ON ACCELERATION AND MOTION OF IONS IN CORONA AND SOLAR WIND

by

JOHANNES GEISS

National Aeronautics and Space Administration
Manned Spacecraft Center, Houston;
and Physikalisches Institut, University of Berne

PETER HIRT

Physikalisches Institut, University of Berne

HEINRICH LEUTWYLER

Institut für Theoretische Physik, University of Berne

June 1969

Abstract

Assuming a stationary, radial, spherically symmetric solar wind and a radial magnetic field direction in the vicinity of the sun, an equation of motion for ions heavier than protons in the solar wind is derived. The general properties of this equation are discussed and the results of numerical integrations are given. These results are based on the assumption of maxwellian velocity distribution functions for electrons, protons and ions, but the effects of first order deviations from such distributions are also presented and discussed. It is shown that dynamical friction, i.e. momentum transfer from protons to heavier ions accounts for the observed fact that heavier ions - if accelerated at all - normally reach the same velocity as the protons in the solar wind. Because of the non-linear relation between dynamical friction and proton-ion velocity difference a minimum proton flux is required to carry a certain ion species in the solar wind. Formulae comparing the minimum fluxes for different ions are given. It is shown that elements up to and beyond iron will be carried along in the solar wind as long as helium is carried along. Substantial isotopic fractionation is possible, in particular in the case of helium. The effects of ion motion and escape on abundances in the corona and in the outer convective zone of the sun are discussed.

1. INTRODUCTION

It has been established for several years (SNYDER and NEUGEBAUER, 1964; WOLFE et al., 1966; HUNDHAUSEN et al., 1967) that $^4He^{++}$ ions are present in the solar wind and that their relative abundance is highly variable, values for He/H between 0.01 and 0.25 having been observed (HUNDHAUSEN et al., 1967). The average He/H ratio in the solar wind is about 0.04 - 0.05 (HUNDHAUSEN et al., 1967; SNYDER and NEUGEBAUER, 1966; OGILVIE and WILKERSON, 1969; ROBBINS et al., 1969), which is lower than the value of 0.1 often assumed for the sun (cf. BISWAS and FICHTEL, 1964). At least during stationary conditions the bulk velocities of H and He are normally the same within a few percent (HUNDHAUSEN et al., 1967; OGILVIE et al., 1968; ROBBINS et al., 1969). Recently, also the presence of He^3 and oxygen has been reported in the solar wind (BAME et al., 1968). However, as these ion species are observed only under very favourable conditions, no figures on their abundance averages and variations can be given. Abundances of still heavier elements such as Ca, Fe, and Ni have been determined in the corona (POTTASH, 1963), and it appears that these elements are overabundant relative to the photosphere.

Both the coronal and the solar wind observations indicate that there are processes at work which are substantially influencing the relative abundances of ions. If these processes would be sufficiently understood abundance measurements in the solar wind could be used to gauge conditions in the solar atmosphere. A theoretical understanding of enrichment processes is also important if solar abundances, for instance of isotopes, are to be inferred from solar wind observations.

PARKER (1961) discussed the static non-convective atmosphere formed by a multi-component plasma in a gravitational field. He showed that a mass separation of ions will occur similar but not identical to the mass separation in an atmosphere composed of neutral molecules. Ions with mass/charge ratios larger than that of protons will accumulate at the bottom. This result has sometimes led to the assumption that the abundance of elements of higher mass number may be significantly reduced in the solar wind.

JOKIPII (1965) and DELACHE (1965) used the diffusion equation as given by CHAPMAN and COWLING (1958) for studying the motion of ions in the corona. The interesting result is that in the lower corona heavier ions move markedly slower than protons, producing enhanced number densities of the former. More recently DELACHE (1967) and NAKADA (1969) have extended these studies and have shown that essentially as a result of thermal diffusion, very strong enhancements of heavier elements may occur in the lower corona, unless convective mixing is dominant. Conditions for the occurence or absence of radial convection in the lower corona have been discussed by NAKADA (1969).

The diffusion equation as given by CHAPMAN and COWLING (1958) is an approximation applicable only as long as bulk velocities and their differences are small compared to thermal velocities. Thus its applicability is restricted to the lower corona, since in the coronal expansion protons and in particular heavier ions reach high Mach numbers.

For the purpose of theoretical study, two fundamental problems may be distinguished:

1. Do there exist diffusion processes or perhaps other processes across the photosphere-corona transition region which are enhancing the abundance of heavier elements in the corona?
2. Which are the factors determining acceleration and escape of ions other than hydrogen from the lower corona into interplanetary space?

In this paper we address ourselves mainly to the second problem. We shall derive and discuss a general equation of motion for ions applicable in the solar wind acceleration region.

2. THE PROTON-ELECTRON GAS

The theory of expansion of the solar corona and of acceleration of the plasma to supersonic velocities was developed by PARKER (1958, 1962, 1963, 1964). We shall summarize here those of Parker's results which are of direct relevance to the discussion in this paper. The equation of stationary radial motion is given by

$$V \frac{dV}{dr} + \frac{2}{m_p n} \frac{d}{dr} \left(n k T \right) + \frac{G M_\odot}{r^2} = 0 \tag{2.1}$$

V is the bulk velocity of the plasma, n is the number density of protons which is equal to that of electrons. The conservation of mass can be written as

$$n V r^s = \phi_\oplus r_\oplus^s \tag{2.2}$$

where $\phi_\oplus$ is the proton flux at $r_\oplus$ = 1 a.u. For radial flow s is equal to 2, whereas s > 2 corresponds to a flow diverging more rapidly.

In his original work, PARKER (1958, 1963) used a polytropic law to complete the system of equations for the unknowns n(r), V(r) and T(r):

$$T = \text{const.}\; n^{\alpha - 1} \tag{2.3}$$

By proper choice of α, heat transport can be taken into account. Later PARKER (1962, 1964), NOBLE and SCARF (1963) and WHANG and CHANG (1965) solved both the momentum equation and the energy equation by assuming that heat transport is solely conductive. More recently STURROCK and HARTLE (1966) pointed out that energy partition times between electrons and protons under coronal conditions are large enough as to give substantial temperature differences between these two components. These authors have accounted for this in their two-fluid model (cf. also HARTLE and STURROCK, 1968). For the purpose of this paper, we consider the polytropic model adequate, because the refinements mentioned can be approximated by adjusting α, or even by assuming it to be varying with solar distance. In fact the polytropic model is very convenient for discussing the factors determining ion acceleration and therefore, our discussions will be based on it.

The polytropic law and the continuity equation for the mass imply $nkT \propto (V \cdot r^s)^{-\alpha}$. Inserting this relation in the differential equation (2.1) we obtain

$$m_p V \frac{dV}{dr}\left(1 - \frac{2\alpha kT}{m_p V^2}\right) = 2s\alpha \frac{kT}{r} - \frac{G M_\odot m_p}{r^2} \qquad (2.4)$$

where the temperature may be expressed in terms of the velocity as $T \propto (V \cdot r^s)^{1-\alpha}$. Integrating (2.4) we arrive at the Bernoulli equation

$$\frac{m_p V^2}{2} + \frac{2\alpha}{\alpha - 1} kT - \frac{G M_\odot m_p}{r} = E \quad . \qquad (2.5)$$

In order to determine the admissible values of the constant of integration E we observe with Parker that the differential equation (2.4) is singular if the velocity V reaches the critical value

$$\frac{1}{2} m_p V^2 = \alpha kT \qquad (2.6)$$

As this point the gradient dV/dr becomes infinite unless the right hand side in (2.4) vanishes simultaneously

$$2s\alpha kT = \frac{G M_\odot m_p}{r} \quad . \qquad (2.7)$$

The two relations (2.6) and (2.7) define the so-called critical point (r_c, V_c, T_c). Parker has shown that the physical solution must pass through the critical point if the boundary conditions at $r \rightarrow o$ and at $r \rightarrow \infty$

are to be satisfied, i.e.

$$V(r_c) = V_c = (GM_\odot / s r_c)^{1/2}$$
$$T(r_c) = T_c = GM_\odot m_p / 2s\alpha k r_c \quad . \tag{2.8}$$

This implies that only one particular value of the integration constant E is acceptable. Evaluating (2.5) at the critical point one finds

$$E = \frac{GM_\odot m_p}{s r_c}\left(\frac{1}{2} + \frac{1}{\alpha - 1} - s\right) \quad . \tag{2.9}$$

To obtain a convenient representation of the physical solution we express it in terms of the dimensionless quantities:

$$r = r_c \xi$$
$$V(r) = V_c W(\xi)$$
$$T(r) = T_c \tau(\xi) \quad . \tag{2.10}$$

The dimensionless temperature τ is given by

$$\tau = (W\xi^s)^{1-\alpha} \quad , \tag{2.11}$$

and the velocity W is the monotonically rising solution of the transcendental equation

$$\frac{W^2}{2} + \frac{1}{\alpha - 1}(W\xi^s)^{1-\alpha} - \frac{s}{\xi} = \frac{1}{2} + \frac{1}{\alpha - 1} - s \quad . \tag{2.12}$$

The functions $W(\xi)$ and $\tau(\xi)$ are determined uniquely once the polytrope index α and the divergence coefficient s have been chosen. With $W(\xi)$ and $\tau(\xi)$ known, the physical solution V(r), T(r) is given by (2.10), (2.8) in terms of the position r_c of the critical point which is the essential parameter in our representation of the physical solutions.

For the particular case of isothermal expansion characterized by $\alpha \to 1$ (2.11) and (2.12) reduce to

$$\tau = 1$$

$$\frac{W^2}{2} - \ln(W\xi^s) - \frac{s}{\xi} = \frac{1}{2} - s \qquad (2.13)$$

The following asymptotic relations derived from (2.12) and (2.13) will be needed in our further discussions.

a.) $\xi \ll 1$

$$W = [s(\alpha-1)]^{-\frac{1}{\alpha-1}} \xi^{-s+\frac{1}{\alpha-1}} \{1 + O(\xi)\} \qquad \alpha > 1 \qquad (2.14)$$

$$\tau = s(\alpha-1)\xi^{-1} \qquad \alpha > 1 \qquad (2.15)$$

$$W = \xi^{-s} \exp\left\{-\frac{s}{\xi} - \frac{1}{2} + s\right\} \qquad \alpha = 1 \qquad (2.16)$$

The condition that W should not go to infinity for $\xi \to 0$ places a limit on α:

$$\alpha < \frac{s+1}{s} \qquad \left(\alpha < \frac{3}{2} \text{ for } s = 2\right) \qquad (2.17)$$

b.) $\xi \gg 1$

$$W = W_\infty + W_\infty^{-1}\left\{\frac{s}{\xi} - \frac{1}{\alpha-1}\left(W_\infty \xi^s\right)^{1-\alpha} + \cdots\right\} \qquad (2.18)$$

with

$$W_\infty = \{2s+1 - \alpha(2s-1)\}^{1/2}(\alpha-1)^{-1/2} \qquad (2.19)$$

W_∞ is real only for

$$\alpha < \frac{2s+1}{2s-1} \qquad \left(\alpha < \frac{5}{3} \text{ for } s = 2\right) \qquad (2.20)$$

The temperature decrease is given by

$$\tau \propto \xi^{-s(\alpha-1)} \qquad (2.21)$$

For the limiting case $\alpha = 1$

$$W = (2s \ln \xi)^{1/2} \qquad (2.22)$$

is obtained.

