$ (i = 1, 2, 3) we obtain the scalar equations to be used in the actual computations:

$$\frac{dq_i}{dt} = \frac{1}{2} \frac{m' n a^2}{M + m} (1 + s^2) \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^3 (A'_{13} K_1 + A'_{23} K_2) (\xi_{i1} \cos\lambda + \xi_{i2} \sin\lambda) \, d\lambda.$$

The components of $\vec{s}$ in the inertial system of coordinates and, if necessary, the standard elliptic elements e, ω, i, Ω, can be easily determined from the equation

$$\vec{s} = \Lambda \cdot (s_1 \vec{S}_1 + s_2 \vec{S}_2) .$$

As a check we can compute the derivative of the semi-major axis. Its smallness guarantees the accuracy of the theory. From

$$\frac{da}{dt} = \frac{2 m' n a^3}{M + m} \frac{1}{2\pi} \int_0^{2\pi} (\vec{r}^{\,0} e \sin \epsilon + \vec{n}^{\,0} \sqrt{1 - e^2}) \cdot \vec{F}_0 \, d\epsilon ,$$

and taking into account (40) and (41) and

$$d\epsilon = \frac{r}{a} \cdot \frac{d\lambda}{\sqrt{1 - e^2}} ,$$

we obtain

$$\frac{da}{dt} = \frac{2 m' n a^3}{M + m} \frac{1}{2\pi} \int_0^{2\pi} (\alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3) \, d\lambda ,$$

where

$$\alpha_j = \frac{r}{a} \left(\sqrt{1 + s^2} \, e \sin \epsilon \, \vec{r}^{\,0} \cdot \vec{A}'_j + \vec{n}^{\,0} \cdot \vec{A}'_j \right), \quad (j = 1, 2)$$

$$\alpha_3 = - a \left(\frac{r}{a} \right)^2 \sqrt{1 + s^2} \, e \sin \epsilon ,$$

and

$$\vec{r}^{\,0} \cdot \vec{A}'_j = + A'_{j1} \cos \lambda + A'_{j2} \sin \lambda, \quad \vec{n}^{\,0} \cdot \vec{A}_j = - A'_{j1} \sin \lambda + A'_{j2} \cos \lambda ,$$

r/a is given in terms of s by the equation (42) and we have also

$$e \sin \epsilon = \frac{r}{a}(s_1 \sin \lambda - s_2 \cos \lambda).$$

CONCLUSION

We have developed a set of symmetric formulas for computing the zero-rank perturbations in the framework of Hill's theory. This symmetry facilitates the optimization of programming. The vectorial elements of motion are chosen such that the system can also be used when the eccentricity or the inclination of the orbit becomes small but oscillates in a wide interval.

Hill's and Halphen's methods both become inapplicable if two orbits come too close together. In Hill's method the numerical difficulty caused by the proximity of orbits appears as a small numerical divisor k'. Thus, in Hill's method, the difficulty can be watched more easily and directly than in Halphen's method. Yet we hesitate to give a definite preference to either, partly because we have succeeded in applying Halphen's method to determine the long range effects in the orbits of planets, comets, and space probes. However, Hill's method is appealing to the celestial mechanician because of its geometrical simplicity and elegance. Almost every transformation in Hill's method has a direct geometrical or kinematical meaning.

The methods of numerical averaging presently have certain advantages over purely analytical methods. They can treat a large range of eccentricities and orbital inclinations. They can also treat the free secular oscillations as well as the forced ones, and also their mutual cross-effects. At the present time, no analytical theory can do this completely. Perhaps this is because we are not in possession, and hopefully never will be (for the sake of science), of a general analytical solution of the many body problem.

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