X-641-70-178
PREPRINT

NASA TM X- 63907

A NEW FORMULATION OF CANONICAL PERTURBATION THEORY

DAVID P. STERN

MAY 1970

GSFC

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

A New Formulation of Canonical Perturbation Theory

David P. Stern
Laboratory for Space Physics
Goddard Space Flight Center
Greenbelt, Maryland 20771

Abstract

By means of direct canonical transformations, the Poincare-Von Zeipel perturbation method may be modified into a more useful form. Two particular variants of this approach are examined and explicit formulas are given for carrying them out to at least the 4-th order.

1. INTRODUCTION

In this work, a modification of the Hamilton-Jacobi equation is developed, utilizing the direct formulation of canonical transformations. The resulting perturbation scheme is easy to apply and may be related either to the standard Poincaré-Von Zeipel method or to Hori's expansion, depending on the choice of certain expressions entering it.

The solution of a mechanical problem by the Hamilton-Jacobi equation seeks a canonical transformation to new variables, in which the complexity of the problem is reduced. In particular, let us consider a perturbed periodic motion, with p_1 the action variable of the unperturbed motion and with a time-independent Hamiltonian

$$H = p_1 + \sum_{k=1} \varepsilon^k H^{(k)}(\underline{p}, \underline{q}) \tag{1}$$

Then by the Poincaré-Von Zeipel method (Poincare, 1893; Von Zeipel, 1916; Corben and Stehle, 1960) one seeks a near-identity transformation to new variables $(\underline{P}, \underline{Q})$, given by the generating function

$$\sigma = \sum P_i q_i + \sum_{k=1} \varepsilon^k \sigma^{(k)}(\underline{P}, \underline{q}) \tag{2}$$

via

$$\left.\begin{aligned} \partial\sigma/\partial P_i &= Q_i \\ \partial\sigma/\partial q_i &= p_i \end{aligned}\right\} \tag{3}$$

In this method, one uses the fact that the new Hamiltonian equals the old one, substituting (3) to obtain

$$H(\partial\sigma/\partial \underline{q}, \underline{q}) = H^*(\underline{P}, \partial\sigma/\partial \underline{P}) \tag{4}$$

When this is expanded in orders of ε , a recursive scheme is obtained and both H^* and σ may be derived order by order, subject to the condition that the transformed angle variable Q_1 is absent from the new Hamiltonian H^* . When this is accomplished, not only has Q_1 been removed from the problem, but its conjugate P_1 has become a constant of the motion, so that the motion represented by H^* has two fewer variables to contend with.

Now a near-identity transformation may be represented in other ways than by equations (2)-(3) ; these ways may offer more convenience, since σ in (2) depends on both old and new variables and therefore the relations (3) require further untangling to be useful. In particular, we may consider a <u>direct</u> canonical transformation

$$\underline{z} = \underline{y} + \sum_{k=1} \varepsilon^k \underline{\zeta}^{(k)}(\underline{y}) \qquad (5)$$

where we will use the notation

$$\left.\begin{aligned} \underline{y} &\equiv (\underline{p}, \underline{q}) \\ \underline{z} &\equiv (\underline{P}, \underline{Q}) \end{aligned}\right\} \quad (6)$$

with

$$\left.\begin{aligned} y_1 &= p_1 \\ y_n &= q_1 \end{aligned}\right\} \quad (7)$$

Also, it will be useful to denote by a tilde the vector formed by omitting the component y_n or the one corresponding to it, e.g.

$$\underline{z} \equiv (\tilde{\underline{z}}, z_n) \qquad (8)$$

Then, if one substitutes (5) into the basic relation

$$H(\underline{y}) = H^*(\tilde{\underline{z}}) \qquad (9)$$

and expands order by order, the appropriate transformation (5) may be derived. Hori (1966) used this approach, with (5) given through Lie's formulation of canonical transformations (Lie, 1888). Here a somewhat different approach (Stern, 1970) will be used, which can be made equivalent either to Hori's or to the Poincare-Von Zeipel method, depending on one's choice of the expression $\underline{f}^{(k)}$, defined later.