3. EQUATION OF MOTION FOR IONS

Before we derive an equation of motion for ions in the solar wind a discussion of the role of magnetic fields is called for. The question is, whether heavier ions can be swept along by the magnetic field lines moving with the proton-electron gas. It should be noted that in order to move with the bulk velocity in the solar wind, heavier ions have to acquire a kinetic energy many times their original thermal energy. It is often assumed that the lines of force near the sun in the acceleration region of the solar wind are approximately radial (cf. PARKER, 1963; JOKIPII, 1965). This assumption is probably justified under quiet-day conditions in most of this region. In fact PARKER's (1963) "spiral model" of the interplanetary magnetic field predicts nearly radial field lines in the vicinity of the sun. A radial field, even when divergent, would of course not give an effective radial acceleration. In an exactly transverse field moving with the plasma any ion would have the same mean radial velocity as the plasma, irrespective of its initial velocity. However, in the case of oblique field lines, neglecting collisions the mean radial velocity of an ion would depend on the initial velocity. If for instance an ion is placed into the moving plasma with an initial velocity around zero the radial motion of the ion is slower than that of the plasma and non-radial ion motion occurs. Thus an ordered magnetic field of spiral configuration cannot accelerate ions to full solar wind velocity. This, however, would be in contradiction with the observation that H and He have virtually identical mean radial velocities (HUNDHAUSEN et al., 1967; OGILVIE et al., 1968; ROBBINS et al., 1969). A highly disordered magnetic field would be more efficient in acceleration, but still the magnetic field as the prime cause of acceleration appears to be insufficient, because the He and H velocities are so permanently and closely equal. The conclusion is, that much of the salient features of ion motion in the quiet-day solar wind should be found in a model neglecting the effects of the magnetic field.

We assume a plasma essentially composed of electrons and protons with a small abundance of another ion species in addition. The state of the plasma is described by the partition functions of electrons, protons and ions, $f_e(\vec{x}, \vec{v}, t)$, $f_p(\vec{x}, \vec{v}, t)$ and $f_{Z,A}(\vec{x}, \vec{v}, t)$ respectively. The evolution in time of these partition functions is described by the Boltzmann equation whose explicit form in the present situation is given in the appendix. To simplify the problem, we assume that each of the components of the plasma is in an approximate local thermodynamic equilibrium, i.e., we assume that the velocities of the various components $\nu = (e;\ p;\ Z, A)$ are distributed according to Maxwell's law around the individual bulk velocities $\vec{v}_\nu\ (\vec{x}, t)$ with a spread given by the temperature $T_\nu\ (\vec{x}, t)$:

$$f_\nu(\vec{x}, \vec{v}, t) = n_\nu\,(m_\nu / 2\pi k T_\nu)^{3/2} \exp\{-m_\nu(\vec{v}-\vec{v}_\nu)^2/2kT_\nu\} \qquad (3.1)$$

The quantity $n_\nu\ (\vec{x}, t)$ denotes the particle density of the component ν.

In this approximation thermal diffusion is neglected. Thermal diffusion is proportional to the temperature gradient and - as pointed out by JOKIPII (1965) - its effects are relatively small in the corona outside the sharp temperature rise in the very narrow photosphere corona transition region which is not included in the quantitative theoretical treatment of this paper. However, we shall see in Chapter 9 that also in the corona thermal diffusion may have an effect on the motion of ions.

In order that representation (3.1) of the partition functions be an approximate solution of the Boltzmann equation, the quantities n_ν, $\vec{v}_\nu$, and T_ν, which describe the state of the plasma in this approximation, must satisfy the transport equations for particle number, momentum and energy respectively. The transport equation for the particle number is the familiar conservation law

$$\frac{\partial n_\nu}{\partial t} + \vec{\nabla}\cdot(\vec{v}_\nu n_\nu) = 0 \qquad (3.2)$$

The analogous differential equation describing transport of momentum, i.e., the differential equation for $\vec{v}_\nu\ (\vec{x}, t)$ is derived in the appendix.

The transport equation for $T_\nu(\vec{x}, t)$ is of a similar nature. In the present paper we do not attempt to solve this system of differential equations for n_ν, $\vec{v}_\nu$, and T_ν. Instead, we simplify the problem further by assuming that the energy transfer between the different plasma components is sufficiently effective such that to a good approximation the plasma is characterized by a single local temperature $T_e = T_p = T_{Z,A} = T$. Actually, as mentioned in Chapter 2, a significant difference between electron and proton temperatures in the solar wind cannot be ruled out and the Parker model has been generalized by STURROCK and HARTLE (1966), to account for this difference. There is no reason, however, to expect that this modification strongly affects the motion of the ions. Concerning the equality of proton and ion temperatures we refer the reader to SPITZER's (1962) relation for equipartition times between protons and ions and conclude that under coronal conditions the ion temperature is indeed closely bound to the proton temperature.

To summarize this discussion, we list the approximations involved in our treatment of ion transport:

a) Forces due to the magnetic field are neglected.

b) The velocity distribution of each particle species (electrons, protons, ions) is maxwellian around an individual bulk velocity.

c) The local temperatures of electrons, protons and ions are equal.

d) $n_I \ll n_p = n_e = n$

In this approximation the equations for stationary motion of one particular ion species are

$$n_I v r^2 = const.$$

$$v \frac{dv}{dr} + \frac{1}{A m_p n_I} \frac{d}{dr}\left(n_I k T\right) + \frac{G M_\odot}{r^2} - \frac{Z e E}{A m_p} - C = 0 \qquad (3.3)$$

where v denotes the bulk velocity, n_I the density, Ze the charge and Am_p the mass of the ion species under consideration. The explicit expression for the collision term C is (cf. BURGERS, 1960).

$$C = \frac{4\pi e^4 \ln \Lambda}{m_p} \frac{n}{kT} \frac{Z^2}{A} G\left\{\left(\frac{A}{A+1} \frac{m_p}{2kT}\right)^{1/2} (V - v)\right\} \qquad (3.4)$$

A derivation is given in the appendix. In corona and interplanetary plasma $\ln \Lambda$ is nearly constant, $\ln \Lambda \simeq 22$ (PARKER, 1963).

The function G (x) was first introduced by CHANDRASEKHAR (1943) and is tabulated by SPITZER (1962). It is defined as

$$G(x) = [\phi(x) - x\phi'(x)]/2x^2 \qquad (3.5)$$

$\phi(x)$ being the error function. The asymptotic behaviour

$$G(x) = 0.37\,x \quad \text{for} \quad x \ll 1$$
$$G(x) = 1/2\,x^2 \quad \text{for} \quad x \gg 1$$

shows that "dynamical friction" (CHANDRASEKHAR, 1943), i.e. acceleration due to collisions is proportional to the difference in bulk velocities until V-v approaches the thermal velocity of the protons, beyond which it is actually decreasing.

Equation (3.4) includes only collisions by protons. The momentum transfer by electrons is smaller by a factor $(m_p/m_e)^{1/2} = 43$ for $|V-v| \ll (2KT/m_p)^{1/2}$ and $T_e \gtrsim T_p$. The effects of electrons become dominant only when the argument of G in (3.4) is larger than 3.8.

If we neglect the perturbation of the proton and electron motion due to collisions with ions and use $n_e = n_p = n$, $v_e = v_p = V$ then the transport equations for protons and electrons take the form

$$\begin{aligned} V\frac{dV}{dr} + \frac{1}{m_p n}\frac{d}{dr}(nkT) + \frac{GM_\odot}{r^2} - \frac{eE}{m_p} &= 0 \\ \frac{1}{n}\frac{d}{dr}(nkT) + eE &= 0 \end{aligned} \qquad (3.6)$$

In the electron transport equation we have neglected the kinetic term $\sim V\,dV/dr$ since it is of order m_e/m_p as compared to the remainder. Elimination of the electrical field E in (3.6) leads to the basic plasma equation (2.1). On the other hand, we may use (3.6) to determine the electric field

$$\frac{eE}{m_p} = \frac{1}{2}\left\{ \frac{GM_\odot}{r^2} + V\frac{dV}{dr} \right\} \tag{3.7}$$

Inserting this expression in the ion equation (3. 3) we obtain

$$v\frac{dv}{dr}\left(1 - \frac{kT}{Am_p v^2}\right) = \frac{k}{Am_p}\left(\frac{sT}{r} - \frac{dT}{dr}\right) - \frac{2A-Z}{2A}\frac{GM_\odot}{r^2} + \frac{Z}{2A}V\frac{dV}{dr} + C \tag{3.8}$$

This equation, supplied with suitable boundary conditions at $r \longrightarrow 0$ and at $r \longrightarrow \infty$, determines the bulk velocity of the ions, once the proton velocity V(r) and the temperature T(r) are known from a model of the main solar wind components. In the following we restrict ourselves to Parker's polytropic model for the proton-electron plasma. We use the representation (2. 10) of this model and introduce the dimensionless ion velocity $w(\xi)$ as

$$v(r) = V_c\, w(\xi) \tag{3.9}$$

where V_c is the velocity of the proton gas at the critical point. In terms of this variable the differential equation (3. 8) may be rewritten as

$$\frac{dw}{d\xi}\left(2w - \frac{1}{\alpha A}\frac{T}{w}\right) = \frac{1}{\alpha A}\left(\frac{sT}{\xi} - \frac{dT}{d\xi}\right) + \frac{Z}{A}W\frac{dW}{d\xi} - \frac{2A-Z}{A}\frac{s}{\xi^2} + \frac{Z^2}{A}\frac{f}{w\tau\xi^2}G(\Delta w) \tag{3.10}$$

The dimensionless quantity f denotes the reduced flux

$$f = \frac{\phi_a}{\phi_\odot}\,\alpha\left(\frac{s}{2}\right)^{5/2}\left(\frac{r_c}{r_\odot}\right)^{3/2-s} \tag{3.11}$$

where ϕ_0 is a convenient flux unit

$$\phi_0 = (16\pi e^4 \ln \Lambda)^{-1} m_p^2 r_\odot^{-7/2} (G M_\odot / 2)^{5/2}$$
$$= 1.3 \times 10^5 \ \mathrm{cm}^{-2} \mathrm{sec}^{-1} \qquad (3.12)$$

and the symbol ΔW stands for essentially the ratio of the velocity difference to the thermal velocity

$$\Delta W = \sqrt{\frac{\alpha A}{\tau(A+1)}} \left(W - w \right) \qquad (3.13)$$

Note that the reduced flux f depends on the critical radius of the proton gas and thus on the temperature in the corona.

Equation (3.10) represents the basic equation for ion transport in the framework of Parker's polytropic model of the proton-electron plasma; the following chapters contain a discussion of the properties of this equation.

