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SOLUTIONS OF TWO HEAT-TRANSFER PROBLEMS WITH APPLICATION TO HYPERSONIC CRUISE AIRCRAFT

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16. Abstract

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SYMBOLS

a_n	Fourier coefficients, °F
C	insulation specific heat, Btu/lb-°F
C_B	tank specific heat, Btu/lb-°F
h_{fg}	hydrogen heat of transformation, Btu/ft^3
I_i	defined by equation (44), °F-ft
I_{ij}	defined by equation (45), ft
k	diffusivity, ft^2/hr
K	insulation conductivity, Btu/hr-ft-°F
$K_B = \frac{K}{C_B L_B \rho_B L}$, 1/hr	
L	insulation thickness, ft
L_B	tank thickness, ft
L_{BO}	boil-off thickness, ft
L_{SS}	steady-state insulation thickness, ft
Q	heat transfer, Btu/ft^2
Q_{SS}	steady-state heat transfer, Btu/ft^2
t	time, hr
T	temperature-time function
T_n	temperature-time function associated with λ_n
T_B	maximum structural temperature, °F
$\overline{T}$	insulation properties temperature, °F
t_f	cruise time, hr
T_H	liquid hydrogen fuel temperature, °F
T_o	initial exterior surface temperature, °F
T_S	cruise exterior surface temperature, °F

u temperature distribution, °F

U_{SS} steady-state temperature distribution, °F

v transformed temperature distribution, °F

W wet tank weight function, lb/ft^2

W_{SS} steady-state weight, lb/ft^2

x distance, ft

λ separation constant, $1/ft^2$

λ_n eigenvalues, $1/ft^2$

Φ temperature-distance function

Φ_n eigenfunctions

ρ insulation density, lb/ft^3

ρ_B tank density, lb/ft^3

ρ_H hydrogen fuel density, lb/ft^3

τ dummy time variable, hr

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SUMMARY

Solutions are obtained for two initial-boundary value problems of one-dimensional heat conduction. The problems concern insulation systems for liquid-hydrogen-fueled hypersonic aircraft. The solutions are obtained by standard analytical techniques and are used to develop procedures for estimating the weight of insulation systems for such aircraft. Numerical results are presented and compared with results of a finite difference analysis, and agreement is found to be excellent. Finally, the solutions are compared with steady-state approximations, and it is concluded that these approximations provide convenient weight estimating formulas.

INTRODUCTION

Studies of loads and weights of liquid-hydrogen-fueled hypersonic aircraft are required as part of comparative mission performance analyses. This report presents solutions of two heat-transfer problems encountered in estimating insulation weight of fuselage thermal protection systems. The analysis, however, is general and may be applied to other cases, such as insulated wings and fins.

Previous studies (refs. 1-6) have indicated that hypersonic cruise aircraft are likely to be liquid-hydrogen (LH_2) fueled and will probably carry a major fraction of the fuel in the fuselage. Since the exterior surfaces of such aircraft are at high temperatures and their interior structure is at the cryogenic temperature of the fuel, their fuselages will require thermal protection systems that will significantly increase the gross weight of the vehicles. These systems will be required both to prevent excessive boiloff of fuel (called the "wet tank" problem in this report) and to limit structural temperatures ("dry tank" problem).

In the past, the weights of thermal protection systems have been determined either from steady-state heat conduction analyses or, in more detailed studies, from numerical solution of the transient heat conduction equation. It appears that analytical solutions of the heat equation with application to hypersonic aircraft thermal protection systems are not available in the literature. This report presents such analytic solutions for the wet tank and dry tank problems.

To illustrate the solutions of the two problems, a representative numerical example is presented. The results are compared with results obtained with a finite difference heat-transfer computer program. The transient solutions obtained in the present analysis are also used to investigate the validity of steady-state approximations. These approximations are useful for preliminary weight estimates because they may be solved for the insulation thicknesses in closed form. Both the magnitudes and the sensitivities of these approximations are compared with the transient results.

PROBLEM FORMULATION

A typical hypersonic aircraft configuration is shown in figure 1 and the principal elements of the fuselage of such an aircraft are shown schematically in figure 2. The fuselage bending loads may be carried either by the exterior structure (nonintegral tankage) or by the tank structure (integral tankage). Because fuel is used, not all of the tank circumference will be in contact with the LH_2 for the entire flight. Consequently, there are two extreme or limiting cases to be considered:

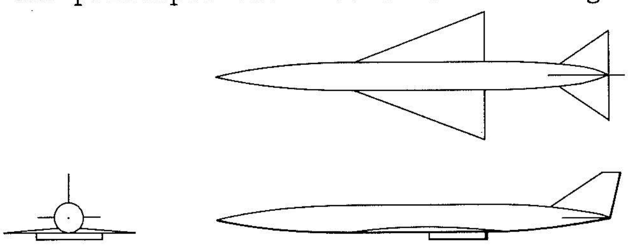


Figure 1.- Hypersonic aircraft configuration.

