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THEORY OF SEPARATION OF VARIABLES FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER IN TWO INDEPENDENT VARIABLES

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16. Abstract

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THEORY OF SEPARATION OF VARIABLES FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER IN TWO INDEPENDENT VARIABLES

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SUMMARY

Necessary and sufficient conditions which any linear second-order partial differential equation in two independent variables must meet whenever it can be transformed into a separable equation are given. These conditions are used to develop a calculational procedure for determining whether any given equation of this type can be transformed into a separable equation and also to develop a procedure for determining the various changes of variable which will lead to separable equations.

INTRODUCTION

Perhaps the most useful way of obtaining solutions to linear partial differential equations is the method of separation of variables. Unfortunately this method is only applicable to a small number of equations. However, the applicability of this method can be increased somewhat if the variables in the differential equation are transformed before the method is applied. It is therefore useful to be able to tell whether a given partial differential equation can be transformed into a separable equation (that is, an equation which can be solved by the method of separation of variables) by changing both its dependent and independent variables.

We shall therefore give necessary and sufficient conditions which the coefficients of any second-order linear partial differential equation in two independent variables must satisfy in order that the equation be transformable into a separable equation. These conditions require that the coefficients (or more precisely, certain combinations of the coefficients) be expressible in certain functional forms. Since it may not always be easy to tell in practice simply by inspection whether a given set of coefficients can be expressed in this way, alternate forms of these conditions are given which allow the coefficients of a given equation to be tested by direct calculation. In order to use the procedures developed for this purpose, it is required in the worst situation that the solution to an ordinary differential equation be found.

In addition, if it is found that a given equation can be transformed into a separable equation, then the formulas developed herein can be used to calculate all the possible transformations which will bring the equation into separable form.

Some incomplete or limited studies along these lines have already been carried out. Thus, in references 1 and 2, all the possible conformal transformations which transform the separable equation

$$\nabla^2 U + k^2 U = 0$$

into another separable equation have been enumerated. (∇^2 will always be used herein to denote the two-dimensional laplacian.) In reference 1, k^2 is taken as a constant and, in reference 2, k^2 is taken to be a constant divided by y^2 where y is one of the original independent variables. In reference 3, sufficient conditions for the equation

$$\frac{\partial^2 U}{\partial x \, \partial y} = \lambda U$$

to be transformable into a certain type of separable equation are given. Boussinesq (ref. 4) showed that the equation

$$\nabla^2 U + \nabla\varphi \cdot \nabla U = 0$$

where φ is a given harmonic function, can always be transformed into a separable equation by transforming both the dependent and independent variables, the tranformation of the independent variables being a conformal transformation. A slight generalization of Boussinesq's result is given in reference 5. It was shown there that the potential could be a slightly more general function.

All these results will emerge as special cases of the general theory developed herein. Limited results for special equations in more than two independent variables are given in references 1 and 5 to 9.

We begin by finding the restrictions that are imposed on an equation by the requirement that it be separable. It turns out that it is convenient to distinguish between those equations for which the method of separation of variables leads to two ordinary differential equations of the highest possible order consistent with the type of partial differential equation and those for which it does not. Only the former type of equations are called separable herein. The latter type are called "weakly separable but not separable." However, since the method of separation of variables often leads to useful results even when an equation is weakly separable but not separable (ref. 10), this case is also considered in detail.

The various algebraic transformations which can be applied to a differential equation are discussed. It is then determined what restrictions are imposed on these transformations by the requirement that they transform a given partial differential equation into a separable equation. Once this is done, the restrictions imposed on the coefficients of the original equation can be found. It turns out that this is best accomplished by considering the elliptic, parabolic, and hyperbolic equations separately.

We shall assume that all the functions which are encountered can be differentiated as many times as is necessary. We shall say that the function f of two variables is not equal to zero and write $f \neq 0$ if it takes on the value zero only at discrete points (or, at most, along line segments).

SEPARABLE EQUATIONS

The most general second-order linear and homogeneous partial differential equation in two independent variables has the form

$$\alpha \frac{\partial^2 U}{\partial \xi^2} + 2\beta \frac{\partial^2 U}{\partial \xi \, \partial \eta} + \gamma \frac{\partial^2 U}{\partial \xi^2} + A \frac{\partial U}{\partial \xi} + B \frac{\partial U}{\partial \xi} + CU = 0 \tag{1}$$

where the coefficients α, β, γ, A, B, and C are real functions of ξ and η. We shall suppose that α, β, and γ are not all zero, for this would imply that equation (1) was in reality a first-order equation. We shall also require that α, β, and A (or γ, β, and B) are not all zeros. Otherwise, equation (1) could be essentially an ordinary differential equation.

Equations of this type are further classified by the sign of the discriminant $\beta^2 - \alpha\gamma$. Thus, if

$$\beta^2 - \alpha\gamma > 0 \tag{2}$$

the equation is said to by hyperbolic. If

$$\beta^2 - \alpha\gamma = 0 \tag{3}$$

the equation is said to be parabolic, and if

$$\beta^2 - \alpha\gamma < 0 \tag{4}$$

the equation is said to be elliptic. We shall always assume that the domain of definition of the equation has been restricted in such a way that the sign of the discriminant does not change. This assumption will simplify the following presentation but will not affect its generality. It is well known that the type of boundary value problems which can be solved by an equation of the form (1) depends only upon whether this equation is hyperbolic, parabolic or elliptic.

When the coefficients of equation (1) satisfy certain restrictions, this equation can be solved by the method of separation of variables. This method consists of substituting the trial solution

$$U(\xi, \eta) = \Xi(\xi)H(\eta) \tag{5}$$

into equation (1) to obtain

$$\frac{1}{\Xi}(\alpha\Xi'' + A\Xi') + \frac{1}{H}(\gamma H'' + BH') + C + \frac{1}{\Xi H} 2\beta\Xi' H' = 0 \tag{6}$$

where the primes denote differentiation of the functions with respect to their arguments. Now suppose it is possible after division by $(H'/H)^n(\Xi'/\Xi)^m f$ for some nonzero function f of ξ and η and $n, m = 0$ or 1, to write equation (6) as the sum of two terms, one of which is a function of ξ only and the other a function of η only, for all choices of the functions Ξ and H. Then since ξ and η are independent variables, we can conclude that the trial solution (5) satisfies equation (1) only if each of these terms is equal to a constant (called the separation constant). This, in turn, implies (in view of the assumptions made about the vanishing of the coefficients) that Ξ and H must each satisfy an ordinary differential equation and that, if Ξ and H do satisfy these equations, then equation (5) is indeed a solution of equation (1). Each of these ordinary differential equations is of, at most, second order. Hence, Ξ and H can each involve two arbitrary constants of integration. Since the separation constant is also arbitrary, it is clear that the solution (5) can contain at most four arbitrary constants. Thus, if the separation of variables method works, it will lead to an, at most, four-fold infinite family of solutions to equation (1). If the family of solutions obtained by this method is sufficiently large, it is possible to express any reasonable solutions as a linear combination of members of this family. The family is then said to be complete. If equation (1) is either hyperbolic or elliptic, it is possible in some cases to obtain two second-order ordinary differential equations when the method is applied.

Equation (1) will be called separable only if the method of separation of variables can lead to two ordinary differential equations of the highest possible order consistent with the type of partial differential equation. Thus, if equation (1) is either hyperbolic or elliptic, it is said to be separable only if the method of separation of variables leads to two second-order ordinary differential equations. It will be shown below that in the parabolic case the method can lead to, at most, one first-order and one second-order ordinary differential equation. Therefore, a parabolic equation will be called separable if the method of separation of variables leads to one first-order and one second-order ordinary differential equation.

The method of separation of variables is sometimes useful (see ref. 10) even when it does not lead to ordinary differential equations of the highest possible order. Therefore, we shall call equation (1) weakly separable if the method leads to two ordinary differential equations regardless of their order. Notice that separation solutions to weakly separable equations involve at least two arbitrary constants and that every separable equation is weakly separable.

In order that equation (1) be weakly separable, it is first necessary that, for arbitrary Ξ and H, equation (6) can be written as the sum of two terms one of which is a function of ξ only and the other a function of η only. It is clear that this cannot occur unless at least one of the coefficients α, β, or γ is equal to zero.

First, suppose that equation (1) is elliptic, then condition (4) implies that $\alpha \neq 0$ and $\gamma \neq 0$. This implies that equation (1) is both elliptic and weakly separable only if $\beta = 0$. In addition, the coefficients of Ξ'' and H'' in equation (6) never vanish and therefore the method of separation of variables, if it works, will always lead to two second-order ordinary differential equations. Thus, an elliptic equation is separable if, and only if, it is weakly separable.

Next, suppose that equation (1) is parabolic. Condition (3) shows that, if $\beta \neq 0$, then $\alpha \neq 0$ and $\gamma \neq 0$. Hence, when equation (1) is parabolic and weakly separable, we can conclude that (since one of these coefficients must be zero) $\beta = 0$. But if $\beta = 0$, condition (3) shows that either $\alpha = 0$ or $\gamma = 0$. Thus, one of the second derivatives must be missing from equation (6). This proves that in the parabolic case the method of separation of variables can lead, at most, to one first-order and one second-order ordinary differential equation. If equation (1) were weakly separable but not separable, the preceding remarks show that we would have $\alpha = \beta = \gamma = 0$. But this is contrary to hypothesis. Hence, we conclude that a parabolic equation is separable if, and only if, it is weakly separable.

Finally, suppose that equation (1) is hyperbolic. In this case, equation (1) can still be weakly separable even if $\beta \neq 0$. However, if $\beta \neq 0$, then at least one of the coefficients α and γ must be zero if equation (1) is to be weakly separable. Thus, at least one of the second derivatives will not occur in equation (6); and, therefore, the method of separation of variables cannot lead to two second-order ordinary differential equations.

This shows that equation (1) cannot be hyperbolic and separable unless $\beta = 0$. If β were zero, then condition (2) shows that $\alpha \neq 0$ and $\gamma \neq 0$. Hence, if equation (1) is hyperbolic and $\beta = 0$, then it is weakly separable if, and only if, it is separable.

The preceding discussion allows us to arrive at the following conclusions:

(C1) In all cases equation (1) is separable only if $\beta = 0$.

(C2) Equation (1) is weakly separable but not separable only if it is hyperbolic and $\beta \neq 0$.

(C3) If equation (1) is parabolic and separable or if it is weakly separable but not separable, then either $\alpha = 0$ or $\gamma = 0$.

Now suppose that $\beta = 0$. Then equation (6) becomes

$$\frac{1}{\Xi}(\alpha\Xi'' + A\Xi') + \frac{1}{H}(\gamma H'' + BH') + C = 0 \tag{7}$$

It is clear that, for arbitrary Ξ and H, this equation can be written as the sum of two terms, one of which depends on ξ and the other only on η if, and only if, the coefficients α, γ, A, B, and C can be expressed in the following forms:

$$\alpha(\xi, \eta) = f(\xi, \eta)\, d_1(\xi) \tag{8}$$

$$\gamma(\xi, \eta) = f(\xi, \eta)\, e_1(\eta) \tag{9}$$

$$A(\xi, \eta) = f(\xi, \eta)\, d_2(\xi) \tag{10}$$

$$B(\xi, \eta) = f(\xi, \eta)\, e_2(\eta) \tag{11}$$

$$C(\xi, \eta) = f(\xi, \eta)\left[d_3(\xi) + e_3(\eta)\right] \tag{12}$$

where $f \neq 0$, d_1 and d_2 are not both zero, e_1 and e_2 are not both zero, and d_1 and e_1 are not both zero. These restrictions follow from the restrictions placed on the vanishing of the coefficients of equation (1). Thus, the conditions (8) to (12), together with the condition

$$\beta = 0 \tag{13}$$

imply that equation (7) is weakly separable, and conclusion (C2) shows that they also impl that equation (1) is separable.

Conversely, suppose that equation (1) is separable. Then conclusion (C1) shows that condition (13) holds. Thus, equation (7) must be expressible as the sum of two terms one of which depends only on ξ and the other only η. But this implies that conditions (8) to (12) hold. Hence, we arrive at the following conclusion:

(C4) Equation (1) is separable if, and only if, its coefficients satisfy conditions (8) to (13).

There are additional restrictions imposed on the functions d_1 and e_1 by the sign of the discriminant $\beta^2 - \alpha\gamma$ of equation (1). The conditions (8), (9), and (13) show that if equation (1) is separable, then

$$\beta^2 - \alpha\gamma = -f^2 d_1 e_1 \tag{14}$$

Suppose first that equation (1) is hyperbolic. Then equations (2) and (14) show that

$$d_1(\xi)e_1(\eta) < 0$$

at each point (ξ, η) of the domain of equation (1) This shows that the sign of $d_1(\xi)$ is different from the sign of $e_1(\eta)$ at each point. Since ξ and η are independent variables, this, in turn, implies either that

$$d_1 > 0 \quad \text{and} \quad e_1 < 0$$

or that

$$d_1 < 0 \quad \text{and} \quad e_1 > 0$$

However, in view of the symmetry of equation (1) and of conditions (8) to (13), no generality will be lost if we assume that the first of these always holds.

Next suppose that equation (1) is parabolic. Then equations (3) and (14) show that

$$d_1(\xi)e_1(\eta) = 0$$

at each point (ξ, η) of the domain of equation (1). Again since ξ and η are independent variables, this shows that either

$$e_1 = 0$$

or

$$d_1 = 0$$

We have already indicated that both these conditions cannot hold simultaneously. Hence, in view of the symmetry of equation (1) and of conditions (8) to (13), no generality will be lost if we assume

$$e_1 = 0 \quad \text{and} \quad d_1 \neq 0$$

Finally, suppose that equation (1) is elliptic. Then equations (4) and (14) show that

$$d_1(\xi)e_1(\eta) > 0$$

at each point (ξ, η) of the domain of equation (1) This shows that the sign of $d_1(\xi)$ is the same as the sign of $e_1(\eta)$ at each point (ξ, η). Since ξ and η are independent variables, this, in turn, implies either that

$$d_1 > 0 \quad \text{and} \quad e_1 > 0$$

or that

$$d_1 < 0 \quad \text{and} \quad e_1 < 0$$

It can be seen, however, from conditions (8) to (13) that no generality will be lost if we assume that a minus sign has been absorbed into the function f. We therefore assume that the first of these conditions always holds.

We have therefore shown that the functions d_1 and e_1 satisfy the following restrictions:

$$d_1 > 0 \quad \text{and} \quad e_1 < 0 \tag{15}$$

if equation (1) is <u>hyperbolic</u>,

$$d_1 \neq 0 \quad \text{and} \quad e_1 = 0 \tag{16}$$

if equation (1) is <u>parabolic</u>, and

$$d_1 > 0 \quad \text{and} \quad e_1 > 0 \tag{17}$$

if equation (1) is <u>elliptic</u>.

Now suppose that $\beta \neq 0$, $\gamma \neq 0$, and $\alpha = 0$. Then equation (6) becomes

$$\frac{H}{\Xi} A \frac{\Xi'}{H'} + \frac{1}{H'} (\gamma H'' + BH' + CH) + \frac{1}{\Xi} 2\beta\Xi' = 0 \tag{18}$$

Since γ and β are not zero, it is clear that, for arbitrary Ξ and H, this equation can be written as the sum of two terms, one of which depends only on ξ and the other only on η if and only if the coefficients α, β, γ, A, B, and C can be expressed in the following form:

$$\alpha(\xi,\eta) = 0 \tag{19a}$$

$$\beta(\xi,\eta) = f(\xi,\eta)d_1(\xi) \tag{20a}$$

$$\gamma(\xi,\eta) = f(\xi,\eta)e_1(\eta) \tag{21a}$$

$$A(\xi,\eta) = 0 \tag{22a}$$

$$B(\xi,\eta) = f(\xi,\eta)\left[e_2(\eta) + d_2(\xi)\right] \tag{23a}$$

$$C(\xi,\eta) = f(\xi,\eta)e_3(\eta) \tag{24a}$$

where $f \neq 0$ and $d_1 \neq 0$. Similarly, if $\beta \neq 0$, $\alpha \neq 0$, and $\gamma = 0$, the resulting form of equation (6) can, for arbitrary Ξ and H, be written as the sum of two terms with one of them depending only on ξ and the other only on η if and only if the coefficients α, β, γ, A, B, and C can be expressed in the form

$$\alpha(\xi,\eta) = f(\xi,\eta)d_1(\xi) \tag{19b}$$

$$\beta(\xi,\eta) = f(\xi,\eta)e_1(\eta) \tag{20b}$$

$$\gamma(\xi,\eta) = 0 \tag{21b}$$

$$A(\xi,\eta) = f(\xi,\eta)\left[d_2(\xi) + e_2(\eta)\right] \tag{22b}$$

$$B(\xi,\eta) = 0 \tag{23b}$$

$$C(\xi,\eta) = f(\xi,\eta)d_3(\xi) \tag{24b}$$

where $f \neq 0$ and $e_1 \neq 0$. Finally, if $\beta \neq 0$, $\alpha = 0$, and $\gamma = 0$, the resulting form of equation (6) can, for arbitrary Ξ and H, be written as the sum of two terms with one of them depending only on ξ and the other only on η if, and only if, the coefficients α, β, γ, A, B, and C can be expressed either in the form (19a) to (24a) or in the form (19b) to (24b). Thus, either the conditions (19a) to (24a) or the conditions (19b) to (24b) taken together with the condition

$$\beta \neq 0 \tag{25}$$

imply that equation (1) is weakly separable and conclusion (C1) shows that they also imply that equation (1) is not separable.

Conversely, suppose that equation (1) is weakly separable but not separable. Then conclusion (C2) shows that condition (25) holds, and conclusion (C3) shows that either $\alpha = 0$ or $\gamma = 0$. Thus, the appropriate form of equation (6) (depending on whether $\alpha = 0$ or $\beta = 0$) must be expressible as the sum of two terms with one of them depending on ξ only and the other on η only. But this implies that either conditions (19a) to (24a) or conditions (19b) to (24b) hold. This shows that the following conclusion holds:

(C5) Equation (1) is weakly separable but not separable if, and only if, its coefficients satisfy condition (25) and either conditions (19a) to (24a) or conditions (19b) to (24b).

The conditions obtained above show that only a very small percentage of all the second-order linear partial differential equations in two independent variables are even weakly separable. However, a somewhat larger percentage of the second-order linear partial differential equations can be transformed into weakly separable equations by changing either their dependent or independent variables. Hence, the usefulness of the method of separation of variables can be extended by using it in conjunction with a change of variables. We therefore develop a procedure for determining whether a given second-order linear partial differential equation can be transformed (by changing either the dependent or independent variables) into a weakly separable equation. In addition, whenever a given equation can be so transformed, a method for calculating the appropriate change of variables will be given.

ALGEBRAIC TRANSFORMATIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Only a change in the dependent variable which is of the form

$$V(\xi,\eta) = \lambda(\xi,\eta)U(\xi,\eta) \tag{26}$$

where λ is any nonzero function of ξ η, will transform the linear homogeneous differential equation (1) into another linear homogeneous equation. Hence, only transformations of this type are appropriate for our purposes. The allowable transformations of the independent variables are much less restricted. Any change in the independent variables of the form

$$\left.\begin{aligned} x &= x(\xi,\eta) \\ y &= y(\xi,\eta) \end{aligned}\right\} \tag{27}$$

where x and y are any functions of ξ and η such that

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} \neq 0 \tag{28}$$

will transform equation (1) into another linear homogeneous partial differential equation. Hence, all transformations of the independent variable which satisfy condition (28) will be considered.

It is important to notice (refs. 3 and 11) that both a change of variable of the type (26) and one of the type (27) will leave the sign of the discriminant invariant. Thus, any change of variable which is of interest for the present purpose will always transform hyperbolic equations into hyperbolic equations, elliptic equations into elliptic equations, etc. This fact is very useful in the following analysis.

Up to this point it has been convenient to consider any two differential equations which have different coefficients as being completely different equations. We shall sometimes change this point of view slightly and consider two differential equations which can be transformed into one another by a transformation of the type (26) or of the type (27) as being different forms of the same equation.

Recall now that every second-order linear and homogeneous partial differential equation can, by a change of variable of the type (27) be transformed into one and only one of three canonical forms, depending on the sign of the discriminant. Thus, every <u>hyperbolic</u> equation can be put in the form (ref. 11)

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} + cU = 0 \tag{29}$$

Every <u>parabolic</u> equation can be put in the form

$$\frac{\partial^2 U}{\partial x^2} + a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} + cU = 0 \tag{30}$$

and every <u>elliptic</u> equation can be put in the form

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + a\,\frac{\partial U}{\partial x} + b\,\frac{\partial U}{\partial y} + cU = 0 \tag{31}$$

where the coefficients a, b, and c can be any functions of x and y.

Hence, no generality will be lost if we assume that this transformation has already been carried out and therefore that the differential equation under investigation is already in one of these three forms. We also see from the preceding remarks that even though it is necessary to specify four functions (since eq. (1) can always be divided through by one of the coefficients of its highest derivatives which cannot all be zero) in order to characterize any second-order linear homogeneous partial differential equation in two independent variables, at most only three functions (namely, the three coefficients a, b, and c appearing in the canonical form of the equation) need be known to determine whether the equation can be transformed into a weakly separable or a separable equation. We shall see that in fact only two functions need be known for this purpose.

If the coefficient b in equation (30) is zero, this equation is essentially an ordinary differential equation and can be solved as such. We therefore exclude this case by imposing the restriction $b \neq 0$. With this restriction it is easy, though somewhat tedious, to verify that equations (29) to (31) can never be transformed by a change of variable of the type (26) or of the type (27) into an equation whose coefficients violate the restrictions listed directly after equation (1). Since the only equations which occur herein arise as a result of applying transformations of these types to equations of the form (29) to (31), these restrictions will always be met.

It will be proved subsequently that a change of dependent variable of the form (26) applied to any equation in one of the canonical forms (29) to (31) transforms this equation into one which has the same canonical form (only the coefficients a, b, and c are changed). Since the order in which the transformations (26) and (27) are applied to a given equation is immaterial, the combined effect of applying a transformation of the type (26) and one of the type (27) to a given equation can be analyzed as follows. First, determine what conditions must be satisfied by the coefficients of an equation which is in one of the canonical forms (29) to (31) and which in addition can be transformed into a weakly separable or a separable equation by a change of variable of the type (27). Once this is done, it is only necessary to decide which of the differential equations having this canonical form can be transformed by a change of variable of the type (26) into an equation whose coefficients satisfy these conditions in order to determine which of the equations in this canonical form can be transformed by a combined change of variable into a weakly separable or a separable equation.

TRANSFORMATIONS OF THE DEPENDENT VARIABLES

Apropos of these remarks, we now turn to a discussion of how equations of the types (29) to (31) transform under a change of variable of the type (26). This material is entirely equivalent to that given in reference 3. However, it is convenient to rederive some of the results in a form which is more suitable for our purposes.

First, consider the hyperbolic equation (29). Substituting

$$V(x,y) = \lambda(x,y)U(x,y)$$

into this equation yields

$$V_{xx} - V_{yy} + \tilde{a}V_x + \tilde{b}V_y + \tilde{c}V = 0 \tag{32}$$

where

$$\tilde{a} = a - \frac{2}{\lambda}\lambda_x \tag{33}$$

$$\tilde{b} = b + \frac{2}{\lambda}\lambda_y \tag{34}$$

$$\tilde{c} = c - \frac{1}{\lambda}(\lambda_{xx} - \lambda_{yy}) - \frac{\lambda_x}{\lambda}\left(a - \frac{2\lambda_x}{\lambda}\right) - \frac{\lambda_y}{\lambda}\left(b + \frac{2\lambda_y}{\lambda}\right) \tag{35}$$

This proves that, when a transformation of the type (26) is applied to an equation of the type (29), the form of the equation is unaltered. Evidently,

$$\tilde{c} - \frac{1}{4}(\tilde{a}^2 - \tilde{b}^2) = c - \frac{1}{4}(a^2 - b^2) - \frac{1}{\lambda}\left(\lambda_{xx} - \frac{\lambda_x^2}{\lambda}\right) + \left(\frac{\lambda_{yy}}{\lambda} - \frac{\lambda_y^2}{\lambda^2}\right)$$

$$\tilde{a}_x = a_x - 2\left(\frac{\lambda_{xx}}{\lambda} - \frac{\lambda_x^2}{\lambda^2}\right)$$

$$\tilde{b}_y = b_y + 2\left(\frac{\lambda_{yy}}{\lambda} - \frac{\lambda_y^2}{\lambda^2}\right)$$

$$\tilde{a}_y = a_y - 2\left(\frac{1}{\lambda}\lambda_x\right)_y$$

$$\tilde{b}_x = b_x + 2\left(\frac{1}{\lambda}\lambda_y\right)_x$$

Hence,

$$\tilde{c} - \frac{1}{2}(\tilde{a}_x + \tilde{b}_y) - \frac{1}{4}(\tilde{a}^2 - \tilde{b}^2) = c - \frac{1}{2}(a_x + b_y) - \frac{1}{4}(a^2 - b^2) \tag{36}$$

and

$$\tilde{a}_y + \tilde{b}_x = a_y + b_x \tag{37}$$

Now define

$$\mathscr{I}_H = a_y + b_x \tag{38}$$

$$\mathscr{J}_H = c - \frac{1}{2}(a_x + b_y) - \frac{1}{4}(a^2 - b^2) \tag{39}$$

Equations (36) and (39) then show that, unlike the coefficients themselves, the quantities $\mathscr{I}_H$ and $\mathscr{J}_H$ are unaltered when a tranformation of the type (26) is applied to an equation which has the form (29). They will therefore be called the <u>canonical invariants</u> for an equation of the hyperbolic type.