4. DISCUSSION OF EQUATION 3.10 FOR LIMITING CASE OF f=0

In this Chapter we shall discuss the limit $f \rightarrow 0$, in which case the collision term in equation (3.10) becomes negligible. (3.10) is reduced to

$$\frac{dw}{d\xi}\left(2w - \frac{1}{\alpha A}\,\frac{\tau}{w}\right) = \frac{1}{A}\,g(\xi) \tag{4.1}$$

with

$$\begin{aligned} g(\xi) &= \frac{1}{\alpha}\left(\frac{\varsigma\tau}{\xi} - \frac{d\tau}{d\xi}\right) + z\,W\frac{dW}{d\xi} - (2A - z)\frac{\varsigma}{\xi^2} \\ &= \frac{dW}{d\xi}\left\{(z+1)W - \frac{\tau}{\alpha W}\right\} - (2A - z - 1)\frac{\varsigma}{\xi^2} \end{aligned} \tag{4.2}$$

$W(\xi)$ and $\tau(\xi)$ are known from the polytropic model, thus g is a known function of ξ, and (4.1) can be integrated. $w(\xi)$ determined in this way includes one free integration constant.

a) Isothermal expansion ($\alpha = 1$).

As an illustration we shall first discuss the isothermal case. Here equation (4.1) can be integrated in closed form:

$$w^2 - \frac{1}{A}\ln w = \frac{z+1}{2A}\,W^2 - \frac{1}{A}\ln W + \frac{2A-z-1}{A}\,\frac{\varsigma}{\xi} + \frac{K}{A} \tag{4.3}$$

K is the integration constant. Generally for a given K, there exist two values of w for each ξ, thus the solution is split into an upper branch w_+ and a lower branch w_-. Physical boundary conditions are determining the choice of K: w should be limited for $\xi \ll 1$, and n_I should go to zero for $\xi \rightarrow \infty$. In order to apply these conditions, the asymptotic behaviour of $w(\xi)$ has to be analyzed. From (4.3) we obtain for the upper and lower branches for $\xi \ll 1$

$$W_+ \propto \xi^{-1/2} \tag{4.4}$$

$$W_- = \xi^{-s} \exp\left\{-(2A - z)\frac{s}{\xi} - K + s - \frac{1}{2}\right\} \tag{4.5}$$

of which only the lower branch is acceptable.
For $\xi \rightarrow \infty$ we obtain

$$W_+ = \sqrt{\frac{s(z+1)}{A} \ln \xi} \tag{4.6}$$

$$W_- \propto \xi^{-s(z+1)} (\ln \xi)^{1/2} \tag{4.7}$$

From $n_I \rightarrow 0$ follows $(w\,\xi^{s}) \rightarrow \infty$, a condition which is fulfilled only by the upper branch. Thus the solution of physical interest has to cross over from the lower branch at $\xi \ll 1$ to the upper at $\xi \rightarrow \infty$ and there exists only one solution, the critical solution, with one specific value of K, passing through the critical point. At this point (ξ_c, w_c) the left and right sides of (4.1) vanish simultaneously, yielding the relations

$$w_c = (2A)^{-1/2} \tag{4.8}$$

$$g(\xi_c) = 0 \tag{4.9}$$

$$\xi_c = \left\{\frac{s(2A - z - 1)}{\frac{dW}{d\xi}\left[(z+1)W - \frac{1}{W}\right]}\right\}^{1/2} \tag{4.10}$$

The critical point for protons falls of course on W(ξ), but it is not identical with the critical point in Parker's theory of the proton-electron gas (W = 1, ξ = 1). There is no contradiction in this result, because the only solution of physical significance of the families of curves defined by (4.3) is the critical solution, and for protons this is identical with W(ξ) as given in (2.13).

We conclude that for isothermal expansion, heavier ions are accelerated to supersonic velocities. In the case of low solar wind flux their velocities remain, however, lower than the solar wind velocity W. From (2.22) and (4.6) we obtain

$$\lim_{\xi \to \infty} \frac{w}{W} = \left(\frac{Z+1}{2A} \right)^{1/2} \tag{4.11}$$

Formally there is no limit for A. However, for $A \to \infty$ and $Z/A \to 0$ the critical point ξ_c goes to infinity,

$$\xi_c \longrightarrow \frac{2A}{Z} \tag{4.12}$$

i.e., it moves to a solar distance where an isothermal approximation becomes meaningless.

b) Polytropic expansion ($\alpha > 1$).

The question to be discussed is whether equation (4.1) has a solution for which w is monotonously rising with ξ. In this case, the left side of (4.1) goes through zero for a critical velocity w_c which is reached at critical point ξ_c:

$$w_c = \left\{ \frac{T(\xi_c)}{2\alpha A} \right\}^{1/2} \tag{4.13}$$

$g(\xi)$ as defined by (4.2) vanishes at ξ_c. In order to establish the existance of a monotonously rising solution $w(\xi)$, it is sufficient to show that $g(\xi)$ is negative for $\xi < \xi_c$, and positive for $\xi > \xi_c$, and that solutions $w(\xi)$ exist which meet the physical boundary conditions. As in the isothermal case we have generally two branches $w_+(\xi)$ and $w_-(\xi)$ for a single integration constant. From (4.2) and the approximations given in Chapter 2 we obtain for $\xi \ll 1$:

$$\begin{aligned} g(\xi) &= -s\, g_0\, \xi^{-2} \\ g_0 &= 2A - Z - (s+1)\frac{\alpha - 1}{\alpha} \end{aligned} \tag{4.14}$$

For (2 A-Z)>>1, g_o is positive and g - as required - negative, as long as

$$\alpha < (s+1)/s \qquad \left(\alpha < \tfrac{3}{2} \text{ for } s=2\right) \tag{4.15}$$

The upper and lower branches of w for $\xi \ll 1$ are:

$$\begin{aligned} w_+ &\propto \xi^{-1} \\ w_- &\propto \xi^{\frac{\alpha}{\alpha-1} g_o} \end{aligned} \tag{4.16}$$

The lower branch meets the boundary condition, because $g_o > 0$.
For $\xi \gg 1$ equation (4.2) gives

$$g(\xi) = s(z+1)\, W_\infty^{1-\alpha}\, \xi^{-s(\alpha-1)-1} - 2As\,\xi^{-2} \quad . \tag{4.17}$$

$g(\xi)$ is positive, again as long as condition (3.15) for α is fulfilled.
The upper and lower branches of $w(\xi)$ for $\xi \gg 1$ are:

$$\begin{aligned} w_+ &= W_\infty - \frac{1}{W_\infty}\,\frac{z+1}{2A}\,\frac{W_\infty^{1-\alpha}}{\alpha-1}\,\xi^{-s(\alpha-1)} \\ w_- &\propto \xi^{-\alpha s(z+1)} \end{aligned} \tag{4.18}$$

Like in the isothermal case the lower branch is not compatible with $n_I \rightarrow 0$, and the upper branch is the physically acceptable solution, as $n_I \rightarrow \xi^{-s(\alpha-1)}$. We conclude that a monotonously rising solution $w(\xi)$ exists as long as $\alpha < (s+1)/s$. This condition is identical with the limit placed on α for polytropic expansion of the proton-electron gas (cf . 2.17).

5. HIGH SOLAR WIND FLUX

If we let the solar wind flux increase, dynamical friction, i.e. the collision term in equation (3.10) forces w closer to W. In the limiting case of very high fluxes ($f \gg 1$), W-w will be small compared to W, and w can be approximated by W in equation (3.10) everywhere but in the collision term:

$$\Delta W = \frac{2A - Z - 1}{Z^2} \frac{F(\xi)}{f} \qquad (5.1)$$

where the function $F(\xi)$ is known from the proton-electron model:

$$F(\xi) = \frac{3\sqrt{\pi}}{2} W \tau \xi^5 \left\{ W \frac{dW}{d\xi} + \frac{5}{\xi^2} \right\} = -\frac{3\sqrt{\pi}}{2(\alpha - 1)} \tau^{\frac{\alpha-2}{\alpha-1}} \frac{d\tau}{d\xi} \qquad (5.2)$$

In this approximation the original differential equation for the ion velocity degenerates into an algebraic equation for which no boundary conditions have to be supplied. Because equation (5.1) is based on the assumption of maxwellian velocity distributions it does not contain a thermal diffusion term. In this respect it deviates from the diffusion equation as given by CHAPMAN and COWLING (1958). The consequences of this deviation will be discussed in Chapter 9.

6. DISCUSSION OF THE GENERAL SOLUTION

The equation of motion (3.10) can be written as:

$$\frac{dw}{d\xi} = \frac{\frac{1}{A}\, g(\xi) + \frac{\xi^2}{A}\, \frac{1}{W\tau\xi^r}\, G(\Delta w)}{2w - \frac{\tau}{\alpha A w}} \tag{6.1}$$

$g(\xi)$ is defined by (4.2), and Δw by (3.13). The singularities of equation (6.1) (simultaneously vanishing nominator and denominator on the right side), determine the topology of the family of solutions. In some cases, one such singularity exists, and the corresponding family of solutions is sketched in Figure 1. In other cases there are three singular points, and the family of solutions has a topology as shown in either Figure 2 or Figure 3. In any case only the one solution passing through the critical point P_c is compatible with the physical boundary conditions.

The critical velocity w_c is given by equation (4.13). Inserting $w = w_c$ in the collision term, ξ_c can be determined. If f is increased P_c moves towards the left along the curve $w = \{\tau(\xi)/2\alpha A\}^{1/2}$. Thus the collision term has the general effect of lowering ξ_c, and bringing w closer to W.

However, even for strong fluxes the velocity of the ions does not necessarily approach the bulk velocity of the solar wind for large solar distances. In the limit $\xi \to \infty$ there are two conditions to be met, if w is to approach W.

1.) For $\xi \to \infty$ the collision term C must dominate the function $g(\xi)$. Using the approximations (2.19), (2.21), and the fact that the function G is limited we obtain

$$C \propto \xi^{-s(2-\alpha)} \tag{6.2}$$

Comparing this with the asymptotic behaviour of $g(\xi)$ as given in (4.17), we obtain

$$\alpha - 1 > \max\left\{\frac{1}{2}\,\frac{s-1}{s}\,,\ \frac{s-2}{s}\right\} \qquad (\alpha > 1.25 \text{ for } s=2) \qquad (6.3)$$

2.) In order to have an effective collision term, C must rise with rising W-w (cf. ~~3.9 and~~ 3.5). This is assured only if W-w approaches zero faster than $\tau^{1/2}$, in order that Δw (cf. 3.13) remains smaller than unity.

We can study the consequences of this condition by solving the equation of motion (3.10) for large solar distances. If we assume an asymptotic behaviour $W - w < \tau^{1/2}$, and introduce

$$\delta = W - w \qquad (6.4)$$

we obtain (cf. 5.1, 5.2 ~~and 5.3~~)

$$\frac{d\delta}{d\xi} = \frac{z^2}{A}\,\frac{1}{2W^2\tau\,\xi^s}\left\{\frac{2A - z - 1}{z^2}\,F(\xi) - f\,G(\Delta w)\right\} \qquad (6.5)$$

For $\xi \gg 1$ and $\Delta w \propto \delta/\tau^{1/2} \ll 1$ (6.5) is approximated by (cf. 2.21)

$$\frac{d\delta}{d\xi} = -\frac{a}{\xi^{n+1}}\,\delta + \frac{b}{\xi^{m+1}} \qquad (6.6)$$

a and b are positive constants, and

$$\begin{aligned} n &= \frac{s}{2}\,(s - 3\alpha) - 1 \\ m &= s\,(\alpha - 1) \end{aligned} \qquad (6.7)$$

It is easily shown that this first order linear equation has a solution with

$$\lim_{\xi \to \infty} \delta = 0 \tag{6.8}$$

only when $n < 0$. From (6.7) follows

$$\alpha > 1 + \frac{2}{3}\,\frac{s-1}{s} \qquad (\alpha > 1.33 \text{ for } s = 2) \tag{6.9}$$

It can be shown that condition (6.9) is applicable not only for PARKER's (1963) solar wind model but also for the two-fluid model of STURROCK and HARTLE (1966), if α is replaced by α_p, the effective polytropic index of the protons. According to (6.9) the natural increase of α (respectively α_p) with larger solar distances (cf. PARKER, 1963; STURROCK and HARTLE, 1966), will force heavier ions to travel with the velocity W of the proton-electron gas, unless somewhere at intermediate distances the velocity deficit W-w was large enough to make $\Delta W \gtrsim 1$, in which case the collision term became ineffective (cf. 3.5).

