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ANALYTICAL STUDY OF STRAIN RATE EFFECTS
IN HYPERVELOCITY IMPACTS

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16. Abstract For the very high deformation rates involved in hypervelocity impacts of small particles, rate effects can introduce important nonlinear size scaling effects in both cratering and ballistic limit phenomena. In this report, the consequences of rate phenomena are examined using results of two-dimensional STEEP code solutions which incorporate a general set of deformation rate sensitive constitutive relations. A phenomenological hydrodynamic-elastic-viscoplastic model valid in three dimensions is used. Detailed numerical solutions of 2024-T3 aluminum spheres impacting finite targets of like material at 7.5 km/sec and at 15 km/sec are obtained. Non-linear size scaling effects are examined for the 7.5 km/sec impact case by comparing the solutions from two configurations which differ in size by a factor of 100 ($D = .32$ cm and $D = 32\mu$). Impact velocity effects are examined by comparing the 15 km/sec impact case and the 7.5 km/sec impact case ($D = .32$ cm). Spatial plots of the velocity and principal stress fields at selected problem times are presented. A rate sensitive model for estimating the critical ballistic limit thickness at a given velocity is presented and applied to the two impact velocities and thicknesses treated. The rate sensitive constitutive relations (including the rate sensitive ballistic limit model) lead to penetration depth predictions and ballistic limit predictions which size scale non-linearly. As the problem scale size is reduced, the penetration depth and ballistic limit thickness decreases more rapidly than would be predicted on the basis of linear size scaling. Good agreement is obtained with the limited amount of available experimental data. However, more experimental data and numerical solutions are needed before the final form of the rate sensitive constitutive equations and the rate sensitive ballistic limit model can be determined for 2024-T3 aluminum and other materials.					
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The basic STEEP code formulation (without rate sensitivity and work hardening) which provided the foundation for the present numerical studies was developed by H. E. Read in 1967 - 1968. For the convenience of readers, the original STEEP formulation is incorporated as an Appendix to this report.

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LIST OF SYMBOLS

d_{ij}	Deformation rate tensor
d_{ij}^p	Plastic component of deformation rate tensor
d_{ij}^e	Elastic component of deformation rate tensor
D	Projectile diameter
e	Specific internal energy
F	A measure of the rate sensitive increase in the "effective" yield strength
G	Shear modulus
M	Mass
P	Penetration depth, or pressure
P'	Penetration depth not including thermal effects on strength
r	Radial distance
T	Target thickness
t	Time
T_j^{i*}	A deviator
U_0	Impact velocity
Y	Static yield strength in uniaxial stress
Z	Axial depth into target
Z_0	Axial depth at $t = 0$

GREEK SYMBOLS

ϵ_p	Generalized plastic strain
σ_f	Fracture stress

$\overset{\wedge}{\sigma}_{\dot{j}}^i$	Jaumann-Noll stress rate
ρ	Density
$\phi(\mathbf{F})$	Rate sensitivity material function
$\bar{\phi}$	Rate sensitivity parameter
γ	Rate sensitivity parameter
μ	Shear modulus
$\omega_{\dot{j}}^i$	Spin tensor

Section 1

INTRODUCTION

1.1 PROBLEM AREA

Hypervelocity impacts produce extreme stresses and high deformation rates. Typically, the stresses are initially of the order of a megabar or larger and the materials behave essentially hydrodynamically. As the impact stress waves and the free surface relief waves propagate into the target and projectile, the initial stresses decay, eventually reaching values comparable to the yield strengths of the materials. The material strengths thus become important in determining the stress and velocity fields at late times, and therefore establish final target deformations.

A two-dimensional Eulerian code called STEEP (an acronym for Shock Two-Dimensional Eulerian Elastic Plastic) was developed during an early phase of this study. STEEP includes strength effects in the form of hydrodynamic-elastic-plastic constitutive relations. While these constitutive relations were the most comprehensive included in any two-dimensional numerical code, they were not sufficiently general to accommodate strain rate effects upon material properties. (For simplicity, the term "strain rate" will be used in place of the more precise "deformation rate".) The strain rates involved in small-scale hypervelocity impacts reach levels which are many orders of magnitude larger than the strain rates achievable in uniaxial stress experiments. Thus, if a material shows a strain rate sensitivity in uniaxial stress tests involving relatively low strain rates, it is likely that even greater rate sensitivity will occur at the higher rates found in hypervelocity impact problems. In this report, we investigate the effects of including strain rate sensitivity in the constitutive relations used for hypervelocity impact numerical calculations. In particular, note that once the constitutive relations are made strain rate sensitive, hypervelocity impacts will no longer linearly size scale. This follows from the fact that a small projectile impact involves larger strain rates than does a large projectile impacting at the same velocity.

1.2 APPROACH

A phenomenological hydrodynamic-elastic-viscoplastic model was formulated and programmed into the STEEP code. The model is strain rate sensitive. Published results of dynamic uniaxial stress experiments were used in determining the rate sensitive material parameters for 2024-T3 aluminum up to rates of approximately 10^4 sec^{-1} . In impacts

of .32 cm dia projectiles at .75 cm/ μ sec, strain rates up to 10^6 sec^{-1} occur. This value increases to 10^8 sec^{-1} with 32 μ projectiles. Uni-axial stress data are not available for these high strain rates. It was therefore necessary to resort to a semi-empirical approach in order to estimate the strain rate sensitive parameters at the higher strain rates. This was done by comparing numerical solutions with penetration data. The strain rate parameters were adjusted to give satisfactory agreement.

The hydrodynamic-elastic-viscoplastic constitutive relations were used in the modified STEEP code to examine the five impact cases indicated in Table 1.

TABLE 1. IMPACT CASES TREATED

Case	Impact Velocity U_0 (cm/ μ sec)	Projectile Diameter D (cm)	Plate Thickness T/D	Rate Sensitivity Parameter $\bar{\phi}$
1	.75	.3175	4	10^4
2	.75	$.3175 \times 10^{-2}$ (32 μ)	4	10^4
3	.75	.3175	4	10^6
4	.75	$.3175 \times 10^{-2}$ (32 μ)	4	10^6
5	1.5	.3175	8	10^6

All the cases involved impacts of 2024-T3 aluminum spheres on finite targets of like material. Size scaling effects were examined in the .75 cm/ μ sec cases, in that two configurations which differed in size by a factor of 100 were used. The large configuration involved a .3175 cm (1/8-inch) projectile; the small configuration used a $.3175 \times 10^{-2}$ cm (31.75 μ) projectile. Impact velocity effects were examined by including a 1.5 cm/ μ sec case. The .75 cm/ μ sec cases used a target thickness to projectile diameter ratio (T/D) of 4. This value corresponds to the nominal ballistic limit thickness for single plate incipient failure for 1/8-in dia projectile impacts, as determined experimentally. In the 1.5 cm/ μ sec impact cases, $T/D = 8$ was used. This large value was chosen so that we could examine the possibility of using one numerical calculation of an impact into a very thick target to determine the critical thickness for incipient failure. (Section 6 contains this failure model.)

The parameter $\bar{\phi}$ listed in Table 1 determines the strain rate sensitive characteristics of the constitutive relations for the very high strain rates, with small values of $\bar{\phi}$ corresponding to greater rate sensitivity. $\bar{\phi}$ must be obtained by comparing theoretical predictions with experimental data involving strain rates of the order of 10^6 sec^{-1} and higher. Uniaxial stress data are not available at these high strain rates. The only available alternative was to use hypervelocity impact penetration data, which do involve these high strain rates, and to compare the experimental results with numerically calculated solutions. Cases 1 and 2 used our initial best estimate for $\bar{\phi}$ (i. e. $\bar{\phi} = 10^4$). However, comparison of the numerical calculations with experimental data demonstrated that $\bar{\phi} = 10^4$ produces an unrealistically high rate sensitivity. The estimate for $\bar{\phi}$ was therefore revised to 10^6 , and this value was then used in the remaining 3 cases.

The constitutive relations for 2024-T3 aluminum were chosen to fit as much of the available theoretical and experimental data as possible. At the same time, the form of these relations was kept intentionally general to permit new data to be incorporated in the model and to permit other strain rate sensitive materials to be fit by the model.

Section 2

CONSTITUTIVE RELATIONS

2.1 THE NEED FOR STRAIN RATE SENSITIVE CONSTITUTIVE RELATIONS IN HYPERVELOCITY IMPACT NUMERICAL CALCULATIONS

Hypervelocity impact events have been studied using two-dimensional numerical codes since the late 1950's¹. Initially, the numerical codes were purely hydrodynamic. Codes were later developed with strength effects included in the form of elastic-plastic or hydrodynamic-elastic-plastic constitutive relations. However, all of these models are insensitive to the rate of deformation. The insensitivity is reflected in the fact that the effective strength of the material (say, as measured by the radius of the yield surface) is not affected by the magnitude of the components of the strain rate tensor. Thus, these models cannot predict variations in uniaxial stress versus strain curves as a function of strain rate. In addition, these models linearly size scale and therefore they do not predict the non-linear size scaling which has been observed in hypervelocity impact events.

2.1.1 Experimental Strain Rate Data

Figure 1 shows the variation with strain rate of experimental uniaxial stress versus strain curves for 2024-0 and 2024-T4 aluminum. Many other metals²⁻⁵ and geological materials⁶ exhibit this type of strain rate sensitivity in the experimentally observable regime up to about 10^4 sec^{-1} for uniaxial stress vs strain experiments. As Figure 1 indicates, the strength of the material appears to increase as the strain rate is increased. Thus, the higher strain rate curves show larger effective yield strengths. (In uniaxial stress experiments, the yield strength is equal to the stress for states beyond the elastic limit.) The solid curves on this figure were obtained by fitting the material parameters in the rate sensitive constitutive relations used in this study to the experimental data. The empirical fit to the rate sensitive data is described in Appendix A. No such fit is possible with the rate insensitive models discussed previously since these models have a fixed strength, independent of the strain rate. Thus, these earlier models would predict one stress strain curve for all strain rates.

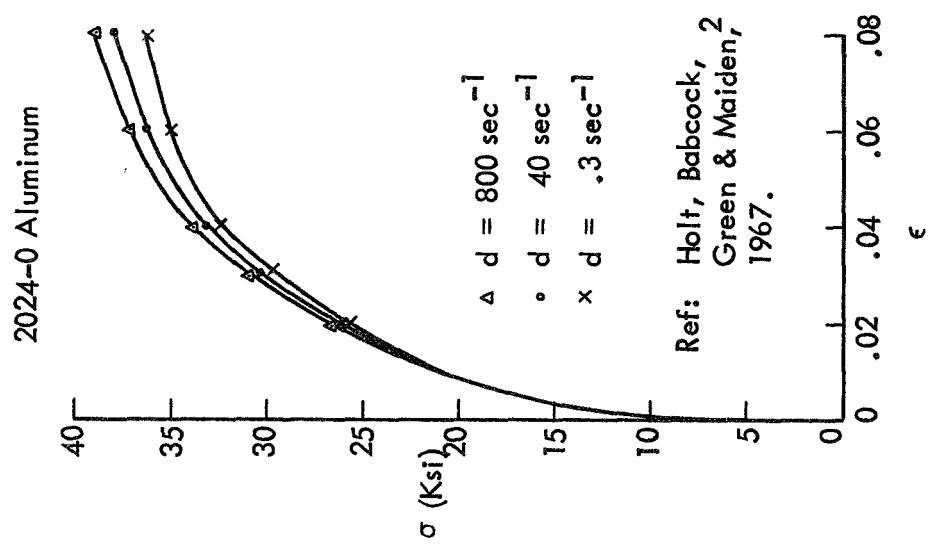
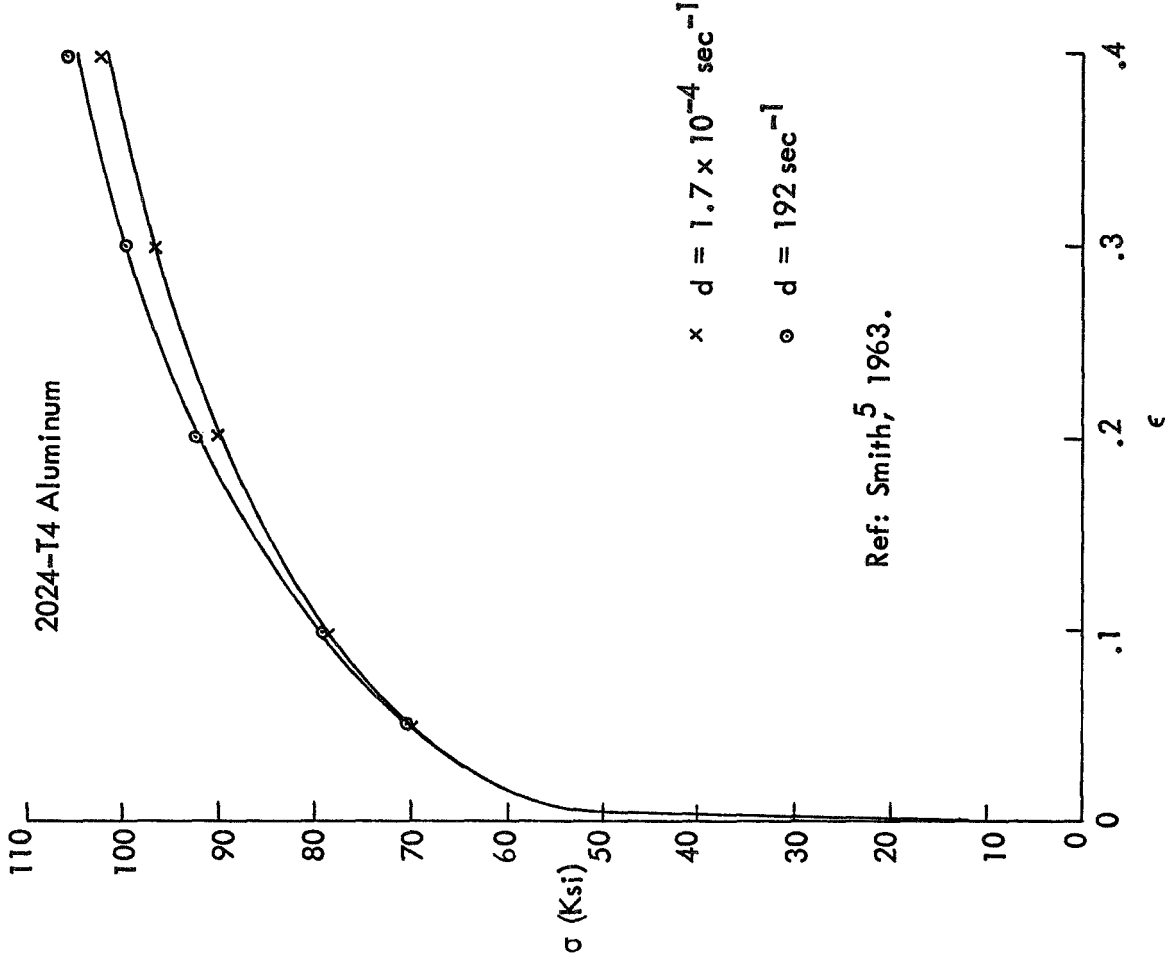


Figure 1 Experimental Uniaxial Stress vs Strain Curves at Various Strain Rates for 2024-0 Aluminum and 2024-T4 Aluminum.

2. 1. 2 Non-Linear Size Scaling in Hypervelocity Impacts

There is another type of experimental data which suggests the existence and the importance of rate effects: In hypervelocity impact experiments it has been found that results do not always linearly size scale. Some of the non-linear size scaling results of a study performed by Denardo and Nysmith⁷ are shown in Figures 2 and 3. In that study, 2017-T4 Aluminum spheres with diameters of 1/16, 1/8, 1/4 and 1/2-inch were impacted into semi-infinite targets of 2024-T4 aluminum at velocities up to 24,000 ft/sec. Figure 2 shows the target momentum multiplication factor (Final Target Momentum/Initial Projectile Momentum) as a function of impact velocity for the various size projectiles. Note the significant effect projectile size has on the experimental results. At 12,000 ft/sec the momentum multiplication factor for a 1/2-inch projectile impact is approximately 1.6, as compared to a value of 1.1 for a 1/16-inch projectile impact. Thus, the 1/2-inch projectile produces a momentum multiplication factor which is 1.45 times greater than the 1/16-inch projectile. If in place of the target momentum we consider the scaled front surface ejecta momentum

$$\left(\frac{M U_{\text{ejecta}}}{M_p U_o} = 1 - \frac{M U_{\text{target}}}{M_p U_o} \right), \text{ then these}$$

results show that the 1/2-inch projectile produces a scaled ejecta momentum factor which is 6 times greater than the smaller 1/16-inch projectile.

The lack of linear size scaling in some hypervelocity impact experiments is also apparent in terms of crater dimensions. For example, Denardo and Nysmith observed a size scaling effect on the normalized depth of penetration, P/D. They observed that the smaller the projectile diameter (D), the smaller the resulting P/D value. Figure 3 is their correlation of the scaled penetration depth as a function of impact velocity using a 1/18 dependence on projectile diameter. Thus,

$$P/D = f(U_o) D^{1/18}$$

The 1/18 diameter dependence has been observed by other investigators^{8,9} for diameters as small as 30 μ . Two possible mechanisms for these departures from linear size scaling are the effects of strain rate upon material properties, and the effects of transfer of energy by conduction. Intuitively, we feel that the effects of conduction are very small. However, until a definitive study of such effects is made, conduction must remain as a possible mechanism.

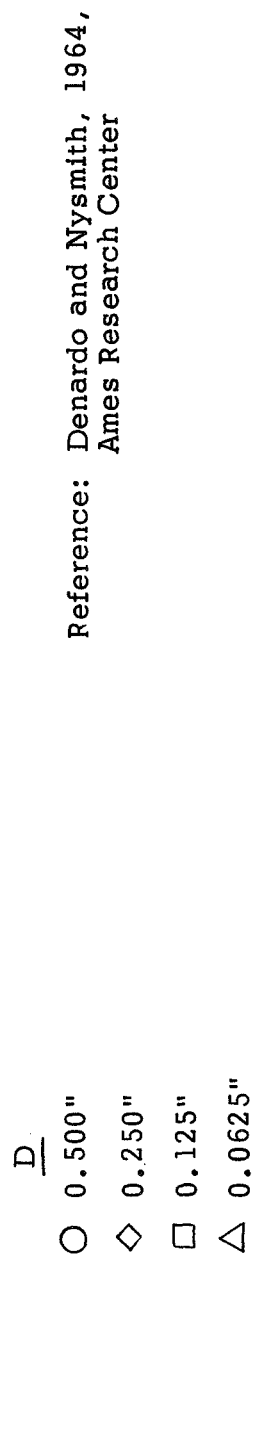


Figure 2. Scaled Target Momentum as a Function of Impact Velocity for 2017-T4 Aluminum Spheres Impacting Semi-Infinite 2024-T4 Aluminum Targets.

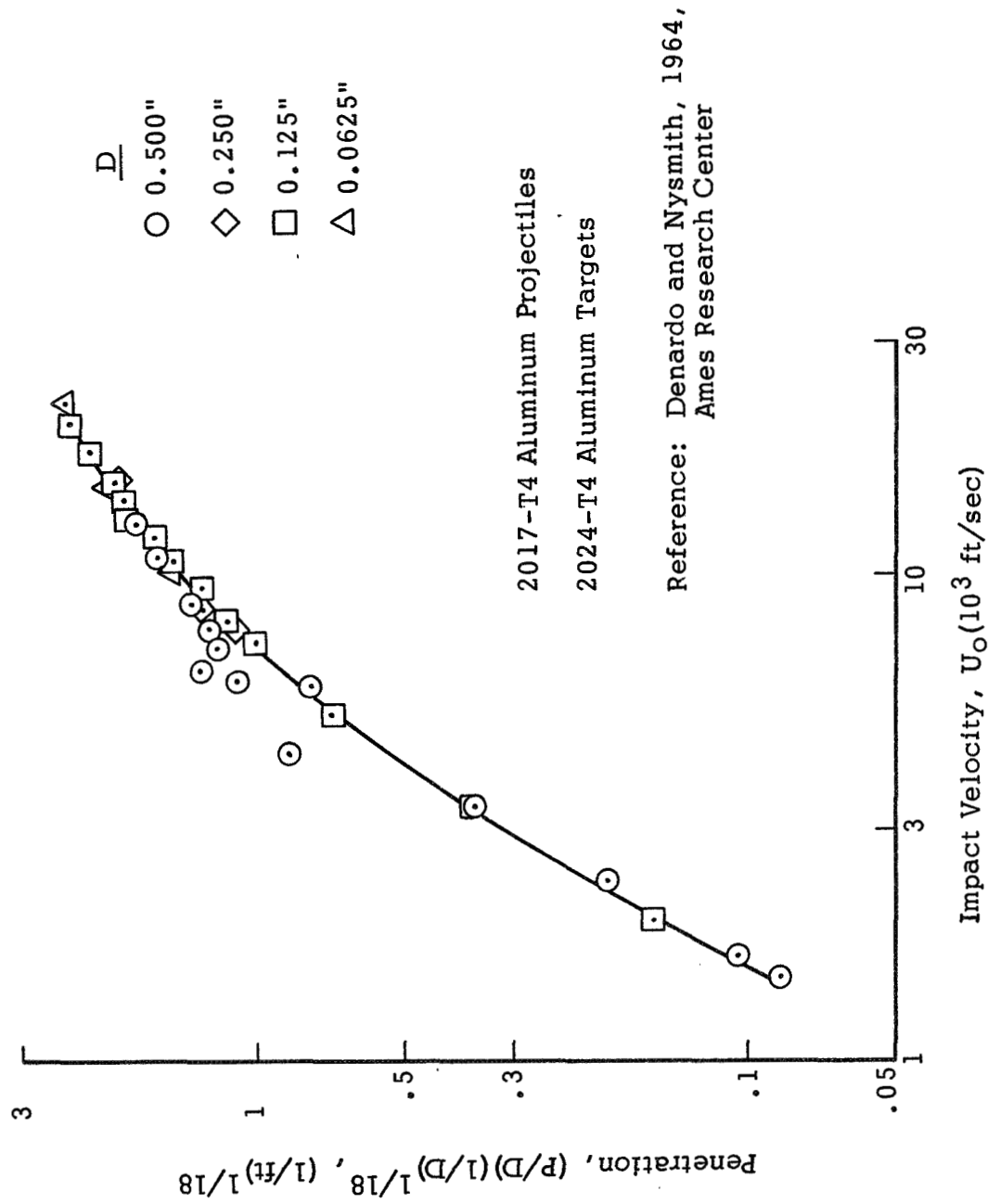


Figure 3. Scaled Penetration Depth as a Function of Impact Velocity Including Correction Factor of $1/D^{1/18}$.

Strain rate effects appear to be the more likely cause of non-linear size scaling. In this study, we have therefore developed a set of strain rate sensitive constitutive relations for use in a 2-D numerical code. Solutions for two hypervelocity impact events which are initially the same in every respect except that they differ in size by a factor of 100 were then obtained. These solutions demonstrate that the experimentally observed size scaling effects can be explained with a strain rate sensitive model.

2.2 HYDRODYNAMIC-ELASTIC-VISCOPLASTIC CONSTITUTIVE RELATIONS

Many approaches have been suggested for describing rate sensitive elastic-plastic materials. However, in most cases the constitutive relations are in a form which is valid only for one-dimensional problems. An important exception is the approach taken by Perzyna^{10,11}, who has developed a general three-dimensional elastic-viscoplastic model for rate sensitive materials. These equations form the basis for the hydrodynamic-elastic-viscoplastic constitutive relations used in this study.

The hydrodynamic-elastic-viscoplastic constitutive relations are indicated below

$$d_{ij} = d_{ij}^e + d_{ij}^p \quad (1)$$

$$\sigma_{ij}^* = 2\mu d_{ij}^e \quad (2)$$

$$\frac{\sigma_m^m}{3} = P = \pi(\rho, e) \quad (3)$$

$$d_{ij}^p = \gamma \phi(F) \frac{\sigma_{ij}^*}{\sqrt{\sigma_n^{*m} \sigma_m^{*n} / 2}} \quad (4a)$$

$$F \equiv \sqrt{\frac{\sigma_n^{*m} \sigma_m^{*n}}{\frac{2}{3} Y^2}} - 1 \quad (4b)$$

$$\gamma = \text{Material Constant } [1/\text{time}]$$

$$\phi(F) = \text{Material Function with } \phi(F) = 0 \text{ if}$$

$$F \leq 0$$

The deformation rate tensor (d_{ij}) is broken up into elastic (d_{ij}^e) and plastic (d_{ij}^p) components. The stress rate deviator is proportional to the elastic deformation rate deviator. A deviator is defined by

$$T_j^{i*} = T_j^i - \frac{T_m^m}{3} \delta_j^i$$

The Jaumann-Noll definition of stress rate

$$\dot{\sigma}_j^{i*} = \frac{D \sigma_j^{i*}}{Dt} + \sigma_m^{i*} w_j^m - w_m^i \sigma_j^{m*}$$

(where w_j^i is the spin tensor.)

is used in the present formulation to satisfy the principal of material frame indifference. The non-deviatoric portion of the stress ($-\frac{\sigma_m^m}{3}$) is related to the hydrodynamic equation of state, $\pi(\rho, e)$, where ρ is the density and e is the specific internal energy. The viscoplastic equation (4) relates the plastic deformation rate to the stress deviator through a material constant (γ) and material function (ϕ).

The viscoplastic equation (4) distinguishes the hydrodynamic-elastic-viscoplastic model from the hydrodynamic-elastic-plastic model. In the hydrodynamic-elastic-plastic model the plastic deformation rate is proportional to the stress deviator. However, the proportionality factor is chosen such that the stress deviator is constrained by a yield surface. For example, the often used von Mises yield condition is

$$\sigma_m^{n*} \sigma_n^{m*} = \frac{2}{3} Y^2 \quad (5)$$

Thus, regardless of the magnitude of the deformation rate, the hydrodynamic-elastic-plastic model limits the deviatoric stress according to equation (5). In contrast, the viscoplastic equation (4) implies a variable "effective" yield condition which can be obtained from the definition of F (equation 4b). Thus

$$\sigma_m^{n*} \sigma_n^{m*} = \frac{2}{3} ([1 + F] Y)^2 = \frac{2}{3} Y_{\text{eff}}^2 \quad (6)$$

where F can be shown to be related to the plastic deformation rate by

$$\phi(F) = \sqrt{(d_n^m P - d_m^n P/2)/\gamma} \quad \text{or equivalently} \quad (7)$$

$$F = \phi^{-1} \left(\sqrt{(d_n^{mP} d_m^{nP} / 2)} / \gamma \right) \quad (7)$$

As a simple example, consider the uniaxial stress condition. For this special case we have

$$\begin{aligned} |\sigma_{11}| &= (1 + F) Y = Y_{\text{eff}}. \\ |d_{11}^P| &= 2/\sqrt{3} \gamma \phi \equiv \gamma' \phi(F) \quad \text{or equivalently} \\ F &= \phi^{-1} (|d_{11}^P| / \gamma') \end{aligned}$$

This example illustrates that $\gamma' \phi$ is a generalized deformation rate. In general $\gamma' \phi = \sqrt{\frac{2}{3} d_n^{mP} d_m^{nP}}$. The example also shows that $F = 0$ is equivalent to the von Mises yield condition. However, if $F > 0$, the material is viscoplastic with an effective yield strength of $Y_{\text{eff}} = (1 + F) Y$. By evaluating uniaxial stress data (F vs d_{11}^P), γ and ϕ can be determined for experimentally accessible strain rates. The details for determining γ and ϕ from uniaxial stress experiments is discussed in Appendix A. Unfortunately, uniaxial stress data only extend to strain rates of approximately 10^4 sec^{-1} . However, the size scaling effects in hypervelocity impact experiments can provide some information concerning much higher deformation rates.

2.3 MATERIAL PARAMETERS FOR 2024-T3 ALUMINUM

Experimental uniaxial stress-strain versus strain rate data for 2024-T3 aluminum and for most other hardened alloys are scarce. Where available, these data do not treat strain rates above 10^3 or 10^4 sec^{-1} . Also, strain rate data from the various investigators do not agree with one another due to differences in experimental techniques. However, the data do indicate relatively small strain rate effects in many hardened metal alloys at the strain rates investigated. Indeed, the strain rate effect in hardened alloys has often been called insignificant. While this is true at the relatively low strain rates examined, note that strain rate effects may become very significant at the much higher strain rates involved in small projectile hypervelocity impacts. Thus, a material which exhibits a small strain rate sensitivity at strain rates of 10^4 sec^{-1} may be significantly strain rate sensitive at 10^8 sec^{-1} .

In order to estimate the form of the behavior of the 2024-T3 aluminum under very high strain rate loadings, it is necessary to extrapolate, based on the form of data obtained with much softer aluminum alloys. Relatively soft aluminum alloys and high purity aluminum do

display significant strain rate sensitivity for strain rates of the order of 10^4 sec^{-1} . We have assumed that the general nature of the strain rate sensitivity observed in these alloys at relatively low strain rates will be found in harder alloys at high strain rates. References 12. and 13 indicate that for high purity aluminum ($> 99.9\%$) at a given strain level, the stress is a linear function of the logarithm of the strain rate below a critical strain rate level ($\bar{d}$) and that the stress increases roughly linearly with strain rate for strain rate values greater than the critical value. Thus,

$$\sigma = Y + C_1 \ln(d/\gamma' + 1) \quad d \leq \bar{d} \quad (7.1)$$

$$\sigma = C_3 + C_4 d \quad d > \bar{d}$$

where Y, C_1, γ', C_3 , and C_4 can depend on the strain and Y is the stress corresponding to $d = 0$ (Quasi-static experiment).

The strain rate sensitivity form (Eq. 7.1) was determined from one dimensional experiments on soft aluminum alloys for strain rates on the order of 10^4 sec^{-1} and smaller. Since corresponding data are not available for the harder aluminum alloys, including 2024-T3, Eq. 7.1 (with some modifications) will be assumed to also hold for these alloys. To incorporate this strain rate sensitivity into the hydrodynamic-elastic-viscoplastic model, it is necessary to determine the corresponding form for the material function $\phi = \phi(F)$ which appear in the constitutive equation 4. Since in one-dimensional high strain rate problems

$$F = \frac{\sigma}{Y} - 1 \quad (7.2)$$

and $d \approx d^P = \gamma' \phi$

equations 7.1 implies the following general form relating F and ϕ

$$\begin{aligned} F &= \theta \ln(\phi + 1) \\ \phi &= e^{F/\theta} - 1 \end{aligned} \quad \begin{aligned} \phi &\leq \bar{\phi} = \frac{\bar{d}}{\gamma'} \\ \phi &= \bar{\phi} + \frac{\bar{\phi} + 1}{\theta} (F - \bar{F}) \quad \phi > \bar{\phi} = \frac{\bar{d}}{\gamma'} \end{aligned} \quad (7.3)$$

where $\theta = \frac{C_1}{Y}$ is a new material function which can depend on the generalized plastic strain, and $\bar{F} = \theta \ln(\bar{\phi} + 1)$.

The strain rate sensitivity will depend on $\theta(\epsilon_p)$, γ' , and $\bar{d}$. $\theta(\epsilon_p)$ and γ' can be estimated from relatively low strain rate data ($d < 10^3 \text{ sec}^{-1}$). For example, the data in Figure 1 were used to determine $\theta(\epsilon_p)$ and γ' for 2024-T3 and 2024-0 Aluminum (see Appendix A for the details of the fit). However, the important critical strain rate parameter, $\bar{d}$, is much more difficult to determine, especially in hard alloys, since high strain rate measurements ($d > \bar{d}$) are necessary. Note that the lower the strain rate sensitivity, the higher the value of $\bar{d}$. This follows from the fact that the strain rate sensitivity is small in the logarithmic region of $d < \bar{d}$, and the sensitivity rapidly increases for $d > \bar{d}$. In soft, nearly pure, aluminum ($> 99.9\%$ pure Al), $\bar{d}$ is approximately 10^3 sec^{-1} .¹² For the harder aluminum alloys, the strain rate sensitivity is smaller than for the softer alloys² and thus $\bar{d}$ will be greater than 10^3 sec^{-1} . However, $\bar{d}$ has not been measured for the hard aluminum alloys such as 6061-T6 or 2024-T3.

Butcher and Karnes³ have used equations similar to equation 7.1 for describing the strain rate sensitivity of 6061-T6 aluminum. They used $\bar{d} = 3 \times 10^3 \text{ sec}^{-1}$ as the critical strain rate in the constitutive equations and performed 1-D numerical calculations. The calculations "compared favorably with the experimentally observed dynamic behavior of 6061-T6 aluminum impacted at $322 \text{ m/sec} = .0322 \text{ cm}/\mu\text{sec}$ ".

For 2024-T3 aluminum, there are no experimental data concerning the value of $\bar{d}$. In order to perform the numerical calculation, a value of $\bar{d}$ had to be selected. Following Butcher and Karnes, we chose $\bar{d}$ to be $3 \times 10^3 \text{ sec}^{-1}$ as a first estimate of the critical strain rate in 2024-T3 aluminum. This value was used in two impact calculations, Cases 1 and 2. These solutions indicated that the first set of strain rate parameters cause an excessive amount of strain rate sensitivity at rates on the order of 10^8 sec^{-1} . Case 2, involving a 32μ projectile "bounced" off the target due to the large strain rate effect which caused the material to behave almost elastically. (Cases 1 and 2 are illustrated in Appendix C.) The excessively large strain rate effect was due to the linear extrapolation of $F = \frac{Y_{\text{eff}}}{Y} - 1$ for strain rates above $\bar{d} = 3 \times 10^3 \text{ sec}^{-1}$. For example, using this value of $\bar{d}$ and a generalized plastic strain rate ($dP = \gamma' \phi$) of 10^8 sec^{-1} , F can become greater than 100. Thus, $Y_{\text{eff}} \sim 100 Y$ and the material behaves nearly elastically. Plastic strain rates of 10^8 sec^{-1} did occur in Case 2 and caused the projectile to bounce elastically off the target. However, these results did demonstrate a size scaling effect and they did enable a better estimate to be made of the critical transition strain rate, $\bar{d}$. For 2024-T3 aluminum, a value of $\bar{d} = 3 \times 10^5 \text{ sec}^{-1}$ was used in the remaining 3 impact cases and physically realistic solutions were obtained. (These 3 impact cases will be discussed in detail in Sections 3, 4 and 5.)

The final forms and constants used for the 2024-T3 aluminum viscoplastic equations are indicated in Figure 4. Also on this figure is a plot of the generalized plastic deformation rate $(\dot{\gamma} \phi)$ as a function of F for various values of the generalized plastic strain, ϵ_p .

$$\epsilon_p = \int_0^t \sqrt{\frac{2}{3}} \dot{d}_m^n \dot{d}_n^m dt \quad (8)$$

The uniaxial stress data indicates that ϕ depends on both F and ϵ_p . Y was also made a function of ϵ_p to account for work hardening effects. The elastic and hydrodynamic material properties for 2024-T3 aluminum are discussed in Appendix B.

Figure 4 contains the strain rate sensitivity information about 2024-T3 aluminum. For example, in uniaxial stress loading, $\dot{\gamma} \phi = \dot{d}_{11}^P \approx \dot{d}_{11}$

and $F = \frac{\sigma_{11}}{Y(\epsilon_p)} - 1$. Thus, Figure 4 shows the fractional increase in effective yield strength versus strain rate at various plastic strain values.

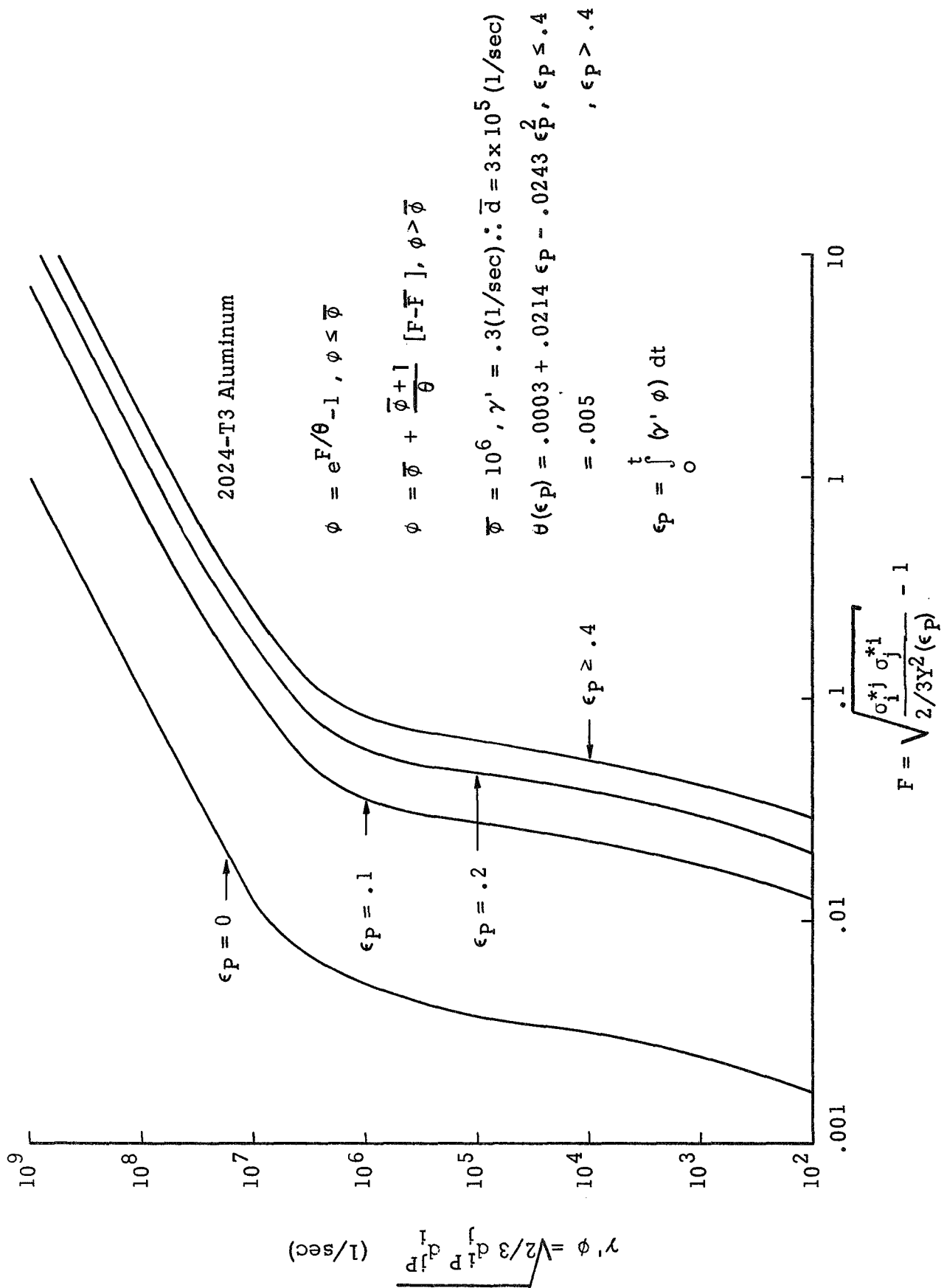


Figure 4. $\gamma' \phi$ versus F With Generalized Plastic Strain as a Parameter

Section 3

GENERAL FEATURES OF HYPERVELOCITY IMPACTS ON FINITE PLATES

The hydrodynamic-elastic-viscoplastic constitutive relations described in Section 2 were used in a 2-D numerical code to examine size scaling and impact velocity effects in hypervelocity impact events. The basic numerical technique is the Eulerian formulations described in Appendix D. This code, developed earlier in the program, used a hydrodynamic-elastic-plastic model. It was modified to accept the new constitutive relations and five solutions were obtained for the hypervelocity impact cases listed in Table I. The numerical solutions provide detailed, time-resolved stress, velocity and specific internal energy fields. However, before discussing the solutions quantitatively, the qualitative features common to hypervelocity impacts on finite plates will be illustrated with the aid of the velocity and principal stress field plots from Case 3 ($D = .3175$ cm, $U_0 = .75$ cm/ μ sec, $\phi = 10^6$). The corresponding plots for Cases 1, 2, 4 and 5 are in Appendix C. The field plots were generated by the SC4020 plotter using the detailed numerical integration data.

Figure 5 is the velocity field at initial conditions. The base of each velocity vector is located at the center of a computational cell. The magnitude and direction of a vector refers to the average velocity of the mass in its cell. In the upper left corner of each velocity plot, a scale bar is shown in units of cm/ μ sec. In all the spatial plots which follow, the horizontal scale is the axis of cylindrical symmetry, while the vertical scale corresponds to radius. The computational cell spacing was chosen such that $\Delta r = \Delta z$ and such that there were 10 cells across a projectile diameter. (Case 5 used 5 cells across the projectile diameter.) In this and all the early time plots, horizontal and vertical grid lines are shown at spacings corresponding to $1/2$ a projectile diameter. In the later spatial plots the grid lines are shown at spacings equivalent to 1 projectile diameter. The small circles connected by straight lines represent the material free surfaces. The x-symbols on the plots are used to trace the motions of specific mass particles in the target and the projectile. The initial field of view does not include the target rear surface. (The target was chosen such that target thickness/projectile diameter = 4.) As the active region enlarges and shifts to the right and upward, the field of view will be correspondingly altered.

Figure 6A and 6B show the velocity and principal stress fields at a time of .12 μ sec. The principal stress field plots indicate the magnitude, sign, and direction of the three principal stress components for each computational cell. In the stress plots, the two principal directions in

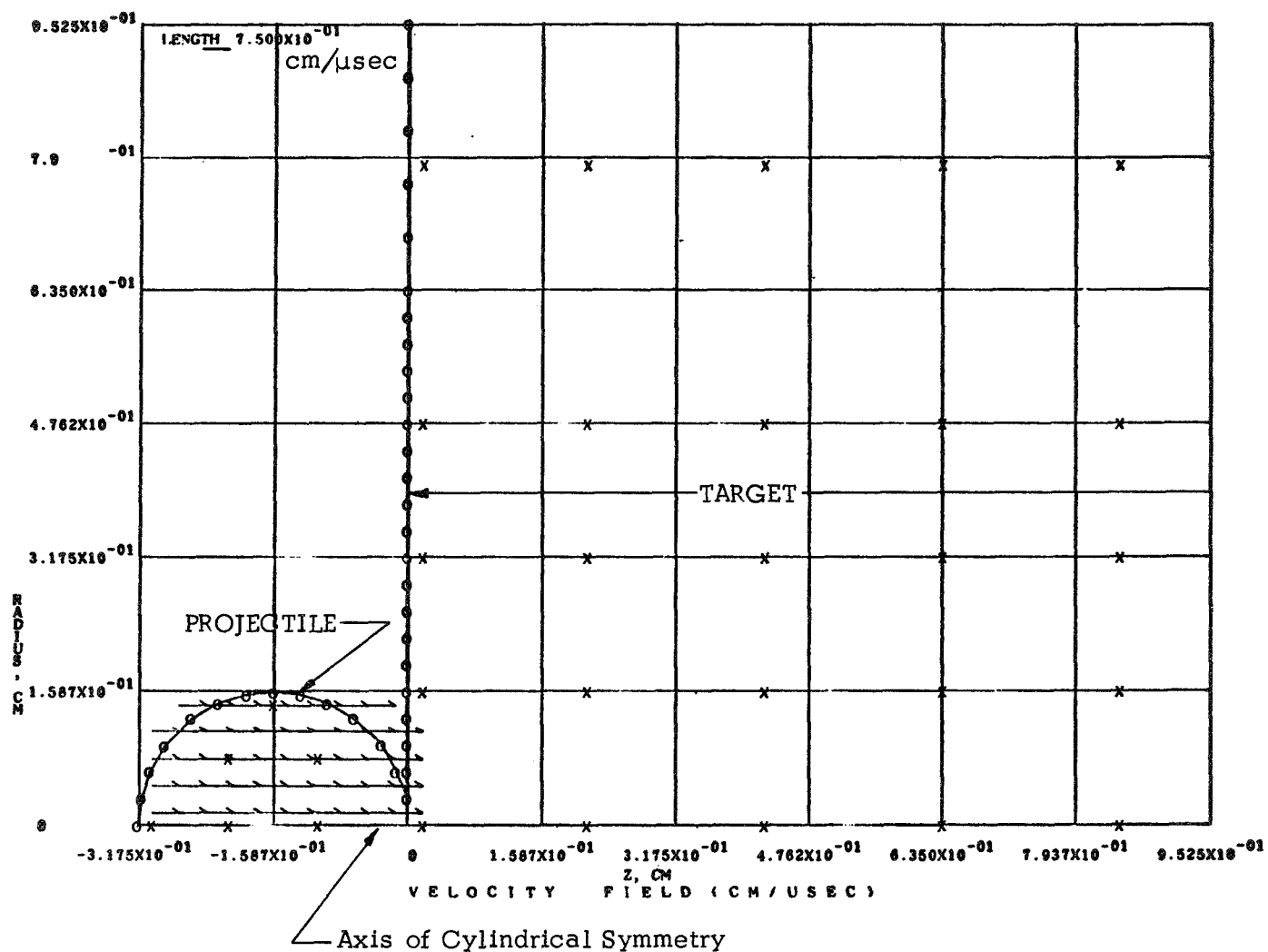


Figure 5. Velocity Field at Initial Conditions for the Large Configuration (Case 3, $U_0 = .75$ cm/ μ sec, $D = .3175$ cm = $1/8$ inch, $\bar{\phi} = 10^6$).

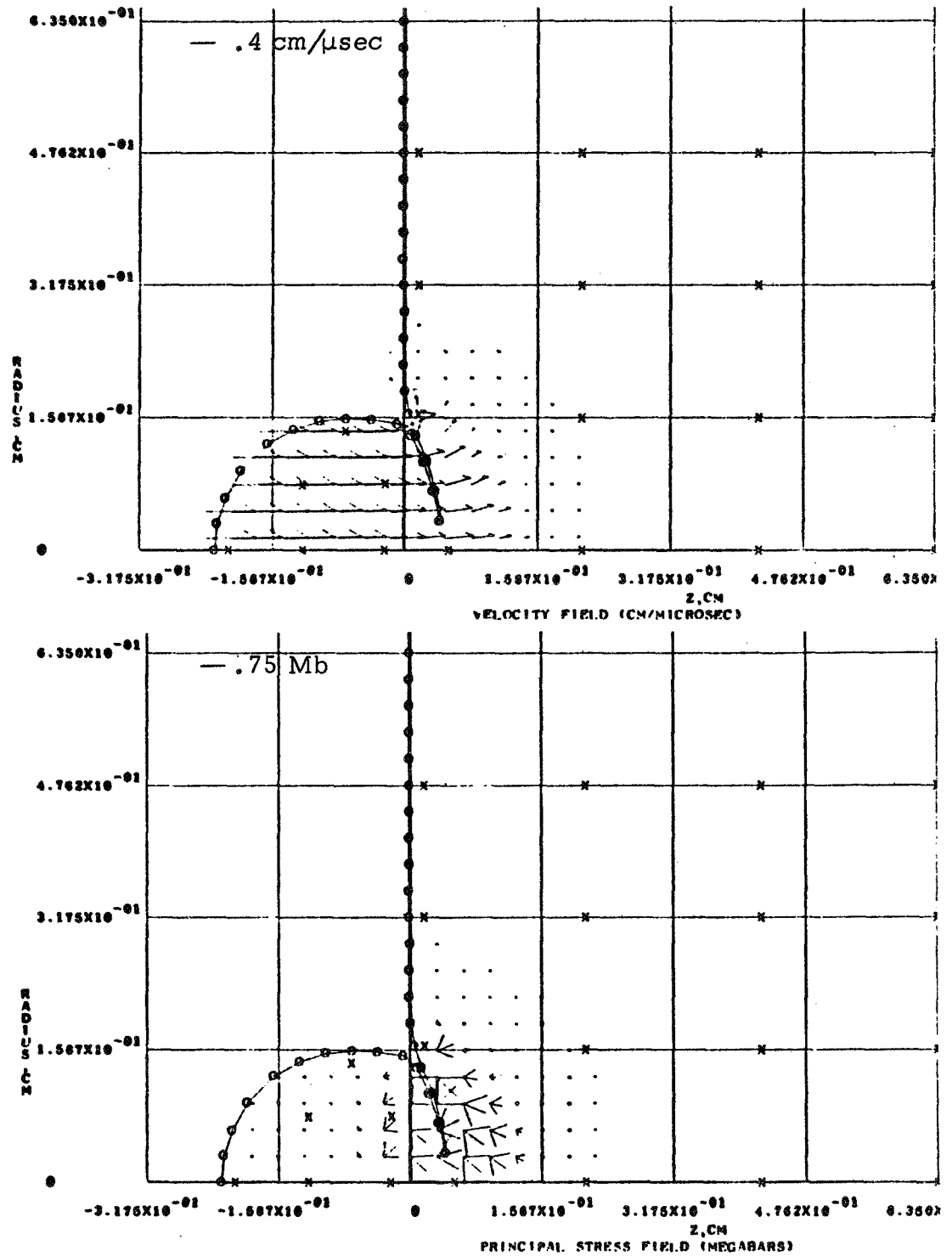


Figure 6. Velocity and Principal Stress Fields at $t = .12 \mu\text{sec}$ ($\frac{U_0 t}{D} = .28$) for the Large Configuration (Case 3, $U_0 = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$).

the r - z plane are identified, being always orthogonal to each other. The third principal direction is always in the azimuthal (θ - θ) direction for axially symmetric problems. The azimuthal stress is plotted along the line bisecting the principal directions in the r - z plane. In each principal direction, the magnitude of the stress is indicated by the length of the vector according to the scale bar shown in the upper left of the plot. The scale bar units are megabars. Any principal stress vector pointing to the right or straight down is compressive. Those pointing to the left or straight up are tensile.

After impact, a shock wave travels from the interface into the target, and another travels from the interface back into the projectile. By $.12 \mu\text{sec}$ the deformation of the target near the impact point is evident. Near the axis of symmetry the behavior is nearly one dimensional, and thus the particle velocity, pressure, and shock wave speeds satisfy the Hugoniot jump conditions. The stress tensor at these early times is essentially hydrodynamic in nature, since the shearing components are negligible. This can be seen in Figure 6B by noticing the equality of the three principal stress components.

Figures 7A and 7B show the velocity and principal stress fields at $.36 \mu\text{sec}$. At this time, the formation of the front surface splash is evident. The front surface splash arises from rarefaction waves propagating into the target from the target front surface. Rarefaction waves have also propagated into the projectile-target system from the projectile free surfaces. (Note that the circles which indicate free surfaces have been removed from where the projectile and target mass have merged.) These relief waves cause the peak pressures to decrease and cause the projectile and target mass near the impact point to diverge from the axis. By $.36 \mu\text{sec}$, the peak pressures have been reduced to about 600 kb from an initial impact pressure of approximately 1 Mb.

Figures 8A and 8B show the velocity and principal stress fields at $.715 \mu\text{sec}$. The beginning of the crater formation is now evident. The velocity field shows that the crater will grow both in depth and width. The crater surface is composed of material which originally formed the projectile rear surface. The peak velocities occur at the crater base near the axis of symmetry. On the other hand, the peak stresses occur in a well-defined stress pulse traveling into the target. At $.715 \mu\text{sec}$ this pulse is located approximately 1.5 projectile diameters into the target along the axis. The pulse is spherical in nature, but decays near the target front surface due to relief at this free surface. The peak stress in this pulse decays as the pulse travels into the target. The stress decay characteristics will be quantitatively discussed in Section 4.

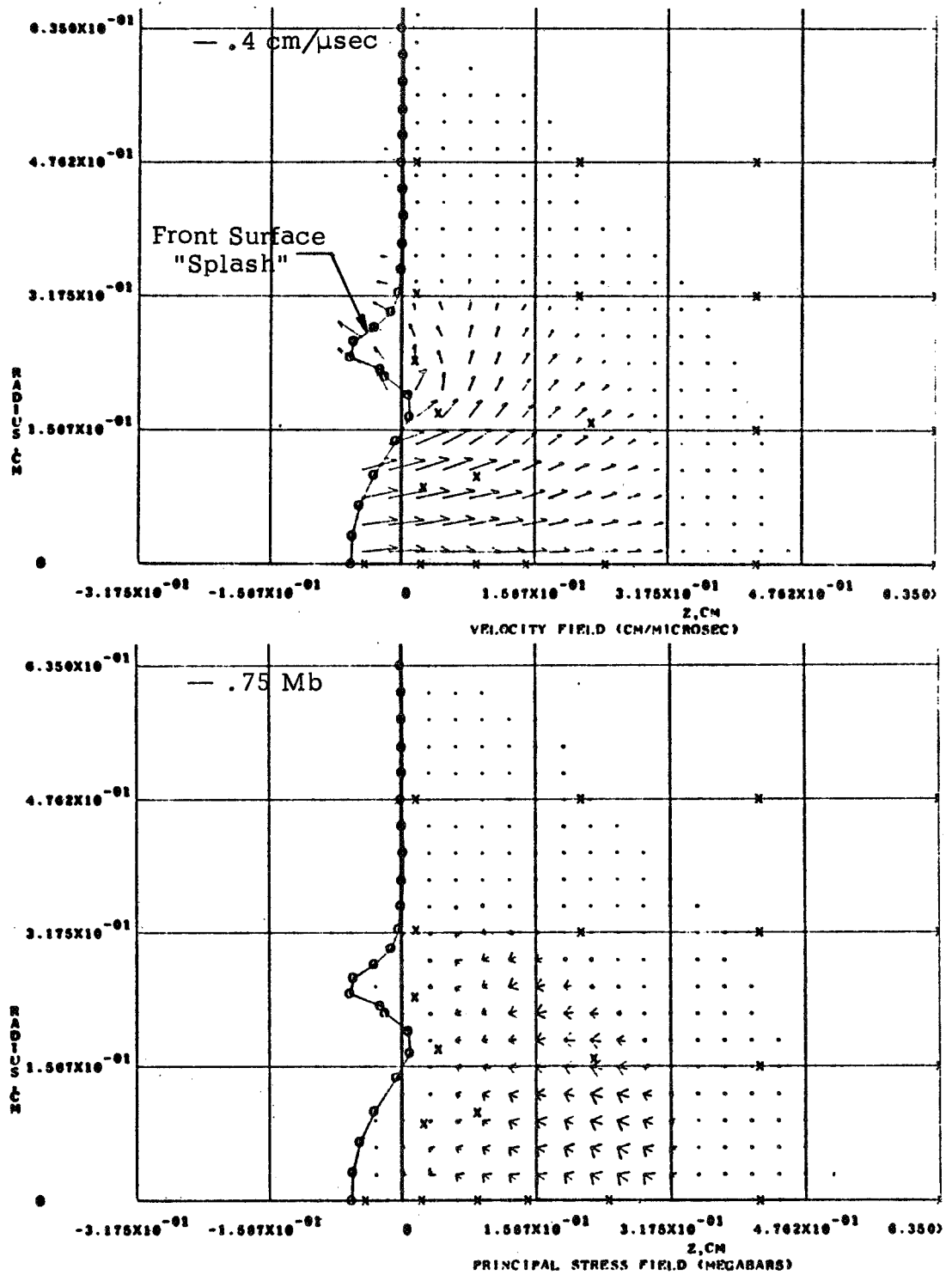


Figure 7. Velocity and Principal Stress Fields at $t = .36 \mu\text{sec}$ ($\frac{U_0 t}{D} = .85$) for the Large Configuration (Case 3, $U_0 = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$).

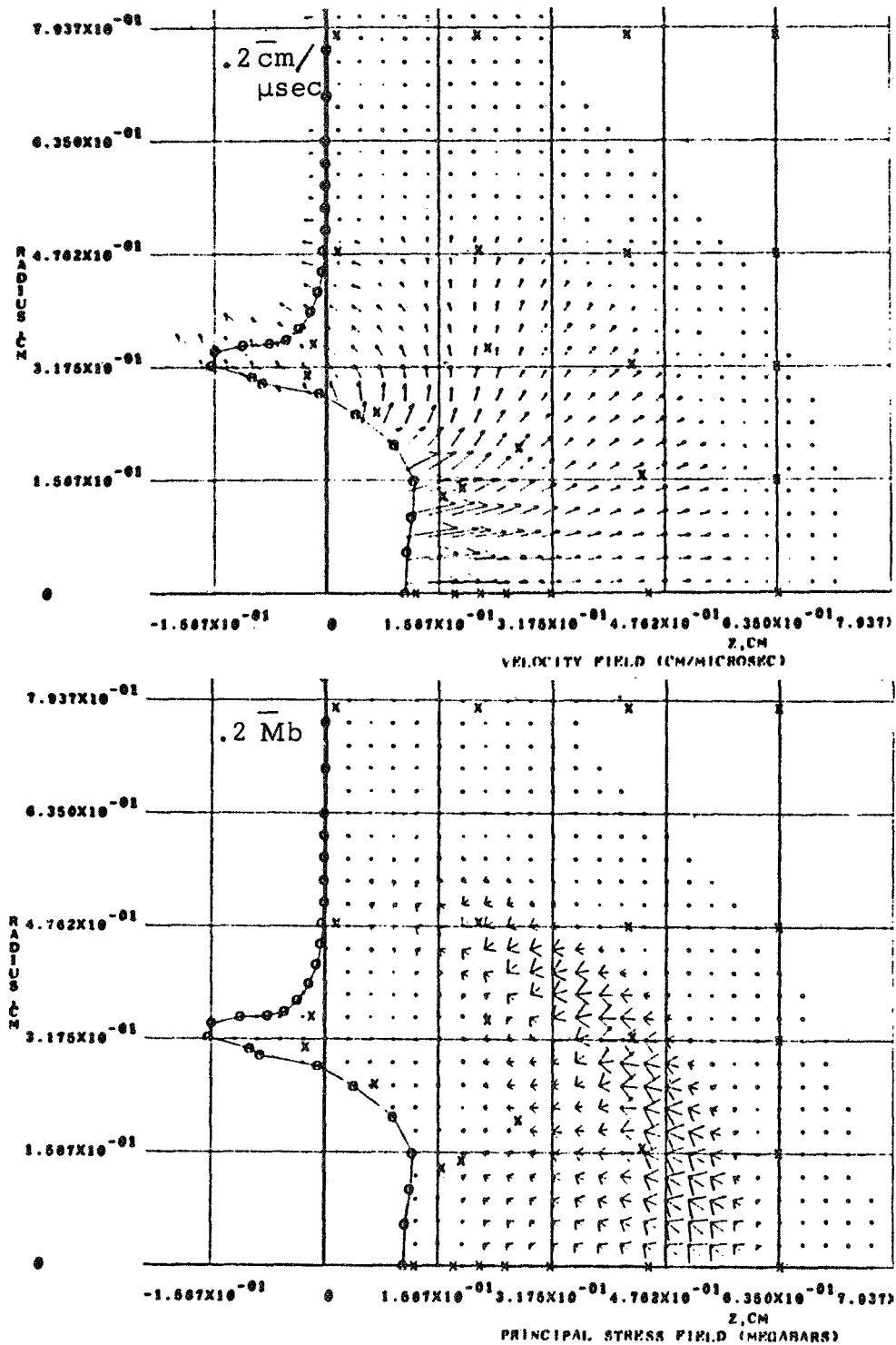


Figure 8. Velocity and Principal Stress Fields at $t = .715 \mu\text{sec}$ ($\frac{U_0 t}{D} = 1.69$) for the Large Configuration (Case 3, $U_0 = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$).

At approximately $1.7 \mu\text{sec}$ the pulse traveling into the target reaches the target rear surface. Figures 9A and 9B show the velocity and principal stress fields at a time of $1.74 \mu\text{sec}$. Note that the rear surface near the axis is now active. The velocity of the rear surface will quickly reach a peak value and then begin to slow down as tensile stresses form near the free surface. The well-defined crater and splash can also be seen in Figure 9. Note that the velocity and stress scale lengths have been reduced considerably from the earlier plots, and the grid lines are now spaced one projectile diameter apart. Figure 9B shows the stress pulse impinging upon the target rear surface, and it is seen that the stress pulse has a width of approximately 0.5 projectile diameters at this time. The peak stress interacting with the rear surface in this case is approximately 50 kb.

Figures 10 and 11 show the velocity and principal stress fields at $2.23 \mu\text{sec}$. By this time, the impact-induced stress pulse system has interacted with the target rear surface to about 2 projectile diameters from the axis. Initially this interaction results in a relief wave propagating back into the target from the free surface, producing acceleration of the surface and reducing the compression. Interaction of this rear surface relief wave with the corresponding relief wave from the front surface eventually produces tension in the material, which causes deceleration and possibly fracture (spallation) of the material near the free surface. The peak velocity of the rear surface, and its deceleration path depend on the peak pressure and the profile of the shock. These relationships are quantitatively discussed in subsequent sections of this report.

Figure 11 also shows the configurations at $t = 2.23 \mu\text{sec}$ of the projectile and of several surfaces which were originally parallel to the surface at various depths. The displacement of projectile and target mass is determined by tracer points (indicated by either o or x symbols on the plots) initially scattered throughout the problem configuration. These massless tracer points flow through the Eulerian grid and indicate the approximate positions of mass elements initially located at spatial coordinates (r_0, z_0) at $t = 0$. Thus, the marker points provide a limited Lagrangian description of the problem configuration.

As indicated in Figure 11, the projectile has undergone gross deformation to form a thin surface layer which coats both the crater and the inner surface of the splash material. The outer surface of that splash is made up of target material which has flowed out of the cratering region.