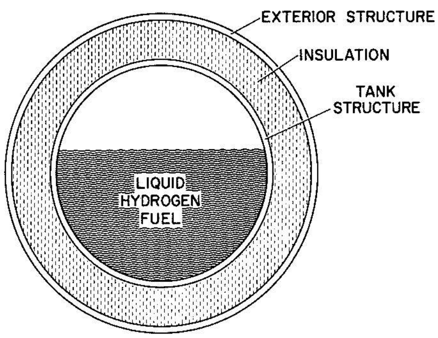


Figure 2.- Fuselage cross section.

(1) The wet tank case. The temperature at the wall is held to that of the LH_2 throughout the flight. (This case corresponds to the bottom of the last tank to be emptied.)

(2) The dry tank case. The heat transferred through the insulation is absorbed by the tank structure and the tank wall temperature is allowed to rise accordingly. (This case corresponds to the top of the first tank to be emptied.)

The insulation thicknesses for each case are given by the solution to the two heat-transfer problems associated with these two cases. Vehicle geometry and tank sequencing are then considered in estimating the fraction of vehicle surface area over which the individual thicknesses apply in the weight calculation.

The major assumptions of the analysis are as follows: (1) heat transfer is by conduction only; (2) circumferential heat transfer is negligible compared with radial; (3) thickness of insulation is small compared with fuselage radius; (4) conductivity of all structural elements is infinitely large compared with conductivity of insulation; (5) insulation is continuous and homogeneous; (6) thermal constants (e.g., conductivity of insulation, specific heat of insulation, specific heat of structure) are independent of position, time, and temperature; (7) exterior is exposed to a square temperature pulse (or, equivalently, a step pulse). These assumptions imply that initial-boundary value problems of the one-dimensional heat equation are to be solved. These solutions are obtained by the standard analytical techniques of separation of variables and eigenfunction expansions (cf. ref. 7).

For application to hypersonic cruise aircraft, all of the above assumptions are reasonably well satisfied. Since both radiative and convective heat transfer at the inner tank surface are relatively small for the tank temperatures of interest, their effects may be included in the conductivity; thus assumption (1) is appropriate. Assumptions (2) and (3) are valid since for such vehicles the fuselage diameter is much larger than the structural and the insulation thicknesses. For materials usually considered for hypersonic vehicles, namely metals (cover panels, tank, load bearing structure) and quartz fiber type insulations, assumption (4) is reasonable. Thus the exterior structure, and in the wet tank case, the tank structure, may be neglected in the heat-transfer analysis. Assumption (5) will be violated in actual insulation systems because of heat leaks caused by cover panel supports and edge effects of finite-dimensional insulation blankets. However, these effects may be accounted for by using an overall insulation conductivity. Although insulation properties will typically be a function of temperature (fig. 3 shows conductivity data for a representative insulation material), representative values may be used in order to comply with assumption (6). Later in this paper a numerical example shows that solutions obtained using a representative value of conductivity agree reasonably with a variable conductivity calculation. Assumption (7) requires that the ascent and descent phases of the flight take negligible time compared with the cruise time. Although this is not strictly true, the actual temperature history may be approximated reasonably well by a square pulse as shown in figure 4. The "actual" pulse shown on this figure was obtained from a detailed trajectory computation. The "idealized" pulse is determined by equating the actual pulse area to the idealized pulse area. Finally, it should be remarked that the solutions may only be used for single-layer insulations.

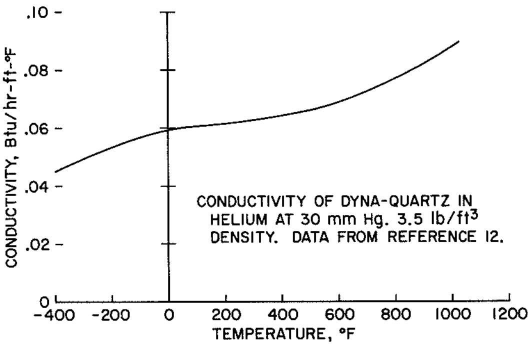


Figure 3.- Variation of conductivity with temperature.

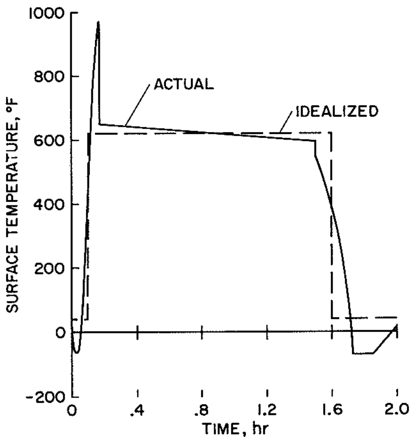


Figure 4.- Typical exterior temperature pulse.

Since the variables of interest (the insulation thicknesses) cannot be obtained explicitly, it is necessary to solve for the thicknesses by iteration. The thicknesses (weights) are optimized in the sense that the weight of insulation plus LH_2 boiloff is minimized. The entire analysis is programmed for a digital computer. To obtain vehicle thermal protection system weight, items such as cover panels, attachments, and "nonoptimum" weight must be added to the insulation weight.