2. NOTATION

If the position vector $\underline{y}$ in phase space is defined as in (6), let its conjugate $\bar{\underline{y}}$ be defined

$$\bar{\underline{y}} = (\underline{q}, -\underline{p}) \qquad (10)$$

from which, by (7)

$$\left.\begin{aligned} \bar{y}_1 &= y_n \\ \bar{y}_n &= - y_1 \end{aligned}\right\} \qquad (11)$$

Also, if ∇ operates in y-space, we can define there a conjugate gradient operator $\bar{\nabla}$, the i-th component of which is $\partial/\partial \bar{y}_i$.

To express functions of $\underline{z}$ in terms of $\underline{y}$, we shall use Taylor expansion operators $T^{(k)}$ (Musen, 1965), with operation denoted (for clarity) by an asterisk:

$$\begin{aligned} G(\underline{z}) &= G(y + \textstyle\sum \varepsilon^k \underline{\zeta}^{(k)}) \\ &= \exp(\sum_{k=1} \varepsilon^k \underline{\zeta}^{(k)} \cdot \nabla) * G(\underline{y}) \\ &= \sum_{k=0} \varepsilon^k T^{(k)} * G(\underline{y}) \qquad (12) \end{aligned}$$

The $T^{(k)}$ may be obtained by expanding the exponential and regrouping terms according to their order in ε; the first few of them are

$$\left.\begin{aligned}
T^{(0)} &= 1 \\
T^{(1)} &= \underline{\zeta}^{(1)} \cdot \nabla \\
T^{(2)} &= \underline{\zeta}^{(2)} \cdot \nabla + \tfrac{1}{2}\, \underline{\zeta}^{(1)} \underline{\zeta}^{(1)} : \nabla \nabla \\
T^{(3)} &= \underline{\zeta}^{(3)} \cdot \nabla + \underline{\zeta}^{(1)} \underline{\zeta}^{(2)} : \nabla \nabla + (1/6)\, \underline{\zeta}^{(1)} \underline{\zeta}^{(1)} \underline{\zeta}^{(1)} \therefore \nabla \nabla \nabla
\end{aligned}\right\} \quad (13)$$

and so forth. In general one can write

$$T^{(k)} = \underline{\zeta}^{(k)} \cdot \nabla + N^{(k)} \qquad (14)$$

where $N^{(k)}$ is an operator involving ∇ at least twice.

3. THE TRANSFORMATION

The relation (5) holds for any near-identity transformation. If this is to be a canonical transformation, $\underline{\zeta}^{(k)}$ must satisfy additional requirements, and it may be shown (Stern, 1970) that these always have the general form

$$\underline{\zeta}^{(k)} = \underline{f}^{(k)}(\underline{\zeta}) + \bar{\nabla} \chi^{(k)} \qquad (15)$$

where $\underline{f}^{(k)}(\underline{\zeta})$ are expressions involving lower orders and $\chi^{(k)}$ is the k-th order of a generalization of the generating function. Many alternative choices of $\underline{f}^{(k)}$ are possible (one can often construct lower order expressions that have the form of a conjugate gradient and add them to $\underline{f}^{(k)}$) but we shall be concerned with two in particular. One may choose

$$f_i^{(k)} = - \sum_{m=1}^{k-1} S^{(m)} * \left(\partial \chi^{(k-m)} / \partial \bar{y}_i \right) \qquad (16)$$

where $S^{(m)}$ are operators resembling the $T^{(m)}$ of (13) but with $\underset{\sim}{\zeta}^{(j)}$ everywhere replaced by

$$\underline{\pi}^{(j)} = (\zeta_1^{(j)}, \ldots \zeta_{(n/2)}^{(j)}, 0, \ldots 0) \tag{17}$$