7. ESTIMATE OF "MINIMUM FLUXES"

In this Chapter we shall discuss the general conditions under which heavier ions will travel with the bulk velocity in the solar wind. It is assumed that α increases gradually from a value 'ose to 1 near the sun until it approaches a value around 1.4 to 1.5 (cf. STURROCK and HARTLE, 1966). At a certain solar distance ξ_a the value $\alpha = 4/3$ (6.9) will be surpassed. If, because of insufficient flux strength, W-w has become so large at ξ_a that $\Delta w > 1$, then the collision term becomes ineffective beyond ξ_a, and the ion velocity w will not converge towards the bulk velocity W. A minimum flux f_{min} sufficient to bring the velocity of a given ion species up to the solar wind velocity can be calculated by numerical integration of equation (3.10). The actual values of f_{min} will of course depend on the details of the chosen solar wind model, and in particular the choice of s will greatly affect the minimum flux f_{min}. However, the relative magnitudes of f_{min} for different ion species will depend much less on such details.

As shown above an approximate condition for obtaining ions travelling with W is

$$\Delta w(\xi_a) \lesssim 1 \tag{7.1}$$

If we use the high flux approximation (5.1), we obtain

$$f_{min} \simeq \frac{2A - Z - 1}{Z^2} F(\xi_a) \tag{7.2}$$

Because $F(\xi_a)$ is model dependent, equation (7.2) does not readily yield absolute values for minimum fluxes. However, (7.2) should be a valid approximation if used for calculating <u>ratios</u> of minimum fluxes of different ion species:

$$f_{min} \propto \Gamma^{(A)}(Z, A) = \frac{2A - Z - 1}{Z^2} \tag{7.3}$$

In the lower corona the times for an atom to approach equilibrium ionization are short compared to the characteristic times of bulk motion (JOKIPII, 1965). Hence it can be assumed that all ions have approximately attained an equilibrium ionization corresponding to the temperature in the lower corona. For a number of elements and isotopes, these degrees of ionization are given in Table 1 assuming $T_o = 10^6\ {}^oK$. They were calculated by the method of ELWERT (1952) (cf. BILLINGS, 1966) using the ionization potentials given by LOTZ (1967). Also given is the factor $\Gamma^{(1)}(Z, A)$ which in the approximation discussed here determines the minimum flux. According to Table 1 isotopic discrimination in solar wind acceleration can be appreciable for light elements. Particularly the He^3/He^4 ratio could be changed in favour of He^3, because the $\Gamma^{(1)}$ values of these two isotopes differ by a factor of 1.7. The important and perhaps somewhat surprising conclusion is that if medium and heavy elements are present in the lower corona, they are accelerated to solar wind velocity, as long as He^4 is accelerated. With the exception of deuterium, which should not exist in the sun anyway, He^4 is no more likely to be accelerated to solar wind velocity than all the stable nuclear species up to and beyond iron. We have not exactly calculated $\Gamma^{(1)}$ for heavy elements, because LOTZ's (1967) table of ionization potentials ends with zinc. However, estimates based on the Thomas-Fermi model indicate that $\Gamma^{(1)}$ for very heavy elements will not be higher than 1.25, the value for helium.

If, as asserted in this Chapter, a minimum proton flux is necessary to carry an ion species in the solar wind, one would expect to observe a correlation between the He/H ratio in the solar wind and the flux. Such a correlation has not been found (ROBBINS et al., 1969) in the Los Alamos Vela 3 data (cf. HUNDHAUSEN et al., 1967). It should be remembered, however, that the minimum flux is dependent on a number of parameters such as temperature in the corona, effective radial expansion index s, or polytropic index α . Moreover, the model presented here is stationary, and non-stationary processes are likely to affect the He/H ratio. The He abundance in the solar wind may in part also reflect the variable influence of thermal diffusion between photosphere and coronal base. All these effects would contribute in masking a correlation between abundance of one ion species and proton flux. However, these effects should cancel out to a large extend when the behaviour of different ion species is compared.

8. RESULTS OF NUMERICAL INTEGRATION

Numerical integrations of equation (3.10) were performed for $s = 2$ and uniform polytropic index α, choosing $\alpha = 1.1$ and 1.4. Actually, for quiet solar conditions the effective value of α will vary from about 1.1 near the sun to about 1.4 to 1.5 at larger solar distances (cf. the α_p values for protons in the two-fluid model of STURROCK and HARTLE, 1966, 1968). Thus, our results with $\alpha = 1.1$ should give a realistic picture of the physical situation near the sun, while the solutions with $\alpha = 1.4$ should be a better approximation for larger solar distances.

In Figure 4 the flux values for $^{4}He^{2+}$, $^{20}Ne^{8+}$ and $^{56}Fe^{14+}$ and $\alpha = 1.4$ are plotted versus the corresponding critical radii ξ_c. The significance of this plot will be explained using neon as an example. For $f = 0$ the solutions have the topology of Figure 1, and ξ_c is very high (3450). With increasing f the critical point moves continuously towards the sun until the flux f reaches a value of 3.4. For $3.4 < f < 4.8$ three singular points exist for a given f. If f is increased above 3.4 the topology of the solutions is initially of the type shown in Figure 2, i.e. the critical point remains at very high solar distances ξ. Somewhere between $f = 3.4$ and $f = 4.8$ the topology changes to that given in Figure 3. Plots of the direction field of the differential equation and the numerical integrations show that this always happens for a flux which is only slightly above the relative minimum in the $f - \xi_c$ plot. Thus the critical point ξ_c jumps discontinuously from a very high value to a place slightly to the left of the relative minimum in the $f - \xi_c$ plot, in the case of neon from $\xi_c > 200$ down to $\xi_c \simeq 2$. Figure 4 clearly demonstrates the idea of a minimum flux for a given ion species. Proton fluxes leading to a ξ_c on the right side of the discontinuity are not able to carry the ions in the solar wind stream. If the flux is high enough to place the critical point to the left of the discontinuity (left of the relative minimum in the $f - \xi_c$ plot) the collision term in equation (3.10) becomes effective and pulls the ion velocity w towards the proton velocity W. The flux value at the relative minimum in the $f - \xi_c$ plot is therefore a good approximation for the minimum flux f_{min} defined in Chapter 7. This is born out by the numerical integrations. Thus for $\alpha = 1.4$ minimum fluxes can be read off in good approximation directly from the relative minimum in the f - ξ_c plot (Figure 4). When the proton flux is below this minimum value, the corresponding ions are virtually not accelerated. If the flux reaches the minimum value the ions are suddenly "locked-in" with the protons.

The behaviour of the ions described here is related to the runaway phenomenon of electrons in a strong electrical field (DREICER, 1958). As discussed by TUCK (1960), ALLIS (1960) and BURGERS (1960) runaway of electrons occurs when dynamical friction decreases with increasing drift velocity.

Figure 5 shows results of the numerical integration for $^4He^{2+}$ and $^{56}Fe^{14+}$. It is seen that the minimum flux for helium in this case is about 6.7 whereas for iron it is about 3.8. Thus, for the $\alpha = 1.4$ expansion model the minimum flux for helium is 1.8 times higher than that for iron. Approximation (7.3) (cf. Table 1) gives 2.5 for the ratio of minimum fluxes for these two ion species, in fair enough agreement with the numerical result. It should be reemphasized that we do not attach great significance to the <u>absolute</u> values for minimum fluxes given here, because they are model dependent. Nevertheless, it is rewarding that physically reasonable numbers are obtained (cf. Caption of figure 5).

Figure 6 gives the ion velocity w as a function of solar distance ξ for a few flux values with $\alpha = 1.1$. Here, of course, critical fluxes cannot be established, because α does not rise above the value 1.33 which is required if ion velocities are to converge towards the proton-electron velocity (6.9). Still it is seen that for a given flux the velocity of iron is much closer than the helium velocity to the proton-electron velocity. Roughly identical $w(\xi)$ are obtained if the flux is 2.5 times higher in the case of helium than in the case of iron. This is in good agreement with equation (5.1).

As pointed out above, $\alpha = 1.1$ should be a good approximation for the lower corona. For $T_c = 10^6$ °C, the solar rim is at $\xi = 0.2$. It is seen that near the sun ion velocities can be substantially lower than proton velocities. For instance for $f = 5$ the iron velocity near the sun is more than 10 times lower than the proton velocity. If α were to rise from 1.1 to about 1.4 in the range

$0.4 \lesssim \xi \lesssim 0.8$, at this flux the iron velocity would still converge towards the solar wind bulk velocity. In this case we would have an enhancement of a factor of 5 to 10 in the number density of iron in the corona as compared to the iron flux abundance in the solar wind at 1 a.u.

9. INFLUENCE OF NON-MAXWELLIAN DISTRIBUTION FUNCTIONS

The equation of motion for ions (3. 3) is derived from the Boltzmann equation under the assumption that the velocity distribution functions are locally maxwellian, which is of course an approximation to the true distribution function. Better approximations of $f(\vec{x}, \vec{V}, t)$ have to be calculated and then used in the pressure and collision terms of (3. 3) in order to arrive at an equation of motion of higher precision. If the mean free path λ is short compared to the characteristic dimension L of the system (L is the length over which an appreciable change of say the temperature T(r) occurs, $L \sim T/\frac{dT}{dr}$), then the plasma is in an approximate local equilibrium, i. e. the distribution functions of the plasma components approximately satisfy Maxwell's law with one and the same temperature and a unique bulk velocity. Of course in this approximation the dynamical friction vanishes. Short λ implies a strong flux, and our high flux expression (5.1) for the difference of the bulk velocities of protons and ions represents a statement about the deviation from local equilibrium. The order of magnitude of the velocity difference given by (5. 1) is $V_{th} \sim \lambda/L$ as to be expected. This small effect is not necessarily exactly described by our simple method, which is based on the assumption that the velocity distribution of the individual plasma components satisfy Maxwell's law. In order to determine the velocity difference correctly to first order in λ/L we have to account for the corrections to the velocity distribution which arise in first order. These modifications may be worked out by expanding the Boltzmann equation in powers of λ/L according to the Enskog-Chapman method given in detail by CHAPMAN and COWLING (1958). Using the general equation of these authors (8.4.7) we arrive at the following expression for the first order difference between the bulk velocities of protons and ions ("diffusion equation"):

$$\Delta W = \frac{1}{z^2}\left\{ 2A - z - 1 - \frac{\alpha - 1}{\alpha}\alpha_T \right\} \frac{F(\xi)}{f} \qquad (9.1)$$

The thermal diffusion factor α_T (CHAPMAN, 1958) can be approximated by

$$\alpha_T \approx - \frac{15}{8} Z^2 \left(\frac{2A}{A+1}\right)^{1/2} \left(1 - \frac{1}{A}\right) \quad , Z \geq 2 \qquad (9.2)$$

It is seen that our high flux approximation (5. 1) is equivalent to the Chapman-Cowling diffusion equation, except for the thermal diffusion term. This correction results from the deviations from maxwellian velocity distributions which were neglected in the derivation of (3. 10) and (5. 1).