Figure 12 shows the velocity field at a time of $3.21 \mu\text{sec}$. The crater is still growing, but the rate of growth is diminishing. Figure 13A shows the velocity field at $4.56 \mu\text{sec}$. By this time the bottom of

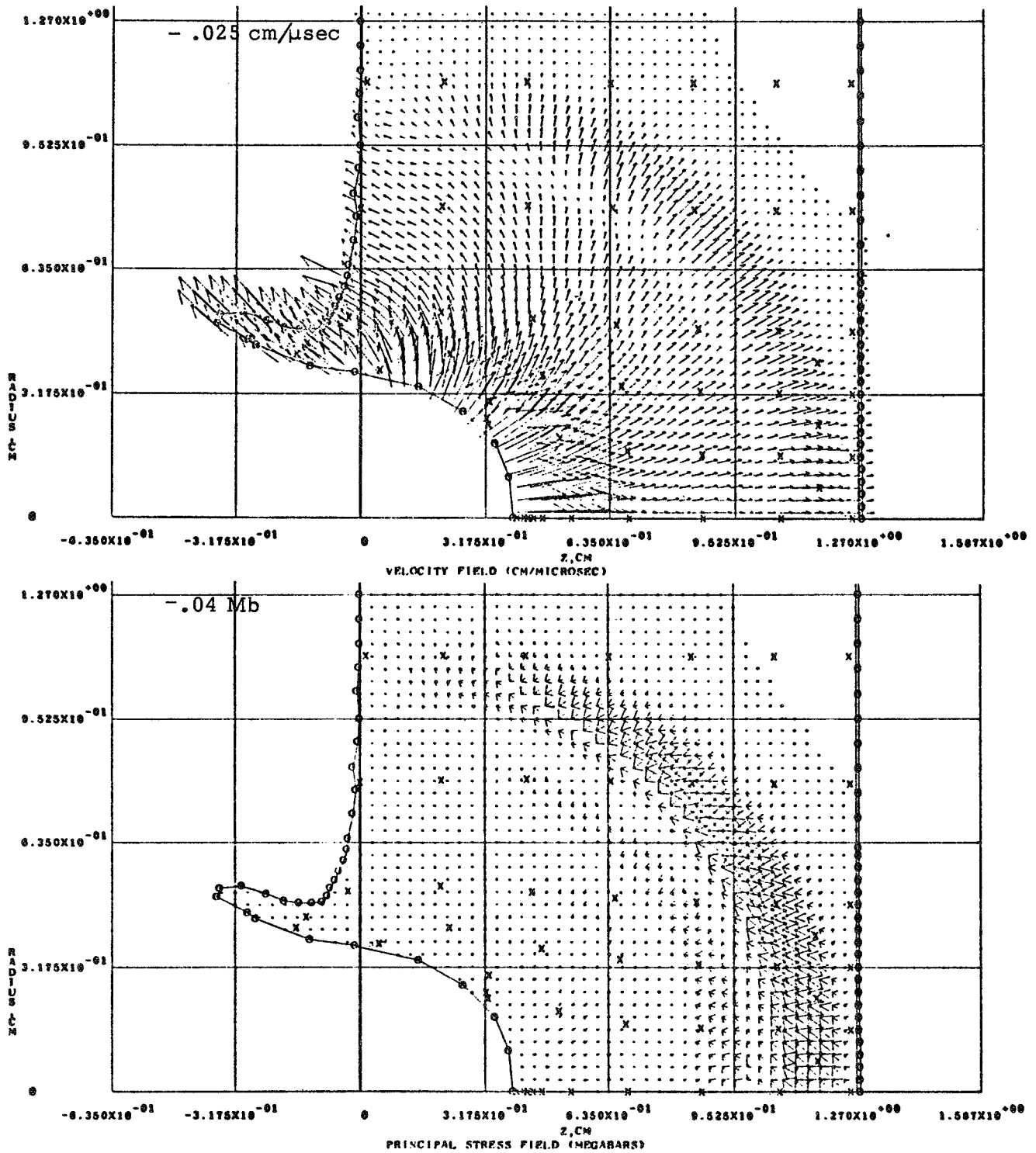


Figure 9. Velocity and Principal Stress Fields at $t = 1.74 \mu\text{sec}$ ($\frac{U_0 t}{D} = 4.11$) for the Large Configuration (Case 3, $U_0 = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$).

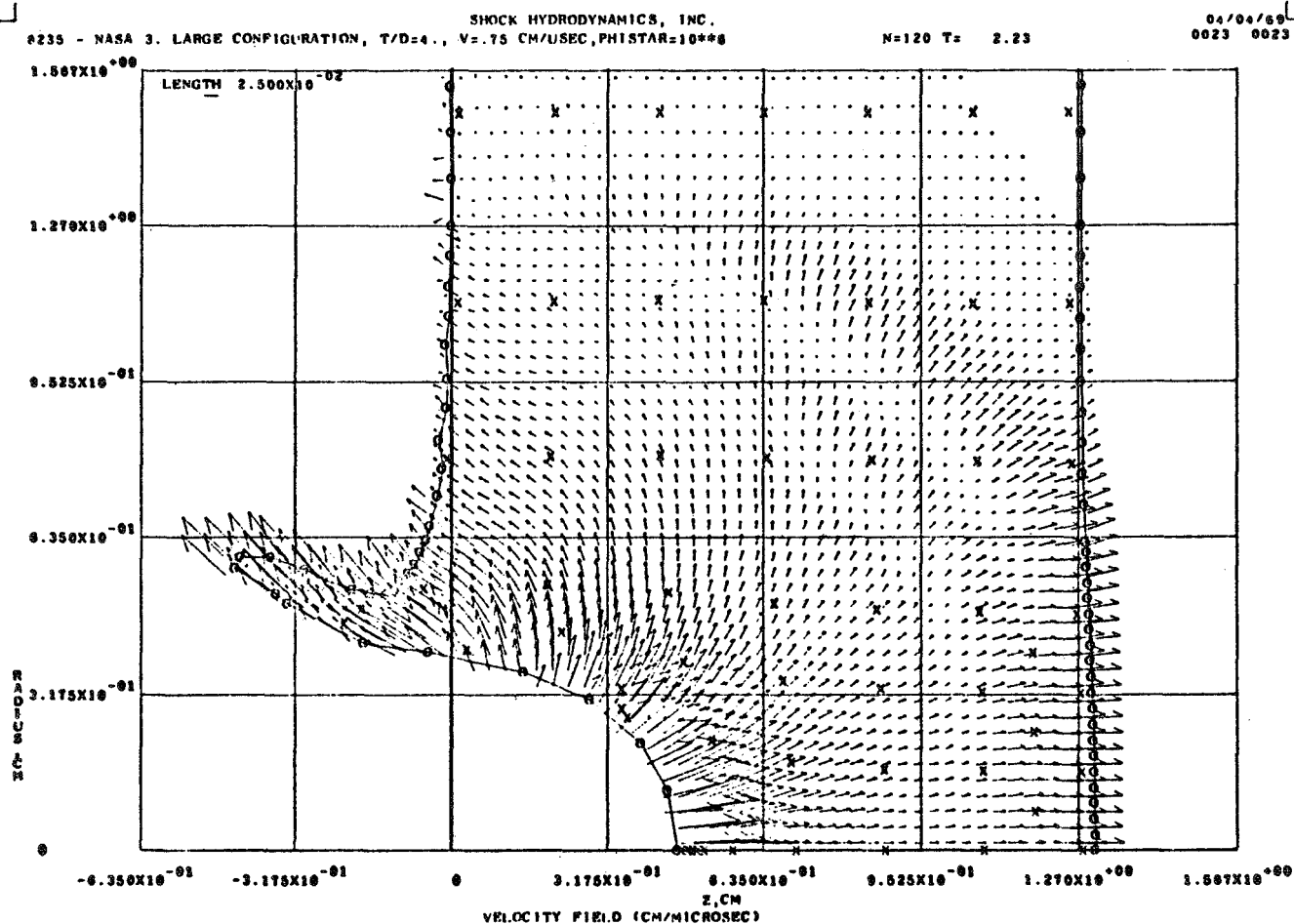


Figure 10. Velocity Field at $t = 2.23 \mu\text{sec}$ ($\frac{U_{ot}}{D} = 5.27$) for the Large Configuration
(Case 3, $U_0 = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$).

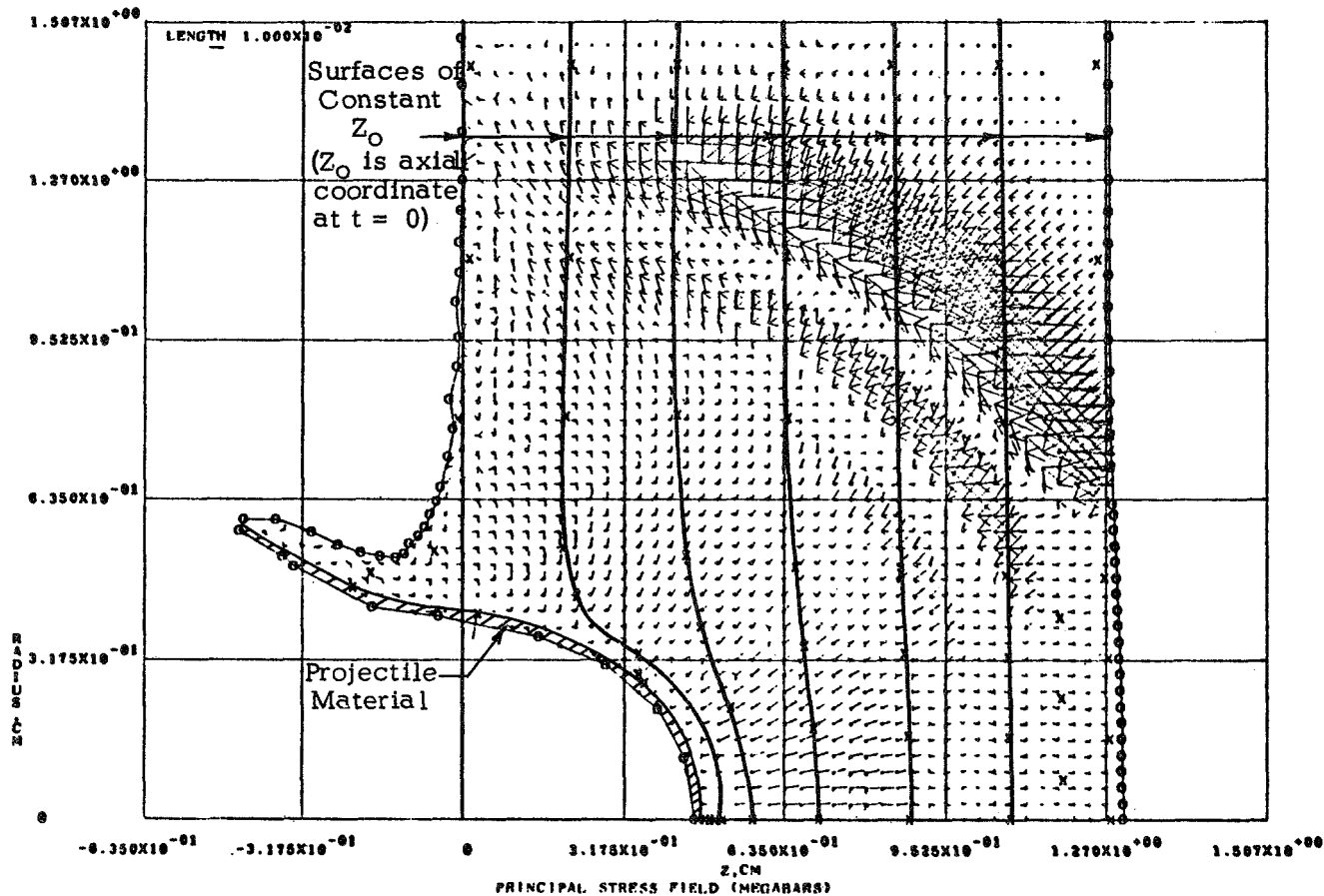


Figure 11. Principal Stress Field at $t = 2.23 \mu\text{sec}$ ($\frac{U_0 t}{D} = 5.27$) for the Large Configuration (Case 3), and the Current Location of Projectile Material and Surfaces of Constant Z_0 in the Target.

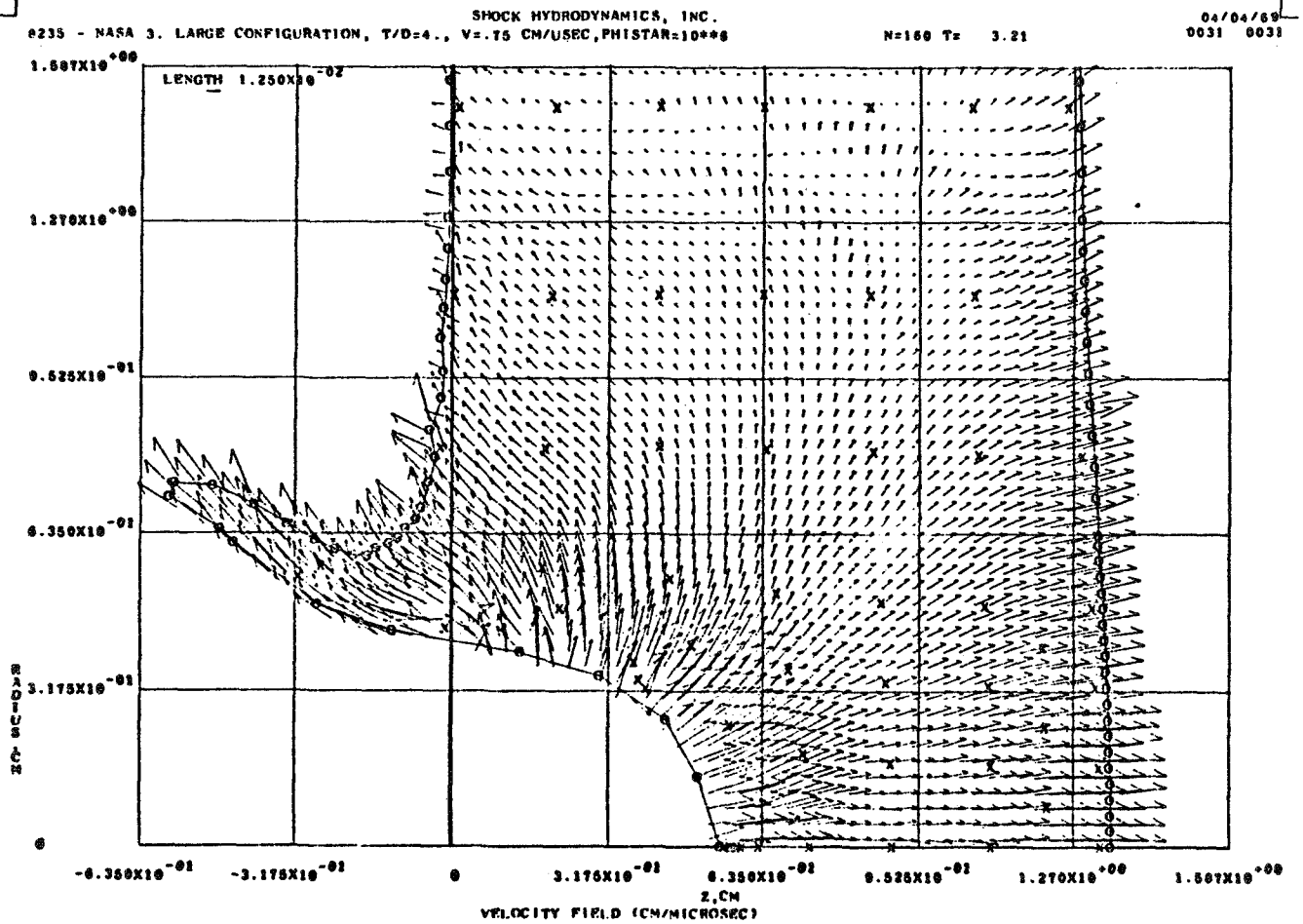


Figure 12. Velocity Field at $t = 3.21 \mu\text{sec}$ ($\frac{U_0 t}{D} = 7.58$) for the Large Configuration
(Case 3, $U_0 = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$).

the crater (on axis) has essentially stopped growing, and the final penetration is obtained. However, the crater diameter is still increasing at $t = 4.56 \mu\text{sec}$. The details of the crater growth will be quantitatively discussed in Section 4.

The relatively large velocity vectors which appear near the crater boundary are spurious. They arise from small amounts of mass rebounding at unreal velocities. The large velocities then cause large spurious oscillations in neighboring cells. This situation could have been corrected if the cells with very low mass had been removed. However, the solution had been carried sufficiently far to obtain the desired information, and therefore, the solution was terminated at this time.

The qualitative features of hypervelocity impacts on finite targets have been discussed for the representative Case 3 solution of a .3175 cm diameter sphere impacting at .75 cm/ μsec . Predicted solutions for other diameter projectiles at this impact velocity would be determined by linear size scaling if the hydrodynamic, elastic-plastic, or hydrodynamic-elastic-plastic constitutive relations were applicable. However, the hydrodynamic-elastic-viscoplastic relations used in this study do not linearly size scale. An illustration of non-linear size scaling is shown in Figure 13 in terms of a velocity field plot comparison. Figure 13A is the Case 3 (large configuration impact) at $t = 4.56 \mu\text{sec}$. Figure 13B is the corresponding small configuration impact (Case 4) velocity field plot at $t = 4.57 \times 10^{-2} \mu\text{sec}$. The two cases were initially identical except that all initial dimensions in Case 4 were 1/100 of the corresponding dimensions in Case 3. Linear size scaling would predict identical velocity field plots at times which differed by a factor of 1/100 (in other words, the solutions would be identical in terms of non-dimensional spatial coordinates and time. Convenient non-dimensional spatial coordinates and times are r/D , z/D and $U_0 t/D$. Note that the U_0 term is included to make the scaling factor dimensionless, and does not imply velocity scaling.) However, as demonstrated in Figure 13, the two solutions exhibit significant quantitative differences with respect to the crater dimensions and the front surface splash characteristics. In the small impact, the crater dimensions and the splash mass and momentum are smaller than predicted by linear size scaling. The differences are due to the strain rate sensitivity included in the constitutive relations.

In the next section, the non-linear size scaling effects will be quantitatively analysed. In Section 5, the effects of impact velocity are discussed. And in Section 6, a failure criterion is applied to the numerical solutions to predict ballistic limit thicknesses.

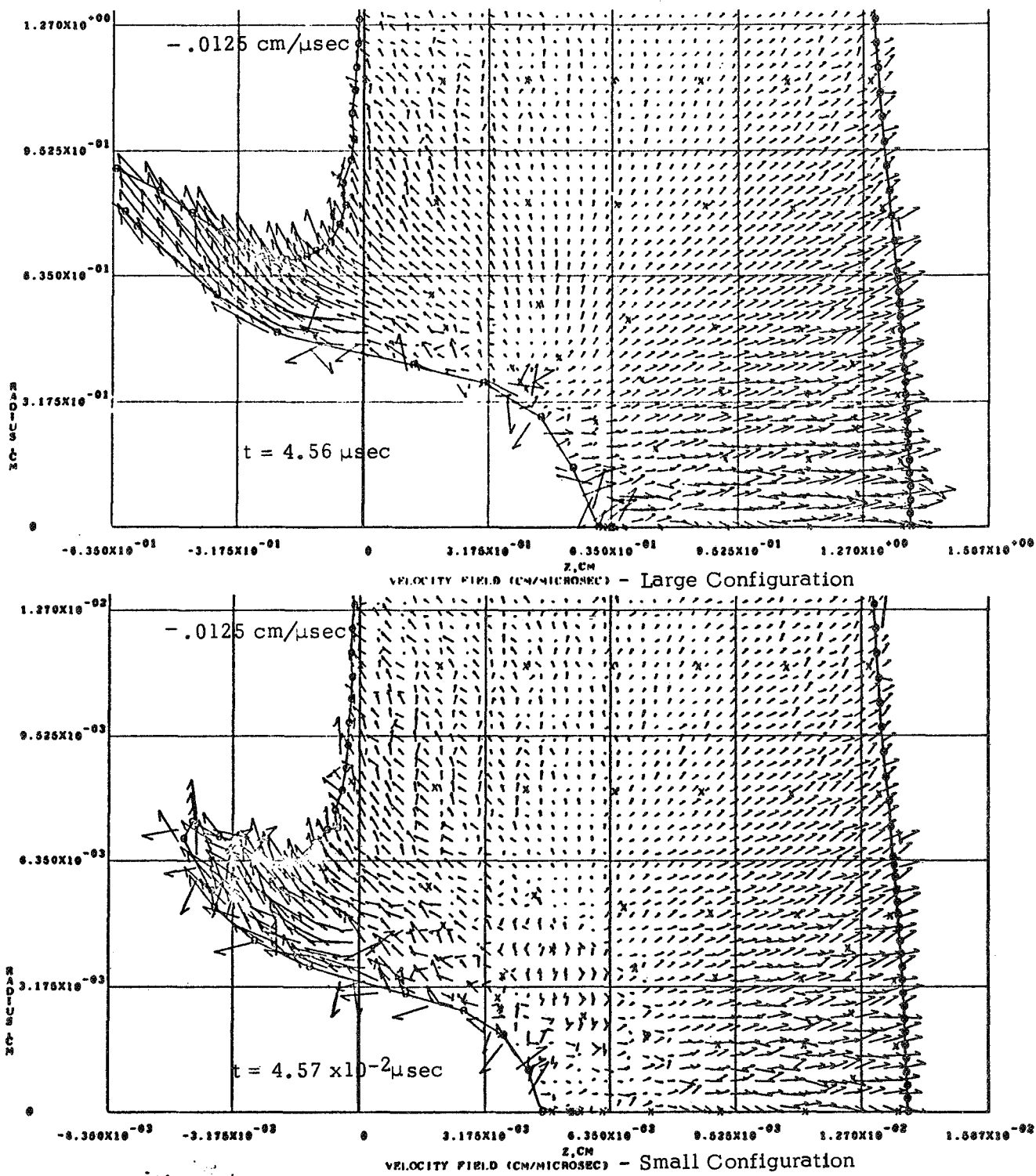


Figure 13. Velocity Fields for the Large (Case 3) and Small (Case 4) Configurations at a Scaled Time of $U_0 t/D = 10.8$.

Section 4

SIZE SCALING EFFECTS

Size scaling effects were examined using the hydrodynamic-elastic viscoplastic model by comparing two pairs of numerical solutions of impact events which were initially the same in every respect except for a size scale factor of 100 (i. e. , Cases 1 and 2 and Cases 3 and 4 in Table 1).

The first pair (1 and 2) used the strain rate properties derived on the basis of the Sandia estimate of the strain rate sensitive properties of 6061-T6 Al⁵. These solutions produced unrealistic results, in that the strength properties of the target were raised to such an extent by the high strain rates that the projectile "bounced" off the target in the small scale case (#2). Velocity and principal stress fields from Cases 1 and 2 are included in Appendix C to this report, but the solutions will not be further discussed here. The extrapolation of experimental strain rate data was revised to reflect less sensitivity of strength on strain rate for 2024-T3 aluminum. These revised properties were then used in a second pair of solutions (3 and 4). Plausible results were obtained from Cases 3 and 4, and these results will be quantitatively compared to indicate the influence of strain rate parameters upon crater growth, front surface splash, peak stress decay, and specific internal energy contours in impacts of different scale.

4.1 CRATER AND FRONT SURFACE SPLASH CHARACTERISTICS

Figure 14 shows comparisons of the crater and splash profiles for the large (Case 3) and small (Case 4) configuration cases at two scaled times. $\frac{U_0 t}{D}$ is used as the non-dimensional time variable. If the solutions did linearly size scale, then for all $\frac{U_0 t}{D}$ values the scaled profiles would be identical.

The small configuration crater and splash show the effects of the strain rate sensitivity in the form of less rapid crater growth, smaller crater dimensions, and lower velocity splash material. Thus, the material in the small configuration behaves stronger and more elastically than the same material in the large configuration.

The front surface splash mass, axial momentum, and energy are listed in Table 2 at the time corresponding to the last numerical integration cycle ($U_0 t/D = 10.8$) for both the large and small configurations. Note, however, that these values were still increasing when the solutions were terminated.

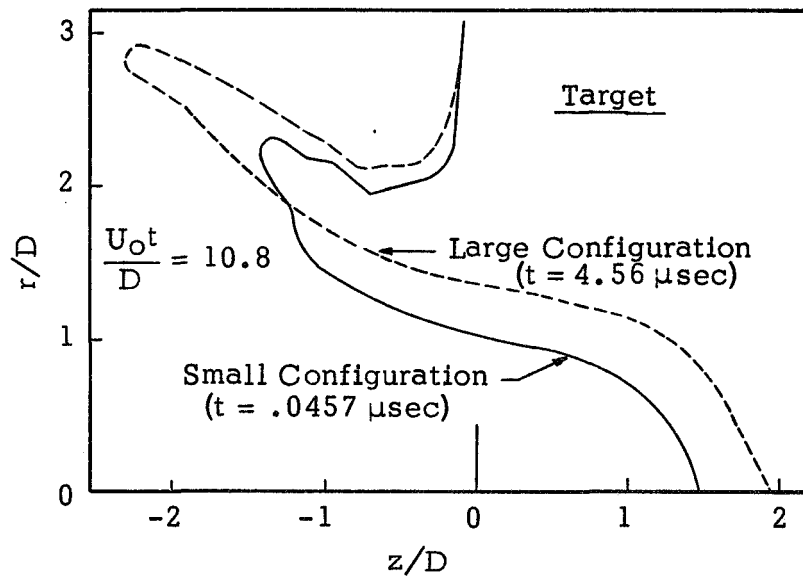
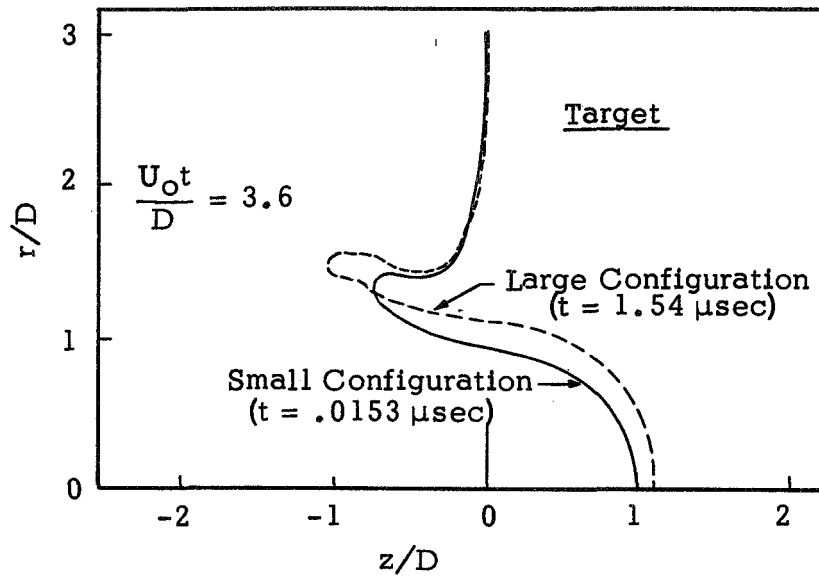


Figure 14. Comparison of Crater and Splash Profiles at Scaled Times of $\frac{U_0 t}{D} = 3.6$ and 10.8.

TABLE 2. COMPARISON OF LARGE AND SMALL CONFIGURATION
SPLASH CHARACTERISTICS AT THE LAST CYCLE
($U_0 t/D = 10.8$)

Case	Diameter D (cm)	Projectile Mass M (gm)	Impact Velocity U (cm/ μ sec)	$\frac{M_{\text{splash}}}{M}$	$\frac{(MU)_{\text{splash}}}{M U_0}$	$\frac{\frac{1}{2} M (U^2 + V_{\text{splash}}^2)}{\frac{1}{2} M U_0^2}$
3 Large Configuration	.3175	.0453	.75	11.3	- .45	.072
4 Small Configuration	.3175 $\times 10^{-2}$	.0453 $\times 10^{-6}$	.75	10.0	- .215	.021

The large configuration case has a slightly greater splash mass and considerably greater splash momentum and kinetic energy than the small configuration case. This indicates that the particle velocities in the splash are much larger in the large configuration case.

In the above tabulation, "splash" has been considered as all mass above the original front surface out to 3 projectile diameters. Note that this may be somewhat different from "ejecta", which we consider as being composed of the splash material which actually detaches from the target. Fracture mechanisms are important in determining the ejecta characteristics. Thus, the rate aspects of fracture (see Section 6) may become significant in determining the ejecta characteristics.

The time resolved axial penetration for the two cases appear in Figure 15. The two curves show the scaled penetration depth (P'/D) versus the scaled time ($U_0 t/D$). P' represents the depth, not including thermal effects on strength. Thermal effects will be discussed in Section 4.3. At early times (corresponding to essentially hydrodynamic conditions) the rates of crater growth are similar. However, as soon as strength effects become important, the small configuration solution shows slower crater growth than the large configuration solution. The slower crater growth in the small configuration is due to the increased material strength caused by the plastic strain rate sensitivity.

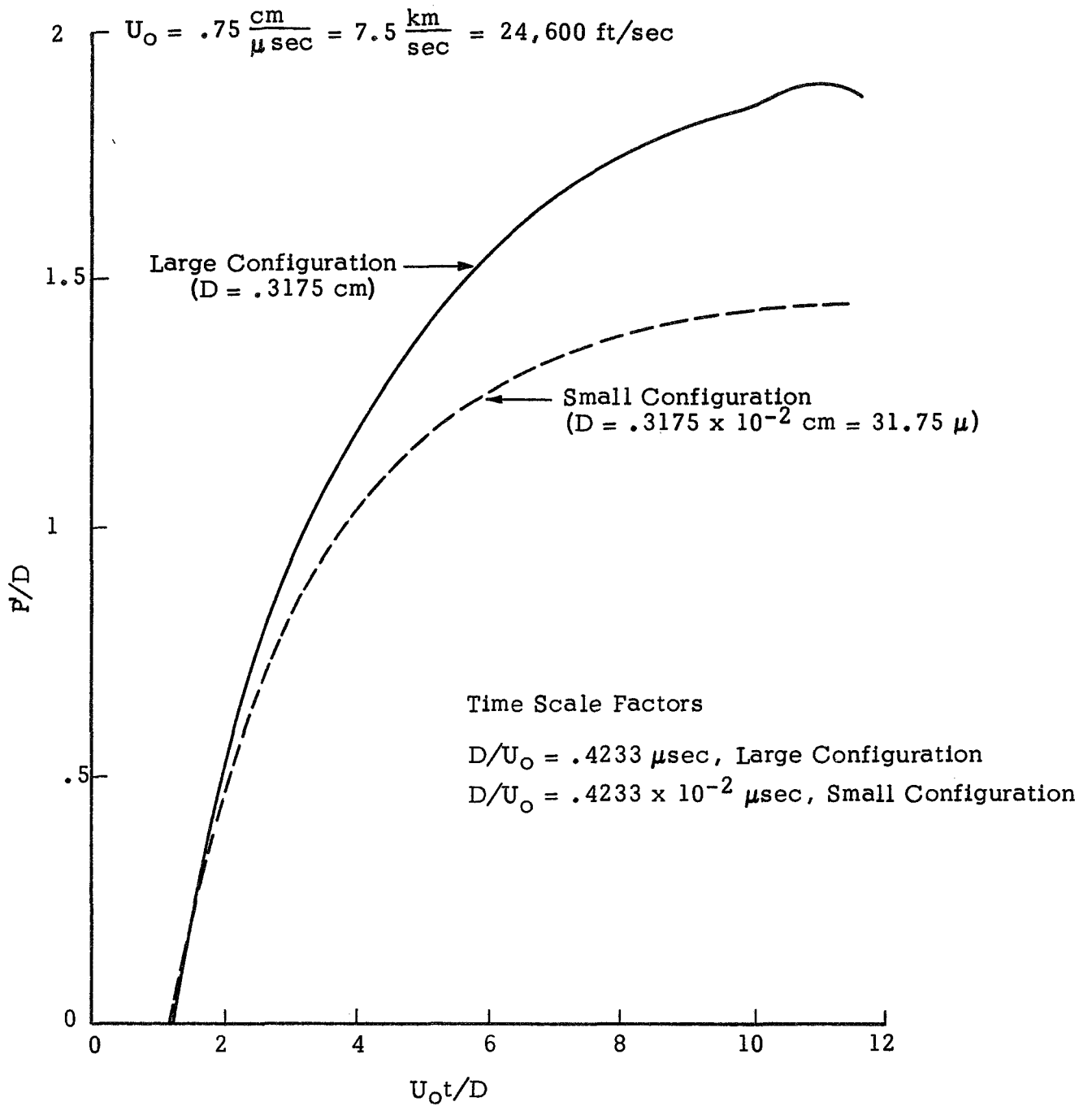


Figure 15. Scaled Penetration Depth (P/D) versus Scaled Time ($U_0 t/D$) for the Large and Small Configurations Not Including Thermal Softening Effects.

In both cases, crater growth in the numerical solutions essentially stopped by a scaled time of $U_0 t/D = 10.8$ which corresponds to $4.6 \mu\text{sec}$ in the large configuration and $.046 \mu\text{sec}$ in the small configuration. The scaled penetration depths differ by over 20%. However, note that the crater depths indicated in Figure 15 are not the final predicted depths since the numerical solutions did not include the effects of thermal softening (reduction in material strength due to heating). Thermal softening will cause the crater depths to be larger than indicated. Thermal effects are discussed and final crater depths (P/D) are predicted in Section 4.3.

The plastic strain rates involved in the two impact solutions are indicated in Figure 16. The peak generalized plastic deformation rate ($\dot{\gamma}' \phi$) along the axis is plotted versus scaled time ($U_0 t/D$). Also on this figure are the corresponding values of $F = \sqrt{\frac{\sigma_m^{n*} \sigma_m^{n*}}{\frac{2}{3} Y}} - 1$. F is a measure of the rate sensitive increase in the effective yield strength. In fact, the "effective" yield strength is $Y_{\text{eff}} = (1+F)Y$ where Y is the "static" yield strength in uniaxial stress. F is therefore the fractional increase in effective yield, $F = \frac{Y_{\text{eff}} - Y}{Y}$. For the large configuration, the peak plastic strain rates are approximately $4 \times 10^6 \text{ sec}^{-1}$. In the small configuration the peak rates are roughly $4 \times 10^8 \text{ sec}^{-1}$. The one hundred-fold increase in the strain rates corresponds, of course, to the one hundred-fold decrease in scale size.

F for the large configuration reached a peak value of about .1. Thus, a 10% increase occurred in the effective yield strength. In the small configuration, F reached a maximum value of 4, meaning a 400% increase in the effective yield strength.

4.2 PEAK STRESS AND PARTICLE VELOCITY ALONG THE AXIS

Upon impact, a shock front propagates into the target. Behind this front, there is a well defined stress pulse. This pulse was illustrated in Section 3, e. g., see Figures 8B and 9B. The peak stress in the pulse decreases in magnitude as the pulse propagates outward. The peak stress vs scaled distance along the axis for the large and small configurations are given in Table 3. Figure 17 is a plot of these data. The two configurations do not show any significant differences. This indicates that no appreciable size scaling effect occurs with respect to the peak stress decay characteristics.

Associated with the peak stress is a peak velocity in the detached shock wave pulse. The peak stress and velocity immediately behind the shock front are related by the Hugoniot jump conditions. As the strength

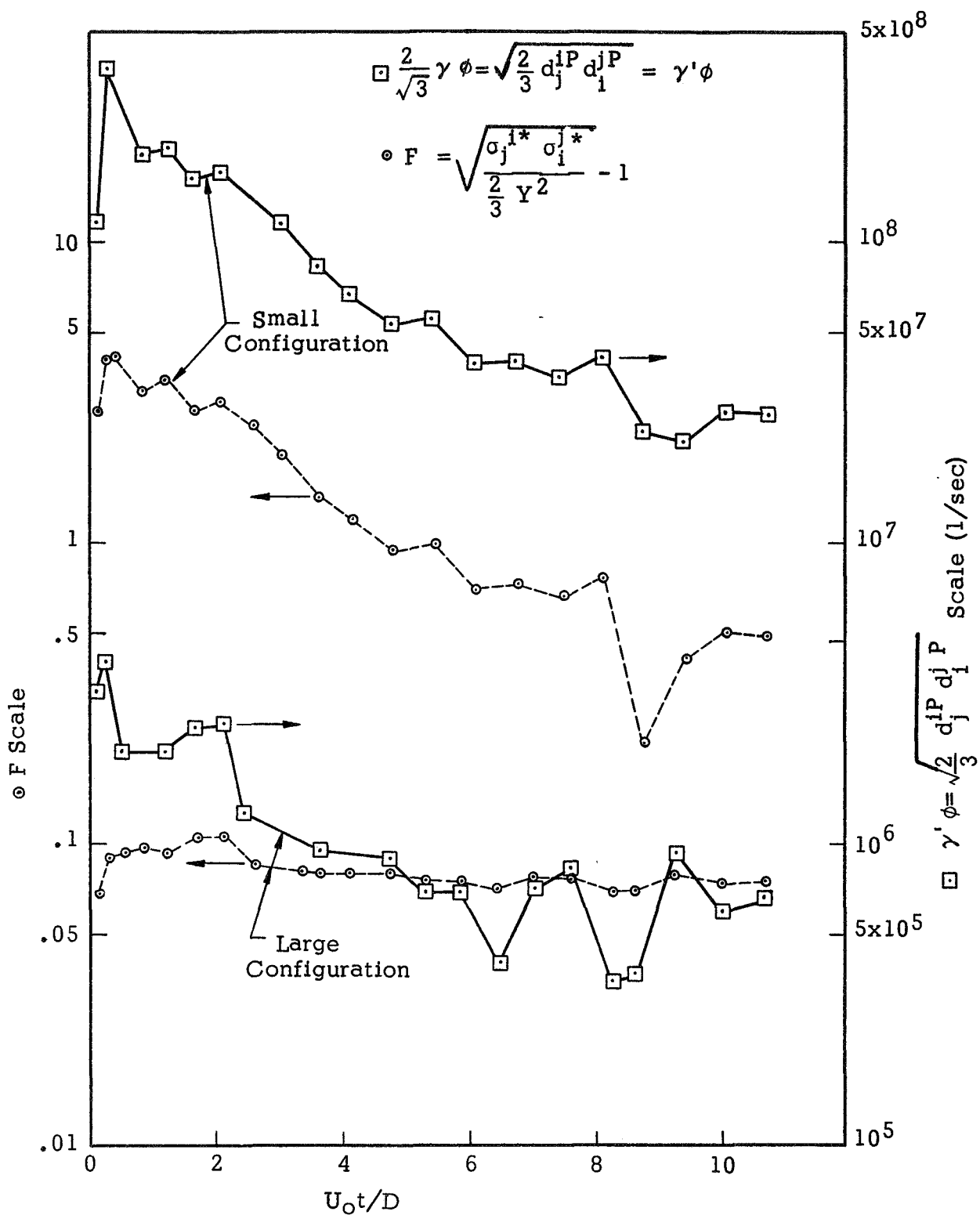


Figure 16. Peak Values of F and $\gamma' \phi$ versus the Scaled Time for the Large (Case 3) and Small (Case 4) Configurations.

TABLE 3. PEAK AXIAL STRESS VERSUS DEPTH AND TIME

Case 3 ($U_0 = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$)					Case 4 ($U_0 = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \times 10^{-2} \text{ cm}$)				
t (μsec)	z (cm)	z/D	$-\sigma_{zz}^{\text{max}}$ (Mb)	t (μsec)	z (cm)	z/D	$-\sigma_{zz}^{\text{max}}$ (Mb)	t (μsec)	$-\sigma_{zz}^{\text{max}}$ (Mb)
.1195	.03175	.1	.75773	.1195 $\times 10^{-2}$	.03175 $\times 10^{-2}$	.1	.76949	.1195 $\times 10^{-2}$	.76949
.2187	.1270	.4	.53917	.2187 $\times 10^{-2}$	.1270 $\times 10^{-2}$	.4	.55604	.2187 $\times 10^{-2}$	.55604
.3619	.2540	.8	.36648	.3619 $\times 10^{-2}$	.2540 $\times 10^{-2}$	.8	.37628	.3619 $\times 10^{-2}$	.37628
.5208	.3810	1.2	.25033	.5213 $\times 10^{-2}$	.3810 $\times 10^{-2}$	1.2	.25455	.5213 $\times 10^{-2}$	.25455
.7146	.5080	1.6	.18145	.6976 $\times 10^{-2}$	.5080 $\times 10^{-2}$	1.6	.18480	.6976 $\times 10^{-2}$	.18480
.9084	.6350	2.0	.13680	.8824 $\times 10^{-2}$	.6033 $\times 10^{-2}$	1.9	.13793	.8824 $\times 10^{-2}$	.13793
1.1022	.7620	2.4	.10692	1.0791 $\times 10^{-2}$	.7303 $\times 10^{-2}$	2.3	.10814	1.0791 $\times 10^{-2}$	.10814
1.3008	.8573	2.7	.08621	1.3047 $\times 10^{-2}$	.8890 $\times 10^{-2}$	2.8	.08640	1.3047 $\times 10^{-2}$	.08640
1.5431	1.0160	3.2	.07083	1.5303 $\times 10^{-2}$	1.0160 $\times 10^{-2}$	3.2	.07052	1.5303 $\times 10^{-2}$	.07052
1.7472	1.1430	3.6	.05900	1.7559 $\times 10^{-2}$	1.1430 $\times 10^{-2}$	3.6	.05742	1.7559 $\times 10^{-2}$	.05742

Case 5 ($U_0 = 1.5 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$)

t (μsec)	z (cm)	z/D	$-\sigma_{zz}^{\text{max}}$ (Mb)
.0862	.0635	.2	1.75050
.2692	.2540	.8	.99104
.5404	.5080	1.6	.37371
.8929	.7620	2.4	.19738
1.0074	.8255	2.6	.16880
1.4242	1.0795	3.4	.11202
1.7991	1.3335	4.2	.08457
2.1533	1.5240	4.8	.06813
2.5195	1.7145	5.4	.05614
2.8868	1.9685	6.2	.04645
3.2720	2.1590	6.8	.04022
3.6572	2.3495	7.4	.02508

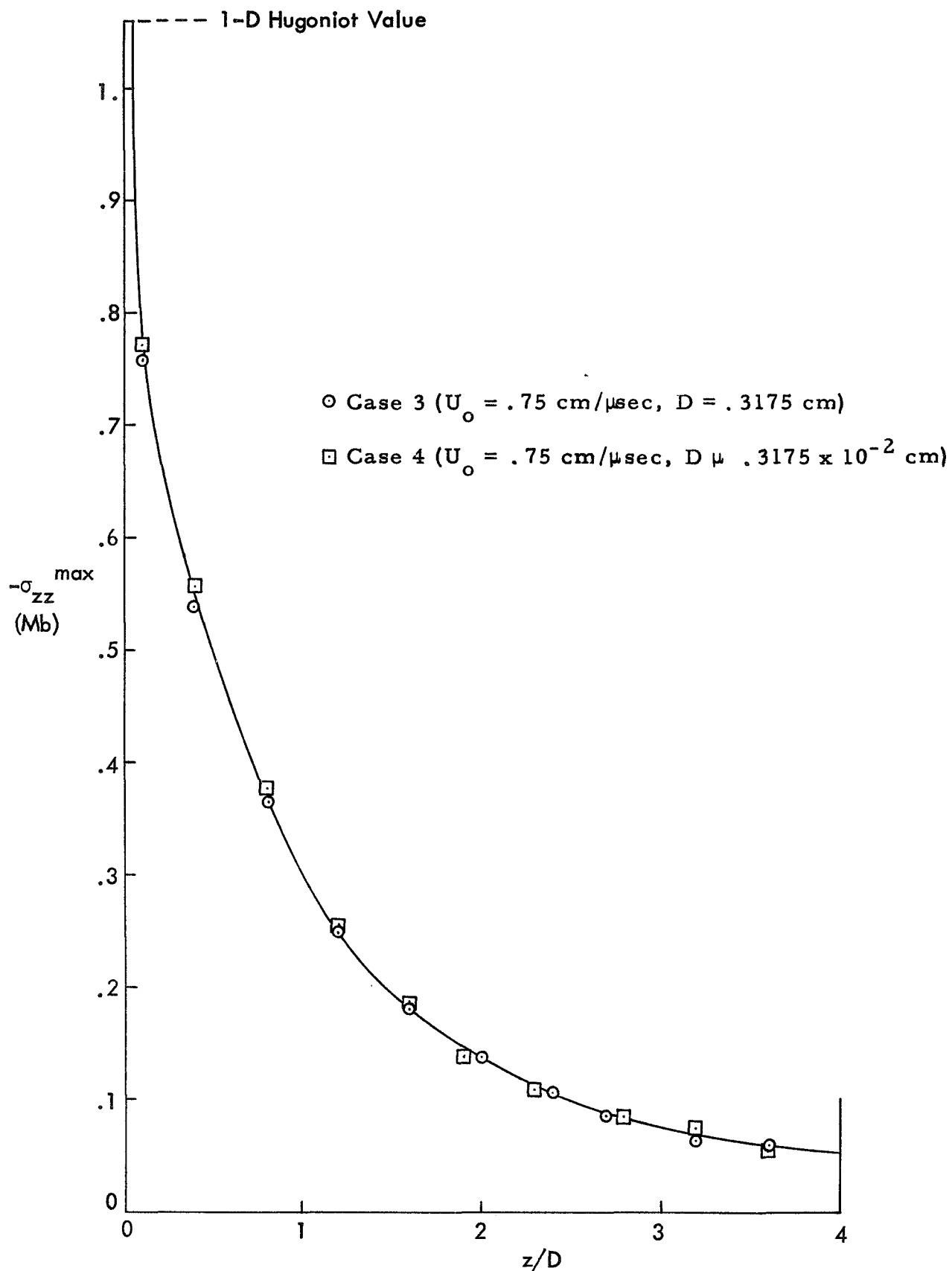


Figure 17. Peak Axial Stress on the Axis as a Function of Scaled Distance for the Large (Case 3) and Small (Case 4) Configurations

of the shock wave decreases, both the peak stress and associated particle velocity decrease in magnitude. By the time the impact shock wave reaches the target rear surface at $z/d = 4$ in both impacts, the peak stress behind the shock front has decayed to 52 kb. The associated particle velocity is .032 cm/ μ sec. The interaction of the shock wave with the target rear surface accelerates that surface to approximately twice the particle velocity, or .064 cm/ μ sec. Relief waves propagating into the target from both the front and rear surfaces subsequently decelerate the rear surface. Figure 18 shows the velocity-time profile of the target rear surface at the axis for both the large and small configurations. At a scaled time of 4.72 (2 μ sec and .02 μ sec in the large and small configurations, respectively), a peak rear surface velocity of .062 cm/ μ sec is attained, or approximately twice the Hugoniot particle velocity in the incident shock. No significant size scaling effects are observed in the target rear surface velocity.

While the stress and velocity fields near the target free surface are similar in the large and small configurations, the strain rates in the two cases are not. To the extent that rear surface fracture (spallation) is rate dependent, the two impacts could significantly differ with respect to whether or not spallation occurs. Fracture and ballistic limit failure phenomena will be discussed in detail in Section 6.

Figure 19 indicates the reason the stress and velocity fields are so similar near the target rear surface. This figure shows the profile of F versus z/D along the axis at a scaled time of 4.1 (about the time the impact shock wave reaches the target rear surface). Recall that F represents the fractional increase of the effective yield strength over the static yield strength $F = \frac{Y_{\text{eff}} - Y}{Y}$. Figure 19 indicates, at $\frac{U_0 t}{D} = 4.1$, the small configuration experiences a 100% ($F = 1$) increase in effective yield strength near the crater bottom but only a 5% ($F = .05$) increase near the target rear surface. This 5% increase in the small configuration is not significantly greater than the 1% increase in the large configuration. Thus the two solutions are similar near the target rear surface.

4.3 SPECIFIC INTERNAL ENERGY CONTOURS AND PENETRATION PREDICTIONS

In addition to the principal stress and velocity field plots, specific internal energy contour plots can be obtained from the numerical solutions. Specific internal energy contours for .0005, .001, .004, .0067, and .01 $\times 10^{12}$ ergs/gm are drawn in Figures 20 and 21 for the large and small configurations. Figure 22 shows the relationship between specific internal energy, material state, and temperature for pure aluminum at one atmosphere.¹⁴

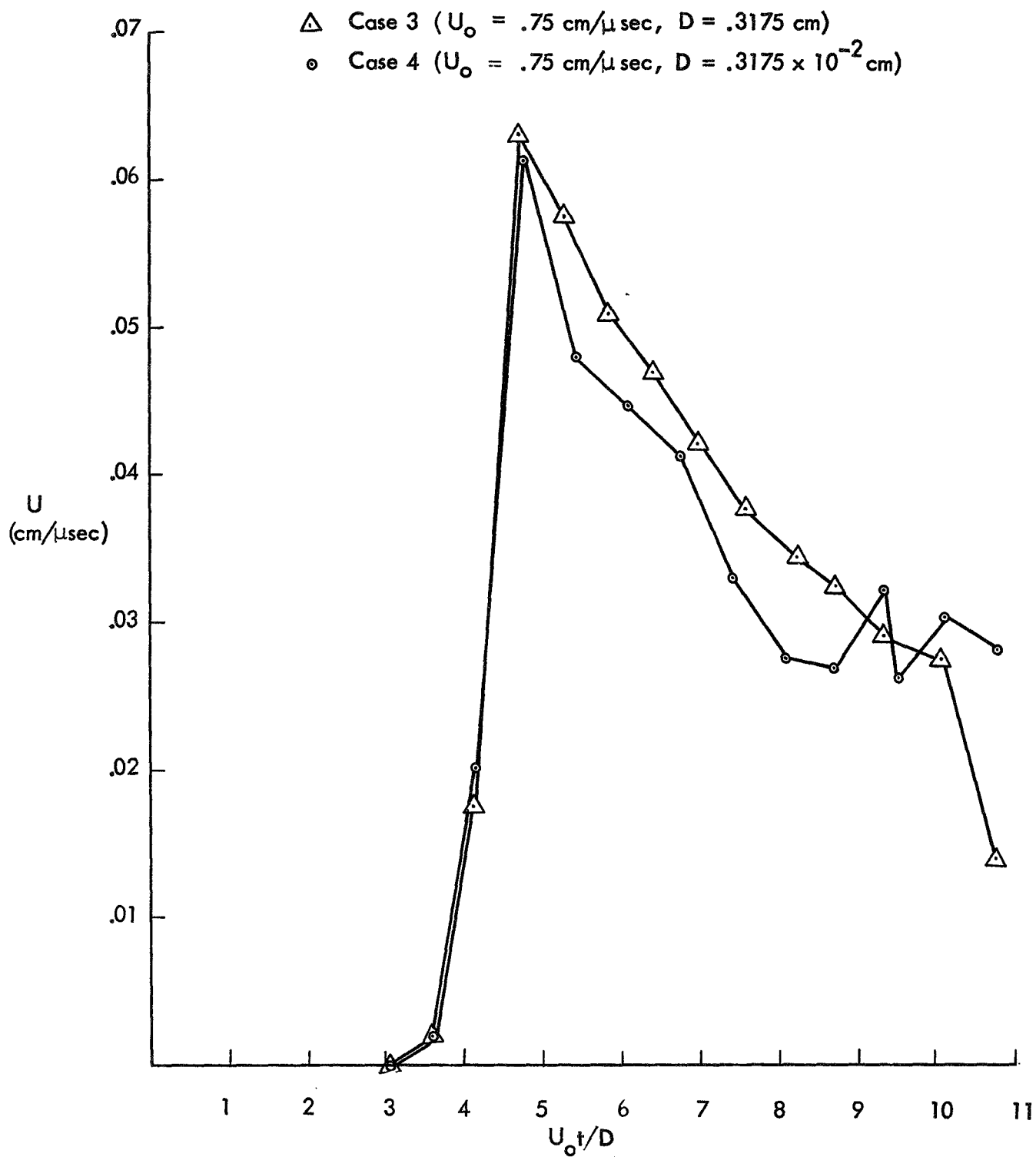


Figure 18. Rear Surface Particle Velocity on the Axis versus Scaled Time for the Large (Case 3) and Small (Case 4) Configurations.

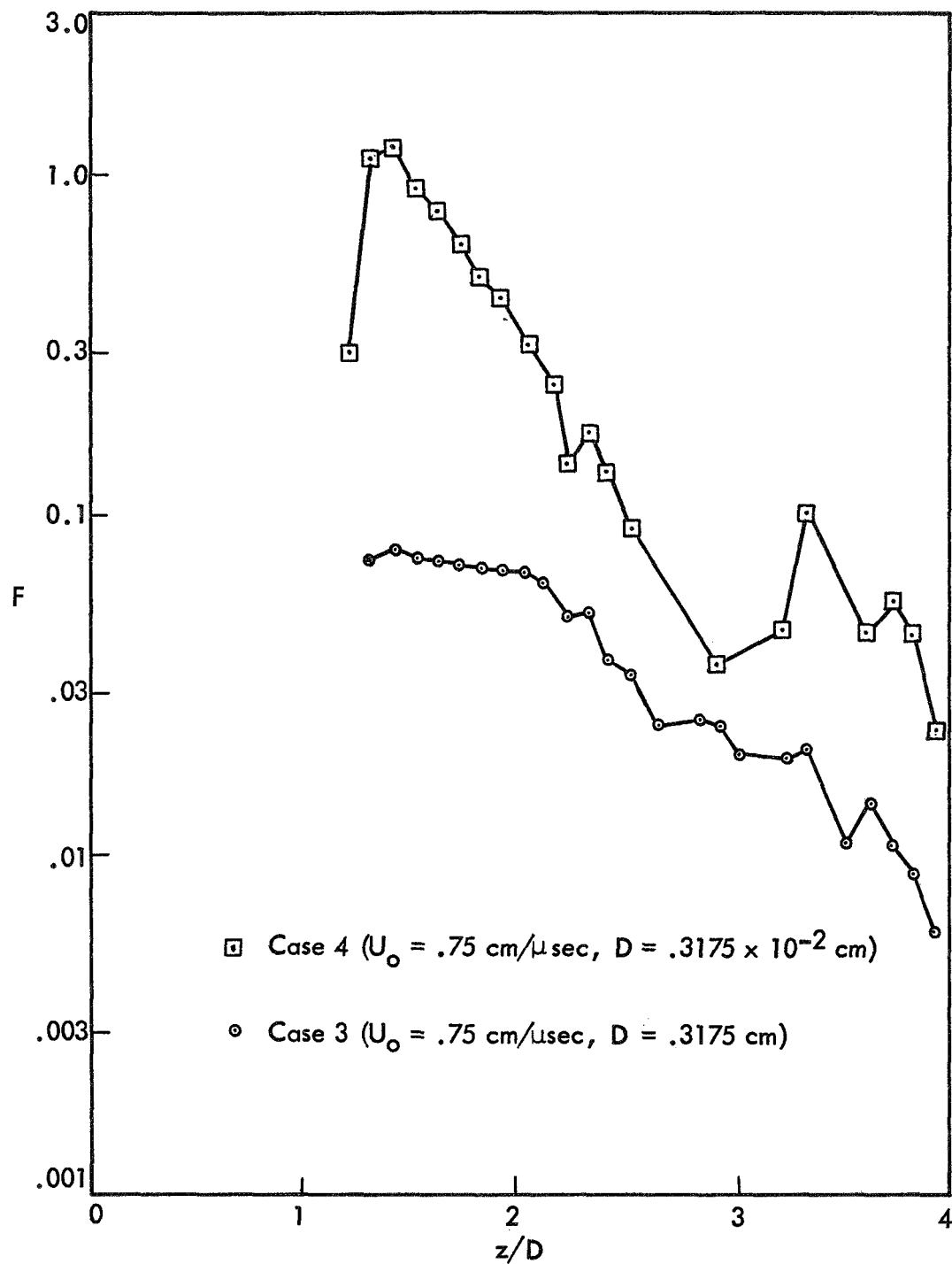


Figure 19. Fractional Increase in the Effective Yield Strength Along the Axis versus Scaled Depth at a Scaled Time of $U_0 t/D = 4.09$ for the Large (Case 3) and Small (Case 4) Configurations.

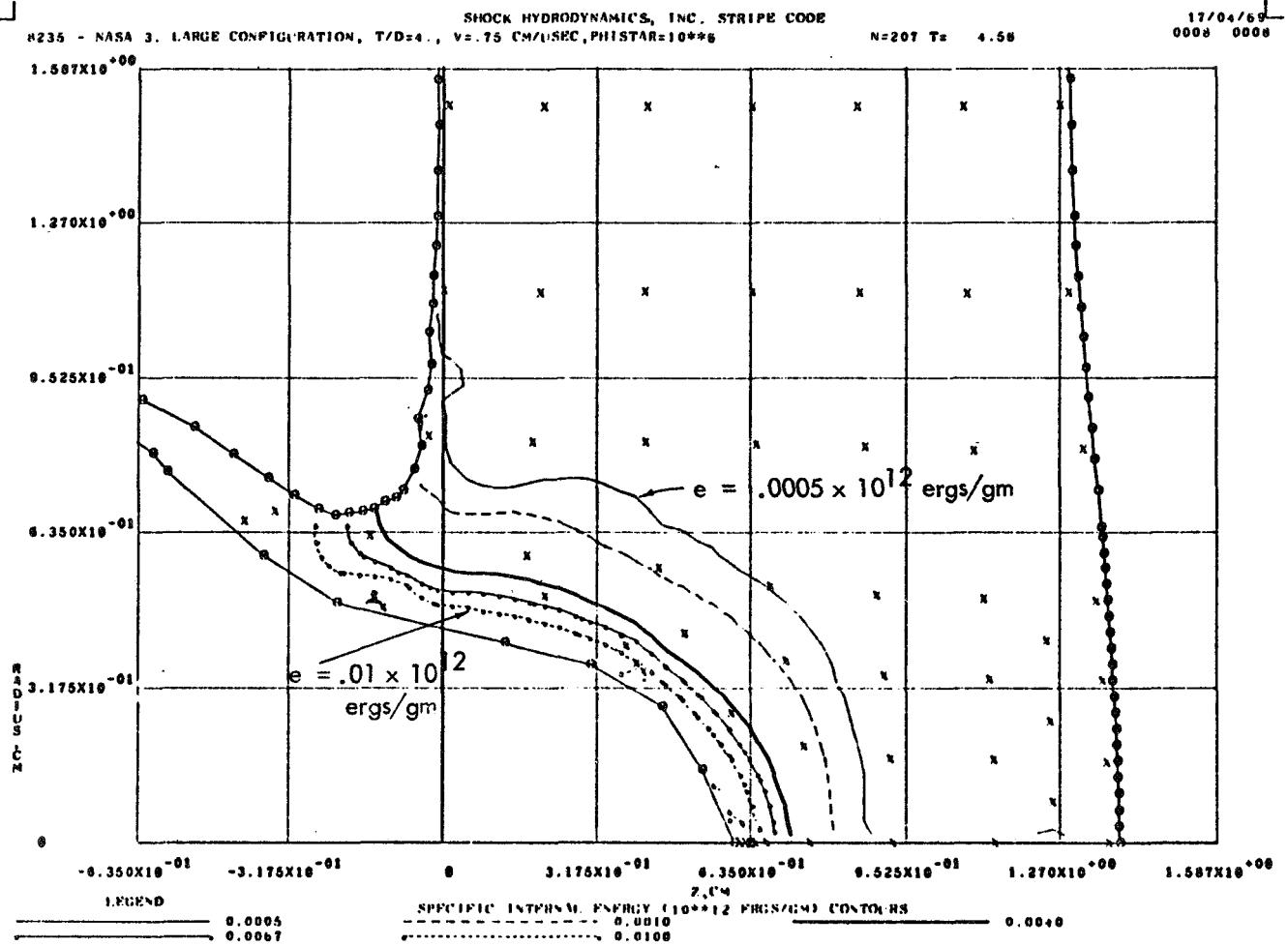


Figure 20. Specific Internal Energy Contour Plot for the Large Configuration (Case 3, $U_0 = .75$ cm/ μ sec, $D = .3175$ cm) at $t = 4.56 \mu$ sec.

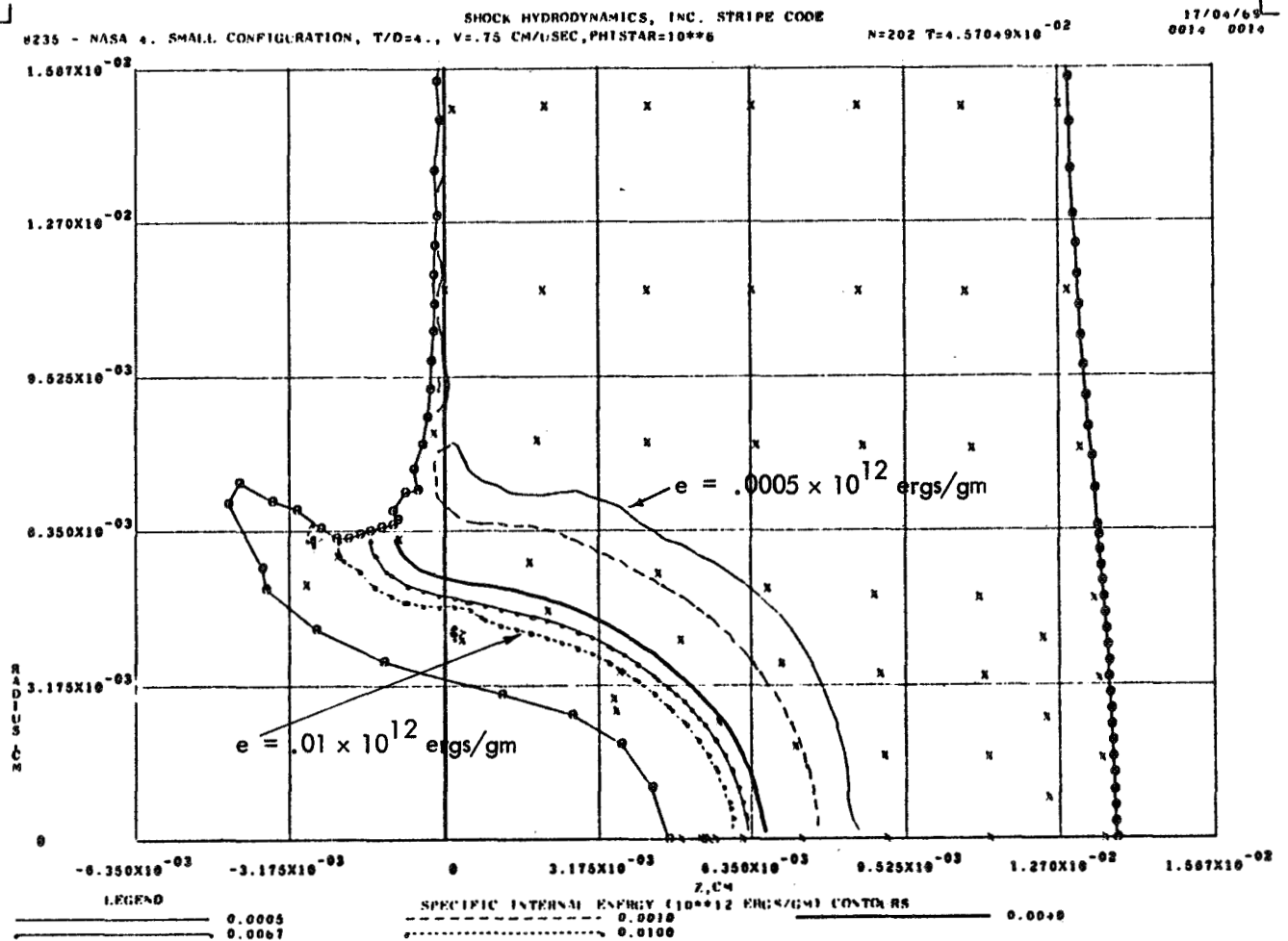


Figure 21. Specific Internal Energy Contour Plot for the Small Configuration (Case 4, $U_0 = .75$ cm/ μ sec, $D = .3175 \times 10^{-2}$ cm) at $t = 4.57 \times 10^{-2}$ μ sec.