that is, with vectors in which the momentum-like parts are retained and the coordinate-like parts are replaced by zeros. It may then be shown (see Stern, 1970, where this choice of $\underline{f}^{(k)}$ is denoted by $\underline{\rho}^{(k)}$) that in this case

$$\chi^{(k)} = \sigma^{(k)} \tag{18}$$

with $\sigma^{(k)}$ the k-th order "conventional" generating function of equation (2) which describes the same transformation. Since $S^{(m)}$ contains orders of $\underset{\sim}{\zeta}$ and of χ lower than the k-th, equation (16) must be derived recursively order by order.

Alternatively, $f_i^{(k)}$ may be derived without breaking up $\underset{\sim}{\zeta}^{(k)}$ into momentum-like and coordinate-like parts. The general method for doing this (Stern, 1970) is too lengthy to be described here, and we shall merely give the lowest orders of the result:

$$\left.\begin{aligned}
\underline{f}^{(1)} &= 0 \\
\underline{f}^{(2)} &= \tfrac{1}{2}\, \underset{\sim}{\zeta}^{(1)} \cdot \nabla \underset{\sim}{\zeta}^{(1)} \\
\underline{f}^{(3)} &= \underset{\sim}{\zeta}^{(2)} \cdot \nabla \underset{\sim}{\zeta}^{(1)} \\
\underline{f}^{(4)} &= \underset{\sim}{\zeta}^{(3)} \cdot \nabla \underset{\sim}{\zeta}^{(1)} + \tfrac{1}{2}\, \underset{\sim}{\zeta}^{(2)} \cdot \nabla \underset{\sim}{\zeta}^{(2)} + \\
&\quad + \tfrac{1}{4} \left[(\underset{\sim}{\zeta}^{(1)} \cdot \nabla \underset{\sim}{\zeta}^{(1)}) \cdot \nabla \underset{\sim}{\zeta}^{(2)} - \underset{\sim}{\zeta}^{(2)} \cdot \nabla (\underset{\sim}{\zeta}^{(1)} \cdot \nabla \underset{\sim}{\zeta}^{(1)}) \right] \\
&\quad - \tfrac{1}{4} \left[(\underset{\sim}{\zeta}^{(1)} \cdot \nabla \underset{\sim}{\zeta}^{(1)}) \cdot \nabla \underset{\sim}{\zeta}^{(1)} \right] \cdot \nabla \underset{\sim}{\zeta}^{(1)}
\end{aligned}\right\} \tag{19}$$

With this choice it may be shown (Stern, 1970) that

$$\chi^{(k)} = - S^{(k)} \tag{20}$$

where $S^{(k)}$ is the k-th order generating function defined by Hori (1966).

4. THE PERTURBATION METHOD

Expanding (9), and using (1) and (12), gives

$$\begin{aligned} y_1 + \sum_{k=1} \varepsilon^k H^{(k)}(\underset{\sim}{y}) &= \sum_{k=0} \varepsilon^k H^{*(k)}(\underset{\sim}{\tilde{z}}) \\ &= \sum_{k=0} \varepsilon^k H^{*(k)}(\underset{\sim}{\tilde{y}} + \sum \varepsilon^k \underset{\sim}{\tilde{\zeta}}^{(k)}) \\ &= \sum_{k=0} \varepsilon^k \sum_{m=0}^{k} T^{(m)} * H^{*(k-m)}(\underset{\sim}{\tilde{y}}) \end{aligned} \tag{21}$$

This may be equated order by order, the zeroth order being simply

$$H^{*(0)}(\underset{\sim}{\tilde{y}}) = H^{(0)}(\underset{\sim}{y}) = y_1 \tag{22}$$