In Chapter 7 we have used the high flux approximation (5. 1) in order to arrive at an expression for the minimum flux f_{min} (7. 3). Using the improved high flux approximation (9. 1) in the same way we obtain

$$f_{min} \propto \Gamma^{(2)}(Z,A) = \frac{2A-Z-1}{Z^2} + \frac{\alpha-1}{\alpha} \frac{15}{8} \left(\frac{2A}{A+1}\right)^{1/2} \left(1 - \frac{1}{A}\right) \qquad (9.3)$$

In as much as (9. 1) is a better approximation than (5. 1) to the first order difference between the proton and ion velocities, equation (9. 3) should better than equation (7. 3) approximate the relation between minimum fluxes for different ion species.

$\Gamma^{(2)}(Z, A)$ factors for several types of ions are listed in Table 1. $\alpha = 1.25$ is chosen as a reasonable average. For Z = 1 the exact expression of α_T given by CHAPMAN (1958) has been used. It is seen that the $\Gamma^{(2)}$'s vary less than the $\Gamma^{(1)}$'s, i.e. differences between minimum fluxes for different ions are somewhat smaller when using (9.4) instead of (7. 3). Nevertheless the general conclusions drawn in the preceeding chapters remain valid. Thus He^4 still has the highest minimum flux among all ions up to and beyond krypton, except again for the heavy isotopes of hydrogen. Isotopes have appreciable differences in minimum fluxes. Thus isotopic fractionation has to be expected. In particular the differences between the minimum fluxes of He^3 and He^4 are quite large and therefore considerable variations of the He^3/He^4 ratio might occur in the solar wind.

Clearly for those regions where the solar wind flux is strong in the sense that the mean free path λ is indeed small compared to the characteristic geometrical length L the diffusion equation is the appropriate tool to investigate the behaviour of the ion velocity. An analysis of this equation in the context of ion transport in the lower corona has been given by JOKIPII, (1965), DELACHE (1965, 1967) and NAKADA (1969). Our approach, on the other hand, provides a method to analyze the general case of an arbitrary solar wind flux. The basic assumption of our method, viz. the assumption that the velocity distribution of the individual plasma components may be approximated by a maxwellian distribution must be regarded as a phenomenological parametrization of the real velocity distribution in terms of its center (bulk velocity) its height (particle density) and its width (temperature). Accordingly we expect our method to describe the main features of ion transport only; to improve the validity of our approximation we would have to use a more refined parametrization of the velocity distribution which allows for deviations from Maxwell's law.

10. SUMMARY OF CONCLUSIONS

The model discussed in the preceeding chapters which is restricted to stationary conditions and assumes spherically symmetric flow and radial magnetic field lines near the sun, leads to the following conclusions:

For a species of ions heavier than protons there exists a minimum solar wind flux above which the ion velocity is virtually equal to the proton-electron velocity in the solar wind at large solar distances. A condition for this is, however, that α_p, the effective polytropic index for protons rises above 1.33 with increasing solar distance. During quiet solar conditions α_p will rise beyond this value at some distance from the sun (one to a few solar radii) because of slow heat transfer between electrons and protons (STURROCK and HARTLE, 1966). For stationary proton fluxes below the minimum value ions will normally form a static atmosphere.

The minimum flux for an ion species depends on the temperature in the corona, the radial expansion index s, and the polytropic index α.

If at times α_p should remain below 1.33 throughout the region effective in solar wind ion acceleration, ion velocities would not reach the proton velocity. In this case, the effects of the interplanetary magnetic field moving with the proton-electron gas have to be considered. If this field has an ordered spiral configuration (PARKER, 1963), then ions travelling along the field lines with radial velocities smaller than the proton velocity might occur in interplanetary space.

The critical fluxes for medium heavy elements are smaller than for helium. Thus under stationary conditions these elements should be present in the solar wind as long as helium is present.

In the lower corona ions heavier than protons may travel considerably slower than protons, giving rise to enhanced number densities for these ions, compared to their abundance in the solar wind. In addition there exists the mechanism of an enrichment of heavier ions in the corona by thermal diffusion between photosphere and corona. This mechanism, if not obliterated by convection, would not only further enhance number densities of heavier ions in the corona, but also increase the abundances of these ions in the solar wind flux escaping from the sun. It is important to distinguish these two possible mechanisms for the enrichment of heavier elements in the corona, because only the latter mechanism will deplete heavier elements in the sun. Comparison of abundances in photosphere and corona with those in the solar wind should bring about this distinction and decide, whether the solar wind could have significantly changed the abundances in the outer convective zone of the sun.

ACKNOWLEDGEMENTS

This study was in part carried out while one of the authors (J.G.) was a visitor at the NASA Manned Spacecraft Center at Houston on a Research Associateship granted by the National Research Council of the United States. The hospitality at MSC offered by Wilmot N. Hess, his support of this work and his suggestions are gratefully acknowledged. Robert M. Myers and Melvin G. Arldt of the Computation and Analysis Division at the Manned Spacecraft Center developed the programs for the numerical integration. We would like to thank them very much for their contribution to o[illegible]rk. The numerical calculations were carried out on one of the Univ[illegible]8 Digital Computers at MSC. We acknowledge many discussions we had with R. Bachmann, H. Debrunner, P. Eberhardt and M. Schlaeppi of Physikalisches Institut, University of Berne. This work was in part supported by Swiss National Science Foundation Grants No. 5094.2 and 5107.2.

REFERENCES

ALLIS, W.P.: 1960, Symposium of Plasma Dynamics, Chapter 3 - 4 (Ed. M.H. Mittleman). Addison-Wesley, Reading.

BAME, S.J., HUNDHAUSEN, A.J., ASBRIDGE, J.R., and STRONG, I.B.: 1968, Phys. Rev. Lett. 20, 393.

BILLINGS, D.E.: 1966, A Guide to the Solar Corona, Academic Press, New York, London.

BISWAS, S., and FICHTEL, C.E.: 1964, Astrophys. J. 139, 941.

BURGERS, J.M.: 1960, Symposium of Plasma Dynamics, Chapter 5. Addison-Wesley, Reading.

CHAPMAN, S.C.: 1958, Proc. Phys. Soc. 72, 353.

CHAPMAN, S.C., and COWLING, T.G.: 1958, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press.

CHANDRASEKHAR, S.: 1943, Astrophys. J. 97, 255.

DELACHE, P.: 1965, Comp. Rend. 261, 643.

DELACHE, P.: 1967, Ann. d'Astr. 30, 827.

DREICER, H.: 1958, Proc. 2. U.N. International Conf. on the Peaceful Uses of Atomic Energy, Geneva.

ELWERT, G.: 1952, Z. Naturforschg. 7a, 432.

HARTLE, R.E., and STURROCK, P.A.: 1968, Astrophys. J. 151, 1155.

HUANG, K.: 1963, Statistical Mechanics, John Wiley + Sons, New York.

HUNDHAUSEN, A.J., ASBRIDGE, J.R., BAME, S.J., GILBERT, H.E., and STRONG, I.B.: 1967, J. Geophys. Res. 72, 87.

JOKIPII, J.R.: 1965, California Institute of Technology Thesis; The Solar Wind, R.J. Mackin and M. Neugebauer Ed., Pergamon Press, New York 1966, p. 215.

LOTZ, W.: 1967, J. Opt. Soc. Am. 57, 873.

NAKADA, M.P.: Preprint 1969.

NOBLE, L.M., and SCARF, F.L.: 1963, Astrophys. J. 138, 1169.

OGILVIE, K.W., BURLAGA, L.F., and WILKERSON, T.D.: 1968, Plasma Observations on Explorer 34, Preprint.

OGILVIE, K.W., WILKERSON, T.D.: 1969, Helium Abundance in the Solar Wind, Preprint.

PARKER, E.N.: 1958, Astrophys. J. 128, 664.

PARKER, E.N.: 1962, Planetary Space Sci. 9, 461.

PARKER, E.N.: 1963, Interplanetary Dynamical Processes, Interscience, New York - London.

PARKER, E.N.: 1964, Astrophys. J. 139, 93.

POTTASH, S.R.: 1963, Mon. Not. Roy. Astrom. Soc. 125, 543.

ROBBINS, D.E., HUNDHAUSEN, A.J., and BAME, S.J.: Preprint 1969.

SNYDER, C.W., and NEUGEBAUER, M.: 1964, Proc. Intern. Space Sci. Symp. 4, 89.

SNYDER, C.W., and NEUGEBAUER, M.: 1966, In the Solar Wind, R.J. Mackin and M. Neugebauer Ed., Pergamon Press, New York, p. 3.

SPITZER, L.: 1962, Physics of Fully Ionized Gases, 2nd Ed., Interscience, New York - London.

STURROCK, P.A., and HARTLE, R.E.: 1966, Phys. Rev. Lett. 16, 628.

TRUBNIKOV, B.A.: 1965, Rev. Plasma Physics 1, 105.

TUCK, J.: 1960, Symposium of Plasma Dynamics, Chapter 1 - 4 (Ed. R.F. Post). Addison-Wesley, Reading.

WHANG, Y.C., and CHANG, C.C.: 1965, J. Geophys. Res. 70, 4175.

WOLFE, J.H., SILVA, R.W., McKIBBIN, D.D., and MATSON, R.H.: 1966, J. Geophys. Res. 71, 3329.

TABLE 1

Minimum proton flux factors Γ for various ion species. Degree of ionisation Z is calculated after ELWERT (1952) and LOTZ (1967) (cf. BILLINGS, 1966); for Kr it is estimated. $\Gamma^{(1)}$ is obtained from equation (7.3), and $\Gamma^{(2)}$ from (9.4). It is seen that in both approximations minimum fluxes for heavier ions are not higher than for $^4He^{2+}$.

Nuclide	Degree of Ionisation Z	Minimum Flux Factor $\Gamma^{(1)}$	$\Gamma^{(2)}$ (α=1.25)
H^1	1	0.00	0.00
H^2	1	2.00	2.19
H^3	1	4.00	4.28
He^3	2	0.75	1.06
He^4	2	1.25	1.61
C^{12}	5	0.72	1.19
O^{16}	6	0.69	1.17
Ne^{20}	8	0.48	0.97
Ne^{22}	8	0.55	1.05
Si^{28}	10	0.45	0.95
Ar^{36}	9	0.77	1.28
Fe^{56}	14	0.49	1.01
Kr^{84}	14	0.78	1.30

Figure 1. Topology of solutions of equation of motion for ions (3.10) in the case of one singular point. ξ, W and w are defined by (2.10) and (3.9). The heavy line crossing the critical point (singular point) P_c is the only solution for $w(\xi)$ compatible with the boundary conditions.