Although application to hypersonic cruise aircraft fuselages is exclusively considered in this paper, other possible applications should be mentioned. The dry tank solution is directly applicable to insulated structures such as wings and fins on cruise vehicles. Its application to vehicles other than cruise vehicles may be limited by assumption (7) above. For example, launch vehicles, which have no cruise time, may have exterior temperature pulses that cannot be approximated by a square pulse. Application to reentry vehicles is discussed in reference 8.

ANALYSIS

Wet Tank

The partial differential equation to be solved is

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{1}$$

subject to boundary conditions

$$u(0,t) = T_S \tag{2}$$

$$u(L,t) = T_H \tag{3}$$

and the initial condition

$$u(x,0) = \frac{T_H - T_o}{L} x + T_o \tag{4}$$

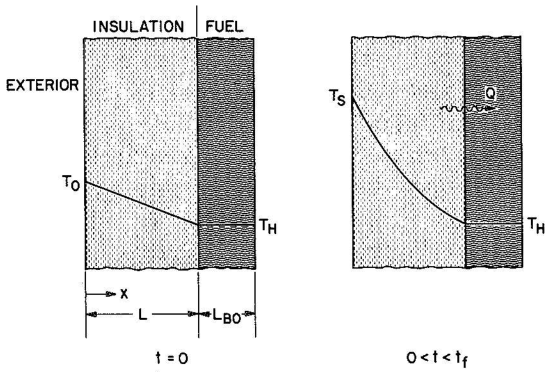


Figure 5.- Wet tank.

where

$$k = \frac{K}{C\rho} \tag{5}$$

This problem is depicted in figure 5. Introducing the transformation

$$u = v + \frac{T_H - T_S}{L} x + T_S \tag{6}$$

into equations (1) - (4) results in the following homogeneous version of the problem:

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \tag{7}$$

$$v(0,t) = 0 \tag{8}$$

$$v(L,t) = 0 \tag{9}$$

$$v(x,0) = \frac{T_H - T_o}{L} x + T_o - \frac{T_H - T_S}{L} x - T_S \tag{10}$$

Setting

$$v(x,t) = T(t)\Phi(x) \tag{11}$$

in equation (7) results in the separated equations

$$\frac{dT}{dt} + \lambda kT = 0 \tag{12}$$

$$\frac{d^2\Phi}{dx^2} + \lambda\Phi = 0 \tag{13}$$

Solving equation (13) subject to boundary conditions (8) and (9) gives for the eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 ; \qquad n = 1, 2, \ldots \tag{14}$$

$$\Phi_n = \sin\left(x\sqrt{\lambda_n}\right); \qquad n = 1, 2, \ldots \tag{15}$$

respectively. Since the solution of equation (12) is

$$T_n = e^{-\lambda_n kt} ; \qquad n = 1, 2, \ldots \tag{16}$$

$v(x,t)$ may now be written as

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n kt} \sin\left(x\sqrt{\lambda_n}\right) \tag{17}$$

Since the eigenfunctions are orthogonal for this problem, equations (10) and (17) imply

$$a_n = \frac{2}{L}\int_0^L \left(\frac{T_H - T_o}{L}x + T_o - \frac{T_H - T_S}{L}x - T_S\right)\sin\left(x\sqrt{\lambda_n}\right)dx$$

$$a_n = \frac{2}{L\sqrt{\lambda_n}}(T_o - T_S) ; \qquad n = 1, 2, \ldots \tag{18}$$

The temperature distribution is then obtained from equations (6), (14), (17), and (18)

$$u(x,t) = \frac{T_H - T_S}{L} x + T_S + 2(T_o - T_S) \sum_{n=1}^{\infty} \frac{1}{n\pi} e^{\frac{-n^2\pi^2 kt}{L^2}} \sin \frac{n\pi x}{L} \tag{19}$$

Since this function is a "strict" solution of the problem it may easily be shown that it is also a unique solution.

The heat transferred through the tank wall to the LH_2 fuel is

$$Q = -\int_0^{t_f} K \frac{\partial u(L,\tau)}{\partial x} d\tau$$

$$Q = \frac{Kt_f(T_S - T_H)}{L} + \frac{2KL(T_S - T_o)}{\pi^2 k} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(1 - e^{\frac{-n^2\pi^2 kt_f}{L^2}}\right) \tag{20}$$

An analytical verification of equation (20) is provided by reference 8 which gives the solution for the special case of $T_o = T_H$ (applicable for re-entry vehicles). Setting $T_o = T_H$ in equation (20) gives

$$Q\big|_{T_o=T_H}$$

$$= \rho CL \frac{\rho K}{C} \frac{t_f}{L^2\rho^2} (T_S - T_H) \left\{1 + \frac{2}{\pi^2 \frac{\rho K}{C} \frac{t_f}{L^2\rho^2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left[1 - e^{-(n\pi)^2 \frac{\rho K}{C} \frac{t_f}{L^2\rho^2}}\right]\right\} \tag{21}$$

which is equation (2) of reference 8. The heat absorbed by the fuel is

$$Q = h_{fg} L_{BO} \tag{22}$$

where L_{BO} is the "boiloff thickness" or volume of LH_2 boiled off during flight per unit surface area. Hence, combining equations (20) and (22) gives

$$L_{BO} = \frac{Kt_f(T_S - T_H)}{Lh_{fg}} + \frac{2KL(T_S - T_o)}{\pi^2 kh_{fg}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(1 - e^{\frac{-n^2\pi^2 kt_f}{L^2}}\right) \tag{23}$$