Now consider the parabolic equation (30). Substituting

$$V(x, y) = \lambda(x, y)U(x, y)$$

into this equation yields

$$V_{xx} + \tilde{a}V_x + \tilde{b}V_y + \tilde{c}V = 0 \tag{40}$$

where

$$\tilde{a} = a - \frac{2}{\lambda}\lambda_x \tag{41}$$

$$\tilde{b} = b \tag{42}$$

$$\tilde{c} = c - a\frac{1}{\lambda}\lambda_x - b\frac{1}{\lambda}\lambda_y - \frac{\lambda_{xx}}{\lambda} + \frac{2}{\lambda^2}\lambda_x^2 \tag{43}$$

This proves that when a transformation of the type (26) is applied to an equation of the type (30) the form of the equation is unaltered. Evidently,

$$\tilde{c} - \frac{1}{4}\tilde{a}^2 - \frac{1}{2}\tilde{a}_x = c - \frac{1}{4}a^2 - \frac{1}{2}a_x - b\frac{1}{\lambda}\lambda_y$$

Hence,

$$\left[\frac{1}{\tilde{b}}\left(2\tilde{c} - \frac{1}{2}\tilde{a}^2 - \tilde{a}_x\right)\right]_x - \tilde{a}_y = \left[\frac{1}{b}\left(2c - \frac{1}{2}a^2 - a_x\right)\right]_x - a_y$$

Therefore, it follows from equation (42) that

$$\left(\frac{1}{\tilde{b}}\left\{2\tilde{c} - \frac{1}{2}\left[\tilde{a}^2 - \left(\frac{\tilde{b}_x}{2\tilde{b}}\right)^2\right] - \left(\tilde{a} + \frac{\tilde{b}_x}{2\tilde{b}}\right)_x\right\}\right)_x - \left(\tilde{a} + \frac{\tilde{b}_x}{\tilde{b}}\right)_y$$

$$= \left(\frac{1}{b}\left\{2c - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right] - \left(a + \frac{b_x}{2b}\right)_x\right\}\right)_x - \left(a + \frac{b_x}{b}\right)_y \tag{44}$$

Now define

$$\mathscr{I}_P = b \tag{45}$$

$$\mathscr{J}_P = \frac{\partial}{\partial x}\left(\frac{1}{b}\left\{2c - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right] - \left(a + \frac{b_x}{2b}\right)_x\right\}\right) - \frac{\partial}{\partial y}\left(a + \frac{b_x}{b}\right) \tag{46}$$

Equations (42) and (44) show that the quantities $\mathscr{I}_P$ and $\mathscr{J}_P$ are unaltered when a transformation of the type (26) is applied to an equation which is in the form (30). They shall be called the canonical invariants for the parabolic equations.

Finally, consider the elliptic equation (31). Substituting

$$V(x,y) = \lambda(x,y)U(x,y)$$

into this equation yields

$$V_{xx} + V_{yy} + \tilde{a}V_x + \tilde{b}V_y + \tilde{c}V = 0 \tag{47}$$

where

$$\tilde{a} = a - \frac{2}{\lambda}\lambda_x \tag{48}$$

$$\tilde{b} = b - \frac{2}{\lambda}\lambda_y \tag{49}$$

$$\tilde{c} = c - \frac{1}{\lambda}(\lambda_{xx} + \lambda_{yy}) - \frac{\lambda_x}{\lambda}\left(a - 2\frac{\lambda_x}{\lambda}\right) - \frac{\lambda_y}{\lambda}\left(b - 2\frac{\lambda_y}{\lambda}\right) \tag{50}$$

This proves again that, when a transformation of the type (26) is applied to an equation of the type (31), the form of the equation is unaltered.

As before, it again follows easily from equations (47) to (50) that the quantities

$$\mathscr{I}_E = a_y - b_x \tag{51}$$

$$\mathscr{J}_E = c - \frac{1}{2}(a_x + b_y) - \frac{1}{4}(a^2 + b^2) \tag{52}$$

are unaltered when a transformation of the type (26) is applied to an equation which has the form (31). They will be called the canonical invariants for the equations of the elliptic type.

We have therefore shown that

(C6a) An equation which is in one of the canonical forms (29) to (31) is transformed into an equation of the same form by any change of dependent variable of the type (26).

(C6b) For each of the canonical forms (29) to (31), there are two quantities, called the canonical invariants, which are unaltered when a transformation of the type (26) is applied to an equation which has that canonical form.

TRANSFORMATIONS OF THE INDEPENDENT VARIABLES

Having discussed the effects of transformations of the type (26) on the canonical forms (29) to (31), we now turn to a discussion of the effects of transformations of the type (27) on these canonical forms. To this end notice that the three canonical forms (29) to (31) can be combined into the single equation

$$\frac{\partial^2 U}{\partial x^2} + j\frac{\partial^2 U}{\partial y^2} + a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} + cU = 0 \qquad \text{for } j = -1,\ 0,\ 1 \tag{53}$$

It is convenient to define the linear operator $L^{(j)}$ for $j = -1, 0, 1$ by

$$L^{(j)}(U) \equiv \frac{\partial^2 U}{\partial x^2} + j\frac{\partial^2 U}{\partial y^2} + a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} \qquad \text{for } j = -1,\ 0,\ 1 \tag{54}$$

When the general transformation

$$\left.\begin{aligned} \xi &= \varphi(x,y) \\ \eta &= \psi(x,y) \end{aligned}\right\} \quad \frac{\partial(\varphi,\omega)}{\partial(x,y)} \neq 0 \tag{55}$$

of the type (27) is applied to equation (53), an equation of the form (1) with discriminant $D = \beta^2 - \alpha\gamma$ is obtained. The coefficients of this equation depend only on the coefficients of equation (53) and the functions φ and ψ. In order to determine this dependence, it is only necessary to apply the chain rule and then substitute the results into equation (53). Thus,

$$U_x = U_\xi \varphi_x + U_\eta \psi_x$$

$$U_y = U_\xi \varphi_y + U_\eta \psi_y$$

$$U_{xx} = U_{\xi\xi}\varphi_x^2 + 2U_{\xi\eta}\varphi_x\psi_x + U_{\eta\eta}\psi_x^2 + U_\xi\varphi_{xx} + U_\eta\psi_{xx}$$

$$U_{yy} = U_{\xi\xi}\varphi_y^2 + 2U_{\xi\eta}\varphi_y\psi_y + U_{\eta\eta}\psi_y^2 + U_\xi\varphi_{yy} + U_\eta\psi_{yy}$$

Substituting these results into equation (53) and collecting terms reveals that the coefficients in equation (1) are determined by the following equations:

$$\alpha = \varphi_x^2 + j\varphi_y^2 \tag{56}$$

$$\beta = \varphi_x\psi_x + j\varphi_y\psi_y \tag{57}$$

$$\gamma = \psi_x^2 + j\psi_y^2 \tag{58}$$

$$A = L^{(j)}(\varphi) \tag{59}$$

$$B = L^{(j)}(\psi) \tag{60}$$

$$C = c \tag{61}$$

TRANSFORMATIONS WHICH LEAD TO SEPARABLE EQUATIONS

Equation (1) is separable if and only if its coefficients satisfy conditions (8) to (13). Hence, it follows from equations (56) to (61) that equation (53) can be transformed into a separable equation by a change of variable of the type (55) if, and only if,

$$fd_1 = \varphi_x^2 + j\varphi_y^2 \tag{62}$$

$$0 = \varphi_x\psi_x + j\varphi_y\psi_y \tag{63}$$

$$fe_1 = \psi_x^2 + j\psi_y^2 \tag{64}$$

$$fd_2 = L^{(j)}(\varphi) \tag{65}$$

$$fe_2 = L^{(j)}(\psi) \tag{66}$$

$$f(d_3 + e_3) = c \tag{67}$$

where the d_k for $k = 1, 2, 3$ are functions of $\xi = \varphi(x, y)$ only and the e_k for $k = 1, 2, 3$ are functions of $\eta = \psi(x, y)$ only. It has already been pointed out that the sign of the discriminant of the equation of type (1) which results from the transformation (55) must be the same as the sign of the discriminant of equation (53). Hence, it follows from conditions (15) to (17) that

$$d_1 > 0 \quad \text{and} \quad e_1 < 0 \tag{68}$$

if equation (53) is hyperbolic (i.e., if $j = -1$),

$$d_1 \neq 0 \quad \text{and} \quad e_1 = 0 \tag{69}$$

if equation (53) is parabolic (i.e., if $j = 0$), and

$$d_1 > 0 \quad \text{and} \quad e_1 > 0 \tag{70}$$

if equation (53) is elliptic (i.e., if $j = 1$). These conditions are, in fact, direct consequences of equations (62) to (64).

On the other hand conclusion (C2) shows that equation (1) can be weakly separable but not separable only if it is hyperbolic. Since the type of equation cannot be changed by a transformation of the form (27), we conclude that equation (53) can be transformed into a weakly separable but not separable equation by a transformation of the type (27) only if it is hyperbolic. Now equation (1) is weakly separable but not separable if, and only if, its coefficients satisfy condition (25) and either conditions (19a) to (24a) or conditions (19b) to (24b). In view of the symmetry of these conditions, however, we can assume without loss of generality that only conditions (19a) to (24a) are relevant. Hence, it follows from equations (56) to (61) that equation (53) can be transformed into an equation which is weakly separable but not separable by a change of variable of the type (55) if, and only if, it is hyperbolic and

$$0 = \varphi_x^2 - \varphi_y^2 \tag{71}$$

$$fd_1 = \varphi_x\psi_x - \varphi_y\psi_y \neq 0 \tag{72}$$

$$fe_1 = \psi_x^2 - \psi_y^2 \tag{73}$$

$$0 = L^{(-1)}(\varphi) \tag{74}$$

$$f(e_2 + d_2) = L^{(-1)}(\psi) \tag{75}$$

$$fe_3 = c \tag{76}$$

At this point, it is convenient to consider the three equations (29) to (31) individually. We discuss the hyperbolic equation (29) first.

EQUATIONS OF HYPERBOLIC TYPE ($j = -1$) - SEPARABLE CASE

Functional Form of Invariants

First, suppose that equation (29) can be transformed into a separable equation by a change of independent variable of the type (55). Then the functions φ and ψ must satisfy conditions (62) to (67) with $j = -1$ and $d_1 > 0$ and $e_1 < 0$. It is therefore permissible to introduce the (real) functions

$$\frac{1}{\sqrt{d_1}}$$

and

$$\frac{1}{\sqrt{-e_1}}$$

and to define a (real) function $\tilde{u}$ of ξ only and a (real) function $\tilde{v}$ of η only by

$$\tilde{u}(\xi) = \int \frac{1}{\sqrt{d_1(\xi)}} d\xi \tag{77}$$

$$\tilde{v}(\eta) = \int \frac{1}{\sqrt{-e_1(\eta)}} d\eta \tag{78}$$

Thus, there are functions u and v of x and y such that

$$u(x,y) = \tilde{u}\left[\varphi(x,y)\right] = \tilde{u}(\xi)$$

$$v(x,y) = \tilde{v}\left[\psi(x,y)\right] = \tilde{v}(\eta)$$

We shall suppose that these equations can always be solved for φ and ψ (if necessary the domain of the differential equations can be divided into a series of subdomains in which these equations can be solved) to obtain

$$\xi = \varphi(x,y) = \tilde{\varphi}\left[u(x,y)\right] \tag{79}$$

and

$$\eta = \psi(x,y) = \tilde{\psi}\left[v(x,y)\right] \tag{80}$$

Therefore, it follows from equation (77) that

$$\frac{\partial \varphi}{\partial x} = \frac{d\tilde{\varphi}}{du}\,\frac{\partial u}{\partial x} = \left(\frac{d\tilde{u}}{d\xi}\right)^{-1}\frac{\partial u}{\partial x} = \sqrt{d_1}\,\frac{\partial u}{\partial x} \tag{81}$$

In a similar way, we find

$$\frac{\partial \varphi}{\partial y} = \sqrt{d_1}\,\frac{\partial u}{\partial y} \tag{82}$$

$$\frac{\partial \psi}{\partial x} = \sqrt{-e_1}\,\frac{\partial v}{\partial x} \tag{83}$$

$$\frac{\partial \psi}{\partial y} = \sqrt{-e_1}\,\frac{\partial v}{\partial y} \tag{84}$$

Hence,

$$\begin{aligned}\frac{\partial^2 \varphi}{\partial x^2} &= \sqrt{d_1}\,\frac{\partial^2 u}{\partial x^2} + \frac{d\sqrt{d_1}}{d\xi}\,\sqrt{d_1}\left(\frac{\partial u}{\partial x}\right)^2 \\ &= \sqrt{d_1}\,\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\,d_1'\left(\frac{\partial u}{\partial x}\right)^2 \end{aligned} \tag{85}$$

Similarly,

$$\frac{\partial^2 \varphi}{\partial y^2} = \sqrt{d_1}\,\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\,d_1'\left(\frac{\partial u}{\partial y}\right)^2 \tag{86}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \sqrt{-e_1}\,\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}\,e_1'\left(\frac{\partial v}{\partial x}\right)^2 \tag{87}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \sqrt{-e_1}\,\frac{\partial^2 v}{\partial y^2} - \frac{1}{2}\,e_1'\left(\frac{\partial v}{\partial y}\right)^2 \tag{88}$$

It now follows from definition (54) and equations (81) to (88) that

$$L^{(-1)}(\varphi) = \frac{1}{2} d_1' \left[\left(\frac{\partial u}{\partial x} \right)^2 - \left(\frac{\partial u}{\partial y} \right)^2 \right] + \sqrt{d_1} L^{(-1)}(u) \tag{89}$$

$$L^{(-1)}(\psi) = -\frac{1}{2} e_1' \left[\left(\frac{\partial v}{\partial x} \right)^2 - \left(\frac{\partial v}{\partial y} \right)^2 \right] + \sqrt{-e_1}\, L^{(-1)}(v) \tag{90}$$

Substituting equations (81) to (84), (89), and (90) into conditions (62) to (67) with $j = -1$ yields

$$f = u_x^2 - u_y^2 \tag{91}$$

$$0 = u_x v_x - u_y v_y \tag{92}$$

$$-f = v_x^2 - v_y^2 \tag{93}$$

$$f d_2 = \frac{1}{2} d_1' \left(u_x^2 - u_y^2 \right) + \sqrt{d_1}\, L^{(-1)}(u) \tag{94}$$

$$f e_2 = -\frac{1}{2} e_1' \left(v_x^2 - v_y^2 \right) + \sqrt{-e_1}\, L^{(-1)}(v) \tag{95}$$

$$f(d_3 + e_3) = c \tag{96}$$

Upon substituting equation (91) into equations (94) and (96) and equation (93) into equation (95) we find

$$L^{(-1)}(u) = d_4 \left(u_x^2 - u_y^2 \right) \tag{97}$$

$$L^{(-1)}(v) = -e_4 \left(v_x^2 - v_y^2 \right) \tag{98}$$

$$c = (d_3 + e_3) \left(u_x^2 - u_y^2 \right) \tag{99}$$

where the function d_4 of ξ only and the function e_4 of η only are defined by

$$d_4 \equiv \frac{d_2 - \frac{1}{2} d_1'}{\sqrt{d_1}}$$

$$e_4 \equiv \frac{e_2 - \frac{1}{2} e_1'}{\sqrt{-e_1}}$$

In view of equations (79) and (80) we see that there are functions p_1 and p_2 of u only and functions q_1 and q_2 of v only such that

$$p_1(u) = d_3(\xi) = d_3\left[\tilde{\varphi}(u)\right]$$

$$p_2(u) = d_4\left[\tilde{\varphi}(u)\right]$$

$$q_1(v) = e_3\left[\tilde{\psi}(v)\right]$$

$$q_2(v) = e_4\left[\tilde{\psi}(v)\right]$$

Substituting these into equations (97) to (99) yields

$$L^{(-1)}(u) = \left(u_x^2 - u_y^2\right)p_2(u) \tag{100}$$

$$L^{(-1)}(v) = -\left(v_x^2 - v_y^2\right)q_2(v) \tag{101}$$

$$c = \left[p_1(u) + q_1(v)\right]\left(u_x^2 - u_y^2\right) \tag{102}$$

Equations (91) and (93) show that

$$u_x^2 - u_y^2 = v_y^2 - v_x^2 \tag{103}$$

or multiplying both sides by v_y^2

$$v_y^2 u_x^2 - \left(v_y u_y\right)^2 = v_y^2\left(v_y^2 - v_x^2\right)$$

Eliminating $v_y u_y$ between this equation and equation (92) yields

$$v_y^2 u_x^2 - u_x^2 v_x^2 = v_y^2\left(v_y^2 - v_x^2\right)$$

or

$$u_x^2\left(v_y^2 - v_x^2\right) = v_y^2\left(v_y^2 - v_x^2\right)$$

and since $v_y^2 - v_x^2 = f \neq 0$, this implies

$$u_x^2 = v_y^2$$

Hence,

$$u_x = \pm v_y \tag{104}$$

Substituting this result into equations (92) and (103) shows that

$$v_x = \pm u_y \tag{105}$$

where the plus sign in equation (105) must be associated with the plus sign in equation (104), and the minus sign in equation (105) must be associated with the minus sign in equation (104).

Differentiating equation (104) with respect to x, differentiating equation (105) with respect to y, and subtracting the results yields

$$u_{xx} - u_{yy} = 0 \tag{106}$$

Similarly,

$$v_{xx} - v_{yy} = 0 \tag{107}$$

The most general solutions of these two equations are

$$u = \frac{1}{2} F(x + y) + \frac{1}{2} G(x - y) \tag{108}$$

$$v = \frac{1}{2} \tilde{F}(x + y) + \frac{1}{2} \tilde{G}(x - y) \tag{109}$$

where F and $\tilde{F}$ are any functions of $x + y$, and G and $\tilde{G}$ are any functions of $x - y$. However, upon substituting equations (108) and (109) into equations (104) and (105) we find that

$$F' + G' = \pm(\tilde{F}' - \tilde{G}')$$

and

$$F' - G' = \pm(\tilde{F}' + \tilde{G}')$$

or

$$\tilde{F}' = \pm F'$$

and

$$-\tilde{G}' = \pm G'$$

Hence,

$$v = \pm\left[\frac{1}{2} F(x + y) - \frac{1}{2} G(x - y)\right] \tag{110}$$

Since the general form of conditions (100) to (102) remains unaltered if we replace v by $-v$, no generality will be lost if we assume that only the plus sign holds in equation (110). Hence, we conclude that the functions u and v must have the following form:

$$u = \frac{1}{2} F(\sigma) + \frac{1}{2} G(\tau) \tag{111}$$

$$v = \frac{1}{2} F(\sigma) - \frac{1}{2} G(\tau) \tag{112}$$

where

$$\sigma \equiv x + y \tag{113}$$

$$\tau \equiv x - y \tag{114}$$

and F and G are any nonconstant functions of their arguments. (The functions F and G are nonconstant because $f = \left(u_x^2 - u_y^2\right) = F'G' \neq 0$.)

It follows from definition (54) and equations (100), (101), (103), (106), and (107) that

$$au_x + bu_y = p_2(u)\left(u_x^2 - u_y^2\right)$$

$$av_x + bv_y = q_2(v)\left(u_x^2 - u_y^2\right)$$

Multiplying the first of these equations by v_y and the second by u_y and then subtracting the resulting expressions yields

$$a\left(u_x v_y - v_x u_y\right) = \left(p_2 v_y - q_2 u_y\right)\left(u_x^2 - u_y^2\right)$$

By eliminating v_y and u_y from this equation by using equations (104) and (105) we find that

$$a\left(u_x^2 - u_y^2\right) = \left(p_2 u_x - q_2 v_x\right)\left(u_x^2 - u_y^2\right)$$

or since $f = u_x^2 - u_y^2 \neq 0$

$$a = p_2(u)u_x - q_2(v)v_x \tag{115}$$

In a similar way, we find that

$$b = -\left[p_2(u)u_y - q_2(v)v_y\right] \tag{116}$$

Equations (102) and (111) to (116), in which p_1, p_2, q_1, q_2, F, and G can be any functions of their arguments, now give the most general form that the coefficients of equation (29) can have if this equation is to be transformable into a separable equation by a change of variable of the type (55). For the present purpose, however, it is more convenient to work with the canonical invariants of equation (29) rather than with its coefficients themselves. To this end we differentiate equations (115) and (116) with respect to x and y to obtain

$$a_x = p_2(u)u_{xx} - q_2(v)v_{xx} + p_2'(u)u_x^2 - q_2'(v)v_x^2$$

$$a_y = p_2(u)u_{xy} - q_2(v)v_{xy} + p_2'(u)u_x u_y - q_2'(v)v_x v_y$$

$$b_x = -\left[p_2(u)u_{xy} - q_2(v)v_{xy} + p_2'(u)u_x u_y - q_2'(v)v_x v_y\right]$$

$$b_y = -\left[p_2(u)u_{yy} - q_2(v)v_{yy} + p_2'(u)u_y^2 - q_2'(v)v_y^2\right]$$

These equations together with equations (103), (106), and (107) show that

$$a_y + b_x = 0 \tag{117}$$

and

$$a_x + b_y = \left[p_2'(u) + q_2'(v)\right]\left(u_x^2 - u_y^2\right) \tag{118}$$

Also equations (92), (103), (115), and (116) show that

$$a^2 - b^2 = \left\{\left[p_2(u)\right]^2 - \left[q_2(v)\right]^2\right\}\left(u_x^2 - u_y^2\right) \tag{119}$$

Substituting these results together with equation (102) into the definitions (38) and (39) of the canonical invariants of equation (29) shows that

$$\mathcal{J}_H = 0 \tag{120}$$

and

$$\mathcal{I}_H = \left[\left(p_1 - \frac{1}{2}p_2' - \frac{1}{4}p_2^2\right) + \left(q_1 - \frac{1}{2}q_2' + \frac{1}{4}q_2^2\right)\right]\left(u_x^2 - u_y^2\right) \tag{121}$$

Or upon defining the function Φ of u and the function Ψ of v by

$$\Phi(u) = p_1(u) - \frac{1}{2}p_2'(u) - \frac{1}{4}\left[p_2(u)\right]^2$$

$$\Psi(v) = q_1(v) - \frac{1}{2}q_2'(v) + \frac{1}{4}\left[q_2(v)\right]^2$$

equation (121) becomes

$$\mathcal{I}_H = \left[\Phi(u) + \Psi(v)\right]\left(u_x^2 - u_y^2\right) \tag{122}$$

Thus, equation (29) can be transformed (by changing its independent variables) into a separable equation only if there are functions Φ and Ψ of u and v, respectively, such that its canonical invariants satisfy conditions (120) and (122).

Now suppose that equation (29) can be transformed into a separable equation by changing both its dependent and independent variables. Since the order in which these transformations are performed is immaterial, it must first be possible to transform this equation by a change of variable of the type (26) into an equation whose coefficients satisfy the conditions (102), (115), and (116). But this implies that equation (29) can be transformed into an equation whose canonical invariants satisfy conditions (120) and (122). Since the canonical invariants of equation (26) are unaltered by a transformation of the type (26), we conclude that if equation (29) can be transformed into a separable equation by changing both its dependent and independent variables, then it is necessary that its canonical invariants satisfy conditions (120) and (122).

In order to see that these conditions are also sufficient, suppose that there exist functions Φ and Ψ such that conditions (120) and (122) hold with u and v given by equations (111) and (112) for some functions F and G. Then it follows from definition (38) that condition (120) implies that there exists a function ω such that

$$\left.\begin{aligned} a &= \omega_x \\ b &= -\omega_y \end{aligned}\right\} \tag{123}$$

and therefore definition (39) shows that

$$\mathscr{I}_H = c - \frac{1}{2}(\omega_{xx} - \omega_{yy}) - \frac{1}{4}\left(\omega_x^2 - \omega_y^2\right)$$

It is now easy to see from equations (32) to (35) that in this case the change of variable (ref. 3)

$$V = e^{\omega/2}U \tag{124}$$

transforms equation (29) into the equation

$$V_{xx} - V_{yy} + \mathscr{I}_H V = 0 \tag{125}$$

or, substituting equation (122),

$$V_{xx} - V_{yy} + \left[\Phi(u) + \Psi(v)\right]\left(u_x^2 - u_y^2\right)V = 0 \tag{126}$$

Upon introducing the new independent variables u and v defined by equations (111) to (114) we find that

$$V_{xx} - V_{yy} = (V_{uu} - V_{vv})F'G'$$

and

$$u_x^2 - u_y^2 = F'G' \tag{127}$$

Substituting these results into equation (126) shows (since $F' \neq 0$ and $G' \neq 0$) that

$$V_{uu} - V_{vv} + \left[\Phi(u) + \Psi(v)\right] V = 0 \tag{128}$$

and this equation is certainly separable.