Figure 2. Topology of solutions of (3.10) in the case of three singular points. Here the critical point P_c is the singular point on the right, and the only solution $w(\xi)$ compatible with the boundary conditions passes through P_c.

Figure 3. Again three singular points, but now the critical point is on the left. The solutions change from the topology in Figure 2 to that in Figure 3 through increase of proton flux.

Figure 4. Dependency of the position of the critical point ξ_c on the proton flux f (cf. 3.11) for polytropic expansion with $\alpha = 1.4$. As shown in Chapter 8, the flux at the relative minimum in the f - ξ_c plot corresponds approximately to the minimum flux defined in Chapter 7.

Figure 5. Proton velocity $W(\xi)$ and ion velocities $w(\xi)$ for different proton fluxes f and $\alpha = 1.4$. Proton fluxes in protons/cm^2 sec at 1 a.u. ($\xi = 51.5$) are given by $\Phi_{\oplus} = 0.34 \times 10^8 f \text{ cm}^{-2} \text{ sec}^{-1}$ for $T = 10^6$ °K at $\xi = 1$. W, w = 1 corresponds to V, v = 152 km/sec.

Figure 6. Proton velocity $W(\xi)$ and ion velocities $w(\xi)$ for different proton fluxes f and $\alpha = 1.1$. Proton fluxes in protons/cm^2 sec at 1 a.u. ($\xi = 40.5$) are given by $\Phi_{\oplus} = 0.3 \times 10^8 f \text{ cm}^{-2} \text{ sec}^{-1}$ for $T = 10^6$ °K at $\xi = 1$. W, w = 1 corresponds to V, v = 135 km/sec. Under the same conditions the solar rim is at $\xi = 0.19$.