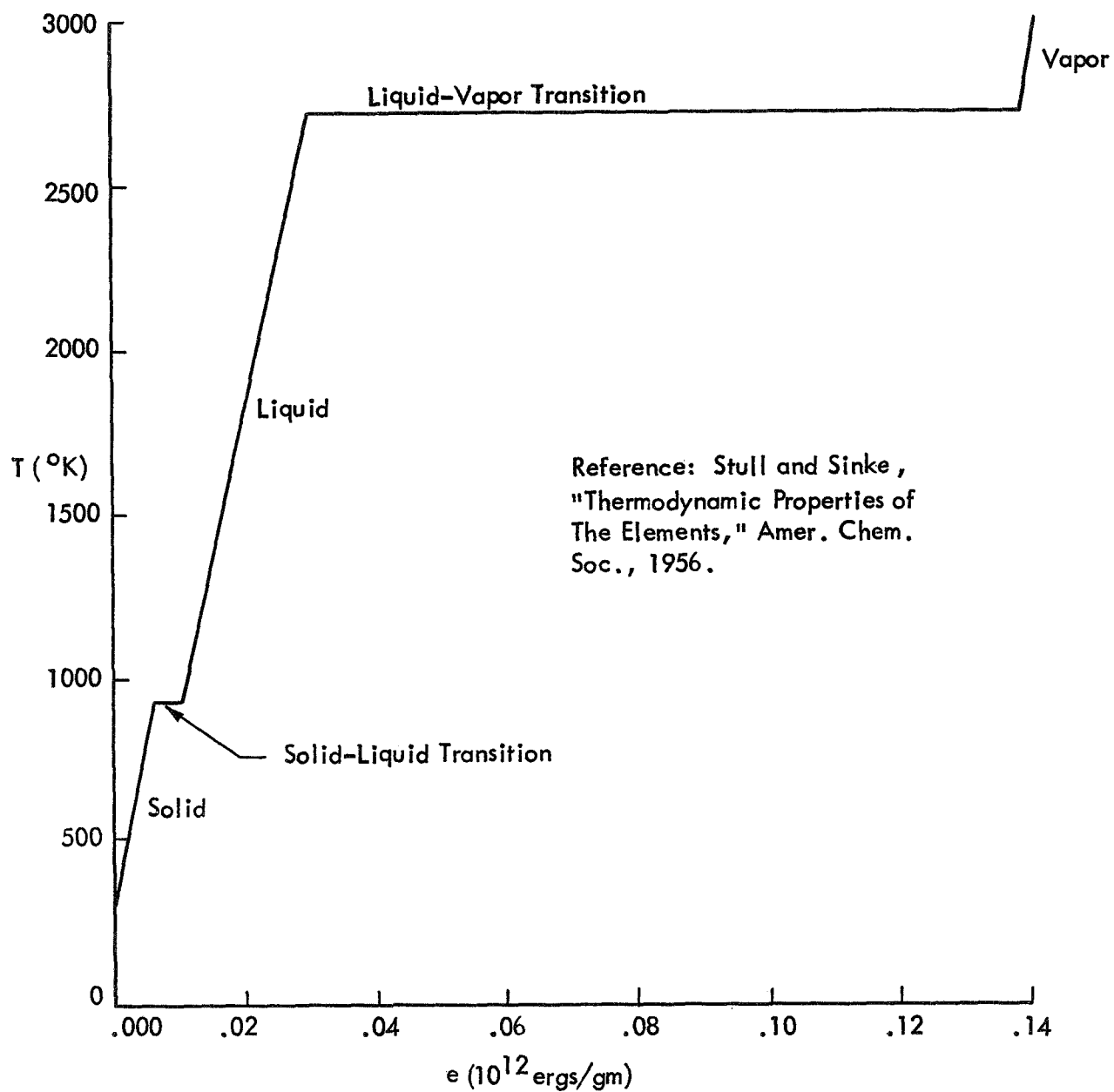


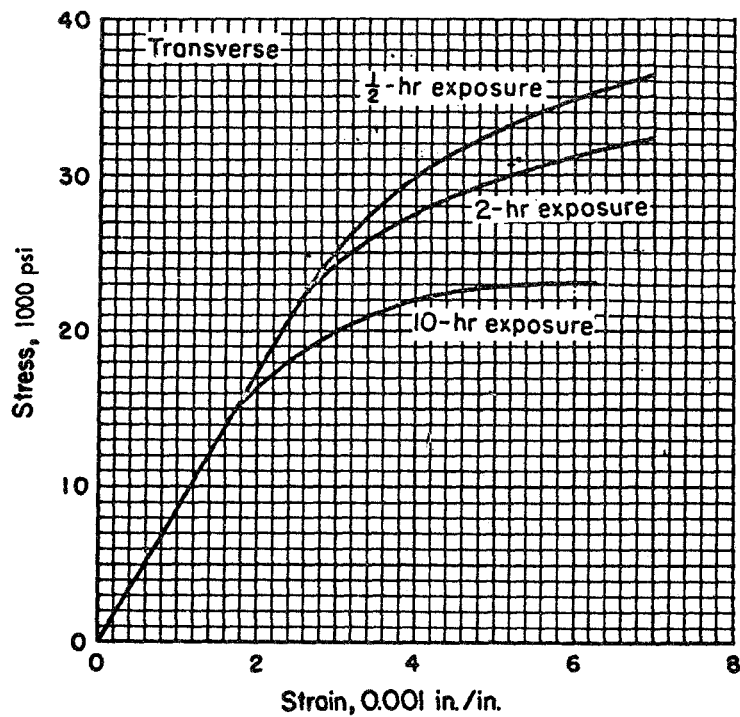
Figure 22. Specific Internal Energy versus Temperature for Pure Aluminum at One Atmosphere of Pressure.

Two energy contours are of special interest. The $.0067 \times 10^{12}$ ergs/gm contour corresponds to the specific internal energy necessary for incipient melting, and the $.01 \times 10^{12}$ ergs/gm contour corresponds to the energy necessary for complete melting (for a pressure of 1 atmosphere). As indicated in Figures 20 and 21, complete melting occurred in the large configuration to a depth of about 2.1 diameters, and in the small configuration to a depth of about 1.9 diameters. However, the peak specific internal energy near the base of the crater is $.01 \times 10^{12}$ ergs/gm in the large configuration as compared to $.028 \times 10^{12}$ ergs/gm in the small configuration. The internal energies near the crater boundary are greater in the small configuration due to the smaller final deformations and to the fact that less work is expended on material expansion in the small configuration (i. e. , more of the shock energy remains in the form of specific internal energy in the small configuration). Farther from the crater, the deformations are relatively small in both configurations and the specific internal energy contours are nearly identical.

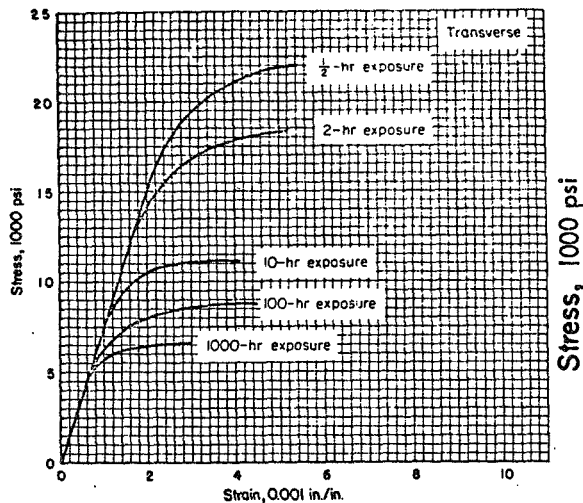
In the present study, the strength of the material (measured by the yield value Y , the shear modulus μ , and the fracture strength) was not made a function of temperature or pressure. At high stress levels (> 1 Mb), variations in strength due to these factors are not important since the stresses are essentially hydrodynamic in nature. However, at lower stress levels these factors may become important. Some material strength data as a function of temperature are available. These data show that the duration of the temperature exposure is important in determining the degradation of material strength. For example, Figure 23 shows quasi-static stress versus strain curves for various exposure times on 2024-T3 aluminum at temperatures of 500, 600, and 700°F¹⁵. The exposure time refers to the duration of the temperature before the stress-strain test is performed. In general, the following two qualitative relationships hold for the material yield strength, temperature, and duration:

1. for a fixed exposure time (duration) the static yield value (Y) decreases with an increase in temperature, and
2. for a fixed temperature the static yield value (Y) increases with a decrease in exposure time.

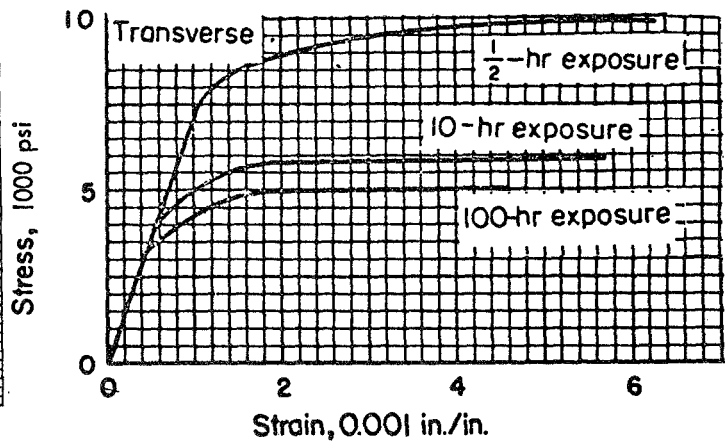
Unfortunately, the experimental exposure times considered were from 1/2 hour to 1000 hours, many orders of magnitude longer than the durations of interest in hypervelocity impact experiments (i. e. , 5 to 10 μ sec for a .3175 cm dia projectile impact). Thus, the amount of yield strength decay for a duration experienced in a hypervelocity impact event is expected to be much smaller than the corresponding decay measured for durations of 1/2 hour or more.



Typical compressive stress-strain curves for clad 2024-T3 aluminum alloy (sheet) at 500 F.



Typical compressive stress-strain curves for clad 2024-T3 aluminum alloy (sheet) at 600 F.



Typical compressive stress-strain curves for clad 2024-T3 aluminum alloy (sheet) at 700 F.

Figure 23. Quasi-Static Stress versus Strain Curves for Various Exposure Times at 500, 600 and 700° F for 2024-T3 Aluminum¹⁵.

Additional experimental data are needed to determine the yield strength as a function of temperature, pressure, and duration. Once such data are available, then an appropriate yield model which depends on temperature, pressure, and duration can be included in the numerical calculations. However, in the present calculations, the yield strength was assumed to be independent of temperature, pressure, and duration since the durations of interest were so small. This assumption, of course, is not valid for liquid or vapor states, and thus the model overestimates the strength of the melted material near the crater boundary. The extent of the melted region along the axis for the large and small configurations is indicated in Figure 24.

Figure 24 shows the specific internal energy versus the scaled depth along the axis for Cases 3, 4, and 5. The impact velocity effects evident in this figure, by comparing Case 3 ($U_o = .75 \text{ cm}/\mu\text{sec}$) to Case 5 ($U_o = 1.5 \text{ cm}/\mu\text{sec}$) will be discussed in Section 5. The size scaling effects relative to specific internal energy are also evident by comparing Case 3 ($D = .3175 \text{ cm}$) to Case 4 ($D = .3175 \times 10^{-2} \text{ cm} = 31.75 \mu$). The energies are higher in the small configuration. However, in both cases, the specific internal energies are sufficiently high to cause melting down to some depth below the crater bottom. The depths at which incipient melting occurs for the various cases are indicated in Table 4.

TABLE 4. DEPTHS FOR INCIPIENT MELTING AND PREDICTED PENETRATION

Case	U_o cm/ μ sec	D cm	Depth P cm	P/D
3	.75	.3175	.69	2.17
4	.75	$.3175 \times 10^{-2}$	$.62 \times 10^{-2}$	1.95
5	1.5	.3175	.97	3.06

For depths greater than indicated in Table 4, the material remains solid. However, at the shallower depths, the material experiences a phase change (solid-to-liquid) and the material strength is expected to nearly vanish. When this occurs all non-solid material may be expected to flow out of the crater, with perhaps a very thin layer of melted material remaining to coat the solid crater walls.

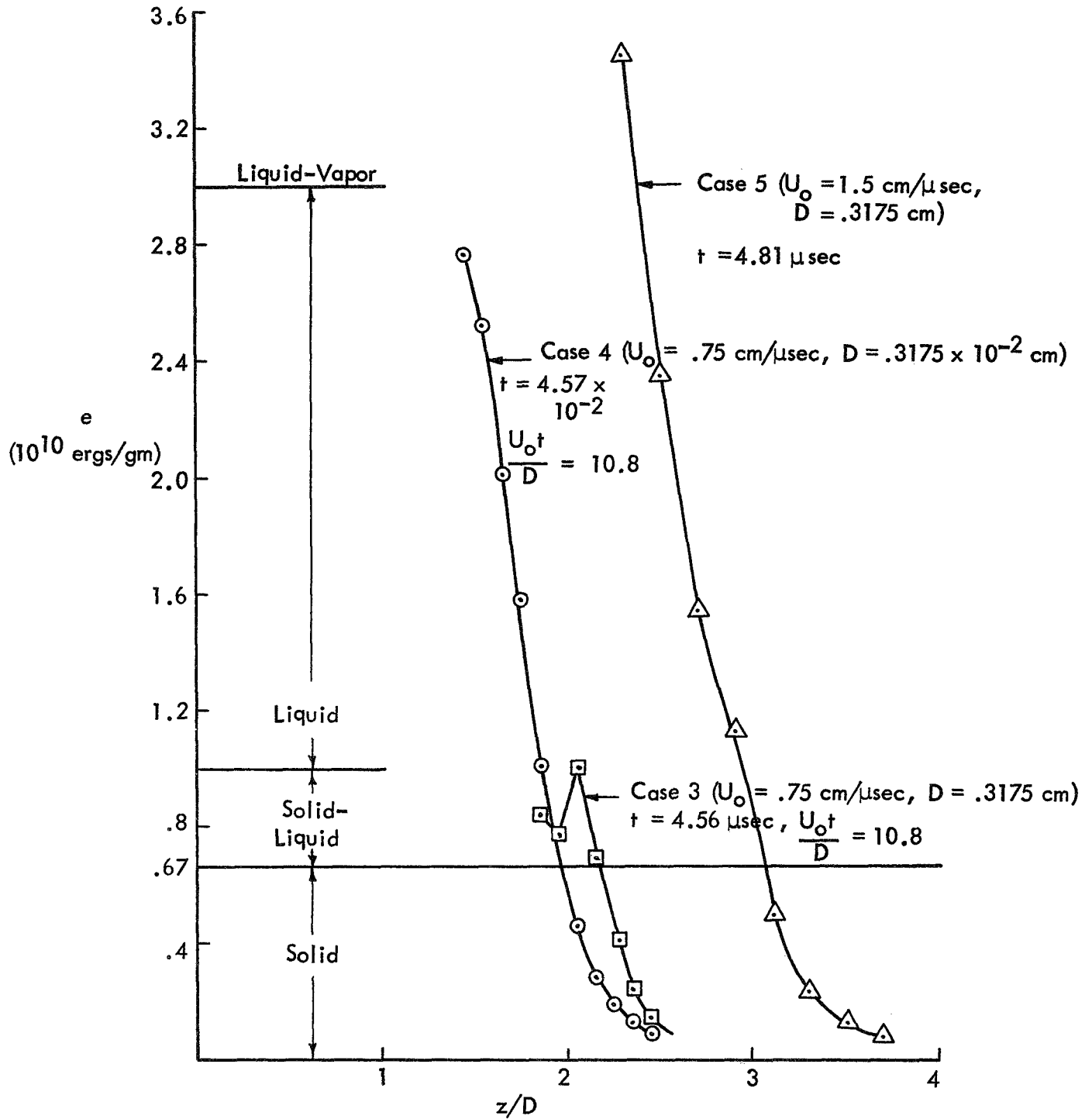


Figure 24. Specific Internal Energy versus Scaled Depth Along the Axis for Cases 3, 4, and 5.

The final crater depth prediction therefore is the depth corresponding to incipient melting. Thus, the penetration depth (P) predicted by the numerical solutions are indicated in Table 4. The size scaling effect is evident. The small configuration impact case (Case 4) has a smaller penetration depth than would be predicted by linear size scaling on the basis of the large configuration impact case (Case 3).

Section 5

IMPACT VELOCITY EFFECTS

Comparison of Impact Cases 3 and 5 illustrates the effects of varying the impact velocity. In Case 3, the impact velocity was .75 cm/ μ sec, while in Case 5 the impact velocity was 1.5 cm/ μ sec.

The initial pressures, particle velocities, densities and specific internal energies which develop near the impact point are determined by the impact velocity, according to the material shock Hugoniot characteristics. Table 5 lists these variables for Cases 3 and 5.

TABLE 5. STATE VARIABLES ON THE HUGONIOT

	Particle Velocity (cm/ μ sec)	Pressure (Mb)	Compression (ρ_H / ρ_o)	Energy (10^{12} ergs/ gm)
Case 3 ($U_o = .75$ cm/ μ sec)	.375	1.06	1.56	.0703
Case 5 ($U_o = 1.5$ cm/ μ sec)	.75	3.12	1.95	.2813

The initial state variables are, of course, much higher for the higher impact velocity case and these higher pressures and particle velocities produce greater deformation and more rapid crater growth. The crater and splash boundary profiles are compared for the two cases at two times in Figure 25. Note that the higher impact velocity case has a deeper open crater and a larger ejecta mass above the original surface.

The time resolved details of the crater growth on-axis are compared in Figure 26. The higher impact velocity case shows a more rapid crater growth at early times. However, note that both cases reach their maximum depth at approximately the same time. The penetration depths indicated on Figure 26 are not the final predicted crater depths, inasmuch as additional growth will result from the residual melting of material near the crater boundaries. For example, in the 1.5 cm/ μ sec impact

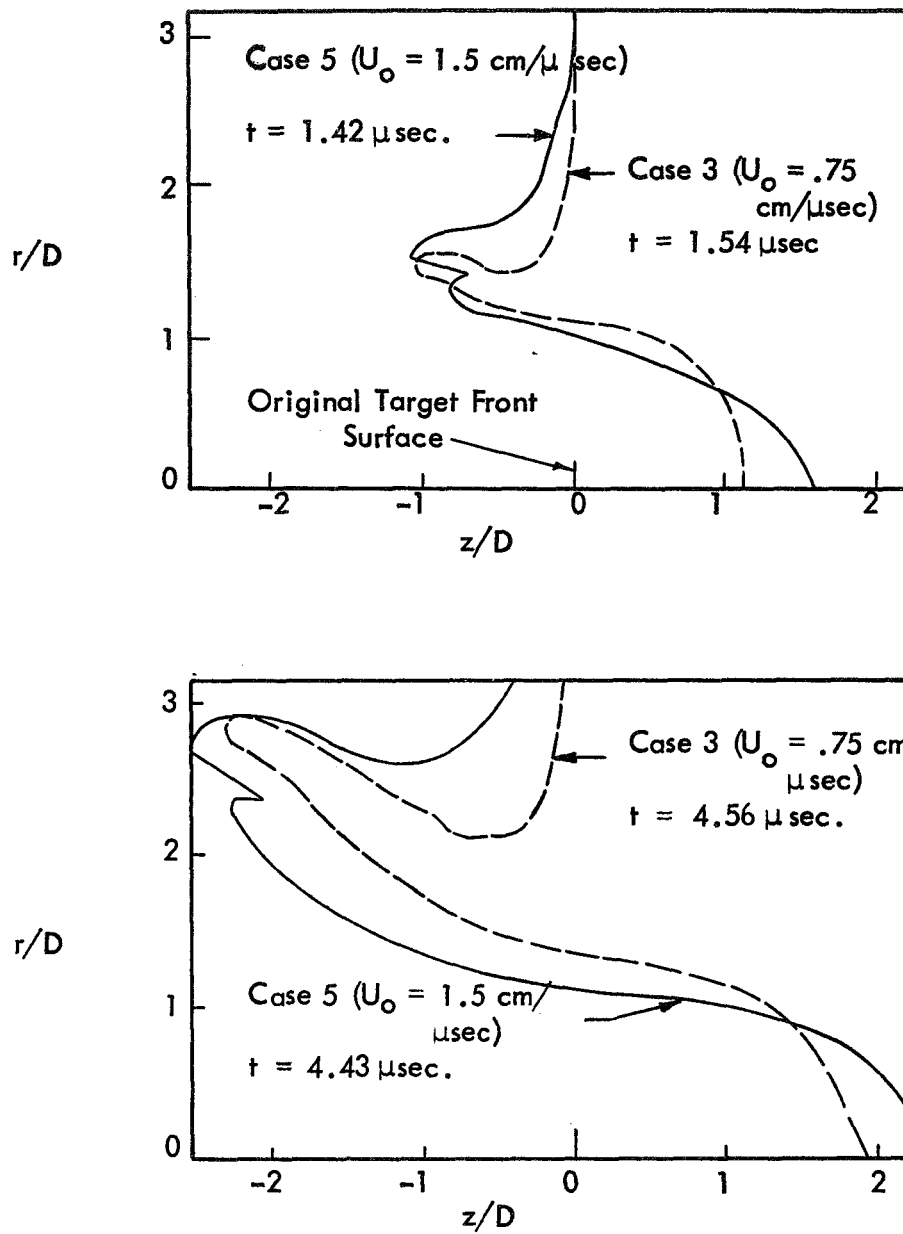


Figure 25. Comparison of Crater and Splash Profiles for the $.75 \text{ cm}/\mu\text{sec}$ and the $1.5 \text{ cm}/\mu\text{sec}$ Impact Velocity Cases, $D = .3175 \text{ cm}$.

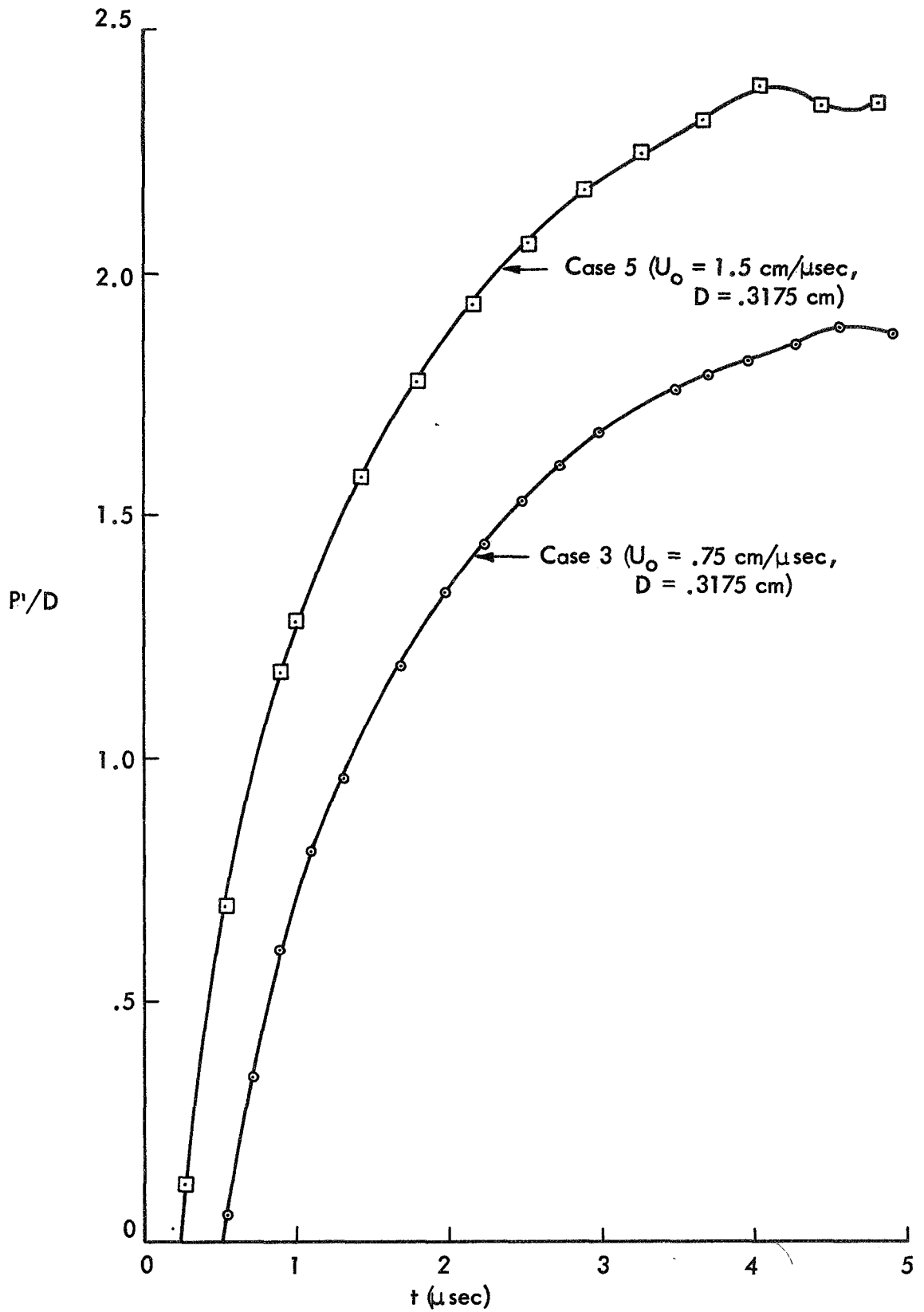


Figure 26. Scaled Penetration Depth (P'/D) versus Time for the $.75 \text{ cm}/\mu\text{sec}$ and $1.5 \text{ cm}/\mu\text{sec}$ Impact Cases ($D = .3175 \text{ cm}$) Not Including Thermal Softening Effects.

case, the material near $P/D = 2.4$ is in the liquid-vapor state. Such material has no strength and will surely flow out of the crater. The final penetration depth (P) is assumed to be at the maximum depth at which incipient melting occurs. Figure 27 is the Case 5 specific internal energy contour plot at the last integration cycle. This plot can be compared with the corresponding plot for Case 3 in Figure 20. The specific internal energy along the axis is indicated in Figure 24. The peak residual specific internal energy in the high velocity case is 3.4×10^{10} ergs/gm as compared to 1×10^{10} ergs/gm in the low velocity case. These energies correspond to release temperatures of 2720°K and 913°K , respectively. Also, the residual specific internal energies (energies at 1 atmosphere) at any depth are greater for the $1.5 \text{ cm}/\mu\text{sec}$ impact case as compared to the corresponding $.75 \text{ cm}/\mu\text{sec}$ impact case. The energy level necessary to produce incipient melting therefore occurs at a greater depth in the high velocity case. As indicated in Table 4 in Section 4.3, the $1.5 \text{ cm}/\mu\text{sec}$ impact velocity case resulted in a normalized penetration depth (P/D) of 3.06 as compared to $P/D = 2.17$ for the corresponding $.75 \text{ cm}/\mu\text{sec}$ impact velocity case.

5.1 STRESS ON THE AXIS

Figure 28 shows the peak axial stress in the traveling shock wave as a function of distance along the axis. (The numerical peak stress data are contained in Table 3.) At any distance, the stress remains greater in the higher impact case and therefore the residual energies will be greater. Similarly, Figure 28 shows that the stress decays to a given value at a greater depth in the higher impact velocity case. For example, in the low impact velocity case the stress decayed to 60 kb at a depth of $z/D \cong 3.6$, while in the high impact case, the stress decayed to 60 kb at a depth of $z/D \cong 5.4$.

As a first approximation (which will be improved in Section 6) it seems reasonable to hypothesize that a free surface placed at $z/D = 5.4$ in the high velocity case would respond in a similar fashion to a free surface placed at $z/D = 3.6$ in the low velocity case. If this is indeed true, then incipient failure of the target (the ballistic limit) could be predicted for any impact velocity. The predictions would rely on experimental data to determine the peak stress incident on the target rear surface at incipient failure (this peak stress could also be obtained using a numerical calculation of the ballistic limit configuration). Then, ballistic limits could be determined by performing numerical solutions of other impacts at various velocities into semi-infinite targets. Such solutions would determine the equivalent depth at which the stress decays to the experimentally determined value.

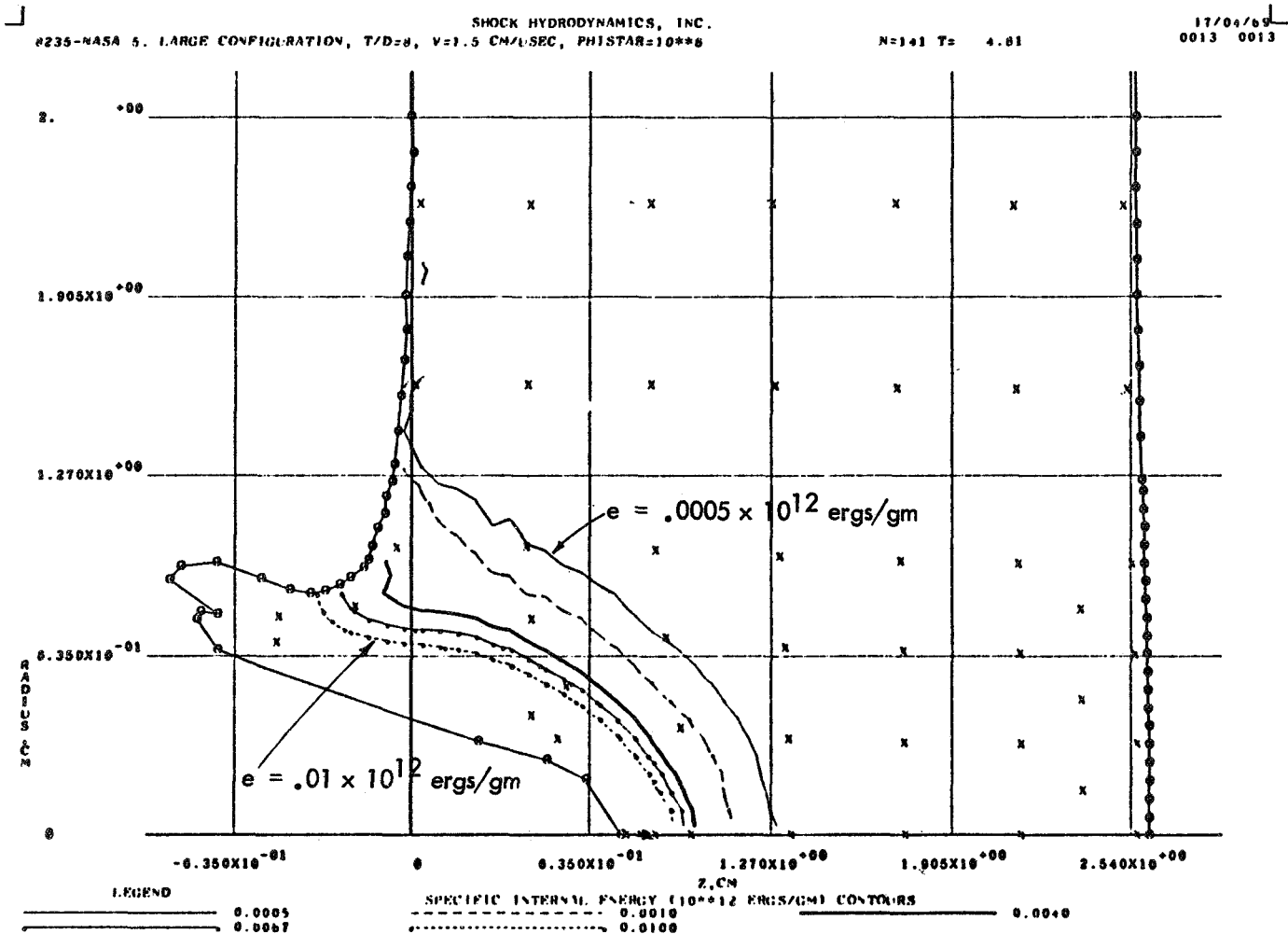


Figure 27. Specific Internal Energy Contour Plot for the High Impact Velocity Case (Case 5, $U_0 = 1.5$ cm/ μ sec, $D = .3175$ cm) at $t = 4.81$ μ sec.

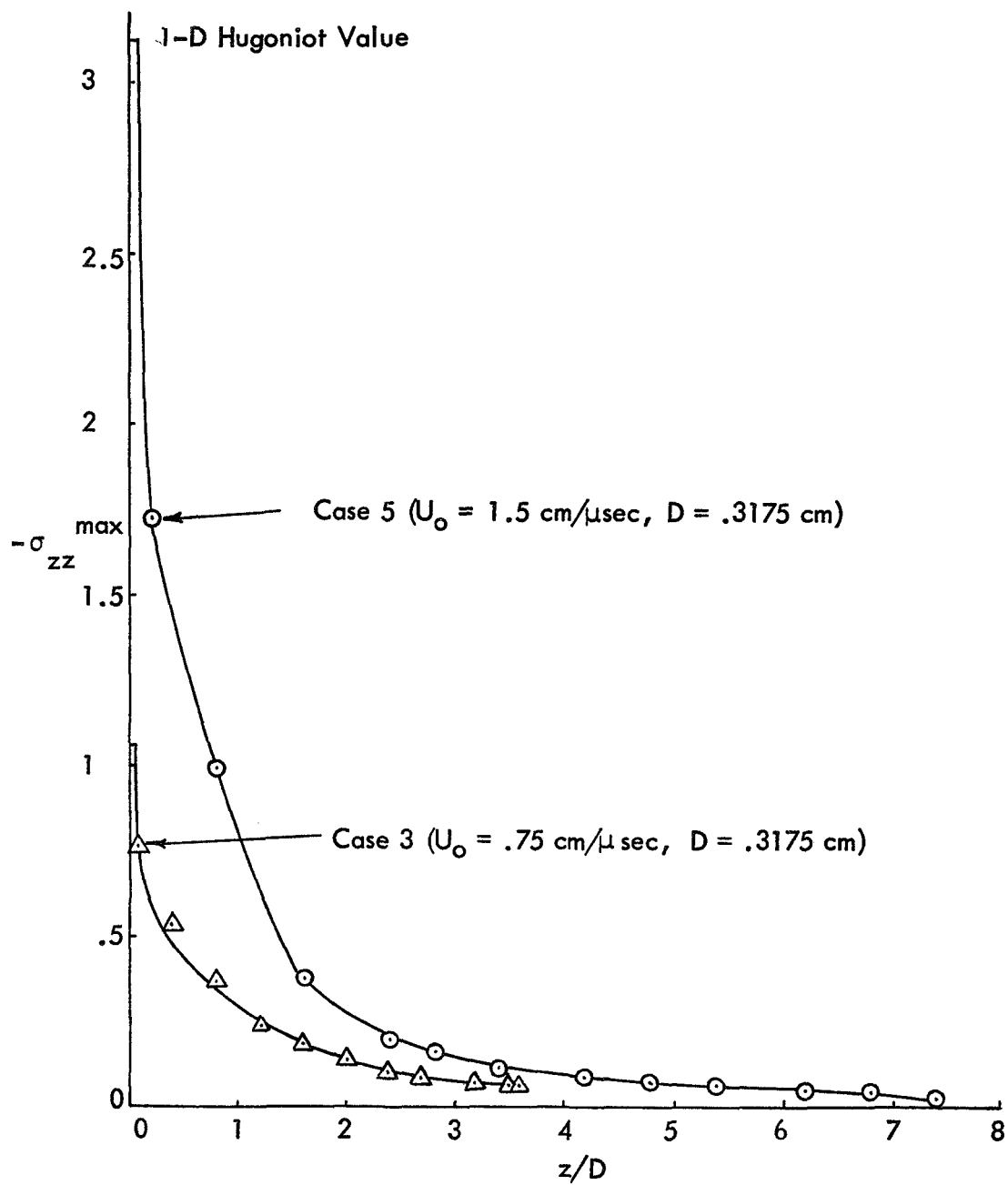


Figure 28. Peak Axial Stress on the Axis as a Function of Scaled Depth for Cases 3 and 5.

In the next section, we examine the similarity of the Case 3 and Case 5 numerical solutions near corresponding depths where the stress has decayed to about 60 kb. (60 kb was chosen because in Case 3 this was the last peak stress level available from the numerical solution before the shock wave interacted with the target rear surface.)

5.2 COMPARISONS OF STRESS AND VELOCITY FIELDS AT "EQUIVALENT DEPTHS"

From Figure 28, the depths at which the peak stress drops to about 60 kb on the axis can be determined. The term "equivalent depths" will be used to refer to depths where a given peak stress level occurs in different velocity impacts. The equivalent depths for 60 kb are $z/D = 3.6$ and $z/D = 5.4$ for the low and high impact velocity cases, respectively. Figures 29 and 30 show quantitative comparisons of the stress and velocity at times corresponding to the arrival of the peak stress at the indicated depth. The entire stress and velocity fields at these times can be examined and compared in Figures 9 and C-25.

Figure 29 shows a comparison for the two cases of axial stress versus radius at the two equivalent depths of interest. Figure 30 is the corresponding comparison of the axial particle velocity versus radius. Figures 29 and 30 demonstrate that the stress and particle velocity characteristics at equivalent depths are nearly the same out to radii of about $1/2$ a projectile diameter. At greater radii, the stress and velocity are greater for the higher impact velocity case.

A significant difference between the two impact velocity cases is illustrated in Figure 31, which shows the axial stress profiles along the axis at various times for both impact velocities. The detached stress wave pulse is evident in the stress profiles. Note that the pulse width is larger at equivalent depths for the higher impact velocity case. Thus, target rear surfaces placed at equivalent depths for different impact velocity cases will behave differently due to variations in the incident stress wave pulse width.

The equivalent depth concept for determining incipient failure thicknesses at various impact velocities can thus be used only as a first order approximation. As Figures 29, 30, and 31 indicate, there are quantitative differences in the stress and velocity fields at equivalent depths. In particular, the increase in the shock pulse width with impact velocity will be important in modifying the first order equivalent depth approach to ballistic limit determinations. Such equivalent depth predictions will underestimate the thicknesses for incipient failure at high impact velocities. The low estimate would result from not including the effects of the greater shock pulse width. In Section 6, the effects of stress pulse width on ballistic limits will be examined.

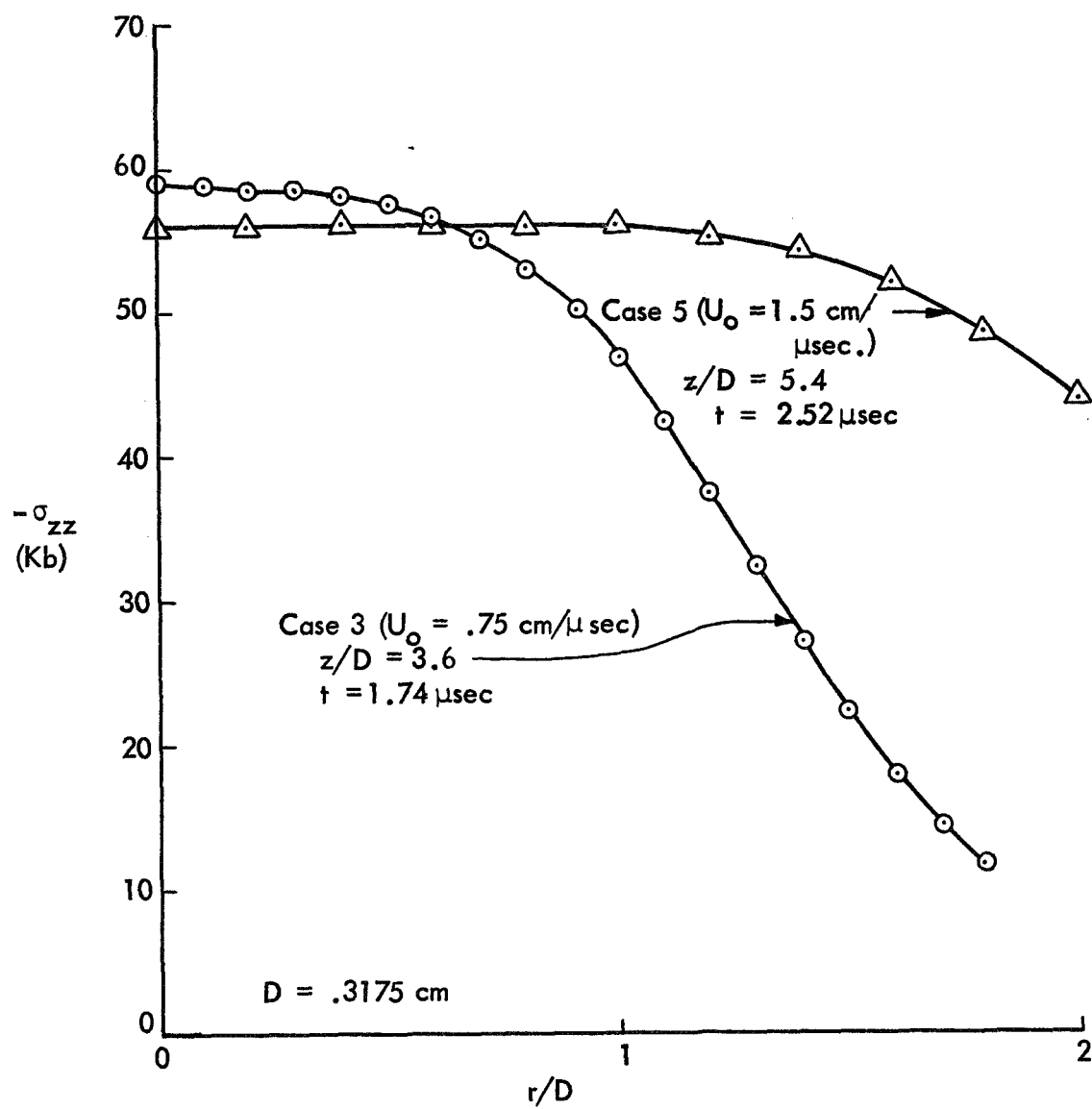


Figure 29. Axial Stress versus Scaled Radius at Approximately "Equivalent Depths" Corresponding to a Stress of ~ 60 Kb.

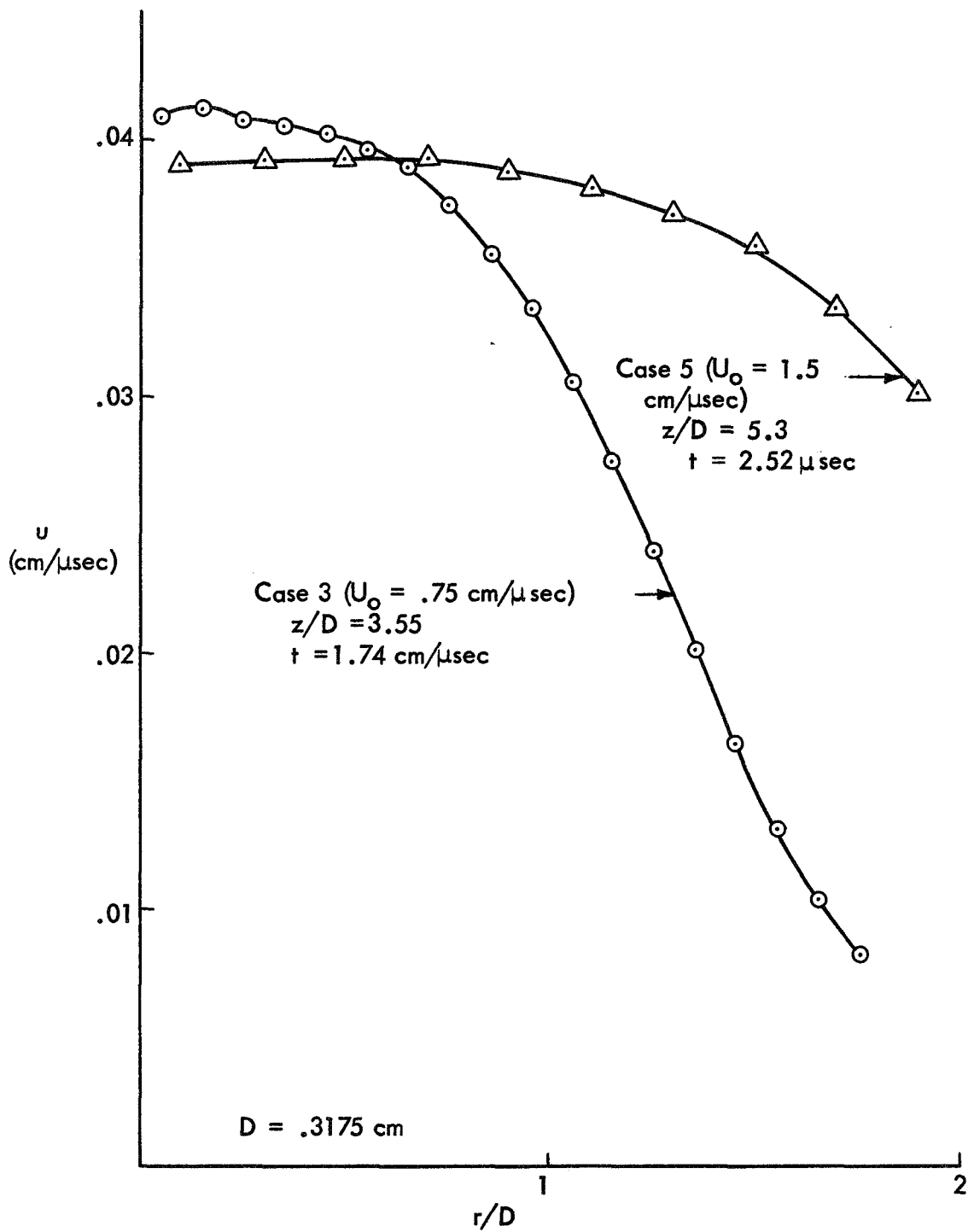


Figure 30. Particle Velocity versus Scaled Radius at Approximately "Equivalent Depths" Corresponding to a Stress of $\sim 60 \text{ Kb.}$

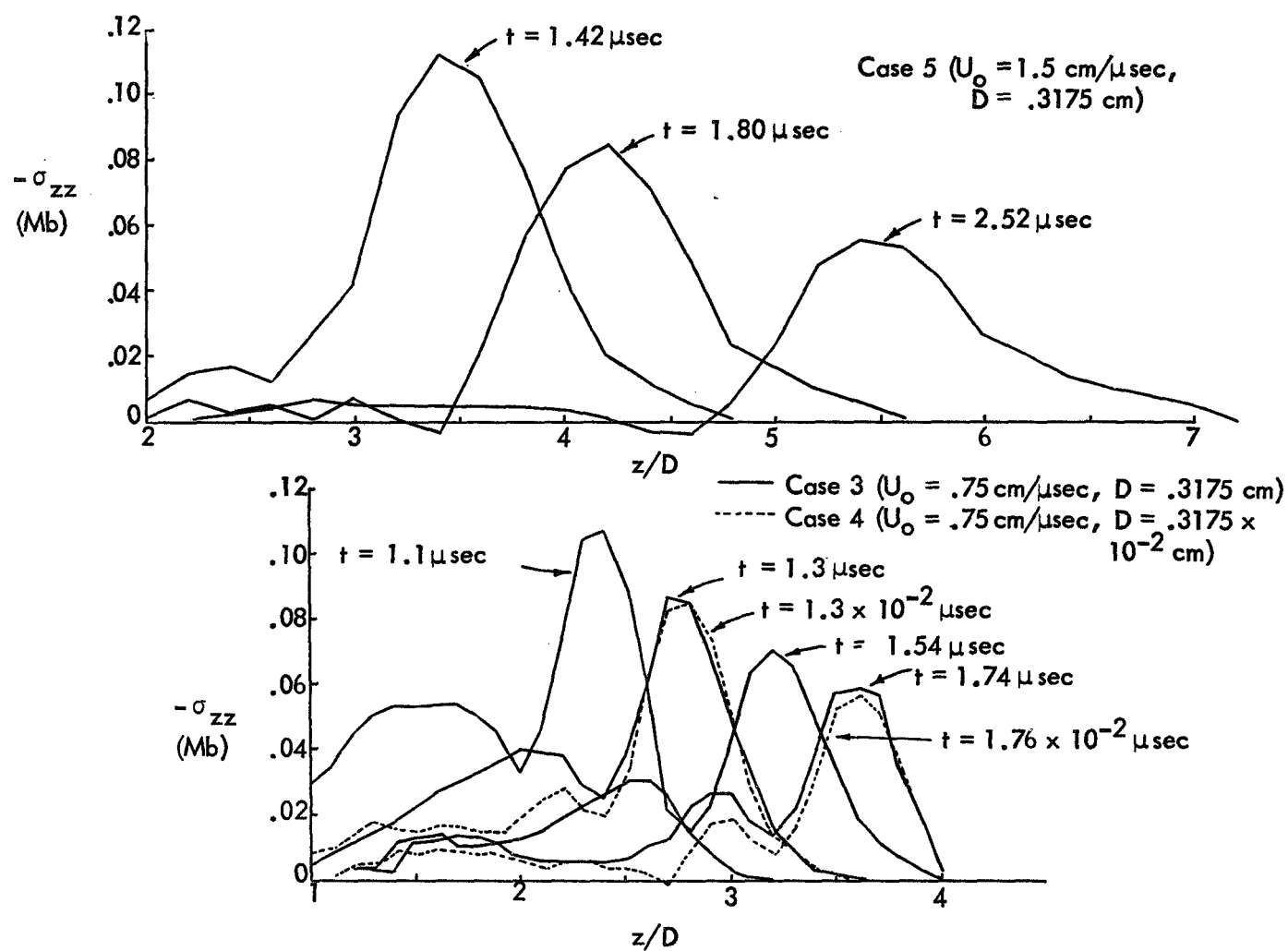


Figure 31. Axial Stress versus Scaled Distance at Various Times for Cases 3, 4, and 5.

Section 6

FRACTURE

6.1 THE DYNAMIC TENSILE STRENGTH OF METALS

Fracture phenomena depend not only on the material properties, but also upon the dynamics of the stress field, including the nature and duration of wave interactions. While the wave propagation aspects of material failure can be handled by computer codes, a major problem exists in the establishment of criteria for the onset and propagation of fracture which are both physically meaningful and compatible with the numerical technique. This problem arises largely because the strength of a material is not a fixed or definite property of the material, but rather depends on the temperature and pressure of the material, and on the duration (and rate of change) of the applied load. That is, the fracture of a material is a rate process, and in general the effective strength of a material increases with a decrease in the duration of the loading. The relative importance of the rate aspects of fracture to a specified problem depends, of course, on the specific material involved, and on the characteristics of the wave motion. Thus, for many situations the rate aspects of fracture are of no consequence, and some classical quasi-static criterion, such as the octahedral plastic strain criterion, can be used to specify the onset of material failure. On the other hand, for most situations involving shock (impulsive) loading, the tensile strength of the material is a sensitive function of the time-dependent properties of the shock wave interactions. For example, spallation studies involving the dynamic tensile strengths of metals have been reported in References 16 to 23.

In particular, recent studies of the dynamic tensile strength of metals subjected to explosive loading, and utilizing flash radiographic techniques to measure spall thickness, have been conducted at the Los Alamos laboratories²⁰. Near-triangular stress waves were used, and the results were correlated on the basis of stress gradient. The fracture stress of copper, lead, and aluminum were found to correlate with the square root of the fracture stress gradient, i. e. ,

$$\sigma_f = A (\partial \sigma / \partial x)^{\frac{1}{2}} + B \quad (9)$$

A and B are experimental constants.

Spallation studies have also been conducted during the past few years at Sandia²¹⁻²³. These investigators discussed the relationship between the time duration of the tension that is produced by the reflection

of triangular and square shock waves at a free surface. They obtained time-dependent spallation data on 6061-T6 aluminum using flat topped waves, and compared it to the 1100 aluminum data²⁰ by expressing the stress gradient (Eq. 9) in terms of time. The fracture stress was found to increase with decrease in loading time²³. The data on the 6061-T6 and 1100 aluminum samples differed, and interestingly enough the softer and weaker 1100 Al (static tensile strength about 15,000 psi) was stronger than the harder 6061 Al (tensile strength about 40,000 psi) under short time impact loads. This difference was largely attributed to differences in experimental conditions, but the effects may be real. The crossing of the two individual data curves of the high impulse data shows that the 1100 Al would be the weaker under long duration loads. Also, hypervelocity impact penetration data indicate that 1100 Al can behave harder than 2024-T3 Al for very small projectile impacts⁹.

It was also shown²³ that the stress gradient expression (Eq. 9) is approximately equivalent to a simple impulse criterion. An empirical fracture criterion

$$\Delta t \sigma^\lambda \geq k \quad (10)$$

was combined with the time τ ($\tau = \Delta t + z/U$) at which fracture occurs at any plane z from the surface. λ , k are constants and U is tensile wave propagation velocity, and Δt = duration of tensile stress σ . The first plane that fractures is given by the minimum τ . Combining the above terms and using $d\tau/dz = 0$ gives the corresponding stress, $\sigma = (k\lambda U(d\sigma/dz))^{1/(1+\lambda)}$. For $\lambda = 1$ this expression is the form of Eq. (9) (assuming B is very small compared to the first term, which is often found experimentally). Also the fracture criterion becomes $\Delta t \sigma \geq k$, which has the form of a simple impulse criterion.

Tuler and Butcher²³ also modified the above simple criterion into a more general one, namely

$$\int_0^t (\sigma_0 - \sigma)^\lambda dt \geq k \quad (11)$$

σ_0 is the minimum stress required to fracture the material for long time duration loads.

There have been a variety of other approaches and investigations on the fracture of solids. Some of these involve detailed discussions regarding formation or the movement of dislocations to form microcracks. In the end, however, a rate expression is required for the dislocation motion (or its effect), and these expressions are usually introduced empirically. Certain constants in these equations always require experimental evaluation, and hence the models cannot make apriori predictions.

6.2 BALLISTIC LIMIT FAILURE PREDICTIONS

Numerical solutions can be used with the various fracture theories to make ballistic limit predictions. However, ballistic limit predictions for very small particles are sensitive to the assumed form of the fracture criterion. Also, ballistic limit predictions depend on the definition of target failure adopted (e. g. , material detachment, pressure leak, visible cracks, etc.).

Experimental ballistic limit data have been obtained for 2024-T4 aluminum by Fish and Summers²⁴ at Ames. In their study, the ballistic limit is defined "as that point at which a given specimen is sufficiently damaged that it will no longer sustain a pressure differential of a few atmospheres without leaking". Their data is for 0.159 cm dia 2017-T4 aluminum spheres impacting 2024-T4 aluminum single sheet targets. Impact velocities up to 6.5 km/sec were treated. We have extrapolated these data to obtain an experimental ballistic limit for an impact velocity of .75 cm/μsec. The extrapolation indicates that the ballistic limit for a .75 cm/μsec impact of a 0.159 cm dia sphere is $T_b/D = 4$. For a projectile diameter of $D = .3175$ cm, the ballistic limit value will not change significantly (if even measurably). Therefore, the ballistic limit value (using the Ames definition of failure) for Case 3 ($U_o = .75$ cm/μsec, $D = .3175$ cm) is taken as $T_b/D = 4$.

In the following section, the numerical solutions, and a tensile stress failure criterion will be used to predict 2024-T3 aluminum ballistic limits for $U_o = .75$ and 1.5 cm/μsec and $D = .3175$ cm and 31.75 μ.

6.2.1 Stress Gradient Failure Criterion

Breed, Mader, and Venable²⁰ have related the peak tensile stress for fracture (σ_f) to the tensile stress gradient. Their experimental data is correlated by

$$\sigma_f = A \left(\frac{\partial \sigma}{\partial z} \right)^{\frac{1}{2}} + B$$

For aluminum, their latest data²⁵ indicate the following best fit

$$\begin{aligned} A &= .0714 \text{ Mb}^{\frac{1}{2}} \text{ cm}^{\frac{1}{2}} \\ B &= 0 \end{aligned} \quad \text{if } \left(\frac{\partial \sigma}{\partial x} \right)^{\frac{1}{2}} < .7 \left(\frac{\text{Mb}}{\text{cm}} \right)^{\frac{1}{2}}$$

or

$$\begin{aligned} A &= 0 \\ B &= .05 \text{ Mb} \end{aligned} \quad \text{if } \frac{\partial \sigma}{\partial z} \geq .7 \left(\frac{\text{Mb}}{\text{cm}} \right)^{\frac{1}{2}}$$

However, note that the data only extend to gradients of $\frac{\partial \sigma}{\partial x} = 1$ Mb/cm. At gradients considerably above this value, σ_f could increase to values greater than 50 kb. In fact, for extremely high gradients, the tensile strength (σ_f) could approach the theoretical strength of a perfectly homogeneous solid, i. e. , the single crystal fiber strength. This limiting theoretical strength should be the atomic or intermolecular bond strength of the solid which experimentally corresponds to the sublimation energy. In aluminum, the sublimation energy (e_s) is about $.12 \times 10^{12}$ ergs/gm, corresponding to a theoretical bond strength of about $\rho_0 e_s = 320$ kb. Figure 32 indicates the best fit to the available data and two possible extensions to higher stress gradients.

Note that the fracture data was obtained from essentially 1-D spallation experiments. The onset of spallation was observed to have the σ_f vs $\frac{\partial \sigma}{\partial z}$ dependence indicated in Figure 32. The mechanism for failure in ballistic limit experiments is more complex than the mechanism for spallation. Nevertheless, the peak tensile stress and the axial stress gradient near the target rear surface are expected to be important in ballistic limit determinations. Therefore, the failure criterion for the ballistic limit of a single sheet target is assumed to depend on the peak axial tensile stress and stress gradient near the target rear surface.

Thus, the peak axial tensile stress, σ , and the corresponding stress gradient, $\frac{\partial \sigma}{\partial z}$, are used as a measure of the effects of a shock wave load incident on the target rear surface. Other failure models using pulse duration equations similar to Eqs 10 and 11 in Section 6.1 could be formulated. However, these formulations would exhibit the same general characteristics as the stress gradient approach due to the dependence of σ and $\frac{\partial \sigma}{\partial z}$ on pulse duration. Thus, the details of the failure mechanism (e. g. , tensile pulse duration, tensile stress rate, and crack propagation) are not explicitly considered, but are assumed to be related to the peak σ and $\frac{\partial \sigma}{\partial z}$ near the target rear surface.

Using this assumption, the extrapolation curves on Figure 32 will be interpreted as possible ballistic limit failure curves. Points $(\sigma, \frac{\partial \sigma}{\partial z})$ above the ballistic limit failure curve imply failure, points below imply that the target will not fail. Critical values of $(\sigma, \frac{\partial \sigma}{\partial z})$ for incipient failure will, of course, depend on the definition of ballistic limit adopted.

□ Case 5 ($U_o = 1.5 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$)

○ Case 3 ($U_o = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$)

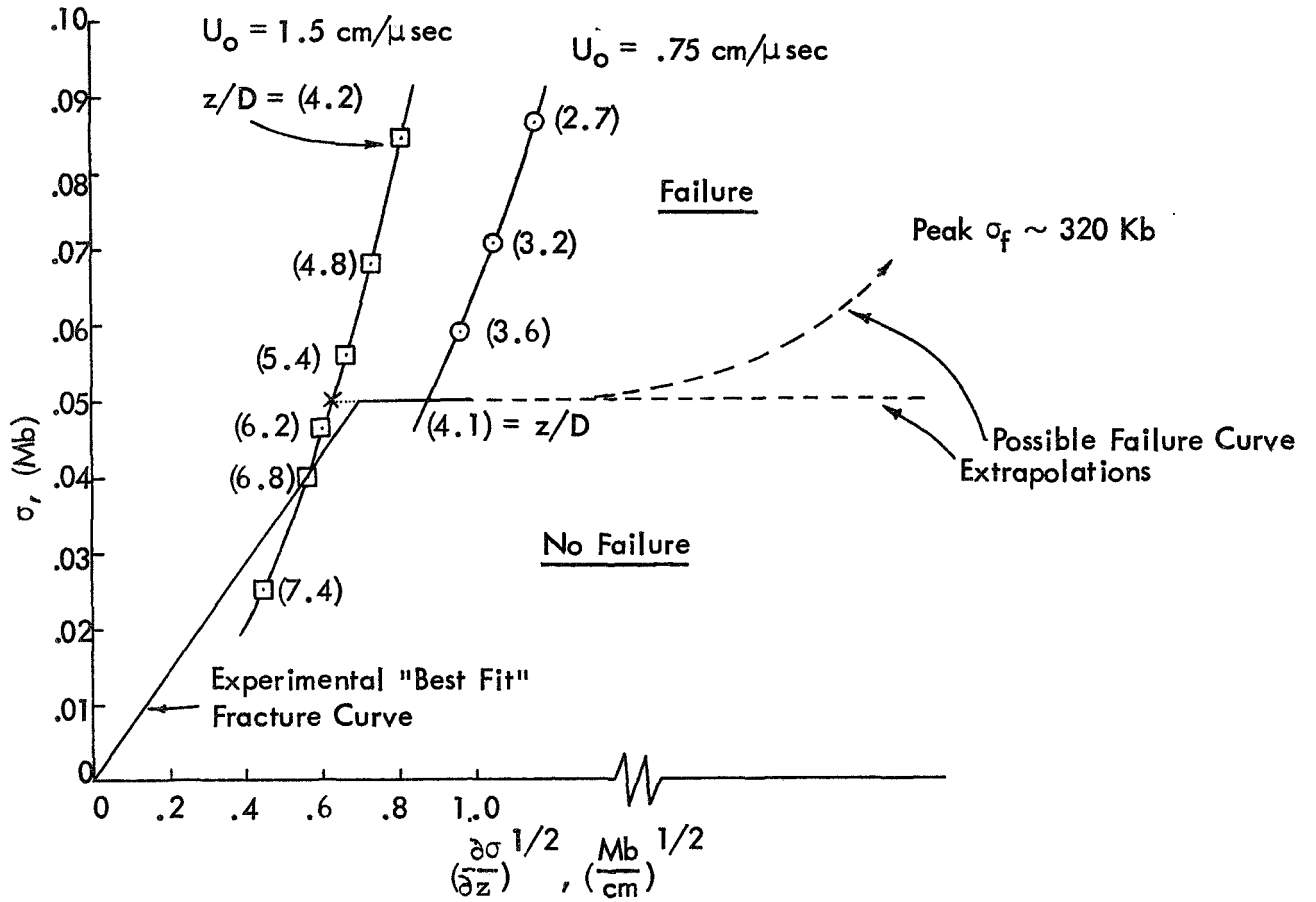
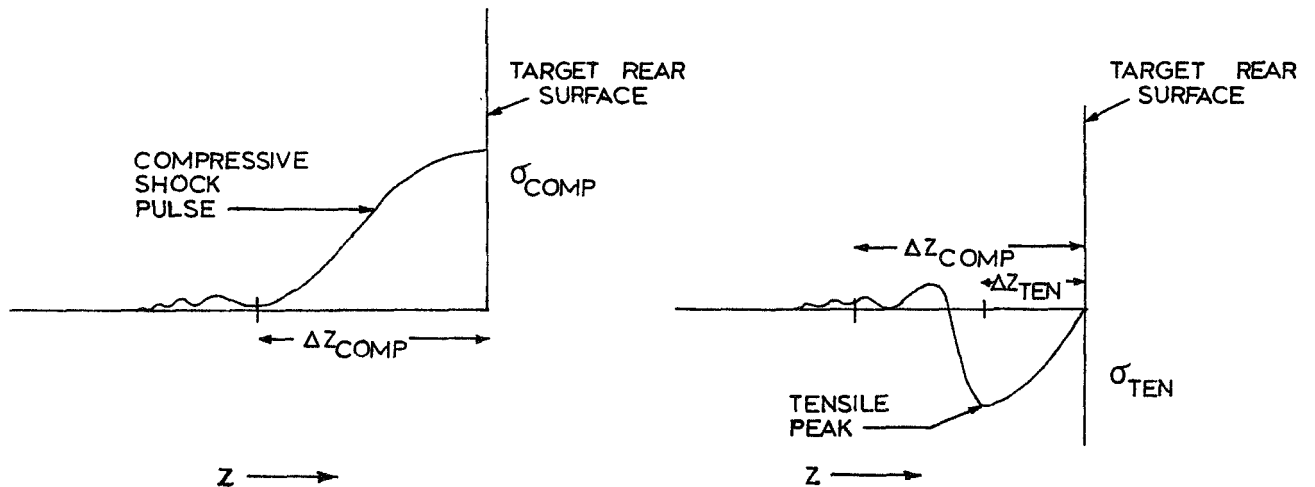


Figure 32. Possible Ballistic Limit Failure Curves, σ_f vs $(\frac{\partial \sigma}{\partial z})^{1/2}$, and Numerical Solution Data Points for the .3175 cm diameter Impact Cases.