If any functions φ and ψ which satisfy equations (79) and (80) were used as the new independent variables in place of u and v, then instead of equation (128) we would have arrived at the equation

$$\frac{1}{\tilde{u}'(\xi)} \frac{\partial}{\partial \xi}\left(\frac{1}{\tilde{u}'(\xi)} V_\xi\right) - \frac{1}{\tilde{v}'(\eta)} \frac{\partial}{\partial \eta}\left(\frac{1}{\tilde{v}'(\eta)} V_\eta\right) + \left\{\Phi\left[\tilde{u}(\xi)\right] + \Psi\left[\tilde{v}(\eta)\right]\right\} V = 0 \tag{129}$$

where u and v are the solutions of equations (79) and (80), respectively, for u and v in terms of ξ and η, respectively. It is easy to see that this equation is also separable. We have therefore established the following conclusions:

(C7) <u>The canonical hyperbolic differential equation (29) can be transformed into a separable equation by changing both the dependent and independent variables if, and only if, there exist functions Φ and Ψ and nonconstant functions F and G such that the canonical invariants $\mathscr{I}_H$ and $\mathscr{J}_H$ satisfy the following conditions:</u>

$$\mathscr{I}_H = 0 \tag{130}$$

$$\left.\begin{aligned} \mathscr{J}_H &= \left[\Phi(u) + \Psi(v)\right]\left(u_x^2 - u_y^2\right) \\ &\text{where} \\ u &= \frac{1}{2} F(\sigma) + \frac{1}{2} G(\tau) \\ v &= \frac{1}{2} F(\sigma) - \frac{1}{2} G(\tau) \\ \sigma &= x + y \quad \text{and} \quad \tau = x - y \end{aligned}\right\} \tag{131}$$

(C8) If the canonical invariants of equation (29) do satisfy conditions (130) and (131), then this equation can always be transformed into a separable equation by introducing both a new dependent variable defined by

$$V = e^{\omega/2} U$$

where w is determined to within an unimportant constant by

$$a = \omega_x$$

$$b = -\omega_y$$

and new independent variables ξ and η defined by

$$\xi = \tilde{\varphi}(u)$$

$$\eta = \tilde{\psi}(v)$$

where $\tilde{\varphi}$ and $\tilde{\psi}$ are any convenient nonconstant functions (i.e., $\tilde{\varphi}' \neq 0$ and $\tilde{\psi}' \neq 0$) and u and v are determined from condition (131).

Direct Calculational Procedure for Testing $\mathcal{I}_H$

In practice, it may not always be easy to tell simply by inspection whether the invariant $\mathcal{I}_H$ of a given equation can be put in the form (131). Also, there may be several ways in which a given function $\mathcal{I}_H$ can be expressed in the form (131). Each of these ways will lead to a different "coordinate system" in which the equation is separable. For these reasons, it is useful to give an alternative form of condition (131) which supplies the means of testing the invariant $\mathcal{I}_H$ by direct calculation and which, in addition, gives a procedure for calculating all of the functions u and v which determine the new independent variables.

To this end suppose first that $\mathcal{I}_H$ satisfies condition (131). Then equation (127) implies that the first equation (131) can be put in the form

$$\frac{\mathcal{I}_H}{F'G'} = \Phi(u) + \Psi(v) \tag{132}$$

There exist functions Φ and Ψ such that equation (132) holds if, and only if,

$$\frac{\partial^2}{\partial u \, \partial v}\left(\frac{\mathscr{I}_H}{F'G'}\right) = 0 \tag{133}$$

But it follows from the middle two equations (131) that

$$\frac{\partial}{\partial \sigma} = \frac{1}{2} F'(\sigma)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)$$

and

$$\frac{\partial}{\partial \tau} = -\frac{1}{2} G'(\tau)\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)$$

Hence,

$$\frac{\partial}{\partial u} = \frac{1}{F'(\sigma)} \frac{\partial}{\partial \sigma} + \frac{1}{G'(\tau)} \frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial v} = \frac{1}{F'(\sigma)} \frac{\partial}{\partial \sigma} - \frac{1}{G'(\tau)} \frac{\partial}{\partial \tau}$$

Using these in equation (133) shows that

$$\frac{1}{F'} \frac{\partial}{\partial \sigma}\left[\frac{1}{F'} \frac{\partial}{\partial \sigma}\left(\frac{\mathscr{I}_H}{F'G'}\right)\right] = \frac{1}{G'} \frac{\partial}{\partial \tau}\left[\frac{1}{G'} \frac{\partial}{\partial \tau}\left(\frac{\mathscr{I}_H}{F'G'}\right)\right]$$

Hence,

$$\frac{\partial}{\partial \sigma}\left[\frac{1}{F'} \frac{\partial}{\partial \sigma}\left(\frac{\mathscr{I}_H}{F'}\right)\right] = \frac{\partial}{\partial \tau}\left[\frac{1}{G'} \frac{\partial}{\partial \tau}\left(\frac{\mathscr{I}_H}{G'}\right)\right]$$

Upon introducing the function S of σ only and the function T of τ only defined by

$$\left.\begin{aligned} S(\sigma) &\equiv \left[\frac{1}{F'(\sigma)}\right]^2 \\ T(\tau) &\equiv \left[\frac{1}{G'(\tau)}\right]^2 \end{aligned}\right\} \tag{134}$$

this equation becomes

$$2S\left(\mathscr{J}_H\right)_{\sigma\sigma} + 3S'\left(\mathscr{J}_H\right)_{\sigma} + S''\mathscr{J}_H = 2T\left(\mathscr{J}_H\right)_{\tau\tau} + 3T'\left(\mathscr{J}_H\right)_{\tau} + T''\mathscr{J}_H \tag{135}$$

This equation was first introduced by Darboux in a less general context (ref. 3).

Thus, if $\mathscr{J}_H$ satisfies condition (131), then there exist strictly positive functions S of σ only and T of τ only such that $\mathscr{J}_H$ satisfies equation (135). Conversely, if there are strictly positive functions S of σ and T of τ such that $\mathscr{J}_H$ satisfies equation (135), then it is an easy matter to reverse the steps carried out above to show that equation (134) can be used to define nonconstant functions F and G and equation (133) can be used to introduce functions Φ and Ψ such that $\mathscr{J}_H$ satisfies condition (131). Hence, we conclude that

(C9) There exist nonconstant functions F and G and functions Φ and Ψ such that $\mathscr{J}_H$ satisfies condition (131) if, and only if, there exists a strictly positive function S of σ only and a strictly positive function T of τ only such that $\mathscr{J}_H$ satisfies equation (135).

(C10) If strictly positive function S and T can be found such that equation (135) holds, then the functions F and G for which condition (131) is satisfied can be calculated from equation (134).

Thus, if the functions S and T are known, conclusions (C8) and (C10) give a procedure for calculating a change in the independent variables which will transform equation (29) into a separable equation.

Now equation (135) always possesses at least one solution. (Note that equation (135) is satisfied by taking $S = T = 0$.) We shall subsequently develop a procedure which will yield expressions for all the solutions S and T to equation (135) provided that $\mathscr{J}_H$ is not one of two special types of functions. These expressions will involve, at most, two undetermined constants. To determine what restrictions, if any, must be placed on these constants in order that these expressions satisfy equation (135), they must be substituted back into that equation. If after this is done the constants can still be adjusted so that S and T are positive functions, then we can conclude that equation (29) can be transformed into a separable equation, and we can use the expressions for S and T to calculate the new independent variables which will accomplish this.

Since this procedure will not work when $\mathscr{J}_H$ is one of two special types of functions, before establishing this procedure we shall prove that, if $\mathscr{J}_H$ is one of these types of functions, the condition (131) is always satisfied and that the functions F and G can easily be determined.

First, suppose that $\mathscr{J}_H = 0$. Then it is easy to see that condition (131) can always be satisfied by taking

$$\Phi = \Psi = 0$$

and that any nonconstant functions F and G can be used for determining u and v. Next assume that $\mathcal{J}_H \neq 0$. We may then define the function j_H in terms of $\mathcal{J}_H$ by

$$j_H \equiv \frac{1}{\mathcal{J}_H} \frac{\partial}{\partial\sigma}\left(\frac{1}{\mathcal{J}_H} \frac{\partial \mathcal{J}_H}{\partial \tau}\right) \quad \text{for } \mathcal{J}_H \neq 0 \tag*{(1}$$

Suppose there is a constant c_o such that

$$j_H = c_o \tag{137}$$

If $c_o = 0$ then

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{\mathcal{J}_H} \frac{\partial \mathcal{J}_H}{\partial \tau}\right) = 0$$

But this implies that $\mathcal{J}_H$ has the form

$$\mathcal{J}_H = \gamma(\sigma)\lambda(\tau)$$

for sone nonzero functions γ and λ.

It is clear from equation (127) that condition (131) will always be satisfied if we take

$$\Phi = 1$$

$$\Psi = 1$$

and

$$F' = \gamma$$

$$G' = \lambda$$

If $c_o \neq 0$, definition (136) and equation (137) show that

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{\mathcal{J}_H} \frac{\partial \mathcal{J}_H}{\partial \tau}\right) = c_o \mathcal{J}_H$$

But this is Liouville's equation. It is shown in reference 3 that the most general solution to this equation is

$$\mathcal{J}_H = -\frac{2}{c_o} \frac{\gamma'(\sigma)\lambda'(\tau)}{\left[\gamma(\sigma) - \lambda(\tau)\right]^2} \tag{138}$$

for all nonconstant functions γ and λ. Now in view of equation (127), it is clear (if we take $\gamma = F$, $\lambda = G$, and $\Psi(v) = -1/2c_o v^2$) that equation (138) satisfies condition (131). There are many other ways of choosing the functions F and G such that equation (138) can be written in the general form (131) and each of these choices, of course, leads to a different change of variable (coordinate system) which will transform equation (29) into a separable equation. To show this, notice that if γ and λ are bounded below we can always choose these functions in such a way that they are both positive, for in this case there exists a finite number M such that

$$M = \left| \underset{\sigma, \tau \in D}{\mathrm{g\ell b.}} \left\{\gamma(\sigma), \lambda(\tau)\right\} \right|$$

where D is the domain of definition of the differential equation. If we put

$$\gamma_1 = M + \gamma$$

$$\lambda_1 = M + \lambda$$

then

$$\mathcal{J}_H = -\frac{2}{c_o} \frac{\gamma_1'(\sigma)\lambda_1'(\tau)}{\left[\gamma_1(\sigma) - \lambda_1(\tau)\right]^2}$$

and

$$\gamma_1 \geq 0$$

$$\lambda_1 \geq 0$$

A similar argument holds if both γ and λ are bounded above. Suppose that γ and λ are either both bounded above or both bounded below. Then no generality will be lost if we assume that γ and λ are both positive. Hence, if we define the nonconstant functions F and G by

$$
\left.\begin{aligned}
F(\sigma) &\equiv 2 \int_{\gamma(\sigma)}^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}}\, dt \\
G(\tau) &= 2 \int_{\lambda(\tau)}^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}}\, dt
\end{aligned}\right\} \tag{139}
$$

for any suitable positive constants g_2 and g_3, then it is shown in reference 12 (p. 438) that

$$
\gamma(\sigma) = \wp\left[\frac{1}{2} F(\sigma)\right]
$$

$$
\lambda(\tau) = \wp\left[\frac{1}{2} G(\tau)\right]
$$

where $\wp$ is the Weienstrass $\wp$-function (ref. 12). Substituting these equations into equation (138) shows that

$$
\mathcal{J}_H = -\frac{2}{c_o} \frac{\frac{d}{d\sigma}\wp\left(\frac{1}{2} F\right) \frac{d}{d\tau}\wp\left(\frac{1}{2} G\right)}{\left[\wp\left(\frac{1}{2} F\right) - \wp\left(\frac{1}{2} G\right)\right]^2} = -\frac{1}{2c_o} F'G' \frac{\wp'\left(\frac{1}{2} F\right)\wp'\left(\frac{1}{2} G\right)}{\left[\wp\left(\frac{1}{2} F\right) - \wp\left(\frac{1}{2} G\right)\right]^2}
$$

and the results of example 1 in reference 12 (p. 456) whos that this can be written as

$$
\mathcal{J}_H = \frac{F'G'}{2c_o} \left[\wp\left(\frac{1}{2} F + \frac{1}{2} G\right) - \wp\left(\frac{1}{2} F - \frac{1}{2} G\right)\right]
$$

It is now clear from equation (127) that $\mathcal{J}_H$ satisfies condition (131) with $\Phi = \wp/2c_o$ and $\Psi = -\wp/2c_o$. In addition, the functions F and G can be calculated from equation (139). There is usually no difficulty in telling simply by inspection if a given function is a solution to Liouville's equation. Hence, we have shown that

(C11) The condition (121) can always be satisfied if either the invariant $\mathcal{J}_H$ or the function j_H defined by equation (136) is equal to a constant.

Now suppose that $\mathcal{J}_H \neq$ constant, $j_H \neq$ constant and that the functions S of σ only and T of τ only are any two simultaneous solutions of equation (135). Then dividing equation (135) through by $\mathcal{J}_H$ and applying the differential operator

$$\frac{\partial^2}{\partial\sigma\,\partial\tau} - 3j_H\mathcal{J}_H$$

to both sides of the resulting expression, we find (after some manipulation, which is carried out in appendix A) that S and T must satisfy the equation

$$2\frac{\partial k_1}{\partial\sigma}S + 5k_1S' = 2\frac{\partial k_2}{\partial\tau}T + 5k_2T' \tag{140}$$

where we have put

$$\left.\begin{aligned} k_1 &\equiv \mathcal{J}_H^4\frac{\partial}{\partial\sigma}j_H \\ k_2 &\equiv \mathcal{J}_H^4\frac{\partial}{\partial\tau}j_H \end{aligned}\right\} \tag{141}$$

Notice that equation (140) can also be written more compactly as

$$5k_1^{3/5}\frac{\partial}{\partial\sigma}\left(k_1^{2/5}S\right) = 5k_2^{3/5}\frac{\partial}{\partial\tau}\left(k_2^{2/5}T\right) \tag{142}$$

(If in any equation, functions which are negative are raised to fractional powers, it will always be possible to eliminate the fractional exponents by carrying out the indicated differentiations to obtain an equation in which all terms are real. Therefore, no difficulty will be encountered when this more compact notation is used.)

Equation (140) can be used to obtain an equation which involves T only and not S and an equation which involves S only and not T. Actually, this can be done in several ways. We shall discuss one of these ways in detail and also indicate briefly how one of the alternative methods can be carried out.

If k_2 were zero, definition (141) would imply (since by hypotheses $\mathcal{J}_H \neq 0$) that

$$\frac{\partial j_H}{\partial \tau} = 0$$

or equivalently, that there exists a function γ of σ only (which by hypotheses is non-constant) such that

$$j_H = \gamma$$

Therefore, equation (141) would show that

$$k_2 = \mathcal{J}_H^4 \gamma'$$

Substituting this expression together with the condition $k_2 = 0$ into equation (140), however, yields (after some manipulation)

$$2S \frac{4}{\mathcal{J}_H} \frac{\partial \mathcal{J}_H}{\partial \sigma} = 2S \frac{\left(\mathcal{J}_H^4\right)_\sigma}{\mathcal{J}_H^4} = -\frac{2\gamma''S + 5\gamma'S'}{\gamma'}$$

If $S \neq 0$, then

$$\frac{8}{\mathcal{J}_H} \frac{\partial \mathcal{J}_H}{\partial \sigma} = -\frac{\gamma''S + 5\gamma'S'}{\gamma'S}$$

Now the right side is a function of σ only; hence, this expression implies that

$$j_H = \frac{\partial}{\partial \tau}\left(\frac{1}{\mathcal{J}_H} \frac{\partial \mathcal{J}_H}{\partial \sigma}\right) = 0$$

which is contrary to hypothesis. We therefore conclude that $k_1 = 0$ implies $S = 0$. A similar argument shows that $k_1 = 0$ implies $T = 0$.

We have therefore established the following conclusion:

(C12) Whenever $\mathcal{J}_H \neq$ constant and $j_H \neq$ constant, $\left(j_H\right)_\tau = 0$ implies that equation (131) is satisfied only if $S = 0$, and $\left(j_H\right)_\sigma = 0$ implies that equation (131) is satisfied only if $T = 0$.

Since if $S = 0$ there is no need to calculate T and the reverse, we shall now make the additional assumption that $k_1 \neq 0$ and $k_2 \neq 0$. Since $k_1 \neq 0$, we can divide it into both sides of equation (140). Differentiating the result with respect to τ yields

$$2\left(\frac{1}{k_1}\frac{\partial k_1}{\partial \sigma}\right)_\tau S = 2\left(\frac{1}{k_1}\frac{\partial k_2}{\partial \tau}\right)_\tau T + \left\{2\frac{1}{k_1}\left(k_2\right)_\tau + \frac{5}{k_1}\left(k_2\right)_\tau - 5\frac{k_2}{k_1}\left[\frac{1}{k_1}\left(k_1\right)_\tau\right]\right\}T' + 5\frac{k_2}{k_1}T''$$

$$= 2\left[\frac{1}{k_1}\left(k_2\right)_\tau\right]_\tau T + \frac{5}{k_2^{2/5}}\left\{\frac{\partial}{\partial \tau}\left[\left(\frac{k_2}{k_1}\right)k_2^{2/5}\right]\right\}T' + 5\frac{k_2}{k_1}T'' \tag{143}$$

If $\left[1/k_1\left(k_1\right)_\sigma\right]_\tau = 0$, equation (143) is essentially a second-order ordinary differential equation for T. (Notice, however, that the coefficients in this equation are, in general, functions of the two variables σ and τ and not just of τ as is ordinarily the case.) If $\left[1/k_1\left(k_1\right)_\sigma\right]_\tau \neq 0$, we can divide it into both sides of equation (143) to obtain

$$S = \frac{\left[\frac{1}{k_1}\left(k_2\right)_\tau\right]_\tau}{\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau}T + \frac{5}{2}\,\frac{k_2^{-2/5}\frac{\partial}{\partial \tau}\left[\left(\frac{k_2}{k_1}\right)k_2^{2/5}\right]}{\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau}T' + \frac{5}{2}\,\frac{\left(\frac{k_2}{k_1}\right)}{\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau}T'' \tag{144}$$

Equation (144) can now be substituted into equation (142) to obtain, in this case also, what is essentially a second-order linear ordinary differential equation for T. In either case then, it follows from equations (142) to (144) that if we define $t^{(0)}$, $t^{(1)}$, and $t^{(2)}$ by

$$t^{(0)} = \begin{cases} k_1^{3/5}\dfrac{\partial}{\partial \sigma}\left\{\dfrac{k_1^{2/5}}{K_1}\left[\dfrac{1}{k_1}\left(k_2\right)_\tau\right]_\tau\right\} - \dfrac{2}{5}\left(k_2\right)_\tau & \text{for } K_1 \neq 0 \\[2ex] \left[\dfrac{1}{k_1}\left(k_2\right)_\tau\right]_\tau & \text{for } K_1 = 0 \end{cases} \tag{145}$$

$$t^{(1)} = \begin{cases} \frac{5}{2} k_1^{3/5} \frac{\partial}{\partial\sigma} \left\{ \frac{1}{K_1} \left(\frac{k_1}{k_2} \right)^{2/5} \left[\left(\frac{k_2}{k_1} \right) k_2^{2/5} \right]_\tau \right\} - k_2 & \text{for } K_1 \neq 0 \\ \frac{5}{2} \frac{1}{k_2^{2/5}} \frac{\partial}{\partial\tau} \left[\left(\frac{k_2}{k_1} \right) k_2^{2/5} \right] & \text{for } K_1 = 0 \end{cases} \tag{146}$$

$$t^{(2)} = \begin{cases} \frac{5}{2} k_1^{3/5} \frac{\partial}{\partial\sigma} \left[\frac{k_1^{2/5}}{K_1} \left(\frac{k_2}{k_1} \right) \right] & \text{for } K_1 \neq 0 \\ \frac{5}{2} \frac{k_2}{k_1} & \text{for } K_1 = 0 \end{cases} \tag{147}$$

with

$$K_1 \equiv \left[\frac{1}{k_1} \left(k_1 \right)_\sigma \right]_\tau$$

then

$$t^{(0)}T + t^{(1)}T' + t^{(2)}T'' = 0 \tag{148}$$

In view of the symmetry of equation (140), it is easy to see by interchanging σ and τ that if we define $s^{(0)}$, $s^{(1)}$, and $s^{(2)}$ by

$$s^{(0)} = \begin{cases} k_2^{3/5} \frac{\partial}{\partial\tau} \left\{ \frac{k_2^{2/5}}{K_2} \left[\frac{1}{k_2} \left(k_1 \right)_\sigma \right]_\sigma \right\} - \frac{2}{5} \left(k_1 \right)_\sigma & \text{for } K_2 \neq 0 \\ \left[\frac{1}{k_1} \left(k_2 \right)_\tau \right]_\tau & \text{for } K_2 = 0 \end{cases} \tag{149}$$

$$
s^{(1)} = \begin{cases} \dfrac{5}{2} k_2^{3/5} \dfrac{\partial}{\partial \tau} \left\{ \dfrac{1}{K_2} \left(\dfrac{k_2}{k_1} \right)^{2/5} \left[\left(\dfrac{k_1}{k_2} \right) k_1^{2/5} \right]_\sigma \right\} - k_1 & \text{for } K_2 \neq 0 \\[2ex] \dfrac{5}{2} \dfrac{1}{k_1^{2/5}} \dfrac{\partial}{\partial \sigma} \left[\left(\dfrac{k_1}{k_2} \right) k_1^{2/5} \right] & \text{for } K_2 = 0 \end{cases} \tag{150}
$$

$$
s^{(2)} = \begin{cases} \dfrac{5}{2} k_2^{3/5} \dfrac{\partial}{\partial \tau} \left[\dfrac{k_2^{2/5}}{K_2} \left(\dfrac{k_1}{k_2} \right) \right] & \text{for } K_2 \neq 0 \\[2ex] \dfrac{k_1}{k_2} & \text{for } K_2 = 0 \end{cases} \tag{151}
$$

with

$$
K_2 \equiv \left[\frac{1}{k_2} \left(k_2 \right)_\tau \right]_\sigma
$$

then

$$
s^{(0)}S + s^{(1)}S' + s^{(2)}S'' = 0 \tag{152}
$$

There is another way in which equations of the form (148) and (152) can be derived. This procedure will lead to equations whose coefficients $t^{(i)}$ and $s^{(i)}$ $(i = 0, 1, 2)$ are, in general, different from those obtained above. To this end, equation (140) is differentiated first with respect to σ and then with respect to τ to obtain the following two equations:

$$
2\left(k_1\right)_{\sigma\sigma} S + 7\left(k_1\right)_\sigma S' + 5k_1 S'' = 2\left(k_2\right)_{\sigma\tau} T + 5\left(k_2\right)_\sigma T' \tag{153}
$$

and

$$
2\left(k_1\right)_{\sigma\tau} S + 5\left(k_1\right)_\tau S' = 2\left(k_2\right)_{\tau\tau} T + 7\left(k_2\right)_\tau T' + 5k_2 T'' \tag{154}
$$

Since $k_1 \neq 0$, $k_2 \neq 0$, and $\mathcal{J}_H \neq 0$, in none of the equations (135), (140), (153), and (154) can the coefficients of T and its derivatives and S and its derivatives all be equal to zero. Hence, S, S', and S'' can be eliminated among these equations. This procedure will result in an equation for T which has the same form as equation (148). Similarly, T, T', and T'' can be eliminated between equations (135), (140), (153), and (154) to obtain an equation for S which has the form (152). This procedure has the advantage over the preceding one that one less differentiation is required to calculate the coefficients of the equations. However, the algebra involved is more tedious.

We have now proven that if $\mathcal{J}_H$ is not of the form covered by conclusions (C11) and (C12), then all simultaneous solutions T and S of equation (135) must satisfy equations (148) and (152), respectively. We shall show how these equations can be used to determine S and T provided that the coefficients of equation (148) and the coefficients of equation (152) do not all vanish.

In appendix B, it is shown that the following four statements are equivalent:

(1) All the coefficients of equation (148) vanish.

(2) All the coefficients of equation (152) vanish.

