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NUMERICAL CONSTRUCTION OF THE HILL FUNCTIONS

by

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1.1. INTRODUCTION

In the theory of approximation of functions and solution of ordinary or partial differential equations functions are often approximated by finite sums of the form

$$f(x) = \sum_k c_k \varphi\left(\frac{x-kh}{h}\right) \tag{1.1}$$

where the function $\varphi(x)$ must fulfill certain conditions. Frequently these functions are given by a convolution formula

$$\varphi_n(x) = \varphi_{n-1}(x) * \varphi_1(x)$$

where $\varphi_1(x) = 1$ for $x \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$, $\varphi_1(x) = 0$ otherwise.

Higher dimensions are treated in a manner analogous to the one-dimensional case. (See [1], [2], [3].)

An algorithm for computing $\varphi_n(x)$ for higher n is needed, because these functions are useful for solving differential equations of higher orders.

This paper is concerned with a numerical construction of these functions. An algorithm with reasonable stability is designed using local coordinates and an expansion in Legendre polynomials. Two possible local coordinate systems are shown and discussed. In the conclusion some remarks about the accuracy of results are stated.

Tables of coefficients of the functions up to order 10 in both local coordinate systems are included as well as the graphs of these functions and a FORTRAN IV version of an algorithm.

1.2. DEFINITION AND SOME PROPERTIES OF THE FUNCTIONS $\varphi_n(x)$

Let us consider the functions given by the convolution formula in the one-dimensional case in this way:

$$\varphi_1(x) = 1 \quad \text{for } x \in \langle -\tfrac{1}{2}, \tfrac{1}{2} \rangle$$

$$\varphi_1(x) = 0 \quad \text{otherwise}$$

$$\varphi_n(x) = \varphi_{n-1}(x) * \varphi_1(x) = \int_{-\infty}^{\infty} \varphi_{n-1}(t)\,\varphi_1(x-t)\,dt \tag{1.2}$$

These functions have the properties of an approximation (see [1], [2], [3]). Some other properties of the functions $\varphi_n(x)$ follow from the convolution expression.

1. $\varphi_n(x)$ has a compact support $\langle -\frac{n}{2}, \frac{n}{2} \rangle$.
2. $\varphi_n(x)$ is a piecewise polynomial of degree $n-1$. More precisely, by using the convolution formula we get different polynomials (of degree $n-1$) for the function $\varphi_n(x)$ in the intervals: $\left(-\frac{n}{2}, \frac{-n+2}{2}\right), \left(\frac{-n+2}{2}, \frac{-n+4}{2}\right), \ldots, \left(\frac{n-2}{2}, \frac{n}{2}\right)$.
3. The function $\varphi_n(x)$ has continuous derivatives of order $0, 1, \ldots, n-2$ everywhere in $(-\infty, \infty)$. Therefore the function $\varphi_n(x)$ has its zeroes of multiplicity $n-1$ at the boundary points of its compact support, i. e. in the points $-\frac{n}{2}, \frac{n}{2}$.

Examples of the $\varphi_n(x)$:

$$\varphi_1(x) = 1 \quad \text{for } x \in \langle -\tfrac{1}{2}, \tfrac{1}{2} \rangle$$

$$\varphi_1(x) = 0 \quad \text{otherwise}$$

$$\varphi_2(x) = x+1 \quad \text{for } x \in \langle -1, 0 \rangle$$

$$\varphi_2(x) = -x+1 \quad \text{for } x \in \langle 0, 1 \rangle$$

$$\varphi_2(x) = 0 \quad \text{otherwise}$$

$$\varphi_3(x) = \frac{1}{2}\left(x + \frac{3}{2}\right)^2 \quad \text{for } x \in \langle -\frac{3}{2}, -\frac{1}{2} \rangle$$

$$\varphi_3(x) = -x^2 + \frac{3}{4} \quad \text{for } x \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$$

$$\varphi_3(x) = \frac{1}{2}\left(x - \frac{3}{2}\right)^2 \quad \text{for } x \in \langle \frac{1}{2}, \frac{3}{2} \rangle$$

$$\varphi_3(x) = 0 \quad \text{otherwise}$$

See the graphs in the appendix.

1.3. SOME PROBLEMS IN THE COMPUTATION OF THE $\varphi_n(x)$

Let us try to find an algorithm to compute the $\varphi_n(x)$. First of all we must find the expressions for the polynomials which form the function.

Every function $\varphi_n(x)$ is represented by a polynomial in every subinterval of its compact support and these expressions are different. There are many possible ways to express the polynomials. For example we may choose different systems of functions to express them. Representing them with the basis of polynomials $1, x, x^2, \ldots$ we get for example $\varphi_2(x)$ represented by coefficients 1, 1 in the interval $\langle -1, 0 \rangle$ and coefficients -1, 1 in the $\langle 0, 1 \rangle$. Let us consider further this basis. As mentioned above, the function $\varphi_n(x)$ has its zeroes at the points $-\frac{n}{2}, \frac{n}{2}$ of multiplicity n-1. This implies that $\varphi_n(x)$ has the formula

$$\varphi_n(x) = a\left(x+\frac{n}{2}\right)^{n-1} = a\Big(x^{n-1} + (n-1)\cdot\frac{n}{2}x^{n-2} + \dots + \binom{n-1}{k}\left(\frac{n}{2}\right)^k x^{n-k-1} + \dots + \left(\frac{n}{2}\right)^{n-1}\Big)$$

in the first interval of its support (i. e. in the $\langle -\frac{n}{2}, \frac{-n+2}{2} \rangle$), where a is a certain constant. However it is almost impossible to compute the values of the polynomial given in this way because of the magnitude of coefficients for higher n. This difficulty is partially removed if we put the origin at the center of each interval, where the coefficients are to be computed.

There are two possible ways to divide the support of the function $\varphi_n(x)$ into subintervals where the coefficients are sought.

Case I: We may compute the coefficients of the polynomials for $\varphi_n(x)$ in the intervals $\langle -\frac{n}{2}, \frac{-n+2}{2} \rangle$, $\langle \frac{-n+2}{2}, \frac{-n+4}{2} \rangle, \dots, \dots, \langle \frac{n-2}{2}, \frac{n}{2} \rangle$ of the length 1 where the expressions are different. In this case the functions and corresponding intervals are:

$\varphi_1(x)$ .... $\langle -\frac{1}{2}, \frac{1}{2} \rangle$

$\varphi_2(x)$ .... $\langle -1, 0 \rangle, \langle 0, 1 \rangle$

$\varphi_3(x)$ .... $\langle -\frac{3}{2}, -\frac{1}{2} \rangle, \langle -\frac{1}{2}, \frac{1}{2} \rangle, \langle \frac{1}{2}, \frac{3}{2} \rangle$

$\varphi_4(x) \ldots \langle -2,-1\rangle, \langle -1,0\rangle, \langle 0,1\rangle, \langle 1,2\rangle$ etc.

The boundaries where expressions change for $\varphi_n(x)$ where n is an odd integer are the points $\frac{k}{2}$ (k an odd integer) and for $\varphi_n(x)$ where n is an even integer the points k (k an integer).

Case II: It is possible to compute the coefficients of the polynomials comprising $\varphi_n(x)$ in intervals of the length $\frac{1}{2}$, i. e. $\langle -\frac{n}{2}, \frac{-n+1}{2}\rangle, \langle \frac{-n+1}{2}, \frac{-n+2}{2}\rangle, \ldots, \langle \frac{n-1}{2}, \frac{n}{2}\rangle$.
Then the division into subintervals is the same for all the functions $\varphi_n(x)$ for all n.

However, when using these local coordinate systems the computation of the value $\varphi_n(x)$ would not be too stable because of the magnitudes of coefficients which occur in the expression of the polynomials in the basis $1, x, x^2, \ldots$. For this reason it is very convenient to express the polynomials which form the function $\varphi_n(x)$ in the basis of Legendre polynomials. In the Legendre polynomial expansion of a sufficiently smooth function the coefficients of the first Legendre polynomials prevail over the higher coefficients which is advantageous from the point of view of numerical stability.

2.1. COMPUTATION OF COEFFICIENTS OF THE FUNCTION $\varphi_n(x)$ IN THE BASIS OF LEGENDRE POLYNOMIALS

As mentioned above the Legendre polynomial expansion is advantageous from the point of view of round-off errors. Moreover, using Legendre polynomials the computation of the convolution formula as well as other computation become very simple with these functions. Let us consider both possible cases of local coordinate systems.

In the case I, we express the polynomials which form the function $\varphi_n(x)$ in the basis of Legendre polynomials orthogonal on the interval $\langle -\frac{1}{2}, \frac{1}{2} \rangle$. In the case II, we use the Legendre polynomials orthogonal on the interval $\langle -\frac{1}{4}, \frac{1}{4} \rangle$.

The computations in both cases are very similar and each of them has certain advantages. Considering the first possibility, the computation is shorter and the storage requirements are less, while in the case II a certain consistency of the system of subintervals is preserved.

In case I, each function $\varphi_n(x)$ is represented by polynomials, i. e. by their coefficients of the Legendre expansion in n intervals $\langle \frac{k}{2}, \frac{k}{2}+1 \rangle$ where $k=-n,-n+2,\dots,n-2$ and we put the origin of coordinates at the center of each interval so that we eventually treat n intervals which may be considered as $\langle -\frac{1}{2}, \frac{1}{2} \rangle$ in these local coordinate systems.

In case II, each function $\varphi_n(x)$ is represented by Legendre polynomials in $2n$ intervals $\langle \frac{k}{2}, \frac{k+1}{2} \rangle$ where $k=-n,-n+1,\dots$ $\dots,n-1$. If we put the origin of coordinates into the center of each of them we get $2n$ intervals of the form $\langle \frac{-1}{4}, \frac{1}{4} \rangle$ considered in these local coordinate systems.