The numerical solution for Case 3 corresponds to the ballistic limit experiment described in Section 6.1. Case 3 can be used to estimate σ_f and $\frac{\partial \sigma}{\partial z}$ near the target rear surface ($T/D = 4$). Thus a comparison with the experimental results and predictions using the stress gradient failure criterion are possible. After this comparison, the numerical solutions for Cases 4 and 5 will be used for predicting ballistic limits for other size projectiles and impact velocities.

Case 3 ($U_0 = .75$ cm/ μ sec, $D = .3175$, $T/D = 4$) indicates that a 52 kb shock wave interacts with the target rear surface. The compressive shock wave has a relatively constant pulse width at the depth of interest for the .75 cm/ μ sec impacts of about $\frac{\Delta z_{comp}}{D} = .4$ ($\Delta z_{comp} = .127$ in Case 3). The compressive wave pulse will be "reflected" from the free surface as a tensile pulse. The peak tension will occur when the tail of the compressive pulse interacts with the lead wavelet from the reflected rarefaction fan. The lead rarefaction wavelet emanates from the initial interaction between the peak stress in the shock wave and the free surface. The stress behavior on-axis is qualitatively indicated in the following sketch at the time of the initial interaction and at the time when the peak tension occurs. Note that the tensile peak occurs a distance $\Delta z_{ten} < \Delta z_{comp}$ behind the target rear surface (Δz_{ten} represents the tensile pulse width).



Generally, the tail of the compressive pulse is traveling with nearly the same velocity as the head of the tensile pulse. Thus, $\Delta z_{\text{ten}} = \Delta z_{\text{comp}}/2$ is a good approximation. The relationship between the peak compressive stress incident on the free surface and the resulting peak tensile stress is not as easily obtained. However, a fair approximation, before failure occurs, is to set the peak compressive stress equal to the peak tensile stress $|\sigma_{\text{comp}}| = \sigma_{\text{ten}}$. Thus, the spatial tensile gradient can be approximated by

$$\frac{\partial \sigma}{\partial z} = \frac{\sigma_{\text{ten}}}{\Delta z_{\text{ten}}} \approx \frac{2 \sigma_{\text{comp}}}{\Delta z_{\text{comp}}} \quad (12)$$

This expression is useful because the tensile gradient $\frac{\partial \sigma}{\partial z}$ can now be approximated by compressive values available from numerical calculations without introducing a target free surface into the calculation. Thus, a series of numerical calculations using various target thicknesses need not be performed. The series of calculations can be replaced by one impact calculation using a semi-infinite target. Then at any depth into the target, Eq. 12 can be used to estimate $\frac{\partial \sigma}{\partial z}$. The numerical solutions also provide σ_{comp} at any depth. Thus, using $\sigma_{\text{ten}} = |\sigma_{\text{comp}}|$ a point on Figure 34 is determined and whether failure occurs at the given depth can be determined.

For example, applying the method in Case 3 at the depth of $z/D = 3.6$, where $\sigma_{\text{comp}} = 59 \text{ kb}$, we have

$$\sigma_{\text{ten}} = \sigma_{\text{comp}} = .059 \text{ Mb}$$

$$\Delta z_{\text{ten}} = \frac{\Delta z_{\text{comp}}}{2} = \frac{.127 \text{ cm}}{2} = .0635 \text{ cm}$$

$$\frac{\partial \sigma}{\partial z} = \frac{.059}{.0635} = .929 \text{ and } \left(\frac{\partial \sigma}{\partial z}\right)^{\frac{1}{2}} = .964 \left(\frac{\text{Mb}}{\text{cm}}\right)^{\frac{1}{2}}$$

The point $\sigma_{\text{ten}} = 59 \text{ kb}$, $\left(\frac{\partial \sigma}{\partial z}\right)^{\frac{1}{2}} = .964$ is indicated on Figure 32. Note that it lies above the experimental curve, indicating that failure would occur for $T/D = 3.6$. To get incipient failure, the method requires a thicker target such that σ_{comp} would be 50 kb. Extrapolating the peak stress versus depth data indicated on Figure 17 and Table 3 implies a depth of $z/D = 4.1$ for a peak stress of 50 kb. Thus, the method implies a ballistic limit prediction of $T_b/D = 4.1$ which agrees very well with the experimental ballistic limit measure of $T_b/D = 4$.

A ballistic limit prediction for a .3175 cm diameter projectile impacting at 1.5 cm/μsec can be made by applying the stress gradient failure criterion to Case 5. In this case, the numerical solution indicates that the compressive pulse width has a relatively constant value of .254 cm at the depths of interest. Thus, the tensile pulse width, Δz_{ten} is .127 cm. If a free surface were placed at the 50 kb peak stress location ($z/D = 6.05$) there would be a tensile stress gradient of $\frac{\partial \sigma}{\partial z} = \frac{.050}{.127} = .3937 \frac{\text{Mb}}{\text{cm}}$. The point corresponding to $\sigma_{\text{ten}} = 50 \text{ kb}$, $\frac{\partial \sigma}{\partial z} = .3937$ is plotted with an x on Figure 32. This point lies above the fracture curve and thus the target would fail if the rear surface were stress loaded with this 50 kb pulse. In a similar fashion, other $(\sigma_{\text{ten}}, \frac{\partial \sigma}{\partial z})^{\frac{1}{2}}$ points from the Case 5 numerical solution were calculated and plotted on Figure 32. The target thickness corresponding to the $(\sigma, \frac{\partial \sigma}{\partial z})$ points plotted are indicated in Figure 32 in terms of z/D (in parenthesis).

Incipient failure occurs when the target $(\sigma, \frac{\partial \sigma}{\partial z})$ point lies on the failure curve. For a 1.5 cm/μsec impact of a 2024-T3 aluminum .3175 cm sphere on 2024-T3 aluminum, Figure 32 indicates that the incipient failure occurs for a 40 kb stress wave incident on the target free surface. The corresponding scaled target thickness is $T_b/D = 6.8$. However, the uncertainties in the data and in the stress gradient failure approach indicate that this value could be in error by about 10%. For example, if the peak tensile stress is 50 kb for this impact, then $T/D = 6.05$. (Also, note that a slightly smaller projectile would cause the Case 5 curve to shift to the right and then the incipient failure prediction would correspond to $\sigma_f = 50 \text{ kb}$, $T/D = 6.05$.)

Increasing the impact velocity results in lower stress gradients at failure. This effect can be observed by comparing the data points for Cases 3 and 5 in Figure 32. For most materials, a smaller stress gradient implies a lower tensile strength. For aluminum, this is true for stress gradients less than .49 Mb/cm. However, between .49 Mb/cm and 1 Mb/cm, the tensile strength remains constant. Beyond gradients of 1 Mb/cm, the tensile failure strength dependence on stress gradient is uncertain. For hypervelocity impacts of relatively large projectiles, e.g., diameters greater than 1/8-inch, the tensile strength behavior for stress gradients much greater than 1 Mb/cm are not important. However, for small projectiles, the tensile stress gradients of interest become considerably larger than 1 Mb/cm.

In Section 4, it was shown that linear size scaling is nearly satisfied far from the crater boundaries. Case 3 ($D = .3175$ cm) and Case 4 ($D = .3175 \times 10^{-2}$ cm) had nearly identical stress and velocity fields (in the linear size scaling sense) near the target rear surface located at $T/D = 4$. Thus, the stress fields are the same near the target rear surface in both cases, if the stress is expressed in terms of non-dimensional variables. For example, $\sigma = \bar{\sigma} \left(\frac{r}{D}, \frac{z}{D}, \frac{U_o t}{D} \right)$ is the same in both cases. However, note that the axial stress gradient $\left(\frac{\partial \sigma}{\partial z} \right)$ at any non-dimensional time $\frac{U_o t}{D}$ is not the same in the two cases. In fact,

$\frac{\partial \sigma}{\partial z} = \frac{1}{D} \frac{\partial \bar{\sigma}}{\partial (z/D)}$. Thus, the $.3175 \times 10^{-2}$ cm projectile solution (Case 4) has spatial stress gradients which are 100 times larger than the $.3175$ cm projectile solution (Case 3).

The ballistic limit prediction for the small projectile impact case depends on the nature of the stress gradient failure curve. This is illustrated in Figure 33, which indicates two possible extrapolations for the experimental fracture curve. Also, on the curve are numerical stress versus stress gradient^{1/2} data obtained from Cases 3 and 4. The log of σ_f has been plotted versus the log of $\left(\frac{\partial \sigma}{\partial z} \right)^{\frac{1}{2}}$. (The square root is not considered physically significant at very high stress gradients, however, this form is convenient since the experimental data is expressed in these terms.) In this form size scaling effects can be observed by simple horizontal translations. For example, Case 4 has gradients which are 100 times greater than in Case 3. Thus a factor of 10 separates the two cases in Figure 33.

If the failure curve is assumed to remain level at 50 kb (extrapolation curve 1), then the ballistic limit prediction for the 32μ particle (Case 4) would be $T/D \sim 4$. Thus, in this case, the ballistic limit would linearly size scale. This extrapolation curve is considered unlikely because in the limit of very small tensile pulse widths, the material strength should approach the perfectly homogeneous (no dislocations) theoretical strength. For example, if we assume that the dislocation density (the number of dislocation lines that intersect a unit area) is $10^{10}/\text{cm}^2$, then the spacing between dislocations is roughly 10^{-5} cm or $.1 \mu$. Thus, tensile stress pulses of less than this width are expected to cause failure only if the peak tensile stress equals the theoretical strength of the material. Therefore, if we assume 320 kb as the ultimate theoretical strength for the aluminum and if we assume 10^{-5} cm to be the dislocation spacing, then a critical tensile stress gradient is

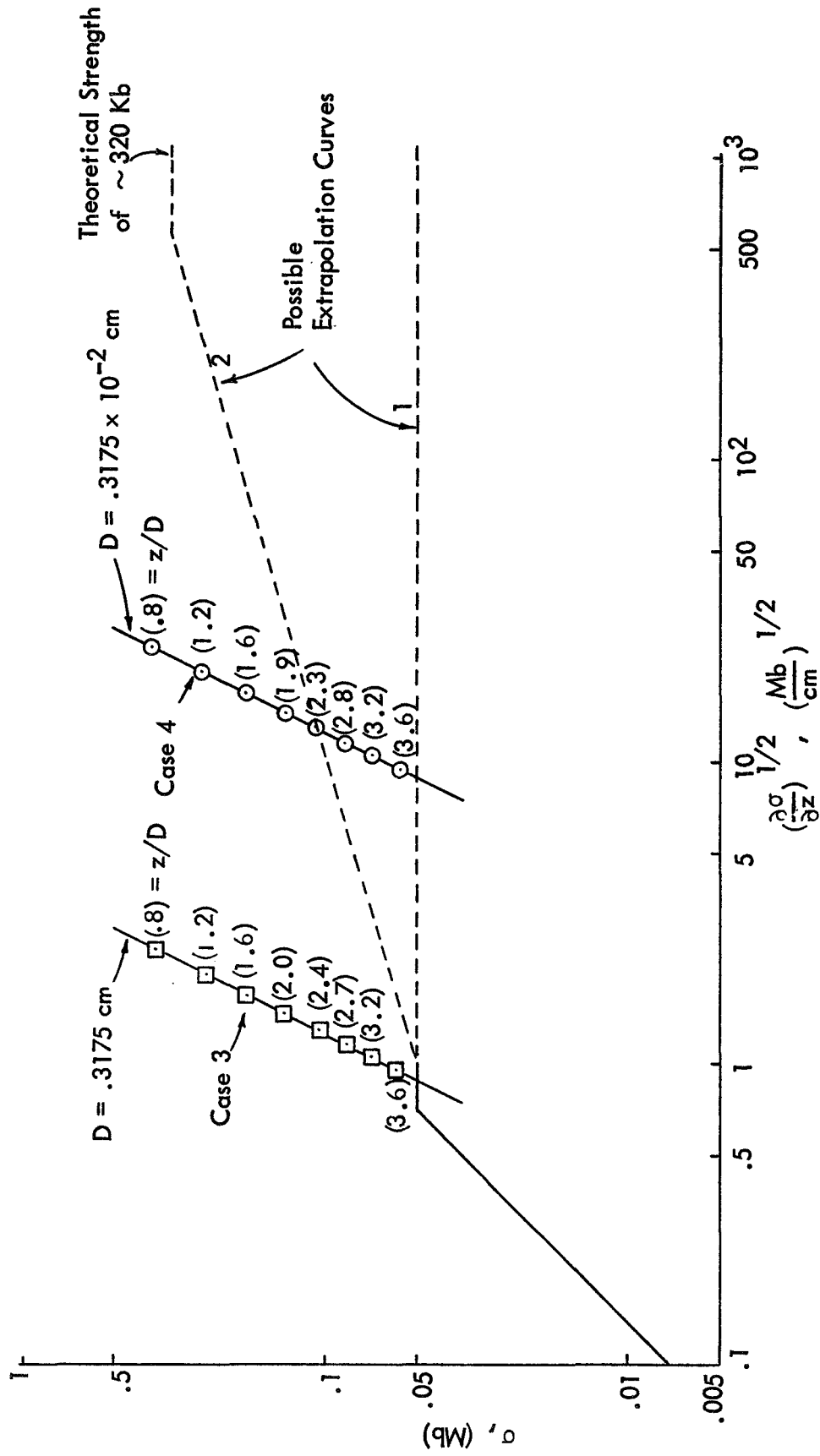


Figure 33. Possible Ballistic Limit Failure Curves, σ_f versus $(\frac{\partial \sigma}{\partial z})^{1/2}$, and the Numerical Data for the .75 cm/ μ sec Impact Cases.

$$\frac{\partial \sigma}{\partial z} = \frac{.320 \text{ Mb}}{10^{-5} \text{ cm}} = 3.2 \times 10^4 \text{ Mb/cm},$$

or

$$\left(\frac{\partial \sigma}{\partial z}\right)^{\frac{1}{2}} \cong 570 \left(\frac{\text{Mb}}{\text{cm}}\right)^{\frac{1}{2}}$$

Thus, using the above assumptions, the tensile strength of aluminum will level off at approximately 320 kb for all tensile stress gradients greater than about $3.2 \times 10^4 \text{ Mb/cm}$. These estimated values have been incorporated into the ballistic limit extrapolation curve (curve 2 in Figure 33) relating σ_f and $\left(\frac{\partial \sigma}{\partial z}\right)^{\frac{1}{2}}$. The extrapolation curve 2 implies that for a fixed velocity, the ballistic limit ratio T_b/D will decrease as the projectile diameter decreases until the tensile stress gradients exceed

$$\frac{\partial \sigma}{\partial z} = 3.2 \times 10^4 \frac{\text{Mb}}{\text{cm}} \quad \text{or} \quad \left(\frac{\partial \sigma}{\partial z}\right)^{\frac{1}{2}} = 570 \left(\frac{\text{Mb}}{\text{cm}}\right)^{\frac{1}{2}}$$

Tensile stress gradients in excess of $3.2 \times 10^4 \text{ Mb/cm}$ are not important physically, e. g., a projectile diameter of about $.1 \mu$ is needed to get these high stress gradients in impacts at $.75 \text{ cm}/\mu\text{sec}$. Also, note that for these very small projectiles, the classical concepts of continuum mechanics (thermal equilibrium, etc.) begin to break down.

A fracture curve similar to the extrapolation curve 2 in Figure 3 is expected to govern ballistic limits for 2024-T3 aluminum as well as many other metals. Experimental ballistic limit data involving small projectiles ($D < .3175 \text{ cm}$) could be used to determine the details of the failure σ_f vs $\frac{\partial \sigma}{\partial z}$ curve. However these data are not available for

2024-T3 aluminum. Therefore, the extrapolation curve 2 will be used for 2024-T3 aluminum as a current estimate until additional ballistic limit data become available.

Using the extrapolation curve 2 in Figures 33 and 34, and the results of the numerical solutions, the ballistic limit predictions for impacts of $.3175 \text{ cm}$ and $.3175 \times 10^{-2} \text{ cm}$ projectiles with impact velocities of $.75 \text{ cm}/\mu\text{sec}$ and $1.5 \text{ cm}/\mu\text{sec}$ were obtained and are listed in Table 6.

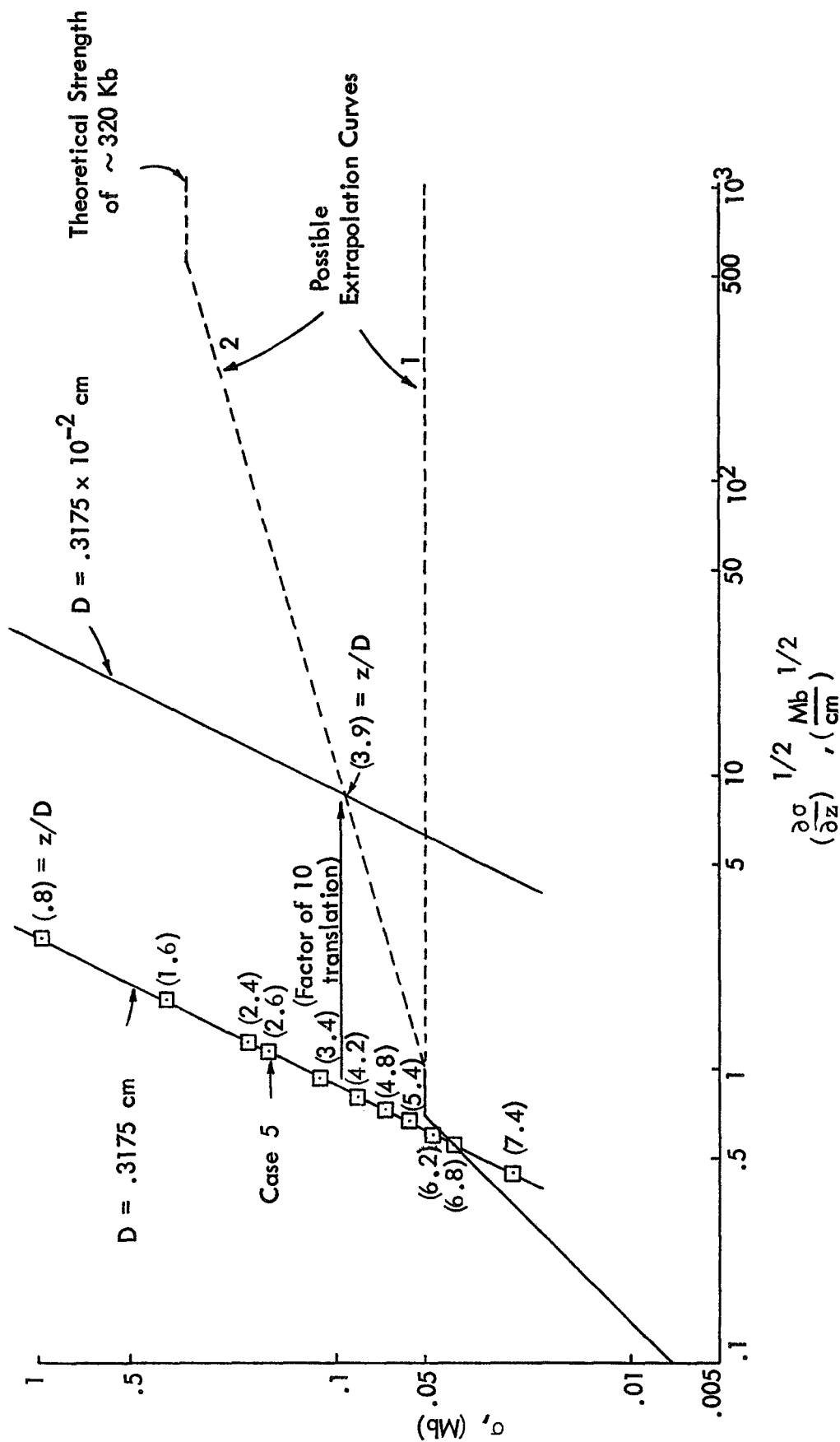


Figure 34. Possible Ballistic Limit Failure Curves, σ_f versus $(\frac{\partial \sigma}{\partial z})^{1/2}$, and the Numerical data for the 1.5 cm/ μ sec Impact Case.

TABLE 6. BALLISTIC LIMIT PREDICTIONS (T_b/D)

Case	D (cm)	U_o (cm/ μ sec)	T_b/D
3	.3175	.75	4.1
4	.3175 $\times 10^{-2}$ (31.75 μ)	.75	2.3
5	.3175	1.5	6.8
	.3175 $\times 10^{-2}$ (31.75 μ)	1.5	3.9

Figure 33 shows the graphical determination for the ballistic limit predictions for the .75 cm/ μ sec impact cases. Figure 34 shows the corresponding determination for the 1.5 cm/ μ sec impacts. Note that the prediction for the 32 μ projectile impacting at 1.5 cm/ μ sec involved a factor of 10 translation of the stress gradients $(\partial \sigma / \partial z)^{\frac{1}{2}}$ from the Case 5 numerical data. The factor of 10 is the square root of the projectile diameter scale factor of 100. Other diameter projectiles can be treated by using the corresponding translation factors in Figure 33 or 34.

Section 7

SUMMARY AND RECOMMENDATIONS

For the very high deformation rates involved in hypervelocity impacts of small particles, rate effects are important. Rate effects can cause non-linear size scaling in both cratering and ballistic limit phenomena. These phenomena have been investigated using a two dimensional numerical code (STEEP) which was modified to solve a general set of deformation rate sensitive constitutive relations. Table 7 contains a summary of crater penetration and ballistic limit predictions for the two impact velocities and two projectile sizes considered.

Experimental comparisons are available for the lower impact velocity and larger projectile configuration ($U_o = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$, $T/D = 4$) considered. These values are indicated in Table 7, and the agreement is good. Experimental data corresponding to the other finite target cases are not available. However, many penetration depth extrapolation equations have been developed for hypervelocity impacts into semi-infinite targets. For example, see References 8 and 26. The single sheet ballistic limit (T_b/D) extrapolation equation is usually related to the semi-infinite penetration depth (P_∞/D) by a factor of approximately 2. For 2024-T3 aluminum, the penetration formulas usually contain a $1/18$ dependence on projectile diameter and a $2/3$ dependence on impact velocity.

All of the empirical fits for 2024-T3 aluminum predict P_∞/D near 2 for $U_o = .75 \text{ cm}/\mu\text{sec}$ and $D = .3175 \text{ cm}$. In this study the ballistic limit for $U_o = .75 \text{ cm}/\mu\text{sec}$, $D = .3175 \text{ cm}$ has been taken as $T_b/D = 4$ on the basis of the experimental results in Reference 24. Thus, fitting $P_\infty/D = 2$ and $T_b/D = 4$, the semi-infinite penetration depth and the ballistic limit target thickness for 2024-T3 aluminum impacts on 2024-T3 targets can be predicted by the empirical expressions

$$P_\infty/D = K D^{.056} U_o^{.67} \quad (13a)$$

and

$$T_b/D = 2 P_\infty/D \quad (13b)$$

where

$$K = .553 \text{ cm}^{-.056} (\text{cm/sec})^{-.67}$$

TABLE 7

SUMMARY OF PENETRATION AND BALLISTIC LIMIT PREDICTIONS

Impact Conditions		Target Thickness Ratio	Penetration Depth *	Predictions	
Impact Velocity	Projectile Diameter			Ballistic Limit Plate Thickness for U_o and D	Ballistic Limit/Semi-Infinite Penetration Depth
U_o (cm/ μ sec)	D (cm)	T/D	$P_{T/D}/D$	T_b/D	T_b/P_∞
.75	.3175	4	$\frac{2.17}{[2.2]}^{27}$	$\frac{4.1}{(4)}^{22}$	2.05 (2)
.75	$.3175 \times 10^{-2}$ (= 31.75 μ)	4	1.95	2.3 (3.1)	1.48 (2)
1.5	.3175	8	3.06	6.8 (6.4)	2.12 (2)
1.5	$.3175 \times 10^{-2}$ (= 31.75 μ)	-	-	3.9 (4.9)	1.59 (2)

* $P_{T/D}/D$ represents the penetration depth for the plate thickness T/D considered in the numerical calculation.

[] Experimental value from the indicated reference.

() Predicted values using Equation 13.

The ballistic limit predictions using Eq. 13 appear in Table 7. Also, equation 13 was used to obtain the ballistic limit ratios, $T_b/P_\infty = 2$. This ratio is predicted to be near 2 for the .3175 cm projectile impacts, but T_b/P_∞ is predicted to decrease to about 1.6 for the 31.75 μ projectile impacts according to the model used in this study. Thus, the model predicts that the ballistic limit thickness decreases faster than the semi-infinite penetration depth as the projectile size decreases.

Table 7 shows that the physical model used in this study predicts non-linear size scaling for impact craters and ballistic limit thicknesses. As the projectile diameters decrease, the non-linearity is such that crater dimensions and ballistic limit target thicknesses become smaller than would be predicted by linear size scaling.

The failure model discussed in Section 6.2.1 can be used to estimate ballistic limit thicknesses for any size projectile using the results of one numerical solution at the impact velocity of interest. The method relies heavily on a failure curve which relates the tensile stress at failure to the stress gradient. Unfortunately, the experimental data defining this curve do not extend to the stress gradients of interest in small particle impacts. The ballistic limit predictions indicated in Table 7 resulted from an extrapolation of the existing data. This extrapolation involved an estimation of the limiting value of the tensile stress. The ballistic limit predictions were lower than those obtained using a simple $T_b/D = 2 P_\infty/D$ relationship. Whether this is indeed the case must be decided by experimentation. However, the method is general and a few well chosen ballistic limit experiments on small particles ($D \ll .3175$ cm) can determine the proper shape of the tensile stress for failure vs the stress

gradient failure curve, σ_f vs $\frac{\partial \sigma}{\partial z}$.

The physical model used in the numerical method contains approximations arising out of incomplete knowledge of material properties and out of limitation in current numerical technology. For this study, the model assumed a continuum material which behaves according to a hydrodynamic-elastic-viscoplastic constitutive relation. Such properties as temperature effects and non-equilibrium effects are consequently not included.

No major problems are involved in including any reasonable description of these effects into the Eulerian STEEP code, but considerable uncertainty exists in the measurement of these properties under appropriate dynamic loading conditions.

Use of the Eulerian continuum model for solving hypervelocity impact problems, of course, precludes the formation or identification of individual particles or chunks of mass which separate from the main body of the material. This constitutes an important limitation in current numerical technology. However, work is currently being performed to partially remove this limitation by permitting portions of the numerical solution to use a Lagrangian grid which can separate along grid lines.

From comparisons with experimental observations, the adequacy and limitations of the present physical model for determining hypervelocity impact characteristics can be determined. Where significant discrepancies occur, this may indicate important physical mechanisms that are missing from the physical model. Continued interactions between theory and experiment will lead to improvements in the model, and ultimately to a high degree of confidence in the numerical method as a means for predicting data which are experimentally unattainable.

Finally, the following recommendations are made regarding future work in this area.

1. Experimental-numerical comparisons should be aggressively pursued, and the results utilized to make any indicated improvements in the numerical technique.
2. In particular, very small projectile ($D \ll .3175$ cm) hypervelocity impact experiments should be performed to determine non-linear size scaling effects and thereby determine material rate sensitivity, for example
 - a. small particle impacts onto semi-infinite targets of various materials,
 - b. small particle ballistic limit determinations for single and double sheet configurations, and
 - c. small particle impacts onto layered (multi-material) targets.
3. Additional numerical studies should be undertaken to evaluate other impact situations, including:

- a. impacts where the projectile and target are of different materials,
- b. additional impact velocities,
- c. additional projectile and target sizes, and
- d. two sheet target configurations for various sheet thicknesses and spacings.

APPENDIX A

THE RATE SENSITIVE CONSTITUTIVE EQUATIONS SPECIALIZED TO THE CASE OF UNIAXIAL STRESS

The set of hydrodynamic-elastic-viscoplastic equations are listed below for reference:

$$\begin{aligned}\hat{S}_{ij} &= 2G (d_{ij}^* - \gamma \Phi(F) \frac{S_{ij}}{\sqrt{J_2}}), \quad \text{for } \sqrt{J_2} > \tau \\ \hat{S}_{ij} &= 2G d_{ij}^*, \quad \text{for } \sqrt{J_2} \leq \tau \quad (A-1) \\ P &= \pi(\rho, e) = -\frac{\sigma}{3} \frac{\alpha}{\alpha}\end{aligned}$$

The symbolism which appears above is defined as follows:

$$\begin{aligned}S_{ij} &= \text{deviatoric component of the stress tensor,} \\ S_j^i &\equiv \sigma_j^{i*} \equiv \sigma_j^i - \frac{1}{3} \sigma \frac{\alpha}{\alpha} \delta_j^i \\ \hat{S}_j^i &= \frac{DS_j^i}{Dt} + S_m^i \omega_j^m - S_j^m \omega_m^i = \text{Jaumann-Noll Stress Rate} \\ \omega_j^i &= \text{spin tensor} \\ d_{ij}^* &= \text{deviatoric component of the deformation rate tensor, } d_{ij} \\ \gamma &= \text{material constant} \\ G &= \text{shear modulus} \\ \Phi &= \text{material function of } F, \text{ where } F = \frac{\sqrt{J_2}}{\tau} - 1 \\ \tau &= \text{yield stress in pure shear } (\sqrt{3}\tau = Y(\epsilon_p) \text{ where } Y \text{ is the yield stress in uniaxial stress.}) \\ \epsilon_p &= \int \sqrt{\frac{2}{3} d_j^i P d_i^j P} dt = \text{generalized plastic strain} \\ \sqrt{J_2} &= \sqrt{\frac{1}{2} S_{ij} S^{ij}} \\ \pi(\rho, e) &= \text{hydrodynamic equation of state relating pressure to density and specific internal energy.}\end{aligned}$$

For large pressures, such as those obtained in hypervelocity impacts, a hydrodynamic equation of state ($\pi(\rho, e)$) must be used to properly describe the material properties. However, for low pressures, such as those obtained in uniaxial stress experiments, an elastic equation of state can be used to describe material properties. The elastic equation of state has the form

$$P = \pi(\rho) = -K \left(\frac{\rho_0}{\rho} - 1 \right) = -K \epsilon^\alpha_\alpha$$

or

$$\frac{1}{P} = -K d^\alpha_\alpha \quad (A-2)$$

where

K is the bulk modulus (assumed constant) and

ϵ^α_α is the trace of the strain tensor

Uniaxial stress experimental data at various strain rates is available for certain aluminum alloys. The experimental data shows a definite dependence on strain rate. This strain rate dependence is contrary to the predictions of a hydrodynamic or an elastic-plastic model.

To indicate how the strain rate sensitivity of the constitutive relations (Eq. (A-1)) arise, these relations will be specialized to uniaxial stress and to an elastic equation of state. Then, exact analytical solutions relating stress, strain, and strain rate will be obtained for special choices of $\phi = \phi(F)$.

A-2. STRAIN RATE DEPENDENCE IN UNIAXIAL STRESS

The constitutive relations of Eq. (A-1) when specialized to an elastic equation of state (Eq. (A-2)), to uniaxial stress, and to the condition $Y = Y_0$ (a constant), can be put in the following form:

$$d = \frac{1}{E} \frac{d\sigma}{dt} + \gamma' \phi(F) = d^e + d^p \quad (A-3)$$

where

$$F = \frac{\sigma}{Y} - 1$$

$d = d_{11}$, and d^e , d^p are the elastic and viscoplastic contribution, respectively

$$\sigma = \sigma_{11}$$

$$\gamma' = \frac{2}{\sqrt{3}} \gamma$$

$$E = \text{Young's modulus}$$

we also define

$$\epsilon = \epsilon_{11}$$

and we assume for simplicity that σ , d , and ϵ are greater than zero. For a uniaxial stress experiment at a constant strain rate, d , we have

$$\frac{d\sigma}{dt} = \frac{d\sigma}{d\epsilon} \frac{d\epsilon}{dt} = \frac{d\sigma}{d\epsilon} d \quad (\text{A-4})$$

Thus

$$\gamma' \phi(F) = d \left(1 - \frac{1}{E} \frac{d\sigma}{d\epsilon} \right) \quad (\text{A-5})$$

or

$$d\epsilon = \frac{\frac{1}{E} d\sigma}{1 - \frac{\gamma'}{d} \phi(F)}$$

Eq. (A-5) is used to determine γ' and $\phi(F)$ from experimental data. In general $F = \frac{\sigma}{Y} - 1$, where Y is a variable which depends on the magnitude of the generalized plastic strain. However, if we assume $Y = Y_0$, then Eq. (A-5) implies

$$\frac{\epsilon}{\epsilon_y} - 1 = \int_0^F \frac{dF}{1 - \frac{\gamma'}{d} \phi(F)} \quad (\text{A-6})$$

where

$$F = \frac{\sigma}{Y_0} - 1$$

$$\epsilon_y = \frac{Y_0}{E} = \text{strain at yield point.}$$

The solution to Eq. (A-6) has been obtained for two choices of $\phi(F)$. One form is $\phi(F) = F$, and the second form is $\phi(F) = e^{F/\theta_0} - 1$ where θ_0 is a material parameter. These are discussed in the sections which follow.

A-3. $\phi(F) = F$

The solution to Eq. (A-6) for $\phi(F) = F$ is

$$\frac{\epsilon}{\epsilon_y} - 1 = - \frac{d}{\gamma'} \ln \left(1 - \frac{F}{d/\gamma'} \right) \quad (A-7)$$

or

$$F = \frac{\sigma}{Y_0} - 1 = \frac{d}{\gamma'} \left\{ 1 - e^{-\left(\frac{\epsilon}{\epsilon_y} - 1 \right) / (d/\gamma')} \right\}$$

This equation shows that the modified Perzyna constitutive equations do predict that σ will be greater than Y_0 for $d > 0$. The elastic-perfectly plastic theory predicts that $\sigma = Y_0$ for all d . Eq. (A-7) also indicates that for $\phi(F) = F$, F has an asymptotic value for any d , namely

$$F_A = \frac{d}{\gamma'}$$

There is experimental, and also some theoretical, evidence that suggest F_A should vary as the logarithm of d for relatively low strain rates. (For example see Reference 2.) The following choice for $\phi(F)$ does predict such a dependence.

A-4. $\phi(F) = e^{F/\theta_0} - 1$

The solution to Eq. (A-6) for $\phi(F) = e^{F/\theta_0} - 1$ is

$$\frac{\epsilon}{\epsilon_y} - 1 = \frac{\theta}{1 + \frac{\gamma'}{d}} \left[\frac{F}{\theta} - \ln \left\{ 1 + \frac{\gamma'}{d} - \frac{\gamma'}{d} e^{F/\theta} \right\} \right] \quad (A-8)$$

The asymptotic value of F corresponds to the vanishing of the argument of the logarithm. Thus

$$1 + \frac{\gamma'}{d} = \frac{\gamma'}{d} e^{F_A/\theta}$$

or

$$F_A = \theta_0 \ln \left(1 + \frac{d}{\gamma'} \right) \quad (A-9)$$

The asymptotic equation for F_A can, in general, be obtained from the following relationship

$$F_A = \phi^{-1} \left(\frac{d}{\gamma'} \right)$$

where

$$\phi^{-1} \text{ is the inverse function of } \phi, \text{ i.e., } \phi^{-1}[\phi(x)] = x$$

This relationship follows from

$$\phi(F) = \frac{dP}{\gamma'} \text{ for uniaxial stress}$$

and $dP \rightarrow d$ in the limit $\epsilon \rightarrow \infty$.

When work hardening is included in the form $Y(\epsilon_p)$, then the differential Eq. (A-5) must be solved using a numerical technique. A numerical technique was developed for solving Eq. (A-5) for arbitrary $\phi(F)$, γ , and $Y(\epsilon_p)$. Therefore, experimental data can be used to obtain the forms and magnitudes of the strain rate sensitive parameters $\phi(F)$ and γ . The numerical technique was used to fit the strain rate experimental data for 2024-0 and 2024-T3 aluminum. The form for $\phi(F)$ which best fits the data in both materials is $\phi(F) = e^{F/\theta} - 1$. The data indicates that the parameter θ is a function of the generalized plastic strain, ϵ_p . Thus, both θ and Y were made functions of ϵ_p in the empirical fit. The empirical fit to the uniaxial stress experimental data for the two aluminum alloys is indicated in Figure A-1. The agreement is good in the uniaxial stress experimental range. (See Figure 1 in main text for comparison.)

However, for strain rates above 10^6 sec^{-1} , a more rate sensitive $\phi(F)$ function is needed to fit very small projectile hypervelocity impact experiments. The relatively strain rate insensitive form

$$\phi(F) = e^{F/\theta} - 1 \text{ implies}$$

$$F = \frac{\sigma}{Y} - 1 = \theta \ln \left(1 + \frac{dP}{\gamma'} \right)$$

Therefore, F increases with only the logarithm of the plastic strain rate. However, if $\phi(F) = \bar{\phi} + \bar{S} F$ is assumed, where $\bar{\phi}$ and $\bar{S}$ are constants, then

$$F = \frac{\sigma}{Y} - 1 = \frac{\frac{dP}{\gamma'} - \bar{\phi}}{\bar{S}}$$

$$\phi(F) = e^{F/\theta(\epsilon_p)} - 1$$

$$F = \frac{\sigma}{Y(\epsilon_p)} - 1$$

$$\epsilon_p = \int_0^t \sqrt{\frac{2}{3} \dot{d}_j^i \mathbf{P} \dot{d}_i^j \mathbf{P}^T} dt$$

2024-0 Aluminum

$$Y(\epsilon_p) = 11 + 112.5 \sqrt{\epsilon_p} \text{ (KSI)}, \quad \epsilon_p < .03693$$

$$Y(\epsilon_p) = 11 + 112.5 \sqrt{\epsilon_p} - 190 (\epsilon_p - .03693)^{1.0814},$$

$$.03693 < \epsilon_p < .08$$

$$\theta(\epsilon_p) = .002 + .105 \epsilon_p, \quad \epsilon_p < .08$$

$$\gamma' = .3 \text{ sec}^{-1}, \quad \bar{\phi} = ?$$

2024-T3 Aluminum

$$Y(\epsilon_p) = 50 + 94.7 \sqrt{\epsilon_p} \text{ (KSI)}, \quad \epsilon_p < .0926$$

$$Y(\epsilon_p) = 50 + 94.7 \sqrt{\epsilon_p} - 54.07 (\epsilon_p - .0926)^{1.6355},$$

$$.0926 < \epsilon_p < .4$$

$$\theta(\epsilon_p) = .0003 + .02138 \epsilon_p - .02429 \epsilon_p^2, \quad \epsilon_p < .4$$

$$\gamma' = .3 \text{ sec}^{-1}, \quad \bar{\phi} = 10^6; \quad \bar{d} = 3 \times 10^5 \text{ sec}^{-1}$$

Figure A-1. Uniaxial Stress Versus Strain Relationships for 2024-0 and 2024-T3 Aluminum Below Rates of $\bar{d} = \gamma' \bar{\phi}$

and F depends linearly on the plastic strain rate. A form similar to this linear form was used to define the rate sensitivity of 2024-T3 aluminum at strain rates above $3 \times 10^5 \text{ sec}^{-1}$.

The strain rate sensitive model predicts non-linear size scaling. A simple non-dimensional analysis argument can demonstrate this fact. For simplicity we will do the analysis in the uniaxial stress state. Consider the stress rate dependence using the strain rate sensitive model, i. e. ,

$$\frac{d\sigma}{dt} = E (d - \gamma' \phi) \quad (\text{A-11})$$

This relationship is just Eq. (A-3) rearranged. To determine if Eq. (A-11) will size scale, we need to put this equation in non-dimensional form using a set of characteristic problem parameters. One such convenient set is

$$\begin{aligned} X_o & \quad \text{Characteristic spatial dimension} \\ U_o & \quad \text{Characteristic velocity} \\ \rho_o & \quad \text{Characteristic density} \\ T_o & \quad \text{Characteristic temperature} \end{aligned} \quad (\text{A-12})$$

Using the set of characteristic parameters, a complete set of non-dimensional dependent and independent variables can be formed, e. g. ,

$$\begin{aligned} \bar{X} &= X/X_o \\ \bar{t} &= U_o t/X_o \\ \bar{U} &= U/U_o \\ \bar{\sigma} &= \sigma / \rho_o U_o^2 \\ \bar{d} &= \frac{\chi_o d}{U_o} \\ \bar{T} &= T/T_o \\ \bar{m} &= m/\rho_o X_o^3 \end{aligned} \quad (\text{A-13})$$

Then, the non-dimensional variable relationships (Eq. A-13) can be used to rewrite Eq. (A-11) in terms of non-dimensional variables and characteristic parameters. Thus, Eq. (A-11) becomes

$$\frac{d(\rho_o U_o^2 \bar{\sigma})}{d(X_o \bar{t}/U_o)} = E \left(\frac{U_o \bar{d}}{X_o} - \gamma' \phi \right)$$

or

$$\frac{d\bar{\sigma}}{d\bar{t}} = \frac{E}{\rho_o U_o^2} \left(\bar{d} - \frac{X_o \gamma'}{U_o} \phi \right) \quad (A-14)$$

The fact that Eq. A-14 contains a term with a factor X_o demonstrates that this equation will not linearly size scale.

The lack of linear size scaling in this case is related to the presence of a material parameter with the dimensions of (1/time), i. e., γ' . The dimensions of this parameter could only be made non-dimensional by using the factor X_o/U_o ($\bar{\gamma}' = \gamma' X_o/U_o$, where $\bar{\gamma}'$ is non-dimensional). In general, equations which will linearly size scale contain material parameters which can be made non-dimensional without using X_o . Similarly, equations which will not linearly size scale usually contain material parameters which can only be made non-dimensional by using X_o . (A possible exception to this later statement would be a case where two material parameters containing X_o appear as a product and the X_o factors cancel.) Thus, other constitutive relations will not linearly size scale if they contain material parameters which involve a factor X_o (involve in the sense that an X_o is necessary to put the material parameter in the non-dimensional form). Three other examples of material parameters of this type are listed below

$$\begin{aligned} \mu & \text{ Coefficient of viscosity } [m X^{-1} t^{-1}] \\ \bar{\mu} & = \mu / \rho_o X_o U_o = 1/\text{Reynolds Number} \end{aligned}$$

$$\begin{aligned} K & \text{ Coefficient of thermal conductivity } [m X t^{-3} T^{-1}] \\ \bar{K} & = \frac{K T_o}{\rho_o U_o^3 X_o} \end{aligned}$$

$$\begin{aligned} D & \text{ Diffusivity of Heat } [X^2 t^{-1}] \\ \bar{D} & = D / X_o U_o \end{aligned}$$

Thus, constitutive equations which include strain rate effects, viscous effects, thermal conductivity, or diffusion will not linearly size scale.

APPENDIX B

THE HYDRODYNAMIC AND ELASTIC MATERIAL PROPERTIES FOR 2024-T3 ALUMINUM

In the numerical solutions, the hydrodynamic properties of a material are specified by a caloric equation of state in the form

$$P = \pi(\rho, e)$$

P is the pressure,
 ρ is the density,
 e is the specific internal energy, and
 π is the function relating P to ρ and e .

Two different equations of state were used in this study. For the low impact velocity cases ($U_0 = .75$ cm/ μ sec), an equation developed at Los Alamos was used. For the high impact velocity case (Case 5, $U_0 = 1.5$ cm/ μ sec) it was necessary to use a more accurate and computationally time consuming tabular equation of state.

There have been many equation of state formulations proposed for aluminum. The various formulations agree on the hydrodynamic material properties as long as the state variables (ρ, e) lie near the shock Hugoniot. However, for states lying far from the Hugoniot, the various equations of state predict very different material behavior. For example, consider the two sets of aluminum release adiabats (isentropes) illustrated in Figures B1 and B2.

Figure B1 shows 6 release isentropes centered on the Hugoniot using a metal equation of state developed at Los Alamos²⁸. (This equation along with the aluminum constants are listed in Table B1.) This equation of state was not intended to account for the vapor state characteristics of metals. For example, both the $U_0 = 1.5$ and 2 cm/ μ sec isentropes should approach a vapor state with a large volume and therefore a small compression ratio. This behavior is not included in this Los Alamos metal equation of state. However for impact problems not involving vapor states, this equation of state can be used to describe the hydrodynamic material properties. (Reference 29 contains an estimation of the impact velocities necessary to produce incipient vaporization in several metals. In aluminum the value is about $U_0 = 1$ cm/ μ sec.)

The .75 cm/ μ sec impact numerical solutions obtained in this study did use the Los Alamos metal equation of state. However, this equation of state would not have been adequate for the 1.5 cm/ μ sec impact case (Case 5, $U_0 = 1.5$ cm/ μ sec), since this case does involve vaporization of material near the impact point. Instead, the Shock Hydrodynamics' Aluminum tabular equation of state was used for this high impact velocity case. Entries in this tabulation were generated by consideration of porous Hugoniot data, the critical point, boiling point, and shear modulus, as well as the Thomas-Fermi theory and ideal gas behavior. A complete description of this tabular equation of state is found in Reference 30. Figure B2 shows 6 isentropes centered on the Aluminum Hugoniot using this Tabular equation of state. Also on Figure B2 is the corresponding Hugoniot from Figure B1 using the Los Alamos Metal equation of state. The two equations of state are nearly identical for all isentropes that intersect the Hugoniot at pressures below about 1.5 Mb. Isentropes centered on the Hugoniot above this pressure show the differences inherent in the two equations of state. For example, the $U_0 = 2$ and 1.5 cm/ μ sec isentropes do expand to the vapor state using the tabular equation of state.

The elastic properties of a material are specified, in general, by two elastic constants. However, once the hydrodynamic equation of state is specified, one elastic constant (the bulk modulus, K) is already determined. The bulk modulus is related to the hydrodynamic equation of state by

$$K = \rho C_{\text{hydro}}^2$$

where

$$C_{\text{hydro}}^2 = \left(\frac{dP}{d\rho} \right)_S = \left(\frac{\partial \pi}{\partial \rho} \right)_e + \frac{P}{\rho^2} \left(\frac{\partial \pi}{\partial e} \right)_\rho$$

Thus, only one additional elastic constant need be specified.

In this study, a shear modulus of $G = .275$ Mb was used as the second elastic constant.

TABLE B-1. LASL METAL FIT

$$\begin{aligned}
 P &= \frac{A + B\epsilon + C\epsilon^2}{\epsilon_0 + \epsilon} \\
 \epsilon &= \rho_0 e \\
 A &= \mu(a_1 + a_2 |u|) \\
 B &= b_0 + b_1 \mu + b_2 \mu^2 \\
 C &= c_0 + c_1 \mu \\
 c^2 &= \frac{1}{\rho_0 (\epsilon_0 + \epsilon)} \left\{ A' + B'\epsilon + C'\epsilon^2 + \frac{P}{\eta^2} (B + 2C\epsilon - P) \right\} \\
 A' &= a_1 + 2a_2 |u| \\
 B' &= b_1 + 2b_2 \mu \\
 C' &= c_1
 \end{aligned}$$

<u>Constants</u>	<u>Aluminum</u>
ρ_0	2.702
a_1	1.1867
a_2	0.07630
b_0	3.445
b_1	1.545
b_2	0.9643
c_0	0.4338
c_1	0.5487
ϵ_0	1.5

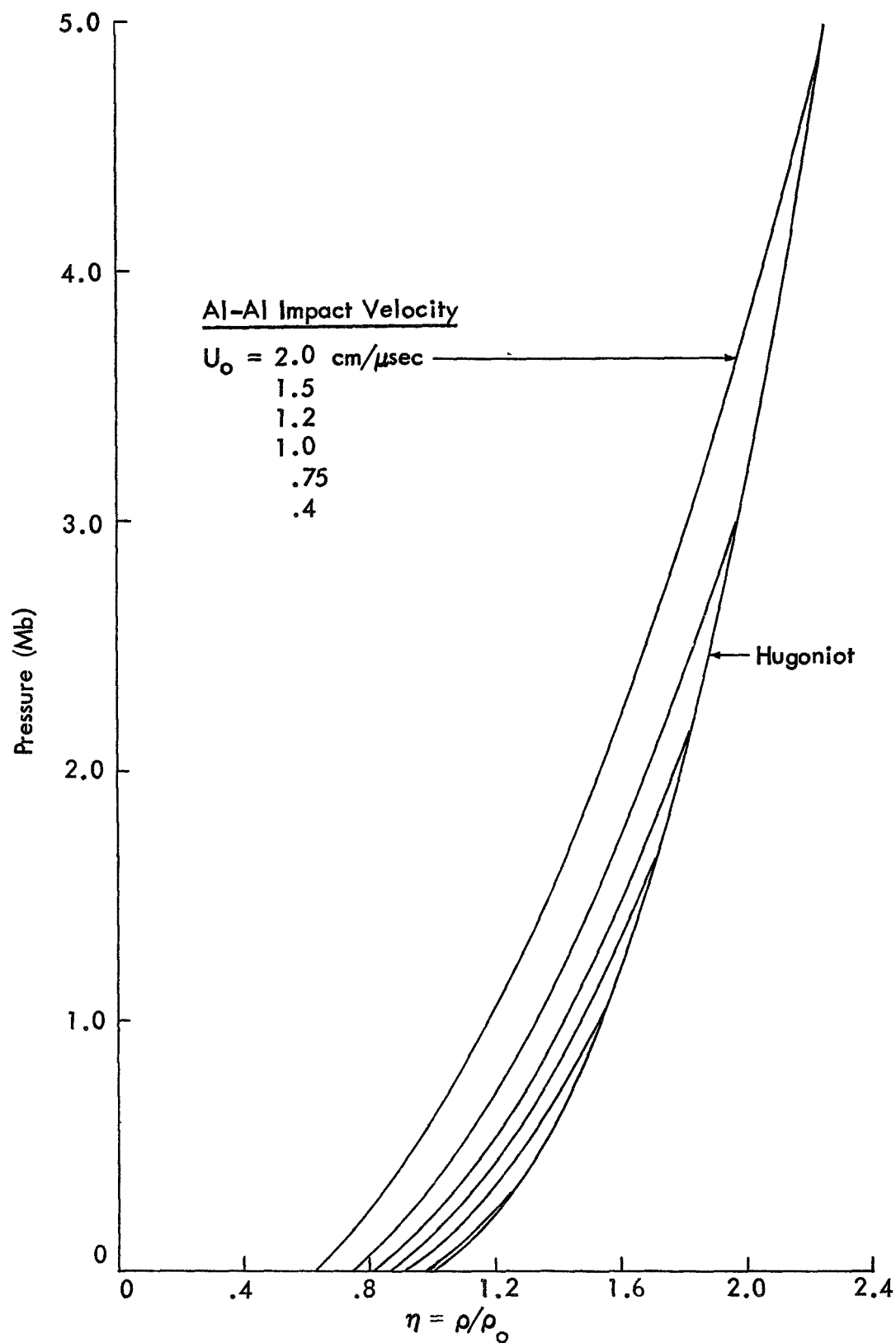


Figure B1. Pressure-Compression Release Isentropes Using the Los Alamos Equation of State for Aluminum.

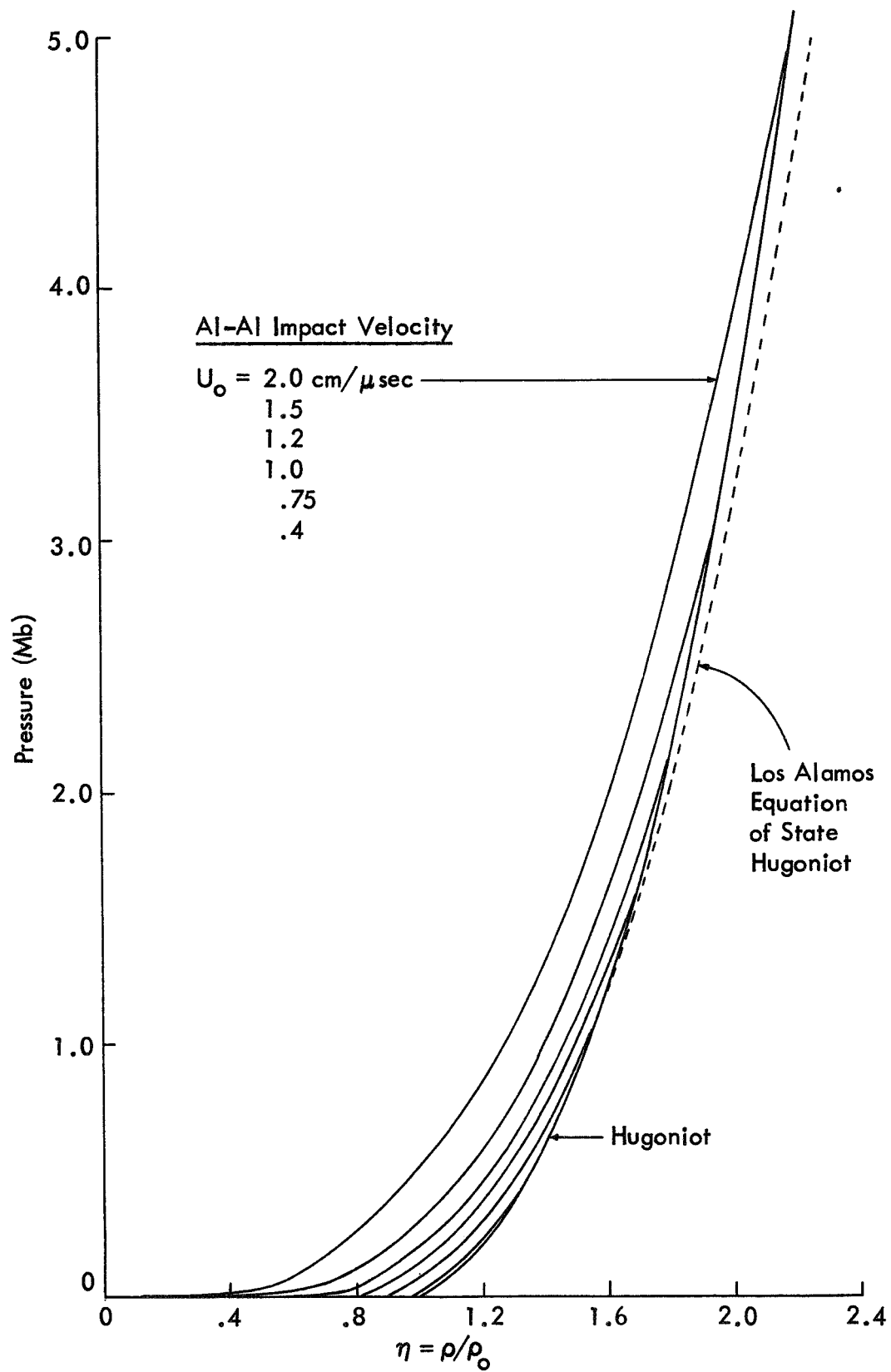


Figure B2. Pressure-Compression Release Isentropes Using the SHI Tabular Equation of State for Aluminum.

APPENDIX C

This Appendix contains the particle velocity and principal stress field plots for Cases 1, 2, 4, and 5. The corresponding field plots for Case 3 are discussed in Section 3.

Particle Velocity and Principal Stress
Field Plots for Case 1

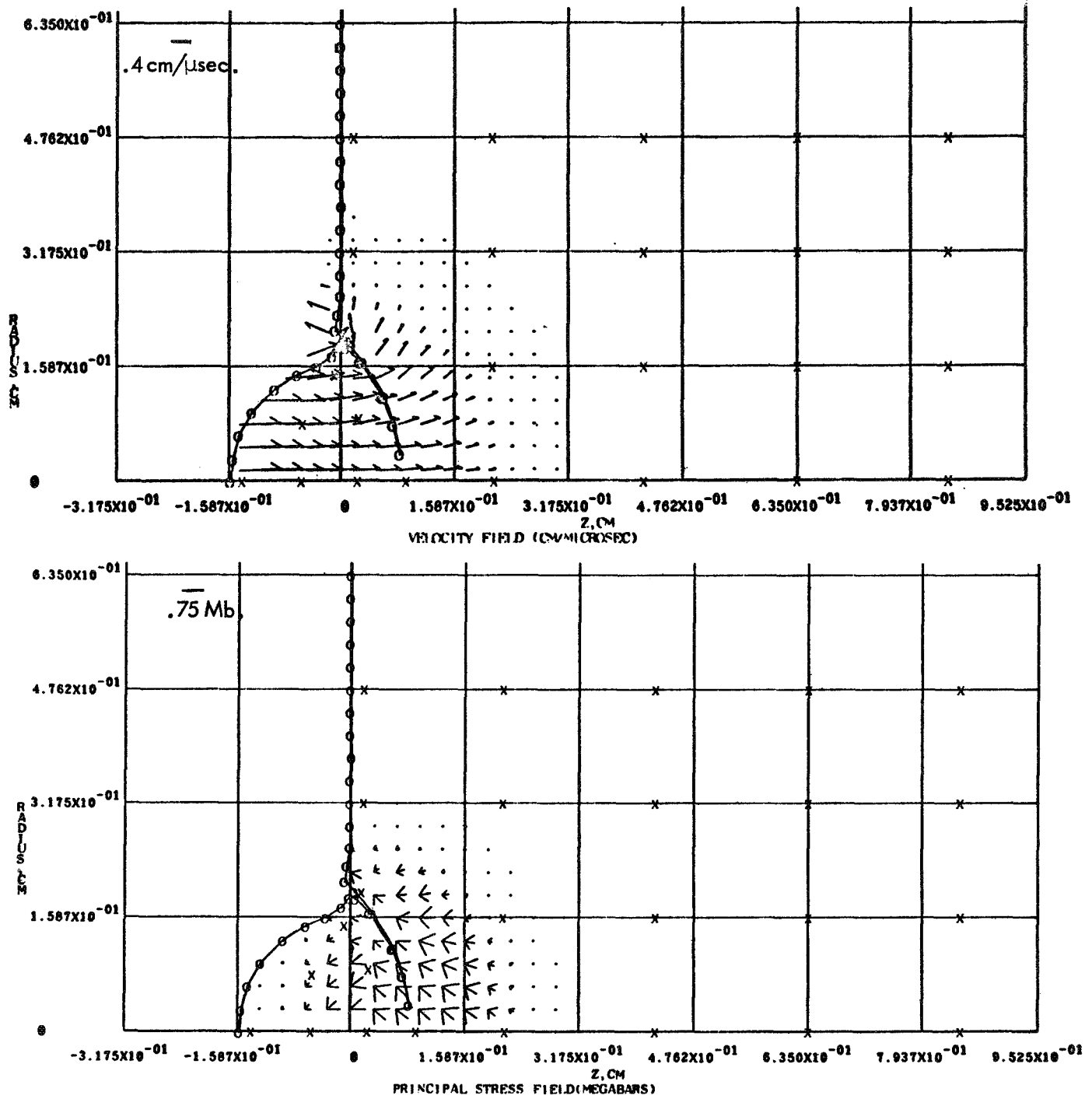


Figure C1. Velocity and Principal Stress Fields at $t = .216 \mu\text{sec}$
 $(\frac{U_0 t}{D} = .510)$ for the Large Configuration (Case 1,
 $U_0 = .75 \text{ cm}/\mu\text{sec.}, \bar{\phi} = 10^4$).