In the $O(\varepsilon^k)$ relation we separate the terms with m equal to 0 and k from the rest and use (14) and (22). This gives

$$\begin{aligned} H^{(k)}(\underset{\sim}{y}) = H^{*(k)}(\underset{\sim}{\tilde{y}}) + \underset{\sim}{\zeta}^{(k)} \cdot \nabla y_1 + N^{(k)} * y_1 + \\ + \sum_{m=1}^{k-1} T^{(m)} * H^{*(k-m)}(\underset{\sim}{\tilde{y}}) \end{aligned} \tag{23}$$

In the last equation, $N^{(k)}$ may be dropped, since it contains ∇ twice or more and its action on y_1 thus yields zero. Also, by (11)

$$\underset{\sim}{\zeta}^{(k)} \cdot \nabla y_1 = \zeta_1^{(k)} = f_1^{(k)} + \partial \chi^{(k)} / \partial y_n \qquad (24)$$

The final result, which is the basic relation of our perturbation scheme, may thus be written

$$\partial \chi^{(k)} / \partial y_n + H^{*(k)}(\tilde{\underset{\sim}{y}}) = \Lambda^{(k)}(\underset{\sim}{y}) \qquad (25)$$

where

$$\Lambda^{(k)}(\underset{\sim}{y}) = H^{(k)}(\underset{\sim}{y}) - f_1^{(k)}(\underset{\sim}{\xi}) - \sum_{m=1}^{k-1} T^{(m)}{}_* H^{*(k-m)}(\tilde{\underset{\sim}{y}}) \qquad (26)$$

If the variable y_n enters only as an angle with basic period unity, $\partial \chi^{(k)} / \partial y_n$ is clearly periodic, since any nonperiodic part of $\chi^{(k)}$ is removed by the differentiation. If one defines an averaging operator

$$\langle \Lambda^{(k)} \rangle = \int_0^1 \Lambda^{(k)} \, dy_n \qquad (27)$$

then clearly

$$H^{*(k)}(\tilde{\underset{\sim}{y}}) = \langle \Lambda^{(k)} \rangle \qquad (28)$$

$$\chi^{(k)} = \int_0^{y_n} \left(\Lambda^{(k)} - \langle \Lambda^{(k)} \rangle \right) dy_n' + M^{(k)}(\tilde{\underset{\sim}{y}}) \qquad (29)$$

where $M^{(k)}(\tilde{\underset{\sim}{y}})$ is an arbitrary function, representing the fact that when y_1 and y_n are eliminated, the other variables may also undergo an arbitrary near-identity transformation among themselves. If no other considerations exist, it may be set equal to zero.

Suppose the calculation has been carried out up to and including order (k-1). We now form $\Lambda^{(k)}$ by (26), derive $H^{*(k)}$ and $\chi^{(k)}$ by (28) and (29), and then obtain $\underset{\sim}{\zeta}^{(k)}$ from (15), using the appropriate choice of $\underset{\sim}{f}^{(k)}$. The derivation is now complete to the k-th order.

ACKNOWLEDGEMENT

The author wishes to thank Dr. Peter Musen for a stimulating discussion of this work.

REFERENCES

Corben, H.C., and Stehle, P.: 1960, Classical Mechanics, 2nd Ed., John Wiley and Sons; section 75.

Hori, G.-I.: 1966, Theory of General Perturbations with Unspecified Canonical Variables, Publ. Astron. Soc. Japan 18, 287-296

Lie, S.: 1888, Theorie der Transformationgruppen, Teubner, Leipzig

Musen, P.: 1965, On the High Order Effects in the Methods of Krylov-Bogoliubov and Poincaré, J. Astronautical Sciences 12, 129-134

Poincaré, H.,: 1893, Les Methodes Nouvelles de la Mecanique Celeste, vol. II, Gauthier-Villars, Paris; reprinted by Dover Publications, New York, 1957; NASA translation TT F-451, 1967.

Stern, D.: 1970, Direct Canonical Transformations, to appear in J. Math. Physics

Von Zeipel, H.: 1916, Ark. Astr. Mat. Fys. 11, no. 1