First let us consider case I. Let us assume that $\varphi_{n-1}(x)$ is given, i. e. we have

$$\varphi_{n-1,j}(x) = \sum_{i=1}^{n-1} {}^{n-1}a_{i,j} \; P_i(x)$$

where j is the index which denotes the interval $(-\frac{n-1}{2}+j-1, -\frac{n-1}{2}+j)$, $j = 1, \ldots, n-1$. The ${}^{n-1}a_{i,j}$ are the coefficients in the basis of Legendre polynomials in the j-th interval considered in the local coordinate system, $P_i(x)$ is the Legendre polynomial of degree $i-1$ where

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} P_i(t)\, P_k(t)\, dt = 0 \qquad \text{for } i \neq k$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} P_i^2(t)\, dt = \frac{1}{2i-1} \quad .$$

Let us compute the coefficients for the function $\varphi_n(x)$ in the expression

$$\text{(2.1)} \qquad \varphi_{n,j}(x) = \sum_{i=1}^{n} {}^{n}a_{i,j} \; P_i(x) \quad , \quad j = 1, \ldots, n$$

In this notation the index j denotes the interval $(-\frac{n}{2}+j-1, -\frac{n}{2}+j)$. The interval with the index j for the function $\varphi_n(x)$ consists of the second half of the interval with the index $j-1$ and of the first half of the interval with the index j for the function $\varphi_{n-1}(x)$.

$$\varphi_n(x) = \int_{-\infty}^{\infty} \varphi_{n-1}(t)\varphi_1(x-t) = \int_{-\frac{n-1}{2}}^{\frac{n-1}{2}} \varphi_{n-1}(t)\varphi_1(x-t)$$

Then for $j = 2, \dots, n-1$

$$\varphi_{n,j}(x) = \int_{-\frac{1}{2}}^{x} \varphi_{n-1,j}(t)\,dt + \int_{x}^{1/2} \varphi_{n-1,j-1}(t)\,dt$$

is valid and

$$\varphi_{n,1}(x) = \int_{-\frac{1}{2}}^{x} \varphi_{n-1,1}(t)\,dt\,, \quad \varphi_{n,n}(x) = \int_{x}^{\frac{1}{2}} \varphi_{n-1,n-1}(t)\,dt$$

where both the integrals as well as the x are considered in local coordinates for corresponding intervals. Let us set

$$I_1(x) = \int_{x}^{\frac{1}{2}} \varphi_{n-1,j-1}(t)\,dt\,, \quad I_2(x) = \int_{-\frac{1}{2}}^{x} \varphi_{n-1,j}(t)\,dt$$

The following relations for the Legendre polynomials orthogonal on $\langle -\frac{1}{2}, \frac{1}{2} \rangle$ are valid:

$$P_{i-1}(x) = \frac{1}{2} \cdot \frac{1}{2i-3} \left(P_i'(x) - P_{i-2}'(x) \right)$$

$$P_i\left(\tfrac{1}{2}\right) = 1, \quad P_i\left(-\tfrac{1}{2}\right) = (-1)^{i+1}.$$

Then

$$I_1(x) = \sum_{i=1}^{n-1} {}^{n-1}a_{i,j-1} \int_{x}^{\frac{1}{2}} P_i(t)\,dt =$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n-1} \frac{{}^{n-1}a_{i,j-1}}{2i-1} \int_{x}^{\frac{1}{2}} P_{i+1}'(t)\,dt - \sum_{i=2}^{n-1} \frac{{}^{n-1}a_{i,j-1}}{2i-1} \int_{x}^{\frac{1}{2}} P_{i-1}'(t)\,dt \right)$$

$$= \frac{1}{2}\left(- \sum_{i=1}^{n-1} \frac{{}^{n-1}a_{i,j-1}}{2i-1} P_{i+1}(x) + \sum_{i=2}^{n-1} \frac{{}^{n-1}a_{i,j-1}}{2i-1} P_{i-1}(x) + \right.$$

$$+ {}^{n-1}a_{1,j-1} P_2\left(\tfrac{1}{2}\right) + \sum_{i=2}^{n-1} \frac{{}^{n-1}a_{i,j-1}}{2i-1}\left(P_{i+1}\left(\tfrac{1}{2}\right) - P_{i-1}\left(\tfrac{1}{2}\right)\right) =$$

$$= \tfrac{1}{2}\Big(\sum_{i=2}^{n-1} \frac{{}^{n-1}a_{i,j-1}}{2i-1} P_{i-1}(x) - \sum_{i=1}^{n-1} \frac{{}^{n-1}a_{i,j-1}}{2i-1} P_{i+1}(x) +$$

$$+ {}^{n-1}a_{1,j-1} P_1(x)\Big)$$

and analogously

$$I_2(x) = \tfrac{1}{2}\Big(\sum_{i=1}^{n-1} \frac{{}^{n-1}a_{i,j}}{2i-1} P_{i+1}(x) - \sum_{i=2}^{n-1} \frac{{}^{n-1}a_{i,j}}{2i-1} P_{i-1}(x) +$$

$$+ {}^{n-1}a_{1,j} P_1(x)\Big)$$

Then the following relations for the coefficients in (2.1) are valid:

$$ {}^{n}a_{i,j} = c_{i,j-1} + d_{i,j} \quad (j=2,\ldots,n-1)$$

$$ {}^{n}a_{i,1} = d_{i,1}, \quad {}^{n}a_{i,n} = c_{i,n-1}, \quad i=1,\ldots,n$$

where

$$c_{i,j-1} = \tfrac{1}{2}\left(\frac{{}^{n-1}a_{i+1,j-1}}{2i+1} - \frac{{}^{n-1}a_{i-1,j-1}}{2i-3}\right) \quad \text{for } i=2,\ldots,n-2,$$

$$c_{1,j-1} = \tfrac{1}{2}\left(\frac{{}^{n-1}a_{2,j-1}}{3} + {}^{n-1}a_{1,j-1}\right),$$

$$d_{i,j} = \tfrac{1}{2}\left(\frac{{}^{n-1}a_{i-1,j}}{2i-3} - \frac{{}^{n-1}a_{i+1,j}}{2i+1}\right) \quad \text{for } i=2,\ldots,n-2,$$

$$d_{1,j} = \tfrac{1}{2}\left(-\frac{{}^{n-1}a_{2,j}}{3} + {}^{n-1}a_{1,j}\right),$$

$$c_{n,j-1} = -\frac{1}{2} \cdot \frac{{}^{n-1}a_{n-1,j-1}}{2n-3}, \quad d_{n,j} = \frac{1}{2} \cdot \frac{{}^{n-1}a_{n-1,j}}{2n-3},$$

$$c_{n-1,j-1} = -\frac{1}{2} \cdot \frac{{}^{n-1}a_{n-2,j-1}}{2n-5}, \quad d_{n-1,j} = \frac{1}{2} \cdot \frac{{}^{n-1}a_{n-1,j}}{2n-5}, \quad j = 2, \dots, n-1$$

Moreover, the functions $\varphi_n(x)$ are even, so that it is sufficient to compute only half the coefficients, i. e. only ${}^{n}a_{K,1}, {}^{n}a_{K,2}, \dots, {}^{n}a_{K,\frac{n}{2}}$ for n even or ${}^{n}a_{K,1}, {}^{n}a_{K,2}, \dots, {}^{n}a_{K,\frac{n+1}{2}}$ for n odd. In the first case

$${}^{n}a_{K,\frac{n}{2}+\ell} = (-1)^{K+1} \cdot {}^{n}a_{K,\frac{n}{2}+1-\ell}, \quad \ell = 1, \dots, \frac{n}{2}, \; K = 1, \dots, n$$

and in the second case

$${}^{n}a_{K,\frac{n+1}{2}+\ell} = (-1)^{K+1} \cdot {}^{n}a_{K,\frac{n+1}{2}-\ell}, \quad \ell = 1, \dots, \frac{n-1}{2}, \; K = 1, \dots, n$$

and moreover, ${}^{n}a_{K,\frac{n+1}{2}} = 0$ for K even.

The computation in case II is very similar. The function is represented by its components in this way:

$$\varphi_{n,j}(x) = \sum_{i=1}^{n} {}^{n}a_{i,j} P_i(x), \quad j = 1, \dots, 2n \tag{2.2}$$

and the coefficients ${}^{n}a_{i,j}$ can be computed from the coefficients ${}^{n-1}a_{i,j}$ of the components of

$$\varphi_{n-1,j}(x) = \sum_{i=1}^{n-1} {}^{n-1}a_{i,j} P_i(x), \quad j = 1, \dots, 2n-2$$

where $P_i(x)$ is the Legendre polynomial of order $i-1$ and

$$\int_{-\frac{1}{4}}^{1/4} P_i(x) P_K(x) = 0 \quad \text{for } i \neq k,$$

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} P_i^2(x) = \frac{1}{2(2i-1)} .$$

Then

$$\varphi_{n,j}(x) = I_1(x) + I_2(x) + I_3(x) \qquad \text{for } j=3,\ldots,2n-2$$

$$\varphi_{n,1}(x) = I_3(x), \; \varphi_{n,2}(x) = I_2(x) + I_3(x),$$

$$\varphi_{n,2n-1}(x) = I_1(x) + I_2(x), \; \varphi_{n,2n}(x) = I_1(x)$$

where

$$I_1(x) = \frac{1}{4}\Big(\sum_{i=2}^{n-1} \frac{{}^{n-1}a_{i,j-2}}{2i-1} P_{i-1}(x) - \sum_{i=1}^{n-1} \frac{{}^{n-1}a_{i,j-2}}{2i-1} P_{i+1}(x) + {}^{n-1}a_{1,j-2} P_1(x)\Big),$$

$$I_2(x) = \frac{1}{2}\, {}^{n-1}a_{1,j-1} P_1(x),$$

$$I_3(x) = \frac{1}{4}\Big(\sum_{i=1}^{n-1} \frac{{}^{n-1}a_{i,j}}{2i-1} P_{i+1}(x) - \sum_{i=2}^{n-1} \frac{{}^{n-1}a_{i,j}}{2i-1} P_{i-1}(x) + {}^{n-1}a_{1,j} P_1(x)\Big)$$

Using the fact that the functions $\varphi_n(x)$ are even, only the coefficients ${}^{n}a_{k,j}$ $(k=1,\ldots,n, j=1,\ldots,n)$ are computed and for the other intervals we know

$${}^{n}a_{k,n+\ell} = (-1)^{k+1}\, {}^{n}a_{k,n+1-\ell} \qquad \text{for } \ell=1,\ldots,n,\; k=1,\ldots,n$$

Having the coefficients ${}^{n}a_{i,j}$ of (2.1) we can compute the value $\varphi_n(x)$ by evaluating the values of the Legendre polynomials. We can use, for example, one of the following recurrence formulas:

In case I,

$$n P_{n+1}(x) - (2n-1)\cdot 2x P_n(x) + (n-1) P_{n-1}(x) = 0$$

where

$$P_1(x) = 1, \quad P_2(x) = 2x, \quad n = 2, 3, \ldots$$

and in case II,

$$n P_{n+1}(x) - (2n-1) 4x P_n(x) + (n-1) P_{n-1}(x) = 0$$

where $P_1(x) = 1, \quad P_2(x) = 4x, \quad n = 2, 3, \ldots$

2.2. COMPUTATION OF COEFFICIENTS OF DERIVATIVES OF THE $\varphi_n(x)$

In most of the computations the derivatives of the $\varphi_n(x)$ are needed, too. Using the convolution formula (1.2) we can express the first derivative :

$$\varphi_n'(x) = \int_{-\infty}^{\infty} \varphi_{n-1}(t)\, \varphi_1'(x-t)\, dt = \int_{-\infty}^{\infty} \varphi_{n-1}(t)\left(\delta(x-t+\tfrac{1}{2}) - \delta(x-t-\tfrac{1}{2})\right)$$

$$= \varphi_{n-1}(x+\tfrac{1}{2}) - \varphi_{n-1}(x-\tfrac{1}{2}) .$$

Then having $\varphi_n(x)$ represented in system I, we get a formula for the coefficients of $\varphi_n'(x)$ in the basis of Legendre polynomials in the intervals denoted by $j = 1, \ldots, n$:

$$\varphi_{n,j}'(x) = \varphi_{n-1,j}(x) - \varphi_{n-1,j-1}(x). \quad \text{for } j = 2, \ldots, n-1$$

and

$$\varphi'_{n,1}(x) = \varphi_{n-1,1}(x), \quad \varphi'_{n,n}(x) = -\varphi_{n-1,n-1}(x)$$

Similarly we have for the other derivatives:

$$\varphi^{(k)}_{n,j}(x) = \varphi^{(k-1)}_{n-1,j}(x) - \varphi^{(k-1)}_{n-1,j-1}(x) \qquad \text{for } j=2,\dots,n-1$$

and

$$\varphi^{(k)}_{n,1}(x) = \varphi^{(k-1)}_{n-1,1}(x), \quad \varphi^{(k)}_{n,n}(x) = -\varphi^{(k-1)}_{n-1,n-1}(x)$$

or

$$\varphi^{(k)}_{n,j}(x) = \sum_{\substack{i=0 \\ 1 \le j-i \le n-k}}^{k} \binom{k}{i} \varphi_{n-k,j-i}(x) \cdot (-1)^i, \quad k=1,\dots,n-2, \quad j=1,\dots,n$$

Considering case II we get

$$\varphi'_{n,j}(x) = \varphi_{n-1,j}(x) - \varphi_{n-1,j-2}(x), \quad j=3,\dots 2n-2$$

$$\varphi'_{n,1}(x) = \varphi_{n-1,1}(x), \quad \varphi'_{n,2}(x) = \varphi_{n-1,2}(x)$$

$$\varphi'_{n,2n-1}(x) = -\varphi'_{n-1,2n-3}(x), \quad \varphi'_{n,2n}(x) = -\varphi_{n-1,2n-2}(x)$$

and for the other derivatives:

$$\varphi^{(k)}_{n,j}(x) = \varphi^{(k-1)}_{n-1,j}(x) - \varphi^{(k-1)}_{n-1,j-2}(x) \qquad \text{for } j=3,\dots 2n-2$$

$$\varphi^{(k)}_{n,1}(x) = \varphi^{(k-1)}_{n-1,1}(x), \quad \varphi^{(k)}_{n,2} = \varphi^{(k-1)}_{n-1,2}(x),$$

$$\varphi^{(k)}_{n,2n-1}(x) = -\varphi^{(k-1)}_{n-1,2n-3}(x), \quad \varphi^{(k)}_{n,2n}(x) = -\varphi^{(k-1)}_{n-1,2n-2}(x)$$

or

$$\varphi^{(k)}_{n,j}(x) = \sum_{\substack{i=0 \\ 1 \le j-2i \le 2(n-k)}}^{k} \binom{k}{i} \varphi_{n-k,j-2i}(x) \cdot (-1)^i, \quad j=1,\dots,2n, \quad k=1,\dots,n-2$$

The coefficients of the function $\varphi_n(x)$ in the basis of Legendre polynomials for both cases, i. e. on $\langle -\frac{1}{2}, \frac{1}{2} \rangle$ and $\langle -\frac{1}{4}, \frac{1}{4} \rangle$ are given in Table 1 and 2 respectively for $n = 1, \dots, \dots 10$ as well as the graphs of these functions in the appendix. The coefficients for the functions $\varphi_n(x)$ for $n > 10$ as well as the coefficients of the derivatives of the $\varphi_n(x)$ are available. They have been computed on an IBM 7094.

2.3. A REMARK ON NUMERICAL ACCURACY

The computation of the coefficients is sufficiently stable. Computing the coefficients in single and double precision and comparing them we get a favourable estimate of accuracy. The following examples of some coefficients show the actual accuracy. (See p. 15.)

The relative error of the coefficients for the first (and the last) intervals is great, of the magnitude $10^{-3} - 10^{-5}$, the relative error in the other intervals is $10^{-7} - 10^{-8}$, almost the accuracy of the machine. The source of the round-off error in the first interval is the subtraction of numbers which are close to each other. This behavior of the process is caused by the magnitude of coefficients. However, this great relative error has no importance because this error is included in small coefficients.

The results of computations using the functions $\varphi_n(x)$ computed in the way mentioned above were very good. However, if we approximate the derivatives of $f(x)$ given by (1.1) we get

$$f^{(\ell)}(x) = \sum c_K \, \varphi^{(\ell)}\left(\frac{x-kh}{h}\right) \cdot \left(\frac{1}{h}\right)^{\ell}$$

and for the higher ℓ and sufficiently small h the computation be-

$n=21$, computed in case II:

j	i	the value of the coefficient	the absolute error
1	1	0.186662112 E-25	-0.590 E-29
	10	0.416353780 E-26	-0.794 E-31
	21	0.284367284 E-35	0.173 E-42
3	1	0.195216016 E-15	0.131 E-22
	10	0.357746880 E-19	0.622 E-26
	21	-0.568734566 E-34	-0.629 E-41
10	1	0.133112569 E-04	0.251 E-11
	10	-0.527637729 E-14	-0.955 E-21
	21	0.137775950 E-31	0.202 E-38
21	1	0.292622687 E 00	0.688 E-07
	10	0.319777402 E-12	0.617 E-19
	21	0.525385625 E-30	0.772 E-37

comes difficult. It is possible to avoid this difficulty using double precision but only to some extent.

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A P P E N D I X

TABLE 1

Coefficients $\overset{n}{a}_{i,j}$ of the expansion (2.1) for

$n = 1, \ldots, 10, \; i = 1, \ldots, n$

$j = 1, \ldots, \frac{n}{2}$ when n is an even integer,

$j = 1, \ldots, \frac{n+1}{2}$ when n is an odd integer.

```
N =  1

J =  1

  1    0.099999999E 01



N =  2

J =  1

  1    0.500000000E 00    0.500000000E 00



N =  3

J =  1

  1    0.166666664E 00    0.250000000E 00    0.833333330E-01

J =  2

  1    0.666666657E 00    0.                -0.166666664E 00



N =  4

J =  1

  1    0.416666665E-01    0.749999993E-01    0.416666665E-01
  4    0.833333330E-02

J =  2

  1    0.458333321E 00    0.274999991E 00   -0.416666665E-01
  4   -0.249999999E-01
```

N = 5

J = 1

1	0.833333342E-02	0.166666666E-01	0.119047617E-01
4	0.416666665E-02	0.595238089E-03	

J = 2

1	0.216666654E 00	0.216666661E 00	0.357142850E-01
4	-0.833333330E-02	-0.238095236E-02	

J = 3

1	0.549999982E 00	0.	-0.952380933E-01
4	0.	0.357142853E-02	

N = 6

J = 1

1	0.138888895E-02	0.297619053E-02	0.248015870E-02
4	0.115740737E-02	0.297619044E-03	0.330687826E-04

J = 2

1	0.791666638E-01	0.101785697E 00	0.342261889E-01
4	0.254629619E-02	-0.892857133E-03	-0.165343912E-03

J = 3

1	0.419444427E 00	0.179761894E 00	-0.367063479E-01
4	-0.134259255E-01	0.595238089E-03	0.330687824E-03

N = 7

J = 1

1	0.198412723E-03	0.446428607E-03	0.413359798E-03
4	0.231481477E-03	0.811688278E-04	0.165343913E-04
7	0.150312647E-05		

N = 7 (CONTINUED)

J = 2

1	0.238095233E-01	C.357142836E-01	0.163690464E-01
4	0.324074057E-02	C.108225103E-03	-0.661375652E-04
7	-0.901875876E-05		

J = 3

1	0.236309506E 00	C.177232131E 00	0.141369050E-01
4	-0.717592560E-02	-C.116341986E-02	0.826719561E-04
7	0.225468968E-04		

J = 4

1	0.479365058E 00	C.	-0.618386222E-01
4	0.	C.194805187E-02	0.
7	-0.300625293E-04		

N = 8

J = 1

1	0.248015940E-04	C.578703821E-04	0.578703762E-04
4	0.368266001E-04	C.157828276E-04	0.445156672E-05
7	0.751563235E-06	C.578125565E-07	

J = 2

1	0.612599216E-02	C.102099866E-01	0.566302869E-02
4	0.159406551E-02	C.218704892E-03	0.190781429E-05
7	-0.375781616E-05	-C.404687892E-06	

J = 3

1	0.106473200E 00	C.106473200E 00	0.243303562E-01
4	-0.152567189E-03	-C.750811647E-03	-0.718610045E-04
7	0.676406910E-05	C.121406366E-05	

J = 4

1	0.387375973E 00	C.129125327E 00	-0.300512554E-01
4	-0.777041219E-02	C.516323926E-03	0.174882976E-03
7	-0.375781616E-05	-C.202343944E-05	