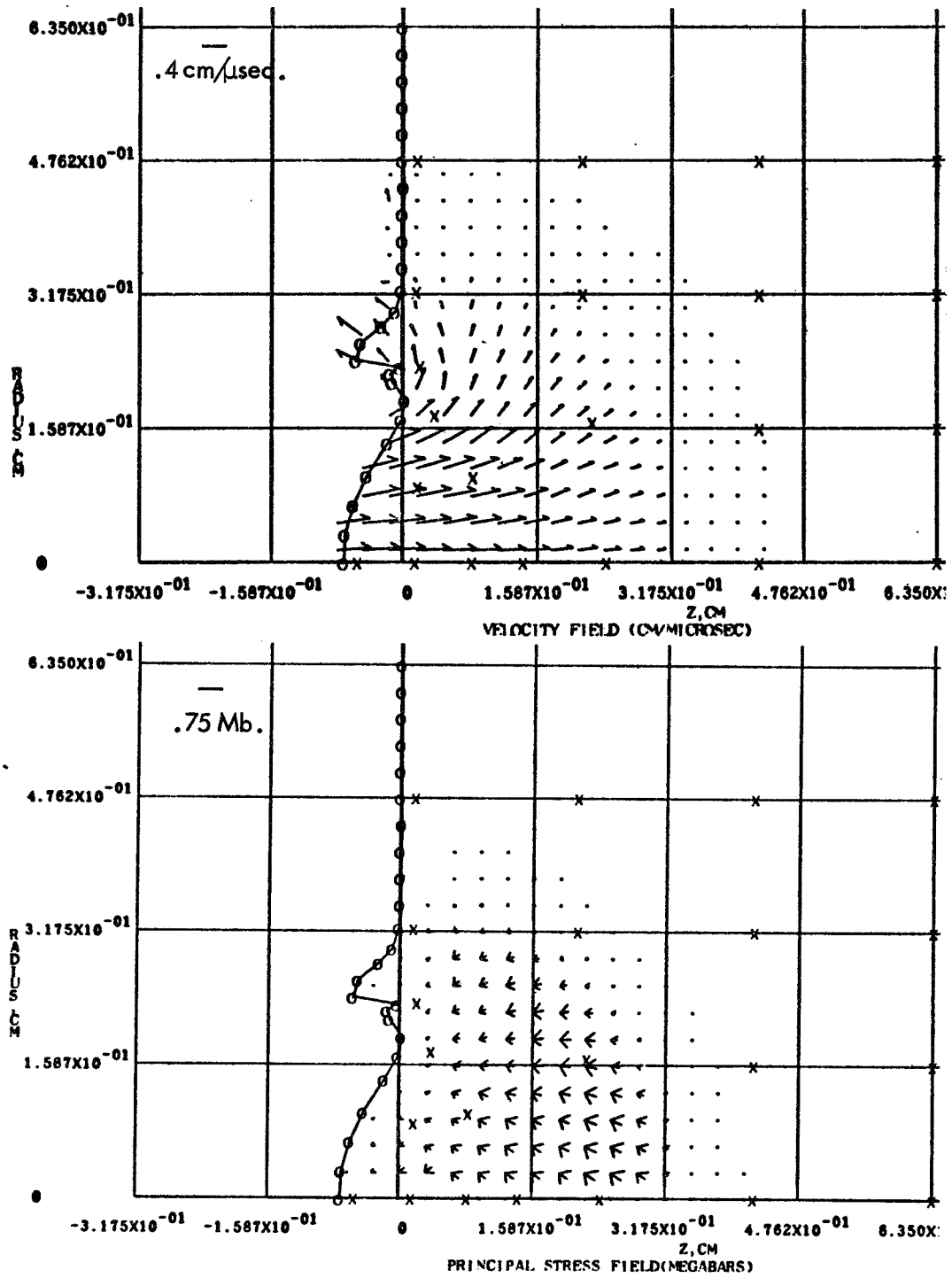


Figure C2. Velocity and Principal Stress Fields at $t = .345 \mu\text{sec}$
 $(\frac{U_0 t}{D} = .815)$ for the Large Configuration (Case 1,
 $U_0 = .75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^4$).

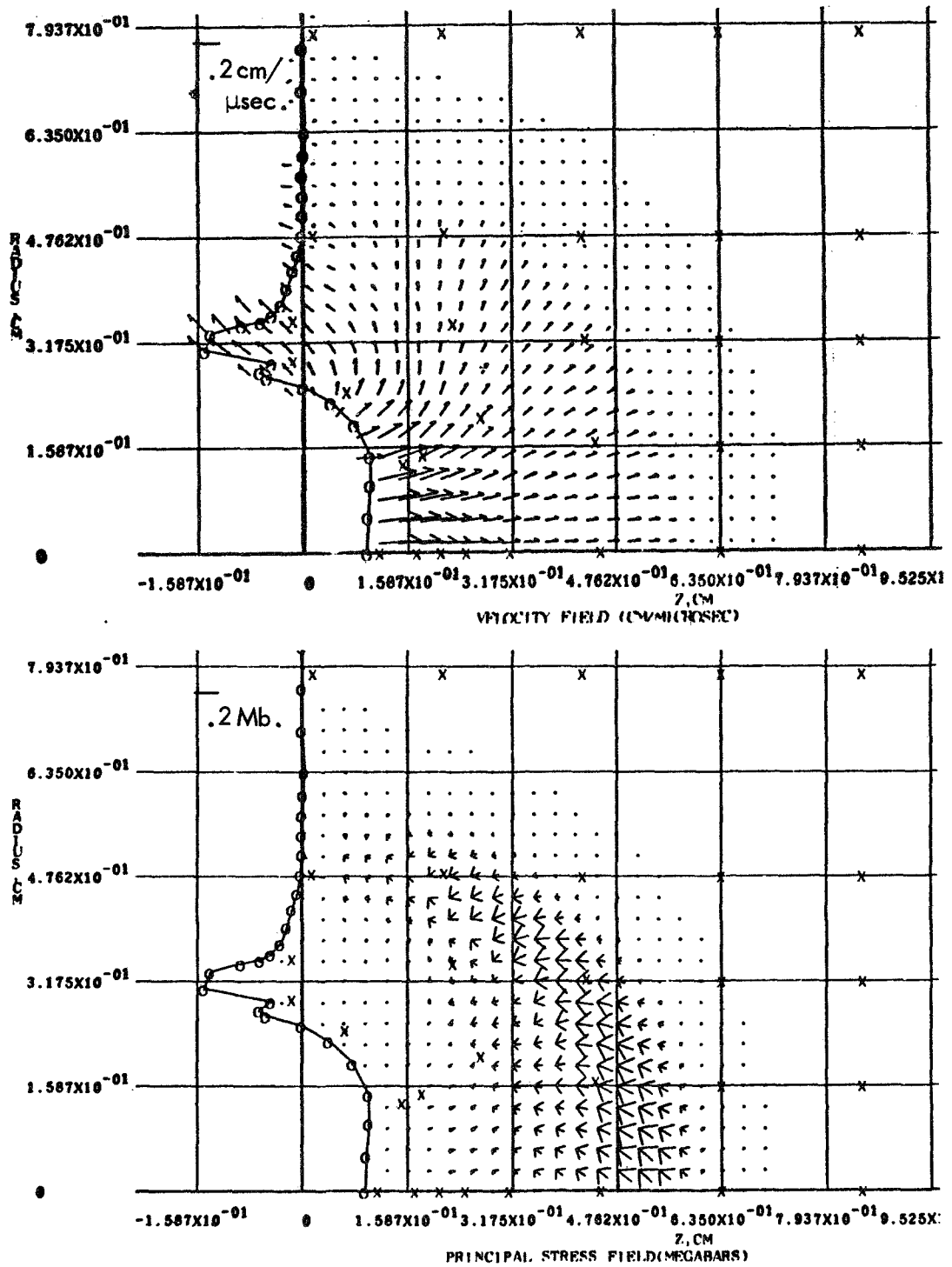


Figure C3. Velocity and Principal Stress Fields at $t = .70 \mu \text{sec}$
 $(\frac{U_0 t}{D} = 1.654)$ for the Large Configuration (Case 1,
 $U_0 = .75 \text{ cm}/\mu \text{sec.}, \bar{\phi} = 10^4$).

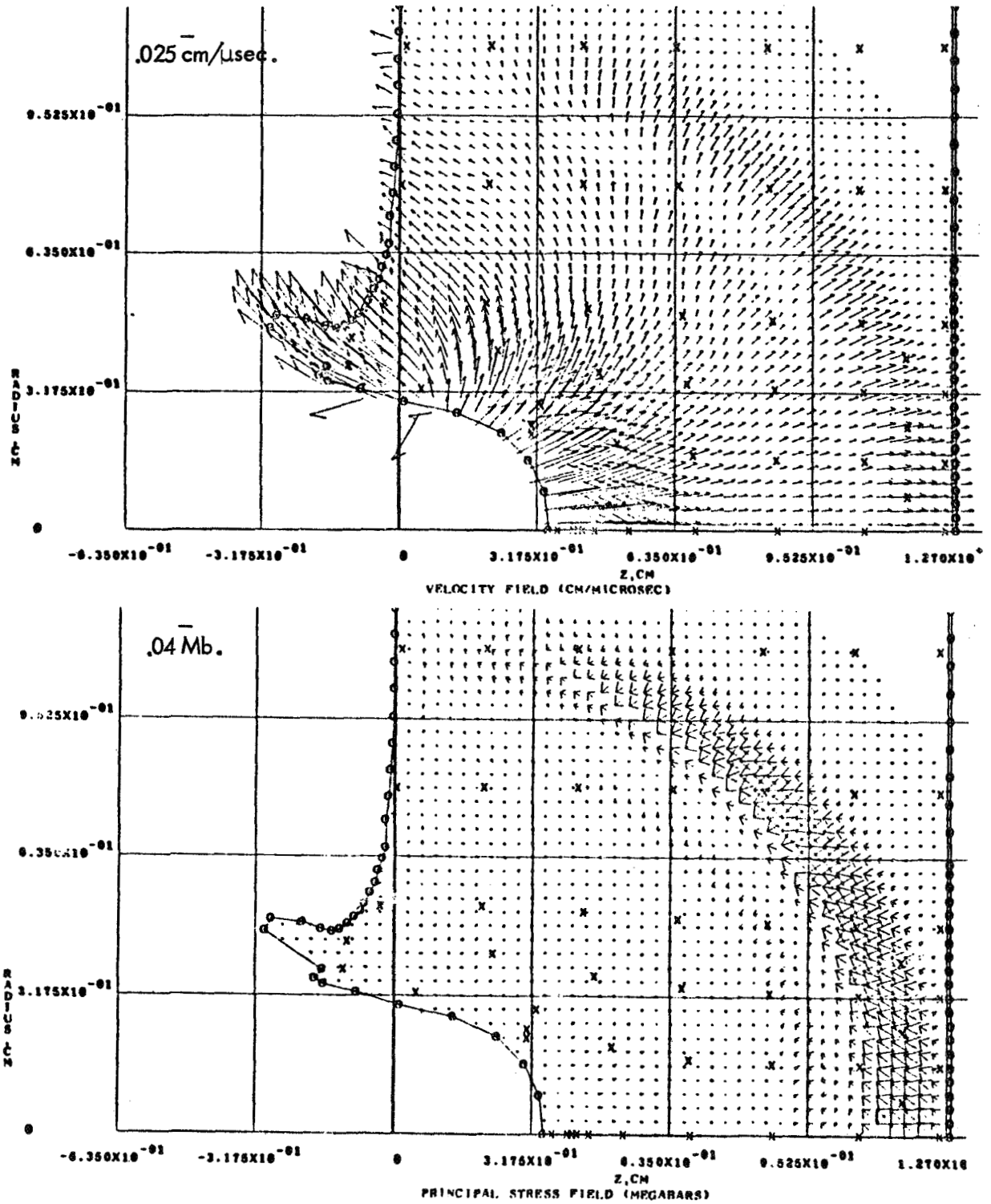


Figure C4. Velocity and Principal Stress Fields at $t = 1.78 \mu\text{sec.}$,

$$\left(\frac{U_0 t}{D}\right) = 4.205 \text{ for the Large Configuration (Case 1,}$$

$$U_0 = .75 \text{ cm}/\mu\text{sec.}, \bar{\phi} = 10^4).$$

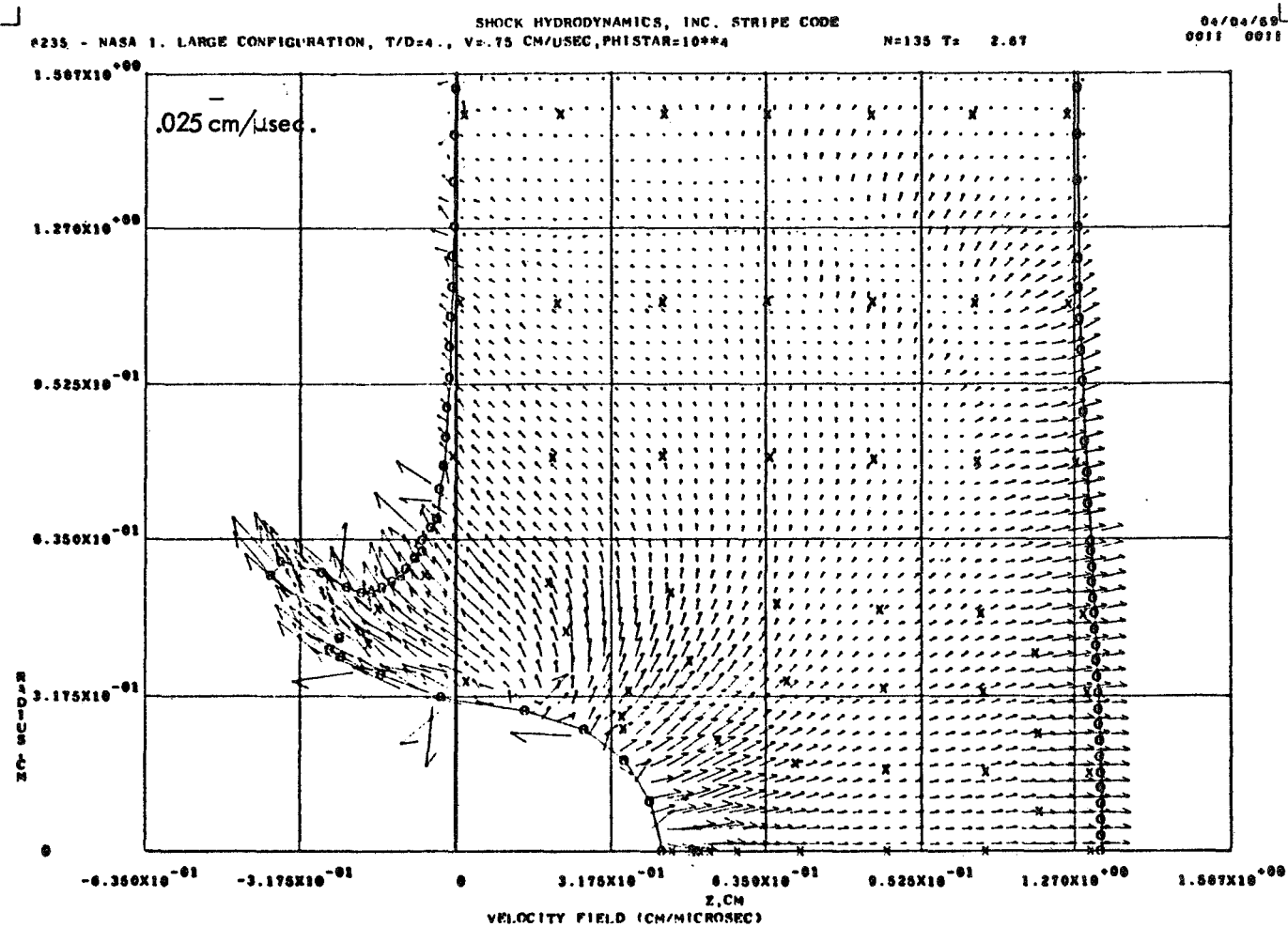


Figure C5. Velocity Field at $t = 2.67 \mu\text{sec}$ ($\frac{U_0 t}{D} = 6.307$)
for the Large Configuration (Case 1, $U_0 =$
.75 cm/ μsec ., $\bar{\varphi} = 10^4$).

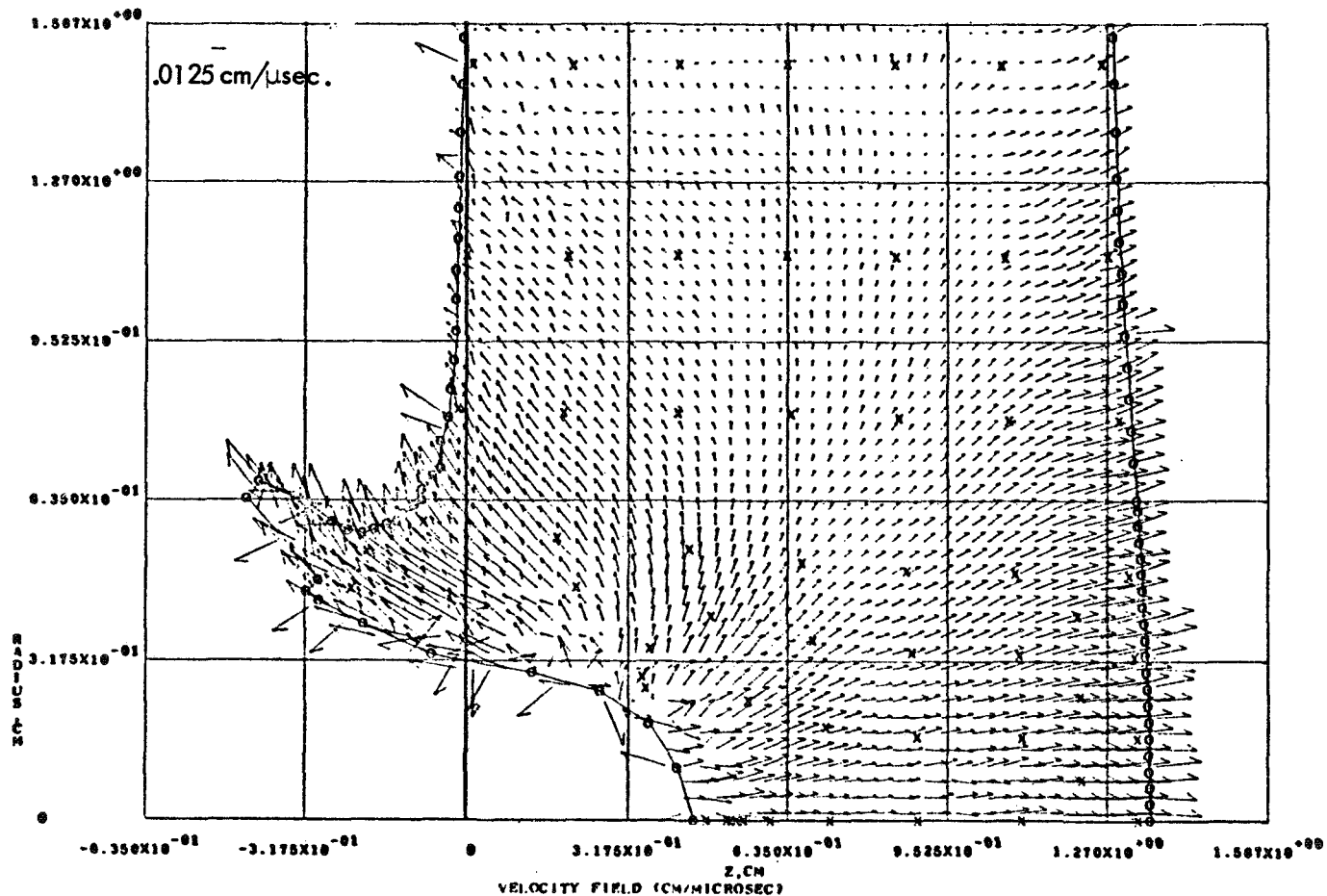


Figure C6. Velocity Field at $t = 3.46 \mu\text{sec}$ ($\frac{U_0 t}{D} = 8.173$)
for the Large Configuration (Case 1, $U_0 =$
.75 cm/ μ sec., $\bar{\phi} = 10^4$).

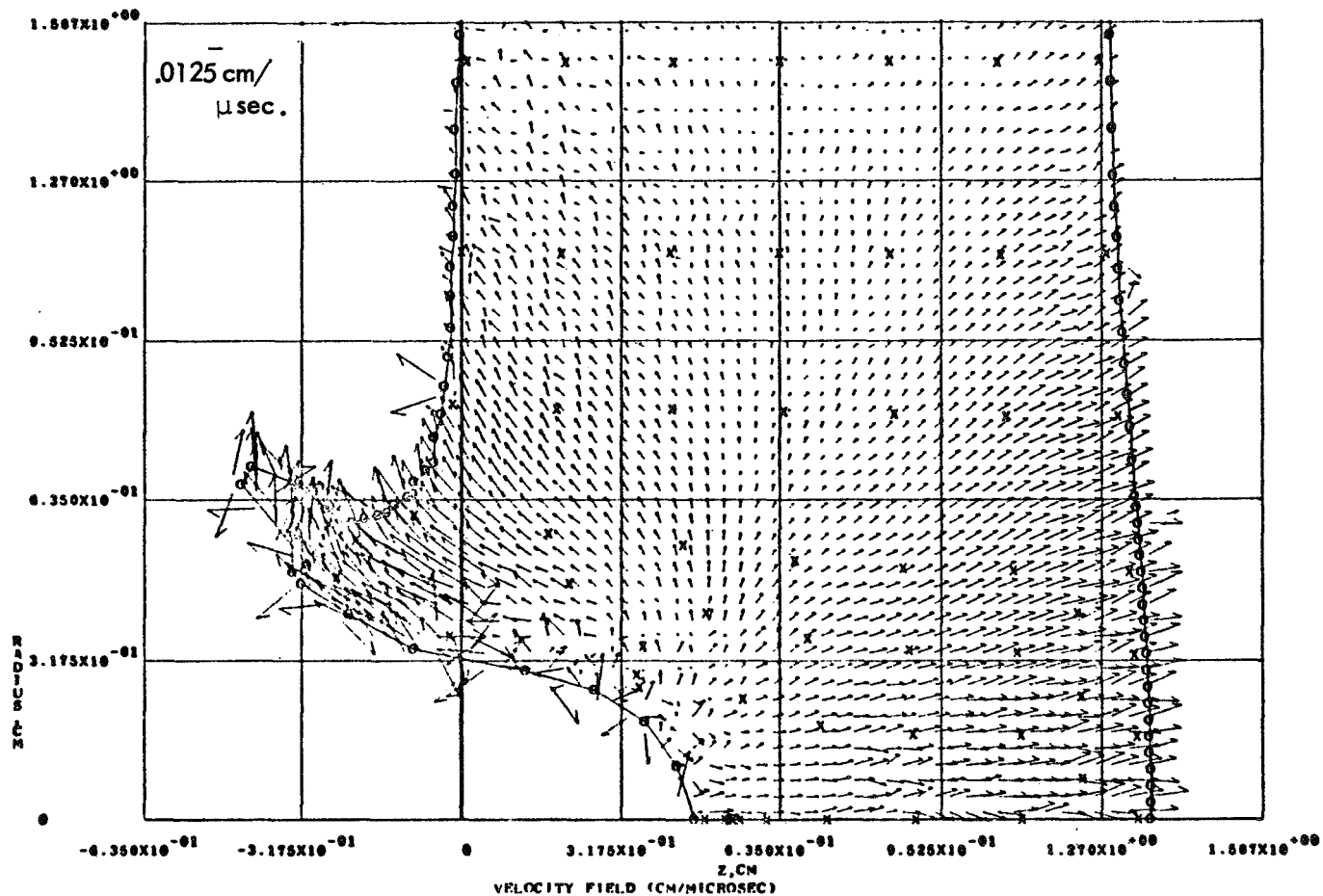


Figure C.7. Velocity Field at $t = 3.97 \mu\text{sec}$ ($\frac{U_0 t}{D} = 9.378$)
for the Large Configuration (Case 1, $U_0 =$
.75 cm/ $\mu\text{sec.}$, $\bar{\phi} = 10^4$).

Particle Velocity and Principal Stress
Field Plots for Case 2

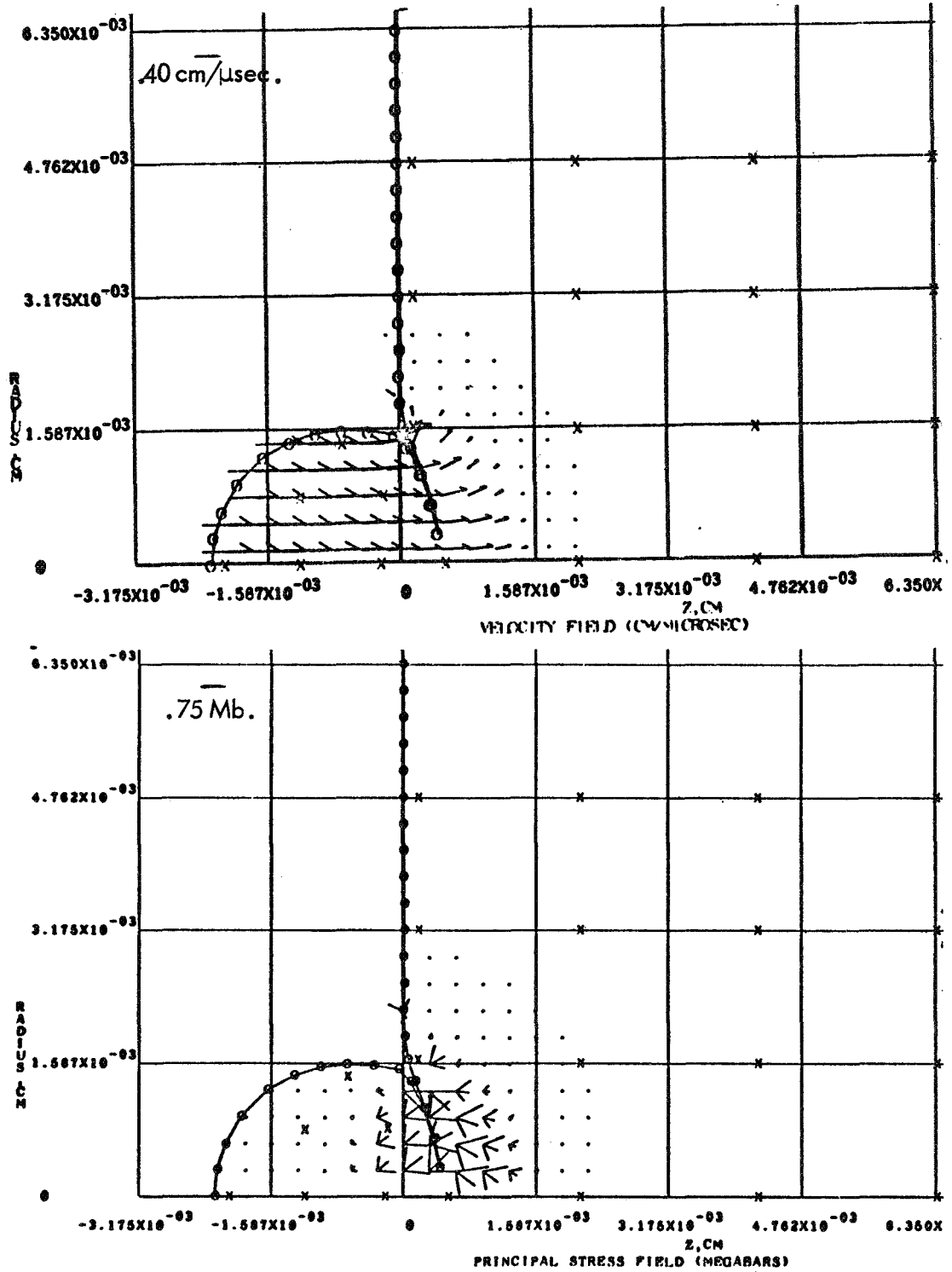


Figure C8. Velocity and Principal Stress Fields at $t = .12 \times 10^{-2} \mu\text{sec}$
 $\left(\frac{U_0 t}{D} = .283\right)$ for the Small Configuration (Case 2,
 $U_0 = .75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^4$).

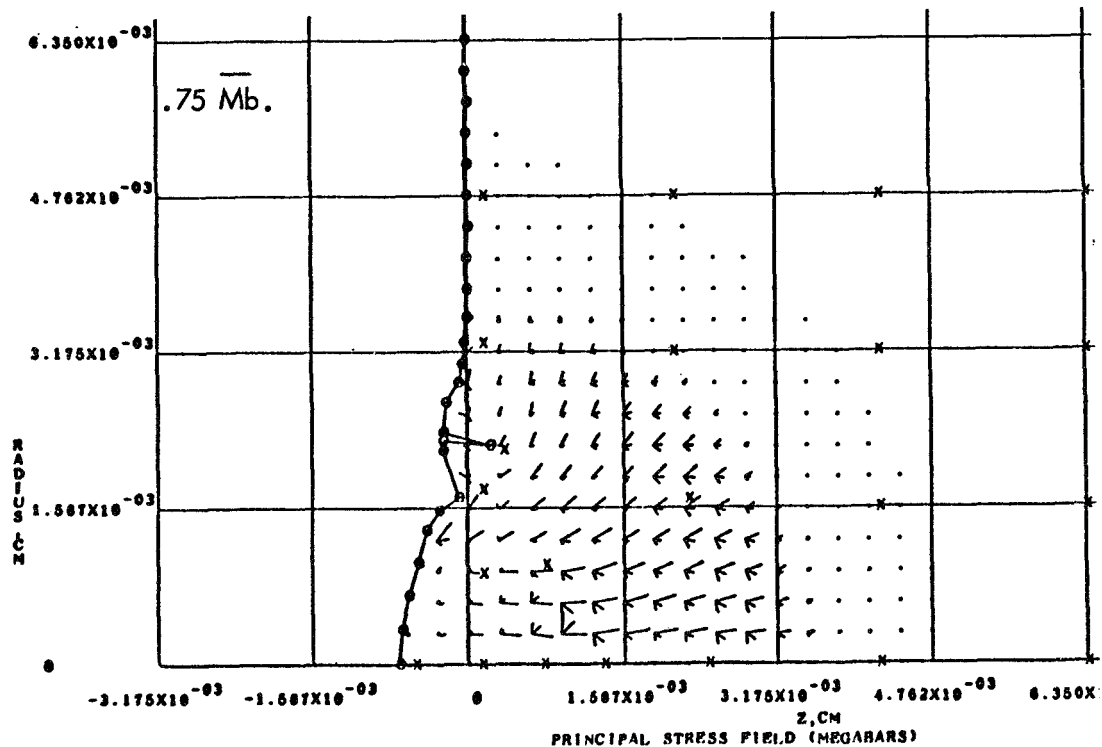
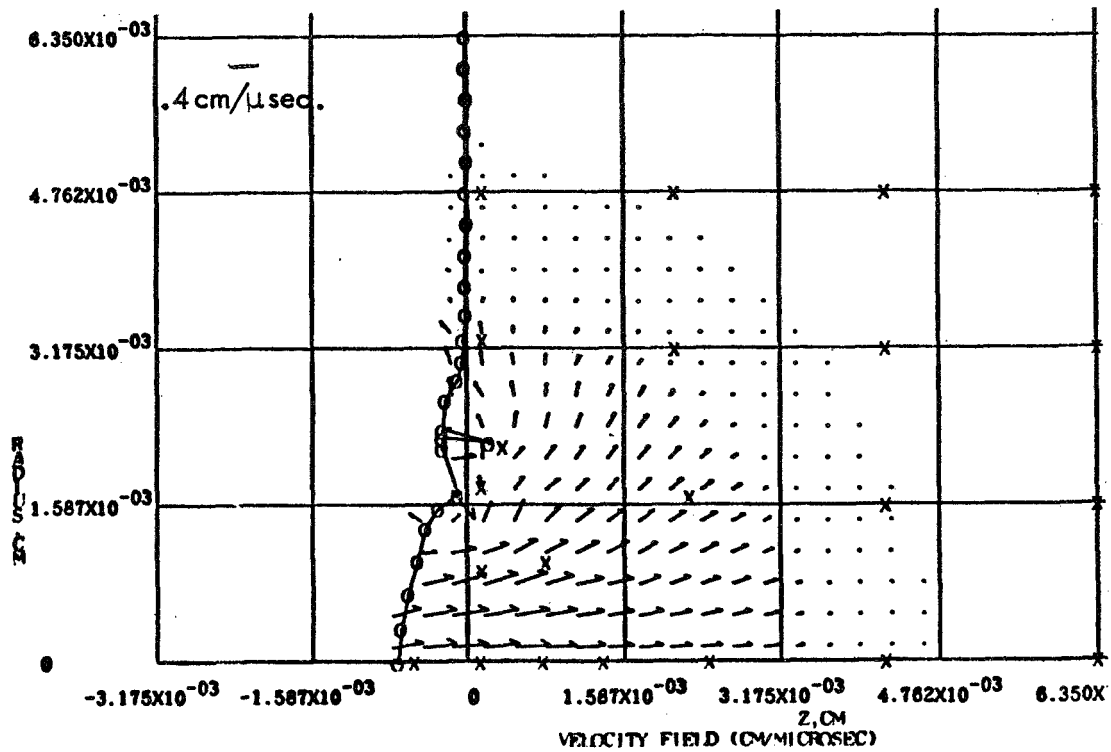


Figure C9. Velocity and Principal Stress Fields at $t = .35 \times 10^{-2} \mu\text{sec}$
 $(\frac{U_0 t}{D} = .827)$ for the Small Configuration (Case 2, $U_0 =$
 $.75 \text{ cm}/\mu\text{sec.}, \bar{\phi} = 10^4$).

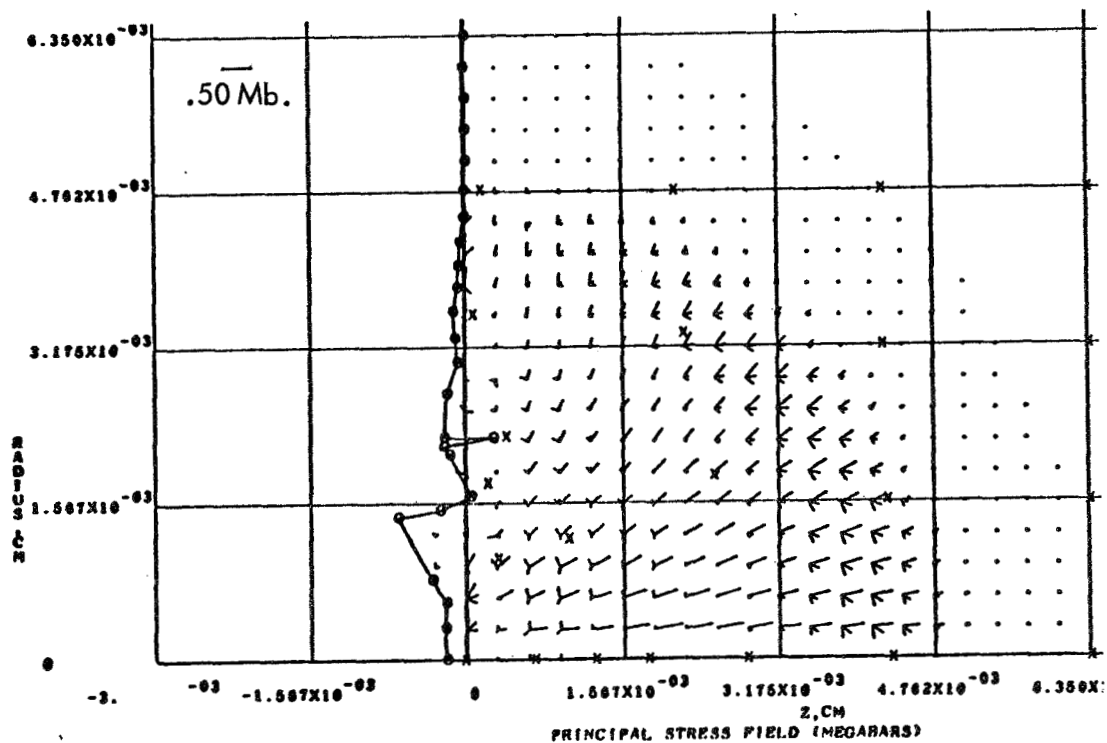
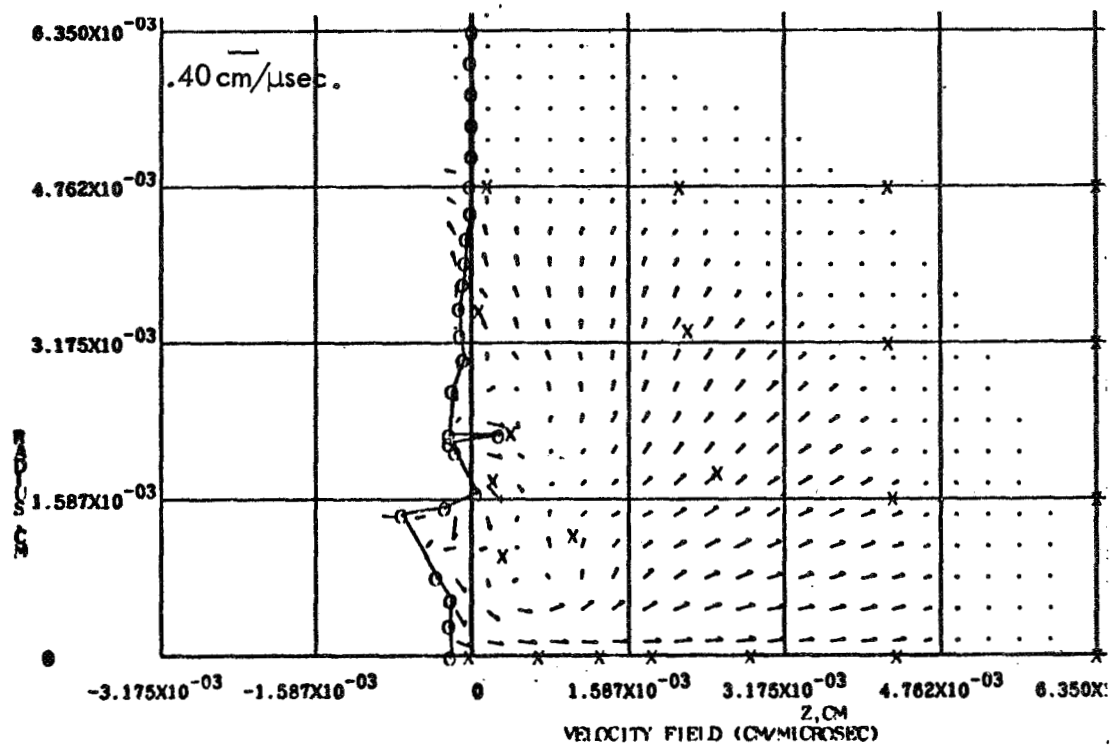


Figure C10. Velocity and Principal Stress Fields at $t = .52 \times 10^{-2} \mu \text{sec}$
 $\left(\frac{U_0 t}{D} = 1.228\right)$ for the Small Configuration (Case 2, $U_0 = .75 \text{ cm}/\mu \text{sec.}$, $\bar{\phi} = 10^4$).

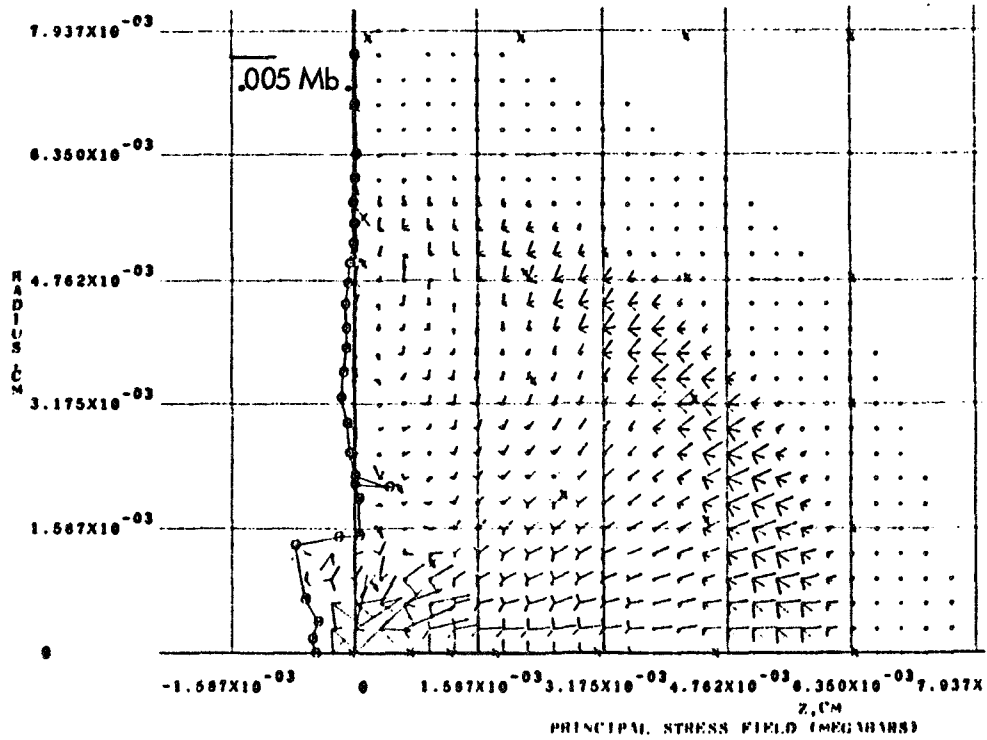
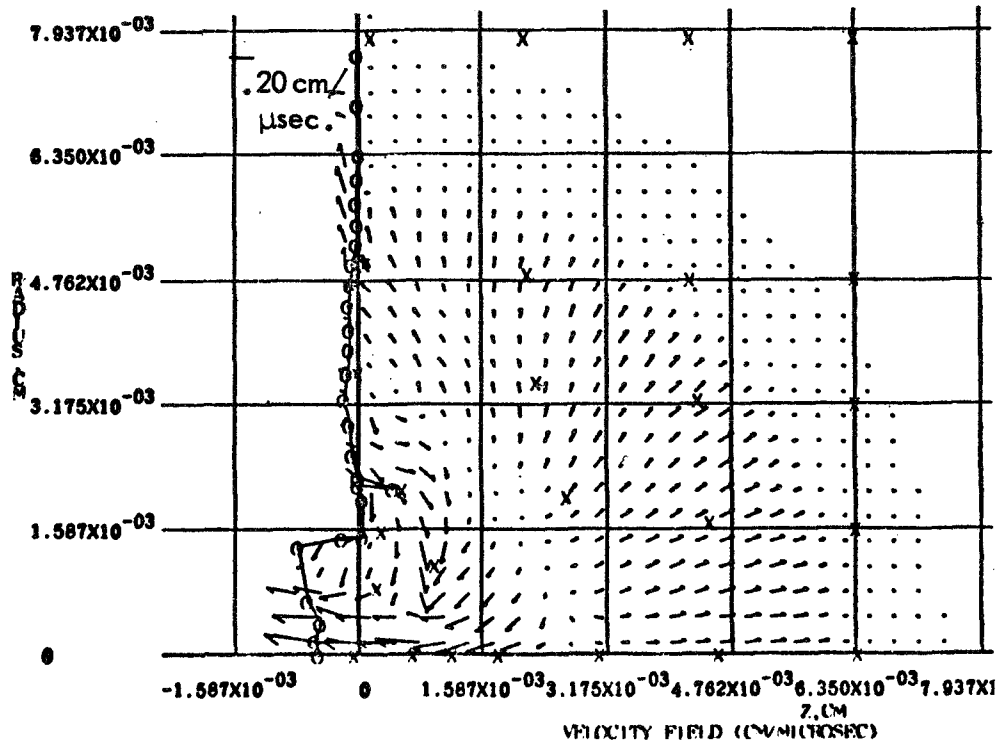


Figure C11. Velocity and Principal Stress Fields at $t = .69 \times 10^{-2} \mu\text{sec}$
 $(\frac{U_0 t}{D} = 1.630)$ for the Small Configuration (Case 2, $U_0 =$
 $.75 \text{ cm}/\mu\text{sec.}, \bar{\phi} = 10^4$).

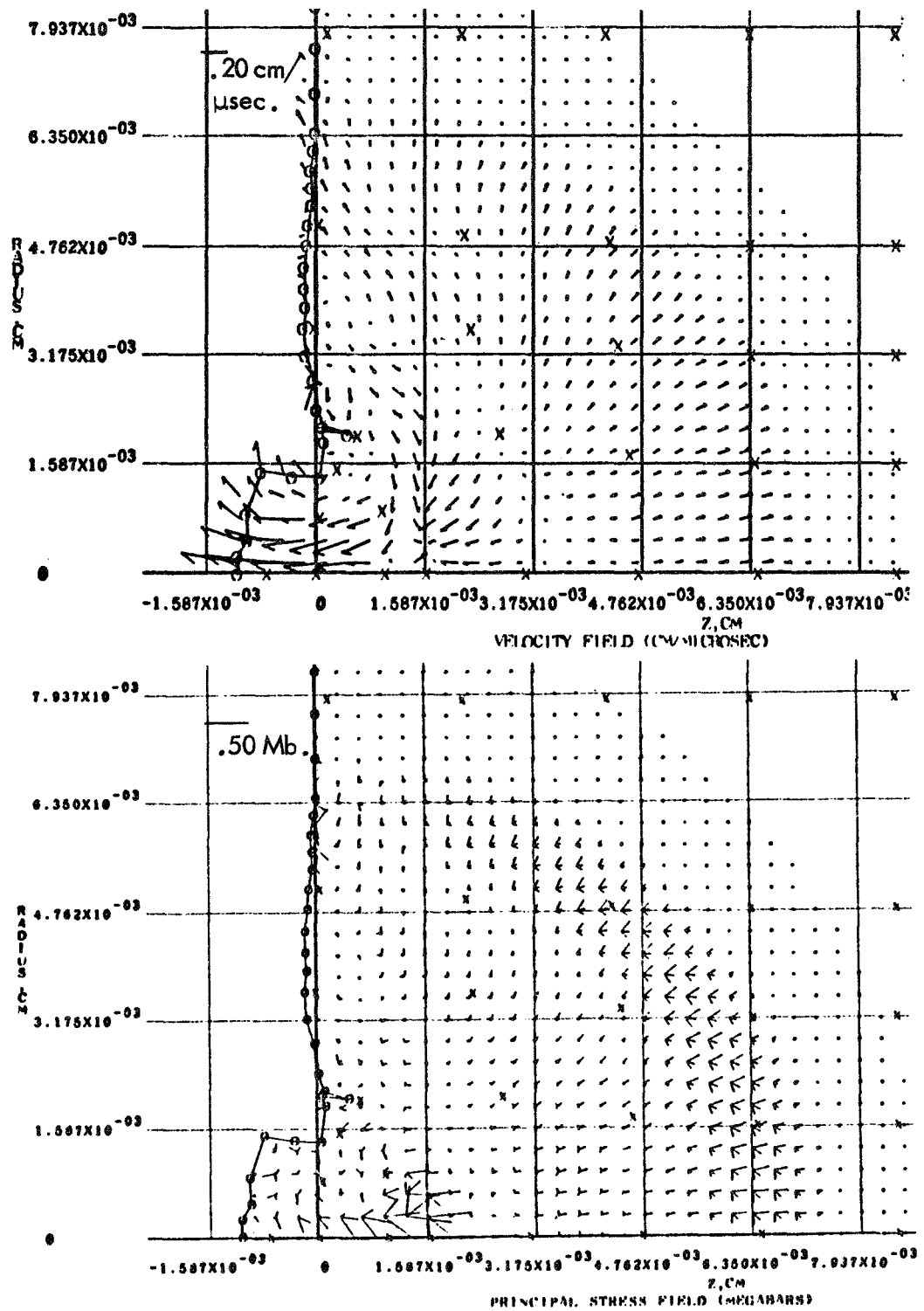


Figure C12. Velocity and Principal Stress Fields at $t = .83 \times 10^{-2} \mu\text{sec}$
 $(\frac{U_0 t}{D} = 1.961)$ for the Small Configuration (Case 2, $U_0 =$
 $.75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^4$).

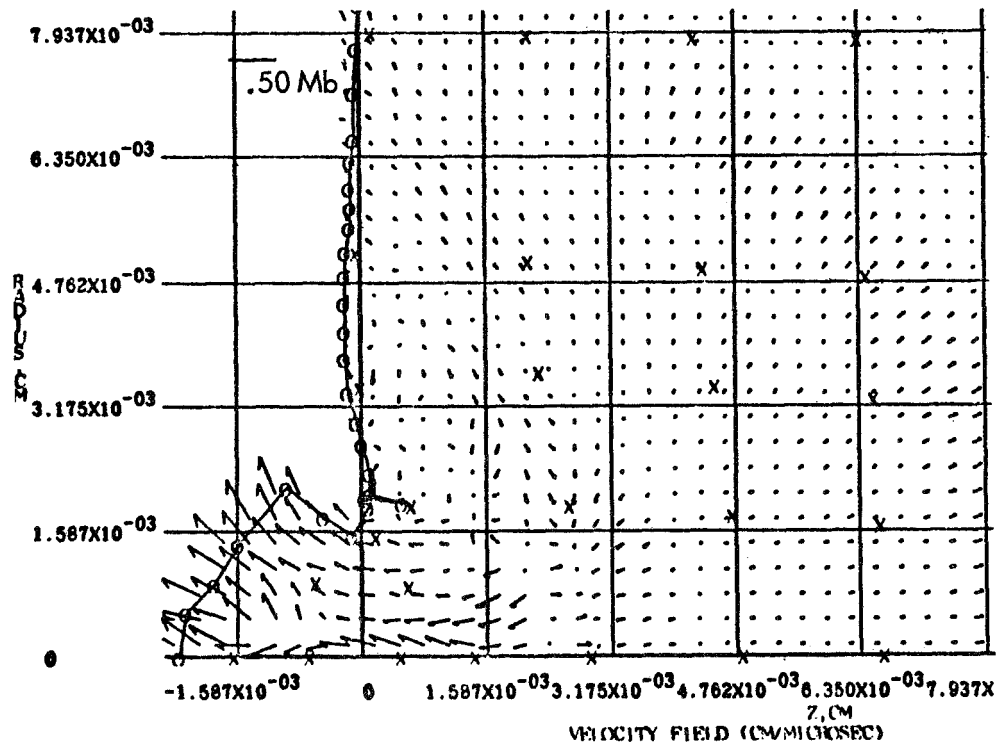
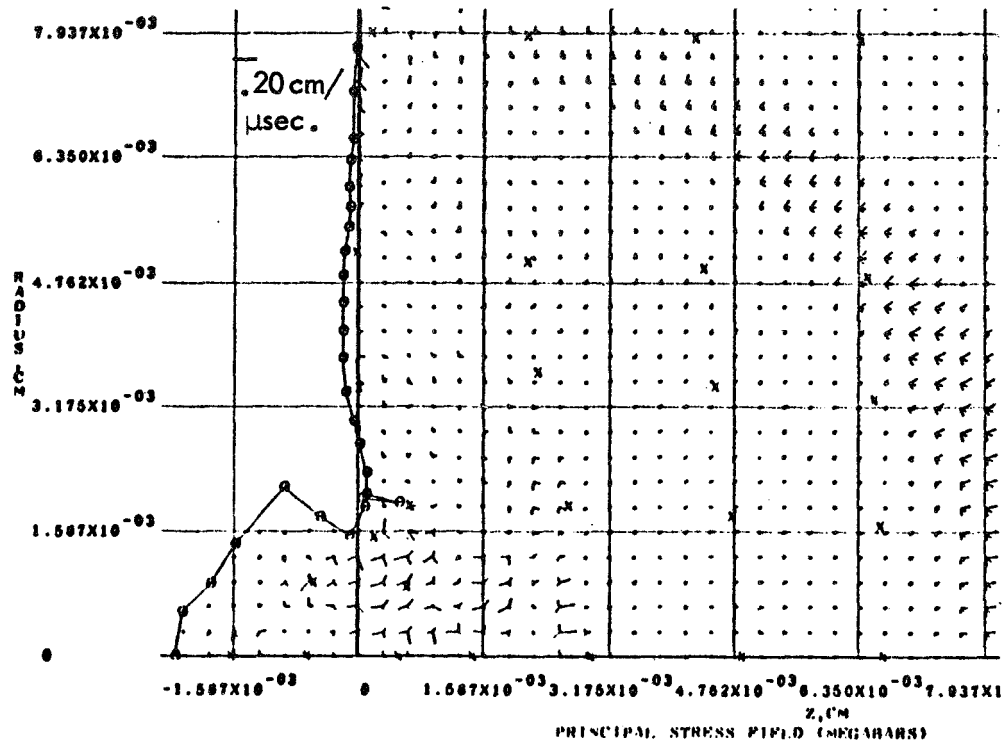


Figure C13. Velocity and Principal Stress Fields at $t = 1.11 \times 10^{-2} \mu$ sec
 $(\frac{U_0 t}{D} = 2.622)$ for the Small Configuration (Case 2, $U_0 =$
 $.75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^4$).

Particle Velocity and Principal Stress
Field Plots for Case 4

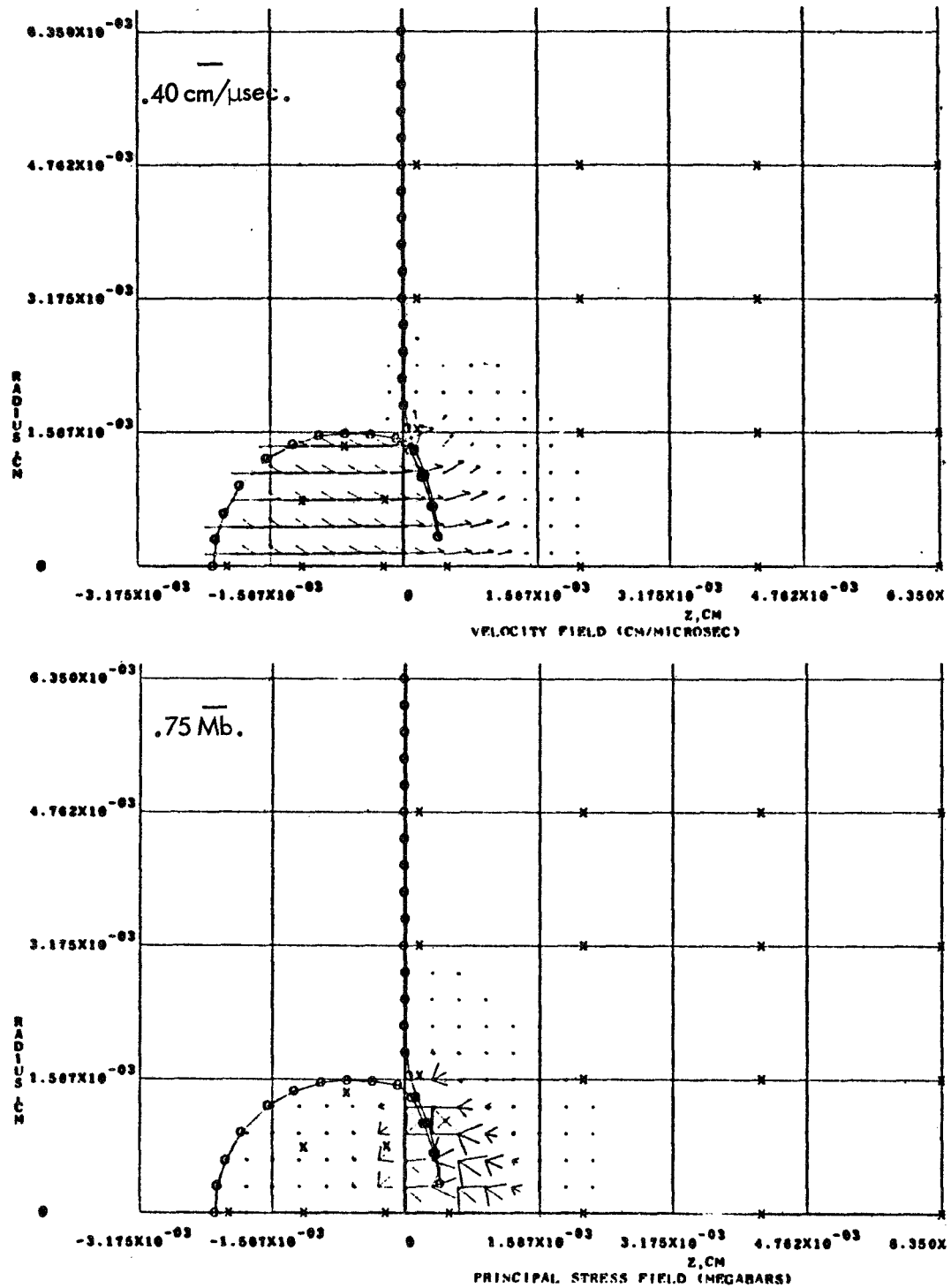


Figure C14. Velocity and Principal Stress Fields at $t = .12 \times 10^{-2} \mu\text{sec}$
 $(\frac{U_0 t}{D} = .283)$ for the Small Configuration (Case 4, $U_0 =$
 $.75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

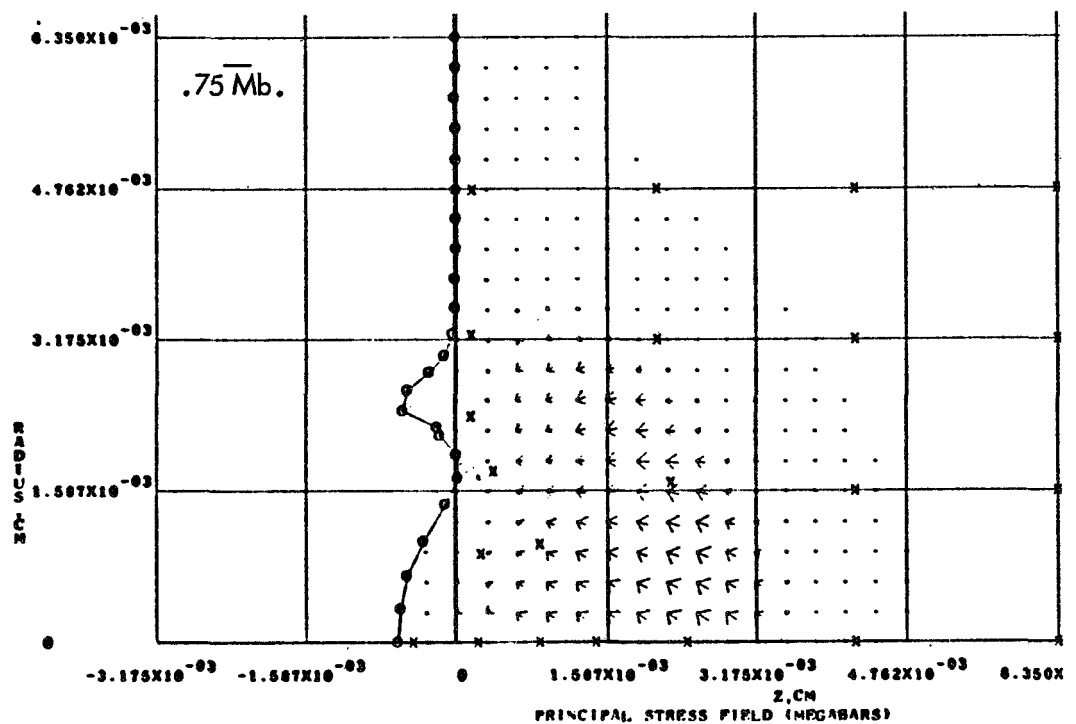
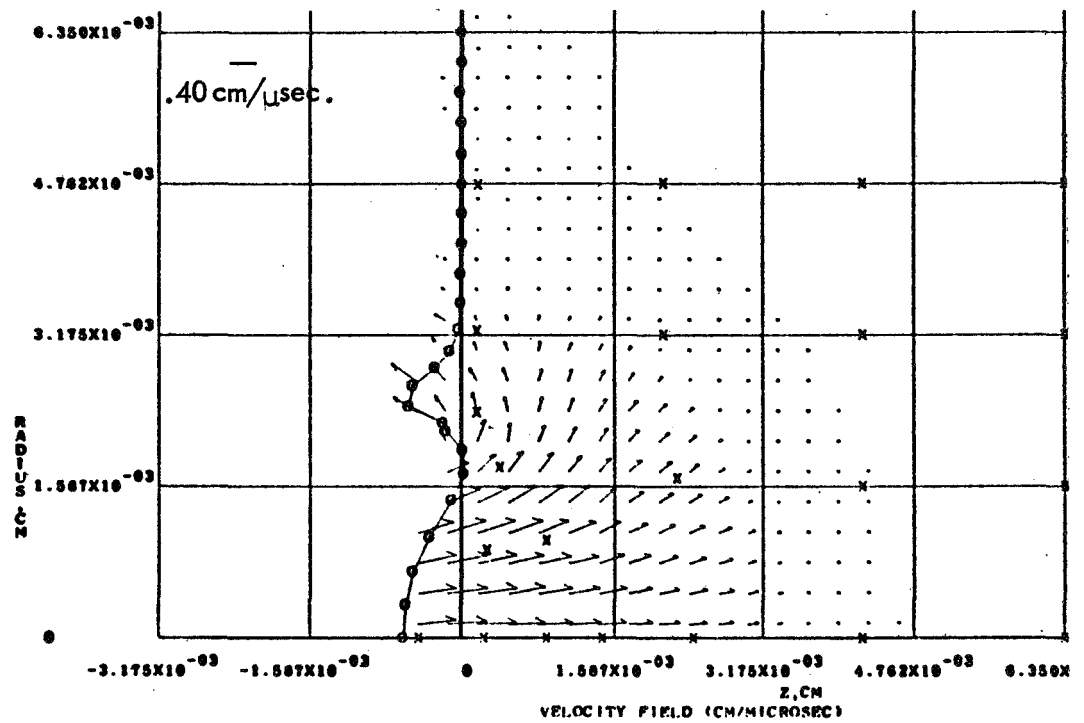


Figure C15. Velocity and Principal Stress Fields at $t = .36 \times 10^{-2} \mu\text{sec}$
 $(\frac{U_0 t}{D} = .850)$ for the Small Configuration (Case 4, $U_0 =$
 $.75 \text{ cm}/\mu\text{sec.}, \bar{\phi} = 10^6$).