(3) There exist functions λ_2 and λ_3 of τ only, functions γ_2 and γ_3 of σ only, and a constant π_1 such that

$$k_1^2 = -\pi_1 \frac{\left[\lambda_2(\tau)\right]^3}{\left[\gamma_2(\sigma)\right]^5} \left\{5 \frac{\lambda_3'(\tau)\gamma_3'(\sigma)}{\left[\gamma_3(\sigma) - \lambda_3(\tau)\right]^2}\right\}^5 \tag{155}$$

and

$$k_2^2 = -\pi_1 \frac{\left[\lambda_2(\tau)\right]^5}{\left[\gamma_2(\sigma)\right]^3} \left\{5 \frac{\lambda_3'(\tau)\gamma_3'(\sigma)}{\left[\gamma_3(\sigma) - \lambda_3(\tau)\right]^2}\right\}^5 \tag{156}$$

(4) There exist a function λ_2 of τ only and a function γ_2 of σ only such that

$$\left.\begin{aligned} \frac{k_2}{k_1} &= \lambda_2(\tau)\gamma_2(\sigma) \\ K_1^5 &= \frac{1}{\pi_1}\,\frac{\gamma_2^5}{\lambda_2^3}\,k_1^2 \end{aligned}\right\} \tag{157}$$

Combining equations (155) and (157) and putting

$$K \equiv K_1 = K_2 \tag{158}$$

shows that

$$K = -5 \frac{\lambda_3'(\tau)\gamma_3'(\sigma)}{\left[\gamma_3(\sigma) - \lambda_3(\tau)\right]^2} \tag{159}$$

Thus, K is a solution of Liouville's equation

$$\left(\frac{1}{K} K_\sigma\right)_\tau = \frac{2}{5} K \tag{160}$$

It is shown in appendix C that in this case equation (142) always possesses nonzero solutions S and T which are given in terms of the functions appearing in equations (155) and (156) by

$$\frac{S\gamma_3'}{\gamma_2} = \frac{\pi_2}{2} \gamma_3^2 + \pi_3\gamma_3 + \pi_4 \tag{161}$$

$$-T\lambda_2\lambda_3' = \frac{\pi_2}{2} \lambda_3^2 + \pi_3\lambda_3 + \pi_4 \tag{162}$$

where π_2 to π_4 are arbitrary constants.

Suppose, therefore, that $t^{(0)}$, $t^{(1)}$, and $t^{(2)}$ are not all zero and, hence, that $s^{(0)}$, $s^{(1)}$, and $s^{(2)}$ are not all zero. If the ratio of each pair of nonzero coefficients of equation (152) is a function σ only, then, after division by a suitable factor, equation (152) becomes just an ordinary differential equation (its coefficients are functions of σ only) which can always, in principle, be integrated to obtain an expression for S as a function of σ only. If this is not the case, then equation (152) can be divided through by a nonzero coefficient to obtain an equation which has the same form as equation (152) but which has the properties that one of its coefficients is equal to unity and at least one of its remaining coefficients is not independent of τ. This equation can then be differentiated with respect to τ to obtain an equation of the form

$$\hat{s}_1 \frac{d^m S}{d\sigma^m} + \hat{s}_2 \frac{d^n S}{d\sigma^n} = 0; \qquad m \neq n \text{ and } m, n = 0, 1 \text{ or } 2 \tag{163}$$

and

$$\hat{s}_1 \neq 0 \tag{164}$$

If $\hat{s}_2/\hat{s}_1$ is a function of σ only, then this equation can be solved to determine S as a function of σ only. (If $m = 0$ and $\hat{s}_2 = 0$, equation (163) would then show that equation (135) possesses only the solution $S' = 0$.) But if $\hat{s}_2/\hat{s}_1$ depends on τ, then this equation shows that

$$\frac{d^m S}{d\sigma^m} = \frac{d^n S}{d\sigma^n} = 0; \qquad m \neq n \text{ and } m, n = 0, 1 \text{ or } 2 \tag{165}$$

If in this case the smaller of the two integers m and n is zero, we conclude that equation (135) has only the solution $S = 0$.

We have therefore shown that, whenever $\mathscr{I}_H$ is not of the form covered by conclusions (C11) and (C12) or by equations (155) and (156), all the solutions S to equation (135) can be determined to within at most two arbitrary constants by solving the appropriate one of the three ordinary differential equations (152), (163), or (165).

Similar considerations, of course, apply to the function T and equation (148). Thus, whenever $\mathscr{I}_H$ is not covered by conclusions (C11) and (C12) or by equations (155) and (156) all the solutions to equation (135) can be obtained by solving ordinary differential equations which are, at most, of second order.

This completes our discussion of the conditions for transforming equation (29) into a separable equation.

EQUATIONS OF HYPERBOLIC TYPE ($j = -1$) - WEAKLY SEPARABLE BUT NOT SEPARABLE

Functional Form of Invariants

Now suppose that equation (29) can be transformed into an equation which is weakly separable but not separable by a change of independent variable of the form (55). Then the functions φ and ψ must satisfy conditions (71) to (73). Differentiating equation (71) with respect to x shows that

$$\varphi_{xx}\varphi_x - \varphi_{xy}\varphi_y = 0 \tag{166}$$

and differentiating equation (71) with respect to y shows that

$$\varphi_x\varphi_{xy} - \varphi_y\varphi_{yy} = 0 \tag{167}$$

Multiplying the first of these equations by φ_x, the second by φ_y and then adding the results shows that

$$\varphi_x^2\varphi_{xx} - \varphi_y^2\varphi_{yy} = 0 \tag{168}$$

or, using equation (71),

$$\left(\varphi_x^2 + \varphi_y^2\right)\left(\varphi_{xx} - \varphi_{yy}\right) = 0 \tag{169}$$

But equation (72) shows that

$$\varphi_x \neq 0 \quad \text{and} \quad \varphi_y \neq 0 \tag{170}$$

We therefore conclude that

$$\varphi_{xx} - \varphi_{yy} = 0 \tag{171}$$

Definition (54) and equation (74) now show that

$$a\varphi_x + b\varphi_y = 0 \tag{172}$$

Equation (72) shows that $d_1 \neq 0$. We can therefore define a nonconstant function $\tilde{u}$ of ξ only by

$$\tilde{u}(\xi) = \int \frac{1}{d_1(\xi)}\, d\xi \tag{173}$$

Thus, there exists a function u of x and y such that

$$u(x, y) = \tilde{u}\left[\varphi(x, y)\right] = \tilde{u}(\xi) \tag{174}$$

Hence,

$$u_x = \tilde{u}'\varphi_x = \frac{1}{d_1}\varphi_x$$

and

$$u_y = \frac{1}{d_1}\varphi_y$$

Using these results in equations (71) and (72) shows that

$$u_x^2 - u_y^2 = 0 \tag{175}$$

and

$$f = u_x\psi_x - u_y\psi_y \neq 0 \tag{176}$$

Substituting equation (176) into equation (73) and then adding $-e_1^2/4$ times equation (175) shows that

$$\psi_x^2 - \psi_y^2 = e_1(u_x\psi_x - u_y\psi_y) - \frac{e_1^2}{4}\left(u_x^2 - u_y^2\right)$$

or, upon collecting terms,

$$\left(\psi_x - \frac{e_1}{2}u_x\right)^2 - \left(\psi_y - \frac{e_1}{2}u_y\right)^2 = 0 \tag{177}$$

Differentiating this equation with respect to x gives

$$\left(\psi_x - \frac{e_1}{2}u_x\right)\left(\psi_x - \frac{e_1}{2}u_x\right)_x - \left(\psi_y - \frac{e_1}{2}u_y\right)\left(\psi_{xy} - \frac{e_1}{2}u_{xy} - \frac{e_1'}{2}\psi_x u_y\right) = 0$$

and differentiating with respect to y shows that

$$\left(\psi_x - \frac{e_1}{2} u_x\right)\left(\psi_{xy} - \frac{e_1}{2} u_{xy} - \frac{e_1'}{2} \psi_y u_x\right) - \left(\psi_y - \frac{e_1}{2} u_y\right)\left(\psi_y - \frac{e_1}{2} u_y\right)_y = 0$$

Multiplying the first of these by $\left[\psi_x - (e_1 u_x/2)\right]$ and the second by $\left[\psi_y - (e_1 u_y/2)\right]$ and then adding the results shows that

$$\left(\psi_x - \frac{e_1}{2} u_x\right)^2 \left(\psi_x - \frac{e_1}{2} u_x\right)_x - \left(\psi_y - \frac{e_1}{2} u_y\right)^2 \left(\psi_y - \frac{e_1}{2} u_y\right)_y$$

$$+ \left(\psi_y - \frac{e_1}{2} u_y\right)\left(\psi_x - \frac{e_1}{2} u_x\right)(\psi_x u_y - u_x \psi_y) \frac{e_1'}{2} = 0$$

Upon using equation (177) this becomes

$$\frac{1}{2}\left[\left(\psi_x - \frac{e_1}{2} u_x\right)^2 + \left(\psi_y - \frac{e_1}{2} u_y\right)^2\right]\left[(\psi_{xx} - \psi_{yy}) - \frac{e_1}{2}(u_{xx} - u_{yy}) - \frac{e_1'}{2}(\psi_x u_x - \psi_y u_y)\right]$$

$$+ \left(\psi_y - \frac{e_1}{2} u_y\right)\left(\psi_x - \frac{e_1}{2} u_x\right) \frac{e_1'}{2}(\psi_x u_y - u_x \psi_y) = 0$$

Hence, upon using equation (175) and adding and subtracting $(e_1/2)u_x$ and $(e_1/2)u_y$ we find

$$\frac{1}{2}\left[\left(\psi_x - \frac{e_1}{2} u_x\right)^2 + \left(\psi_y - \frac{e_1}{2} u_y\right)^2\right]\left[\psi_{xx} - \psi_{yy} - \frac{e_1'}{2}(\psi_x u_x - \psi_y u_y)\right] +$$

$$+ \left[\left(\psi_x - \frac{e_1}{2} u_x\right)^2 \left(\psi_y - \frac{e_1}{2} u_y\right)\right] \frac{e_1'}{2} u_y - \left[\left(\psi_x - \frac{e_1}{2} u_x\right)\left(\psi_y - \frac{e_1}{2} u_y\right)^2\right] \frac{e_1'}{2} u_x = 0$$

Equation (177) now shows that

$$\frac{1}{2}\left[\left(\psi_x - \frac{e_1}{2} u_x\right)^2 + \left(\psi_y - \frac{e_1}{2} u_y\right)^2\right]\left[\psi_{xx} - \psi_{yy} - e_1'(\psi_x u_x - \psi_y u_y)\right] = 0$$

Since equation (72) would be violated if both

$$\psi_x - \frac{e_1}{2} u_x = 0$$

and

$$\psi_y - \frac{e_1}{2} u_y = 0$$

we conclude that

$$\psi_{xx} - \psi_{yy} - e_1'(\psi_x u_x - \psi_y u_y) = 0 \tag{178}$$

Definition (54) shows that, when equations (176) and (178) are substituted into equation (75), we obtain

$$(d_2 + e_2 - e_1')(\psi_x u_x - \psi_y u_y) = a\psi_x + b\psi_y \tag{179}$$

Equation (172) and definition (174) show that

$$au_x + bu_y = 0 \tag{180}$$

Multiplying equation (179) first by u_x and then by u_y shows, after using equation (180), that

$$(d_2 + e_2 - e_1')u_x(\psi_x u_x - \psi_y u_y) = b(u_x \psi_y - \psi_x u_y) \tag{181}$$

and

$$(d_2 + e_2 - e_1')u_y(\psi_x u_x - \psi_y u_y) = a(\psi_x u_y - \psi_y u_x) \tag{182}$$

It follows from equation (175) that

$$u_x(\psi_x u_x - \psi_y u_y) = u_y(\psi_x u_y - \psi_y u_x)$$

and

$$u_y(\psi_x u_x - \psi_y u_y) = u_x(\psi_x u_y - \psi_y u_x)$$

and equation (176) implies that

$$\psi_x u_y - \psi_y u_x \neq 0$$

Substituting these results into equation (181) and (182) shows that

$$b = -u_y(e_2 - e_1' + d_2) \tag{183}$$

$$a = u_x(e_2 - e_1' + d_2) \tag{184}$$

Finally, substituting equation (176) into equation (76) shows that

$$c = (u_x\psi_x - u_y\psi_y)e_3 \tag{185}$$

Equations (183) to (185) with u and ψ determined from equations (175) and (177), respectively, now give the most general forms that the coefficients of equation (29) can have if this equation is to be transformable into an equation which is weakly separable but not separable by a change of variable of the type (55). As before, we shall use these relations to obtain expressions for the canonical invariants. To this end, we differentiate equations (183) and (184) with respect to x and y to obtain

$$a_x = u_{xx}(d_2 + e_2 - e_1') + u_x\psi_x(e_2 - e_1')' + u_x^2 d_2' d_1$$

$$a_y = u_{xy}(d_2 + e_2 - e_1') + u_x\psi_y(e_2 - e_1')' + u_x u_y d_2' d_1$$

$$b_x = -u_{xy}(d_2 + e_2 - e_1') - u_y\psi_x(e_2 - e_1')' - u_y u_x d_2' d_1$$

$$b_y = -u_{yy}(d_2 + e_2 - e_1') - u_y\psi_y(e_2 - e_1')' - u_y^2 d_2' d_1$$

Hence,

$$a_y + b_x = (u_x\psi_y - u_y\psi_x)(e_2 - e_1')'$$

and since the same derivation which was used to derive equation (171) from equation (71) suffices to show that equation (175) implies

$$u_{xx} - u_{yy} = 0$$

we see from equation (175) that

$$a_x + b_y = (u_x\psi_x - u_y\psi_y)(e_2 - e_1')'$$

Finally, equations (175), (183), and (184) show that

$$a^2 - b^2 = 0$$

Using these results together with equation (185) in definitions (38) and (39) shows that

$$\mathcal{J}_H = (u_x\psi_y - u_y\psi_x)(e_2 - e_1')' \tag{186}$$

$$\mathcal{I}_H = (u_x\psi_x - u_y\psi_y)\left[e_3 - \frac{1}{2}(e_2 - e_1')'\right] \tag{187}$$

Upon multiplying equation (177) through by $(u_x)^2$ and using equation (175) we find

$$u_x^2\left(\psi_x - \frac{e_1}{2}u_x\right)^2 - u_y^2\left(\psi_y - \frac{e_1}{2}u_y\right)^2 = 0$$

Hence, upon factoring and using equation (175) again

$$0 = \left[u_x\left(\psi_x - \frac{e_1}{2}u_x\right) + u_y\left(\psi_y - \frac{e_1}{2}u_y\right)\right](u_x\psi_x - u_y\psi_y)$$

$$= \left[u_x\left(\psi_x - \frac{e_1}{2}u_x\right) + u_y\left(\psi_y - \frac{e_1}{2}u_y\right)\right]f$$

Therefore, equation (176) shows that

$$u_x\left(\psi_x - \frac{e_1}{2}u_x\right) + u_y\left(\psi_y - \frac{e_1}{2}u_y\right) = 0 \tag{188}$$

Let e_4 be any convenient nonzero function of η and define the nonzero function e^+ of η by

$$e^+ = \begin{cases} e_1 & \text{for} \quad e_1 \neq 0 \\ \\ e_4 & \text{for} \quad e_1 = 0 \end{cases} \tag{189}$$

It is also convenient to define two new functions e_5 and e_6 of η only by

$$e_5 = e^+(e_2 - e_1')'$$

$$e_6 = e^+\left[e_3 - \frac{1}{2}(e_2 - e_1')'\right]$$

Then since $e^+ \neq 0$, equations (186) and (187) become

$$\mathcal{J}_H = (u_x\psi_y - u_y\psi_x)\frac{e_5}{e^+} \tag{190}$$

$$\mathcal{J}_H = (u_x\psi_x - u_y\psi_y)\frac{e_6}{e^+} \tag{191}$$

And again since $e^+ \neq 0$, we can define a nonconstant function $\tilde{v}$ of η only by

$$\tilde{v}(\eta) = 2\int \frac{1}{e^+(\eta)}\,d\eta$$

This shows that there are functions v and w of x and y such that

$$v(x,y) = \tilde{v}\left[\psi(x,y)\right] = \tilde{v}(\eta) \tag{192}$$

$$w(x,y) = \begin{cases} v(x,y) - u(x,y) & \text{for} \quad e_1 \neq 0 \\ \\ v(x,y) & \text{for} \quad e_1 = 0 \end{cases} \tag{193}$$

Then

$$v_x = \tilde{v}'\psi_x = \frac{2}{e^+}\psi_x$$

$$v_y = \frac{2}{e^+} \psi_y$$

And definitions (189) and (193) show that

$$\frac{e^+}{2} w_x = \psi_x - \frac{e_1}{2} u_x$$

$$\frac{e^+}{2} w_y = \psi_y - \frac{e_1}{2} u_y$$

Substituting these results into equations (188), (190), and (191) shows, after using equation (175) and the fact that $e^+ \neq 0$, that

$$u_x w_x + u_y w_y = 0 \tag{195}$$

$$\mathcal{J}_H = \frac{1}{2} (u_x w_y - u_y w_x) e_5 \tag{196}$$

$$\mathcal{J}_H = \frac{1}{2} (u_x w_x - u_y w_y) e_6 \tag{197}$$

Equations (175) and (195) now show that either

$$u_x = u_y \quad \text{and} \quad w_x = -w_y$$

or

$$u_x = -u_y \quad \text{and} \quad w_x = w_y$$

Hence, if we put

$$\left.\begin{aligned} \sigma &\equiv x + y \\ \tau &\equiv x - y \end{aligned}\right\} \tag{198}$$

then we conclude that there exists a nonconstant function F of σ only and a nonconstant function G of τ only such that

$$u(x,y) = F(\sigma) \quad \text{and} \quad w(x,y) = G(\tau)$$

or (199)

$$u(x,y) = G(\tau) \quad \text{and} \quad w(x,y) = F(\sigma)$$

Upon substituting these results into equations (196) and (197) we find that

$$\left.\begin{array}{ll} \left.\begin{array}{l} \mathcal{J}_H = -F'G'e_5 \\ \\ \mathcal{J}_H = F'G'e_6 \end{array}\right\} & \text{for } u = F \quad \text{and} \quad w = G \\ \\ \left.\begin{array}{l} \mathcal{J}_H = F'G'e_5 \\ \\ \mathcal{J}_H = F'G'e_6 \end{array}\right\} & \text{for } u = G \quad \text{and} \quad w = F \end{array}\right\} \tag{200}$$

Now equation (192) can be solved for η as a function of v to obtain

$$\eta = \tilde{\psi}(v) \tag{201}$$

and equation (174) can be solved for ξ as a function of u to obtain

$$\xi = \tilde{\varphi}(u) \tag{202}$$

We can therefore define functions Φ and Ψ of v only by

$$\Phi(v) \equiv \begin{cases} -e_5(\eta) = -e_5\left[\tilde{\varphi}(v)\right] & \text{for } u = F \quad \text{and} \quad w = G \\ \\ e_5(\eta) = e_5\left[\tilde{\varphi}(v)\right] & \text{for } u = G \quad \text{and} \quad w = F \end{cases}$$

$$\Psi(v) \equiv e_6(\eta) = e_6\left[\tilde{\varphi}(v)\right] \quad \begin{cases} \text{for } u = F \quad \text{and} \quad w = G \\ \\ \text{for } u = G \quad \text{and} \quad w = G \end{cases}$$

Substituting these into equation (200) shows that

$$\left.\begin{aligned} \mathcal{I}_H &= \Phi(v)F'G' \\ \mathcal{J}_H &= \Psi(v)F'G' \end{aligned}\right\} \tag{203}$$

Equations (193) and (199) show that v must be given by one of the three following equations:

$$v = F(\sigma) + G(\tau) \tag{204a}$$

$$v = F(\sigma) \tag{204b}$$

$$v = G(\tau) \tag{204c}$$

and that

$$u = F \quad \text{or} \quad u = G \tag{205a}$$

when equation (204a) holds,

$$u = G \tag{205b}$$

when equation (204b) holds, and

$$u = F \tag{204c}$$

when equation (204c) holds. The new independent variables are then given by equations (201) and (202).

Thus, equation (29) can be transformed into an equation which is weakly separable but not separable (by changing the independent variables) only if there are functions Φ and Ψ of v such that the canonical invariants satisfy equations (203). Now the same argument that was used in the separable case suffices to show that if equation (29) can be transformed into an equation which is weakly separable but not separable by changing both its dependent and independent variables then it is necessary that its canonical invariants satisfy conditions (203).

In order to see that these conditions are also sufficient suppose that there exist functions Φ and Ψ such that conditions (203) hold with v given either by equation (204a) or (204b). Then it is easy to verify by direct calculation that

$$\frac{\partial}{\partial y}\left[\frac{a}{2} - \frac{1}{4} G'(\tau) \int \Phi(v)\, dv\right] = \frac{\partial}{\partial x}\left[-\frac{b}{2} + \frac{1}{4} G'(\tau) \int \Phi(v)\, dv\right]$$

Hence, there exists a function $\omega^{(1)}$ such that

$$\left.\begin{aligned} \omega_x^{(1)} &= \frac{a}{2} - G'(\tau) \int \Phi(v)\, dv \\ \omega_y^{(1)} &= -\frac{b}{2} + \frac{1}{4} G'(\tau) \int \Phi(v)\, dv \end{aligned}\right\} \quad \text{for } v = F + G \text{ and } v = F \tag{206}$$

On the other hand, if conditions (203) hold with v given either by equation (204a) or (204c), then it can again be verified by direct calculation that

$$\frac{\partial}{\partial y}\left[\frac{a}{2} + \frac{1}{4} F'(\sigma) \int \Phi(v)\, dv\right] = \frac{\partial}{\partial x}\left[-\frac{b}{2} + \frac{1}{4} F'(\sigma) \int \Phi(v)\, dv\right]$$

Hence, there exists a function $\omega^{(2)}$ such that

$$\left.\begin{aligned} \omega_x^{(2)} &= \frac{a}{2} + \frac{1}{4} F'(\sigma) \int \Phi(v)\, dv \\ \omega_y^{(2)} &= -\frac{b}{2} + \frac{1}{4} F'(\sigma) \int \Phi(v)\, dv \end{aligned}\right\} \quad \text{for } v = F + G \text{ and } v = G \tag{207}$$

It now follows from equations (32) to (35) that in this case the change in variable

$$V = e^{\omega^{(1)}} U \tag{208}$$

transforms equation (29) into the equation

$$V_{xx} - V_{yy} + \frac{1}{2} G'\left[\int \Phi(v)\, dv\right](V_x + V_y) + \left(\mathscr{J}_H + \frac{1}{2} \mathscr{I}_H\right)V = 0 \tag{209a}$$

when $v = F + G$ and when $v = F$ and that the change in variable

$$V = e^{\omega^{(2)}} U$$

transforms equation (29) into the equation

$$V_{xx} - V_{yy} - \frac{1}{2} F' \left[\int \Phi(v)\, dv \right] (V_x - V_y) + \left(\mathcal{J}_H - \frac{1}{2} \mathcal{J}_H \right) V = 0 \qquad (209b)$$

when $v = F + G$ and when $v = G$. Hence, upon substituting equations (203) we obtain

$$V_{xx} - V_{yy} + \frac{1}{2} G' \left[\int \Phi(v)\, dv \right] (V_x + V_y) + F'G' \left(\Psi + \frac{1}{2} \Phi \right) V = 0 \qquad (210a)$$

for $v = F + G$ and $v = F$, and

$$V_{xx} - V_{yy} - \frac{1}{2} F' \left[\int \Phi(v)\, dv \right] (V_x - V_y) + F'G' \left(\Psi - \frac{1}{2} \Phi \right) V = 0 \qquad (210b)$$

for $v = F + G$ and $v = G$. Upon introducing the new independent variables $v = F + G$ and $u = G$ we find that

$$V_x + V_y = 2F'V_v$$

$$V_{xx} - V_{yy} = 4F'G'V_{uv} + 4F'G'V_{vv}$$

and upon introducing the new independent variables $v = F$ and $u = G$ we find that

$$V_x + V_y = 2F'V_v$$

$$V_{xx} - V_{yy} = 4F'G'V_{uv}$$

Substituting these results into equation (210a) shows that

$$V_{uv} + V_{vv} + \frac{1}{4} \left[\int \Phi(v)\, dv \right] V_v + \frac{1}{4} \left[\Psi(v) + \frac{1}{2} \Phi(v) \right] V = 0 \qquad \text{for } v = F + G \text{ and } u = G \qquad (211a)$$

$$V_{uv} + \frac{1}{4} \left[\int \Phi(v)\, dv \right] V_v + \frac{1}{4} \left[\Psi(v) + \frac{1}{2} \Phi(v) \right] V = 0 \qquad \text{for } v = F \text{ and } u = G. \qquad (211b)$$

Upon introducing the new independent variables $v = F + G$ and $u = F$ we find that

$$V_x - V_y = 2G'V_v$$

$$V_{xx} - V_{yy} = 4F'G'V_{uv} + 4F'G'V_{vv}$$

and upon introducing the new independent variables $v = G$ and $u = F$ we find that

$$V_x - V_y = 2G'V_v$$

$$V_{xx} - V_{yy} = 4F'G'V_{uv}$$

Substituting these results into equation (210b) shows that

$$V_{uv} + V_{vv} - \frac{1}{4}\left[\int \Phi(v)\, dv\right]V_v + \frac{1}{4}\left[\Psi(v) - \frac{1}{2}\Phi(v)\right]V = 0 \qquad \text{for } v = F + G \text{ and } u = F \quad (211c)$$

and

$$V_{uv} - \frac{1}{4}\left[\int \Phi(v)\, dv\right]V_v + \frac{1}{4}\left[\Psi(v) - \frac{1}{2}\Phi(v)\right]V = 0 \qquad \text{for } v = G \text{ and } u = F \quad (211d)$$

It is clear that equations (211a) to (211d) are weakly separable but not separable. It is not hard to show that if, instead of taking u and v as the new independent variables, we had chosen any nonconstant functions of u and v as the new independent variables, we would have again transformed equation (29) into an equation which is weakly separable but not separable.