N = 9

J = 1

1	0.275573336E-05	0.661375941E-05	0.701459220E-05
4	0.491021382E-05	0.242812746E-05	0.847917477E-06
7	0.200416855E-06	0.289062783E-07	0.192708521E-08

J = 2

1	0.138337752E-02	0.249007941E-02	0.158078800E-02
4	0.549242373E-03	0.111346974E-03	0.114468855E-04
7	-0.100208426E-06	-0.173437668E-06	-0.154166815E-07

J = 3

1	0.402557291E-01	0.483068740E-01	0.161686293E-01
4	0.192059478E-02	-0.121406348E-03	-0.542667171E-04
7	-0.340708647E-05	0.404687889E-06	0.539583844E-07

J = 4

1	0.243149228E 00	0.145889543E 00	0.431948510E-02
4	-0.550855749E-02	-0.555347411E-03	0.708011075E-04
7	0.113235521E-04	-0.404687889E-06	-0.107916769E-06

J = 5

1	0.430417746E 00	0.	-0.441518337E-01
4	0.	0.112595731E-02	0.
7	-0.160333482E-04	0.	0.134895961E-06

N = 10

J = 1

1	0.275573456E-06	0.676407467E-06	0.751563668E-06
4	0.566563244E-06	0.312187854E-06	0.127187629E-06
7	0.375781601E-07	0.765166153E-08	0.963542607E-09
10	0.566789766E-10		

J = 2

1	0.279155680E-03	0.532933547E-03	0.375030068E-03
4	0.151326291E-03	0.383990964E-04	0.606260949E-05
7	0.488516065E-06	-0.110523999E-07	-0.674479816E-08
10	-0.510110780E-09		

N = 10 (CONTINUED)

J = 3

1	0.131834208E-01	0.179773914E-01	0.753817853E-02
4	0.147171484E-02	0.100940723E-03	-0.128035524E-04
7	-0.300625274E-05	-0.129228058E-06	0.192708518E-07
10	0.204044309E-08		

J = 4

1	0.125438698E 00	0.102631666E 00	0.167944317E-01
4	-0.116080655E-02	-0.536338623E-03	-0.246743994E-04
7	0.571188019E-05	0.571324044E-06	-0.269791924E-07
10	-0.476103390E-08		

J = 5

1	0.361098409E 00	0.984813888E-01	-0.247083919E-01
4	-0.494053762E-02	0.396686621E-03	0.944580042E-04
7	-0.323172171E-05	-0.105933000E-05	0.134895962E-07
10	0.714155085E-08		

TABLE 2

Coefficients ${}^{n}a_{i,j}$ of the expansion (2.2) for $n = 1, \ldots, 10$, $i = 1, \ldots, n$, $j = 1, \ldots, n$

```
N =  1

J =  1

  1     0.099999999E 01


N =  2

J =  1

  1     0.250000000E 00    0.250000000E 00

J =  2

  1     0.750000000E 00    0.250000000E 00


N =  3

J =  1

  1     0.416666665E-01    0.625000000E-01    0.208333333E-01

J =  2

  1     0.291666664E 00    0.187500000E 00    0.208333333E-01

J =  3

  1     0.666666657E 00    0.125000000E 00   -0.416666665E-01


N =  4

J =  1

  1     0.520833331E-02    0.937499991E-02    0.520833331E-02
  4     0.104166666E-02
```

N = 4 (CONTINUED)

J = 2

1	0.781249991E-01	0.718749985E-01	0.156250000E-01
4	0.104166666E-02		

J = 3

1	0.317708321E 00	0.159374997E 00	0.520833331E-02
4	-0.312499999E-02		

J = 4

1	0.598958321E 00	0.968749989E-01	-0.260416665E-01
4	-0.312499999E-02		

N = 5

J = 1

1	0.520833339E-03	0.104166666E-02	0.744047604E-03
4	0.260416666E-03	0.372023806E-04	

J = 2

1	0.161458331E-01	0.187499996E-01	0.595238077E-02
4	0.781249997E-03	0.372023806E-04	

J = 3

1	0.107291661E 00	0.781249963E-01	0.126488091E-01
4	-0.	-0.148809522E-03	

J = 4

1	0.326041654E 00	0.132291660E 00	0.223214290E-02
4	-0.208333330E-02	-0.148809522E-03	

J = 5

1	0.549999982E 00	0.718749994E-01	-0.215773804E-01
4	-0.156249998E-02	0.223214283E-03	

N = 6

J = 1

1	0.434027797E-04	0.930059541E-04	0.775049593E-04
4	0.361689804E-04	0.930059514E-05	0.103339946E-05

J = 2

1	0.273437495E-02	0.373883921E-02	0.153459817E-02
4	0.296585637E-03	0.279017854E-04	0.103339946E-05

J = 3

1	0.286024297E-01	0.260974690E-01	0.643291132E-02
4	0.600405067E-03	-0.930059514E-05	-0.516699725E-05

J = 4

1	0.129730895E 00	0.776599655E-01	0.956411171E-02
4	-0.180844892E-03	-0.102306545E-03	-0.516699725E-05

J = 5

1	0.327864565E 00	0.112388387E 00	-0.465029498E-03
4	-0.172164345E-02	-0.558035704E-04	0.103339945E-04

J = 6

1	0.511024281E 00	0.571800587E-01	-0.171440965E-01
4	-0.120081015E-02	0.130208329E-03	0.103339945E-04

N = 7

J = 1

1	0.310019880E-05	0.697544698E-05	0.645874684E-05
4	0.361689808E-05	0.126826293E-05	0.258349864E-06
7	0.234863511E-07		

J = 2

1	0.393725193E-03	0.606863825E-03	0.300977586E-03
4	0.759548575E-04	0.105688576E-04	0.775049592E-06
7	0.234863511E-07		

J = 3

1	0.636160700E-02	0.682198635E-02	0.214688727E-02

N = 7 (CONTINUED)

J = 3 (CONTINUED)

4	0.318287013E-03	0.202922065E-04	-0.516699728E-06
7	-0.140918106E-06		

J = 4

1	0.412574382E-01	0.313476543E-01	0.617714488E-02
4	0.405092571E-03	-0.169101718E-04	-0.361689806E-05
7	-0.140918106E-06		

J = 5

1	0.146791287E 00	0.751604317E-01	0.727384025E-02
4	-0.343605283E-03	-0.832825990E-04	-0.129174930E-05
7	0.352295263E-06		

J = 6

1	0.325827733E 00	0.966587560E-01	-0.167023156E-02
4	-0.134186914E-02	-0.367796256E-04	0.645874650E-05
7	0.352295263E-06		

J = 7

1	0.479365051E 00	0.466238814E-01	-0.142350771E-01
4	-0.839120337E-03	0.104843069E-03	0.516699720E-05
7	-0.469727020E-06		

N = 8

J = 1

1	0.193762453E-06	0.452112360E-06	0.452112314E-06
4	0.287707813E-06	0.123303341E-06	0.347778650E-07
7	0.587158777E-08	0.451660598E-09	

J = 2

1	0.494094120E-04	0.833824188E-04	0.478593115E-04
4	0.147552998E-04	0.269505870E-05	0.293127712E-06
7	0.176147634E-07	0.451660598E-09	

J = 3

1	0.122012182E-02	0.148260526E-02	0.556679348E-03
4	0.106492982E-03	0.112558331E-04	0.531604492E-06

```
N =  8    (CONTINUED)


J =  3    (CONTINUED)

  7   -0.176147632E-07  -0.316162416E-08

J =  4

  1    0.110318618E-01   0.992211979E-02   0.254997759E-02
  4    0.294571666E-03   0.118547352E-04  -0.760144736E-06
  7   -0.998169911E-07  -0.316162416E-08

J =  5

  1    0.532220714E-01   0.348510719E-01   0.571850938E-02
  4    0.259224726E-03  -0.236213955E-04  -0.288656273E-05
  7   -0.176147630E-07   0.948487233E-08

J =  6

  1    0.159724340E 00   0.715349410E-01   0.550498319E-02
  4   -0.391816884E-03  -0.626204801E-04  -0.561414133E-06
  7    0.228991919E-06   0.948487233E-08

J =  7

  1    0.321830988E 00   0.842188867E-01  -0.236034882E-02
  4   -0.108067156E-02  -0.178437560E-04   0.524152102E-05
  7    0.146789691E-06  -0.158081206E-07

J =  8

  1    0.452920936E 00   0.390125732E-01  -0.120181121E-01
  4   -0.632176227E-03   0.781567014E-04   0.394977178E-05
  7   -0.264221445E-06  -0.158081206E-07



N =  9


J =  1

  1    0.107645834E-07   0.258349977E-07   0.274007508E-07
  4    0.191805227E-07   0.948487289E-08   0.331217764E-08
  7    0.782878339E-09   0.112915149E-09   0.752767662E-11

J =  2

  1    0.550069933E-05   0.995938740E-05   0.642155982E-05
  4    0.231810282E-05   0.520312994E-06   0.745239950E-07
  7    0.665446576E-08   0.338745448E-09   0.752767662E-11
```

N = 9 (CONTINUED)

J = 3

1	0.206270837E-03	0.277170649E-03	0.119719716E-03
4	0.275021246E-04	0.378175395E-05	0.309688591E-06
7	0.113517354E-07	-0.451660594E-09	-0.602214121E-10

J = 4

1	0.256048390E-02	0.262050715E-02	0.809901328E-03
4	0.124851476E-03	0.100173795E-04	0.256693760E-06
7	-0.238777884E-07	-0.225830296E-08	-0.602214121E-10

J = 5

1	0.163457734E-01	0.127423957E-01	0.277525082E-02
4	0.259060311E-03	0.553239073E-05	-0.968811875E-06
7	-0.778963916E-07	0.346944691E-17	0.210774939E-09

J = 6

1	0.641656835E-01	0.370253692E-01	0.515891553E-02
4	0.149819034E-03	-0.245183930E-04	-0.207507921E-05
7	0.430582947E-08	0.632324826E-08	0.210774939E-09

J = 7

1	0.169511445E 00	0.675561726E-01	0.416183798E-02
4	-0.404103397E-03	-0.480381673E-04	0.157328351E-06
7	0.185150723E-06	0.316162410E-08	-0.421549878E-09