The insulation weight for the wet tank case can now be computed. The weight per unit area is

$$W = \rho L + \rho_H L_{BO} \tag{24}$$

where L_{BO} as a function of L is given by equation (23). The necessary condition for minimum weight is $dW/dL = 0$ which leads to

$$\rho + \rho_H \frac{dL_{BO}}{dL} = 0$$

$$\frac{h_{fg}\rho}{\rho_H} - \frac{Kt_f(T_S - T_H)}{L^2} - \frac{2K(T_S - T_o)}{k}\left[\frac{1}{12} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n^2\pi^2} + \frac{2kt_f}{L^2}\right) e^{\frac{-n^2\pi^2 kt_f}{L^2}}\right] = 0 \tag{25}$$

where $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$ was used. Equation (25) must be solved iteratively for L. After this has been done, other quantities such as L_{BO}, Q, and the temperature distribution $u(x,t)$ may easily be computed. Computationally it was found that all of the infinite series involved converge very rapidly. Thus only the first five terms of each series were retained. It is of interest to note that for fixed materials and temperatures, equation (25) implies that for optimum insulation thickness the ratio t_f/L^2 is fixed (i.e., the optimum insulation thickness is proportional to the square root of cruise time).

The commonly used steady-state equation may be readily obtained from equation (25) by setting $T_o = T_S$ (from eqs. (1), (2), (3), and (4) this gives a linear steady-state temperature distribution) with the result

$$\frac{h_{fg}\rho}{\rho_H} - \frac{Kt_f(T_S - T_H)}{L_{SS}^2} = 0 \tag{26}$$

This may be solved for L_{SS} in closed form. From equations (20) and (23) the steady-state heat transfer and boiloff thicknesses are:

$$Q_{SS} = \frac{Kt_f(T_S - T_H)}{L_{SS}} \tag{27}$$

$$L_{BO_{SS}} = \frac{Kt_f(T_S - T_H)}{L_{SS}h_{fg}} \tag{28}$$

Equation (26) may be deduced from the transient solution in two other ways. First, setting the insulation heat capacity, C, equal to zero (i.e., letting $k \to \infty$) in equation (25) leads directly to equation (26). Secondly, letting $t_f \to \infty$ in equation (23) leads to equation (28) which, by virtue of equation (24), leads to equation (26).

Dry Tank

As shown in figure 6, the initial-boundary value problem to be solved is

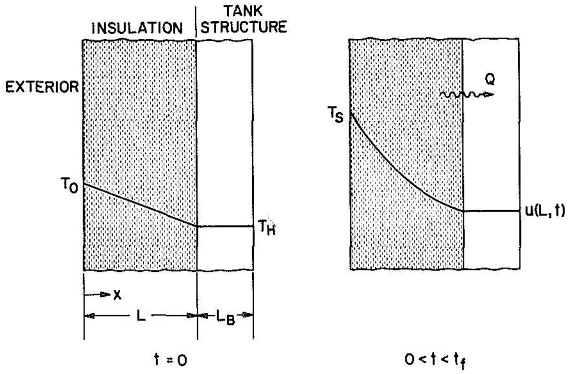


Figure 6.- Dry tank.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{29}$$

$$u(0,t) = T_S \tag{30}$$

$$\int_0^t K \frac{\partial u(L,\tau)}{\partial x} d\tau = -C_B L_B \rho_B [u(L,t) - T_H] \tag{31}$$

$$u(x,0) = \frac{T_H - T_o}{L} x + T_o \tag{32}$$

The second boundary condition, equation (31), is a statement that heat transferred through the insulation at $x = L$ is equal to the heat absorbed by the tank structure. To write this condition in differential form we differentiate equation (31) with respect to time and use equation (29) to obtain

$$LK_B \frac{\partial u(L,t)}{\partial x} + k \frac{\partial^2 u(L,t)}{\partial x^2} = 0 \tag{33}$$

where

$$K_B = \frac{K}{C_B L_B \rho_B L}$$

Making the transformation

$$u = v + T_S \tag{34}$$

in equations (29), (30), (32), and (33) yields a problem with homogeneous boundary conditions

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \tag{35}$$

$$v(0,t) = 0 \tag{36}$$

$$LK_B \frac{\partial v(L,t)}{\partial x} + k \frac{\partial^2 v(L,t)}{\partial x^2} = 0 \tag{37}$$

$$v(x,0) = \frac{T_H - T_o}{L} x + T_o - T_S \tag{38}$$

We proceed as in the wet tank solution by substituting equation (11) into equation (35) to get equations (12) and (13). Applying boundary conditions (36) and (37) to equation (13) gives expressions for the eigenvalues and the eigenfunctions

$$-\frac{LK_B}{k} \cos\left(L \sqrt{\lambda_n}\right) + \sqrt{\lambda_n} \sin\left(L \sqrt{\lambda_n}\right) = 0 \; ; \qquad n = 1, 2, \ldots \tag{39}$$

$$\Phi_n = \sin\left(x \sqrt{\lambda_n}\right); \qquad n = 1, 2, \ldots \tag{40}$$