We have therefore established the following conclusions:

(C13) The canonical hyperbolic differential equation (29) can be transformed into an equation which is weakly separable but not separable by changing both the dependent and independent variables if and only if there exist functions Φ and Ψ and nonconstant functions F and G such that the canonical invariants $\mathcal{I}_H$ and $\mathcal{J}_H$ satisfy the following conditions:

$$\mathscr{I}_H = \Phi(v)F'(\sigma)G'(\tau)$$

$$\mathscr{J}_H = \Psi(v)F'(\sigma)G'(\tau)$$

where

$$v = F(\sigma) + G(\tau)$$

or

$$v = F(\sigma) \tag{212}$$

or

$$v = G(\tau)$$

and

$$\sigma = x + y, \qquad \tau = x - y$$

(C14) If the canonical invariants of equation (29) do satisfy condition (212), then this equation can always be transformed into an equation which is weakly separable but not separable by introducing both the dependent variable V defined either by

$$V = e^{\omega^{(1)}} U \tag{213a}$$

when $v + F + G$ or when $v = F$ or by

$$V = e^{\omega^{(2)}} U \tag{213b}$$

when $v = F + G$ or $v + G$, where $\omega^{(1)}$ and $\omega^{(2)}$ are determined to within an unimportant constant by equations (206) and (207), respectively, and also the new independent variables ξ and η defined by

$$\xi = \widetilde{\varphi}(u)$$

$$\eta = \widetilde{\psi}(v)$$

where $\tilde{\varphi}$ and $\tilde{\psi}$ are any convenient nonconstant functions, v is given in equation (212), $u = F$ when the transformation (213a) is used and $u = G$ when the transformation (213b) is used.

Direct Calculational Procedure for Testing Invariants

We shall now give an alternate form of condition (212) which can be used to test the invariants $\mathcal{I}_H$ and $\mathcal{J}_H$ by direct calculation.

To this end notice that $\mathcal{I}_H$ and $\mathcal{J}_H$ satisfy condition (212) if, and only if, there exists a function F of σ only and a function G of τ only such that

$$v_\sigma\left(\frac{\mathcal{I}_H}{F'G'}\right)_\tau - v_\tau\left(\frac{\mathcal{I}_H}{F'G'}\right)_\sigma = 0$$

$$v_\sigma\left(\frac{\mathcal{J}_H}{F'G'}\right)_\tau - v_\tau\left(\frac{\mathcal{J}_H}{F'G'}\right)_\sigma = 0$$

It is now an easy matter to establish the following conclusions:

(C15) There exist nonconstant functions F and G and functions Φ and Ψ such that $\mathcal{I}_H$ and $\mathcal{J}_H$ satisfy condition (212) if, and only if, there exists a nonzero function T of τ only and/or a nonzero function S of σ only such that either

$$\left.\begin{aligned} &\frac{\partial}{\partial\sigma}(S\mathcal{I}_H) = \frac{\partial}{\partial\tau}(T\mathcal{I}_H) \\ &\text{and} \\ &\frac{\partial}{\partial\sigma}(S\mathcal{J}_H) = \frac{\partial}{\partial\tau}(T\mathcal{J}_H) \end{aligned}\right\} \qquad (214a)$$

or

$$\left.\begin{aligned} &\frac{\partial}{\partial \sigma}(S\mathscr{I}_H) = 0 \\ &\text{and} \\ &\frac{\partial}{\partial \sigma}(S\mathscr{J}_H) = 0 \end{aligned}\right\} \tag{214b}$$

or

$$\left.\begin{aligned} &\frac{\partial}{\partial \tau}(T\mathscr{I}_H) = 0 \\ &\text{and} \\ &\frac{\partial}{\partial \tau}(T\mathscr{J}_H) = 0 \end{aligned}\right\} \tag{214c}$$

(C16) If nonzero functions S and T can be found such that equation (214a) is satisfied, then condition (212) holds with $v = F + G$, and the functions F and G are given by

$$F' = \frac{1}{S} \tag{215a}$$

and

$$G' = \frac{1}{T} \tag{215b}$$

If a nonzero function S can be found such that equation (214b) is satisfied, then condition (212) holds with $v = G$, and the function F is given by equation (215a). If a nonzero function T can be found such that equation (214c) is satisfied, then condition (212) holds with $v = F$, and the function G is given by equation (215b).

It is clear that, if $\mathscr{I}_H \neq 0$ and $\mathscr{J}_H \neq 0$, then equation (214b) holds if, and only if,

$$\frac{\left(\mathscr{I}_H\right)_\sigma}{\mathscr{I}_H} = \frac{\left(\mathscr{J}_H\right)_\sigma}{\mathscr{J}_H} = \text{Function of } \sigma \text{ only}$$

Similar remarks apply to equation (214c). Hence, there is no difficulty in determining whether conditions (214b) and (214c) are satisfied. Determining whether $\mathscr{J}_H$ and $\mathscr{I}_H$ satisfy condition (214a) is slightly more difficult. If $\mathscr{I}_H$ were zero, then it would be possible to transform equation (29) into a separable equation whenever it is possible to transform it into a weakly separable equation (compare condition (212) with conditions (130) and (131) and eq. (127)). Since this case has already been discussed, we shall exclude it here and suppose that $\mathscr{I}_H \neq 0$. If $\left[\left(\mathscr{I}_H\right)_\sigma / \mathscr{I}_H\right]_\tau$ were zero, then there would be a function γ of σ only and a function λ of τ only such that

$$\mathscr{I}_H = \gamma(\sigma)\lambda(\tau)$$

This would imply that either

$$T = \frac{\pi_1}{\lambda}$$

or

$$S = \frac{\pi_2}{\gamma}$$

where π_1 and π_2 are constants and that $\mathscr{I}_H$ satisfies all three of the conditions (214a) to (214c). In this case then, it is only necessary to determine whether $\mathscr{J}_H$ satisfies any one of the following conditions in order to establish that condition (212) holds:

$$\mathscr{J}_H = 0$$

$$\pi_2 \frac{\partial}{\partial\sigma}\left(\frac{1}{\gamma}\mathscr{J}_H\right) = \pi_1 \frac{\partial}{\partial\tau}\left(\frac{1}{\lambda}\mathscr{J}_H\right)$$

$$\frac{\partial}{\partial\sigma}\left(\frac{\mathscr{J}_H}{\gamma}\right) = 0$$

$$\frac{\partial}{\partial\tau}\left(\frac{\mathscr{J}_H}{\lambda}\right) = 0$$

Hence, suppose now that equation (214a) has a nonzero solution and that $\mathscr{I}_H \neq 0$ and

$\left[\left(\mathscr{I}_H\right)_\sigma / \mathscr{I}_H\right]_\tau \neq 0$. We can therefore divide the first equation (214a) through by $\mathscr{I}_H$ and obtain

$$S \frac{\left(\mathscr{I}_H\right)_\sigma}{\mathscr{I}_H} + S' = T \frac{\left(\mathscr{I}_H\right)_\tau}{\mathscr{I}_H} + T'$$

Upon applying the operator

$$\frac{\partial^2}{\partial\sigma\,\partial\tau} - \left[\frac{\left(\mathscr{I}_H\right)_\sigma}{\mathscr{I}_H}\right]_\tau$$

to both sides of this equation we obtain

$$S\left(i_H\right)_\sigma = T\left(i_H\right)_\tau \tag{216}$$

where

$$i_H \equiv \frac{1}{\mathscr{I}_H} \frac{\partial}{\partial\tau}\left[\frac{1}{\mathscr{I}_H}\left(\mathscr{I}_H\right)_\sigma\right] \tag{217}$$

Now by hypotheses $i_H \neq 0$. If there were a nonzero constant c_o such that

$$i_H = c_o$$

then $\mathscr{I}_H$ would satisfy Liouville's equation and would therefore have to have the form of equation (138). Now it is clear (if we take $\gamma = F$, $\lambda = -G$, and $\Phi(v) = 1/2c_o v^2$) that equation (138) satisfies condition (212). Hence, assume that $i_H \neq$ constant. Thus, $\left(i_H\right)_\sigma$ and $\left(i_H\right)_\tau$ do not both vanish. If one of them vanished, say $\left(i_H\right)_\sigma$ for definiteness, then equation (216) would show that $T = 0$; that is, that the first equation (214a) does not have a nonzero solution, which is contrary to hypothesis. We therefore conclude that $\left(i_H\right)_\sigma \neq 0$ and $\left(i_H\right)_\tau \neq 0$. Equation (216) now shows (upon differentiation) that

$$\frac{T'}{T} = \frac{\left({}^{i}H\right)_{\tau}}{\left({}^{i}H\right)_{\sigma}} \frac{\partial}{\partial\tau}\left[\frac{\left({}^{i}H\right)_{\sigma}}{\left({}^{i}H\right)_{\tau}}\right] \equiv \Omega^{(2)} \tag{218}$$

and

$$\frac{S'}{S} = \frac{\left({}^{i}H\right)_{\sigma}}{\left({}^{i}H\right)_{\tau}} \frac{\partial}{\partial\sigma}\left[\frac{\left({}^{i}H\right)_{\tau}}{\left({}^{i}H\right)_{\sigma}}\right] \equiv \Omega^{(1)} \tag{219}$$

These equations show that if equation (214a) has a nonzero solution, then $\Omega^{(1)}$ must be a function of σ only and $\Omega^{(2)}$ must be a function of τ only. Conversely, if these conditions are satisfied, equations (218) and (219) can be integrated to obtain

$$S = \pi_1 e^{\int \Omega^{(1)} d\sigma} \tag{220}$$

and

$$T = \pi_2 e^{\int \Omega^{(2)} d\tau} \tag{221}$$

where π_1 and π_2 are constants, and these results can be substituted back into the two equations (214a) in order to determine whether they possess a nonzero solution.

This completes the discussion of the hyperbolic equation (29). We shall now consider the parabolic equation (30).

EQUATIONS OF THE PARABOLIC TYPE (j = 0)

Functional Form of Invariants

First, suppose that equation (30) can be transformed into a separable equation by a change of variable of the form (55). Then, the functions φ and ψ must satisfy conditions (62) to (67) with $j = 0$, $d_1 \neq 0$, and $e_1 = 0$. In addition, we have already indicated that we shall require that $b \neq 0$ in order to avoid trivialities. Equation (64) now becomes

$$\psi_x = 0 \tag{222}$$

(Incidently, this shows that equation (63) is automatically satisfied.) Equation (62) shows that

$$fd_1 = \varphi_x^2 \neq 0$$

It follows from definition (54) and the fact that $d_1 \neq 0$ that substituting the two preceding equations into equation (66) yields

$$b\psi_y = \frac{e_2}{d_1} \varphi_x^2 \tag{223}$$

Since by hypotheses

$$\frac{\partial(\varphi, \psi)}{\partial(x, y)} \neq 0$$

it follows from equation (222) that

$$\psi_y \neq 0 \tag{224}$$

Hence, equation (223) shows that

$$e_2 \neq 0 \tag{225}$$

Equation (222) shows that ψ is a function of y only and since

$$\frac{\partial e_2}{\partial x} = e_2' \psi_x = 0$$

we can conclude that e_2 is a function of y only. It now follows that there exists a non-zero function v of y only such that

$$v = \frac{e_2}{\psi_y}$$

Hence, equation (223) becomes

$$\frac{b}{v} = \frac{1}{d_1} \varphi_x^2 \neq 0 \tag{226}$$

It can again be concluded from definition (54) and the fact that $d_1 \neq 0$ that substituting equation (223) into equations (65) and (67) yields

$$\varphi_{xx} + a\varphi_x + b\varphi_y = \frac{d_2}{d_1} \varphi_x^2 \tag{227}$$

and

$$c = (d_3 + e_3) \frac{1}{d_1} \varphi_x^2 \tag{228}$$

Upon substituting equation (226) into equation (228) we obtain

$$\frac{c}{b} = \frac{d_2 + e_3}{v} \tag{229}$$

Since equation (226) shows that $\varphi_x \neq 0$, we can divide equation (227) through by φ_x to obtain

$$a + \frac{1}{\varphi_x} \varphi_{xx} = \frac{d_2}{d_1} \varphi_x - b\left(\frac{\varphi_y}{\varphi_x}\right)$$

Equation (226) can be differentiated to obtain

$$\frac{b_x}{b} = 2 \frac{1}{\varphi_x} \varphi_{xx} - \frac{d_1'}{d_1} \varphi_x \tag{230}$$

Upon eliminating φ_{xx}/φ_x between these two equations we obtain

$$\left(a + \frac{1}{2} \frac{b_x}{b}\right) = \left(\frac{d_2 - \frac{1}{2} d_1'}{d_1}\right) \varphi_x - b\left(\frac{\varphi_y}{\varphi_x}\right) \tag{231}$$

Equations (226), (229), and (231), in which d_1, d_2, d_3, e_3, and v can be any function of their arguments, now give the most general form that the coefficients of equation (30) can have if this equation is to be transformable into a separable equation by a change of variable of the type (55). However, it is again more suitable to use these expressions to obtain conditions for the canonical invariants. Since equation (45) shows that $\mathcal{J}_P = b$, we see from equation (226) that this is already partially accomplished. An expression for $\mathcal{I}_P$ can be obtained by substituting equations (226), (229), and (231) into definition (46). After considerable algebraic manipulation, which is carried out in appendix D, we obtain

$$\mathcal{I}_P = \left[\frac{1}{v} R(\varphi)\right]_x - \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x + \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y \tag{232}$$

where the function R of φ is defined in appendix D. Definition (45) and equation (226) show that

$$\mathcal{J}_P = b = \frac{v}{d_1} \varphi_x^2 \neq 0 \tag{233}$$

This shows then that equation (30) can be transformed (by changing the independent variables) into a separable equation only if there is a function φ of x and y, functions d_1 and R of φ only, and a function v of y only such that the canonical invariants satisfy conditions (232) and (233). The argument used for the hyperbolic case now suffices to show that if equation (30) can be transformed into a separable equation by changing both its dependent and independent variables then it is necessary that its canonical invariants satisfy conditions (232) and (233).

In order to see that these conditions are also sufficient suppose that there exist functions φ, R, d_1, and v such that equations (232) and (233) are satisfied, and define the function θ by

$$\theta = \int a\, dx \tag{234}$$

Then,

$$a = \frac{\partial \theta}{\partial x}$$

and definition (46) shows that

$$\mathscr{I}_P = \frac{\partial}{\partial x}\left(\frac{1}{b}\left\{2c - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right] - \left[a + \left(\frac{b_x}{2b}\right)\right]_x\right\} - \left(\theta_y + \frac{b_y}{b}\right)\right)$$

Thus, if we put

$$\Omega \equiv \frac{1}{b}\left\{2c - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right] - \left[a + \frac{b_x}{2b}\right]_x\right\} - \left(\theta_y + \frac{b_y}{b}\right) \tag{235}$$

then definition (46) can be written as

$$\mathscr{I}_P = \frac{\partial \Omega}{\partial x}$$

Using this result in equation (232) gives

$$\frac{\partial}{\partial x}\left[\Omega - \frac{1}{v} R(\varphi) + \frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right] = \frac{\partial}{\partial y}\left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]$$

This shows that there exists a function ω such that

$$\omega_x = b\left(\frac{\varphi_y}{\varphi_x}\right) \tag{236}$$

$$\omega_y = \Omega - \frac{1}{v} R(\varphi) + \frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2 \tag{237}$$

On substituting equation (236) into equation (237) we obtain

$$\Omega - \omega_y + \frac{\omega_x^2}{2b} = \frac{1}{v} R(\varphi) \tag{238}$$

The results of appendix E show that equation (233) implies

$$\left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_x - \frac{1}{2}\frac{b_x}{b}\left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right] = \frac{1}{2}b_y - \frac{1}{2}b\frac{v'}{v}$$

Substituting equation (236) into this expression shows that

$$\omega_{xx} - \frac{1}{2}\frac{b_x}{b}\omega_x = \frac{1}{2}b_y - \frac{1}{2}b\frac{v'}{v} \tag{239}$$

It is shown in appendix F that equations (234), (235), (238), and (239) taken together with equations (40) to (43) imply that the change of variable

$$V = e^{(1/2)\left[\omega + (1/2)\ln|b| + \theta\right]}U \tag{240}$$

transforms equation (30) into the equation

$$V_{xx} - \left(\omega_x + \frac{1}{2}\frac{b_x}{b}\right)V_x + bV_y + \frac{b}{2v}\left[R(\varphi) + \frac{1}{2}v'\right]V = 0 \tag{241}$$

Upon introducing the new independent variables

$$\xi = \varphi(x, y)$$

$$\eta = y$$

we obtain

$$\varphi_x^2 V_{\xi\xi} + \left[\varphi_{xx} + b\varphi_y - \varphi_x\left(\omega_x + \frac{1}{2}\frac{b_x}{b}\right)\right]V_\xi + bV_\eta + \frac{b}{2v}\left[R(\xi) + \frac{1}{2}v'(\eta)\right]V = 0$$

Hence, upon using equation (230) (which follows from differentiating eq. (233)) and equation (236) we find

$$\varphi_x^2 V_{\xi\xi} + \frac{d_1'}{2d_1}\varphi_x^2 V_\xi + bV_\eta + \frac{b}{2v}\left[R(\xi) + \frac{1}{2}v'(\eta)\right]V = 0$$

Finally, upon using equation (226) we obtain

$$d_1(\xi)V_{\xi\xi} + \frac{1}{2} d_1'(\xi)V_\xi + v(\eta)V_\eta + \frac{1}{2}\left[R(\xi) + \frac{1}{2} v'(\eta)\right]V = 0$$

and this equation is certainly separable.

It is easy to see that equation (241) could also have been transformed into a separable equation if η was taken to be any nonconstant function of y.

We have therefore established the following conclusions:

(C17) The canonical parabolic differential equation (30) can be transformed into a separable equation by changing both the dependent and independent variables if and only if, there exist a nonconstant function φ, a nonzero function d_1 of φ only, a function R of φ only, and a nonzero function v or y only such that the canonical invariants $\mathscr{I}_P$ and $\mathscr{J}_P$ satisfy the following conditions:

$$\left.\begin{aligned} \mathscr{I}_P &= \frac{v}{d_1}\varphi_x^2 \neq 0 \\ \mathscr{J}_P &= \left[\frac{1}{v}R(\varphi)\right]_x - \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x + \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y \end{aligned}\right\} \tag{242}$$

(C18) If the canonical invariants of equation (30) satisfy condition (242), then this equation can always be transformed into a separable equation by introducing both a new dependent variable V defined by

$$V = e^{(1/2)\left[\omega + (1/2)\ln|b| + \theta\right]}U$$

where ω is determined to within an unimportant constant by equations (236) and (237) and θ is defined by equation (234), and new independent variables ξ and η defined by

$$\xi = \varphi(x, y)$$

$$\eta = \tilde{\psi}(y)$$

where φ is determined from condition (242) and $\tilde{\psi}$ is any nonconstant function of y only.

Direct Calculational Procedure for Testing $\mathscr{I}_P$

We shall now give an alternate form of condition (242) which can be used to test the canonical invariants $\mathscr{I}_P$ and $\mathscr{I}_P$. To this end notice that (since $\varphi_x \neq 0$) the second equation (242) can be written in the form

$$\frac{v}{\varphi_x}\left\{\mathscr{I}_P + \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x - \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y\right\} = R'(\varphi)$$

But there exists a function R of φ only such that this equation holds if and only if,

$$\varphi_y\left(\frac{v}{\varphi_x}\left\{\mathscr{I}_P + \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x - \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y\right\}\right)_x - \varphi_x\left(\frac{v}{\varphi_x}\left\{\mathscr{I}_P + \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x - \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y\right\}\right)_y = 0$$

Hence, upon differentiating by parts we can conclude that there exists a function R of φ only such that the second equation (242) holds if and only if,

$$\frac{\partial}{\partial y}\left(v\left\{\mathscr{I}_P + \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x - \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y\right\}\right) = \frac{\partial}{\partial x}\left(v\left\{\mathscr{I}_P + \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x - \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y\right\}\frac{\varphi_y}{\varphi_x}\right) \qquad (243)$$

On the other hand, it is shown in appendix E that there exists a function d_1 of φ only such that the first equation (242) holds if and only if,

$$\left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_x - \frac{1}{2}\,\frac{b_x}{b}\left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right] = \frac{1}{2}\,v\left(\frac{b}{v}\right)_y \qquad (244)$$

Upon putting

$$T \equiv v\left(\frac{\varphi_y}{\varphi_x}\right)$$

in equations (243) and (244) we obtain

$$\frac{\partial}{\partial y}\left[v f_P + \left(\frac{b}{2v}T^2\right)_x - v\left(\frac{b}{v}T\right)_y\right] = \frac{\partial}{\partial x}\left[T f_P + \frac{T}{v}\left(\frac{b}{2v}T^2\right)_x - \left(\frac{b}{v}T\right)_y T\right] \tag{245}$$

and

$$T_x + \frac{1}{2}\frac{b_x}{b}T = \frac{1}{2}\frac{v^2}{b}\left(\frac{b}{v}\right)_y \tag{246}$$

Since given T/v (recall that $v \neq 0$) the first-order linear differential equation

$$\varphi_x \frac{T}{v} = \varphi_y \tag{247}$$

always has a nonconstant solution (i.e., a solution φ such that φ_x and φ_y are not both zero) and since equation (247) shows that $\varphi_x = 0$ and implies that $\varphi_y = 0$, we can conclude that equation (247) always has a solution φ such that $\varphi_x \neq 0$. This remark taken together with the preceding results is sufficient to show that there exist functions φ, d_1, R, and $v \neq 0$ such that condition (242) is satisfied if, and only if, there exists a function T and a nonzero function v of y only such that equations (245) and (246) are satisfied.

Multiplying equation (246) by (bT/v) shows that

$$\left(\frac{bT^2}{2v}\right)_x = \frac{1}{2}Tv\left(\frac{b}{v}\right)_y \tag{248}$$

Hence,

$$\frac{T}{v}\left(\frac{b}{2v}T^2\right)_x - \left(\frac{b}{v}T\right)_y = \frac{T^2}{2}\left(\frac{b}{v}\right)_y - \left(\frac{b}{v}T\right)_y T = -\left(\frac{T^2 b}{2v}\right)_y$$

Using this result in equation (245) shows that

$$\frac{\partial}{\partial y}\left[v f_P + \left(\frac{b}{v}T^2\right)_x - v\left(\frac{b}{v}T\right)_y\right] = \left(T f_P\right)_x$$

and, by using equation (248) again to eliminate $(bT^2/v)_x$, we obtain upon differentiating by parts

$$\left(v\mathscr{J}_P\right)_y - \left(bT_y\right)_y = \left(T\mathscr{J}_P\right)_x \tag{249}$$

The steps of this argument can easily be reversed to show that equation (249) taken together with equation (246) is equivalent to equation (245) taken together with equation (246). We have therefore established the following conclusions:

(C19) There exists a function φ and a nonzero function d_1 of φ only, a function R of φ only, and a nonzero function v of y only such that $\mathscr{J}_P$ and $\mathscr{J}_P$ satisfy conditions (242) if, and only if, there exist a function T and a nonzero function v of y only such that

$$\left(v\mathscr{J}_P\right)_y - \left(bT_y\right)_y = \left(T\mathscr{J}_P\right)_x \tag{249}$$

$$T_x + \frac{1}{2}\frac{b_x}{b}T = \frac{1}{2}v\frac{b_y}{b} - \frac{1}{2}v' \tag{250}$$

(C20) If a function T and a nonzero function v of y only can be found such that equations (249) and (250) hold, then a function φ which satisfies condition (242) can be found by solving the first-order linear partial differential equation (247). Thus, φ is any function such that φ = Constant is an integral of the ordinary differential equation

$$\frac{dx}{dy} = -\frac{T}{v}$$

Notice that equations (249) and (250) are linear and homogeneous in T and v. Hence, they always possess the trivial solution $T = v = 0$. However, conclusion (C19) requires that the function v of y only be nonzero. We shall now develop a procedure which will yield expressions for all the solutions T and v to equations (249) and (250). As in the hyperbolic case, these expressions will involve undetermined constants. Whenever these expressions are obtained by this procedure, it is necessary to substitute them back into equations (249) and (250) to determine what restrictions, if any, must be placed on the undetermined constants in order that these equations be satisfied. If after this is done the constants can still be adjusted so that $v \neq 0$, then we can conclude that equation (30) can be transformed into a separable equation, and we can use the expressions for T and v to calculate the new coordinates from equation (251).