J = 8

1	0.316787004E 00	0.741753038E-01	-0.270161298E-02
4	-0.880065229E-03	-0.868678910E-05	0.391996207E-05
7	0.102948501E-06	-0.948487233E-08	-0.421549878E-09

J = 9

1	0.430417731E 00	0.332553750E-01	-0.103304613E-01
4	-0.485554840E-03	0.613820266E-04	0.267458341E-05
7	-0.209419952E-06	-0.790406018E-08	0.526937348E-09

N = 10

J = 1

1	0.538229405E-09	0.132110833E-08	0.146789779E-08

N = 10 (CONTINUED)

J = 1 (CONTINUED)

4	0.110656884E-08	0.609741903E-09	0.248413338E-09
7	0.733948440E-10	0.149446514E-10	0.168191915E-11
10	0.110701127E-12		

J = 2

1	0.550608178E-06	0.105409684E-05	0.747159554E-06
4	0.306624852E-06	0.810956582E-07	0.143251682E-07
7	0.168808140E-08	0.127859794E-09	0.564575746E-11
10	0.110701127E-12		

J = 3

1	0.312253487E-04	0.455804020E-04	0.221138673E-04
4	0.587983050E-05	0.974570597E-06	0.104582003E-06
7	0.697250985E-08	0.204243560E-09	-0.940959561E-11
10	-0.996310121E-12		

J = 4

1	0.527085911E-03	0.598571809E-03	0.213169453E-03
4	0.399101800E-04	0.437205165E-05	0.264394554E-06
7	0.418350599E-08	-0.586162417E-09	-0.432841397E-10
10	-0.996310121E-12		

J = 5

1	0.437948416E-02	0.390209901E-02	0.103049880E-02
4	0.132727922E-03	0.829899193E-05	0.503451090E-07
7	-0.290643560E-07	-0.172029535E-08	0.752767651E-11
10	0.398524041E-11		

J = 6

1	0.219873565E-01	0.151838490E-01	0.286618012E-02
4	0.218410036E-03	0.944693348E-06	-0.959868984E-06
7	-0.531378652E-07	0.538007402E-09	0.143025852E-09
10	0.398524041E-11		

J = 7

1	0.739793321E-01	0.382220899E-01	0.459149911E-02
4	0.708174311E-04	-0.237100119E-04	-0.149312963E-05
7	0.255414019E-07	0.506789716E-08	0.526937344E-10
10	-0.929889434E-11		

J = 8

1	0.176898070E 00	0.635483563E-01	0.313260945E-02
4	-0.393466191E-03	-0.369178315E-04	0.437869790E-06

N = 10 (CONTINUED)

J = 8 (CONTINUED)

7	0.136514403E-06	0.190627331E-08	-0.263468674E-09
10	-0.929889434E-11		

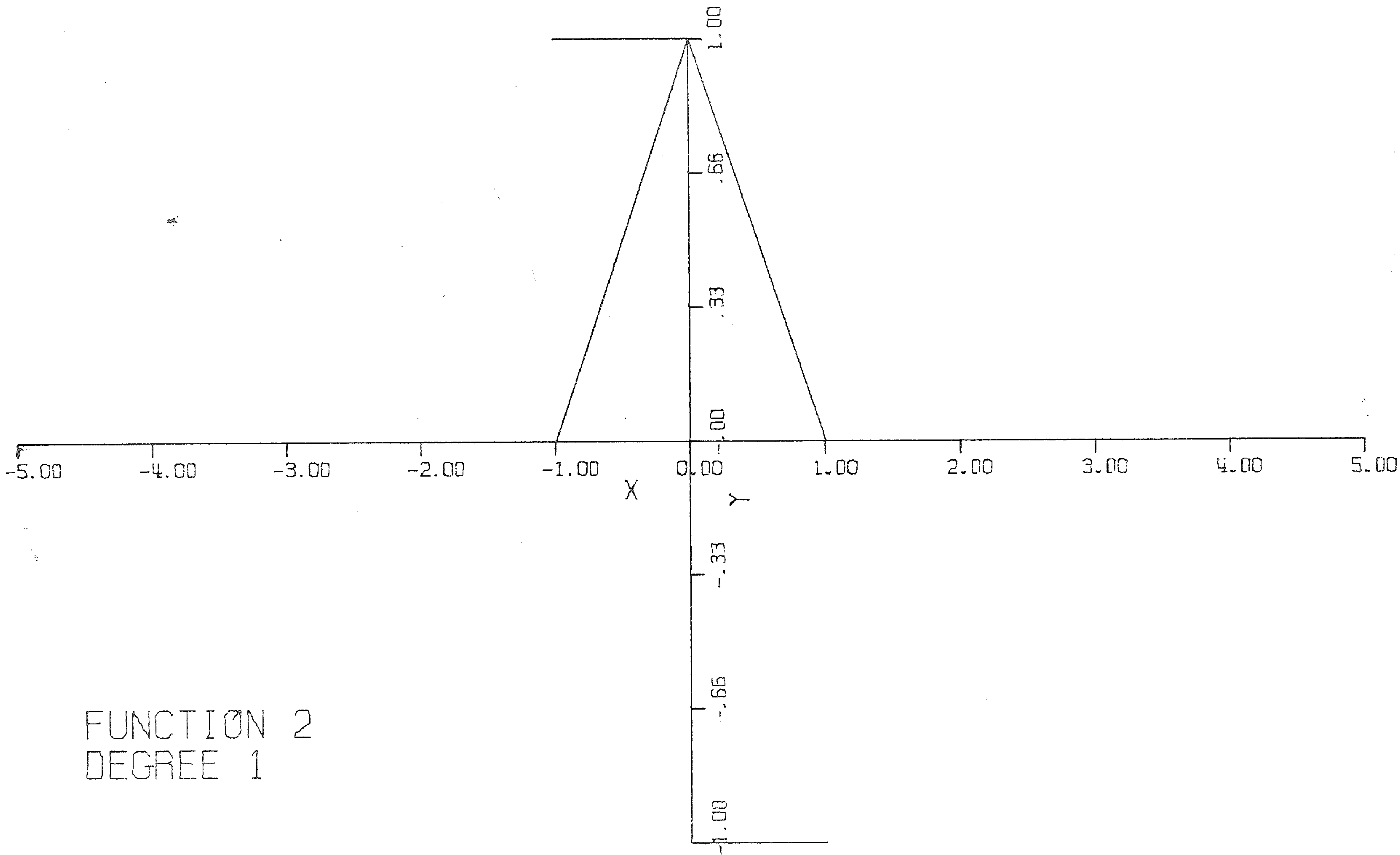
J = 9

1	0.311234191E 00	0.659511853E-01	-0.285549081E-02
4	-0.727654406E-03	-0.296619032E-05	0.304703769E-05
7	0.573947694E-07	-0.760184571E-08	-0.184428070E-09
10	0.139483415E-10		

J = 10

1	0.410962604E 00	0.287891247E-01	-0.900132861E-02
4	-0.383388768E-03	0.489220206E-04	0.195236305E-05
7	-0.150165846E-06	-0.602103384E-08	0.289815540E-09
10	0.139483415E-10		