In this case the eigenvalues cannot be solved for explicitly. Each eigenvalue λ_n may be shown to lie in the interval

$$\pi(n - 1) < L \sqrt{\lambda_n} < \pi\left(n - \frac{1}{2}\right) \tag{41}$$

From equations (11), (16), (34), and (40) the solution may be written as

$$u(x,t) = T_S + \sum_{n=1}^{\infty} a_n e^{-\lambda_n kt} \sin\left(x \sqrt{\lambda_n}\right) \tag{42}$$

and it remains only to find the Fourier coefficients, a_n.

The computation of the Fourier coefficients is not straightforward in this problem because the eigenfunctions are not orthogonal. This may be shown by application of Green's theorem which, in addition, gives a formula for the nonorthogonal values:

$$\int_o^L \left[\left(-\Phi_j''\right)\Phi_K - \Phi_j\left(-\Phi_k''\right)\right] dx = -\left[\Phi_j'\Phi_k - \Phi_j\Phi_k'\right]_o^L \; ; \qquad i \neq j \; ; \quad i, j = 1, 2, \ldots \tag{43}$$

In view of equations (37), (38), and (40), eqaution (43) reduces to

$$\int_o^L \Phi_j\Phi_k \, dx = -\frac{k}{LK_B} \Phi_j(L)\Phi_k(L) \; ; \qquad i \neq j \; ; \qquad i, j = 1, 2, \ldots \tag{44}$$

The nonorthogonality of the eigenfunctions means that the temperature distribution cannot be obtained in explicit form. This situation, however, does not inhibit numerical solution since the problem can be reduced to one of matrix algebra. Next introduce the quantities

$$I_i = \int_o^L v(x,0)\Phi_i(x)dx \; ; \qquad i = 1, 2, \ldots \tag{45}$$

$$I_{ij} = \int_o^L \Phi_i(x)\Phi_j(x)dx \; ; \qquad i, j = 1, 2, \ldots \tag{46}$$

where I_i and I_{ij} are an infinite-dimensional vector and an infinite-dimensional matrix, respectively. Using equations (38), (40), and (44) allows equations (45) and (46) to be expressed as

$$I_i = \frac{k}{LK_B}(T_S - T_H)\sin\left(L\sqrt{\lambda_i}\right) + \frac{1}{L\lambda_i}(T_H - T_o)\sin\left(L\sqrt{\lambda_i}\right) + \frac{1}{\sqrt{\lambda_i}}(T_o - T_S) \; ; \qquad i = 1, 2, \ldots \tag{47}$$

$$\left.\begin{aligned} I_{ii} &= \frac{1}{2}\left[L - \frac{k}{LK_B}\sin^2\left(L\sqrt{\lambda_i}\right)\right] \; ; \qquad i = 1, 2, \ldots \\ I_{ij} &= -\frac{k}{LK_B}\sin\left(L\sqrt{\lambda_i}\right)\sin\left(L\sqrt{\lambda_j}\right) \; ; \qquad i \neq j \; ; \quad i, j = 1, 2, \ldots \end{aligned}\right\} \tag{48}$$

Now evaluate the $v(x,t)$ portion of equation (42) at $t = 0$, multiply both sides by $\Phi_m(x)$, and integrate from $x = 0$ to $x = L$ to get

$$\int_o^L v(x,0)\Phi_m(x)dx = \sum_{n=1}^{\infty} a_n \int_o^L \Phi_n(x)\Phi_m(x)dx$$

$$I_m = \sum_{n=1}^{\infty} a_n I_{nm} \tag{49}$$

where equations (45) and (46) were used. Equation (49) gives the Fourier coefficients for equation (42).

Taking the design point of view for the dry tank weight calculation, it is desired to compute the insulation thickness (weight) which limits the structural temperature $u(L,t)$ to a specified value T_B. Since the maximum $u(L,t)$ occurs at t_f, equation (42) implies

$$T_B = T_S + \sum_{n=1}^{\infty} a_n e^{-\lambda_n k t_f} \sin\left(L\sqrt{\lambda_n}\right) \tag{50}$$

which is to be iteratively solved for L. The numerical procedure for solving the dry tank problem is as follows: (1) Guess an L. (2) Solve for the λ_i from equation (39) with the aid of equation (41). (3) Solve for the a_i from equation (49) with the aid of equations (47) and (48). (4) Test to see if equation (50) is satisfied; if not, repeat the procedure. Although it was found that the infinite series involved in the solution converged relatively slowly, convergence was entirely satisfactory when the solutions obtained by using the first four and the first five terms of these series were averaged. This is not surprising since all of the series involved are alternating. Computationally, the averaging was accomplished by dividing the fifth rows and columns of I_i and I_{ij} by two.