Now suppose that the function T and the function v of y only are any two simultaneous solutions to equations (249) and (250). Then it is shown in appendix G that when the operator

$$\frac{\partial}{\partial x} - \frac{1}{2}\,\frac{b_x}{b}$$

is applied to equation (250) the following equation is obtained:

$$2v'\mathscr{j}_P + v\left(\mathscr{j}_P\right)_y + \frac{v'''}{2} = \left(\mathscr{j}_P\right)_x T \tag{251}$$

where

$$\mathscr{j}_P \equiv \frac{1}{b}\left[\left(\mathscr{J}_P\right)_x - \left(\frac{b_x}{2b}\right)\mathscr{J}_P\right] - \frac{b_{yy}}{2b} + \left(\frac{b_y}{2b}\right)^2 \tag{252}$$

Now if $\left(\mathscr{j}_P\right)_x = 0$, equation (251) is a third-order ordinary differential equation for v since this condition also implies that all the coefficients in this equation are functions of y only. On the other hand, if $\left(\mathscr{j}_P\right)_x \neq 0$, it can be divided into both sides of equation (251) to obtain

$$v\,\frac{\left(\mathscr{j}_P\right)_y}{\left(\mathscr{j}_P\right)_x} + 2v'\,\frac{\mathscr{j}_P}{\left(\mathscr{j}_P\right)_x} + \frac{v'''}{2}\,\frac{1}{\left(\mathscr{j}_P\right)_x} = T \tag{253}$$

Equation (253) can be substituted into equation (250) to obtain, in this case also, what is essentially a third-order ordinary differential equation for v. (Notice, however, that the coefficients in this equation will be, in general, functions of the two variables x and y and not just of y.) In either case then, it follows from equations (250), (251), and (253) that if we define $Y^{(0)}$, $Y^{(1)}$, and $Y^{(3)}$ by

$$Y^{(0)} = \begin{cases} \left(\mathscr{j}_P\right)_y & \text{for } \left(\mathscr{j}_P\right)_x = 0 \\[2ex] \dfrac{1}{\sqrt{b}}\,\dfrac{\partial}{\partial x}\left\{\sqrt{b}\left[\dfrac{\left(\mathscr{j}_P\right)_y}{\left(\mathscr{j}_P\right)_x}\right]\right\} - \dfrac{1}{2}\,\dfrac{b_y}{b} & \text{for } \left(\mathscr{j}_P\right)_x \neq 0 \end{cases} \tag{254}$$

$$
Y^{(1)} = \begin{cases} 2j_P & \text{for } \left(j_P\right)_x = 0 \\ \dfrac{1}{\sqrt{b}} \dfrac{\partial}{\partial x} \left\{ \sqrt{b} \left[\dfrac{2j_P}{\left(j_P\right)_x} \right] \right\} + \dfrac{1}{2} & \text{for } \left(j_P\right)_x \neq 0 \end{cases} \tag{255}
$$

$$
Y^{(3)} = \begin{cases} \dfrac{1}{2} & \text{for } \left(j_P\right)_x = 0 \\ \dfrac{1}{\sqrt{b}} \dfrac{\partial}{\partial x} \left[\dfrac{\sqrt{b}}{2\left(j_P\right)_x} \right] & \text{for } \left(j_P\right)_x \neq 0 \end{cases} \tag{256}
$$

then

$$
Y^{(0)} v + Y^{(1)} v' + Y^{(3)} v''' = 0 \tag{257}
$$

We have now proved that every function v of y only which satisfies equations (249) and (250) for any function T must satisfy equation (257). The coefficients of this equation cannot all be zero. For if $\left(j_P\right)_x = 0$, this conclusion follows directly from equation (256); and, if $\left(j_P\right)_x \neq 0$, it is easy to see from equations (255) and (256) that $Y^{(3)} = 0$ implies that $Y^{(1)} = 5/2$. Equation (257) can therefore be used to determine v in exactly the same way that equation (148) was used to determine T in the hyperbolic case.

Thus, if the ratio of each pair of nonzero coefficients of equation (257) is a function of y only, then after division by a suitable factor equation (152) becomes just an ordinary differential equation (with coefficients now functions of y only) which can always be integrated to obtain an expression for v as a function of y only. On the other hand, if this is not the case, then equation (257) can be divided through by a nonzero coefficient to obtain an equation which has the same form as equation (257) but which has the properties that one of its coefficients is equal to unity and at least one of its remaining coefficients is not independent of x. This equation can then be differentiated with respect to x to obtain an equation of the form

$$
\tilde{Y}_1 \frac{d^m v}{dy^m} + \tilde{Y}_2 \frac{d^n v}{dy^n} = 0 \qquad m \neq n \quad \text{and} \quad m, n = 0, 1 \text{ or } 3 \tag{258}
$$

and

$$\tilde{Y}_1 \neq 0$$

If $\tilde{Y}_2/\tilde{Y}_1$ is a function of y only, then this equation can be solved to determine v as a function y only. But if $\tilde{Y}_2/\tilde{Y}_1$ depends on x, then equation (258) shows that

$$\frac{d^m v}{dy^m} = \frac{d^n v}{dy^n} = 0 \qquad m \neq n \quad \text{and} \quad m, n = 0, 1 \text{ or } 3 \tag{259}$$

We have therefore shown that all the functions v of y which satisfy equations (249) and (250) can be determined to within at most three arbitrary constants by solving the appropriate one of the three ordinary differential equations (257) to (259).

If $\left(\dot{\mathscr{J}}_P\right)_x = 0$, equation (257) is just an ordinary differential equation (its coefficients are functions of y only). And, therefore, there are three linearly independent functions v of y only which satisfy this equation, and any linear combination of these solutions is also a solution. Now it is shown in appendix H that, for each of these infinitely many nonzero functions v of y only which satisfy equation (257), there exists a function T which contains two arbitrary constants, such that v and T satisfy equations (249) and (250). Thus, if $\left(\dot{\mathscr{J}}_P\right)_x = 0$, there is a six-parameter family of solutions to equations (249) and (250). The analysis of appendix H shows that, in general, it may be necessary to use different expressions for T in different parts of the domain of equations (249) and (250). Since the analysis of appendix H is constructive, it can be used to determine T once the solutions v of equation (257) are found. As has already been shown, once T and v are determined the six-parameter family of independent variables which transform equation (30) into a separable can be found.

If $\left(\dot{\mathscr{J}}_P\right)_x \neq 0$, then once v is found by the procedure just described the function T can be determined from equation (253). It clear from the way that equation (257) was derived that, <u>if v satisfies this equation</u>, then T and v will automatically satisfy equation (250). However, it is necessary to substitute these expressions for T and v back into equation (249) to determine what restrictions must be placed on the arbitrary constants which they contain in order that both equations (249) and (250) are simultaneously satisfied.

Since a parabolic equation cannot be transformed into an equation which is weakly separable but not separable, this compleses the discussion of the parabolic equation (30). We shall now consider the elliptic equation (31).

EQUATIONS OF THE ELLIPTIC TYPE (j = 1)

Functional Form of Invariants

First, suppose that equation (31) can be transformed into a separable equation by a change of variable of the form (55). Then, the functions φ and ψ must satisfy conditions (62) to (67) with $j = 1$, $d_1 > 0$, and $e_1 > 0$. It is therefore permissible to introduce the functions

$$\frac{1}{\sqrt{d_1}}$$

and

$$\frac{1}{\sqrt{e_1}}$$

and to define a function $\tilde{u}$ of ξ only and a function $\tilde{v}$ of η only by

$$\tilde{u}(\xi) = \int \frac{1}{\sqrt{d_1(\xi)}}\, d\xi \tag{260}$$

$$\tilde{v}(\eta) = \int \frac{1}{\sqrt{e_1(\eta)}}\, d\eta \tag{261}$$

Thus, there are functions u^+ and v of x and y such that

$$u^+(x,y) = \tilde{u}\left[\varphi(x,y)\right] = \tilde{u}(\xi)$$

$$v(x,y) = \tilde{v}\left[\psi(x,y)\right] = \tilde{v}(\eta)$$

We shall suppose that these equations can always be solved for φ and ψ to obtain

$$\xi = \varphi(x,y) = \tilde{\varphi}\left[u^+(x,y)\right] \tag{262}$$

and

$$\eta = \psi(x,y) = \tilde{\psi}\left[v(x,y)\right] \tag{263}$$

Therefore, it follows from equations (260) and (261) that

$$\frac{\partial \varphi}{\partial x} = \sqrt{d_1}\,\frac{\partial u^+}{\partial x} \tag{264}$$

$$\frac{\partial \varphi}{\partial y} = \sqrt{d_1}\,\frac{\partial u^+}{\partial y} \tag{265}$$

$$\frac{\partial \psi}{\partial x} = \sqrt{e_1}\,\frac{\partial v}{\partial x} \tag{266}$$

$$\frac{\partial \psi}{\partial y} = \sqrt{e_1}\,\frac{\partial v}{\partial y} \tag{267}$$

Hence,

$$\frac{\partial^2 \varphi}{\partial x^2} = \sqrt{d_1}\,\frac{\partial^2 u^+}{\partial x^2} + \frac{1}{2}\,d_1'\left(\frac{\partial u^+}{\partial x}\right)^2 \tag{268}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \sqrt{d_1}\,\frac{\partial^2 u^+}{\partial y^2} + \frac{1}{2}\,d_1'\left(\frac{\partial u^+}{\partial y}\right)^2 \tag{269}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \sqrt{e_1}\,\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\,e_1'\left(\frac{\partial v}{\partial x}\right)^2 \tag{270}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \sqrt{e_1}\,\frac{\partial^2 v}{\partial y^2} + \frac{1}{2}\,e_1'\left(\frac{\partial v}{\partial y}\right)^2 \tag{271}$$

It now follows from definition (51) and equations (264) to (271) that

$$L^{(1)}(\varphi) = \frac{1}{2}\,d_1'\left[\left(\frac{\partial u^+}{\partial x}\right)^2 + \left(\frac{\partial u^+}{\partial y}\right)^2\right] + \sqrt{d_1}\,L^{(1)}(u^+) \tag{272}$$

$$L^{(1)}(\psi) = \frac{1}{2}\,e_1'\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \sqrt{e_1}\,L^{(1)}(v) \tag{273}$$

Upon substituting equations (267) to (269), (272), and (273) into conditions (62) to (67) with $j = 1$ we obtain

$$f = \left(u_x^+\right)^2 + \left(u_y^+\right)^2 \tag{274}$$

$$0 = u_x^+ v_x + u_y^+ v_y \tag{275}$$

$$f = \left(v_x\right)^2 + \left(v_y\right)^2 \tag{276}$$

$$fd_2 = \frac{1}{2} d_1' \left[\left(u_x^+\right)^2 + \left(u_y^+\right)^2\right] + \sqrt{d_1}\, L^{(1)}(u^+) \tag{277}$$

$$fe_2 = \frac{1}{2} e_1' \left[\left(v_x\right)^2 + \left(v_y\right)^2\right] + \sqrt{e_1}\, L^{(1)}(v) \tag{278}$$

$$f(d_3 + e_3) = c \tag{279}$$

After substituting equation (274) into equations (277) and (279) and equation (276) into equation (278), we obtain

$$L^{(1)}(u^+) = d_4 \left[\left(u_x^+\right)^2 + \left(u_y^+\right)^2\right] \tag{280}$$

$$L^{(1)}(v) = e_4 \left[\left(v_x\right)^2 + \left(v_y\right)^2\right] \tag{281}$$

$$c = (d_3 + e_3)\left[\left(u_x^+\right)^2 + \left(u_y^+\right)^2\right] \tag{282}$$

where the function d_4 of ξ only and the function e_4 of η only are defined by

$$d_4 = \frac{d_2 - \frac{1}{2} d_1'}{\sqrt{d_1}}$$

$$e_4 = \frac{e_2 - \frac{1}{2} e_1'}{\sqrt{e_1}}$$

Equations (274) and (276) show that

$$\left(u_x^+\right)^2 + \left(u_y^+\right)^2 = \left(v_x\right)^2 + \left(v_y\right)^2 \tag{283}$$

or multiplying both sides by $\left(v_y\right)^2$

$$\left(v_y\right)^2\left(u_x^+\right)^2 + \left(v_y\right)^2\left(u_y^+\right)^2 = \left(v_y\right)^2\left[\left(v_x\right)^2 + \left(v_y\right)^2\right]$$

Eliminating $v_y u_y^+$ between this equation and equation (275) yields

$$\left(u_x^+\right)^2\left[\left(v_y\right)^2 + \left(v_x\right)^2\right] = \left(v_y\right)^2\left[\left(v_x\right)^2 + \left(v_y\right)^2\right]$$

and, since $\left(v_x\right)^2 + \left(v_y\right)^2 = f \neq 0$, this shows that

$$\left(u_x^+\right)^2 = \left(v_y\right)^2$$

Hence,

$$u_x^+ = \pm v_y \tag{284}$$

Substituting this into equations (275) and (283) shows that

$$u_y^+ = \mp v_x \tag{285}$$

where the minus sign in equation (285) must be associated with the plus sign in equation (284), and the plus sign in equation (285) must be associated with the minus sign in equation (284). Thus, it is convenient to introduce a new function u of x and y by

$$u = \begin{cases} u^+ & \text{if the plus sign holds in equation (284) and} \\ & \quad \text{the minus sign holds in equation (285)} \\ -u^+ & \text{if the minus sign holds in equation (284) and} \\ & \quad \text{the plus sign holds in equation (285)} \end{cases} \tag{286}$$

Equations (284) and (285) now become

Equations (284) and (285) now become

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \tag{287}$$

Equations (262) and (263) and definition (286) now show that we can define functions p_1 and p_2 of u only and functions q_1 and q_2 of v only such that

$$p_1(u) \equiv \begin{cases} d_3\left[\tilde{\varphi}(u)\right] & \text{if } u = u^+ \\ d_3\left[\tilde{\varphi}(-u)\right] & \text{if } u = -u^+ \end{cases}$$

$$p_2(u) \equiv \begin{cases} d_4\left[\tilde{\varphi}(u)\right] & \text{if } u = u^+ \\ -d_4\left[\tilde{\varphi}(-u)\right] & \text{if } u = -u^+ \end{cases}$$

$$q_1(v) \equiv e_3\left[\tilde{\psi}(v)\right]$$

$$q_2(v) \equiv e_4\left[\tilde{\psi}(v)\right]$$

Substituting these definitions together with definition (286) into equations (280) to (282) gives

$$L^{(1)}(u) = p_2(u)\left(u_x^2 + u_y^2\right) \tag{288}$$

$$L^{(1)}(v) = q_2(v)\left(v_x^2 + v_y^2\right) \tag{289}$$

$$c = \left[p_1(u) + q_1(v)\right]\left(u_x^2 + u_y^2\right) \tag{290}$$

Now equations (287) are just the Cauchy-Riemann equations. Thus, u and v are conjugate harmonic functions. Therefore,

$$\nabla^2 u = \nabla^2 v = 0 \tag{291}$$

and there exists an analytic function W of the complex variable

$$z = x + iy \tag{292}$$

such that

$$W(z) = u(x, y) + iv(x, y) \tag{293}$$

In particular, we can conclude from this that

$$\left|\frac{dW}{dz}\right|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2 \neq 0 \tag{294}$$

It follows from definition (54) and equations (288) to (290), (291), and (294) that

$$au_x + bu_y = p_2(u)\left|\frac{dW}{dz}\right|^2 \tag{295}$$

$$av_x + bv_y = q_2(v)\left|\frac{dW}{dz}\right|^2 \tag{296}$$

$$c = \left[p_1(u) + q_1(v)\right]\left|\frac{dW}{dz}\right|^2 \tag{297}$$

It follows from the definition of the derivative of an analytic function that equations (295) and (296) can be written as the following single complex equation:

$$(a + ib)\frac{dW}{dz} = (p_2 + iq_2)\left|\frac{dW}{dz}\right|^2 \tag{298}$$

Now equation (294) shows that $dW/dz \neq 0$. Therefore, it can be divided into both sides of equation (298) to obtain

$$a + ib = (p_2 + iq_2)\left(\frac{dW}{dz}\right)^*$$

Hence, upon taking the real and imaginary parts we find

$$a = p_2(u)u_x + q_2(v)v_x \tag{299}$$

$$b = p_2(u)u_y + q_2(v)v_y \tag{300}$$

upon recalling that the derivative of an analytic function is independent of direction.

Equations (292), (293), (297), (299), and (300), in which p_1, p_2, q_1, and q_2 can be any functions of their arguments and W can be any nonconstant analytic function of the complex variable z, now give the most general forms that the coefficients of equation (31) can have if this equation is to be transformable into a separable equation by a change of variable of the type (55). However, it is again more useful to use these expressions to obtain conditions for the canonical invariants. Upon differentiating equations (229) and (300) with respect to x and y we find, after using equations (291) and (294) to simplify

$$a_y - b_x = 0$$

$$a_x + b_y = \left[p_2'(u) + q_2'(v)\right]\left|\frac{dW}{dz}\right|^2$$

Equations (287), (294), (299), and (300) also show that

$$a^2 + b^2 = \left(p_2^2 + q_2^2\right)\left|\frac{dW}{dz}\right|^2$$

Substituting these results together with equations (297) into the definitions (51) and (52) of the canonical invariants of equation (31) shows that

$$\mathscr{I}_E = 0 \tag{301}$$

and

$$\mathscr{I}_E = \left[\left(p_1 - \frac{1}{2}p_2' - \frac{1}{4}p_2^2\right) + \left(q_1 - \frac{1}{2}q_2' - \frac{1}{4}q_2^2\right)\right]\left|\frac{dW}{dz}\right|^2 \tag{302}$$

Or upon defining the function Φ of u and the function Ψ of v by

$$\Phi(u) \equiv p_1(u) - \frac{1}{2} p_2'(u) - \frac{1}{4}\left[p_2(u)\right]^2$$

$$\Psi(v) \equiv q_1(v) - \frac{1}{2} q_2'(v) - \frac{1}{4}\left[q_2(v)\right]^2$$

equation (302) becomes

$$\mathscr{I}_E = \left[\Phi(u) + \Psi(v)\right]\left|\frac{dW}{dz}\right|^2 \tag{303}$$

Thus, equation (31) can be transformed by a change in the independent variables into a separable equation only if there is an analytic function W of the complex variable z and functions Φ and Ψ of u and v, respectively, such that its canonical invariants satisfy conditions (301) and (303). The argument used in the hyperbolic case now suffices to show that, if equation (31) can be transformed into a separable equation by changing both its dependent and independent variables, then it is necessary that its canonical invariants satisfy conditions (301) and (303).

In order to see that these conditions are also sufficient, suppose that there exists an analytic function W of the complex variable z and functions Φ and Ψ such that equations (301) and (303) hold. Then it follows from definition (51) that condition (301) implies that there exists a function ω such that

$$\left.\begin{aligned} a &= \omega_x \\ b &= \omega_y \end{aligned}\right\} \tag{304}$$

It is easy to see from equations (47) to (50) and equation (52) that the change of variable (refs. 3 and 4)

$$V = e^{\omega/2} U \tag{305}$$

transforms equation (31) into the equation

$$\nabla^2 V + \mathscr{I}_E V = 0 \tag{306}$$

where ∇^2 is the laplacian

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Or substituting for $\mathcal{J}_E$ in equation (303) this becomes

$$\nabla^2 V + \left[\Phi(u) + \Psi(v)\right] \left|\frac{dW}{dz}\right|^2 V = 0 \tag{307}$$

But it is shown in reference 1 that, upon introducing the new independent variables u and v defined by equation (293), this equation transforms into the separable equation

$$V_{uu} + V_{vv} + \left[\Phi(u) + \Psi(v)\right] V = 0 \tag{307}$$

It is also shown in reference 1 that, if any functions φ and ψ which satisfy equations (79) and (80) were used as the new independent variables in place of u and v, then equation (307) would still be transformed into a separable equation.

We have therefore established the following conclusions:

(C21) The canonical elliptic differential equation (31) can be transformed into a separable equation by changing both the dependent and independent variables if, and only if, there exist a nonconstant analytic function W of the complex variable $z = x + iy$ and functions Φ and Ψ such that the canonical invariants $\mathcal{I}_E$ and $\mathcal{J}_E$ satisfy the following conditions:

$$\mathcal{I}_E = 0 \tag{308}$$

where

$$\left.\begin{aligned} \mathcal{J}_E &= \left[\Phi(u) + \Psi(v)\right] \left|\frac{dW}{dz}\right|^2 \\ u &= \mathcal{R}e\, W \\ v &= \mathcal{I}m\, W \end{aligned}\right\} \tag{309}$$

(C22) If the canonical invariants of equation (31) do satisfy conditions (308) and (309), then this equation can always be transformed into a spearable equation by introducing both a new dependent variable V defined by

$$V = e^{\omega/2} U \tag{310}$$

where ω is determined to within an unimportant constant by

$$a = \omega_x$$

$$b = \omega_y$$

and new independent variables ξ and η defined by

$$\left.\begin{aligned} \xi &= \tilde{\varphi}(u) \\ \eta &= \tilde{\psi}(v) \end{aligned}\right\} \tag{311}$$

where $\tilde{\varphi}$ and $\tilde{\psi}$ are any convenient nonconstant functions and u and v are determined from condition (309).

Notice that, by choosing $\tilde{\varphi}(u) = u$ and $\tilde{\psi}(v) = v$, any equation which satisfies conditions (308) and (309) can always be transformed into a separable equation by the conformal transformation

$$z \rightarrow W$$

More generally, the change of variable (311) represents an orthogonal tranformation and it is shown in reference 1 that the particular choice of the functions $\tilde{\varphi}$ and $\tilde{\psi}$ merely serves to modify the scale factors of the coordinate geometry associated with the new independent variables.

Direct Calculational Procedure for Testing $\mathscr{I}_E$

As was done in the previous cases, we shall now give an alternate form of condition (309) which can be used to test the canonical invariant $\mathscr{I}_E$ directly. In order to simplify the following analysis, we shall assume that $\mathscr{I}_E$ is analytic both in its dependence on x and in its dependence on y. Actually, this restriction can be weakened considerably, but it is not felt the additional effort is justified.

Suppose first that $\mathscr{I}_E$ satisfies condition (309). Then, since W is nonconstant, equation (309) can be written as

$$\frac{\mathscr{I}_E}{\left|\frac{dW}{dz}\right|^2} = \Phi(u) + \Psi(v) \tag{312}$$

But there exist functions Φ and Ψ such that equation (312) holds if, and only if,

$$\frac{\partial^2}{\partial u\, \partial v}\left(\frac{\mathscr{I}_E}{\left|\frac{dW}{dz}\right|^2}\right) = 0 \tag{313}$$

We shall be able to greatly simplify the analysis and use many of the results obtained in the hyperbolic case if we perform an analytic continuation of the function $\mathscr{I}_E$ to complex values of the variables x and y and introduce the complex substitution

$$\left.\begin{aligned} z &\equiv x + iy \\ z^* &= x - iy \end{aligned}\right\} \tag{314}$$

It is important to notice that z^* is the complex conjugate of z only when the variables x and y are real. Now when x and y are each allowed to range over the complex plane, the variables z and z^* can certainly be varied independently and we can therefore treat them as independent variables.

Now suppose x and y are real variables and $\chi = h + ig$ is an analytic function of the complex variable $z = x + iy$. If $\chi^* = h - ig$ is the complex conjugate of χ, then when the substitution (314) with x and y real is used to eliminate x and y in the particular formulas for χ and χ^*, it will be found that the expression for χ will contain only z and the expression for χ^* will contain only z^*.

Hence, if the variables x and y are continued analytically into the complex plane, then χ will be a function of the independent variable z only and χ^* will be a function of the complex variable z^* only. However, χ^* will be the complex conjugate of χ only when x and y are real, or equivalently, only when z^* is the complex conjugate of z. If P is any function of the complex variable z, we shall always denote by P^* the function of z^* which is equal to the complex conjugate of P when x and y are real (or equivalently, when z^* is the complex conjugate of z).