The graphs of the functions $\varphi_n(x)$ and their derivatives $\varphi_n'(x)$, $n = 1, \cdots, 10$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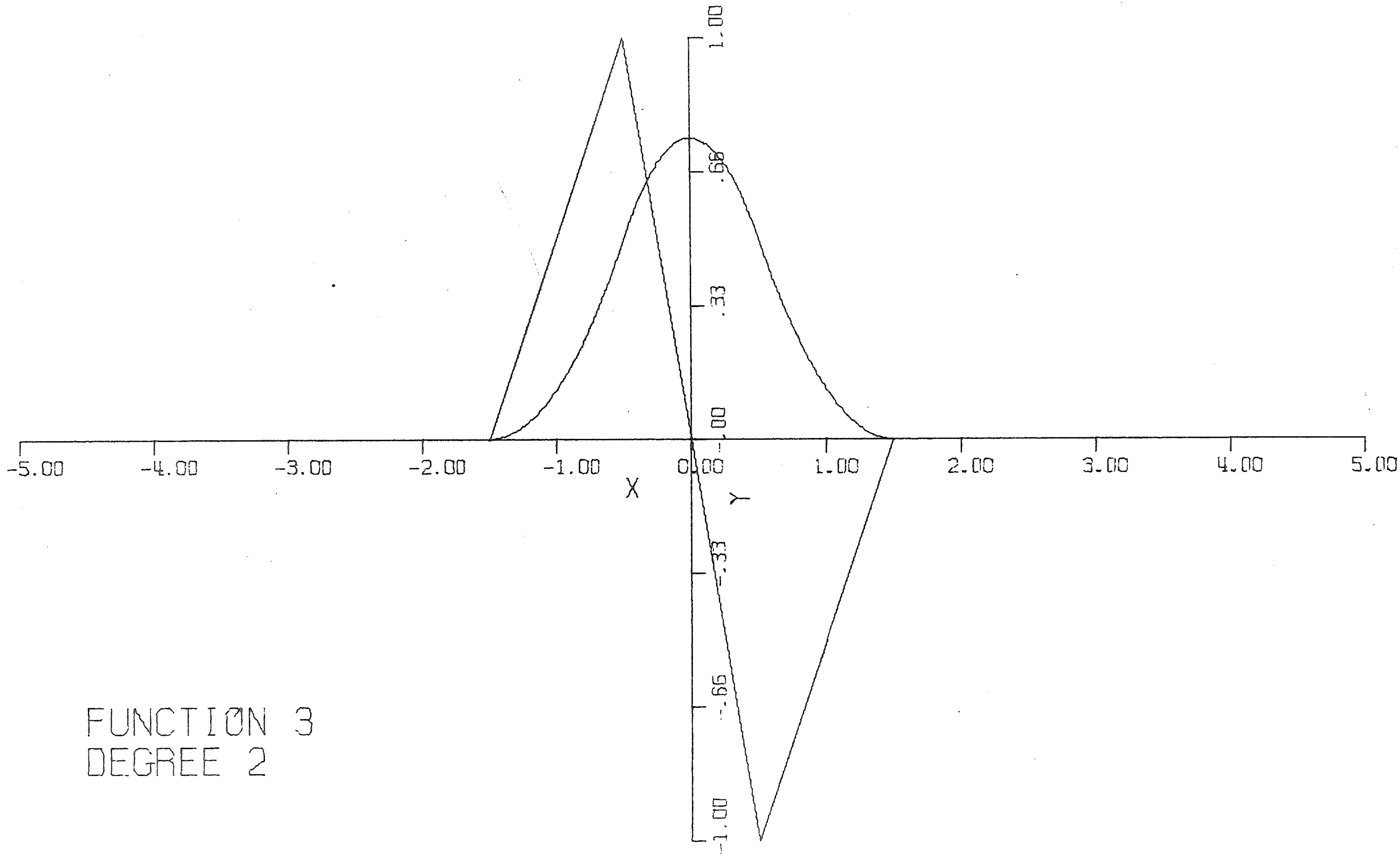
X
Y
FUNCTIØN 2
DEGREE 1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X
Y
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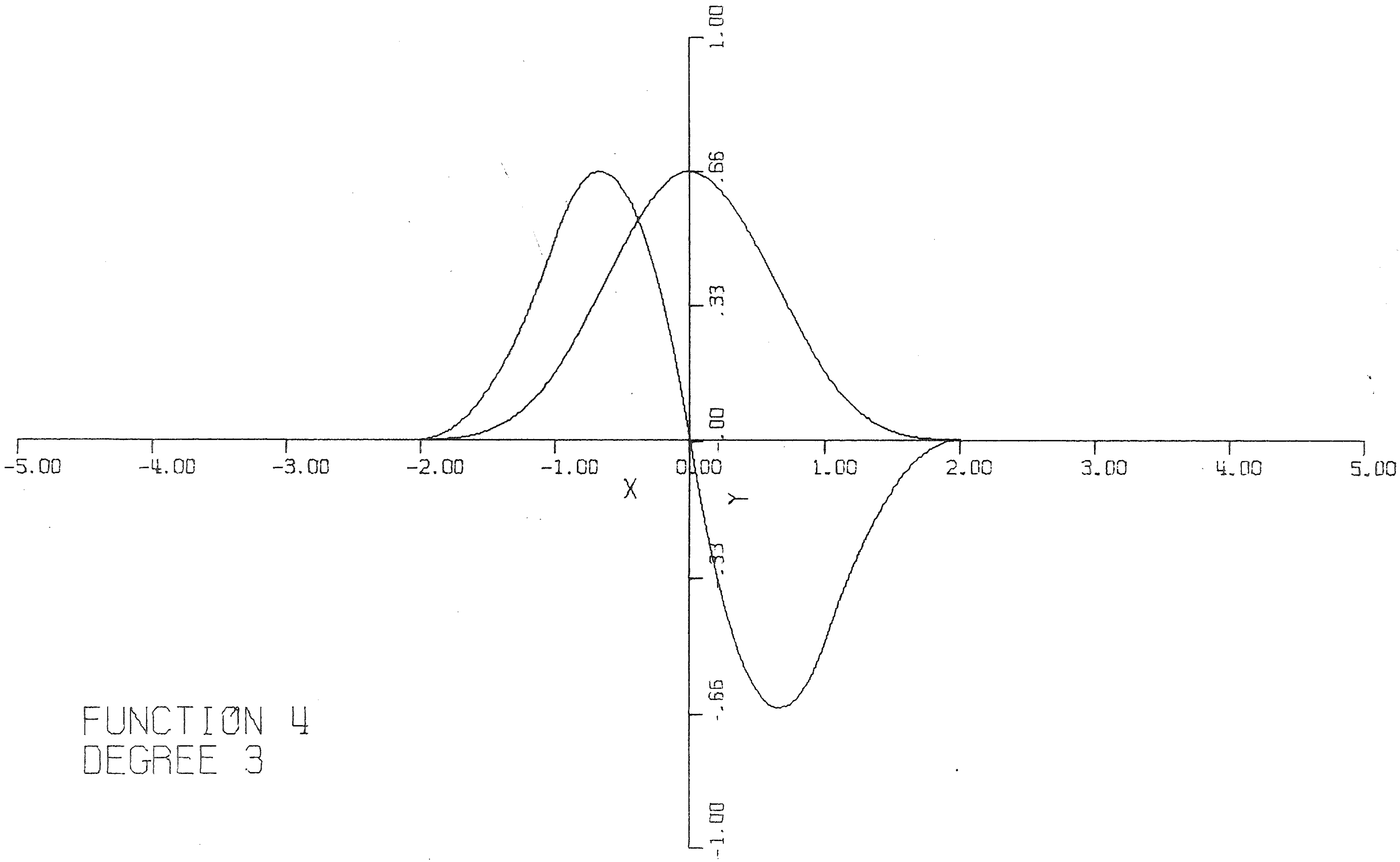
DEGREE 2