Unlike the wet tank solution, there is no readily apparent steady-state approximation to the transient solution, since the boundary condition at $x = L$ makes the problem inherently unsteady. However, a quasi-steady state relation may be obtained if it is assumed that the temperature distribution is a linear function of x at any time t, that is,

$$U_{SS}(x,t) = f_1(t) + f_2(t)x \tag{51}$$

This is equivalent to setting $C = 0$ ($k = \infty$) in equation (29). Since $U_{SS}(x,t)$ must satisfy the boundary and initial conditions (eqs. (30), (31), (32)) with $T_o = T_S$,

$$U_{SS}(x,t) = T_S - \frac{x}{L_{SS}}(T_S - T_H)e^{-\frac{Kt}{C_B L_B \rho_B L_{SS}}} \tag{52}$$

Since $U_{SS}(L_{SS},t_f) = T_B$, equation (52) leads to

$$\frac{Kt_f}{C_B L_B \rho_B L_{SS}} = \ln \frac{T_S - T_H}{T_S - T_B} \tag{53}$$

which gives L_{SS} in closed form. The heat transferred to the structure is given by

$$Q_{SS} = C_B L_B \rho_B (T_B - T_H) \tag{54}$$

To conclude this section, mention should be made of the relation of these two solutions to existing solutions. The solution given in this paper of the relatively straightforward wet tank case may also be obtained from the analysis given in section 3.4 of reference 9. This reference also discusses problems with the boundary condition of equation (33), but does not give the solution of the dry tank case. However, section 3.13 may be used to obtain the solution for the special case $T_H = T_o$. For this case, transformation of equation 3.13(9) of reference 9 gives the temperature distribution

$$u(x,t) = T_S + 2(T_o - T_S) \sum_{n=1}^{\infty} \frac{\left[\lambda_n + (L^2K_B^2/k^2)\right]e^{-k\lambda_n t} \sin\left(\sqrt{\lambda_n}\, x\right)}{\sqrt{\lambda_n}\left\{L[\lambda_n + (L^2K_B^2/k^2)] + (LK_B/k)\right\}} \tag{55}$$

Since it is desired to limit the structural temperature to T_B, this equation gives

$$T_S - T_B + 2(T_o - T_S) \sum_{n=1}^{\infty} \frac{\sqrt{1 + (L^2K_B^2/\lambda_n k^2)}\, e^{-k\lambda_n t_f}}{\left\{L[(\lambda_n k/LK_B) + (LK_B/k)] + 1\right\}} = 0 \tag{56}$$

where λ_n are the roots of equation (39). Equation (56), which is to be solved iteratively for L, has application to the insulated structure of nonfueled portions of hypersonic cruise vehicles.

RESULTS

Comparison With a Finite Difference Solution

In this section an example is presented to illustrate the solutions obtained in the previous section and to provide a comparison with a finite difference method. The data for the example was chosen to be representative of a hypersonic cruise aircraft travelling at Mach 6 with a cruise time of about 1-1/2 hours. The insulation is taken to be quartz fiber, and the tank structure, which carries the vehicle loads, is aluminum alloy, weighs about 3 lb/ft^2, and is limited to 200° F. The temperature distributions for this example for the wet and dry tank cases are shown in figures 7 and 8, respectively. The flight profile for this case yields: T_o = 70° F, $T_{S_{WET}}$ = 952° F, $T_{S_{DRY}}$ = 632° F, and T_H = -424° F. Note that because of angle of attack effects, $T_{S_{WET}}$ (which corresponds to the underside of the vehicle) is greater than $T_{S_{DRY}}$. Average insulation thermal properties are those at the following temperatures:

$$\overline{T}_{WET} = T_H \tag{57}$$

$$\overline{T}_{DRY} = \frac{1}{4}(T_S + T_o + T_B + T_H) \tag{58}$$

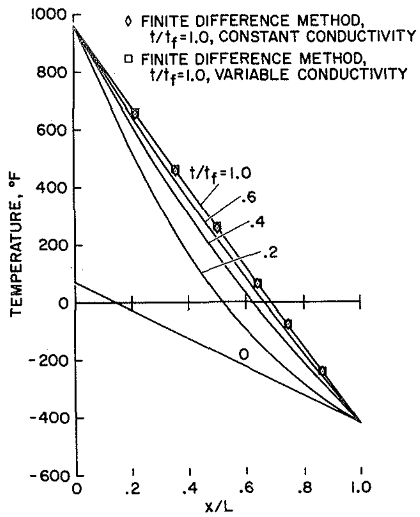


Figure 7.- Wet-tank temperature distributions.

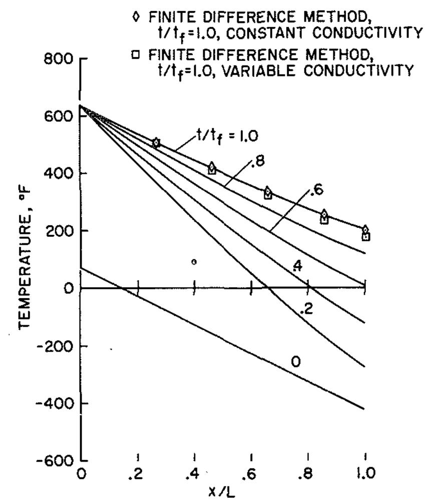


Figure 8.- Dry-tank temperature distributions.