In view of these remarks, we see that when x and y are extended to complex values, then W, dW/dz, etc. are functions of the independent variable z only; and W^*, $(dW/dz)^* = (dW^*/dz^*)$, etc. are functions of the independent variable z^* only; and, for example, (dW^*/dz^*) is the complex conjugate of (dW/dz) only when x and y are restricted to the real line. In addition, the relations

$$\left.\begin{aligned} u &= \frac{1}{2} W + \frac{1}{2} W^* \\ \text{and} \qquad & \\ v &= \frac{W - W^*}{2i} \end{aligned}\right\} \tag{315}$$

still hold when x and y are complex. Since W is a nonconstant function of z only and W^* is therefore a nonconstant function of z^* only, we can also take W and W^* as independent variables; and equation (315) shows that u and v can be treated as independent variables. We shall use the notation W' for dW/dz and $(W^*)'$ for $dW^*/dz^* = (dW/dz)^*$, etc.

Now the principle of analytic continuation shows that equation (313) must still hold when x and y are continued to complex values. Equation (315) shows that

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial W} + \frac{\partial}{\partial W^*}$$

$$\frac{\partial}{\partial v} = i\left[\frac{\partial}{\partial W} - \frac{\partial}{\partial W^*}\right]$$

But

$$\frac{\partial}{\partial W} = \frac{1}{W'} \frac{\partial}{\partial z}$$

and

$$\frac{\partial}{\partial W^*} = \frac{1}{(W^*)'} \frac{\partial}{\partial z^*}$$

Hence,

$$\frac{\partial^2}{\partial u\, \partial v} = \frac{i}{W'} \frac{\partial}{\partial z} \frac{1}{W'} \frac{\partial}{\partial z} - \frac{i}{(W^*)'} \frac{\partial}{\partial z^*} \frac{\partial}{(W^*)'} \frac{\partial}{\partial z^*}$$

Using this in equation (313) we find

$$\frac{1}{W'}\frac{\partial}{\partial z}\left\{\frac{1}{W'}\frac{\partial}{\partial z}\left[\frac{\mathcal{J}_E}{W'(W^*)'}\right]\right\}=\frac{1}{(W^*)'}\frac{\partial}{\partial z^*}\left\{\frac{1}{(W^*)'}\frac{\partial}{\partial z}\left[\frac{\mathcal{J}_E}{W'(W^*)'}\right]\right\}$$

or, since $(W^*)'$ is indpendent of z and W' is independent of z^*,

$$\frac{\partial}{\partial z}\frac{1}{W'}\frac{\partial}{\partial z}\left(\frac{1}{W'}\mathcal{J}_E\right)=\frac{\partial}{\partial z^*}\frac{1}{(W^*)'}\frac{\partial}{\partial z^*}\left[\frac{1}{(W^*)'}\mathcal{J}_E\right]$$

Upon introducing the function S of z only by

$$S\equiv\left(\frac{1}{W'}\right)^2 \tag{316}$$

and according to our convention

$$S^*=\left[\frac{1}{(W^*)'}\right]^2$$

this equation becomes

$$2S\left(\mathcal{J}_E\right)_{zz}+3S'\left(\mathcal{J}_E\right)_z+S''\mathcal{J}_E=2S^*\left(\mathcal{J}_E\right)_{z^*z^*}+3(S^*)'\left(\mathcal{J}_E\right)_{z^*}+(S^*)''\mathcal{J}_E \tag{317}$$

As in the derivation of equation (135) from condition (131), we can show that the steps of this argument can be reversed to establish the following conclusions:

(C23) There exist a nonconstant and analytic function $W=u+iv$ of the complex variable $z=x+iy$ and functions Φ and Ψ such that $\mathcal{J}_E$ satisfies condition (309) if, and only if, if, there exist a nonzero function S of the complex variable z only and a nonzero function S^* of z^* only which is equal to the complex conjugate of S whenever z^* is equal to the complex conjugate of z such that $\mathcal{J}_E$ satisfies equation (317).

(C24) If nonzero functions S and S^* can be found such that equation (317) holds, then the analytic functions W of the complex variable z for which condition (309) is satisfied can be calculated from equation (316).

Thus, if the function S is known, conclusions (C21) and (C22) give a procedure for calculating a change in the independent variables which will transform equation (31) into a separable equation.

Now equation (317) is essentially the same as equation (135) for the hyperbolic case. Therefore, all the solutions to equation (317) can be found by the same procedure as was used for equation (135). The remarks following conclusion (C9) apply to this case also, except that, after the constants have been determined by substituting the solutions into equation (317), it must be possible to adjust the constants so that S is not zero and S^* reduces to the complex conjugate of S when x and y are real.

Again, the two exceptional forms of the function $\mathcal{J}_E$ for which the general procedure will not work must be considered separately. Thus, when $\mathcal{J}_E = 0$, condition (309) can always be satisfied. If $\mathcal{J}_E \neq 0$, we define the function $\dot{\mathcal{J}}_E$ by

$$\dot{\mathcal{J}}_E \equiv \frac{1}{\mathcal{J}_E} \frac{\partial}{\partial z}\left[\frac{1}{\mathcal{J}_E} \frac{\partial}{\partial z^*} \mathcal{J}_E\right] \quad \text{for } \mathcal{J}_E \neq 0 \tag{318}$$

or introducing the variables x and y this becomes

$$\dot{\mathcal{J}}_E = \frac{1}{4\mathcal{J}_E}\left[\frac{\partial}{\partial x}\left(\frac{1}{\mathcal{J}_E} \frac{\partial \mathcal{J}_E}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mathcal{J}_E} \frac{\partial \mathcal{J}_E}{\partial y}\right)\right] \tag{319}$$

In particular, this shows that like $\mathcal{J}_E$ itself the function $\dot{\mathcal{J}}_E$ is real whenever the variables x and y are real.

Now suppose there is a constant c_o such that

$$\dot{\mathcal{J}}_E = c_o \tag{320}$$

Clearly, c_o must be real. (Otherwise, the equation could not hold with x and y real.) For the present purpose, we need consider only real values of x and y. If $c_o = 0$, then

$$\frac{\partial}{\partial x}\left(\frac{1}{\mathcal{J}_E} \frac{\partial \mathcal{J}_E}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mathcal{J}_E} \frac{\partial \mathcal{J}_E}{\partial y}\right) = 0$$

but this implies that there exists a nonzero analytic function χ of the complex variable z such that

$$\mathcal{J}_E = \pi_1 |\chi(z)|^2$$

where π_1 is a constant. It is clear that condition (309) will always be satisfied if we take

$$\frac{dW}{dz} = \chi$$

If $c_o \neq 0$, equations (319) and (320) show that

$$\frac{\partial}{\partial x}\left(\frac{1}{\mathscr{I}_E}\frac{\partial \mathscr{I}_E}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mathscr{I}_E}\frac{\partial \mathscr{I}_E}{\partial y}\right) = 4c_o\mathscr{I}_E$$

It is shown in appendix I that the most general (real) solution tothis equation is

$$\mathscr{I}_E = \begin{cases} -\dfrac{1}{2c_o}\dfrac{(h_x)^2 + (h_y)^2}{\cosh^2 h} & \text{if } \mathscr{I}_E c_o \leq 0 \\[2ex] \dfrac{1}{2c_o}\dfrac{(h_x)^2 + (h_y)^2}{\cos^2 h} & \text{if } \mathscr{I}_E c_o \geq 0 \end{cases} \tag{321}$$

where h can be any harmonic function.

It is easy to see from equation (294) that if we put

$$\Phi(u) = \begin{cases} -\dfrac{1}{2c_o \cosh^2 u} & \text{for } \mathscr{I}_E c_o \leq 0 \\[2ex] \dfrac{1}{2c_o \cos^2 u} & \text{for } \mathscr{I}_E c_o > 0 \end{cases}$$

and $h = u$, then equation (321) satisfies condition (309).

By changing the harmonic function h it is possible to express the solution (321) in many different ways. Several of these are given in appendix I. Thus, the second line of equation (321) can be written either as

$$\frac{1}{2c_o u^2}\left|\frac{dW}{dz}\right|^2$$

or as

$$\frac{1}{2c_o}\left[\wp(u;g_2,g_3)+\wp(v;g_2,-g_3)\right]\left|\frac{dW}{dz}\right|^2$$

where $W = u + iv$ is an analytic function of z. It is easy to see that the first of these satisfies condition (309) with $\Phi(u) = 1/2c_o u^2$ and $\Psi(v) = 0$, and that the second satisfies condition (309) with $\Phi(u) = \wp(u;g_2,g_3)/2c_o$ and $\Psi(v) = \wp(v;g_2,-g_3)/2c_o$. It is shown in reference 2 that all the changes of the independent variable which will transform equation (31) into a separable equation when

$$\mathscr{J}_E = \frac{1}{2c_o u}\left|\frac{dW}{dz}\right|^2$$

are given by equation (I22) which transforms this expression for $\mathscr{J}_E$ into the form (I21).

We have now shown that

(C25) The condition (309) is always satisfied if either the invariant $\mathscr{J}_E$ or the function $\dot{\mathscr{J}}_E$ defined by equation (318) is equal to a constant.

Now as in the hyperbolic case we again suppose that $\mathscr{J}_E \neq$ constant, $\dot{\mathscr{J}}_E \neq$ constant, and that the functions S of z only and S^* of z^* only are any two simultaneous solutions of equation (317). The same procedure as was used in the hyperbolic case now leads to the equation

$$2\frac{\partial k}{\partial z}S + 5kS' = 2\frac{\partial k^*}{\partial z^*}S^* + 5k^*(S^*)' \tag{322}$$

where

$$\left.\begin{aligned} k &\equiv \mathscr{J}_E^4 \frac{\partial}{\partial z}\dot{\mathscr{J}}_E \\ k^* &\equiv \mathscr{J}_E^4 \frac{\partial}{\partial z^*}\dot{\mathscr{J}}_E \end{aligned}\right\} \tag{323}$$

Notice that since $\dot{\mathscr{J}}_E$ is real for x and y real, k^* becomes the complex conjugate of k when x and y are real. This shows that the notation used herein is justified.

Now if k were zero, equation (323) would show that

$$\frac{\partial}{\partial z} j_E = 0$$

Hence, we could conclude that for x and y real

$$\frac{\partial j_E}{\partial x} = \frac{\partial j_E}{\partial y} = 0$$

or j_E = constant. But this is contrary to hypothesis. Similarly, it can be shown that $k^* \neq 0$. Therefore, the procedure used to derive equation (148) can be applied here to show that if we define s_0, s_1, and s_2 by

$$s_0 = \begin{cases} (k^*)^{3/5} \dfrac{\partial}{\partial z^*} \left[\dfrac{(k^*)^{2/5}}{K} \left(\dfrac{1}{k^*} k_z \right)_z \right] - \dfrac{2}{5} k_z & \text{for } K \neq 0 \\ \left(\dfrac{1}{k^*} k_z \right)_z & \text{for } K = 0 \end{cases} \tag{324}$$

$$s_1 = \begin{cases} \dfrac{5}{2} (k^*)^{3/5} \dfrac{\partial}{\partial z^*} \left\{ \left(\dfrac{k^*}{k} \right)^{2/5} \dfrac{1}{K} \left[\left(\dfrac{k}{k^*} \right) k^{2/5} \right]_z \right\} - k & \text{for } K \neq 0 \\ \dfrac{5}{2} \dfrac{1}{k^{2/5}} \dfrac{\partial}{\partial z} \left[\left(\dfrac{k}{k^*} \right) k^{2/5} \right] & \text{for } K = 0 \end{cases} \tag{325}$$

$$s_2 = \begin{cases} \dfrac{5}{2} (k^*)^{3/5} \dfrac{\partial}{\partial z^*} \left[\dfrac{(k^*)^{2/5}}{K} \left(\dfrac{k}{k^*} \right) \right] & \text{for } K \neq 0 \\ \dfrac{5}{2} \dfrac{k}{k^*} & \text{for } K = 0 \end{cases} \tag{326}$$

with

$$K \equiv \left[\frac{1}{k^*} (k^*)_{z^*} \right]_z$$

then S must satisfy the equation

$$s_0 S + s_1 S' + s_2 S'' + 0$$

Similarly, S^* satisfies the equation

$$s_0^* S^* + s_1^* (S^*)' + S_2^* (S^*)'' = 0 \qquad (328)$$

where the coefficients s_0^*, s_1^*, and s_2^* can be obtained from equations (324) to (326) by interchanging the starred and unstarred quantities.

The solutions to equations (327) and (328) can now be obtained by exactly the procedure as used for equations (148) and (152). We shall therefore not repeat it here. Since an elliptic equation cannot be transformed into an equation which is weakly separable but not separable, this completes our discussion of the elliptic case.

SUMMARY AND CONCLUDING REMARKS

Necessary and sufficient conditions which a linear second-order partial differential equation in two independent variables must satisfy if it can be transformed into a separable equation (or into an equation which is weakly separable but not separable) have been obtained. These conditions together with the appropriate changes of variable (which will bring the equation into separable form) are summarized in tables I and II.

In addition, a procedure has been developed for testing by direct calculation whether a given equation of this type can be transformed into a separable equation, and a procedure has been given to calculate the new variables.

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National Aeronautics and Space Administration,
Cleveland, Ohio, January 14, 1970,
129-01.

TABLE I. - TRANSFORMATION OF EQUATIONS INTO SEPARABLE EQUATIONS

		Type of equation		
		Hyperbolic	Parabolic	Elliptic
Canonical form		$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} + cU = 0$	$\frac{\partial^2 U}{\partial x^2} + a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} + cU = 0$	$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y}\ cU = 0$
Canonical form	First invariant	$\mathcal{I}_H = a_y + b_x$	$\mathcal{I}_P = b$	$\mathcal{I}_E = a_y - b_x$
	Second invariant	$\mathcal{I}_H = c - \frac{1}{2}(a_x + b_y) - \frac{1}{4}(a^2 - b^2)$	$\mathcal{I}_P = \frac{\partial}{\partial x}\left(\frac{1}{b}\left\{2c - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right] - \left(a + \frac{b_x}{2b}\right)_x\right\}\right) - \frac{\partial}{\partial y}\left(a + \frac{b_x}{b}\right)$	$\mathcal{I}_E = c - \frac{1}{2}(a_x + b_y) - \frac{1}{4}(a^2 + b^2)$
Functional forms of the invariants for equations which can be transformed into separable equations[a]	First invariant	$\mathcal{I}_H = 0$	$\mathcal{I}_P = \frac{\tilde{v}(y)}{d_1(\varphi)}\varphi_x^2 \neq 0$	$\mathcal{I}_E = 0$
	Second invariant	$\mathcal{I}_H = [\Phi(u) + \Psi(v)](u_x^2 - u_y^2)$ where $u = \frac{1}{2}F(\sigma) + \frac{1}{2}G(\tau)$ $v = \frac{1}{2}F(\sigma) - \frac{1}{2}G(\tau)$ $\sigma = x + y \quad \tau = x - y$ $F' \neq 0 \quad G' \neq 0$	$\mathcal{I}_P = \left[\frac{1}{\tilde{v}(y)}R(\varphi)\right]_x - \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x + \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y$ where $d_1 \neq 0$	$\mathcal{I}_E = [\Phi(u) + \Psi(v)]\left\lvert\frac{dW}{dz}\right\rvert^2$ where $u = \mathcal{Re}\,W$ $v = \mathcal{Im}\,W$ and where W is any analytic function of the complex variable $z = x + iy$ with $dW/dZ \neq 0$
Change of variable which transforms differential equation into a separable equation	Dependent variable	$V = e^{\omega/2}U$ where $a = \omega_x$ $b = -\omega_y$	$V = e^{(1/2)[\omega + (1/2)\ln\lvert b\rvert + \theta]}U$ where $\theta = \int a\,dx$ $\omega_x = b\left(\frac{\varphi_y}{\varphi_x}\right)$ $\omega_y = \Omega - \frac{1}{\tilde{v}(y)}R(\varphi) + \frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2$ $\Omega \equiv \frac{1}{2}\left\{2c - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right] - \left(a + \frac{b_x}{2b}\right)_x\right\} - \left(\theta_y + \frac{b_y}{b}\right)$	$V = e^{\omega/2}U$ where $a = \omega_x$ $b = \omega_y$
	Independent variables[a]	$\xi = \tilde{\varphi}(u)$ where $\tilde{\varphi}' \neq 0$ $\eta = \tilde{\Psi}(v)$ where $\tilde{\Psi}' \neq 0$	$\xi = \varphi(x, y)$ $\eta = \tilde{\Psi}(y)$ where $\tilde{\Psi}' \neq 0$	$\xi = \tilde{\varphi}(u)$ where $\tilde{\varphi}' \neq 0$ $\eta = \tilde{\Psi}(v)$ where $\tilde{\Psi}' \neq 0$

[a] $\tilde{\varphi}$, $\tilde{\Psi}$, Φ, Ψ, F, G, R, $\tilde{v}$, d_1, and φ can be any functions of their arguments.

TABLE II. - TRANSFORMATION OF EQUATIONS INTO EQUATIONS WHICH ARE WEAKLY SEPARABLE BUT NOT SEPARABLE

		Type of equation, hyperbolic
Canonical form		$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + a \frac{\partial U}{\partial x} + b \frac{\partial U}{\partial y} + cU = 0$
Canonical invariants	First invariant	$\mathcal{I}_H = a_y + b_x$
	Second invariant	$\mathcal{J}_H = c - \frac{1}{2}(a_x + b_y) - \frac{1}{4}(a^2 - b^2)$
Functional forms of the invariants for equation which can be transformed into an equation which is weakly separable but not separable[a]		$\mathcal{I}_H = \Phi(v)F'(\sigma)G'(\tau)$ $\mathcal{J}_H = \Psi(v)F'(\sigma)G'(\tau)$ where $v = F(\sigma) + G(\tau), \quad v = F(\sigma), \quad \text{or} \quad v = G(\tau)$ $\sigma = x + y, \quad \tau = x - y$ $F' \neq 0, \quad G' \neq 0, \quad \text{and} \quad \Phi \neq 0$
Change of variable which transforms differential equation which is weakly separable but not separable	Dependent variable	$V = e^{\omega^{(1)}}U$ when $v = F + G$ or when $v = F$ $\left.\begin{aligned} \omega_x^{(1)} &= \frac{a}{2} - \frac{1}{4} G'(\tau) \int \Phi(v)\,dv \\ \omega_y^{(1)} &= -\frac{b}{2} + \frac{1}{4} G'(\tau) \int \Phi(v)\,dv \end{aligned}\right\\}$ for $v = F + G$ or $v = F$ $V = e^{\omega^{(2)}}U$ when $v = F + G$ or when $v = G$ $\left.\begin{aligned} \omega_x^{(2)} &= \frac{a}{2} + \frac{1}{4} F'(\sigma) \int \Phi(v)\,dv \\ \omega_y^{(2)} &= -\frac{b}{2} + \frac{1}{4} F'(\sigma) \int \Phi(v)\,dv \end{aligned}\right\\}$ for $v = F + G$ or $v = G$
	Independent variables[a]	$\xi = \tilde{\varphi}(u)$ where $\tilde{\varphi}' \neq 0$ $\eta = \tilde{\psi}(v)$ where $\tilde{\psi}' \neq 0$

[a] $\tilde{\varphi}$, $\tilde{\psi}$, Φ, Ψ, F, and G can be any functions of their arguments.

APPENDIX A

DERIVATION OF EQUATION (140) FROM EQUATION (135)

For brevity, I will be used in place of $\mathcal{I}_H$ in this appendix. With this notation equation (135) is, after dividing through by I,

$$2\left(\frac{I_{\sigma\sigma}}{I}\right) S + 3\left(\frac{I_\sigma}{I}\right) S' + S'' = 2\left(\frac{I_{\tau\tau}}{I}\right) T + 3\left(\frac{I_\tau}{I}\right) T' + T'' \tag{A1}$$

Differentiating both sides of this expression with respect to σ and τ, we find

$$\frac{\partial^2}{\partial\sigma\,\partial\tau}\left[2\left(\frac{I_{\sigma\sigma}}{I}\right) S + 3\left(\frac{I_\sigma}{I}\right) S'\right] = \frac{\partial^2}{\partial\sigma\,\partial\tau}\left[2\left(\frac{I_{\tau\tau}}{I}\right) T + 3\left(\frac{I_\tau}{I}\right) T'\right]$$

Hence,

$$\left[2\left(\frac{I_{\sigma\sigma}}{I}\right)_\tau S + 3\left(\frac{I_\sigma}{I}\right)_\tau S'\right]_\sigma = \left[2\left(\frac{I_{\tau\tau}}{I}\right)_\sigma T + 3\left(\frac{I_\tau}{I}\right)_\sigma T'\right]_\tau$$

Upon defining μ by

$$\mu \equiv \left(\frac{I_\sigma}{I}\right)_\tau = \left(\frac{I_\tau}{I}\right)_\sigma$$

we find

$$2\left(\frac{I_{\sigma\sigma}}{I}\right)_{\sigma\tau} S + \left[2\left(\frac{I_{\sigma\sigma}}{I}\right)_\tau + 3\mu_\sigma\right] S' + 3\mu S'' = 2\left(\frac{I_{\tau\tau}}{I}\right)_{\sigma\tau} T + \left[2\left(\frac{I_{\tau\tau}}{I}\right)_\sigma + 3\mu_\tau\right] T' + 3\mu T'' \tag{A2}$$

Since

$$\left(\frac{I_{\sigma\sigma}}{I}\right) = \left(\frac{I_\sigma}{I}\right)_\sigma + \left(\frac{I_\sigma}{I}\right)^2$$

and

$$\left(\frac{I_{\tau\tau}}{I}\right) = \left(\frac{I_\tau}{I}\right)_\tau + \left(\frac{I_\tau}{I}\right)^2$$

equation (A2) becomes

$$2\left(\frac{I_{\sigma\sigma}}{I}\right)_{\sigma\tau} S + \left[5\mu_\sigma + 4\left(\frac{I_\sigma}{I}\right)\mu\right] S' + 3\mu S'' = 2\left(\frac{I_{\tau\tau}}{I}\right)_{\sigma\tau} T + \left[5\mu_\tau + 4\left(\frac{I_\tau}{I}\right)\mu\right] T' + 3\mu T'' \qquad \text{(A3)}$$

Now multiplying equation (A1) by 3μ and subtracting the result from equation (A3), we obtain

$$2\mu\left[\frac{1}{\mu}\left(\frac{I_{\sigma\sigma}}{I}\right)_{\sigma\tau} - 3\left(\frac{I_{\sigma\sigma}}{I}\right)\right] S + 5\mu\left(\frac{\mu_\sigma}{\mu} - \frac{I_\sigma}{I}\right) S' = 2\mu\left[\frac{1}{\mu}\left(\frac{I_{\tau\tau}}{I}\right)_{\sigma\tau} - 3\left(\frac{I_{\tau\tau}}{I}\right)\right] T + 5\mu\left(\frac{\mu_\tau}{\mu} - \frac{I_\tau}{I}\right) T' \qquad \text{(A4)}$$

Now

$$\frac{1}{\mu}\left(\frac{I_{\sigma\sigma}}{I}\right)_{\sigma\tau} - 3\left(\frac{I_{\sigma\sigma}}{I}\right) = \frac{1}{\mu}\left\{\left(\frac{I_\sigma}{I}\right)_{\sigma\sigma\tau} + \left[\left(\frac{I_\sigma}{I}\right)^2\right]_{\sigma\tau}\right\} - 3\left[\left(\frac{I_\sigma}{I}\right)_\sigma + \left(\frac{I_\sigma}{I}\right)^2\right]$$

$$= \frac{1}{\mu}\mu_{\sigma\sigma} + \frac{2}{\mu}\left[\left(\frac{I_\sigma}{I}\right)\mu\right]_\sigma - 3\left(\frac{I_\sigma}{I}\right)_\sigma - 3\left(\frac{I_\sigma}{I}\right)^2$$

$$= \left(\frac{1}{\mu}\mu_\sigma\right)_\sigma + \left(\frac{\mu_\sigma}{\mu}\right)^2 + 2\left(\frac{I_\sigma}{I}\right)_\sigma + \frac{2\mu_\sigma}{\mu}\left(\frac{I_\sigma}{I}\right) - 3\left(\frac{I_\sigma}{I}\right)_\sigma - 3\left(\frac{I_\sigma}{I}\right)^2$$

$$= \left(\frac{1}{\mu}\mu_\sigma - \frac{I_\sigma}{I}\right)_\sigma + \left(\frac{\mu_\sigma}{\mu}\right)^2 - 2\frac{\mu_\sigma}{\mu}\left(\frac{I_\sigma}{I}\right) + \left(\frac{I_\sigma}{I}\right)^2 + 4\left(\frac{\mu_\sigma}{\mu} - \frac{I_\sigma}{I}\right)\frac{I_\sigma}{I}$$

$$= \left[\frac{I}{\mu}\left(\frac{\mu}{I}\right)_\sigma\right]_\sigma + \left(\frac{I}{\mu}\right)^2\left[\left(\frac{\mu}{I}\right)_\sigma\right]^2 + 4\frac{I}{\mu}\left(\frac{\mu}{I}\right)_\sigma\left(\frac{I_\sigma}{I}\right)$$

$$= \frac{I}{\mu}\left[\left(\frac{\mu}{I}\right)_{\sigma\sigma} + 4\left(\frac{\mu}{I}\right)_\sigma \frac{I_\sigma}{I}\right]$$

$$= \frac{I}{\mu}\,\frac{1}{I^4}\left[I^4\left(\frac{\mu}{I}\right)_\sigma\right]_\sigma$$

Interchanging σ and τ shows that

$$\frac{1}{\mu}\left(\frac{I_{\tau\tau}}{I}\right)_{\sigma\tau} - 3\left(\frac{I_{\tau\tau}}{I}\right) = \frac{I}{\mu}\,\frac{1}{I^4}\left[I^4\left(\frac{\mu}{I}\right)_\tau\right]_\tau$$

When these results are substituted into equation (A4) we obtain, after multiplying through by I^3,

$$2\left[I^4\left(\frac{\mu}{I}\right)_\sigma\right]_\sigma S + 5\left[I^4\left(\frac{\mu}{I}\right)_\sigma\right]S' = 2\left[I^4\left(\frac{\mu}{I}\right)_\tau\right]_\tau T + 5\left[I^4\left(\frac{\mu}{I}\right)_\tau\right]T' \qquad \text{(A5)}$$

Since we have put $I = \mathcal{J}_H$ and $\mu = (I_\sigma/I)_\tau$, definition (136) shows that

$$\left(\frac{\mu}{I}\right) = j_H$$

and therefore equation (A5) is the same as equation (140).