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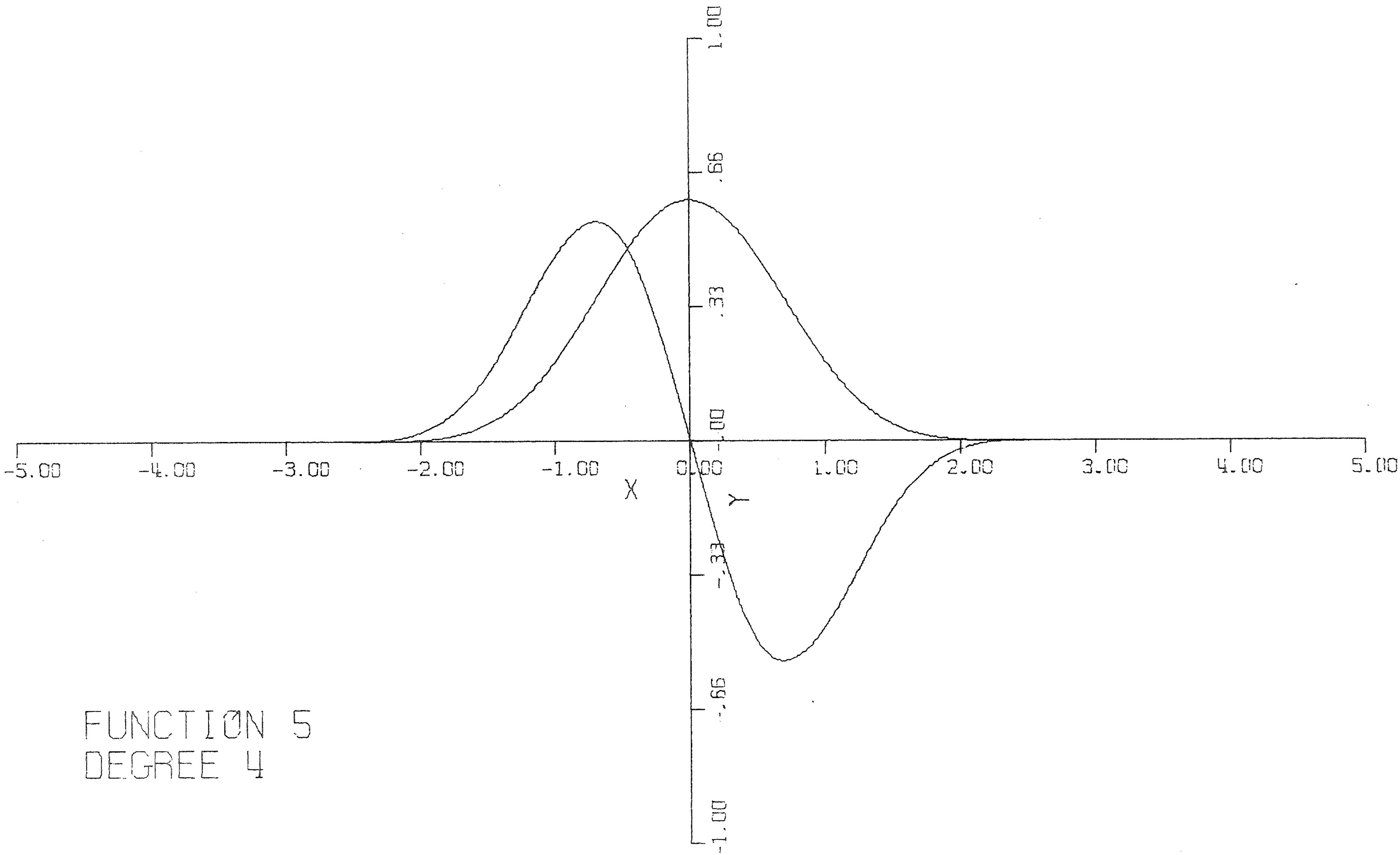
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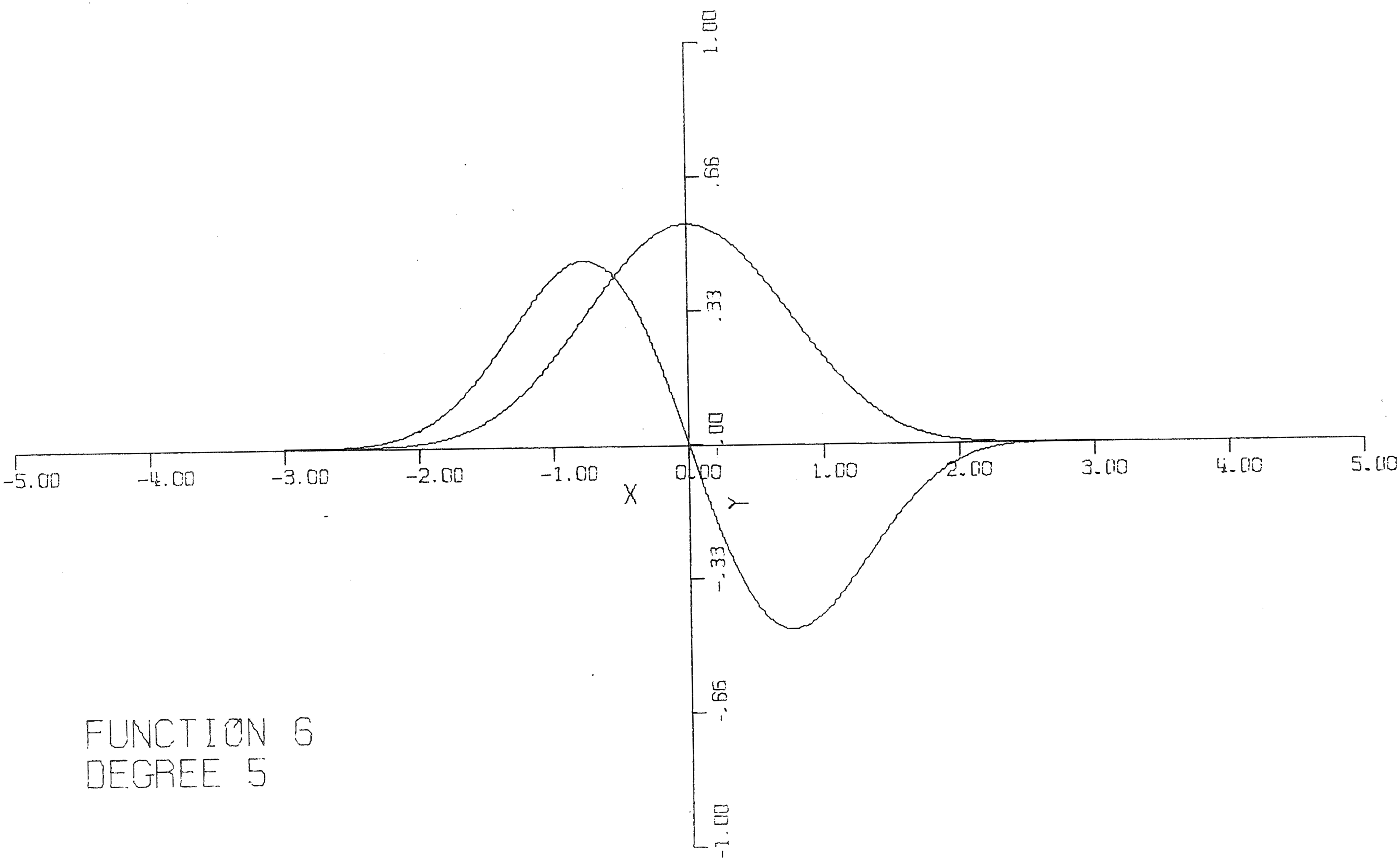
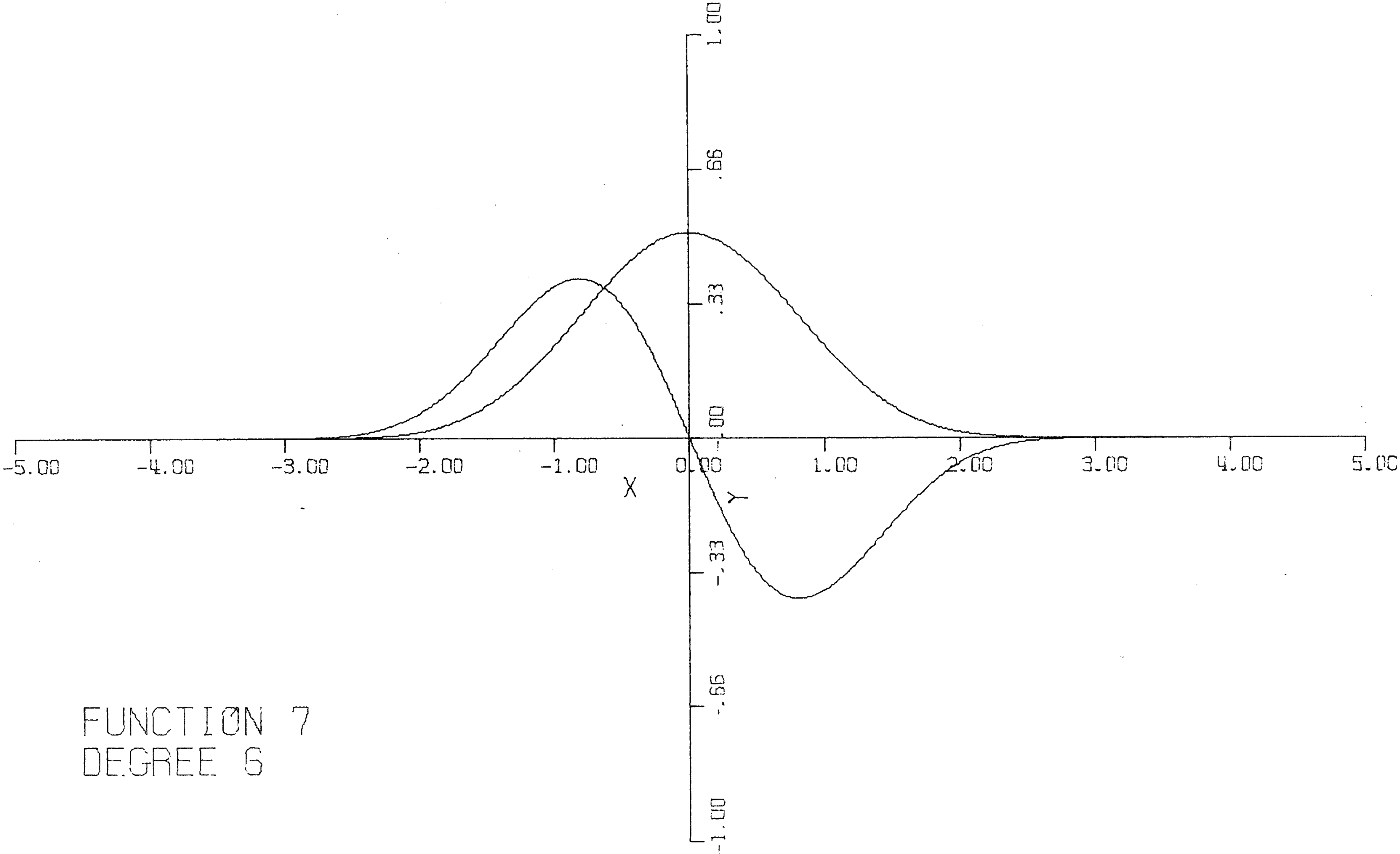
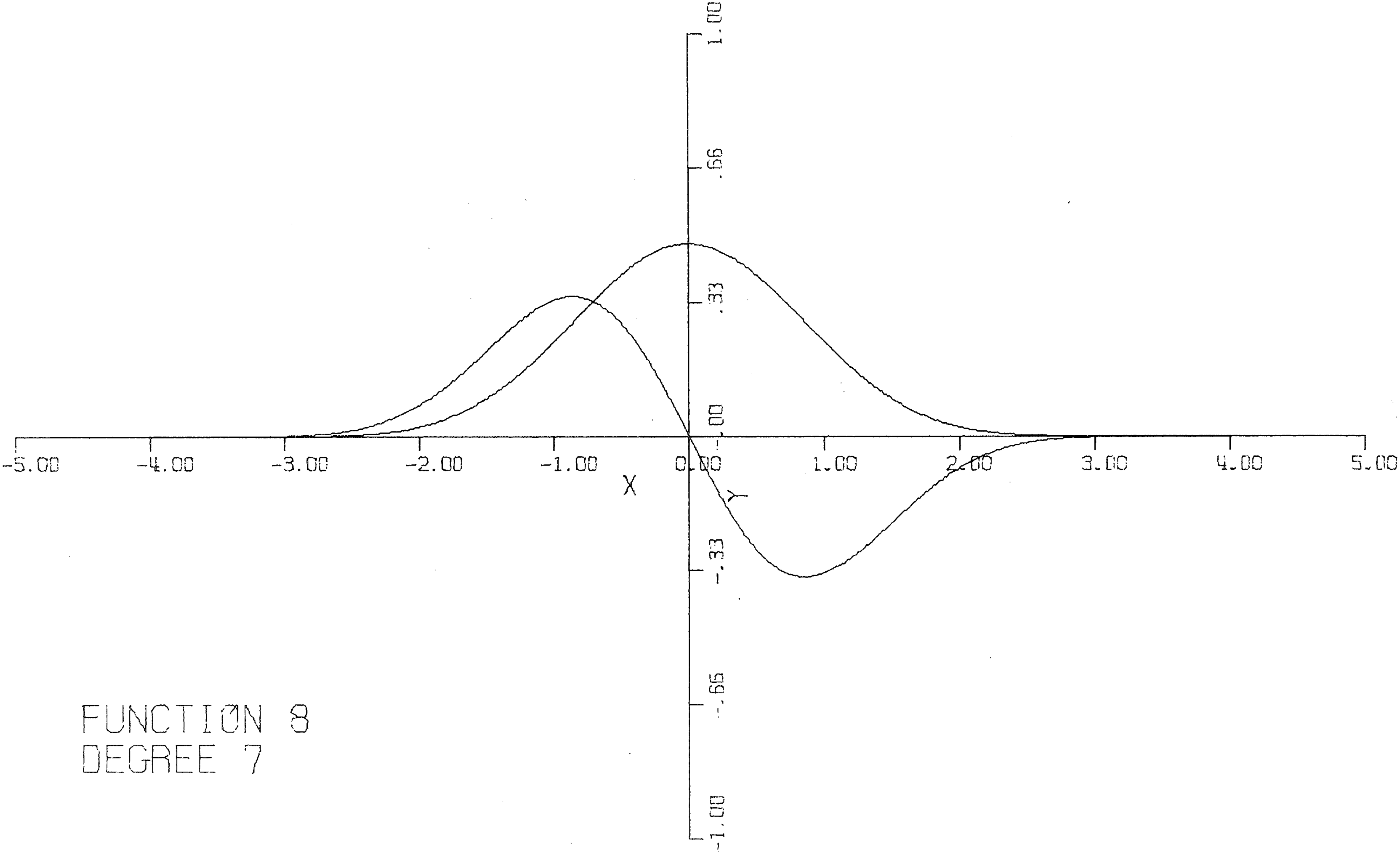
3.00
4.00
5.00
X
Y
FUNCTIØN 5
DEGREE 4