Here T_H is used for $\overline{T}_{WET}$ since the quantity of importance is heat transfer to the LH_2 fuel, and this heat transfer is proportional to the conductivity at temperature T_H. For the wet tank case, the optimum insulation thickness L was 4.89 in.; the boiloff thickness L_{BO} was 3.10 in.; the unit weight of insulation W_I was 1.48 lb/ft^2; the unit weight of boiloff W_{BO} was 1.15 lb/ft^2; and the heat transferred to the fuel was 222 Btu/ft^2. For the dry tank case, the insulation thickness required to limit the tank temperature to 200° F, L, was 2.34 in.; the unit weight of insulation W_I was 0.88 lb/ft^2; and the heat transferred to the structure was 249 Btu/ft^2. These values are summarized in table 1. Qualitatively, as can be seen from figures 7 and 8 the temperature distributions are much as would be expected. As mentioned earlier, total vehicle insulation weight may be estimated from a weighted average of the weights of the wet and dry tank cases.

TABLE 1.- COMPARISON OF RESULTS OF PRESENT ANALYSIS WITH FINITE DIFFERENCE SOLUTION

	Wet tank					Dry tank			
	L, in.	L_{BO}, in.	W_I, lb/ft^2	W_{BO}, lb/ft^2	Q, Btu/ft^2	L, in.	W_I, lb/ft^2	Q, Btu/ft^2	T_B, °F
Present analysis	4.89	3.10	1.84	1.15	222	2.34	0.88	249	200*
Finite difference, constant K	4.89*	3.17	1.84	1.18	227	2.34*	0.88	248	198
Finite difference, variable K	4.89*	3.38	1.84	1.25	242	2.34*	0.88	240	178

*Input values

A finite-difference heat-conduction program was also applied to the above problem. This program solves one-dimensional heat conduction problems by finite difference methods. The insulation thicknesses computed by the analytic solutions of the present paper were entered into the finite difference program and the resulting temperature distributions were compared with the analytically derived ones. The temperature distributions at $t = t_f$ for constant conductivity are indicated by the diamonds on figures 7 and 8. It may be seen that these distributions are in excellent agreement with the transient ones. Since there is no theoretical difference between these two solutions, this agreement indicates a high degree of numerical accuracy. The finite difference method gave 227 Btu/ft^2 for heat transferred to the fuel in the wet tank case (2.0 percent difference as compared with the analytic solution), 198° F for the final wall temperature of the dry tank case (as compared with 200° F), and 248 Btu/ft^2 for heat transfer to the structure in the dry tank case (0.2 percent difference) as shown in table 1.

Since the finite difference method is capable of solving variable property problems, it may be used to test the validity of using constant conductivity. If K is input as a function of temperature (using data from ref. 10), the resulting temperature distributions at $t = t_f$ are denoted by the squares on figures 7 and 8. As shown in table 1, the finite-difference, variable-conductivity solution gave 242 Btu/ft^2 for heat transferred to the fuel in the wet tank case (9.0 percent difference as compared with the analytical solution), 178° F for the final wall temperature of the dry tank case (as compared with 200° F), and 240 Btu/ft^2 for heat transfer to the structure in the dry tank case (3.5 percent difference). Although the wet tank constant-conductivity calculation using $\bar{T}_{WET}$ given by equation (57) underestimates the heat transfer by 9 percent, it only underestimates insulation plus boiloff weight by about 4 percent. The constant-conductivity dry tank calculation with $\bar{T}_{DRY}$ selected from equation (58) gives slightly conservative results.

Comparison With Steady-State Approximations

From equations (24), (26), and (28), the steady-state approximation of the insulation weight (including boiloff) for the wet tank case is

$$W_{SS_{WET}} = 2\sqrt{\frac{\rho_H \rho K t_f (T_S - T_H)}{h_{fg}}} \tag{59}$$

and using equation (53) the dry tank insulation weight is

$$W_{SS_{DRY}} = \frac{\rho K t_f}{C_B L_B \rho_B \; ln[(T_S - T_H)/(T_S - T_B)]} \tag{60}$$

Since, as mentioned earlier, these equations may be viewed as ignoring the heat capacity of the insulation, they will give conservative weights. Equations (59) and (60) are convenient for weight estimation because they give the weight in closed form. In order to assess the validity of these equations both the magnitude of the steady-state weights and the sensitivities of the weights to the independent parameters were compared with the weights and sensitivities obtained from the transient solutions. Table 2 shows the results of applying