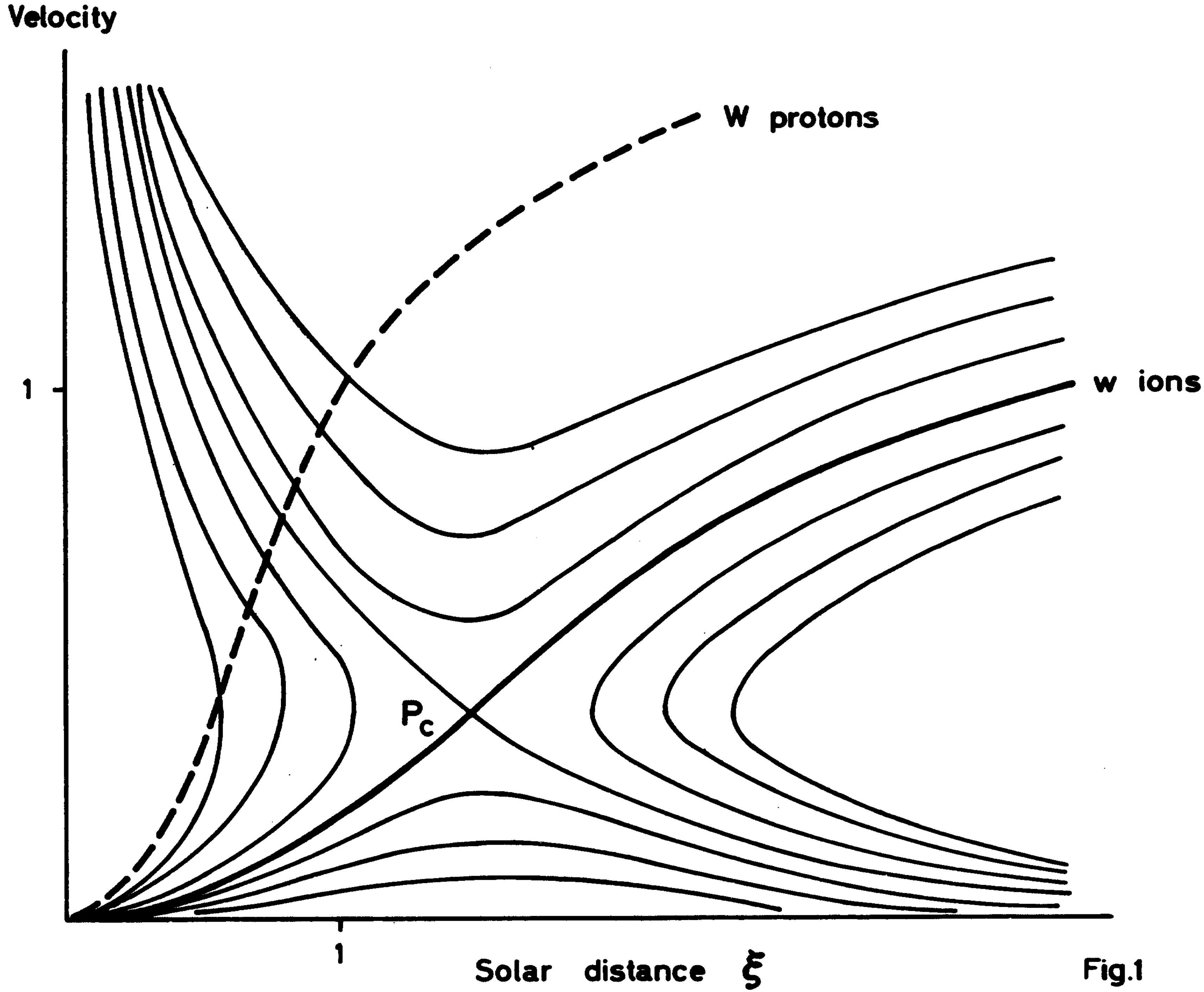


Fig.1

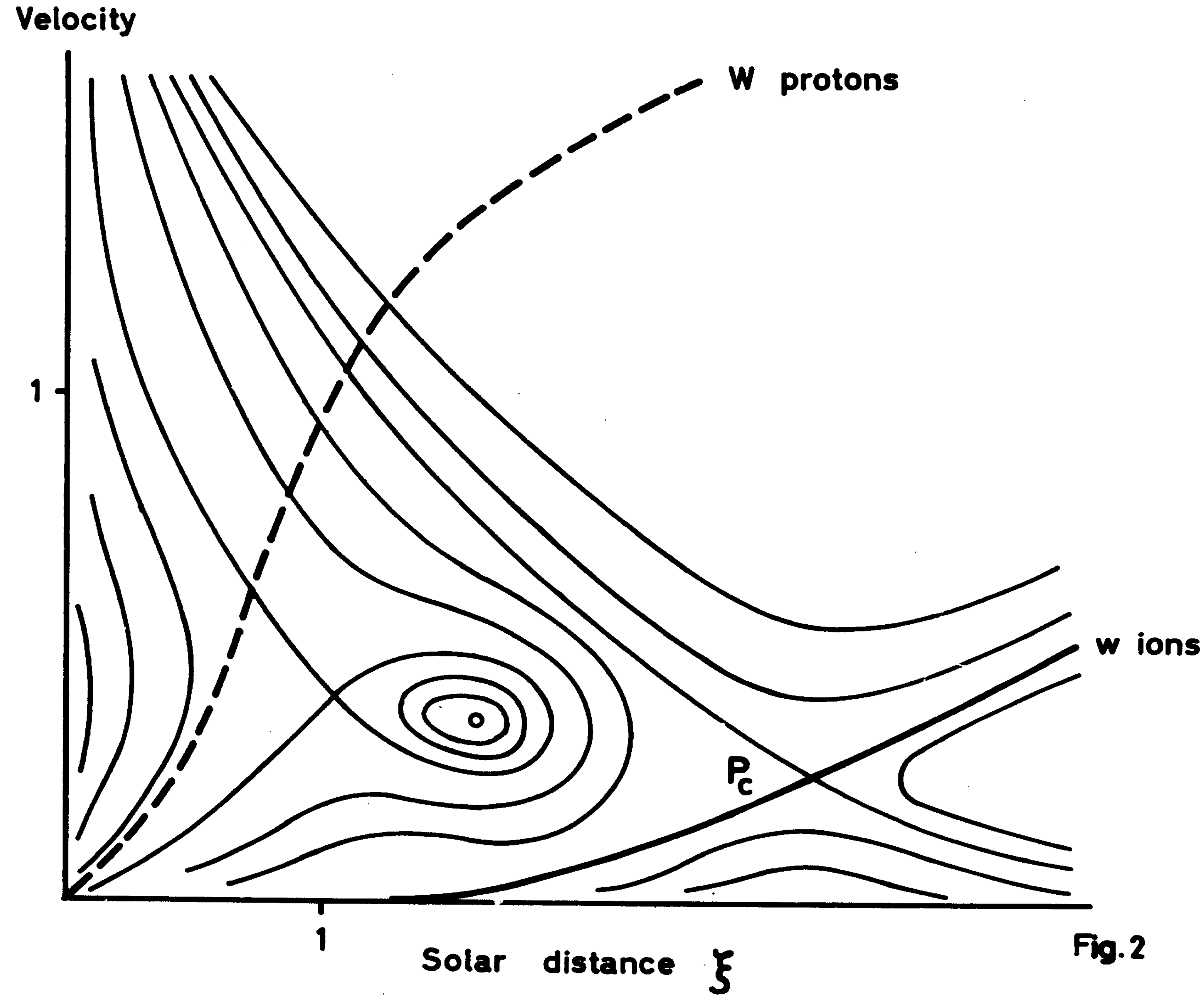


Fig. 2

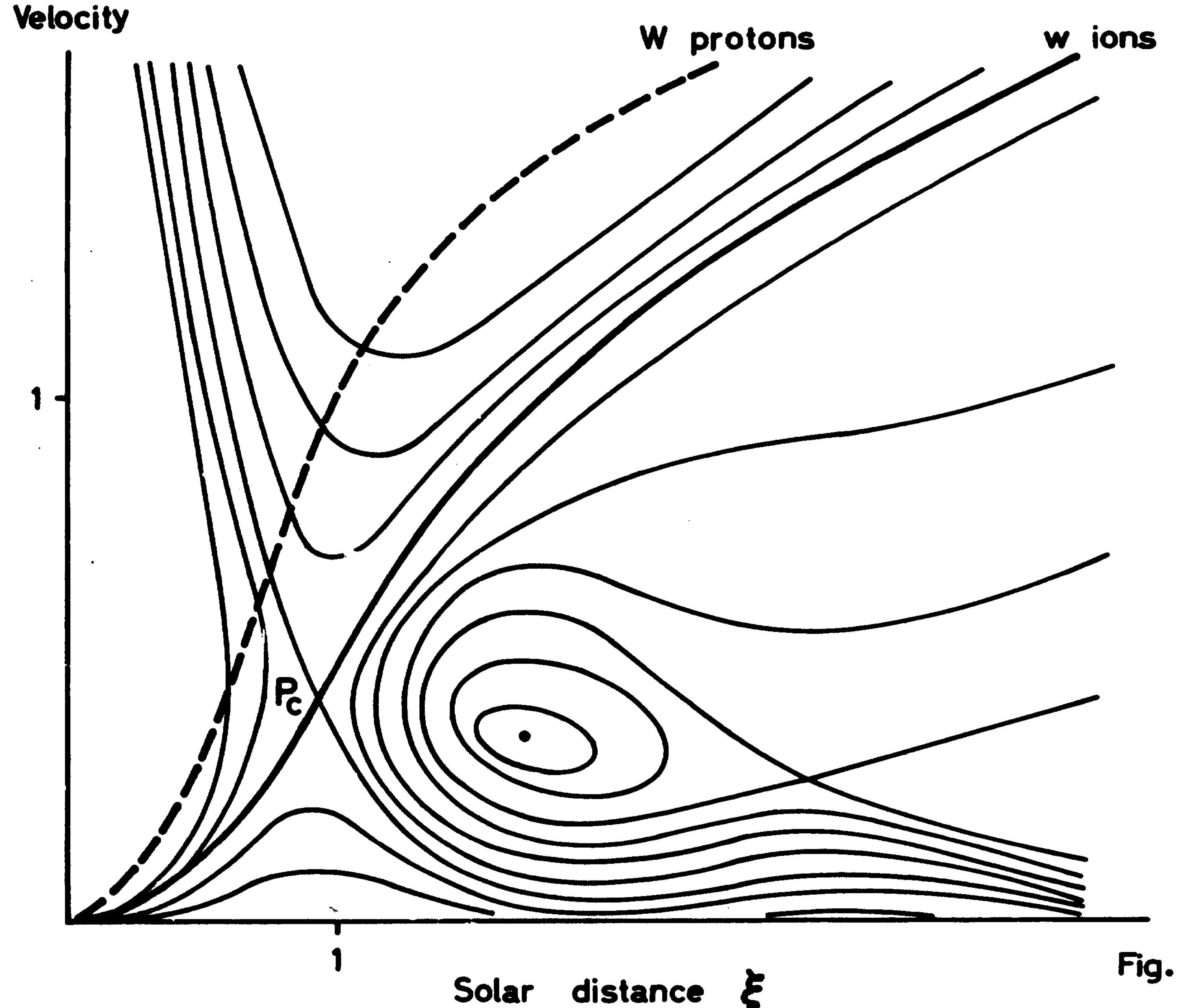


Fig.3

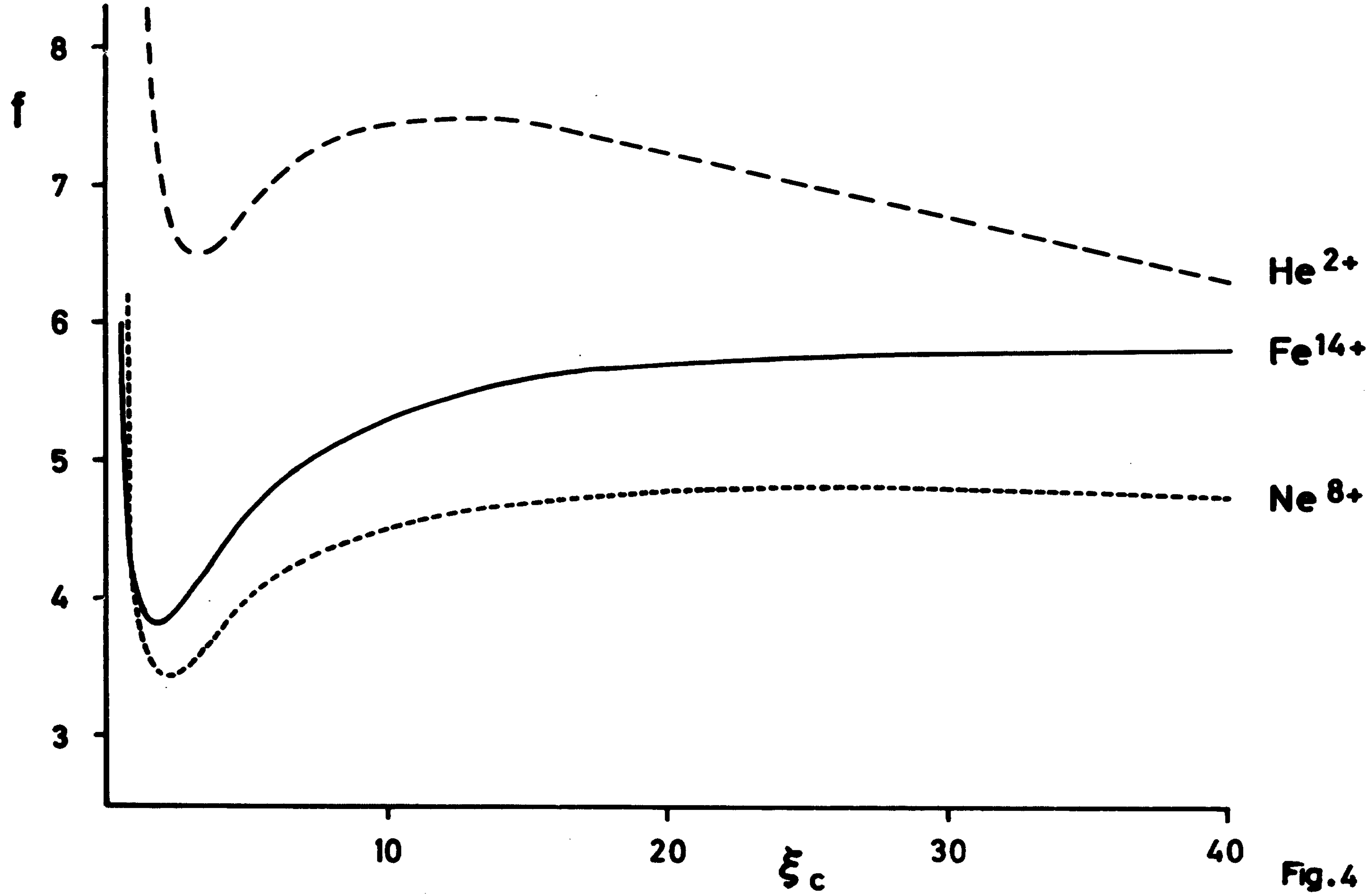
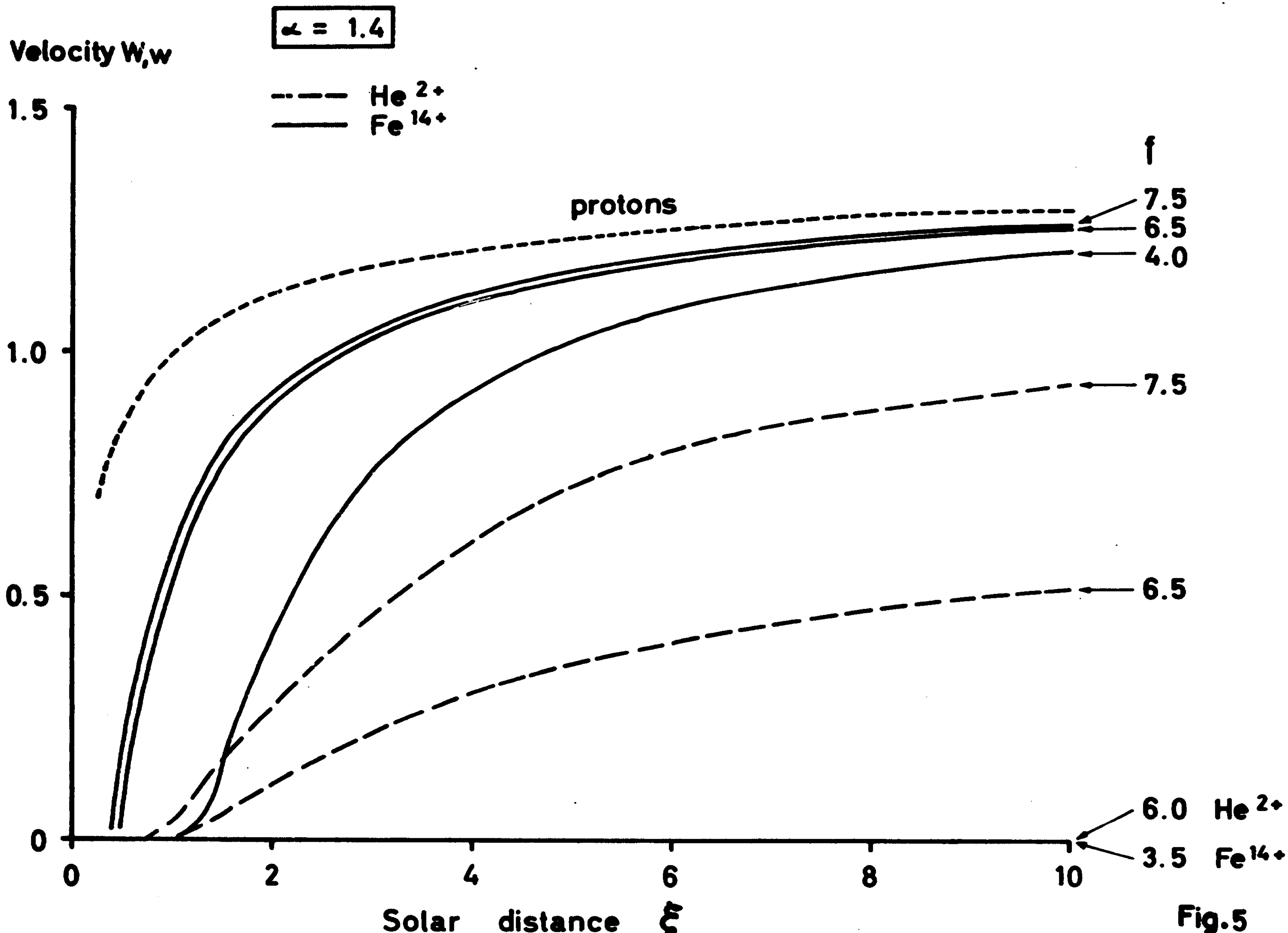
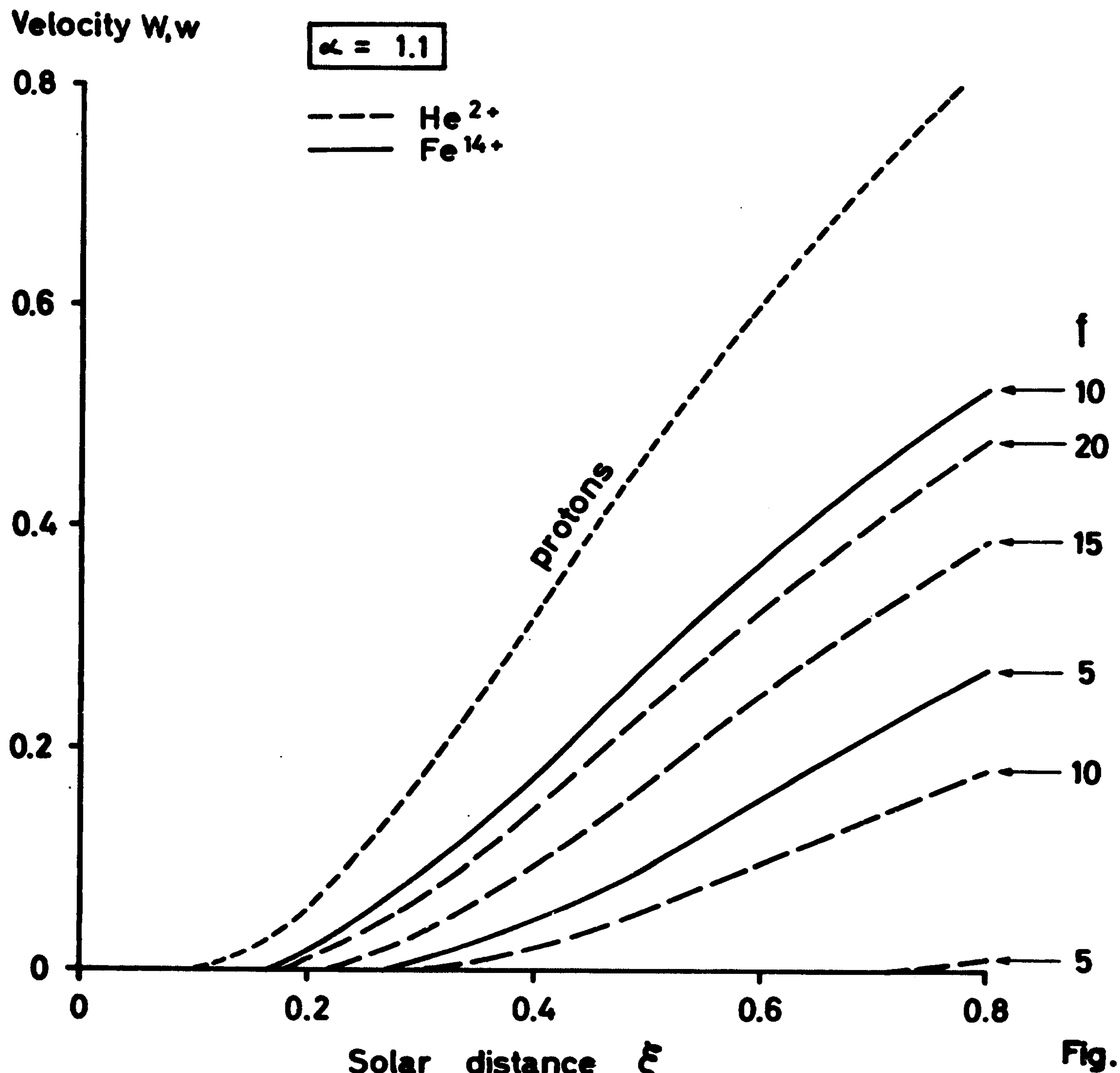