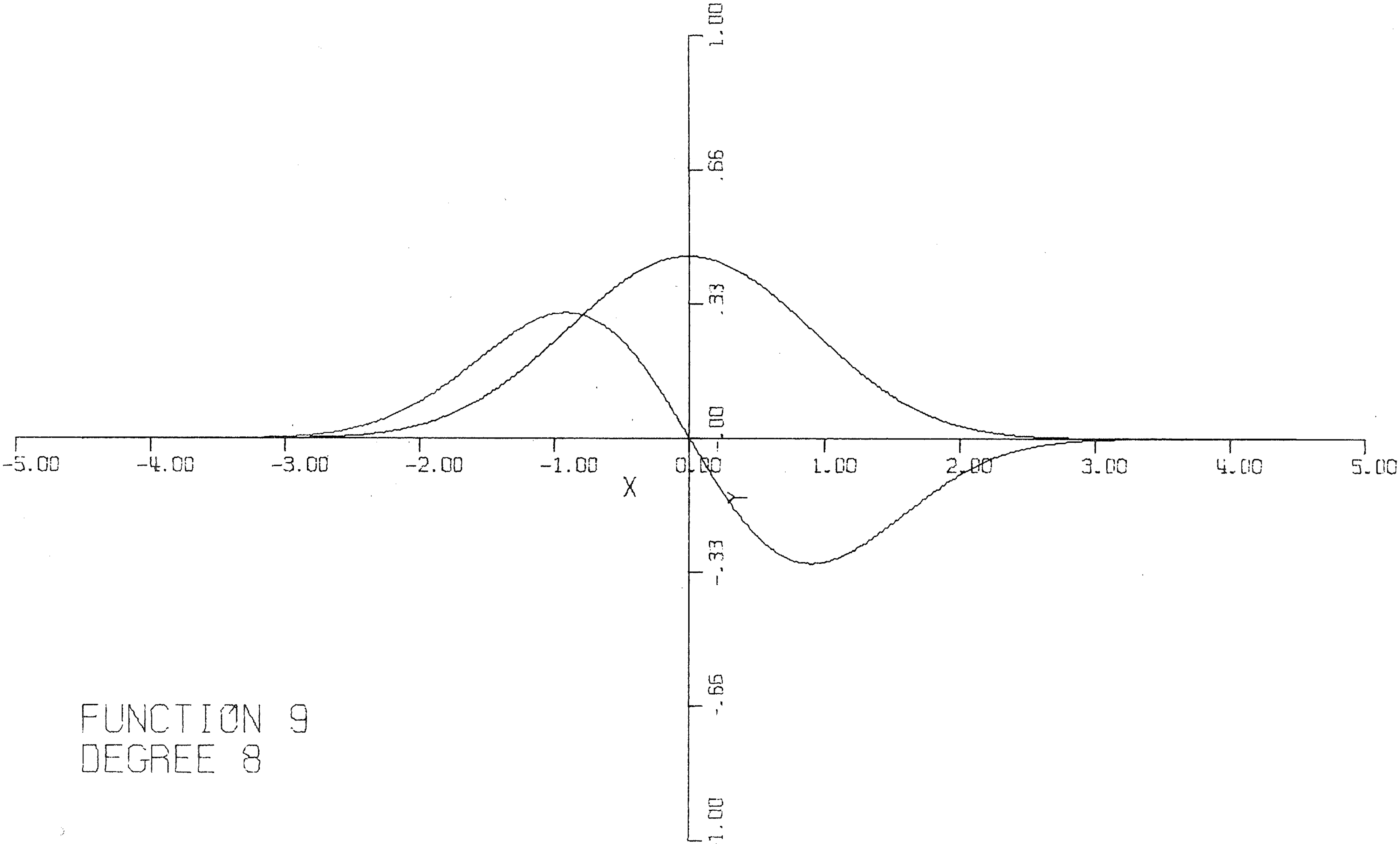
FUNCTIØN 6
DEGREE 5



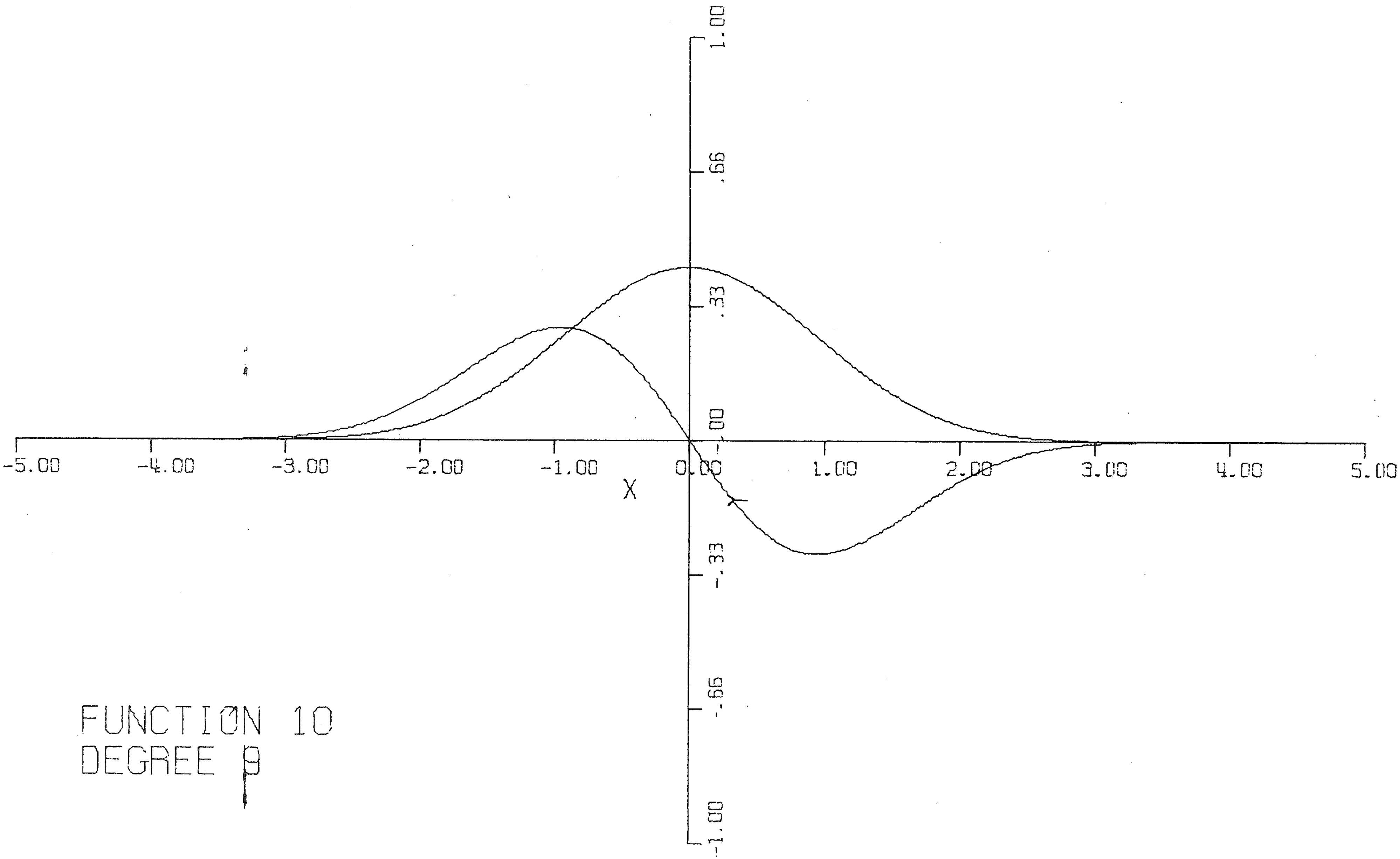
FUNCTIØN 7
DEGREE 6



FUNCTIØN 8
DEGREE 7



1.00
.66
.33
.00
-.33
-.66
-1.00
-5.00
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
5.00
X
Y
FUNCTIØN 9
DEGREE 8



FUNCTION 10
DEGREE 8

The computer subroutine for computation of the

${}^{n}a_{i,j}$ of (2.1)

```
      SUBROUTINE HILLS(M,A)
      DIMENSION A(21,21,42)
C AN ARRAY A IS FILLED WITH THE VALUES A(N,I,J) OF THE CO-
C EFFICIENTS OF THE HILL FUNCTIONS. N IS THE ORDER OF THE
C HILL, N=1,M, M.GT.2. I IS THE INDEX OF SUMMATION IN EACH
C LINEAR COMBINATION WHERE EACH TERM HAS THE FORM
C A(N,I,J)*P(I), I=1,N. J DENOTES A SUBINTERVAL OF THE
C SUPPORT OF THE HILL FUNCTION, J=1,2*N.
      DIMENSION AN(21,42)
      A(1,1,1)=1.0
      A(1,1,2)=1.0
      A(2,1,1)=.25
      A(2,1,2)=.75
      A(2,1,3)=.75
      A(2,1,4)=.25
      A(2,2,1)=.25
      A(2,2,2)=.25
      A(2,2,3)=-.25
      A(2,2,4)=-.25
      DO 14 N=3,M
      NN=N-1
      DO 1 J=1,N
      DO 1 I=1,N
    1 AN(I,J)=.0
      J=1
      II=1
      GO TO 12
    2 J=2
      GO TO 11
    3 II=2
      GO TO 12
    4 IF(J.EQ.N)GO TO 7
      J=J+1
      GO TO 9
    5 GO TO 11
    6 GO TO 12
    7 DO 8 J=1,N
      NPJ=N+J
      NMJ1=N-J+1
      C=-1.0
      DO 8 I=1,N
      C=-C
      A(N,I,J)=AN(I,J)
    8 A(N,I,NPJ)=C*AN(I,NMJ1)
      GO TO 14
    9 AN(1,J)=AN(1,J)+.25*A(NN,1,J-2)
      AN(2,J)=AN(2,J)-.25*A(NN,1,J-2)
      DO 10 I=2,NN
      C=A(NN,I,J-2)/FLOAT(8*I-4)
      AN(I-1,J)=AN(I-1,J)+C
   10 AN(I+1,J)=AN(I+1,J)-C
      GO TO 5
   11 AN(1,J)=AN(1,J)+.5*A(NN,1,J-1)
      GO TO (3,6),II
```

```
12 AN(1,J)=AN(1,J)+.25*A(NN,1,J)
   AN(2,J)=AN(2,J)+.25*A(NN,1,J)
   DO 13 I=2,NN
   C=A(NN,I,J)/FLOAT(8*I-4)
   AN(I-1,J)=AN(I-1,J)-C
13 AN(I+1,J)=AN(I+1,J)+C
   GO TO (2,4), II
14 CONTINUE
   RETURN
   END
```