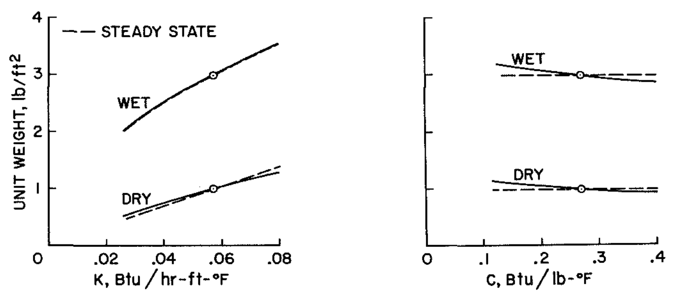
TABLE 2.- COMPARISON OF RESULTS OF PRESENT ANALYSIS WITH STEADY-STATE APPROXIMATIONS

	Wet tank					Dry tank			
	L, in.	L_{BO}, in.	W_I, lb/ft^2	W_{BO}, lb/ft^2	Q, Btu/ft^2	L, in.	W_I, lb/ft^2	Q, Btu/ft^2	T_B, °F
Present analysis	4.89	3.10	1.84	1.15	222	2.64	0.99	205	200*
Steady-state approximation	4.46	4.51	1.67	1.67	323	3.56	1.34	205	200*

*Input values

both the transient analyses of the present paper and the steady-state approximations to an example case. This example uses a different tank material from that used in the example of the previous section. For the wet tank case, the steady-state approximation implies that the weight of insulation will equal the weight of boiled-off fuel for minimum total weight; whereas the transient solution indicates that the insulation weight will be greater than the boiloff weight. Since the transient solution gives 2.99 lb/ft^2 for the weight of insulation plus boiloff and the steady-state approximation gives 3.34 lb/ft^2, the steady-state approximation overestimates the weight by 11.7 percent. The steady-state approximation overestimates the dry tank insulation weight by 35.7 percent.

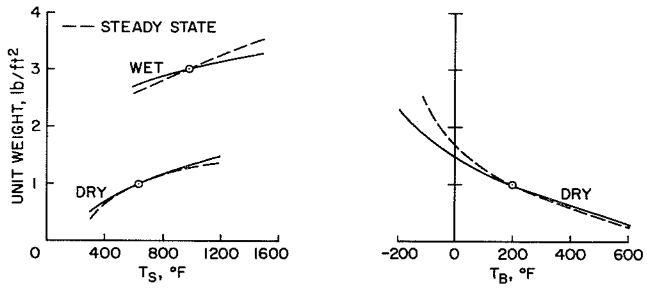
Sensitivities of the weight to four of the independent parameters are shown in figure 9. In order to compare sensitivities (slopes), the steady-state weights were multiplied by 0.894 and 0.741 for the wet and dry tank cases, respectively, to make them pass through the same nominal point as the transient weights. As indicated in figure 9(a), the steady-state sensitivities of the weight to insulation conductivity agree quite well with the transient ones; the wet tank case in fact agrees virtually exactly. It is obvious that this parameter is significant for determining the weight. Figure 9(b) shows the sensitivities to a parameter that appears only in the transient solutions, the insulation heat capacity. The weak influence of this parameter indicates that the insulation weights will be relatively insensitive to



(a) Insulation conductivity.

(b) Insulation heat capacity.

Figure 9.- Sensitivity of insulation weight to selected parameters.



(c) Surface temperature.

(d) Maximum structural temperature.

Figure 9.- Concluded.

changes in initial conditions. The sensitivity to surface temperature, shown in figure 9(c), shows perhaps the poorest agreement between transient and steady-state solutions. This parameter is important since it is a function of the highly important mission parameter, cruise Mach number. An important sensitivity, that of maximum structural temperature, is shown in figure 9(d). Since insulation weight decreases with increasing structural temperature and structural weight increases, this parameter is important for minimization of vehicle weight. It was mentioned earlier that the sensitivities of a fifth parameter, cruise time, of the transient and steady-state solutions is identical for the wet tank case.

It may be concluded from figure 9 that the following equation will give a first-order estimate of the average insulation weight (including boiloff) per unit area if the values of the independent parameters are "reasonably" close to the values of the nominal case:

$$W_{SS} = 0.894\sqrt{\frac{\rho_H \rho K t_f \left(T_{S_{WET}} - T_H\right)}{h_{fg}}} + \frac{0.741\rho K t_f}{2C_B L_B \rho_B \ ln\left[\left(T_{S_{DRY}} - T_H\right)/\left(T_{S_{DRY}} - T_B\right)\right]} \tag{61}$$

In this equation half the surface area of the vehicle is assumed to be governed by the wet tank case and half by the dry tank case. As mentioned earlier, thermal protection system weight is obtained by adding such items as cover panels, attachments, and "nonoptimum" weights to the insulation weight.

CONCLUDING REMARKS

Solutions of two initial-boundary value problems of one-dimensional conduction heat transfer have been obtained. The application of these solutions to hypersonic cruise vehicle thermal protection systems was discussed. A numerical example was considered and it was found that the solutions agreed well with finite difference solutions. The finite difference solution was also used to investigate the validity of the constant conductivity assumption, and it was found that this assumption is a reasonable one. Comparison of weight sensitivities with those of steady-state approximations showed that such approximations give reasonable estimates of the sensitivities, but that these approximations overestimate the weight. An equation for weight estimation based on steady-state approximations was presented.

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