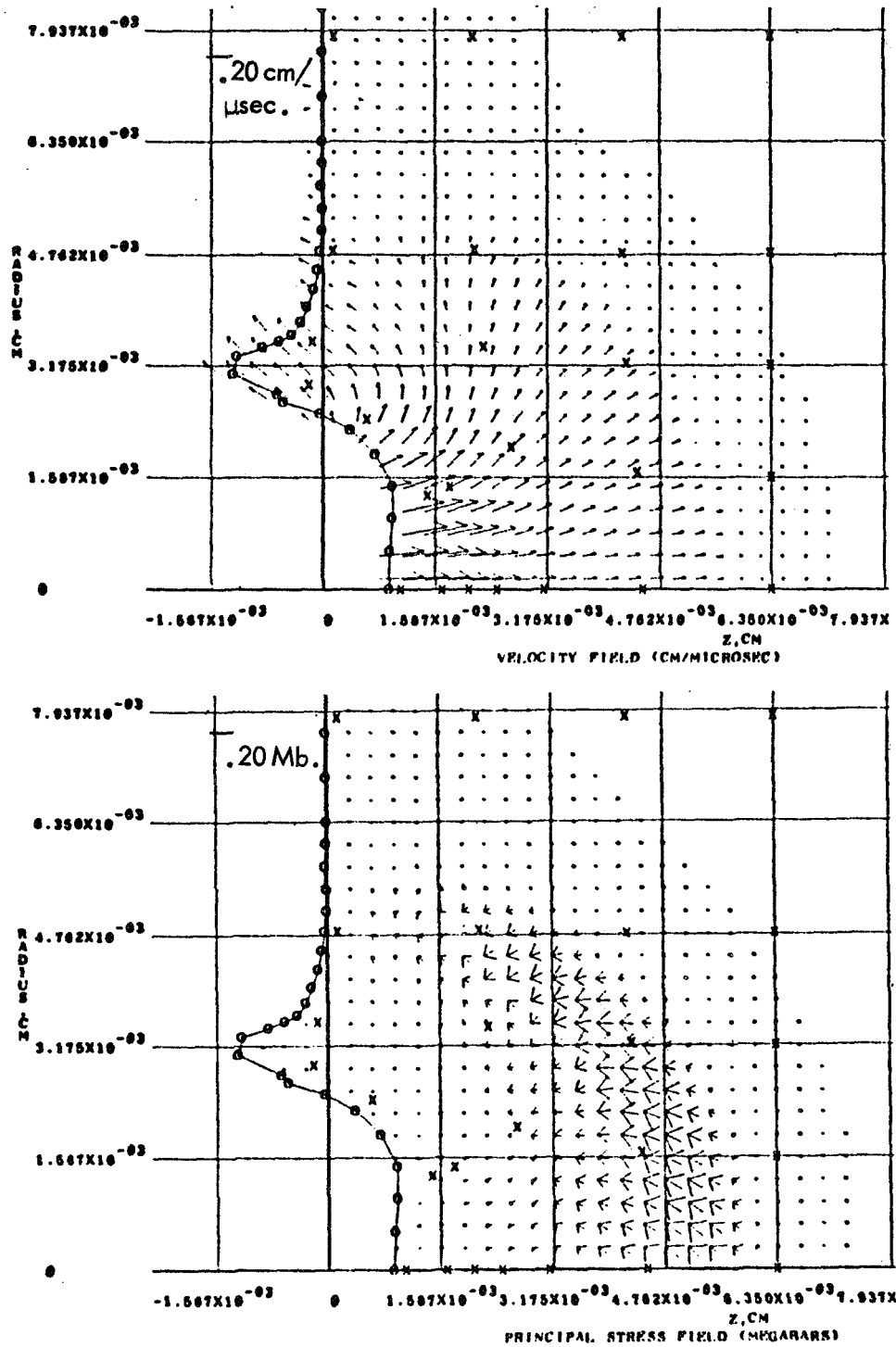


Figure C16. Velocity and Principal Stress Fields at $t = .70 \times 10^{-2} \mu \text{sec}$
 $(\frac{U_0 t}{D} = 1.654)$ for the Small Configuration (Case 4, $U_0 =$
 $.75 \text{ cm}/\mu \text{sec.}, \bar{\phi} = 10^6$).

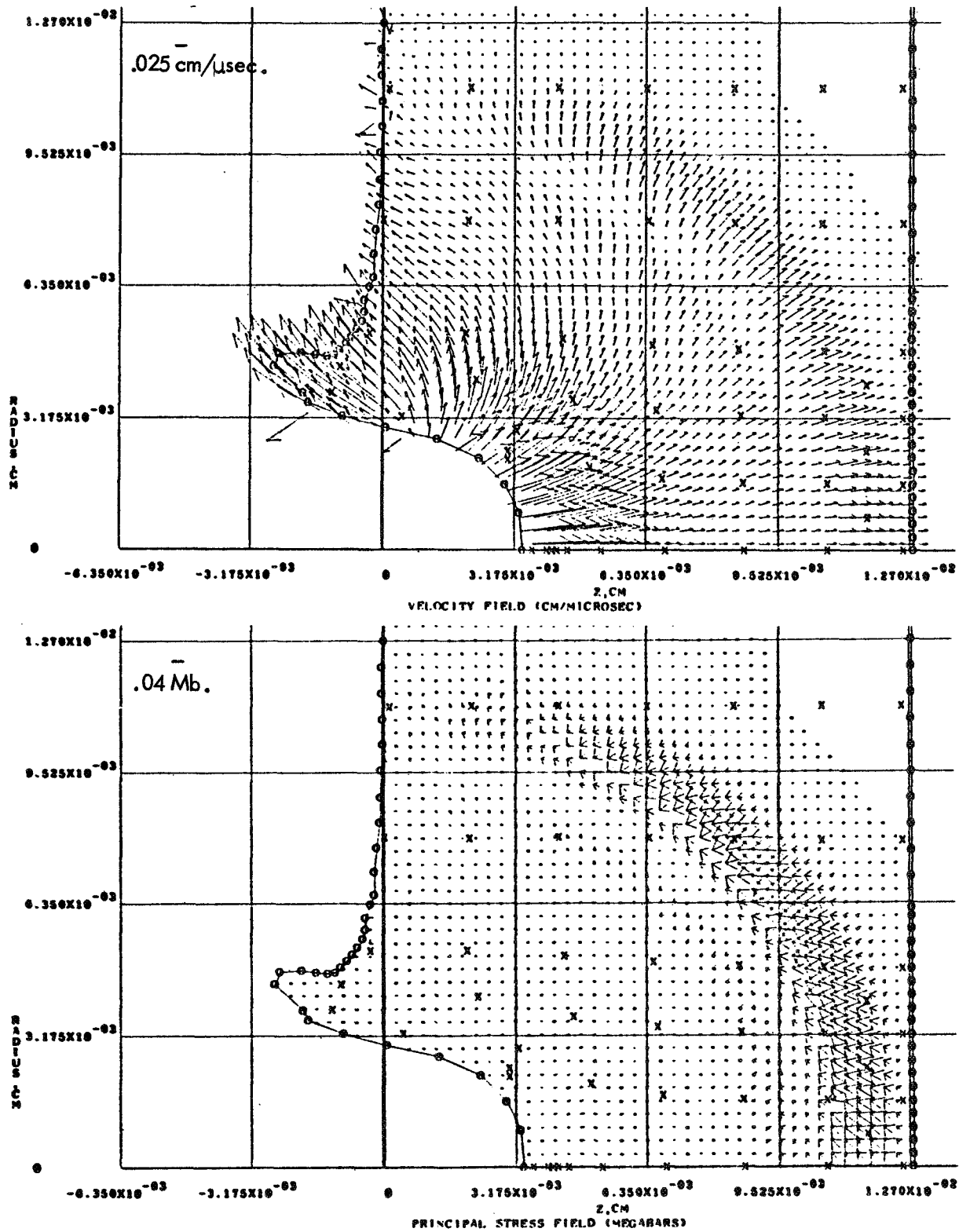


Figure C17. Velocity and Principal Stress Fields at $t = 1.76 \times 10^{-2} \mu\text{sec}$
 $(\frac{U_0 t}{D} = 4.157)$ for the Small Configuration (Case 4, $U_0 =$
 $.75 \text{ cm}/\mu\text{sec.}, \bar{\phi} = 10^6$).

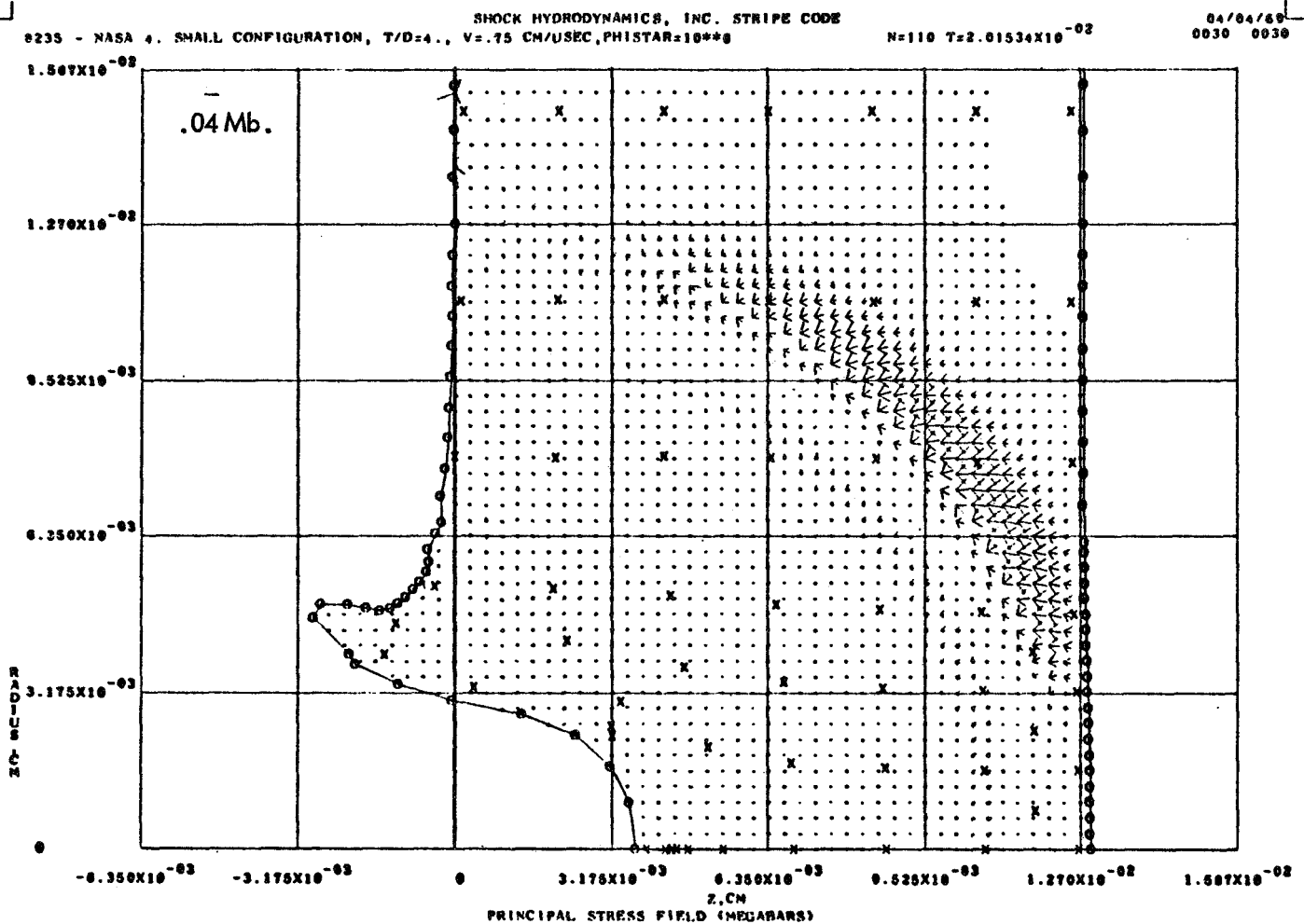


Figure C18. Principal Stress Field at $t = 2.02 \times 10^{-2} \mu\text{sec}$
 $\left(\frac{U_0 t}{D} = 4.772\right)$ for the Small Configuration
 (Case 4, $U_0 = .75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

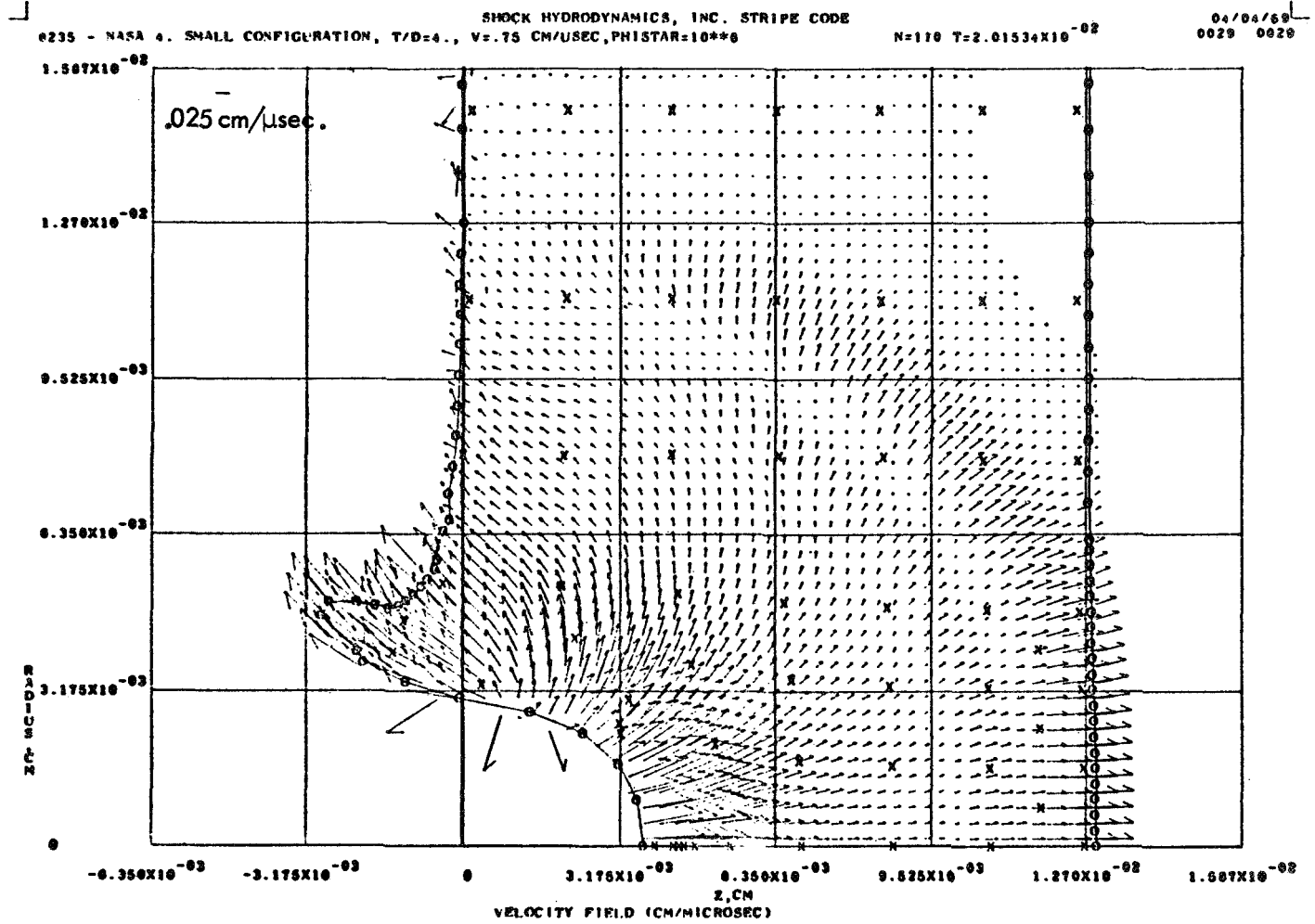


Figure C19. Velocity Field at $t = 2.02 \times 10^{-2} \mu\text{sec}$ ($\frac{U_0 t}{D} = 4.772$) for the Small Configuration (Case 4, $U_0 = .75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

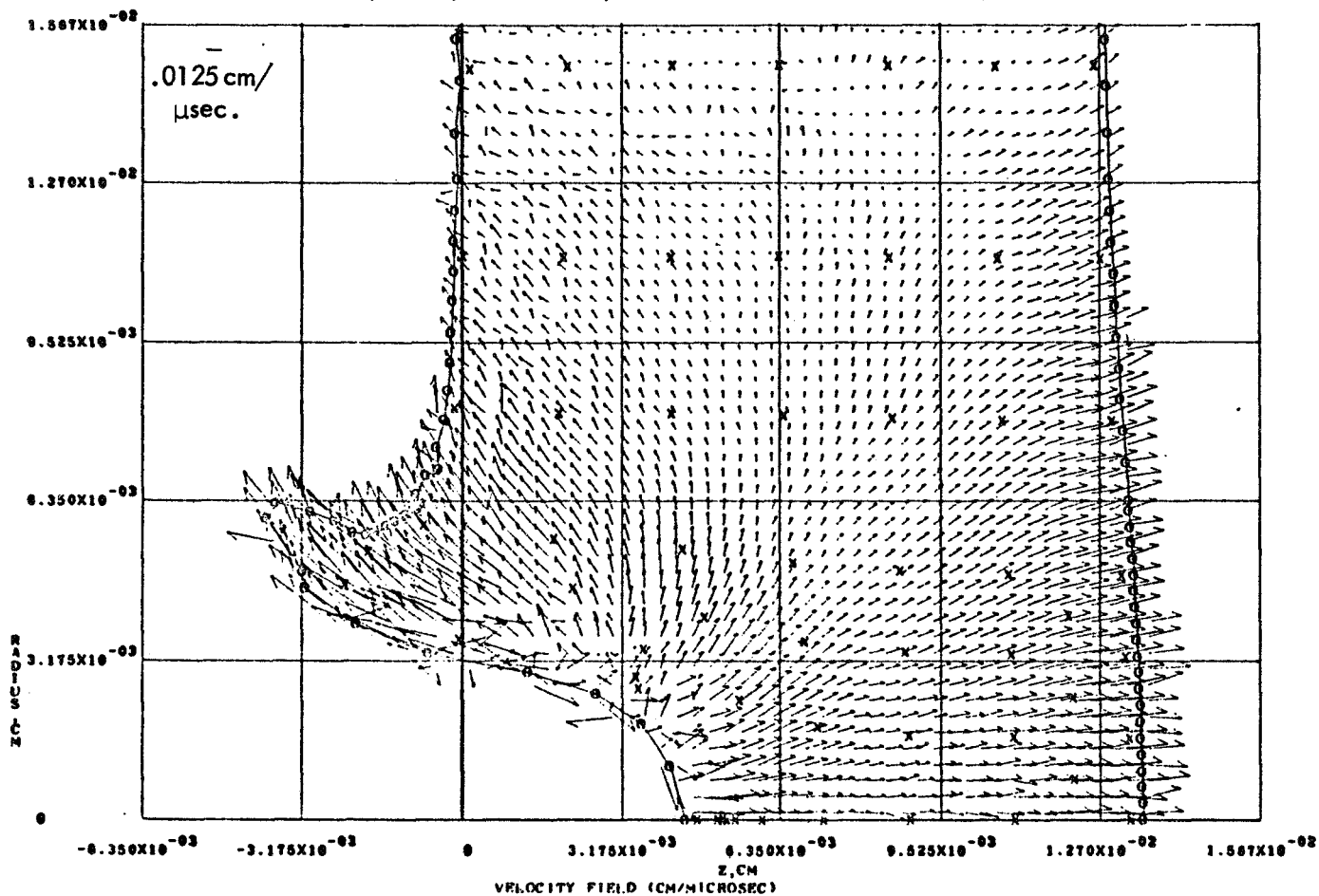


Figure C20. Velocity Field at $t = 3.43 \times 10^{-2} \mu\text{sec}$ ($\frac{U_0 t}{D} = 8.102$) for the Small Configuration (Case 4, $U_0 = .75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

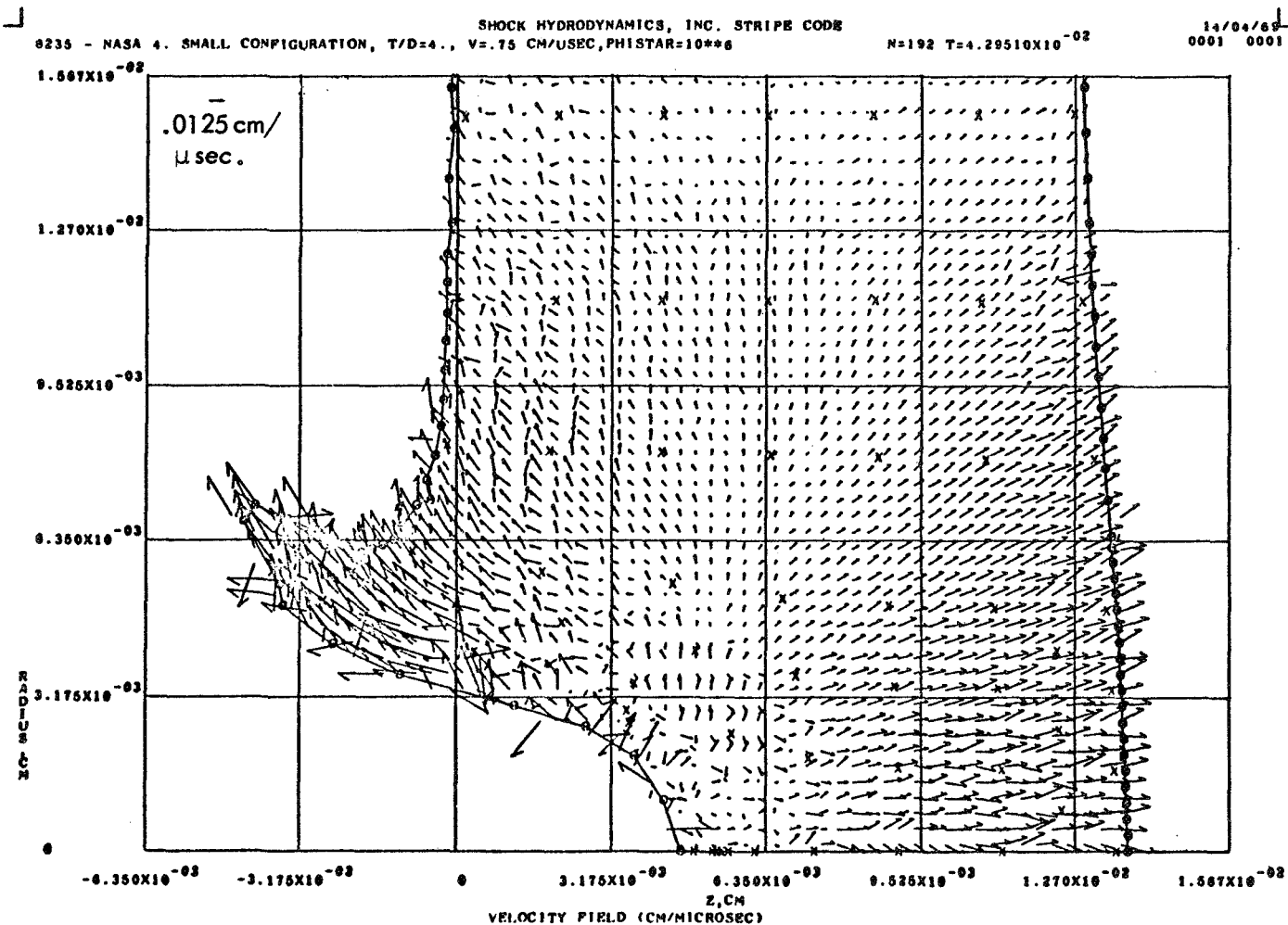


Figure C21. Velocity Field at $t = 4.30 \times 10^{-2} \mu\text{sec}$ ($\frac{U_0 t}{D} = 10.157$) for the Small Configuration (Case 4, $U_0 = .75 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

Particle Velocity and Principal Stress
Field Plots for Case 5

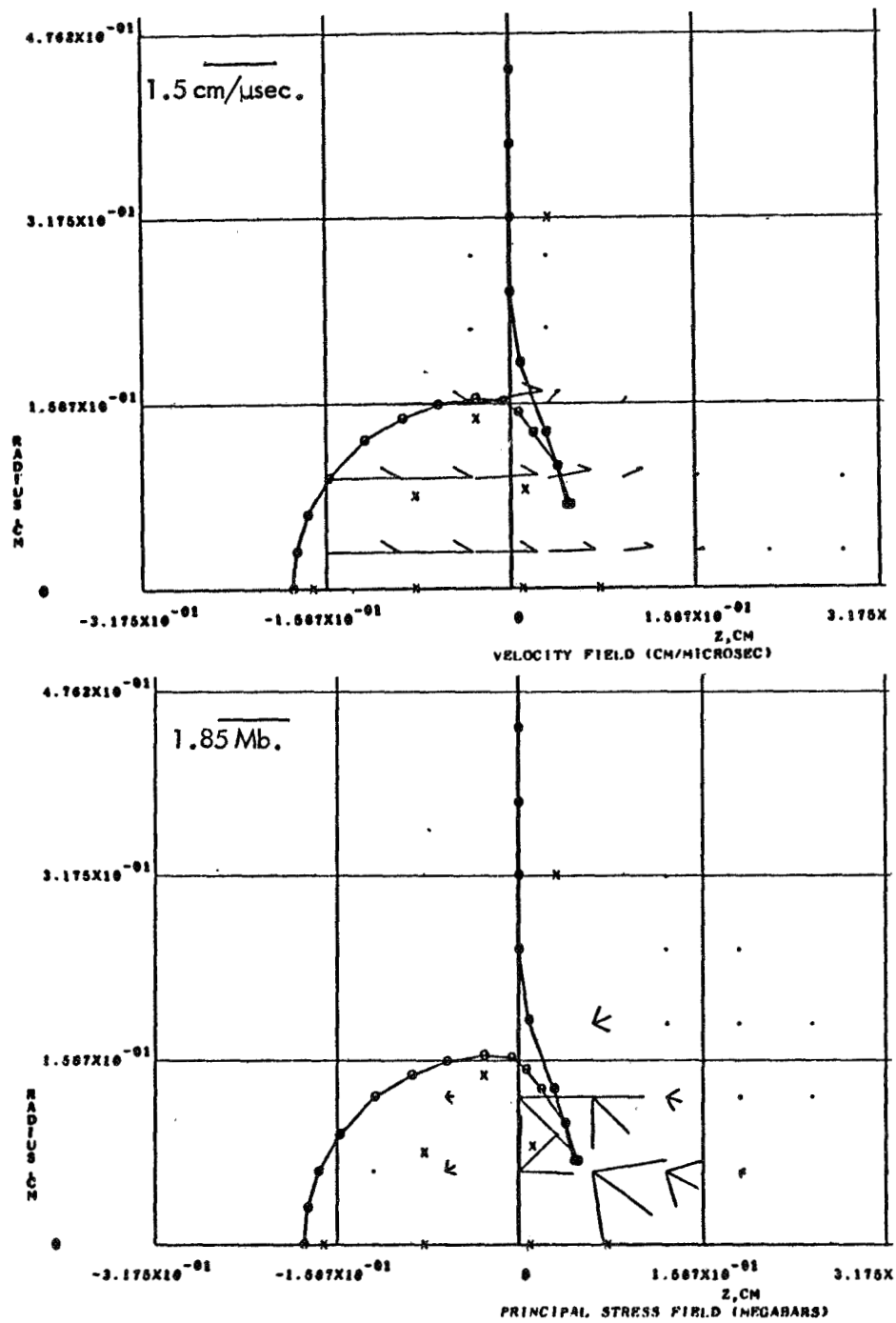


Figure C22. Velocity and Principal Stress Fields at $t = .086 \mu\text{sec}$ for the Large Configuration (Case 5, $U_0 = 1.5 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

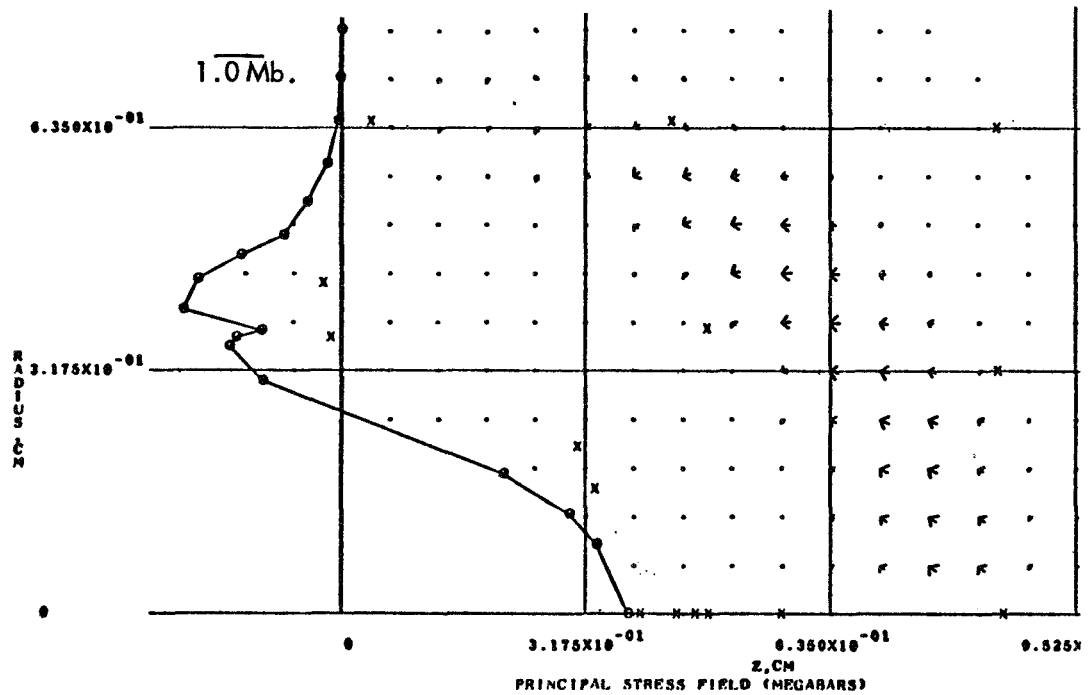
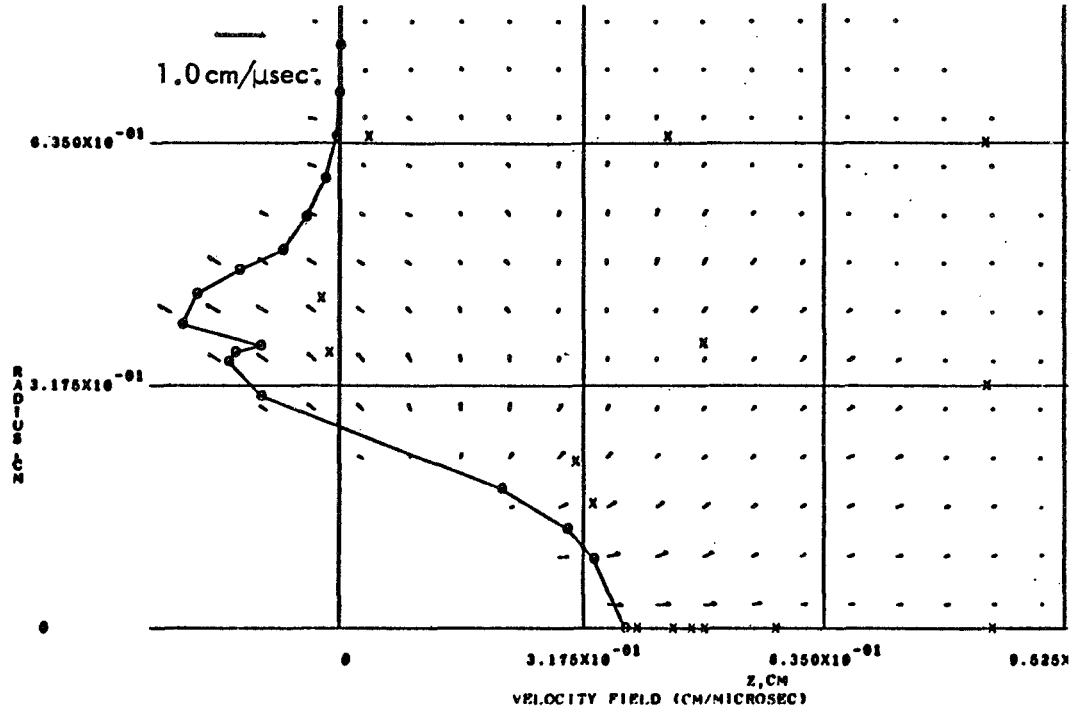


Figure C23. Velocity and Principal Stress Fields at $t = .89\mu\text{sec}$ for the Large Configuration (Case 5, $U_0 = 1.5\text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

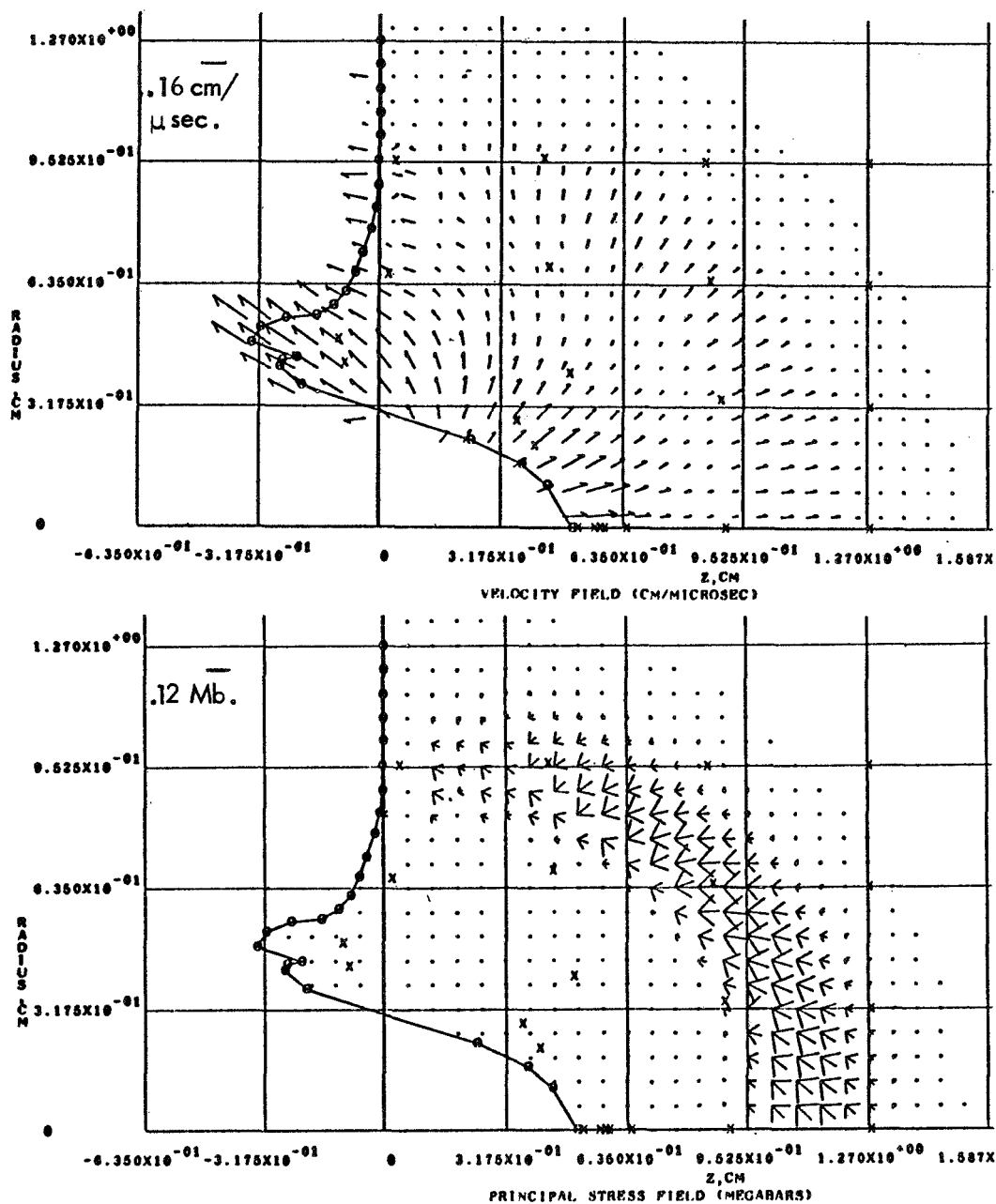


Figure C24. Velocity and Principal Stress Fields at $t = 1.42\mu\text{sec}$ for the Large Configuration (Case 5, $U_0 = 1.5\text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

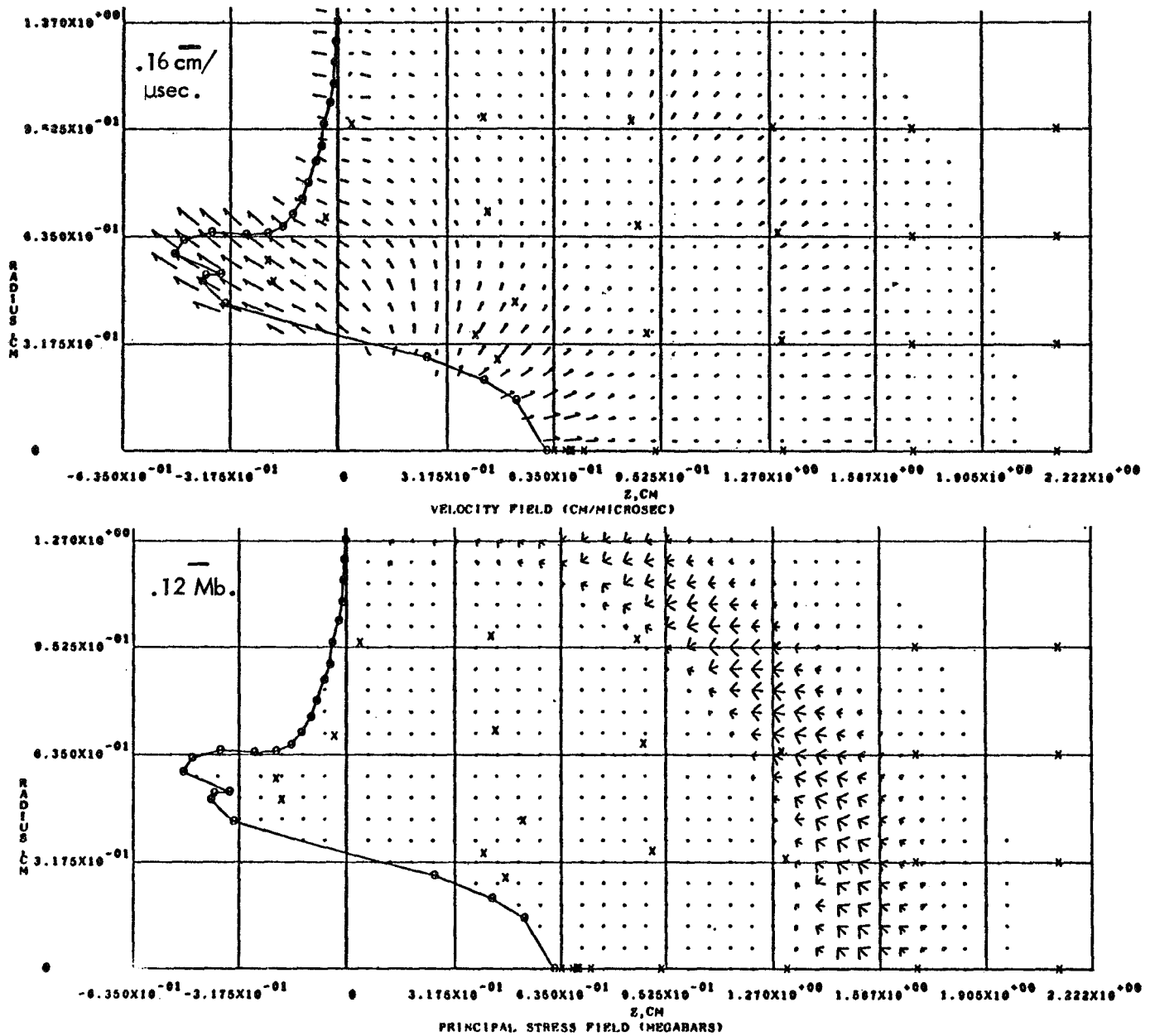


Figure C25. Velocity and Principal Stress Fields at $t = 2.15 \mu\text{sec}$ for the Large Configuration (Case 5, $U_0 = 1.5 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

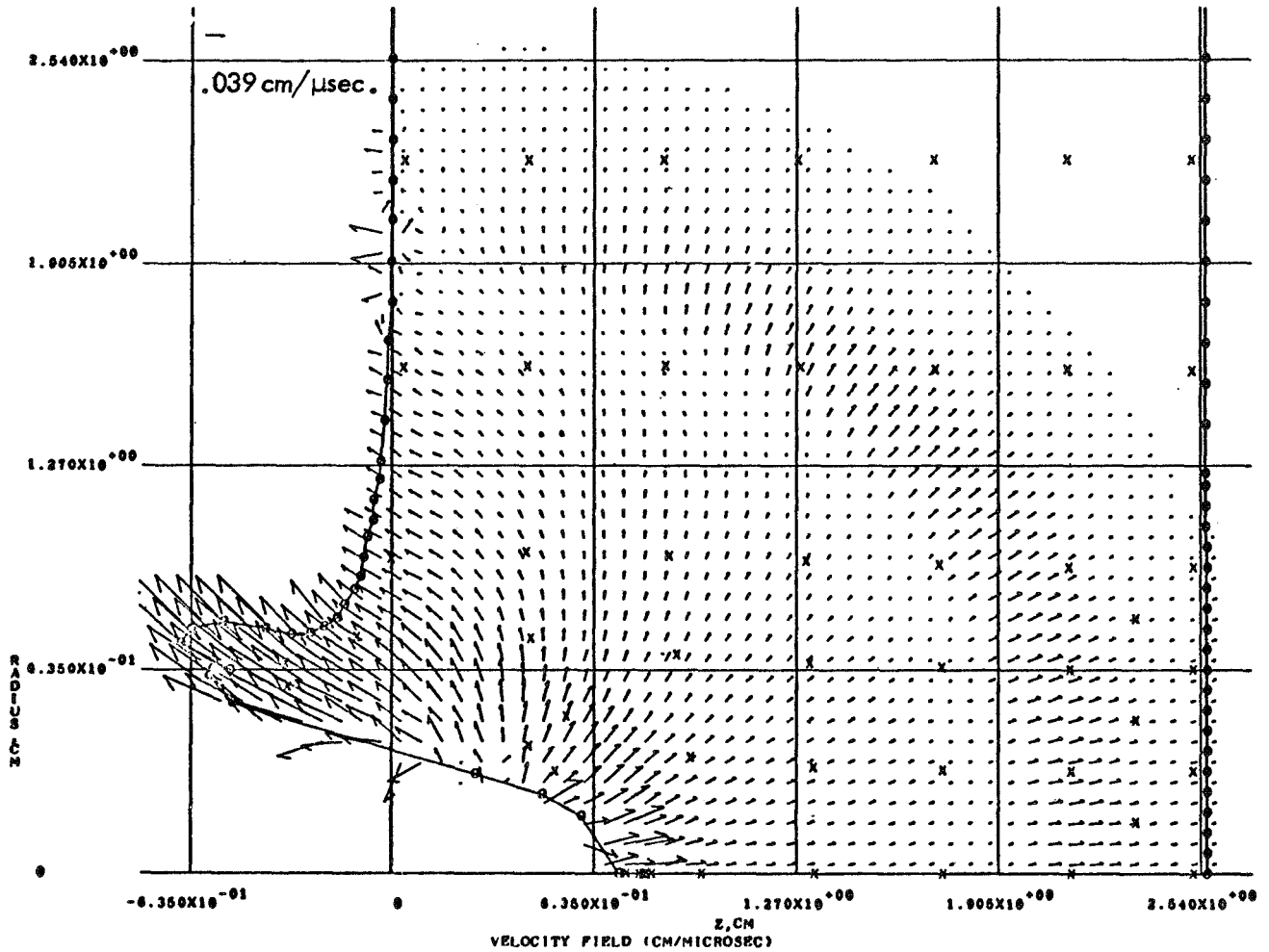


Figure C26. Velocity Field at $t = 3.27 \mu\text{sec}$ for the Large Configuration (Case 5, $U_0 = 1.5 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

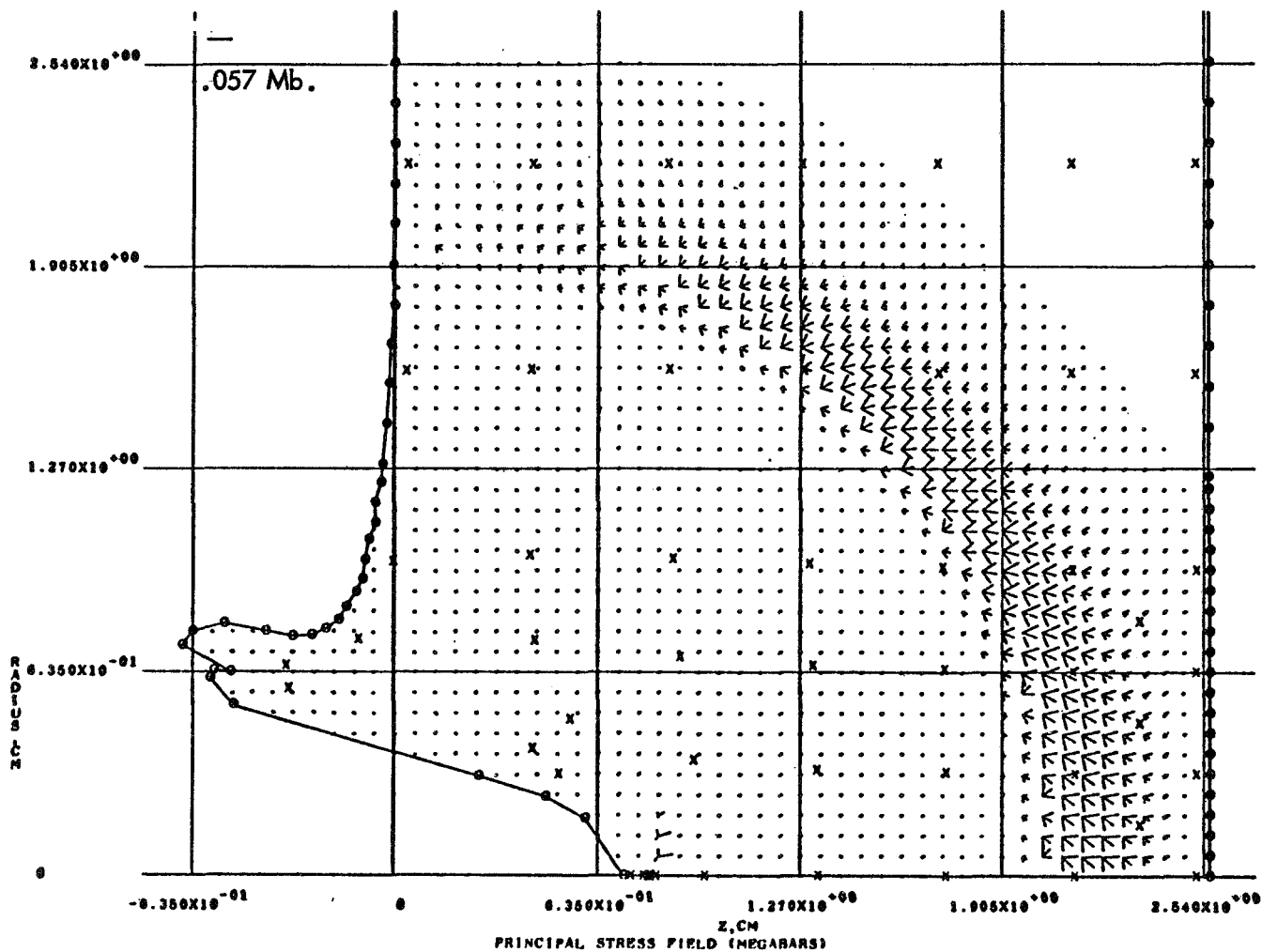


Figure C27. Principal Stress Field at $t = 3.27 \mu\text{sec}$ for the Large Configuration (Case 5, $U_0 = 1.5 \text{ cm}/\mu\text{sec.}$, $\bar{\phi} = 10^6$).

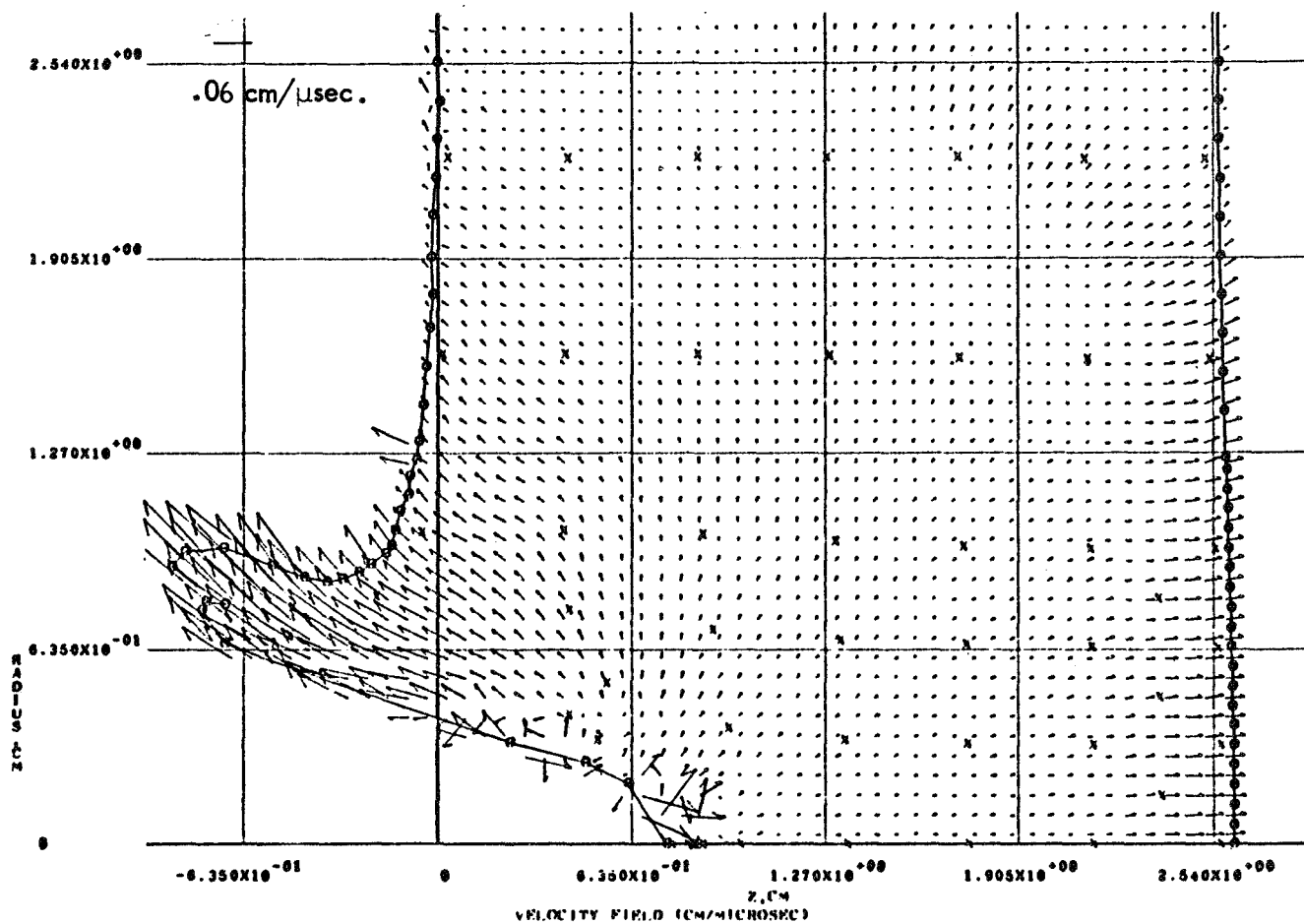


Figure C28. Velocity Field at $t = 4.81 \mu\text{sec}$ for the Large Configuration (Case 5, $U_0 = 1.5 \text{ cm}/\mu\text{sec}$, $\bar{\phi} = 10^6$).

APPENDIX D

AN EULERIAN METHOD FOR HYDRODYNAMIC, ELASTIC, OR ELASTIC-PERFECTLY PLASTIC PROBLEMS*

H. E. Read

A numerical method has been developed for solving the system of equations governing the two-dimensional, time-dependent motion of continuous media. The method is formulated in Eulerian variables, and because of this, its accuracy is not influenced by the magnitude of the material distortion. In order to model the response of a large class of materials over broad ranges of stresses and pressures, a constitutive equation is adopted which is capable of describing either hydrodynamic, elastic or elastic-perfectly plastic behavior. Provision is made for effects arising from large finite compression at very high pressures through a nonlinear equation of state. For solid behavior, distortion is governed by Hooke's law up to the elastic limit, above which it follows the plastic rate theory of Prandtl-Reuss. Plastic yielding is predicted in accordance with von Mises criterion.

The numerical method described herein is particularly applicable to the study of high impulse loading processes and related shock wave phenomena. Shock fronts are treated in the conventional numerical manner by introducing artificial viscosity. In the case of elastic-plastic materials, the method accounts for a (possible) two-wave structure and is, therefore, superior to any of the existing Eulerian numerical techniques in this respect. Unlike the hydrodynamic codes - in which approximate methods must be employed to provide cratering information - the present method permits the material to come to rest in a natural manner yielding the correct crater configuration. A numerical example is given in this report to demonstrate that even at very low stress levels where material rigidity effects become very important the numerical method provides accurate results.

*The development of the basic Eulerian method described herein was completed in 1967. Portions of this development work were supported by Subcontract 24-16219 with Lockheed Missiles and Space Company under Air Force Contract AF04(694)-814. As described in the main body of this report, the basic elastic-perfectly plastic formulation has subsequently been modified to include work hardening and strain rate effects.

D-1. GOVERNING EQUATIONS

D-1-1. General Considerations

In the following discussion, the indicial notation of Cartesian tensor analysis is employed. Any term containing the same letter twice as an index is summed for all values of the index, the indices ranging from 1 to 3. A semicolon is used to denote covariant partial differentiation with respect to spatial variables. Vector quantities are designated by an arrow placed above the symbol, and the usual notation for the basic vector operations is maintained. Any additional symbolism required in the sequel will be defined where it is first introduced.

We turn now to consider a homogeneous, isotropic body at rest in the unstrained state. An arbitrary material particle P of the body is located in a fixed frame of reference by a system of general coordinates X^i . As the body is deformed under the action of applied loading, the particle P moves to a new location at time t, the coordinates of the new location being x^i with respect to the fixed reference system.

The contravariant components of the velocity vector $\vec{v}$ are given by

$$v^i = \frac{dx^i}{dt} \quad (D-1)$$

and by the term deformation rate, we shall mean the tensor d_{ij} defined by the expression

$$d_{ij} = \frac{1}{2} (v_{i;j} + v_{j;i}) \quad (D-2)$$

The components of the stress tensor, which we shall denote by t_j^i , may be decomposed into a deviatoric part, s_j^i , and a spherical part in the usual manner to give

$$t_j^i = \frac{1}{3} t_k^k \delta_j^i + s_j^i \quad (D-3)$$

where δ_j^i denotes the Kronecker delta. If we now introduce the mechanical pressure p, defined by the expression

$$p = - \frac{1}{3} t_k^k \quad (D-4)$$

then Eq. (D-3) may be alternatively written as

$$t_j^i = -p \delta_j^i + s_j^i \quad (D-5)$$

which is a form we shall find convenient in the sequel.

Either one of two viewpoints may be taken in writing the equations which govern the motion of a continuous material body. One may choose to express the dependent variables in terms of the initial (material) coordinates X^i and the time t , or as functions of the current (spatial) coordinates x^i and the time t . The former viewpoint is customarily termed Lagrangian, while the latter is called Eulerian.

In the formulation which follows, the Eulerian viewpoint is adopted. For future reference, we give here the material derivative (following a particle) of a tensor field, $\psi_{n \dots}^m \dots$, which is taken as a function of x^i and t . Denoting, in the usual fashion, the material derivative by the operator D/Dt , we have for a tensor of any order

$$\frac{D \psi_{n \dots}^m \dots}{Dt} = \frac{\partial \psi_{n \dots}^m \dots}{\partial t} + \psi_{n \dots}^m \dots ;_k v^k \quad (D-6)$$

which may be written more compactly in the form

$$\frac{D \psi}{Dt} = \frac{\partial \psi}{\partial t} + \vec{v} \cdot \text{grad } \psi$$

In those instances where it is not suitable to use the symbol D/Dt to designate the material derivative, a dot above a symbol will be used instead.

There are five basic equations which govern the general time-dependent motion of deformable solid bodies, namely, the equations of conservation of mass, linear momentum, angular momentum and energy together with a constitutive equation characterizing the mechanical behavior for a given material. We turn, first of all, to a discussion of the conservation equations.

D-1-2. Conservation Equations

The global, or integral, form of the conservation equations is adopted in the present work in preference to the local form customarily employed in numerical techniques because the continuity requirements on the dependent variables are less stringent, and also because the flux terms for finite regions result as a natural consequence.

The following notation is used in this section: ρ denotes the mass density, $\vec{R}$ is the position vector of a generic particle; $\vec{T}$ represents the stress vector whose components are $T^i = t^i_j n^j$ on a surface having unit outer normal n^j ; and ϵ represents the specific total energy.¹

¹In terms of the notation adopted here, the specific internal energy, which shall be denoted by e , is defined by the relation

$$e = \epsilon - \frac{1}{2} \vec{v} \cdot \vec{v}$$

For a moving, deforming material volume τ enclosed by a surface σ , conservation of mass leads to the expression

$$\frac{d}{dt} \int_{\tau} \rho \, d\tau = 0 \quad (D-8)$$

which simply states that the mass of an identifiable portion of material does not change with time. From the global conservation of linear momentum, one obtains the result

$$\frac{d}{dt} \int_{\tau} \rho \vec{v} \, d\tau = \oint_{\sigma} \vec{T} \, d\sigma \quad (D-9)$$

if body forces and couple stresses are disregarded. The principle of global conservation of angular momentum is given by¹

$$\frac{d}{dt} \int_{\tau} (\vec{R} \times \rho \vec{v}) \, d\tau = \oint_{\sigma} (\vec{R} \times \vec{T}) \, d\sigma \quad (D-10)$$

if body moments and body forces are neglected. Finally, from the principle of global conservation of total energy, we find

$$\frac{d}{dt} \int_{\tau} \rho \epsilon \, d\tau = \oint_{\sigma} \vec{T} \cdot \vec{v} \, d\sigma \quad (D-11)$$

when energy influxes and heat sources are not present.

For future numerical purposes, the spatial domain in which the deformational process takes place is subdivided into a large number of small control volumes by a rigid, stationary space lattice. Attention is focused on a generic control volume--fixed in the reference frame--through which the material flows. The control volume is bounded by a surface S and encloses a volume V .

A useful result which permits a transformation between material volumes and spatial volumes is given by the transport theorem²: If a volume τ of identifiable material instantaneously coincides with a fixed control volume V whose surface is S , the time derivative of an arbitrary tensor function $\psi_{n \dots}^m \dots$ taken over the material volume τ is

¹ According to the way the present numerical method is designed, it does not become necessary to invoke Eq. (D-10). Therefore, no further reference to the principle of conservation of angular momentum will be necessary in the sequel.

² See Reference 31, page 347.

$$\frac{d}{dt} \int_V \psi d\tau = \int_V \frac{\partial \psi}{\partial t} dV + \oint_S \psi \vec{v} \cdot \vec{n} dS \quad (D-12)$$

where $\vec{n}$ denotes the unit vector normal to the surface element dS . The first term on the right hand side of Eq. (D-12) gives the rate of local change in ψ within the stationary control volume, while the second term gives the flux of ψ through the surface of the control volume.

The result given by Eq. (D-12) may now be used to give the form which the conservation equations take for stationary control volumes. Upon combining Eq. (D-12) with Eqs. (D-8), (D-9) and (D-11), we find that the conservation of mass becomes

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_S \rho \vec{v} \cdot \vec{n} dS \quad (D-13)$$

while the conservation of linear momentum is given by the equation

$$\int_V \frac{\partial}{\partial t} (\rho \vec{v}) dV + \oint_S (\rho \vec{v}) \vec{v} \cdot \vec{n} dS = \oint_S \vec{T} dS \quad (D-14)$$

and, finally, the equation

$$\int_V \frac{\partial}{\partial t} (\rho \epsilon) dV + \oint_S (\rho \epsilon) \vec{v} \cdot \vec{n} dS = \oint_S \vec{T} \cdot \vec{v} dS \quad (D-15)$$

expresses the conservation of total energy for a stationary control volume. The conservation equations, in the form given in Eqs. (D-13) to (D-15), are adopted as basic for the development of the present numerical methods.

D-1-3. Constitutive Equations

D-1-3-1. General Discussion

It would be desirable to have at one's disposal a general constitutive equation for a given material which would be capable of describing in a single expression the various aspects of mechanical behavior normally encountered during a high impulse loading process. Such an equation should be capable of describing both elastic and plastic response, and furthermore, provide for smooth transition between these realms of behavior. Unfortunately, however, a single constitutive equation has not been developed for deformable solids which encompasses all aspects of mechanical behavior.¹

¹To this end, however, a number of advances have been made (32), (33) and (34).

Until such an equation becomes available, it is necessary to resort to reasonable idealizations when attempting to model the response of typical materials of interest.

One of two approaches is usually taken in the formulation of constitutive equations for elastic-perfectly plastic bodies. There is the rate theory, on which the Prandtl-Reuss equations are based, and the total strain theory, which has given rise to the Hencky relations. While these theories differ from each other in many aspects, they both view the total deformation as consisting of a small recoverable elastic component and an irrecoverable plastic component.

Extensive experiments have been conducted on a large number of metallic materials to determine whether the rate theory or the total strain theory is more physically realistic. From tests on many materials, it appears that the rate theory follows more closely the experimental results. From a theoretical standpoint, moreover, it has been demonstrated¹ that the total strain theory is not entirely suitable for describing the behavior of metallic materials. It, therefore, appears that the rate theory of plasticity is more physically appropriate, and it will be adopted in the present work.

The rate theory of plasticity is essentially dynamical in nature, the constitutive equations being formulated in terms of rate of stress and rate of deformation. The description of purely elastic behavior in metals, given by Hooke's law for small strain, is not essentially dynamical; it is based on the notion of static strain from an unstrained initial configuration. So long, however, as the elastic strain remains small -- as it does in most metals of practical importance -- a rate form of Hooke's law may be alternatively employed to describe elastic response with no significant loss in generality. In fact, it may be recalled that in classical plasticity theory, the elastic component of strain is described by the rate form of Hooke's law.

From the preceding discussion, it becomes clear that the constitutive equations governing elastic-plastic response in metals are most naturally expressed in rate form. Consequently, it does not become necessary to deal with displacement and strain fields but only with velocity fields; this is fortunate since no satisfactory computational schemes have, to date, been developed for numerically calculating displacements and strains in Eulerian descriptions.

¹See (35), p. 47.

For a large number of metallic solids which exhibit little or no strain-hardening, the classical elastic-perfectly plastic body has proved to be a realistic model for describing mechanical behavior in the neighborhood of the unstrained configuration. As is known, the classical model is characterized by two regions of mechanical response, each of which has its own peculiar constitutive equation; there is the purely elastic region in which Hooke's law applies, and the plastic region where plastic flow equations are employed. A yield condition is then postulated which links the two regions together.