Fig.4



α = 1.4
Velocity W,w
He 2+
Fe 14+
1.5
1.0
0.5
0
protons
f
7.5
6.5
4.0
7.5
6.5
6.0 He 2+
3.5 Fe 14+
0
2
4
6
8
10
Solar distance ξ

Fig.5



Velocity W,w
α = 1.1
He 2+
Fe 14+
protons
f
10
20
15
5
10
5
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
Solar distance ξ

Fig. 6

Appendix

BOLTZMANN EQUATION : SMALL ANGLE SCATTERING OF CHARGED PARTICLES

In this Appendix we briefly sketch the derivation of the transport equation for the bulk velocity used in our discussion of ion transport. We consider a plasma consisting of a number of different components α. The state of the system is described by a set of distribution functions $f_\alpha(\vec{x}, \vec{u}, t)$ satisfying the Boltzmann equation (cf. HUANG, 1963).

$$\left\{\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} + \frac{1}{m_\alpha}\vec{K}_\alpha\cdot\vec{\nabla}_u\right\} f_\alpha(\vec{x},\vec{u},t) = \sum_\beta C_{\alpha\beta}(\vec{x},\vec{u},t) \tag{A. 1}$$

The quantity $C_{\alpha\beta}(\vec{x}, \vec{u}, t)$ represents the collision term for encounters between particles of type α and β and is explicitly given by

$$C_{\alpha\beta}(\vec{x},\vec{u},t) = \int d^3v\, |\vec{u}-\vec{v}|\, d\sigma_{\alpha\beta} \left\{ f_\alpha(\vec{x},\vec{u},t) f_\beta(\vec{x},\vec{v},t) - f_\alpha(\vec{x},\vec{u}',t) f_\beta(\vec{x},\vec{v}',t) \right\} \tag{A. 2}$$

where $d\sigma_{\alpha\beta}$ denotes the differential cross section of the elastic process $\alpha(\vec{u}) + \beta(\vec{v}) \longrightarrow \alpha(\vec{u}') + \beta(\vec{v}')$. The dominating contribution to the differential cross section arises from small angle coulomb scattering, described by the Rutherford formula (c.m. system).

$$d\sigma_{\alpha\beta} = e_\alpha^2\, e_\beta^2 \left(4\mu_{\alpha\beta}^2\, w^4 \sin^4\frac{\theta}{2}\right)^{-1} d\Omega \tag{A. 3}$$

In this expression e_α stands for the charge of the particles of type α, $\mu_{\alpha\beta}$ denotes the reduced mass

$$\mu_{\alpha\beta} = m_\alpha m_\beta / m_\alpha + m_\beta \tag{A. 4}$$

and w is the relative velocity of the two particles, $\vec{w} = \vec{u} - \vec{v}$. If the Rutherford cross section is inserted in (A.2) one obtains an integral that diverges at $\Theta = 0$. As is well-known the singularity of the Rutherford cross section at $\Theta = 0$ reflects the long range nature of the coulomb force valid between isolated particles. In a plasma the long range tail is cut off by shielding effects. One accordingly cuts the integral over the center of mass scattering angle Θ off at a value $\Theta_{min} = \Lambda$. The precise mechanism producing the cut-off is of no relevance here since the collision term depends only logarithmically on the value of the cut-off parameter The order of magnitude is

$$\Lambda \simeq \frac{3}{2e_\alpha e_\beta} \left(\frac{k^3 T^3}{\pi n_e} \right)^{1/2} \tag{A.5}$$

For a more detailed account of the shielding mechanism see e.g. TRUBNIKOV (1965). The values of $\ln\Lambda$ for various temperatures and electron densities are tabulated by SPITZER (1962).

To evaluate the dominating contribution to the collision term which arises from scattering by small angles in the range $\Lambda \leq \Theta \leq 1$ it sufficies to expand the quantity $f_\alpha(\vec{x}, \vec{u}', t)\, f_\beta(\vec{x}, \vec{v}', t)$ appearing in (A. 2) in powers of the velocity transfers $\vec{u}' - \vec{u}$ and $\vec{v}' - \vec{v}$ up to terms of second order. The integral over $d\sigma_{\alpha\beta}$ may then be worked out with the result

$$C_{\alpha\beta}(\vec{x},\vec{u},t) = - g_{\alpha\beta}\, \vec{\nabla}_u \cdot \vec{j}_{\alpha\beta}(\vec{x},\vec{u},t)$$

$$g_{\alpha\beta} = 2\pi \ln\Lambda\, e_\alpha^2 e_\beta^2 / m_\alpha \tag{A.6}$$

$$j^i_{\alpha\beta}(\vec{x},\vec{u},t) = \int \frac{d^3v}{w^3} \left(w^2 \delta^{ik} - w^i w^k \right) \left\{ \frac{1}{m_\beta} f_\alpha(\vec{x},\vec{u},t) \frac{\partial}{\partial v^k} f_\beta(\vec{x},\vec{v},t) - \frac{1}{m_\alpha} \frac{\partial}{\partial u^k} f_\alpha(\vec{x},\vec{u},t) f_\beta(\vec{x},\vec{v},t) \right\}$$

This is the desired approximation for the collision term in the Boltzmann equation.

TRANSPORT EQUATION FOR BULK VELOCITY

The bulk velocity $\vec{v}_\alpha(\vec{x}, t)$ is defined by

$$\vec{v}_\alpha(\vec{x},t) = \frac{1}{n_\alpha(\vec{x},t)} \int d^3u \; \vec{u} \; f_\alpha(\vec{x},\vec{u},t)$$
$$n_\alpha(\vec{x},t) = \int d^3u \; f_\alpha(\vec{x},\vec{u},t) \qquad \text{(A. 7)}$$

The Boltzmann equation leads to the following transport equation for $\vec{v}_\alpha(\vec{x}, t)$:

$$\left\{\frac{\partial}{\partial t} + \vec{v}_\alpha \cdot \vec{\nabla}\right\}\vec{v}_\alpha + \frac{1}{m_\alpha n_\alpha} \vec{\nabla} \cdot \overleftrightarrow{P}_\alpha - \frac{1}{m_\alpha} \vec{K}_\alpha = \frac{1}{n_\alpha} \sum_\beta g_{\alpha\beta} \int d^3u \; \vec{J}_{\alpha\beta} \qquad \text{(A. 8)}$$

where the pressure tensor $\overleftrightarrow{P}_\alpha(\vec{x}, t)$ is defined by

$$P_\alpha^{ik} = m_\alpha \int d^3u \left(u^i - v_\alpha^i\right)\left(u^k - v_\alpha^k\right) f_\alpha(\vec{x},\vec{u},t) \qquad \text{(A. 9)}$$

Making use of partial integrations the collision integral $\int d^3u \; \vec{J}_{\alpha\beta}$ may be rewritten in the form

$$\int d^3u \; \vec{J}_{\alpha\beta} = -\frac{2}{\mu_{\alpha\beta}} \int d^3u \; d^3v \; \frac{\vec{w}}{w^3} \; f_\alpha(\vec{x},\vec{u},t) \, f_\beta(\vec{x},\vec{v},t) \qquad \text{(A. 10)}$$

To proceed with the evaluation of this integral we now <u>assume</u> that the partition functions $f_\alpha(\vec{x}, \vec{u}, t)$ may be approximated by Maxwellian distributions around the bulk velocity $\vec{v}_\alpha(\vec{x}, t)$ characterized by a temperature $T_\alpha(\vec{x}, t)$:

$$f_\alpha(\vec{x},\vec{u},t) = n_\alpha(\vec{x},t)\left(m_\alpha / 2\pi k T_\alpha\right)^{3/2} \exp\left\{-m_\alpha\left(\vec{u}-\vec{v}_\alpha\right)^2 / 2kT_\alpha\right\} \qquad \text{(A. 11)}$$

In this case the pressure tensor takes the well-known form

$$P_\alpha^{ik} = \delta^{ik} n_\alpha k T_\alpha \tag{A. 12}$$

and the collision integral may be evaluated in closed form with the result

$$\int \vec{J}_{\alpha\beta}\, d^3u = \frac{\vec{V}_\beta - \vec{V}_\alpha}{|\vec{V}_\beta - \vec{V}_\alpha|} \cdot \frac{4\, n_\alpha n_\beta}{\mu_{\alpha\beta}\, v_{\alpha\beta}^2}\, G\left\{\frac{|\vec{V}_\beta - \vec{V}_\alpha|}{v_{\alpha\beta}}\right\} \tag{A. 13}$$

where $v_{\alpha\beta}$ denotes a mean thermal velocity

$$v_{\alpha\beta} = \left\{2k\left(\frac{T_\alpha}{m_\alpha} + \frac{T_\beta}{m_\beta}\right)\right\}^{1/2} \tag{A. 14}$$

and the function G(x) is given by

$$G(x) = \frac{\phi(x) - x\,\phi'(x)}{2x^2} \tag{A. 15}$$

in terms of the error function $\phi(x)$.