The classical elastic-perfectly plastic model is not, however, particularly well-suited for realistically describing the behavior of non-hardening materials for large deformations. As Truesdell has pointed out in Reference 33, the current (classical) theories of plasticity are usually limited, if only tacitly, to problems involving very small deformation.

High impulse loading processes in deformable solids are normally characterized by shock waves, large deformations and maximum states of stress greatly exceeding that at which the material yields. The loading process usually occurs so rapidly that heat cannot be transferred from the system, making the process adiabatic. On the other hand, however, the local changes in the mechanical states caused by the loading process take place much slower than the local changes in the thermodynamical variables so that it is reasonable to assume that local thermodynamical equilibrium exists at each instant.

In order to realistically describe the mechanical states experienced during a high impulse loading process, it becomes clear why a constitutive equation more general than that associated with the classical elastic-perfectly plastic must be employed; a suitable material model must be capable of accounting for nonlinear effects stemming from large distortion and finite compression, as well as effects caused by large changes in the thermodynamical variables.

D-1-3-2. Jaumann-Noll Stress Rate

Let us consider now the notion of stress rate. From the principle of material-frame indifference¹, it is necessary that the stress rate satisfy the following requirement in order to be dynamically admissible: If a stressed material undergoes a rigid motion, and the stress field is independent of time when referred to a coordinate system attached to and moving with the material, the stress rate must vanish identically. It can be shown,

¹ See (31), p. 702.

however, that this restriction is not stringent enough to result in a unique expression for the stress rate and, in fact, many expressions may be constructed which qualify as stress rates.¹

Of these, however, Prager has shown³⁷ that only one form of stress rate is suitable for use in constitutive equations for elastic-perfectly plastic materials. Basing his conclusion on the fact that the invariants of the stress tensor and, also, the yield function must be stationary when the stress rate vanishes in elastic-perfectly plastic materials, Prager demonstrated that the Jaumann-Noll expression for stress rate, given by the equation

$$\dot{\hat{t}}_j^i = \dot{t}_j^i + t_k^i \omega_j^k - t_j^k \omega_k^i \quad (\text{D-16})$$

is the only form which meets this requirement. In this expression, ω_j^i denotes the mixed components of the spin tensor, defined by the relation

$$\omega_{ij} = \frac{1}{2} (v_{i;j} - v_{j;i}) \quad (\text{D-17})$$

By noting that the deviatoric component of stress is, like the total stress, an objective tensor, we may define, analogously to Eq. (D-16), the rate of deviatoric stress as²

$$\dot{\hat{s}}_j^i = \dot{s}_j^i + s_k^i \omega_j^k - s_j^k \omega_k^i \quad (\text{D-18})$$

and, from a similar argument, the rate of the spherical component of stress is given by

$$\frac{1}{3} \dot{\hat{t}}_k^k = -\dot{p} \quad (\text{D-19})$$

Therefore, by combining Eqs. (D-16), (D-18) and (D-19) one may write

$$\dot{\hat{t}}_j^i = -\dot{p} \delta_j^i + \dot{s}_j^i \quad (\text{D-20})$$

Wherever the symbol $\hat{}$ appears above a stress component in the sequel, we shall tacitly mean the Jaumann-Noll expressions given above.

¹ In this regard see, for instance, the expressions listed in (36), p. 111.

² Note that Eq. (D-18) defines the rate of the deviatoric stress and not the deviatoric component of the stress rate.

D-1-3-3. Present Material Model and Associated Constitutive Equations

The model adopted in the present work is designed to describe the mechanical behavior, even for large deformation, of materials which exhibit no strain-hardening and show no strain-rate effects. The present model is more general in range of application than the classical elastic-perfectly plastic body, and differs from the classical body in essentially two ways. In the first place, an additional thermodynamical variable is introduced so that the pressure component of the stress tensor is a function of the instantaneous thermodynamic state. No restrictions are placed on the thermodynamic path of the process except that it be adiabatic. Secondly, the deviatoric component of stress is described by an equation which is dynamically valid, regardless of the magnitude of the deformation. The constitutive equations adopted here reduce to those associated with the classical Prandtl-Reuss rate theory under the assumptions implicit in that theory, while permitting a description of mechanical behavior beyond the scope of the classical theory.

The constitutive equations governing the plastic behavior of the material model adopted in the present work are based on the following postulates:

1. The deformation rate tensor may be decomposed into a recoverable elastic component and a permanent plastic component; we, thus write

$$d_j^i = (d_j^i)_e + (d_j^i)_p \quad (D-21)$$

where the subscripts e and p denote, respectively, the elastic and plastic components. It is further assumed that there is no thermomechanical coupling between the elastic and plastic components of deformation.¹

2. All changes in volume are completely elastic, and therefore, recoverable. For this to hold, the plastic component of volume change must vanish and we have

¹ A general approach to finite elastic-plastic behavior, which allows for thermomechanical coupling between the elastic and plastic components of deformation, is found in Reference 38. In this paper, an elastic-plastic theory valid for both finite elastic and finite plastic strains was proposed. The implementation of these notions into Eulerian numerical schemes appear quite remote at the present time.

$$\left(d_k^k\right)_P = 0 \quad (D-22)$$

3. The rate of mechanical pressure $\dot{p}$ may, in general, be decomposed into a recoverable component $\dot{\pi}$ and a dissipative component $\dot{q}$ in the following manner:

$$\dot{p} = \dot{\pi} + \dot{q} \quad (D-23)$$

where the dot placed above a symbol indicates the material time derivative. In compressible media, the recoverable component $\dot{\pi}$ follows a thermodynamic equation of state,¹ which in the present work we take to be of the form

$$\pi = \pi(\rho, e) \quad (D-24)$$

where, as before, ρ denotes the mass density and e is the specific internal energy. The function π is, therefore, not path-dependent. On the other hand, the dissipative component $\dot{q}$ is generally a path-dependent function related to the bulk viscosity and the rate of the change of volume.² Since it is being assumed here that volume changes are elastic and, therefore, completely recoverable, we must disregard the effect of $\dot{q}$ in Eq. (D-23). The mechanical pressure then becomes a function only of the thermodynamic state, i. e. ,

$$\dot{p} = \dot{\pi} \quad (D-25)$$

which may be integrated directly to give

$$p = \pi(\rho, e) \quad (D-26)$$

¹ See, for example, (31), p. 640 and (39). The rate at which local thermodynamic equilibrium is attained is generally much greater than the rate at which a disturbance can be propagated (40); it is then reasonably accurate to assume that local thermodynamic equilibrium exists at each instant.

² The term $\dot{q}$, which characterizes the physical dissipative rate for dilatation, is not to be confused with an artificial (bulk) viscosity introduced later in Section D-2-2-3 to permit automatic treatment of shock phenomena.

since both p and π vanish in the unstrained, initial configuration.

4. Plastic behavior takes place when a function f , termed the yield function, vanishes. The yield function depends on the deviatoric components of stress, s_j^i , as well as on certain physical scalar parameters, k_i . The function f is non-positive, i. e. ,

$$f(s_j^i, k_i) \leq 0 \quad (D-27)$$

and, essentially, limits the states of deviatoric stress. There are two yield criteria which find frequent application in metal plasticity, namely, that due to Tresca and that proposed by von Mises. While both of these criteria lead to very nearly the same critical states of stress for plastic yield, the von Mises criterion is adopted in the present work due to the computational convenience which it affords. The von Mises criterion is based on the notion that a metallic material is only capable of supporting a limited amount of elastic distortional energy, and when this critical value of energy is attained, further distortion produces yielding. Mathematically, this criterion is expressed through the inequality

$$s_j^i s_i^j \leq \frac{2}{3} Y^2 \quad (D-28)$$

from which the yield function is identified as

$$f = \frac{1}{6} s_j^i s_i^j - \frac{1}{3} Y^2 \quad (D-29)$$

In these equations, Y denotes a material parameter, identified as the stress at which the material yields in uniaxial tension. For perfectly plastic materials-- such as we are considering here -- Y is constant for the entire deformation and is not influenced by plastic work.

5. The elastic component of the deformation rate tensor follows the rate of Hooke's law, which may be written in terms of the bulk modulus K and the shear modulus μ as¹

¹The constitutive relation given in Eq. (D-30) also defines the hypo-elastic body of grade zero(41).

$$\hat{t}_j^i = (K - \frac{2}{3}\mu) d_k^k \delta_j^i + 2\mu d_j^i \quad (D-30)$$

It is more convenient for future purposes, however, to decompose Eq. (D-30) into two equations; upon introducing the deviatoric component of the elastic deformation rate

$$(d_j^{i*})_e = (d_j^i)_e - \frac{1}{3} d_k^k \delta_j^i \quad (D-31)$$

and recalling Eq. (D-20), it follows from Eq. (D-30) that the rate of mechanical pressure is given by

$$-\dot{p} = K d_k^k \quad (D-32)$$

while the rate of deviatoric stress obeys the relationship

$$\hat{s}_j^i = 2\mu (d_j^{i*})_e \quad (D-33)$$

In order to extend the notions embodied in Eq. (D-30) to account for effects arising from finite compression, the bulk modulus is taken as a function of the thermodynamic pressure π , i. e. ,

$$K = K(\pi) \quad (D-34)$$

The shear modulus for metallic materials increases with pressure, also, but its effect is considerably less than for the bulk modulus⁴²; moreover, since accurate measurements of the variation of μ with pressure are difficult to obtain, we simply assume that μ is independent of pressure, and maintains its value at small strain throughout the entire deformation.

6. At those points where the yield surface is smooth, i. e. , where the derivative $\frac{\partial f}{\partial s_j^i}$ is defined, the plastic component of the deformation rate tensor is perpendicular to the yield surface. This postulate can be written mathematically as¹

¹For those familiar with plasticity theory, Eq. (D-35) is recognized as the von Mises "plastic potential," relating the yield function f to the constitutive equation for the plastic component of the deformation rate tensor.

$$(d_j^i)_p = \Lambda \frac{\partial f}{\partial s_j^i} \quad (D-35)$$

where Λ is a scalar proportionality factor, which usually varies during the deformation. Upon using the von Mises yield criterion in conjunction with Eq. (D-35), the constitutive equation for the plastic component of deformation rate can be made more explicit, and we obtain

$$(d_j^i)_p = \Lambda s_j^i \quad (D-36)$$

After the notions embodied in the preceding postulates have been combined, and the expanded form of the Jaumann-Noll stress rate, given in Eq. (D-18) is introduced, it can be shown that the deviatoric component of stress rate follows a constitutive equation of the form

$$\frac{D s_j^i}{Dt} + s_k^i \omega_j^k - s_j^k \omega_k^i = 2\mu (d_j^{i*} - \Lambda s_j^i) \quad (D-37)$$

for plastic behavior ($s_j^i s_i^j = \frac{2}{3} Y^2$), while for purely elastic behavior ($s_j^i s_i^j < \frac{2}{3} Y^2$), it is given by the equation

$$\frac{D s_j^i}{Dt} + s_k^i \omega_j^k - s_j^k \omega_k^i = 2\mu d_j^{i*} \quad (D-38)$$

where d_j^{i*} is the deviatoric component of the deformation rate tensor. Eq. (D-37) is the dynamically correct form of the constitutive relation associated with the well-known Prandtl-Reuss theory of plasticity⁴³; it differs from the usual (classical) form of the Prandtl-Reuss equations in essentially two ways, namely, the rate of stress is a function of the current stress, and the rate of stress is modified by nonlinear convective terms in such a manner that Eq. (D-37) is dynamically correct, even for large deformations.

Let us note parenthetically that the numerical method described in the sequel does not make explicit use of Eq. (D-37) in treating plastic behavior. By design, it is necessary only to deal with one constitutive equation for the deviatoric components of stress, namely Eq. (D-38), regardless of whether the mechanical behavior is purely elastic or plastic. When plastic behavior occurs, the numerical method uses Eq. (D-38) to determine a tentative state of deviatoric stress, which is then appropriately modified

in such a manner as to satisfy the von Mises criterion exactly and the Prandtl-Reuss relations, given in Eq. (D-37), up to first order terms in the computational time increment.¹

Turning now to the constitutive equation for the spherical component of the stress rate tensor, the preceding postulates lead to a constitutive relationship for the rate of mechanical pressure which may be expressed by the equation

$$\dot{p} = \dot{\pi} \quad (D-39)$$

or, equivalently, in another form through the equation

$$\dot{p} = -K(\pi) d_k^k \quad (D-40)$$

regardless of whether the instantaneous, gross behavior is elastic or plastic.

From the computational standpoint, however, it is more expedient to deal with the form given in Eq. (D-39), which may be integrated directly to give

$$p = \pi(\rho, e) \quad (D-41)$$

since this eliminates, at the outset, any computational integrations which would be required if Eq. (D-40) were employed. Moreover, by using a constitutive relation for the mechanical pressure in the form of Eq. (D-41), it does not become necessary to explicitly deal with the bulk modulus $K(\pi)$.

D-1-3-4. Global Form of the Constitutive Equation

The equations describing material response which have been discussed in the preceding section are in a form which apply in a local sense, that is, they apply at a point. From the numerical point of view adopted in the present work, it is more convenient to have the constitutive equations given by Eqs. (D-38) and (D-41) expressed in a global form, similar to the conservation equations discussed earlier. In that which follows, a global (or integral) form of the constitutive equation governing the deviatoric stress rate is developed.² As before, attention is directed to a control volume fixed in space which is completely occupied by a single material in motion.

¹ The theoretical proof of this is given in Appendix E.

² The global form of Eq. (D-41) is obtained straightforwardly and does not require detailed consideration.

To begin, the local form of the constitutive equation adopted in the present work is, from Eq. (D-38), given by

$$\frac{D s_j^i}{Dt} + s_k^i \omega_j^k - s_j^k \omega_k^i = 2\mu d_j^{i*}$$

where all symbols have been defined earlier. In order to achieve a form which is more suitable for our present purposes, we add the term $s_j^i v_{;k}^k$ to both sides of the preceding equation to obtain

$$\frac{D s_j^i}{Dt} + s_j^i v_{;k}^k = 2\mu d_j^{i*} - s_k^i \omega_j^k + s_j^k \omega_k^i + s_j^i v_{;k}^k \quad (D-42)$$

For convenience, we now set

$$f_j^i = 2\mu d_j^{i*} - s_k^i \omega_j^k + s_j^k \omega_k^i + s_j^i v_{;k}^k \quad (D-43)$$

so that Eq. (D-42) may be written in the more compact form

$$\frac{D s_j^i}{Dt} + s_j^i v_{;k}^k = f_j^i \quad (D-44)$$

Eq. (D-44) is now integrated over a moving material volume τ which encloses a surface σ to give¹

$$\int_{\tau} \left(\frac{D s_j^i}{Dt} + s_j^i v_{;k}^k \right) d\tau = \int_{\tau} f_j^i d\tau \quad (D-45)$$

Recognizing that

$$\frac{d}{dt} \int_{\tau} s_j^i d\tau = \int_{\tau} \left(\frac{D s_j^i}{Dt} + s_j^i v_{;k}^k \right) d\tau \quad (D-46)$$

for a material volume τ (see Reference 31, p. 347), Eq. (D-45) may be placed in the following form:

$$\frac{d}{dt} \int_{\tau} s_j^i d\tau = \int_{\tau} f_j^i d\tau \quad (D-47)$$

¹In performing the integration of a tensor function over a volume, care must be taken to refer the tensor components at each point in the volume to a constant set of base vectors. In this manner, the integration will be invariant to coordinate changes (14).

Using the result given earlier in Eq. (D-12), and assuming that the material volume τ and surface σ coincide instantaneously with a stationary spatial volume V and surface S , we finally reach the result

$$\int_V \frac{\partial s_j^i}{\partial t} dV + \oint_S s_j^i \vec{v} \cdot \vec{n} dS = \int_V (2\mu d_j^* - s_k^i \omega_j^k + s_j^k \omega_k^i + s_j^i v_{,k}^k) dV \quad (D-48)$$

which is the expression upon which the numerical treatment of the deviatoric components of stress is based in the present method, regardless of whether the mechanical behavior is elastic or plastic.

The first integral on the left hand side of Eq. (D-48) represents the local rate of change of stress within the fixed control volume, while the second integral on the left gives the flux of stress through the fixed surface S .

D-2. THE BASIC COMPUTATIONAL METHOD

D-2-1. General Considerations

As mentioned earlier, the spatial domain in which the deformation process takes place is conceptually partitioned into a large number of small control volume - or simply cells, as we shall call them - by a fixed, rigid space lattice. The lattice is usually constructed so that its elements follow the coordinate directions in the system of coordinates selected. As the deformation process takes place, the material deforms and flows through the stationary space lattice, and the identifiable material instantaneously occupying a generic cell changes in time. At any instant, therefore, there will be cells interior to the material which are completely filled with material, and cells near the material surfaces which are only partly filled with material.

In the present method, the governing equations of continuum mechanics are applied in global form at discrete instants of time to the material occupying each of the spatial cells. The particular system of governing equations which the computational method is designed to integrate in time is listed below:

$$\int_V \frac{\partial \rho}{\partial t} dV + \oint_S \rho \vec{v} \cdot \vec{n} dS = 0 \quad (D-49)$$

$$\int_V \frac{\partial}{\partial t} (\rho \vec{v}) dV + \oint_S (\rho \vec{v}) \vec{v} \cdot \vec{n} dS = \oint_S \vec{T} dS \quad (D-50)$$

$$\int_V \frac{\partial}{\partial t} (\rho \epsilon) dV + \oint_S (\rho \epsilon) \vec{v} \cdot \vec{n} dS = \oint_S \vec{T} \cdot \vec{v} dS \quad (D-51)$$

$$\int_V \frac{\partial s_j^i}{\partial t} dV + \oint_S s_j^i \vec{v} \cdot \vec{n} dS = \int_V (2\mu d_j^{i*} - s_k^i \omega_j^k + s_j^k \omega_k^i + s_j^i d_k^k) dV \quad (D-52)$$

$$t_j^i = -\pi \delta_j^i + s_j^i \quad (D-53)$$

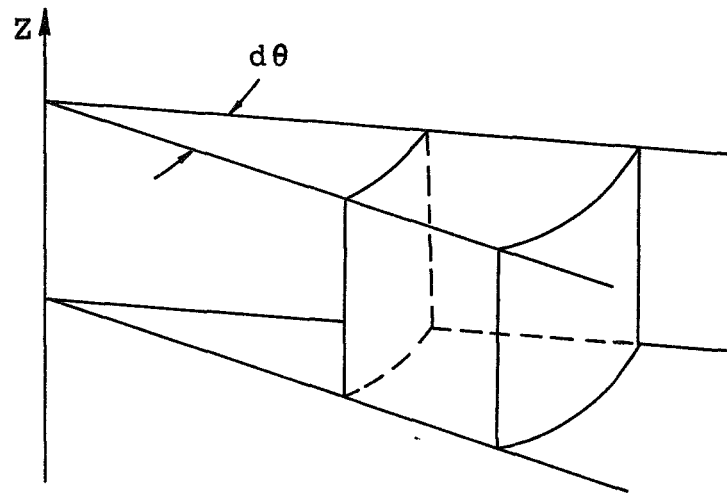
$$s_j^i s_i^j \leq \frac{2}{3} Y^2 \quad (D-54)$$

The notation used in the preceding equations has been defined earlier. The first three equations given above describe the conservation of mass, linear momentum, and total energy relative to a fixed spatial cell, while the last three equations, when appropriately combined, give the constitutive relation for material response. In the form given above, the governing equations are fully invariant, valid in any system of coordinates. Now, in order to be more precise in developing the difference equations, we specialize to cylindrical coordinates (r, θ, z) and, moreover, assume torsionless axially symmetric motion; the discussion which follows is, therefore, confined to two independent space dimensions, namely, the radial coordinate r and the axial coordinate z .

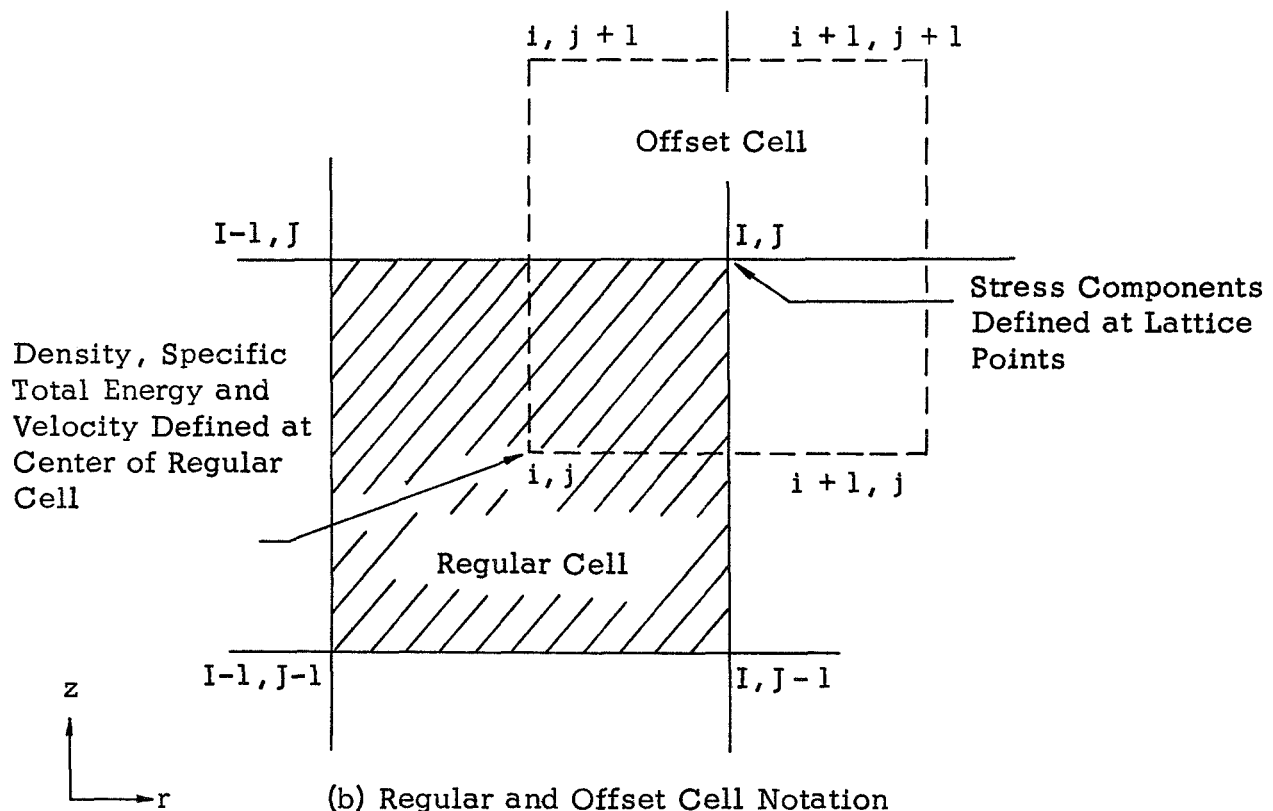
We shall denote the components of the velocity vector $\vec{v}$ in the r and z directions by u and w , respectively. When the physical components of second rank tensors are required, they will be designated for convenience simply by subscripted letters r, θ , and z ; for example, the symbolism $t_{rr}, t_{rz}, t_{\theta\theta}, \dots$ will refer to the physical components of the stress tensor t_j^i .

In the present computational method, two types of computational cells are considered; there are regular cells formed by the elements of the space lattice, and offset cells formed by connecting the centers of volume of the regular cells. A typical cell for cylindrical coordinates is shown in Figure D-1(a). The regular cells are identified by a pair of integers (i, j) , while the offset cells are designated by the notation (I, J) . In both cases, the first integer locates the position of the cell relative to the r -axis and the second integer gives the location of the cell with respect to the z axis. When reference to cell surfaces is required, notation of the form $(I, J - \frac{1}{2})$ - which, in this instance, refers to the right vertical surface of cell (i, j) - will be employed.

In order to make the numerical computation feasible, we shall deal only with mean values of ρ, ϵ and $\vec{v}$ over each of the regular cells, and mean values of the stress components over the offset cells. In a typical regular cell (i, j) , the mean values of ρ, ϵ and $\vec{v}$, which we shall denote by $(\rho)_{i,j}, (\epsilon)_{i,j}$ and $(\vec{v})_{i,j}$, are for computational convenience assumed to act at the



(a) A Typical Computational Cell For Cylindrical Coordinates



(b) Regular and Offset Cell Notation

Figure D1. Description of the Computational Cells

cell center. Similarly, mean values of the stress components in a typical offset cell (I, J) are denoted by $(t_{rr})_{I,J}$, $(t_{rz})_{I,J}$, - and assumed to act at the lattice point (I, J). The computational procedure, therefore, deals with the cell variables $(\rho)_{i,j}$, $(\epsilon)_{i,j}$ and $(\vec{v})_{i,j}$ and the lattice point variables $(t_{rr})_{I,J}$, $(t_{rz})_{I,J}$, . . .

D-2-2. Description of Computational Method

The computational method is designed to integrate with respect to time the system of governing equations subject to prescribed initial and boundary conditions. The calculation is initiated by prescribing initial values of $(\rho)_{i,j}$, $(\epsilon)_{i,j}$ and $(\vec{v})_{i,j}$ to each regular cell occupied by material and similarly, initial values of $(t_{rr})_{I,J}$, $(t_{rz})_{I,J}$, - to each non-vacant offset cell.

In general, by knowing the state of the system at some time t , the state at some later time $t + \Delta t$ is determined by integrating the governing equations over the small time interval Δt , using as initial and boundary conditions the state of the material at time t . To accomplish this, the conservation equations given in Eqs. (D-49) to (D-51) are applied to the regular cells, while the constitutive relations, Eqs. (D-52) to (D-54), are applied to the offset cells.

Unless it is noted otherwise, the discussion which ensues applies to cells which are completely filled with a single material. Specific modifications in the method required when the cells are partially filled with material will be indicated in footnotes.

Assuming that the calculation has advanced to time n , so that the quantities $(\rho)_{i,j}^n$, $(\epsilon)_{i,j}^n$, $(\vec{v})_{i,j}^n$, $(t_{rr})_{I,J}^n$, . . . are known, the state of the material at some later time $n+1$ is determined by a numerical integration, consisting essentially of three phases, which we turn now to discuss. The integration process is modeled, to some extent, on an approach used earlier by Harlow⁴⁵ and Rich⁴⁶ for integrating Eulerian hydrodynamic equations.

D-2-2-1. Phase I

a. Local Changes in the Dependent Variables

In the first phase of the integration process, the local changes in the dependent variables are calculated by neglecting the flux terms in the governing equations, and treating the material in each cell as motionless. Mass flow is, therefore, not permitted during this phase of the integration. Proceeding in the manner just noted, estimated values of the basic dependent variables at time $n+1$, which we shall designate with a tilde above the symbol, are obtained, namely

$$(\tilde{\epsilon})_{i,j}^{n+1}, (\tilde{u})_{i,j}^{n+1}, (\tilde{w})_{i,j}^{n+1}, (\tilde{s}_{rr})_{I,J}^{n+1}, (\tilde{s}_{\theta\theta})_{I,J}^{n+1}, \dots$$

To describe how this is accomplished, we return to Eqs. (D-49) to (D-52). Upon dropping the flux terms in these equations, we are left with the following expressions:¹

$$\int_V \frac{\partial \rho}{\partial t} dV = 0 \quad (D-55)$$

$$\int_V \frac{\partial}{\partial t} (\rho u) dV = \oint_S (\vec{T} \cdot \vec{r}) dS \quad (D-56)$$

$$\int_V \frac{\partial}{\partial t} (\rho w) dV = \oint_S (\vec{T} \cdot \vec{k}) dS \quad (D-57)$$

$$\int_V \frac{\partial}{\partial t} (\rho \epsilon) dV = \oint_S \vec{T} \cdot \vec{v} dS \quad (D-58)$$

$$\int_V \frac{\partial s_j^i}{\partial t} dV = \int_V (2\mu d_j^{i*} - s_k^i \omega_j^k + s_j^k \omega_k^i + s_j^i d_k^k) dV \quad (D-59)$$

where the vector equation of motion, Eq. (D-50), has been decomposed into its scalar components, Eqs. (D-56) and (D-57), in the radial and axial directions.

Equations (D-56) to (D-58) are now integrated over a typical regular cell (i, j) whose volume is $V_{i,j}$ to give

$$(\rho)_{i,j}^n \left(\frac{\partial u}{\partial t} \right)_{i,j}^n V_{i,j}^n = \oint_S \vec{T} \cdot \vec{r} dS \quad (D-60)$$

$$(\rho)_{i,j}^n \left(\frac{\partial w}{\partial t} \right)_{i,j}^n V_{i,j}^n = \oint_S \vec{T} \cdot \vec{k} dS \quad (D-61)$$

¹Equation (D-55) is simply the mathematical statement that the density remains constant during this phase of the integration process.

²Equations (D-60) to (D-62) can, also, be applied to partially filled cells if $V_{i,j}^n$ is interpreted as the volume of material in the cell, and the surface integrals are taken over the surface of the material within the cell (see Section D-2-2-4 a).

$$(\rho)_{i,j}^n \left(\frac{\partial \epsilon}{\partial t} \right)_{i,j}^n V_{i,j}^n = \oint_S \vec{T} \cdot \vec{v} dS \quad (D-62)$$

where use has been made of Eq. (D-55) in removing the density from the time differentiation operation. Similarly, integration of Eq. (D-59) over a typical offset cell (I, J) gives

$$\left(\frac{\partial s_j^i}{\partial t} \right)_{I,J}^n V_{I,J}^n = \int_V (2\mu d_j^{i*} - s_k^i \omega_j^k + s_j^k \omega_k^i + s_j^i d_k^k) dV \quad (D-63)$$

Taylor series expansions in the neighborhood of time n , with the spatial variables held fixed, are now made for the variables u , w , ϵ and s_j^i ; these have the form

$$(\tilde{u})_{i,j}^{n+1} = (u)_{i,j}^n + \left(\frac{\partial u}{\partial t} \right)_{i,j}^n \Delta t + \dots$$

and by combining them with Eqs. (D-60) to (D-63), one obtains

$$(\tilde{u})_{i,j}^{n+1} = (u)_{i,j}^n + \frac{\Delta t}{(\rho)_{i,j}^n V_{i,j}^n} \oint_S \vec{T} \cdot \vec{r} dS \quad (D-64)$$

$$(\tilde{w})_{i,j}^{n+1} = (w)_{i,j}^n + \frac{\Delta t}{(\rho)_{i,j}^n V_{i,j}^n} \oint_S \vec{T} \cdot \vec{k} dS \quad (D-65)$$

$$(\tilde{\epsilon})_{i,j}^{n+1} = (\epsilon)_{i,j}^n + \frac{\Delta t}{(\rho)_{i,j}^n V_{i,j}^n} \oint_S \vec{T} \cdot \vec{v} dS \quad (D-66)$$

$$(\tilde{s}_j^i)_{I,J}^{n+1} = (s_j^i)_{I,J}^n + \frac{\Delta t}{V_{I,J}^n} \int_V (2\mu d_j^{i*} - s_k^i \omega_j^k + s_j^k \omega_k^i + s_j^i d_k^k) dV \quad (D-67)$$

if terms of the order of $(\Delta t)^2$ and higher are disregarded in the Taylor series expansions.

Upon carrying out the evaluation of the surface integrals in Eqs. (D-64) and (D-65) over a typical, completely filled, regular cell (i, j), the following result is obtained for the radial component of estimated velocity

$$(\tilde{u})_{i,j}^{n+1} = (u)_{i,j}^n + \frac{\Delta t}{(\rho)_{i,j}^n V_{i,j}^n} (-F_1^n + F_2^n - F_3^n + F_4^n - F_5^n) \quad (D-68)$$

while for the axial component we find

$$(\tilde{w})_{i,j}^{n+1} = (w)_{i,j}^n + \frac{\Delta t}{(\rho)_{i,j}^n V_{i,j}^n} (-P_1^n + P_2^n - P_3^n + P_4^n) \quad (D-69)$$

In the preceding expressions, we have set, for convenience

$$\begin{aligned} F_1^n &= A [B (t_{rz})_{I-1,J-1}^n + C (t_{rz})_{I,J-1}^n] \\ F_2^n &= A [B (t_{rz})_{I-1,J}^n + C (t_{rz})_{I,J}^n] \\ F_3^n &= \frac{FD}{2} [(t_{rr})_{I-1,J-1}^n + (t_{rr})_{I-1,J}^n] \\ F_4^n &= \frac{FE}{2} [(t_{rr})_{I,J-1}^n + (t_{rr})_{I,J}^n] \\ F_5^n &= \frac{GAF}{4} [(t_{\theta\theta})_{I-1,J-1}^n + (t_{\theta\theta})_{I-1,J}^n + (t_{\theta\theta})_{I,J-1}^n + (t_{\theta\theta})_{I,J}^n] \end{aligned} \quad (D-70)$$

and

$$\begin{aligned} P_1^n &= A [B (t_{zz})_{I-1,J-1}^n + C (t_{zz})_{I,J-1}^n] \\ P_2^n &= A [B (t_{zz})_{I-1,J}^n + C (t_{zz})_{I,J}^n] \\ P_3^n &= \frac{FD}{2} [(t_{rz})_{I-1,J-1}^n + (t_{rz})_{I-1,J}^n] \\ P_4^n &= \frac{FE}{2} [(t_{rz})_{I,J-1}^n + (t_{rz})_{I,J}^n] \end{aligned} \quad (D-71)$$

where t_{rr} , t_{rz} , . . . denote the physical components of the stress tensor and the coefficients A, B, C . . . are defined in Table D-1 for a completely filled cell.

Having an estimate of the velocity components at time $n+1$, we may now estimate the average values of the velocity components over the time interval between n and $n+1$. Letting a bar above a symbol denote the time-averaged value, we then have

$$\begin{aligned} (\bar{u})_{i,j} &= \frac{1}{2} [(u)_{i,j}^n + (\tilde{u})_{i,j}^{n+1}] \\ (\bar{w})_{i,j} &= \frac{1}{2} [(w)_{i,j}^n + (\tilde{w})_{i,j}^{n+1}] \end{aligned} \quad (D-72)$$

Frequent use of these quantities will be made in the sequel.

TABLE D-1. GEOMETRICAL COEFFICIENTS FOR COMPLETELY FILLED CELLS

Coefficient	Definition
A	$r_I - r_{I-1}$
B	$\frac{1}{6}(r_I + 2r_{I-1})$
C	$\frac{1}{6}(2r_I + r_{I-1})$
D	r_{I-1}
E	r_I
F	$(Z_J - Z_{J-1})$
G	1
H	$\frac{1}{8}(r_I^2 - r_{I-1}^2)$

Turning now to the energy equation, Eq. (D-66), the evaluation of the surface and volume integrals is carried out over the regular cell (i, j), using the time-averaged velocity components defined above, to give the result

$$\begin{aligned}
 (\tilde{\epsilon})_{i,j}^{n+1} = & (\epsilon)_{i,j}^n + \frac{\Delta t}{2(\rho)_{i,j}^n V_{i,j}^n} \{ -F_1^n(\bar{u}_{i,j} + \bar{u}_{i,j-1}) \\
 & - P_1^n(\bar{w}_{i,j} + \bar{w}_{i,j-1}) + F_2^n(\bar{u}_{i,j} + \bar{u}_{i,j+1}) + P_2^n(\bar{w}_{i,j} + \bar{w}_{i,j+1}) \\
 & - F_3^n(\bar{u}_{i,j} + \bar{u}_{i-1,j}) - P_3^n(\bar{w}_{i,j} + \bar{w}_{i-1,j}) + F_4^n(\bar{u}_{i,j} + \bar{u}_{i+1,j}) \\
 & + P_4^n(\bar{w}_{i,j} + \bar{w}_{i+1,j}) \}
 \end{aligned} \tag{D-73}$$

which can be applied to partially filled regular cells, as well as to those which are completely filled with material. Eq. (D-73), therefore, provides an estimate of the specific total energy ϵ in the regular cell (i, j) at time $n+1$.

Let us consider finally Eq. (D-67).¹ The integral on the right hand side of this expression may be evaluated over the offset cell (I, J) to yield the result:

$$\begin{aligned} (\tilde{s}_j^i)_{I,J}^{n+1} = & (s_j^i)_{I,J}^n + [2\mu (\bar{d}_j^i)^*_{I,J} - (s_k^i)_{I,J}^n (\bar{\omega}_i^k)_{I,J} \\ & + (s_j^k)_{I,J}^n (\bar{\omega}_k^i)_{I,J} + (s_j^i)_{I,J}^n (\bar{d}_k^k)_{I,J}] \Delta t \end{aligned} \quad (D-74)$$

where, as before, the quantities inside the parenthesis - such as $(\bar{\omega}_j^k)_{I,J}$, for instance - denote volume-averages, and the bar above a symbol denotes the time-average of the quantity over the interval Δt . Restricting attention, now, to cylindrical coordinates and torsionless axially symmetric motions, and introducing the physical components of the tensors appearing in Eq. (D-74), this equation expands into the following equations for the four independent components of deviatoric stress:

$$\begin{aligned} (\tilde{s}_{rr})_{I,J}^{n+1} &= (s_{rr})_{I,J}^n + [2\mu (\bar{d}_{rr}^*)_{I,J} + 2(\bar{\omega})_{I,J} (s_{rz})_{I,J}^n + (s_{rr})_{I,J}^n (\bar{d}_{mm})_{I,J}] \Delta t \\ (\tilde{s}_{\theta\theta})_{I,J}^{n+1} &= (s_{\theta\theta})_{I,J}^n + [2\mu (\bar{d}_{\theta\theta}^*)_{I,J} + (s_{\theta\theta})_{I,J}^n (\bar{d}_{mm})_{I,J}] \Delta t \\ (\tilde{s}_{zz})_{I,J}^{n+1} &= (s_{zz})_{I,J}^n + [2\mu (\bar{d}_{zz}^*)_{I,J} - 2(\bar{\omega})_{I,J} (s_{rz})_{I,J}^n + (s_{zz})_{I,J}^n (\bar{d}_{mm})_{I,J}] \Delta t \\ (\tilde{s}_{rz})_{I,J}^{n+1} &= (s_{rz})_{I,J}^n + \{2\mu (\bar{d}_{rz}^*)_{I,J} - (\bar{\omega})_{I,J} [(s_{rr})_{I,J}^n - (s_{zz})_{I,J}^n] \\ &\quad + (s_{rz})_{I,J}^n (\bar{d}_{mm})_{I,J}\} \Delta t \end{aligned} \quad (D-75)$$

In the preceding equations, the deviatoric components of the deformation rate are given by

$$\begin{aligned} (\bar{d}_{rr}^*)_{I,J} &= (\bar{d}_{rr})_{I,J} - \frac{1}{3} (\bar{d}_{mm})_{I,J} \\ (\bar{d}_{\theta\theta}^*)_{I,J} &= (\bar{d}_{\theta\theta})_{I,J} - \frac{1}{3} (\bar{d}_{mm})_{I,J} \end{aligned} \quad (D-76)$$

¹The deviatoric stress components in a partially filled offset cell (I, J) are determined by the procedure described here only when the lattice point (I, J) is inside the material at both time n and time n+1. Otherwise, special procedures described in Appendix E are followed.

$$\begin{aligned}
(\bar{d}_{zz}^*)_{I,J} &= (\bar{d}_{zz})_{I,J} - \frac{1}{3} (\bar{d}_{mm})_{I,J} \\
(\bar{d}_{rz}^*)_{I,J} &= (\bar{d}_{rz})_{I,J}
\end{aligned}
\tag{D-76}$$

where we have set

$$(\bar{d}_{mm})_{I,J} = (\bar{d}_{rr})_{I,J} + (\bar{d}_{\theta\theta})_{I,J} + (\bar{d}_{zz})_{I,J} \tag{D-77}$$

Moreover, the spin ω is defined according to the relation

$$\omega = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) \tag{D-78}$$

For computational purposes, it is expedient to set

$$\begin{aligned}
A_{I,J} &= \frac{1}{2 \Delta z} [(\bar{u})_{i,j+1} + (\bar{u})_{i+1,j+1} - (\bar{u})_{i,j} - (\bar{u})_{i+1,j}] \\
B_{I,J} &= \frac{1}{2 \Delta r} [(\bar{w})_{i+1,j+1} + (\bar{w})_{i+1,j} - (\bar{w})_{i,j} - (\bar{w})_{i,j+1}]
\end{aligned}
\tag{D-79}$$

The spin, appearing in Eqs. (D-76), is then calculated from the expression

$$(\bar{\omega})_{I,J} = \frac{1}{2} (A_{I,J} - B_{I,J}) \tag{D-80}$$

while the components of the deformation rate are determined in the following manner:

$$\begin{aligned}
(\bar{d}_{rr})_{I,J} &= \frac{1}{2 \Delta r} [(\bar{u})_{i+1,j} + (\bar{u})_{i+1,j+1} - (\bar{u})_{i,j} - (\bar{u})_{i,j+1}] \\
(\bar{d}_{\theta\theta})_{I,J} &= \frac{1}{4 r_I} [(\bar{u})_{i,j} + (\bar{u})_{i+1,j} + (\bar{u})_{i,j+1} + (\bar{u})_{i+1,j+1}] \\
(\bar{d}_{zz})_{I,J} &= \frac{1}{2 \Delta z} [(\bar{w})_{i,j+1} + (\bar{w})_{i+1,j+1} - (\bar{w})_{i,j} - (\bar{w})_{i+1,j}] \\
(\bar{d}_{rz})_{I,J} &= \frac{1}{2} (A_{I,J} + B_{I,J})
\end{aligned}
\tag{D-81}$$

By using Eqs. (D-81) in conjunction with Eqs. (D-76) and D-77), the deviatoric components of deformation rate are determined; these are then used in Eq. (D-75) to give estimated values of the stress at time $n+1$.

b. Procedure For Updating the Position of Free Surfaces

A free surface is defined for computational purposes by a chain of massless points, as shown in Figure D-2. As the computations progress, the points are moved through the stationary space lattice in a Lagrangian sense. Linear elements connect the particles and offer no resistance to stretching. The number of particles employed to describe the free surface is essentially a compromise between free surface resolution and computational efficiency; usually, an initial distribution of one or two points per cell length is sufficient to tie together the mass movement and the free surface movement.

To illustrate the manner in which a free surface is moved, let us consider a typical free-surface configuration at time n , such as shown in Figure D-2, and confine attention to a typical massless point Q . The area of the stress cell face in which Q lies is subdivided into four smaller areas 1, 2, 3 and 4, as shown, by lines parallel to the lattice elements and passing through point Q . If the coordinates of Q at time n are taken as r_Q^n and z_Q^n , then the coordinates at some time Δt later are given by the expressions

$$\begin{aligned} r_Q^{n+1} &= r_Q^n + \bar{u}_Q \Delta t \\ z_Q^{n+1} &= z_Q^n + \bar{w}_Q \Delta t \end{aligned} \tag{D-82}$$

Here, the velocities $\bar{u}_Q$ and $\bar{w}_Q$ are defined as

$$\begin{aligned} \bar{u}_Q &= \frac{A_1 (m\bar{u})_{i+1,j+1} + A_2 (m\bar{u})_{i,j+1} + A_3 (m\bar{u})_{i+1,j} + A_4 (m\bar{u})_{i,j}}{A_1 m_{i+1,j+1} + A_2 m_{i,j+1} + A_3 m_{i+1,j} + A_4 m_{i,j}} \\ \bar{w}_Q &= \frac{A_1 (m\bar{w})_{i+1,j+1} + A_2 (m\bar{w})_{i,j+1} + A_3 (m\bar{w})_{i+1,j} + A_4 (m\bar{w})_{i,j}}{A_1 m_{i+1,j+1} + A_2 m_{i,j+1} + A_3 m_{i+1,j} + A_4 m_{i,j}} \end{aligned} \tag{D-83}$$

where $\bar{u}$ and $\bar{w}$ denote the time-averaged velocity components given in Eq. (D-72), and m is the mass in the regular cell indicated by the subscript. An inspection of Eqs. (D-83) will reveal that the time-averaged velocities at point Q are determined by area-weighting the momenta in neighboring regular cells. This completes the Phase I calculation.

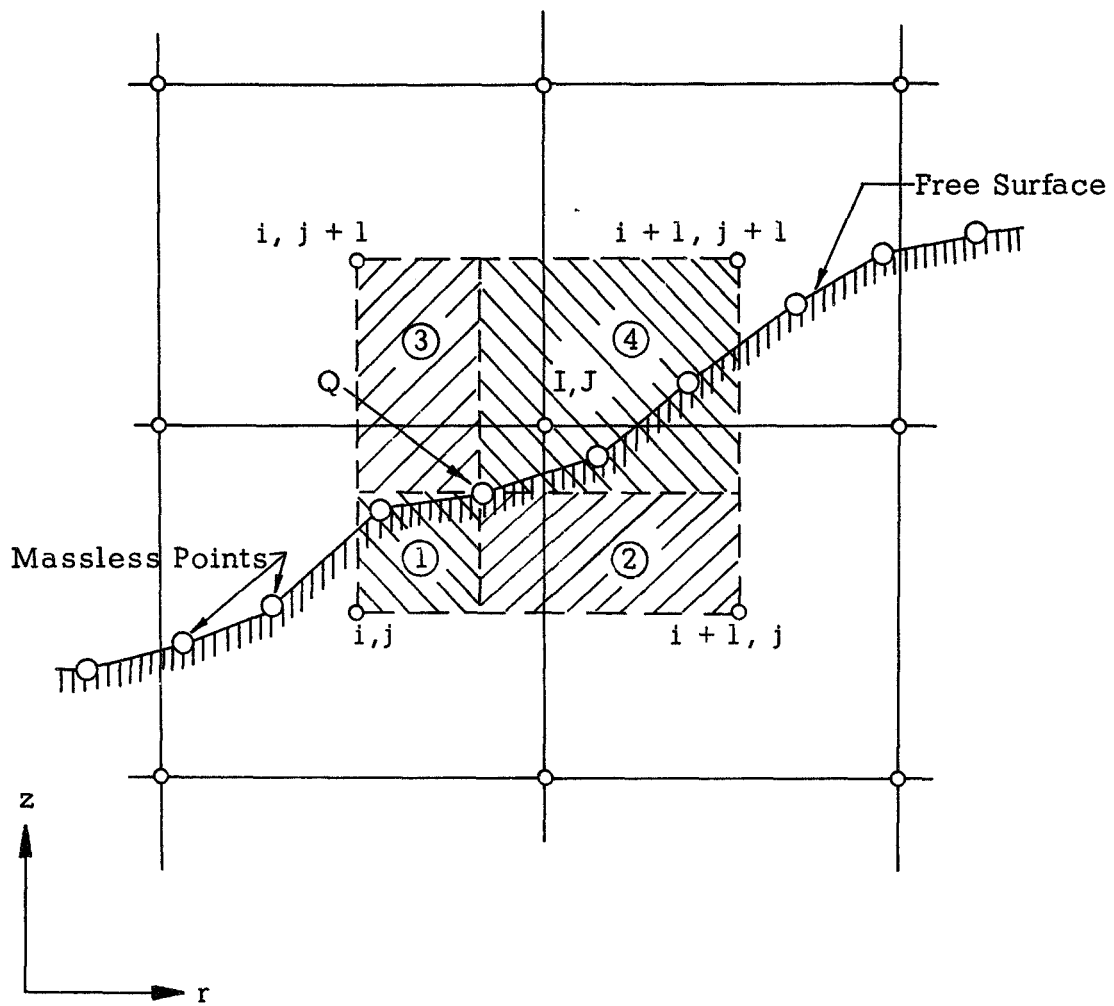


Figure D2. Movement of Free Surface Points

D-2-2-2. Phase II

The movement of the material - which was suppressed in the Phase I part of the integration process - is considered in Phase II. It is clear that when a typical material element changes cells, it transports into the new cell a certain amount of mass, momentum, energy and stress. We shall deal, in particular, with the transport of mass and stress in the discussion which follows.

a. Mass Flow and Repartitioning of Mass

The process of mass flow is governed by the continuity equation, Eq. (D-49), which simply states that the rate of accumulation of mass inside a cell equals the net mass influx. Upon applying Eq. (D-49) to a typical regular cell (i, j) over a small time interval Δt , one may carry out the indicated integrations to obtain the following expression:

$$\begin{aligned}
 (\rho)_{i,j}^{n+1} = (\rho)_{i,j}^n \frac{V_{i,j}^n}{V_{i,j}^{n+1}} + \frac{1}{V_{i,j}^{n+1}} [\Delta m_{I-1, J-\frac{1}{2}} - \Delta m_{I, J-\frac{1}{2}} \\
 + \Delta m_{I-\frac{1}{2}, J-1} - \Delta m_{I-\frac{1}{2}, J}]
 \end{aligned}
 \tag{D-84}$$

where the terms $\Delta m_{I-1, J-\frac{1}{2}}$, etc. denote the amount of mass transported across the indicated surfaces during the time interval Δt . The mass transport terms appearing in Eq. (D-84) are calculated from the relations

$$\begin{aligned}
 \Delta m_{I-1, J-\frac{1}{2}} &= \rho_1^n \bar{u}_1 \Delta t \bar{A}_{I-1, J-\frac{1}{2}} \\
 \Delta m_{I, J-\frac{1}{2}} &= \rho_2^n \bar{u}_2 \Delta t \bar{A}_{I, J-\frac{1}{2}} \\
 \Delta m_{I-\frac{1}{2}, J-1} &= \rho_3^n \bar{w}_3 \Delta t \bar{A}_{I-\frac{1}{2}, J-1} \\
 \Delta m_{I-\frac{1}{2}, J} &= \rho_4^n \bar{w}_4 \Delta t \bar{A}_{I-\frac{1}{2}, J}
 \end{aligned}
 \tag{D-85}$$

where the terms $\bar{A}_{I-1, J-\frac{1}{2}}$, etc. denote the area of the material in contact with the indicated surface, and the bars above the symbols are used, as before, to indicate time-averaged values over the interval Δt . The calculation of the effective densities and effective velocities, which appear in Eqs (D-85) with the numerical subscripts, will be described, first of all, for completely filled cells and, finally, for cells partially filled with material.

We shall consider, now, a completely filled regular cell (i,j) and, for brevity, confine attention to the surface (I, J - $\frac{1}{2}$). It has been found through numerical experimentation that if simple averaging procedures are used for the calculation of the effective densities and velocities small, nevertheless noticeable, computational instabilities generally arise in the numerical method. To avoid such undesirable effects, a number of different ways for evaluating the effective densities and velocities have been numerically examined with regard to accuracy and computational stability. From the results of this experimentation, the method discussed below was found to yield the most favorable results.

The material exchanged between cell (i,j) and cell (i+1,j) during the time interval Δt is located between I and I + δ at time n. The sign of δ depends on the direction of material motion and, in absolute value, δ is usually small compared with a typical cell length.

The time averaged velocity at the location I + δ , which we shall denote by $\bar{u}_2$, must satisfy the relation

$$\bar{u}_2 = - \frac{\delta}{\Delta t} \quad (D-86)$$

in order for the material occupying the region between I and I + δ to be completely expelled from the donor cell in time Δt . By using a Taylor series expansion about I, J - $\frac{1}{2}$, we obtain

$$\bar{u}_2 = \bar{u}_{I, J - \frac{1}{2}} + \delta \left(\frac{\partial \bar{u}}{\partial r} \right)_{I, J - \frac{1}{2}} \quad (D-87)$$

when higher order terms are neglected. Upon setting

$$\bar{u}_{I, J - \frac{1}{2}} = \frac{1}{2} (\bar{u}_{i+1, j} + \bar{u}_{i, j})$$

and

$$\left(\frac{\partial \bar{u}}{\partial r} \right)_{I, J - \frac{1}{2}} = \frac{\bar{u}_{i+1, j} - \bar{u}_{i, j}}{r_{i+1} - r_i}$$

in Eq. (D-87), and combining the result with Eq. (D-86), we finally obtain the equation

$$\bar{u}_2 = \frac{1}{2} (\bar{u}_{i+1, j} + \bar{u}_{i, j}) \left[1 + \left(\frac{\bar{u}_{i+1, j} - \bar{u}_{i, j}}{r_{i+1} - r_i} \right) \Delta t \right]^{-1} \quad (D-88)$$

which gives the time averaged velocity to be used in the second of Eqs. (D-85). With regard to the effective density terms such as, for example, ρ_2^n , it has

been found that the most satisfactory results are obtained by using the density of the cell from which the mass element is transported, i. e., the so-called donor density. We, therefore, have

$$\rho_2^n = \begin{cases} (\rho)_{i,j}^n & \text{if } \bar{u}_2 > 0 \\ (\rho)_{i+1,j}^n & \text{if } \bar{u}_2 < 0 \end{cases} \quad (\text{D-89})$$

Otherwise, simply set $\Delta m_{I,J-\frac{1}{2}} = 0$. The remaining surfaces of the completely filled cell are treated in a similar manner.

For the numerical treatment of mass flow in partially filled regular cells, it is convenient to define a coefficient α for each surface of the cell in the manner shown in Table D-2.

TABLE D-2. EVALUATION OF THE COEFFICIENT α

Time n	Time n+1	α
Material not in contact with surface	Material in contact with surface	1
Material not in contact with surface	Material not in contact with surface	0
Material in contact with surface	Material not in contact with surface	$\frac{1}{2}$
Material in contact with surface	Material in contact with surface	$\frac{1}{2}$

Confining attention, as before, to the surface $(I, J - \frac{1}{2})$, the effective density and velocity at this surface are calculated in the following manner, using the coefficient $\alpha(I, J - \frac{1}{2})$ described above:

$$\begin{aligned} \rho_2^n &= \alpha(I, J - \frac{1}{2}) [(\rho)_{i,j}^n + (\rho)_{i+1,j}^n] \\ \bar{u}_2 &= \alpha(I, J - \frac{1}{2}) [(\bar{u})_{i,j} + (\bar{u})_{i+1,j}] \end{aligned} \quad (\text{D-90})$$

The effective density and velocity on the remaining surfaces of the partially filled cell are calculated in a similar manner.

b. Stress Flow and Repartitioning of Stress

When a material element changes cells, it carries with it into the new cell a certain measure of stress. Letting Δs_j^i represent the net influx of the deviatoric stress s_j^i into a typical offset cell (I, J) during the interval Δt , we have, recalling the second term on the left hand side of Eq. (D-52),

$$(\Delta s_j^i)_{I, J} = - \Delta t \oint_S s_j^i \vec{v} \cdot \vec{n} dS \quad (D-91)$$

We now introduce, as before, the physical components of the deviatoric stress tensor, and confine our attention to the component s_{rr} . The remaining components $s_{\theta\theta}$, s_{zz} and s_{rz} are treated similarly.

If we denote by $(\Delta s_{rr})_{i, j+\frac{1}{2}}$, $(\Delta s_{rr})_{i+1, j+\frac{1}{2}}$, . . . the amount of stress which passes through the respective surfaces of offset cell (I, J) during the time Δt , then the net influx of the stress component s_{rr} may be calculated by the expression¹

$$\begin{aligned} (\Delta s_{rr})_{I, J} = & (\Delta s_{rr})_{i, j+\frac{1}{2}} - (\Delta s_{rr})_{i+1, j+\frac{1}{2}} + (\Delta s_{rr})_{i+\frac{1}{2}, j} \\ & - (\Delta s_{rr})_{i+\frac{1}{2}, j+1} \end{aligned} \quad (D-92)$$

where

$$\begin{aligned} (\Delta s_{rr})_{i, j+\frac{1}{2}} &= (\tilde{s}_{rr})_{i, j+\frac{1}{2}}^{n+1} \bar{u}_{i, j+\frac{1}{2}} \Delta t \bar{A}_{i, j+\frac{1}{2}} \\ (\Delta s_{rr})_{i+1, j+\frac{1}{2}} &= (\tilde{s}_{rr})_{i+1, j+\frac{1}{2}}^{n+1} \bar{u}_{i+1, j+\frac{1}{2}} \Delta t \bar{A}_{i+1, j+\frac{1}{2}} \\ (\Delta s_{rr})_{i+\frac{1}{2}, j} &= (\tilde{s}_{rr})_{i+\frac{1}{2}, j}^{n+1} \bar{w}_{i+\frac{1}{2}, j} \Delta t \bar{A}_{i+\frac{1}{2}, j} \\ (\Delta s_{rr})_{i+\frac{1}{2}, j+1} &= (\tilde{s}_{rr})_{i+\frac{1}{2}, j+1}^{n+1} \bar{w}_{i+\frac{1}{2}, j+1} \Delta t \bar{A}_{i+\frac{1}{2}, j+1} \end{aligned} \quad (D-93)$$

¹The stress flow calculation described here is made for partially filled offset cells only when the lattice point at the center of the cell is inside the material at both times n and $n+1$. For the treatment of stress in other situations, one should refer to Appendix F.

In Eqs. (D-93), the terms $\bar{A}_{i+\frac{1}{2},j}$, $\bar{A}_{i+1,j+\frac{1}{2}}$, . . . represent the area of the material in contact with the respective offset cell surfaces, and the bars, again, indicate time-averaged quantities over an interval Δt .

The effective stresses and velocities at the offset cell surfaces appearing in Eqs. (D-93), are calculated in the following manner and, for brevity, we confine attention to the surface $(i, j + \frac{1}{2})$:

$$\bar{u}_{i,j+\frac{1}{2}} = \frac{1}{2} [(\bar{u}_{i,j}) + (\bar{u}_{i,j+1})]$$

(D-94)

and

$$(\tilde{s}_{rr})_{i,j+\frac{1}{2}}^{n+1} = \frac{1}{2} [(\tilde{s}_{rr})_{I,J}^{n+1} + (\tilde{s}_{rr})_{I-1,J}^{n+1}]^1$$

Otherwise, simply set $(\Delta s_{rr})_{i,j+\frac{1}{2}} = 0$. The remaining surfaces of the offset cell are treated in a similar manner.

By combining the estimated stress given by the first of Eqs. (D-75) with the net influx of stress given by Eq. (D-92), the stress at time $n + 1$, which we shall presently regard as tentative, is determined by the following equation

$$[s_{rr}]_{I,J}^{n+1} = \frac{1}{V_{I,J}^{n+1}} [(\tilde{s}_{rr})_{I,J}^{n+1} V_{I,J}^n + (\Delta s_{rr})_{I,J}] \quad (D-95)$$

where $V_{I,J}^n$ and $V_{I,J}^{n+1}$ represent, respectively, the volume of material in the cell (I, J) at time n and time $n + 1$, and the brackets have been placed around the stress component on the left hand side of the equation to indicate a tentative value. If the mechanical behavior in the neighborhood of point (I, J) has been purely elastic during the interval Δt , the stress component given by Eq. (D-95) is accepted as correct. On the other hand, if the behavior has been plastic, the stress given in Eq. (D-95) provides a tentative stress, which is then appropriately modified in such a manner as to satisfy the von Mises yield criterion exactly, and the Prandtl-Reuss equations up to first order terms in the time increment Δt .

¹In those instances when the lattice point $(I - 1, J)$ lies outside the material at time n , the stress $(\tilde{s}_{rr})_{I-1,J}^{n+1}$ is not defined; when this occurs we set

$$(\tilde{s}_{rr})_{i,j+\frac{1}{2}}^{n+1} = (\tilde{s}_{rr})_{I,J}^{n+1}$$

To illustrate how this is accomplished computationally, let us consider a three-dimensional stress space in which the principal stress t_1 , t_2 , t_3 are taken as Cartesian coordinates. In such a space, the state of stress at a point in a body may be graphically depicted by a vector emanating from the origin. If this vector is decomposed into two components, one representing the deviatoric state of stress and the other representing the pressure, one finds that the vector associated with deviatoric stress always terminates in the plane defined by the expression

$$t_1 + t_2 + t_3 = 0 \quad (D-96)$$

while the vector representing the pressure is always normal to this plane. Following common terminology, the plane defined by Eq. (D-96) will be referred to as the π plane; it passes through the origin of the stress space, and is characterized by a normal which makes equal angles with the coordinate axes. Plastic behavior may be discussed in total by sole reference to the position and movement in the π plane of the vector representing the state of deviatoric stress.

In the stress space considered herein, the von Mises yield criterion, Eq. (D-28), describes a cylinder of infinite length which intersects the π plane in a circle, called the yield circle, and has a radius of $\sqrt{\frac{2}{3}} Y$. For states of stress in which the vector associated with the deviatoric component of stress is located inside of the yield circle, the stress state is below the elastic limit; if the vector contacts the yield circle, plastic behavior takes place. States of stress for which the vector falls outside of the yield circle are not permitted, since this would violate the inequality given in Eq. (D-28).

To be more specific about the present numerical method, let us turn to Figure D-3, where the π plane is shown. In this plane, the various states of deviatoric stress which occur at a fixed spatial position P as the material moves by may be represented by vectors as shown. In Figure D-3, the vector $\vec{OP}$ represents a typical state of deviatoric stress at some generic point P at time n. Given the velocity field in the neighborhood of P, and the state of stress at time n, Eq. (D-95) gives a tentative state of stress at P a small time Δt later, which we shall call time n + 1. In order to determine if the tentative state of stress violates the yield criterion, Eq. (D-28), we form the sum

$$\begin{aligned} Q_p^2 = & [s_{rr}]_p^{n+1} [s_{rr}]_p^{n+1} + [s_{\theta\theta}]_p^{n+1} [s_{\theta\theta}]_p^{n+1} + [s_{zz}]_p^{n+1} [s_{zz}]_p^{n+1} \\ & + 2[s_{rz}]_p^{n+1} [s_{rz}]_p^{n+1} \end{aligned} \quad (D-97)$$

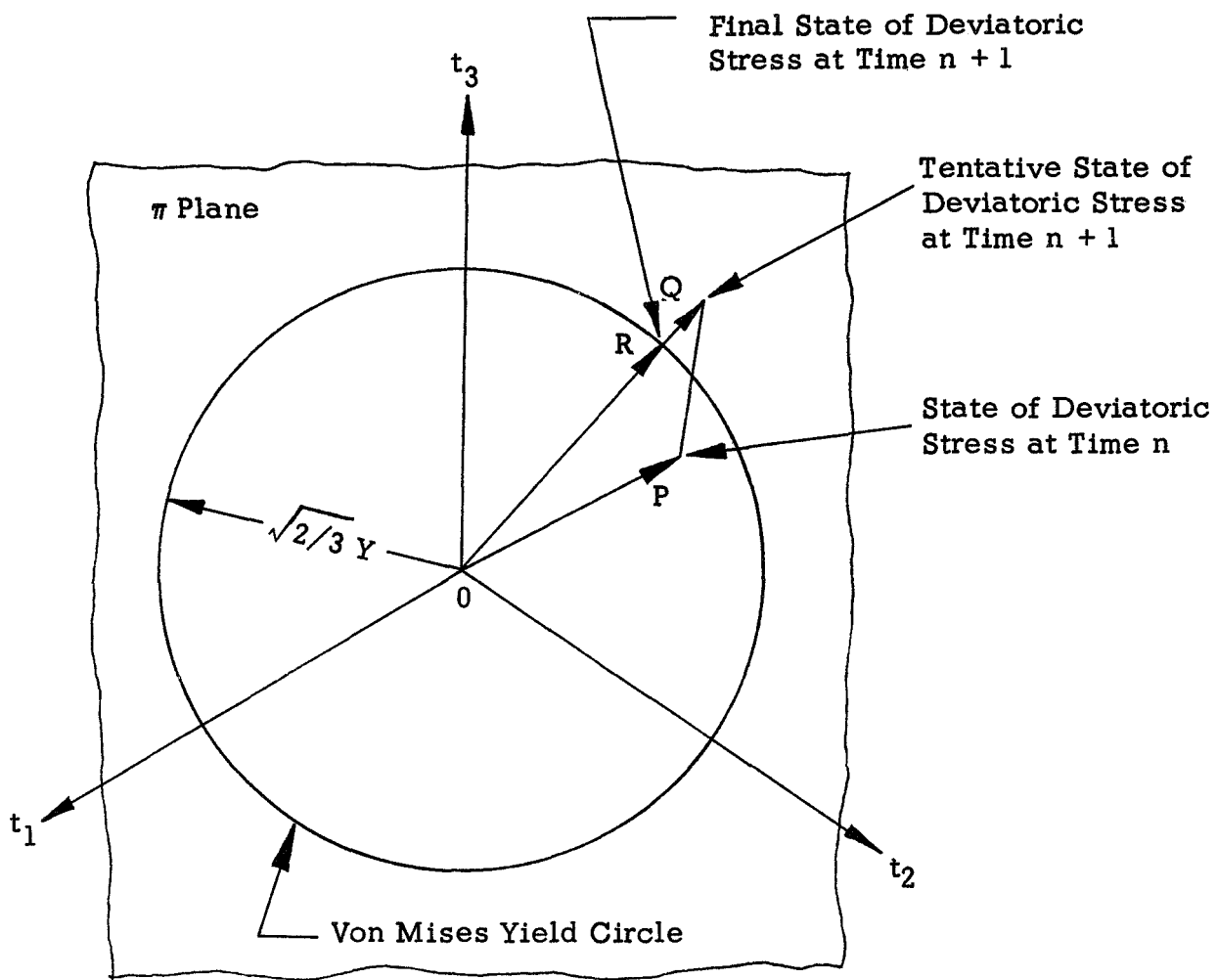


Figure D3. Adjustment of Deviatoric Stress State For Plastic Behavior

where the brackets are used, as before, to indicate tentative stress components.¹ The quantity Q_p^2 , occurring in the preceding equation, is the square of the length of vector $O\vec{Q}$. To continue, let us set

$$K_p = Q_p^2 - \frac{2}{3} Y^2 \quad (D-98)$$

and

$$C_p = \sqrt{\frac{2}{3}} \frac{Y}{Q_p} \quad (D-99)$$

If the vector $O\vec{Q}$, representing the tentative state of stress at $n+1$, falls within the yield circle, then $K_p \leq 0$; this implies that the mechanical behavior during the small time interval has been entirely elastic, and, in this case, the tentative stress components are acceptable and correct. However, if the vector $O\vec{Q}$ falls outside of the yield circle, as in the case depicted in Figure D-3, then elastic-plastic behavior has occurred during the small time interval and $K_p > 0$. The tentative stress components are then used to form acceptable stress components at time $n+1$ in the following manner:

$$(s_{rr})_p^{n+1} = C_p [s_{rr}]_p^{n+1},$$

$$(s_{\theta\theta})_p^{n+1} = C_p [s_{\theta\theta}]_p^{n+1},$$

$$(s_{zz})_p^{n+1} = C_p [s_{zz}]_p^{n+1},$$

$$(s_{rz})_p^{n+1} = C_p [s_{rz}]_p^{n+1},$$

It is easily shown that the stress components on the left hand side of the preceding expressions define a state of stress, represented by $O\vec{E}$ in Figure D-3 which lies on the von Mises yield circle. Moreover, it is proven in Appendix E that the change in the state of stress between times n and $n+1$ as determined by the above procedure is the same, up to terms of the first order in the small time interval, as would be obtained by dealing directly with a differenced form of the Prandtl-Reuss equations. This completes the Phase II part of the integration process.

¹Due to unavoidable numerical scatter, a violation of the yield criterion may occur even though no plastic behavior has taken place. It has been found, however, that the error caused by such scatter does not significantly affect the results.

D-2-2-3. Phase III

In this phase of the integration process, the influx of momentum and energy through each regular cell during the interval Δt is calculated based on the mass flows determined earlier in Phase II. When this has been done, the result is combined with the estimated values of momentum and energy calculated in Phase I to give final values at time $n+1$ of velocity and specific internal energy e . With the density ρ , and specific internal energy, e , known at time $n+1$, the mechanical pressure is calculated through the thermodynamic equation of state. When shock phenomena is expected, an artificial viscosity is determined, and added to the mechanical pressure. Finally, the components of total stress at time $n+1$ are formed by appropriately combining the mechanical pressure and the deviatoric stresses. The notions described in the following sections apply to all cells, regardless of whether they are completely or only partially filled with material.

a. Momentum Flow and Repartitioning of Momentum

In the developments which follow, it proves convenient to define a coefficient $C_{i,j}(k)$ to indicate the direction of material flow at the k^{th} surface of a typical regular cell (i,j) . For this purpose, we set:

$$C_{i,j}(k) = \begin{cases} 1, & \text{if material moves into cell } (i,j) \text{ through the } k^{\text{th}} \text{ surface} \\ 0, & \text{if material passes out of cell } (i,j) \text{ through the } k^{\text{th}} \text{ surface} \end{cases}$$

When the mass elements are transported during Phase II of the integration process, they carry along to their new position a certain amount of momentum. Let us denote by $(\Delta M_r)_{i,j}$ and $(\Delta M_z)_{i,j}$ the net influx of momenta into the regular cell (i,j) during time Δt . Then, by returning to Eq. (D-50) and decomposing the second term on the left hand side of this equation into components in the radial and axial directions, we may write that

$$(\Delta M_r)_{i,j} = - \oint_S u \rho \vec{v} \cdot \vec{n} \Delta t dS$$

and

(D-101)

$$(\Delta M_z)_{i,j} = - \oint_S w \rho \vec{v} \cdot \vec{n} \Delta t dS$$

after some rearrangements. The term $\rho \vec{v} \cdot \vec{n} \Delta t dS$ gives the differential amount of mass transported through the surface dS in time Δt . We now let $(\Delta M_r)_k$ and $(\Delta M_z)_k$ denote the amount of momentum in the radial and axial direction, respectively, which passes through the k^{th} surface of cell (i,j) in time Δt . We may then evaluate the surface integrals in Eqs. (D-101) over the cell (i,j) to obtain the results

$$\begin{aligned}
(\Delta M_r)_{i,j} = & (\Delta M_r)_{I-1, J-\frac{1}{2}} - (\Delta M_r)_{I, J-\frac{1}{2}} + (\Delta M_r)_{I-\frac{1}{2}, J-1} \\
& - (\Delta M_r)_{I-\frac{1}{2}, J}
\end{aligned} \tag{D-102}$$

$$\begin{aligned}
(\Delta M_z)_{i,j} = & (\Delta M_z)_{I-1, J-\frac{1}{2}} - (\Delta M_z)_{I, J-\frac{1}{2}} + (\Delta M_z)_{I-\frac{1}{2}, J-1} \\
& - (\Delta M_z)_{I-\frac{1}{2}, J}
\end{aligned} \tag{D-103}$$

where, by using the mass flows calculated earlier in Phase II and the coefficient $C_{i,j}(k)$, the terms on the right hand sides of Eqs. (D-102) and (D-103) are calculated in the following fashion:

$$\begin{aligned}
(\Delta M_r)_{I-1, J-\frac{1}{2}} = & \{ C_{i,j}(I-1, J) (\tilde{u})_{i-1, j}^{n+1} - [1 - C_{i,j}(I-1, J)] (\tilde{u})_{i, j}^{n+1} \} \\
& | \Delta m_{I-1, J-\frac{1}{2}} | \\
(\Delta M_r)_{I, J-\frac{1}{2}} = & - \{ C_{i,j}(I, J-\frac{1}{2}) (\tilde{u})_{i+1, j}^{n+1} - [1 - C_{i,j}(I, J-\frac{1}{2})] (\tilde{u})_{i, j}^{n+1} \} \\
& | \Delta m_{I, J-\frac{1}{2}} | \\
(\Delta M_r)_{I-\frac{1}{2}, J-1} = & \{ C_{i,j}(I-\frac{1}{2}, J-1) (\tilde{u})_{i, j-1}^{n+1} - [1 - C_{i,j}(I-\frac{1}{2}, J-1)] (\tilde{u})_{i, j}^{n+1} \} \\
& | \Delta m_{I-\frac{1}{2}, J-1} | \\
(\Delta M_r)_{I-\frac{1}{2}, J} = & - \{ C_{i,j}(I-\frac{1}{2}, J) (\tilde{u})_{i, j+1}^{n+1} - [1 - C_{i,j}(I-\frac{1}{2}, J)] (\tilde{u})_{i, j}^{n+1} \} \\
& | \Delta m_{I-\frac{1}{2}, J} |
\end{aligned} \tag{D-104}$$

where the vertical lines around the mass transport terms are used to indicate the absolute value. For brevity, the expressions for the axial components of momentum transport will not be given; they are similar in form to Eqs. (D-104), except that the velocities w replace the velocities u .

The net influx of momenta into the cell (i, j) , given by Eqs. (D-102) and (D-103), is combined with the estimated momentum, calculated in Phase I, to give the final momentum at time $n + 1$. After this repartitioning process has been carried out, one obtains the following expressions for the velocity components at time $n + 1$:

$$(u)_{i,j}^{n+1} = \frac{1}{(\rho)_{i,j}^{n+1} V_{i,j}^{n+1}} [(\rho)_{i,j}^n V_{i,j}^n (\tilde{u})_{i,j}^{n+1} + (\Delta M_r)_{i,j}] \quad (D-105)$$

$$(w)_{i,j}^{n+1} = \frac{1}{(\rho)_{i,j}^{n+1} V_{i,j}^{n+1}} [(\rho)_{i,j}^n V_{i,j}^n (\tilde{w})_{i,j}^{n+1} + (\Delta M_z)_{i,j}]$$

As before, the symbol $V_{i,j}$ represents the volume of material in the cell (i,j) .

b. Energy Flow and Repartitioning of Energy

Let the net influx of energy into the regular cell (i,j) during the time interval Δt be denoted by $(\Delta \epsilon)_{i,j}$. From the second term on the left hand side of Eq. (D-51), we may then write that

$$(\Delta \epsilon)_{i,j} = - \oint_S \epsilon \rho \vec{v} \cdot \vec{n} \Delta t dS \quad (D-106)$$

where, as before, the terms giving the differential amount of mass transported through the surface dS in time Δt have been grouped together inside the integral sign. Using the notation $(\Delta \epsilon)_k$ for the amount of energy transported across the k^{th} surface in time Δt , the surface integral in Eq. (D-106) may be evaluated, and one finds

$$(\Delta \epsilon)_{i,j} = (\Delta \epsilon)_{I-1, J-\frac{1}{2}} - (\Delta \epsilon)_{I, J-\frac{1}{2}} + (\Delta \epsilon)_{I-\frac{1}{2}, J} - (\Delta \epsilon)_{I-\frac{1}{2}, J} \quad (D-107)$$

Using the coefficient $C_{i,j}(k)$ defined earlier, and the mass flows calculated in Phase II, the terms on the right hand side of Eq. (D-107) are calculated in the following manner:

$$\begin{aligned} (\Delta \epsilon)_{I-1, J-\frac{1}{2}} &= \{ C_{i,j}(I-1, J-\frac{1}{2}) (\tilde{\epsilon})_{i-1,j}^{n+1} - [1 - C_{i,j}(I-1, J-\frac{1}{2})] (\tilde{\epsilon})_{i,j}^{n+1} \} \\ &\quad | \Delta m_{I-1, J-\frac{1}{2}} | \\ (\Delta \epsilon)_{I, J-\frac{1}{2}} &= - \{ C_{i,j}(I, J-\frac{1}{2}) (\tilde{\epsilon})_{i+1,j}^{n+1} - [1 - C_{i,j}(I, J-\frac{1}{2})] (\tilde{\epsilon})_{i,j}^{n+1} \} \\ &\quad | \Delta m_{I, J-\frac{1}{2}} | \\ (\Delta \epsilon)_{I-\frac{1}{2}, J-1} &= \{ C_{i,j}(I-\frac{1}{2}, J-1) (\tilde{\epsilon})_{i,j-1}^{n+1} - [1 - C_{i,j}(I-\frac{1}{2}, J-1)] (\tilde{\epsilon})_{i,j}^{n+1} \} \\ &\quad | \Delta m_{I-\frac{1}{2}, J-1} | \end{aligned} \quad (D-108)$$

$$(\Delta \epsilon)_{I-\frac{1}{2}, J} = - \{ C_{i,j}(I-\frac{1}{2}, J) (\tilde{\epsilon})_{i,j+1}^{n+1} - [1 - C_{i,j}(I-\frac{1}{2}, J)] (\tilde{\epsilon})_{i,j}^{n+1} \} / \Delta m_{I-\frac{1}{2}, J} \quad (D-108)$$

The net influx of energy, obtained by combining Eqs. (D-107) and (D-108), is now added to the estimated value of energy, calculated earlier in Phase I, to give the final value of energy at time $n + 1$. After this has been done, the specific energy ϵ at time $n + 1$ is calculated according to the following expression

$$(\epsilon)_{i,j}^{n+1} = \frac{1}{(\rho)_{i,j}^{n+1} V_{i,j}^{n+1}} [(\rho)_{i,j}^n V_{i,j}^n (\tilde{\epsilon})_{i,j}^{n+1} + (\Delta \epsilon)_{i,j}] \quad (D-109)$$

which is based on conserving total energy in the cell during the repartitioning process.

Now that the calculation has progressed up to the point where the specific energy ϵ and the velocity components are known at time $n + 1$, we calculate the specific internal energy e from the equation

$$(e)_{i,j}^{n+1} = (\epsilon)_{i,j}^{n+1} - \frac{1}{2} [(u)_{i,j}^{n+1} (u)_{i,j}^{n+1} + (w)_{i,j}^{n+1} (w)_{i,j}^{n+1}] \quad (D-110)$$

The specific internal energy calculated in the preceding equation is treated as constant over a regular cell and, for computing purposes, is taken to act at the regular cell center.

c. Pressure Calculation

The pressure in a typical offset cell (I, J) is now determined.¹ It will be recalled from the constitutive relation given in Eq. (D-45) that the pressure is a function of the density and the specific internal energy. However, because both the density and the specific internal energy are regular cell variables, effective values of density and specific internal energy are

¹ The procedure described here is applied to a partially filled offset cell only when the lattice point at the cell center is inside the material at both times n and $n + 1$. For the treatment of stress in other situations, consult Appendix F.

calculated in the offset cells, and these are then used in the equation of state to determine the pressure at lattice points.¹

To begin let us denote by $m_{(I,J)}^{n+1}$ the total mass of the material occupying the four regular cells surrounding lattice point (I, J), and let $m_{i,j}^{n+1}$, $m_{i+1,j}^{n+1}$, . . . represent the mass of material in each of these regular cells. We then have that

$$m_{(I,J)}^{n+1} = m_{i,j}^{n+1} + m_{i+1,j}^{n+1} + m_{i,j+1}^{n+1} + m_{i+1,j+1}^{n+1} \quad (D-111)$$

With this in mind, an effective density in the offset cell (I, J) is calculated from the expression

$$\rho_{I,J}^{n+1} = \frac{m_{(I,J)}^{n+1}}{V_{(I,J)}^{n+1}} \quad (D-112)$$

where $V_{(I,J)}^{n+1}$ represents the total volume of material in the four regular cells surrounding point (I, J). In a similar manner, the effective specific internal energy in the offset cell (I, J) is calculated from the expression

$$e_{I,J}^{n+1} = \frac{1}{m_{(I,J)}^{n+1}} \left[m_{i,j}^{n+1} (e)_{i,j}^{n+1} + m_{i+1,j}^{n+1} (e)_{i+1,j}^{n+1} + m_{i,j+1}^{n+1} (e)_{i,j+1}^{n+1} + m_{i+1,j+1}^{n+1} (e)_{i+1,j+1}^{n+1} \right] \quad (D-113)$$

to give the pressure at the lattice point (I, J) at time $n+1$.

d. Artificial Viscosity

When shock waves are likely to occur, the discontinuities in the dependent variables which result are computationally handled by introducing

¹The present approach is preferred to that in which the pressure at a lattice point is calculated by a volume average of the pressures in each of the regular cells surrounding the lattice point; such an approach is not adopted here due to the "leap-frog" differencing which results.

artificial viscosity into the numerical method. In addition to enhancing the stability of the numerical procedure, the artificial viscosity permits an automatic calculation of the variables in the neighborhood of, and across, a shock front without requiring special reference to the jump conditions. The artificial viscosity is formulated in such a way that its effect upon the numerical solution is felt only in regions where the gradients are very large, that is, near shock fronts; outside of the shock regions, the artificial viscosity effect is negligible.

Two different forms of artificial viscosity are employed in the present method to provide viscous dissipation for dilatational changes; first of all, there is the linear form of artificial viscosity originally suggested by Landshoff⁴⁷ and, secondly, the quadratic artificial viscosity introduced by von Neumann and Richtmyer⁴⁸. The basis for deciding which of these forms is to be employed in a given problem usually depends on the magnitudes of the stresses expected. Under certain circumstances, a linear combination of the linear and quadratic forms may be preferred.

For problems in which the stress level remains relatively low, the linear artificial viscosity is usually adopted; it is calculated at a typical interior lattice point (I, J) according to the equation

$$(q)_{I,J}^{n+1} = -C_L (C_S)_{I,J}^{n+1} (\rho)_{I,J}^{n+1} \sqrt{(r_{i+1} - r_i)(z_{j+1} - z_j)} (d_{mm})_{I,J}^{n+1} \quad (D-114)$$

when $(d_{mm})_{I,J}^{n+1} < 0$. Otherwise, when $(d_{mm})_{I,J}^{n+1} \geq 0$, the artificial viscosity is set to zero. In Eq. (D-114), C_L is a positive dimensionless coefficient near unity, whose precise value is determined by numerical experimentation for a specific problem; $(C_S)_{I,J}^{n+1}$ is the local speed of sound in the material relative to an observer moving with the material; $(\rho)_{I,J}^{n+1}$ is the effective density, calculated from Eq. (D-112); and the remaining terms have been defined earlier. The coefficient C_L is normally chosen so that the shock front is smeared over three or four computational cell lengths.

At high stress levels, the quadratic form of artificial viscosity appears to be preferred to the linear form. While the linear form can be adjusted to provide an equivalent smoothing of numerical scatter at high stress levels, it tends to spread the shock front over more cell lengths than the quadratic form. In the present method, the quadratic artificial viscosity is calculated by the equation

$$(q)_{I,J}^{n+1} = C_Q (\rho)_{I,J}^{n+1} (r_{i+1} - r_i)(z_{j+1} - z_j) [(d_{mm})_{I,J}^{n+1}]^2 \quad (D-115)$$

when $(d_{mm})_{I,J}^{n+1} < 0$. Otherwise, it is set to zero. In the preceding expression, C_Q is a positive dimensionless constant and all other symbols have been defined previously.

In addition to the artificial viscosities discussed above, which are associated with purely dilatational changes, we have numerically experimented with introducing artificial viscosity associated with distortional changes, similar in form to that suggested by Herrmann⁴⁹. From the numerical results obtained in these experiments, it was found that the dilatational types of artificial viscosity were sufficient for our purposes and the further addition of distortional artificial viscosity had an adverse effect on the numerical solutions.

It is of interest to note parenthetically that there are first order error terms inherent in the present numerical method which provide a dissipative effect similar in some respects to that stemming from artificial viscosity. These error terms, however, are unlike the artificial viscosity in that they are not Galilean invariant, being dependent upon absolute velocity.¹ Up to the present time, an error analysis has not been carried out to specifically identify the error terms inherent in the present numerical method. However, the computations made to date indicate that these error terms do not appreciably affect the results.

e. Formation of Total Stress Components

With the pressure, artificial viscosity, and the deviatoric stresses having been calculated at time $n+1$, the physical components of total stresses at a typical interior lattice point (I, J) are formed according to Eq. (D-53) after the artificial viscosity has been added to the pressure. As a result, we obtain

$$\begin{aligned}
 (t_{rr})_{I,J}^{n+1} &= - [(p)_{I,J}^{n+1} + (q)_{I,J}^{n+1}] + (s_{rr})_{I,J}^{n+1} \\
 (t_{\theta\theta})_{I,J}^{n+1} &= - [(p)_{I,J}^{n+1} + (q)_{I,J}^{n+1}] + (s_{\theta\theta})_{I,J}^{n+1} \\
 (t_{zz})_{I,J}^{n+1} &= - [(p)_{I,J}^{n+1} + (q)_{I,J}^{n+1}] + (s_{zz})_{I,J}^{n+1} \\
 (t_{rz})_{I,J}^{n+1} &= (s_{rz})_{I,J}^{n+1}
 \end{aligned}
 \tag{D-116}$$

¹ Because of the similarity in integration procedures, it is to be expected that some of the terms in the present method will be of the same form as those discussed in Reference 45.

where the artificial viscosity term may be either of the linear or quadratic form, or a linear combination of these, as discussed earlier.

D-2-2-4. Boundary Conditions

a. Boundary Condition on a Free Surface

The state of stress at each point on a material surface of a continuous medium must be such that its effect equilibrates the externally applied traction. Letting T_j denote the covariant components of the surface traction vector $\vec{T}$, the general condition which must be satisfied at each point on the material surface is

$$T_j = t_j^i n_i \quad (D-117)$$

where, as before, n_i are the covariant components of the unit vector normal to the surface. If the surface is free from externally applied tractions, the components of the surface traction vanish,

$$T_j = 0 \quad (D-118)$$

and Eq. (D-117) reduces simply to

$$t_j^i n_i = 0 \quad (D-119)$$

Therefore, Eq. (D-118), or its alternate form given in Eq. (D-119), is the boundary condition which must be satisfied at every point on a free surface.

To describe how the presence of a free surface manifests itself in the numerical method, it is helpful if we refer to Figure D-4, where a typical cell through which a free surface passes is shown. Let us denote that portion of the cell surface coincident with the material surface by S_m and let V_m represent the volume of the material within the cell. With this in mind, it can be shown that the conservation equations given in Eqs. (D-49) through (D-51) reduce in this case to the following form;

$$\int_{V_m} \frac{\partial \rho}{\partial t} dV = - \oint_{S_m} \rho \vec{v} \cdot \vec{n} dS \quad (D-120)$$

$$\int_{V_m} \frac{\partial}{\partial t} (\rho \vec{v}) dV + \oint_{S_m} (\rho \vec{v}) \vec{v} \cdot \vec{n} dS = \oint_{S_m} \vec{T} dS \quad (D-121)$$

$$\int_{V_m} \frac{\partial}{\partial t} (\rho \epsilon) dV + \oint_{S_m} (\rho \epsilon) \vec{v} \cdot \vec{n} dS = \oint_{S_m} \vec{T} \cdot \vec{v} dS \quad (D-122)$$

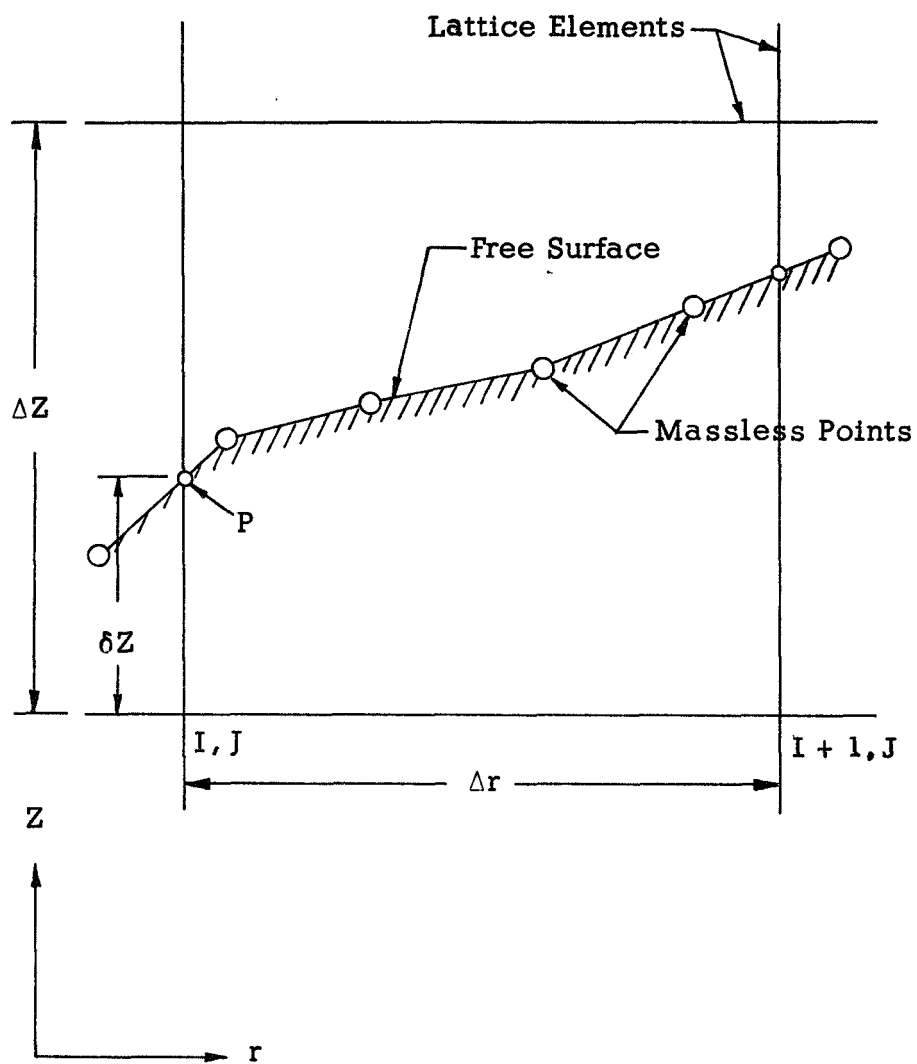


Figure D4. Partially Filled Regular Cell Through Which Free Surface Passes.

if one keeps in mind that the external traction $\vec{T}$ vanishes on the free surface.

Before the surface integrals appearing on the right hand side of Eqs. (D-121) and (D-122) can be evaluated according to the present scheme, one needs to know the state of stress at those points where the free surface intersects a lattice element, such as, for example, point P in Figure D-4. Since P lies on the free surface, the state of stress at this point must satisfy the free surface boundary condition on the stress components given in Eq. (D-119), namely

$$(t_j^i)_P (n_i)_P = 0 \quad (D-123)$$

From a numerical viewpoint, however, it is practically impossible to satisfy Eq. (D-123) on the free surface because there are, usually, more unknown stress components at P than there are equations to deal with. Faced with this situation, the following approach has been adopted for treating the conservation equations in partially filled cells through which a free surface passes:

A Taylor series expansion about the interior lattice point (I, J) nearest to P leads to the following expression for the stress at P:

$$(t_j^i)_P = (t_j^i)_{I, J} + O(\delta z) \quad (D-124)$$

since δz is always smaller in absolute value than the cell length Δz , we neglect the terms of order δz , and simply assume that

$$(t_j^i)_P = (t_j^i)_{I, J} \quad (D-125)$$

Although the stress components at P calculated in this manner are somewhat approximate and do not generally satisfy exactly the free surface boundary condition given in Eq. (D-123), they do provide a good estimate of the state of stress at P without causing undesirable computational complexity.¹

Furthermore, inasmuch as the state of stress at the interior lattice point (I, J) has, when necessary, been appropriately adjusted to satisfy the von Mises inequality, the stress components at P, calculated according to Eq. (D-125), will automatically satisfy this inequality and no special treatment of plastic effects is required at the free surface.

¹ If one desires more accuracy, additional terms in the Taylor series expansions for the stresses at P may be employed.

By adopting the notions described above, the surface integrals in Eqs. (D-121) and (D-122) may be evaluated for any cell through which a free surface passes.

b. Conditions at the Axis of Symmetry

Cells near the axis of symmetry are typically arranged as shown in Figure D-5. Because a cylindrical coordinate system has been employed and, in addition, due to the fact that only torsionless axially symmetric deformations are permitted, there are a number of requirements on the stress components and on the velocity components which must be satisfied. In the first place, for offset cells through which the axis of symmetry passes - such as the offset cell (O, J) shown in Figure D-5 - it is necessary that

$$(t_{rz})_{O,J} = 0$$

and

$$(t_{rr})_{O,J} = (t_{\theta\theta})_{O,J} \tag{D-126}$$

at all times. These conditions may be verified by considering the local form of the equations of motion in cylindrical coordinates at the axis of symmetry and requiring the corresponding accelerations to be finite.

Secondly, let us turn to the regular cells which border on the axis of symmetry. Here, one finds that because of the symmetry of the deformation, the dependent variables in a regular cell, such as (-1, j) shown in Figure D-5, are just the mirrored reflection across the axis of the variables in cell (1, j). The velocity components associated with these cells are then related in the following manner:

$$\begin{aligned} (u)_{-1,j} &= - (u)_{1,j} \\ (w)_{-1,j} &= (w)_{1,j} \end{aligned} \tag{D-127}$$

The first of these equations simply constrains the radial velocity at the axis to vanish.

D-2-2-5. Stability Criterion

In the preceding sections, a method has been described for integrating the system of governing equations over a small time interval Δt between some arbitrary time n and a later time $n+1$. The only restriction placed on the magnitude of Δt should be small enough so that terms of the order of Δt^2 could be reasonably neglected compared with those of order Δt . From the

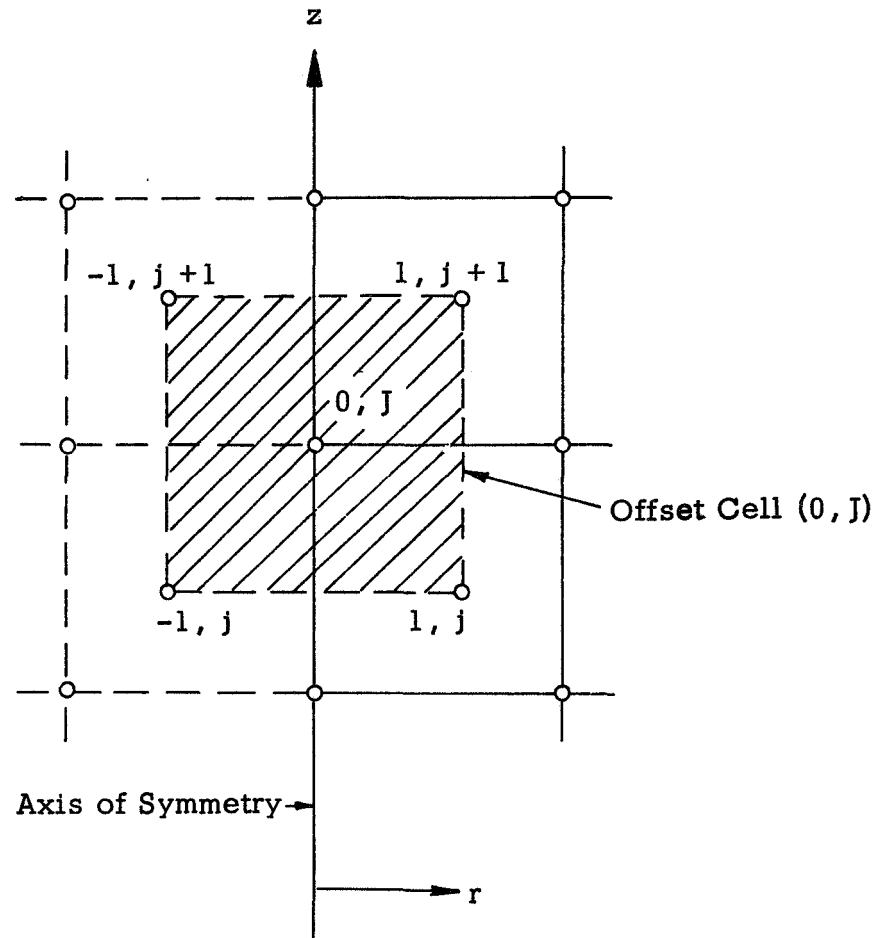


Figure D5. Cells at the Axis of Symmetry

standpoint of numerical stability, however, the time increment Δt cannot be preselected arbitrarily, despite the fact that it may be relatively small, if Δt is chosen too high, the numerical errors, instead of attenuating, will tend to increase in time and the numerical scheme becomes unstable.

In most computational schemes, failure to satisfy the one-dimensional Courant stability criterion, namely

$$\Delta t < \frac{\Delta x}{C_D + |v|} \quad (D-128)$$

generally results in numerical instability. In the preceding expression, Δx represents the length of a computational cell, C_D is the local velocity of sound relative to an observer moving with the material, and $|v|$ denotes the absolute value of the particle velocity in the x-direction.

Although a complete stability analysis of the numerical method presented in this report has not been made, experience has shown that if the computational time step Δt is chosen to satisfy the following criterion - which is simply a generalized form of Eq. (D-128) for two space dimensions - the present method is stable:

$$\Delta t < \min \left\{ \frac{\Delta r}{C_D + |u|}, \frac{\Delta z}{C_D + |w|} \right\} \quad (D-129)$$

In this expression, the notation "min" is used to indicate that a minimum value over the spatial domain of the smaller of the two terms within the braces is sought. At any point in the medium, one may write that

$$\begin{aligned} C_D + |u| &\leq 2 \max \{C_D, |u|\} \\ C_D + |w| &\leq 2 \max \{C_D, |w|\} \end{aligned} \quad (D-130)$$

where the notation "max" refers to the greater of the two quantities inside the braces. With this in mind, one can show that the criterion given in Eq. (D-129) is always satisfied when Δt obeys the condition

$$\Delta t < \frac{1}{2} \min \left\{ \frac{\Delta r}{\max \{C_D, |u|\}}, \frac{\Delta z}{\max \{C_D, |w|\}} \right\} \quad (D-131)$$

In the way the present numerical scheme is designed, the local sound speed C_D and the velocity components (u , w) are not defined at the same points in space. To treat this situation, it becomes necessary to modify Eq. (D-131) to read

$$\Delta t < \frac{1}{4} \min \left\{ \frac{r_i - r_{i-1}}{C_D}, \frac{z_j - z_{j-1}}{C_D}, \frac{r_I - r_{I-1}}{|u|}, \frac{z_J - z_{J-1}}{|w|} \right\} \quad (D-132)$$

where the notation has been described earlier. Eq. (D-132) is the stability criterion adopted for use in conjunction with the present numerical method.

For computational purposes, it becomes convenient to define a stability coefficient S at some arbitrary time $n + 1$ as

$$S^{n+1} = \max \left\{ \frac{C_D \Delta t^{n+1}}{r_i - r_{i-1}}, \frac{C_D \Delta t^{n+1}}{z_j - z_{j-1}}, \frac{|u| \Delta t^{n+1}}{r_I - r_{I-1}}, \frac{|w| \Delta t^{n+1}}{z_J - z_{J-1}} \right\} \quad (D-133)$$

where Δt^{n+1} is the time interval between times n and $n+1$, and the notation "max" refers to the maximum value over the entire spatial domain of the largest of the four quantities within the braces. With this definition in mind, it is then straight-forward to show that the stability criterion given in Eq. (D-132) is satisfied if

$$S^{n+1} < \frac{1}{4} \quad (D-134)$$

at any time $n + 1$.

In order to select a time interval $(\Delta t)^{n+2}$ which optimizes the computational processing, and yet does not violate the stability criterion in Eq. (D-132), the following procedure is followed: First of all, four positive constants S_1 , S_2 , C_1 and C_2 are selected in advance, and these are chosen in such a manner that $S_1 \leq S_2 < \frac{1}{4}$, $C_1 > 1$, and $C_2 < 1$. Then using the stability number S^{n+1} , the time interval $(\Delta t)^{n+2}$ is calculated from the preceding time interval $(\Delta t)^{n+1}$ in the following manner:

$$\begin{aligned} (\Delta t)^{n+2} &= (\Delta t)^{n+1}, \text{ if } S_1 \leq S^{n+1} \leq S_2 \\ (\Delta t)^{n+2} &= C_1 (\Delta t)^{n+1}, \text{ if } S^{n+1} < S_1 \\ (\Delta t)^{n+2} &= C_2 (\Delta t)^{n+1}, \text{ if } S_2 < S^{n+1} < \frac{1}{4} \end{aligned} \quad (D-135)$$

From these equations, it becomes evident that when the stability number falls between the two preassigned constants S_1 and S_2 , no change is made in the time interval. However, if the stability number falls outside of the range bounded by these two constants, the subsequent time interval is either increased or decreased with respect to the preceding time interval, depending on magnitude of S^{n+1} .

D-3. NUMERICAL EXAMPLE OF AN ELASTIC IMPACT PROBLEM

It has been pointed out in the preceding sections that the numerical method described herein has the capability to treat either purely hydrodynamic, purely elastic or elastic-perfectly plastic material behavior. Significant interest has arisen, however, with regard to the ability of the present method to treat purely elastic response. Because of this, the results obtained for an elastic impact problem using a one-dimensional version of the code have been included in this section to demonstrate the reliability of the present method at low stress levels. It is shown that the numerical results are in very excellent agreement with the corresponding theoretical results.

D-3-1. Specification of Problem

We consider two plates, one of which is traveling at a speed of 0.008 cm/ μ sec prior to its normal impact on another plate at rest. Both plates are 30 cm thick and composed of the same aluminum material. For the purposes of the analysis considered here, the deformational process will be assumed to depend of only one spatial variable.

In order to carry out the numerical computations, a uniform grid system was chosen with $\Delta x = \Delta y = 1$ cm. The initial configuration of the system at the time of impact is shown in Figure D-6. The inputs to the one-dimensional version of the STEEP code required for the problem under consideration were selected in the following manner:

a. Material constants for aluminum

$$\text{Initial density} = 2.77 \text{ gm/cm}^3$$

$$\text{Shear modulus} = 0.276 \text{ megabars}$$

b. Constitutive equation for pressure component of the stress tensor.

The Los Alamos equation of state was adopted in the present problem to describe the pressure component of the stress tensor. If the normal material density and the specific internal energy are denoted by ρ_0 and e , respectively, then the compression μ is given by

$$\mu = \frac{\rho}{\rho_0} - 1$$

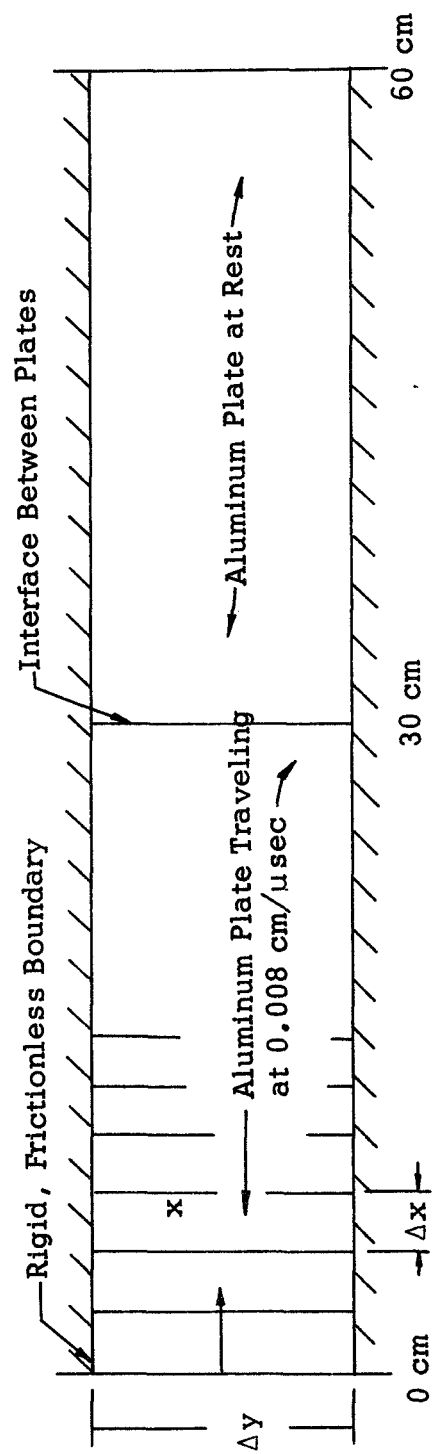


Figure D6. Configuration of Plates at Time of Impact

while the energy density may be defined as $\epsilon = \rho_0 e$. With these definitions in mind, the Los Alamos equation of state is then expressible in the form

$$p = \frac{A + B\epsilon + C\epsilon^2}{\epsilon_0 + \epsilon} \text{ megabar}$$

where

$$A = a_1 \mu + a_2 \mu |\mu|$$

$$B = b_0 + b_1 \mu + b_2 \mu^2$$

$$C = c_0 + c_1 \mu$$

For aluminum, the material constants a_1 , a_2 , b_0 , . . . are evaluated as follows:

$$a_1 = 1.1867$$

$$b_2 = 0.9643$$

$$a_2 = 0.7630$$

$$c_0 = 0.4388$$

$$b_0 = 3.445$$

$$c_1 = 0.5487$$

$$b_1 = 1.545$$

$$\epsilon_0 = 1.5$$

c. Artificial viscosity and stability parameters.

Because the stress levels in the problem under consideration are expected to be relatively low, the linear form of artificial viscosity, given by Eq. (D-114), was employed; the coefficient of linear viscosity, C_L , was set at 1.5. In addition, the stability parameter discussed in Section D-2-2-5 were chosen as:

$$S_1 = 0.15$$

$$S_2 = 0.35$$

$$C_1 = 1.25$$

$$C_2 = 0.50$$

D-3-2. Discussion of Results

To provide the basis with which to compare the numerical results obtained in this study, the important variables of interest for the problem

under consideration were determined theoretically, following the procedure described in Reference 50. In this manner, the magnitude of the longitudinal stress wave was determined as

$$t_{xx} = - 0.00722 \text{ megabars}$$

while the associated lateral stresses are given by

$$t_{yy} = t_{zz} = - 0.00388 \text{ megabars}$$

In addition, the theoretical local speed of the elastic shock front was found to be

$$S_e = 0.657 \text{ cm}/\mu\text{sec}$$

Turning now to the numerical calculations, the problem was studied out to 29.27 μsec after impact, which corresponded to 40 computational cycles. A comparison of the numerical stress wave profiles with the corresponding theoretical results at this time is given in Figure D-7. From an inspection of this figure, it is clear that the numerical solution is in very excellent agreement with the theory; the numerical elastic wave speeds are correct and the intensity of both the longitudinal and lateral stresses agree very well with the theory. While the numerical data shown in Figure D-7 could have been smoothed out more by increasing the magnitude of the coefficient of linear artificial viscosity C_L , no additional refinement was attempted.

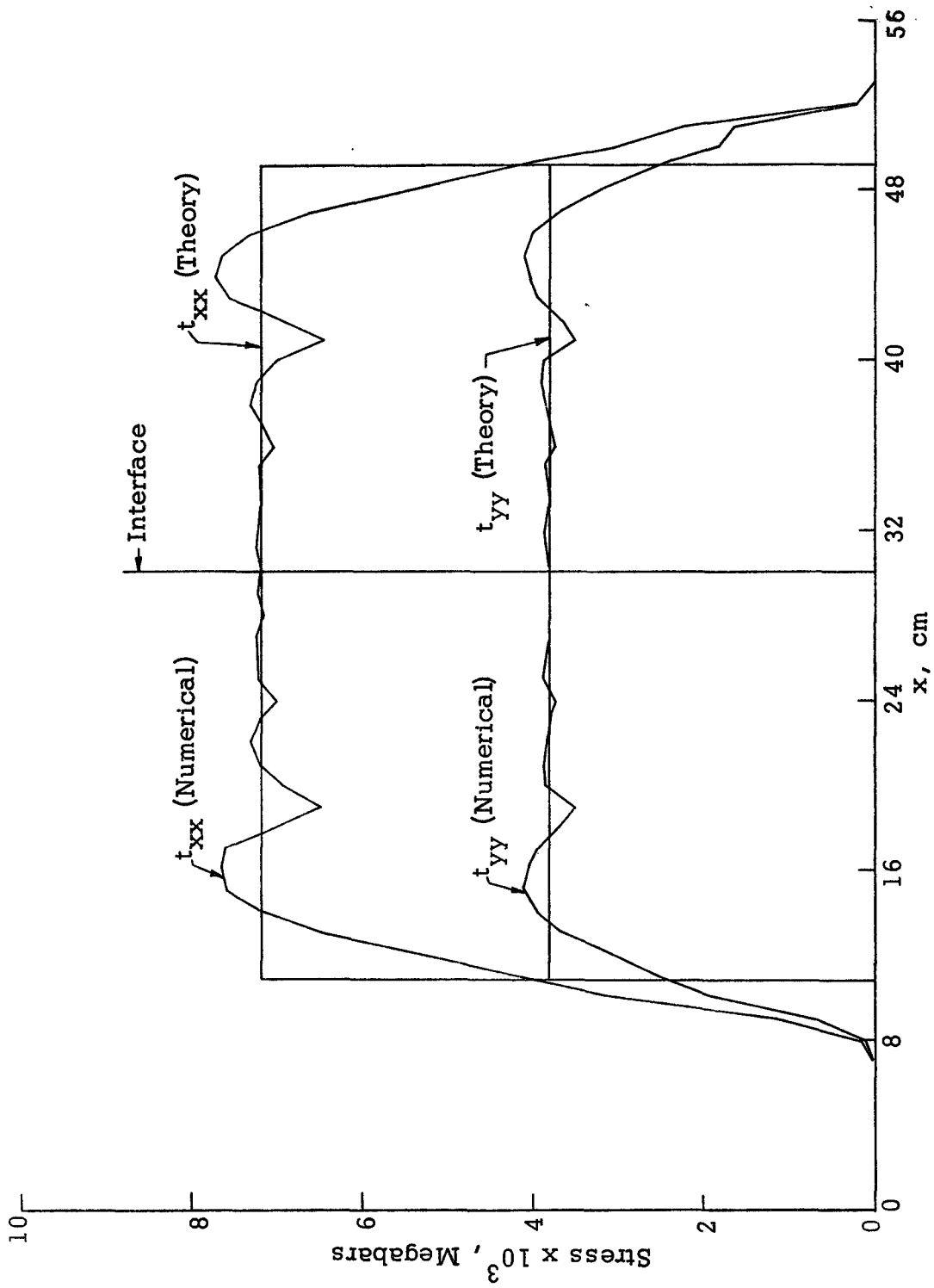


Figure D7. Stress Profiles at 29.268 μsec After Impact

APPENDIX E

THEORETICAL BASIS FOR PRESENT NUMERICAL TREATMENT OF PLASTIC BEHAVIOR

H. E. Read

The numerical procedure employed for the treatment of plastic behavior has been developed specifically for use in Eulerian formulations. The present method circumvents the necessity of dealing directly with the equations governing plastic behavior, namely, the Prandtl-Reuss equations, and as a result provides greater computational efficiency.

In order to keep the following discussion from becoming cumbersome, attention will be confined to homogeneous isotropic solids which do not strain-harden with plastic work. At a fixed spatial location, the mechanical behavior will be studied over a small time interval. For simplicity, it is assumed that the state of deviatoric stress at this fixed location lies on the plastic yield curve at the beginning of the small time interval. Moreover, the mechanical process during the time interval considered is assumed to consist entirely of loading.¹

Under the preceding conditions, the changes in the deviatoric stress components at the fixed spatial location during a subsequent small time interval are determined in the sequel, first of all, from the Prandtl-Reuss equations, and secondly, by the numerical procedure employed in the present method. It is shown that both approaches predict identical results if terms of the order of the square of the time increment are neglected.

E-1. DIRECT APPLICATION OF PRANDTL-REUSS EQUATIONS

The Prandtl-Reuss equations governing the plastic behavior of a material which yields in accordance with von Mises criterion are:

$$\dot{s}_j^i = 2\mu (d_j^{i*} - \Lambda s_j^i) \quad (E-1)$$

$$s_j^i s_i^j \leq \frac{2}{3} Y^2 \quad (E-2)$$

where the symbolism and underlying notions have been discussed earlier. Using the Jaumann-Noll definition of stress rate in Eq. (E-1), this equation

¹ Actually, the restrictions imposed here are not necessary; they are made only to keep the presentation which follows from becoming cumbersome.

may be expanded to read

$$\frac{\partial s_j^i}{\partial t} + v^m s_{j;m}^i + s_m^i \omega_j^m - s_j^m \omega_m^i = 2\mu (d_j^{i*} - \Lambda s_j^i) \quad (E-3)$$

where all of the terms are evaluated at a given spatial location at time n . Both sides of Eq. (E-3) are now multiplied by a small time increment Δt , and after some rearrangement, we obtain the following expression:

$$\begin{aligned} \left(\frac{\partial s_j^i}{\partial t} \right)_p^n \Delta t &= (2\mu d_j^{i*} - v^m s_{j;m}^i - s_m^i \omega_j^m + s_j^m \omega_m^i)_p^n \Delta t \\ &\quad - 2\mu \Lambda \Delta t (s_j^i)_p^n \end{aligned} \quad (E-4)$$

where the sub- and superscripts have been introduced to emphasize that the terms between the parenthesis are evaluated at a fixed spatial position p at time n . Upon introducing the Taylor series expansion for the variable s_j^i , namely

$$(s_j^i)_{p}^{n+1} = (s_j^i)_p^n + \left(\frac{\partial s_j^i}{\partial t} \right)_p^n \Delta t + O(\Delta t^2) \quad (E-5)$$

and setting, for convenience,

$$\begin{aligned} (m_j^i)_p^n &= (2\mu d_j^{i*} - v^m s_{j;m}^i - s_m^i \omega_j^m + s_j^m \omega_m^i)_p^n \Delta t + O(\Delta t^2), \\ \alpha &= 2\mu \Lambda \Delta t, \end{aligned} \quad (E-6)$$

Eq. (E-4) may be placed in the following form:

$$(s_j^i)_{p}^{n+1} = (s_j^i)_p^n + (m_j^i)_p^n - \alpha (s_j^i)_p^n \quad (E-7)$$

Using the preceding result, the product $(s_j^i)_{p}^{n+1} (s_i^j)_{p}^{n+1}$ is formed, keeping in mind the summation convention with repeated indices, and this leads to the following expression:

$$\begin{aligned} (s_j^i)_{p}^{n+1} (s_i^j)_{p}^{n+1} &= (s_j^i)_p^n (s_i^j)_p^n + \alpha^2 (s_j^i)_p^n (s_i^j)_p^n - 2\alpha [(s_j^i)_p^n (s_i^j)_p^n + (s_j^i)_p^n (m_i^j)_p^n] \\ &\quad + 2(s_j^i)_p^n (m_i^j)_p^n + (m_j^i)_p^n (m_i^j)_p^n \end{aligned} \quad (E-8)$$

It will be recalled that we are considering the case in which the states of deviatoric stress at time n and $n+1$ lie on the yield curve; therefore, by

Eq. (E-2), the term on the left hand side of Eq. (E-8) equals the first term on the right hand side. Consequently, Eq. (E-8) reduces, after some rearrangement, to the following quadratic equation in α :

$$\alpha^2 - 2\alpha \left[1 + \frac{(s_j^i)_P (m_i^j)_P^n}{\frac{2}{3} Y^2} \right] + \frac{1}{\frac{2}{3} Y^2} [2(s_j^i)_P (m_i^j)_P^n + (m_j^i)_P (m_i^j)_P^n] = 0 \quad (E-9)$$

The root of Eq. (E-9) which is of interest here is found to be

$$\alpha = \frac{(s_j^i)_P (m_i^j)_P^n}{\frac{2}{3} Y^2} - \frac{1}{2} \left[\frac{(s_j^i)_P (m_i^j)_P^n}{\frac{2}{3} Y^2} \right] + \frac{1}{2} \left[\frac{(m_j^i)_P (m_i^j)_P^n}{\frac{2}{3} Y^2} \right] + O(\Delta t^3) \quad (E-10)$$

Using this expression for α in conjunction with Eq. (E-7), we may determine the state of deviatoric stress at P at time $n+1$ when the velocity and stress fields at time n are given.

E-2. PRESENT NUMERICAL METHOD FOR TREATING PLASTIC BEHAVIOR

Briefly, the numerical method for treating plastic behavior which has been developed for use in the present work may be described as follows: Given the state of stress at a fixed spatial location P together with the velocity field in the neighborhood of P, at some time n , a tentative state of stress at location P a short time later is determined from the constitutive equation associated with purely elastic behavior, regardless of whether the behavior is purely elastic or not. The tentative state of deviatoric stress which results may be located either inside, on, or outside of the yield surface, depending of course on the stress at time n and the nature of the loading process during the subsequent small time interval. If the tentative state of stress at time n falls either inside, or on the yield surface, the tentative state of stress is the correct state of stress at P at time $n+1$. On the other hand, if the tentative state of stress falls outside of the yield surface, then plastic behavior has occurred and the tentative stress components are adjusted so that the state of stress is located once again on the yield surface. Despite the fact that explicit use is made only of Eq. (D-38), it will be shown that the adjusted state of stress coincides with the stress state predicted by direct use of the Prandtl-Reuss equations when higher order terms in the computational time increment are neglected.

To demonstrate this, we begin with the equation governing the deviatoric component of elastic stress, namely,

$$\hat{s}_j^i = 2\mu d_j^{i*} \quad (E-11)$$

Introducing, as before, the Jaumann-Noll stress rate, Eq. (E-11) may be expanded to read

$$\frac{\partial s_j^i}{\partial t} + v^m s_{j;m}^i + s_m^i \omega_j^m - s_j^m \omega_m^i = 2\mu d_j^{i*} \quad (E-12)$$

Each side of Eq. (E-12) is now multiplied by a small time increment Δt , and after some rearrangement we obtain

$$\left(\frac{\partial s_j^i}{\partial t}\right)_P^n \Delta t = (2\mu d_j^{i*} - v^m s_{j;m}^i - s_m^i \omega_j^m + s_j^m \omega_m^i)_P^n \Delta t \quad (E-13)$$

where the sub- and superscripts have again been added to emphasize that the terms within the parenthesis are evaluated at some fixed location P at time n. Upon following the procedure used earlier in this section, Eq. (E-13) may be placed in the following form

$$[s_j^i]_P^{n+1} = (s_j^i)_P^n + (m_j^i)_P^n \quad (E-14)$$

where use has been made of Eqs. (E-5) and (E-6). The stress appearing on the left hand side of Eq. (E-14) has been enclosed in brackets to indicate that it is a tentative stress component, which, depending on the loading process, may or may not have to be suitably adjusted.

We turn now to describe the procedure for adjusting the tentative stress components. It will be recalled that the state of stress at P at time n was taken, for convenience, to lie on the yield surface. Moreover, the mechanical process during the subsequent small time interval Δt was assumed to be pure loading. Under these conditions, the tentative state of deviatoric stress will fall outside of the yield surface, and therefore violate Eq. (E-2). In order to obtain a state of stress at time n+1 which is properly located on the yield surface and does not violate Eq. (E-2), the tentative deviatoric stress components are adjusted according to the expression

$$(s_j^i)_P^{n+1} = (1 - \beta) [s_j^i]_P^{n+1} \quad (E-15)$$

where

$$\beta = 1 - \frac{\sqrt{\frac{2}{3}} Y}{[s_j^i]_P^{n+1} [s_i^j]_P^{n+1}} \quad (E-16)$$

In order to demonstrate that the adjusted stress components given in Eq. (E-15) are identical with the result obtained using the Prandtl-Reuss equations,

let us rewrite Eq. (E-15) in a more suitable form for this purpose. To this end, we combine Eqs. (E-14) and (E-15) to obtain the result

$$(s_j^i)^{n+1}_p = (s_j^i)^n_p + (m_j^i)^n_p - \beta [s_j^i]^{n+1}_p \quad (\text{E-17})$$

Returning now to Eq. (E-16) and making use of Eq. (E-14), one finds after some rearrangement that the parameter β may be expressed as

$$\beta = 1 - \left[1 + \frac{2(s_j^i)^n_p (m_i^j)^n_p}{\frac{2}{3} Y^2} + \frac{(m_j^i)^n_p (m_i^j)^n_p}{\frac{2}{3} Y^2} \right]^{-\frac{1}{2}} \quad (\text{E-18})$$

and if the binomial expansion is applied to the expression inside of the brackets, one reaches finally the equation

$$\beta = \frac{(s_j^i)^n_p (m_i^j)^n_p}{\frac{2}{3} Y^2} - \frac{1}{2} \left[\frac{(s_j^i)^n_p (m_i^j)^n_p}{\frac{2}{3} Y^2} \right]^2 + \frac{1}{8} \left[\frac{(m_j^i)^n_p (m_i^j)^n_p}{\frac{2}{3} Y^2} \right] + O(\Delta t^3) \quad (\text{E-19})$$

which is merely a different form of Eq. (E-16).

E-3. COMPARISON OF THE APPROACHES

Having assumed that the dynamical and stress states at time n together with the subsequent loading process are identical in both approaches, let us now compare the state of stress at time $n+1$ predicted by each method.

From dealing directly with the Prandtl-Reuss equations, we showed in Eq. (E-7) that the correct deviatoric stress components at time $n+1$ were given by the expression

$$[(s_j^i)^{n+1}]_{\text{P. R.}} = (s_j^i)^n_p + (m_j^i)^n_p - \alpha (s_j^i)^n_p \quad (\text{E-20})$$

where the subscript is used signifying that the stress component is calculated via the Prandtl-Reuss equations. Based upon the numerical method employed in the present work for plastic behavior, we showed that the stress components at time $n+1$ were given by the following equation:

$$(s_j^i)^{n+1}_p = (s_j^i)^n_p + (m_j^i)^n_p - \beta [s_j^i]^{n+1}_p \quad (\text{E-21})$$

Upon subtracting Eq. (E-20) from Eq. (E-21), and using the expressions for α and β given, respectively, in Eqs. (E-10) and (E-19), we find the result

$$\begin{aligned}
(s_{j,p}^i)^{n+1} = & [(s_{j,p}^i)^{n+1}]_{P.R.} - \left[\frac{(s_{j,p}^i)^n (m_{i,p}^j)^n}{\frac{2}{3} Y^2} \right] (m_{j,p}^i)^n + \left[\frac{(s_{j,p}^i)^n (m_{i,p}^j)^n}{\frac{2}{3} Y^2} \right]^2 (s_{j,p}^i)^n \\
& + O(\Delta t^3)
\end{aligned} \tag{E-22}$$

Returning now to Eq. (E-6), let us note that

$$(m_{j,p}^i)^n = O(\Delta t) \tag{E-23}$$

Hence, using this fact in conjunction with Eq. (E-22), we finally conclude that

$$(s_{j,p}^i)^{n+1} = [(s_{j,p}^i)^{n+1}]_{P.R.} + O(\Delta t^2) \tag{E-24}$$

We have therefore demonstrated that, under the conditions imposed, the present numerical method for treating plastic behavior yields the same result as would be obtained by dealing directly with the Prandtl-Reuss equations if terms of the order of Δt^2 are neglected.

APPENDIX F

NUMERICAL TREATMENT OF STRESSES IN PARTIALLY FILLED OFFSET CELLS

H. E. Read

The procedure for updating the stresses in partially filled offset cells has been described in the main text for the case in which the lattice point at an offset cell center is inside the material at both times n and $n+1$. For other situations in which the lattice point in a partially filled cell does not meet this requirement, special procedures have been devised for updating the stress components under such conditions, the salient features of which are described in this appendix.

In the numerical treatment of stress in partially filled offset cells, an averaging process proves very useful. For this purpose, a coefficient $A_{I,J}$ is defined to indicate if the lattice point (I, J) in a typical partially-filled offset cell is inside the material at times n and $n+1$, or otherwise. To be specific, we set

$$A_{I,J} = \begin{cases} 1, & \text{if the lattice point } (I, J) \text{ is inside the} \\ & \text{material at times } n \text{ and } n+1 \\ 0, & \text{otherwise} \end{cases}$$

Using the coefficient $A_{I,J}$, the average stress components at lattice point (I, J) are obtained from the equations of the form

$$(t_{rr})_{I,J}^{n+1} = \frac{A_{I,J-1} (t_{rr})_{I,J-1}^{n+1} + A_{I+1,J} (t_{rr})_{I+1,J}^{n+1} + A_{I,J} (t_{rr})_{I,J+1}^{n+1} + A_{I-1,J} (t_{rr})_{I-1,J}^{n+1}}{(4 - N)}$$

where N is the number of coefficients A which vanish in this equation. The remaining components of stress at (I, J) are calculated by expressions similar to the preceding equation.

Using the averaging process just described, the stresses in a typical partially filled offset cell (I, J) are updated from time n to $n+1$ according to the procedure listed in the following table.

TABLE F-1. NUMERICAL PROCEDURE IN PARTIALLY FILLED
OFFSET CELLS

Time n	Time n + 1	Numerical Procedure
1. Point (I, J) outside material	Point (I, J) outside material	Do not perform any stress calculations for offset cell. ¹
2. Point (I, J) outside material	Point (I, J) inside material	Calculate stress components at point (I, J) according to averaging procedure.
3. Point (I, J) inside material	Point (I, J) outside material	Set stress components at time n + 1 equal to zero, and make no further stress calculations.
4. Point (I, J) inside material	Point (I, J) inside material	Follow procedure described in main part of text for updating stresses.
5. Point (I, J) inside material	Point (I, J) on free surface	Use averaging process to obtain the stress components at point (I, J).
6. Point (I, J) outside material	"	
7. Point (I, J) on free surface	"	
8. Point (I, J) on free surface	Point (I, J) inside material	Use averaging process to calculate stress components at point (I, J).
9. Point (I, J) on free surface	Point (I, J) outside material	Set stress components at time n + 1 equal to zero, and make no further stress calculations.

¹It will occasionally happen in cells near a free surface that the stress flow calculations require the stresses in offset cells, whose interior lattice points are outside of the material. When this occurs, simply use the stresses in the offset cells into which the stress is transported for the stress flow calculation.

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