APPENDIX B

CONDITIONS FOR WHICH COEFFICIENTS IN EQUATIONS (148) AND (152) VANISH

Since $k_1 \neq 0$ and $k_2 \neq 0$, it follows from equation (147) that the condition $t^{(2)} = 0$ implies that

$$\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau \neq 0$$

and that there exists a nonzero function λ_1 of τ only such that

$$\frac{k_1^2\left(\dfrac{k_2}{k_1}\right)^5}{\left\{\left[\dfrac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau\right\}^5} = \lambda_1(\tau) \tag{B1}$$

Hence, it follows from equation (146) and equation (B1) that the condition $t^{(1)} = 0$ implies that

$$\frac{5}{2} k_2 \frac{1}{\left[\dfrac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau} \frac{\partial}{\partial\sigma}\left[\frac{1}{k_2^{2/5}}\left(\frac{k_1}{k_2}\right)\left(\frac{k_2}{k_1} k_2^{2/5}\right)_\tau\right] - k_2 = 0$$

but this shows that

$$\frac{\partial}{\partial\sigma}\left[\frac{k_1}{k_2} \frac{\partial}{\partial\tau}\left(\frac{k_2}{k_1}\right)\right] = 0$$

It follows from this that there exist a nonzero function λ_2 of τ only and a nonzero function γ_2 of σ only such that

$$\frac{k_2}{k_1} = \lambda_2(\tau)\gamma_2(\sigma) \tag{B2}$$

and that

$$\left[\frac{1}{k_2}\left(k_2\right)_\tau\right]_\sigma = \left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau \tag{B3}$$

It now follows from equations (145), (B2), and (B1) that the condition $t^{(0)} = 0$ implies that

$$\frac{k_2}{\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau} \frac{\partial}{\partial\sigma}\left\{\frac{1}{\lambda_2}\left[\lambda_2 \frac{\left(k_2\right)_\tau}{k_2}\right]_\tau\right\} - \frac{2}{5}\left(k_2\right)_\tau = 0$$

Hence, upon using equation (B3), this shows that

$$\frac{1}{\lambda_2\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau} \frac{\partial}{\partial\tau}\left\{\lambda_2\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau\right\} - \frac{2}{5}\frac{\left(k_2\right)_\tau}{k_2} = 0$$

This now shows that there exists a nonzero function γ_4 of σ only such that

$$\frac{\lambda_2^5}{k_2^2}\left\{\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau\right\}^5 = \gamma_4(\sigma)$$

Hence, upon substituting for k_1 and k_2 from equations (B1) and (B2), we find that

$$\frac{\left[\lambda_2(\tau)\right]^8}{\lambda_1(\tau)} = \frac{\gamma_4(\sigma)}{\left[\gamma_2(\sigma)\right]^3}$$

We conclude that there exists a nonzero constant π_1 such that

$$\gamma_4 = \frac{\gamma_2^3}{\pi_1}$$

and

$$\lambda_1 = \lambda_2^8 \pi_1$$

Hence, equation (B1) becomes

$$\left\{\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau\right\}^5 = \frac{k_1^2}{\pi_1 \lambda_2^8}\left(\frac{k_2}{k_1}\right)^5$$

or using equation (B2)

$$\left\{\left[\frac{1}{k_1}\left(k_1\right)_\sigma\right]_\tau\right\}^5 = \frac{\gamma_2^5}{\pi_1 \lambda_2^3} k_1^2 \tag{157}$$

This equation can also be written as

$$\left\{\frac{\partial}{\partial \tau}\left[\left(\frac{\gamma_2^5 k_1^2}{\lambda_2^3}\right)^{-1}\left(\frac{\gamma_2^5 k_1^2}{\lambda_2^3}\right)_\sigma\right]\right\}^5 = \frac{2^5}{\pi_1}\left(\frac{\gamma_2^5}{\lambda_2^3} k_1^2\right)$$

But this is essentially Liouville's equation, and therefore its most general solution is (see eq. (138))

$$k_1^2 = -\pi_1 \frac{\lambda_2^3(\tau)}{\gamma_2^5(\sigma)}\left\{5 \frac{\lambda_3'(\tau)\gamma_3'(\sigma)}{\left[\gamma_3(\sigma) - \lambda_3(\tau)\right]^2}\right\}^5 \tag{155}$$

and equation (B2) now shows that

$$k_2^2 = -\pi_1 \frac{\lambda_2^5(\tau)}{\gamma_2^3(\sigma)}\left\{5 \frac{\lambda_3'(\tau)\gamma_3'(\sigma)}{\left[\gamma_3(\sigma) - \lambda_3(\tau)\right]^2}\right\}^5 \tag{156}$$

Thus, $t^{(0)} = t^{(1)} = t^{(2)} = 0$ only if k_1 and k_2 satisfy conditions (155) and (156). It is an easy matter to verify that the steps of the preceding argument can be reversed to show that equations (155) and (156) are also sufficient conditions. It now follows from the symmetry between equations (145) to (148) and equations (149) to (151) and between equations (155) and (156) that $t^{(0)} = t^{(1)} = t^{(2)} = 0$ if and only if, $s^{(0)} = s^{(1)} = s^{(2)} = 0$. It is also clear that equations (B2) and (157) are necessary and sufficient conditions for the coefficients in equations (148) and (152) to vanish.

APPENDIX C

CALCULATION OF S AND T WHEN COEFFICIENTS OF EQUATIONS (148) AND (150) ALL VANISH

Upon substituting equations (155) and (156) into equation (142), we get

$$\frac{\partial}{\partial\sigma}\left\{\left(\frac{S}{\gamma_2}\right)\left[\frac{\lambda_3'\gamma_3'}{(\gamma_3 - \lambda_3)^2}\right]\right\} = \frac{\partial}{\partial\tau}\left\{(\lambda_2 T)\left[\frac{\lambda_3'\gamma_3'}{(\gamma_3 - \lambda_3)^2}\right]\right\}$$

Carrying out the indicated differentiations and rearranging gives

$$\frac{\gamma_3 - \lambda_3}{\gamma_3'}\left(\frac{S\gamma_3'}{\gamma_2}\right)' - 2\left(\frac{S}{\gamma_2}\gamma_3'\right) = \frac{\gamma_3 - \lambda_3}{\lambda_3'}(T\lambda_2\lambda_3')' + 2(T\lambda_2\lambda_3') \tag{C1}$$

Differentiating equation (C1) with respect to both σ and τ gives

$$-\lambda_3'\left[\frac{1}{\gamma_3'}\left(\frac{S\gamma_3'}{\gamma_2}\right)'\right]' = \gamma_3'\left[\frac{1}{\lambda_3'}(T\lambda_2\lambda_3')'\right]'$$

or

$$\frac{1}{\gamma_3'}\left[\frac{1}{\gamma_3'}\left(\frac{S\gamma_3'}{\gamma_2}\right)'\right]' = -\frac{1}{\lambda_3'}\left[\frac{1}{\lambda_3'}(T\lambda_2\lambda_3')'\right]'$$

since the left side is a function of σ only and the right side is a function of τ only there exists a constant, say π_2, such that

$$\frac{1}{\gamma_3'}\left[\frac{1}{\gamma_3'}\left(\frac{S\gamma_3'}{\gamma_2}\right)'\right]' = \pi_2$$

$$\frac{1}{\lambda_3'}\left[\frac{1}{\lambda_3'}(\lambda_2\lambda_3'T)'\right]' = -\pi_2$$

Upon integrating these two equations, we get

$$\frac{S\gamma_3'}{\gamma_2} = \frac{\pi_2}{2}\gamma_3^2 + \pi_3\gamma_3 + \pi_4 \tag{161}$$

$$(T\lambda_2\lambda_3') = -\frac{\pi_2}{2}\lambda_3^2 + \pi_5\lambda_3 + \pi_6$$

where π_3 through π_6 are constants.

Upon substituting these results back into equation (C1), we find that

$$\pi_5 = -\pi_3$$

$$\pi_4 = -\pi_6$$

so

$$T\lambda_2\lambda_3' = -\frac{\pi_2}{2}\lambda_3^2 - \pi_3\lambda_3 - \pi_4 \tag{162}$$

APPENDIX D

DERIVATION OF EXPRESSION FOR $\mathscr{J}_P$

We first differentiate equation (231) with respect to x to obtain

$$\frac{\partial}{\partial x}\left(a + \frac{b_x}{2b}\right) = \left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)'(\varphi_x)^2 + \left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)\varphi_{xx} - \left(b\frac{\varphi_y}{\varphi_x}\right)_x$$

Next squaring both sides of equation (231) gives

$$\frac{1}{2}\left(a + \frac{b_x}{2b}\right)^2 = \frac{1}{2}\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)^2(\varphi_x)^2 - b\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)\varphi_y + \frac{b^2}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2$$

Finally, multiplying equation (231) by $-b_x/2b$ and using equation (230) gives

$$-\frac{b_x}{2b}\left(a + \frac{1}{2}\frac{b_x}{b}\right) = -\left(\frac{d_2 - \frac{1}{2}d_2'}{d_1}\right)\varphi_{xx} + \frac{1}{2}\frac{d_1'}{d_1}\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)\varphi_x^2 + \frac{b_x}{2}\left(\frac{\varphi_y}{\varphi_x}\right)$$

Upon adding these three equations, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x}\left(a + \frac{b_x}{2b}\right) + \frac{1}{2}\left(a + \frac{b_x}{2b}\right)^2 - \frac{b_x}{2b}\left(a + \frac{b_x}{2b}\right) &= \left[\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)' + \frac{1}{2}\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)^2 + \frac{1}{2}\frac{d_1'}{d_1}\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)\right]\varphi_x^2 \\
&\quad - \left(b\frac{\varphi_y}{\varphi_x}\right)_x + \frac{b^2}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2 + \frac{1}{2}b_x\left(\frac{\varphi_y}{\varphi_x}\right) - b\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)\varphi_y \\
&= \left\{\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)' + \frac{1}{2}\left[\left(\frac{d_2}{d_1}\right)^2 - \left(\frac{d_1'}{2d_1}\right)^2\right]\right\}\varphi_x^2 + b\left(\frac{\varphi_y}{\varphi_x}\right)\frac{\varphi_{xx}}{\varphi_x} \\
&\quad + \frac{1}{2}b_x\left(\frac{\varphi_y}{\varphi_x}\right) + \frac{b^2}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2 - \frac{1}{\varphi_x}(b\varphi_y)_x - b\varphi_y\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)
\end{aligned}$$

Using equation (230) to eliminate φ_{xx}/φ_x yields

$$\frac{\partial}{\partial x}\left(a + \frac{b_x}{2b}\right) + \frac{1}{2}\left(a + \frac{b_x}{2b}\right)^2 - \frac{b_x}{2b}\left(a + \frac{b_x}{2b}\right) = \left\{\left(\frac{d_2 - \frac{1}{2} d_1'}{d_1}\right)' + \frac{1}{2}\left[\left(\frac{d_2}{d_1}\right)^2 - \left(\frac{d_1'}{2d_1}\right)^2\right]\right\}\varphi_x^2 + \frac{1}{2}\frac{d_1'}{d_1} b\varphi_y$$

$$- \left(\frac{d_2 - \frac{1}{2} d_1'}{d_1}\right) b\varphi_y + b_x\left(\frac{\varphi_y}{\varphi_x}\right) - \frac{(b\varphi_y)_x}{\varphi_x} + \frac{b^2}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2$$

$$= \left\{\left(\frac{d_2 - \frac{1}{2} d_1'}{d_1}\right)' + \frac{1}{2}\left[\left(\frac{d_2}{d_1}\right)^2 - \left(\frac{d_1'}{2d_1}\right)^2\right]\right\}\varphi_x^2$$

$$+ \left[\frac{1}{2}\frac{d_1'}{d_1} - \left(\frac{d_2 - \frac{1}{2} d_1'}{d_1}\right)\right] b\varphi_y - b\frac{\varphi_{xy}}{\varphi_x} + \frac{b^2}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2 \tag{D1}$$

After differentiating equation (226) with respect to y, we obtain

$$\frac{b_y}{b} = \frac{v'}{v} + 2\frac{\varphi_{xy}}{\varphi_x} - \frac{d_1'}{d_1}\varphi_y$$

Upon using this equation to eliminate φ_{xy}/φ_x and equation (226) to eliminate $(\varphi_x)^2$ in equation (D1), we obtain, after rearranging the left side and dividing by b,

$$\frac{1}{b}\left\{\frac{\partial}{\partial x}\left(a + \frac{b_x}{2b}\right) + \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right]\right\} = \frac{1}{v} d_1\left\{\left(\frac{d_2 - \frac{1}{2} d_1'}{d_1}\right)' + \frac{1}{2}\left[\left(\frac{d_2}{d_1}\right)^2 - \left(\frac{d_1'}{2d_1}\right)^2\right]\right\}$$

$$- \left(\frac{d_2 - \frac{1}{2} d_1'}{d_1}\right)\varphi_y + \frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2 + \frac{v'}{2v} - \frac{b_y}{2b}$$

Subtracting this result from equation (229) shows that

$$\frac{1}{b}\left\{2c - \frac{\partial}{\partial x}\left(a + \frac{b_x}{2b}\right) - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right]\right\} - \frac{b_y}{2b} = \frac{1}{v}R(\varphi) + \left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)\varphi_y - \frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2 - \frac{v'}{2v} + \frac{2e_3}{v}$$

where we have defined the function R of φ only by

$$R \equiv 2d_3 - d_1\left\{\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)' + \frac{1}{2}\left[\left(\frac{d_2}{d_1}\right)^2 - \left(\frac{d_1'}{2d_1}\right)^2\right]\right\}$$

After differentiating this result with respect to x, we obtain, upon recalling that e_3 depends (implicitly through ψ) only on y,

$$\left(\frac{1}{b}\left\{2c - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right] - \frac{\partial}{\partial x}\left(a + \frac{b_x}{2b}\right)\right\} - \frac{b_y}{2b} - \frac{1}{v}R\right)_x = \left[\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)\varphi_y\right]_x - \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x \tag{D2}$$

Differentiating equation (231) with respect to y shows (since d_1 and d_2 are functions of φ only) that

$$\left(a + \frac{b_x}{2b}\right)_y = \left[\left(\frac{d_2 - \frac{1}{2}d_1'}{d_1}\right)\varphi_y\right]_x - \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y \tag{D3}$$

Upon recalling definition (46) and noting that $(b_y/b)_x = (b_x/b)_y$, we find, after subtracting equation (D3) from equation (D2), that

$$\mathcal{I}_P = \left[\frac{1}{v}R(\varphi)\right]_x + \left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_y - \left[\frac{b}{2}\left(\frac{\varphi_y}{\varphi_x}\right)^2\right]_x \tag{D4}$$

which is the desired result.

APPENDIX E

DERIVATION OF ALTERNATE FORM OF CONDITION (233)

Suppose first that $\varphi_v \neq 0$. Now there exists a function d_1 of φ only such that equation (233) holds if and only if,

$$\varphi_x \frac{\partial}{\partial y}\left(\frac{b}{v\varphi_x^2}\right) - \varphi_y \frac{\partial}{\partial x}\left(\frac{b}{v\varphi_x^2}\right) = 0$$

Upon differentiating by parts, shows that this is, in turn, equivalent to

$$\frac{\partial}{\partial y}\left(\frac{b}{v\varphi_x}\right) - \frac{\partial}{\partial x}\left(\frac{b}{b\varphi_x^2}\varphi_y\right) = 0$$

Another differentiation by parts, we find that this is equivalent to

$$\frac{\partial}{\partial x}\left[b\left(\frac{\varphi_y}{\varphi_x}\right)^2\right] = v\left(\frac{\varphi_y}{\varphi_x}\right)\frac{\partial}{\partial y}\left(\frac{b}{v}\right)$$

Hence, we can conclude from this that, if $\varphi_y \neq 0$, there exists a function d_1 of φ only such that equation (233) holds if and only if,

$$\left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right]_x - \frac{1}{2}\,\frac{b_x}{b}\left[b\left(\frac{\varphi_y}{\varphi_x}\right)\right] = \frac{1}{2}\,v\left(\frac{b}{v}\right)_y \tag{E1}$$

Now suppose that $\varphi_y = 0$. This implies that φ is a function of x only. If there exists a function d_1 such that equation (233) holds, then this implies (since d_1 depends on x or y only implicitly through φ) that d/v is a function of x only. Hence,

$$\left(\frac{b}{v}\right)_y = 0$$

But this implies that equation (E1) holds. Conversely, if equation (E1) holds, then the fact that $\varphi_y = 0$ shows that

$$\left(\frac{b}{v}\right)_y = 0$$

since $v \neq 0$. Thus, b/v and φ_x are functions of x only. Hence,

$$\frac{b}{v\varphi_x^2}$$

is a function of x only and therefore (since φ is a function of x only) a function of φ only. This shows that there exists a function d_1 of φ only such that equation (233) holds. We can therefore conclude that there exists a function d_1 of φ only such that equation (233) holds if and only if, equation (E1) is satisfied.

APPENDIX F

APPLICATION OF TRANSFORMATION (240) TO EQUATION (30)

It follows from equations (41) to (43) and (234) that under the change of variable (240) the coefficients $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ of equation (40) are given by

$$\tilde{a} = -\left(\omega_x + \frac{1}{2}\frac{b_x}{b}\right) \tag{F1}$$

$$\tilde{b} = b \tag{F2}$$

and

$$2\tilde{c} = 2c - a\left(\omega_x + \frac{1}{2}\frac{b_x}{b} + a\right) - b\left(\omega_y + \frac{1}{2}\frac{b_y}{b} + \theta_y\right) - \left(\omega_x + \frac{1}{2}\frac{b_x}{b} + a\right)_x + \frac{1}{2}\left(\omega_x + \frac{1}{2}\frac{b_x}{b} + a\right)^2$$

$$= 2c + \frac{1}{2}\left(\omega_x + \frac{1}{2}\frac{b_x}{b}\right)^2 - \frac{1}{2}a^2 - b\left(\omega_y + \frac{1}{2}\frac{b_y}{b} + \theta_y\right) - \left(\omega_x + \frac{1}{2}\frac{b_x}{b} + a\right)_x$$

$$= 2c - \frac{1}{2}\left[a^2 - \left(\frac{b_x}{2b}\right)^2\right] - \left(a + \frac{1}{2}\frac{b_x}{b}\right)_x - b\left(\theta_y + \frac{b_y}{b}\right) + \frac{\omega_x^2}{2} + \frac{1}{2}\frac{b_x}{b}\omega_x - b\omega_y + \frac{1}{2}b_y - \omega_{xx}$$

Upon substituting in definition (235) this becomes

$$2\tilde{c} = b\left(\Omega + \frac{\omega_x^2}{2b} - \omega_y\right) - \left(\omega_{xx} - \frac{1}{2}\frac{b_x}{b}\omega_x - \frac{1}{2}b_y\right)$$

Hence, equations (238) and (239) now show that

$$\tilde{c} = \frac{b}{2v}\left[R(\varphi) + \frac{1}{2}v'\right] \tag{F3}$$

Substituting equations (F1) to (F3) into equation (40) yields

$$V_{xx} - \left(\omega_x + \frac{1}{2}\,\frac{b_x}{b}\right)V_x + bV_y + \frac{b}{2v}\left[R(\varphi) + \frac{1}{2}\,v'\right]V = 0 \tag{241}$$

APPENDIX G

DERIVATION OF EQUATION (251) FROM EQUATION (249)

Upon differentiating equation (249) with respect to x, we get

$$\left(vfP\right)_{xy} - bT_{yyx} - b_xT_{yy} - b_yT_{xy} - b_{xy}T_y = \left(TfP\right)_{xx}$$

This equation can be rearranged to obtain

$$\left(vfP\right)_{xy} - b\left(T_x + \frac{1}{2}\frac{b_x}{b}T\right)_{yy} - \frac{1}{2}b_xT_{yy} + b\left(\frac{b_x}{b}\right)_y T_y + \frac{b}{2}\left(\frac{b_x}{b}\right)_{yy} T$$

$$- b_y\left(T_x + \frac{1}{2}\frac{b_x}{b}T\right)_y + \frac{1}{2}\frac{b_yb_x}{b}T_y + \frac{b_y}{2}\left(\frac{b_x}{b}\right)_y T - b_{xy}T_y = \left(TfP\right)_{xx}$$

Hence,

$$\left(TfP\right)_{xx} = \left(vfP\right)_{xy} - \frac{\partial}{\partial y}\left[b\left(T_x + \frac{1}{2}\frac{b_x}{b}T\right)_y\right] - \frac{1}{2}b_xT_{yy} - \frac{1}{2}\frac{b_xb_y}{b}T_y + \frac{1}{2}\left[b_y\left(\frac{b_x}{b}\right)_y + b\left(\frac{b_x}{b}\right)_{yy}\right]T$$

$$= \left(vfP\right)_{xy} - \frac{\partial}{\partial y}\left[b\left(T_x + \frac{1}{2}\frac{b_x}{b}T\right)_y\right] - \frac{b_x}{2b}\frac{\partial}{\partial y}\left(bT_y\right) + \frac{1}{2}\left[b\left(\frac{b_x}{b}\right)_y\right]_y T$$

Upon using equation (249) to eliminate $\left(bT_y\right)_y$ in this equation we obtain

$$\left(vfP\right)_{xy} - \frac{b_x}{2b}\left(vfP\right)_y = \left(TfP\right)_{xx} - \frac{b_x}{2b}\left(TfP\right)_x + \left[b\left(T_x + \frac{b_x}{b}T\right)_y\right]_y - \frac{1}{2}\left[b\left(\frac{b_x}{b}\right)_y\right]_y T$$

Substituting equation (250) gives

$$\left(vfP\right)_{xy} - \frac{b_x}{2b}\left(vfP\right)_y = \left(TfP\right)_{xx} - \frac{b_x}{2b}\left(TfP\right)_x + \left[\frac{b}{2}\left(v\frac{b_y}{b} - v'\right)_y\right]_y - \frac{1}{2}\left[b\left(\frac{b_x}{b}\right)_y\right]_y T$$

or

$$\left(v\mathcal{J}_P\right)_{xy} - \frac{b_x}{2b}\left(v\mathcal{J}_P\right)_y + \left[\frac{b}{2}\left(v' - v\,\frac{b_y}{b}\right)_y\right]_y = T_{xx}\mathcal{J}_P + \left[2\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\,\mathcal{J}_P\right]T_x$$

$$+ \left\{\left(\mathcal{J}_P\right)_{xx} - \frac{b_x}{2b}\left(\mathcal{J}_P\right)_x - \frac{1}{2}\left[b\left(\frac{b_x}{b}\right)_y\right]_y\right\}T \qquad \text{(G1)}$$

Differentiating equation (250) with respect to x yields

$$T_{xx} = -\frac{1}{2}\left(\frac{b_x}{b}\right)_x T - \frac{1}{2}\,\frac{b_x}{b}\,T_x + \frac{1}{2}\,v\left(\frac{b_y}{b}\right)_x$$

Substituting this result into equation (G1) gives

$$\left[v\left(\mathcal{J}_P\right)_x\right]_y - \left(v\,\frac{b_x}{2b}\,\mathcal{J}_P\right)_y + \left[\frac{b}{2}\left(v' - v\,\frac{b_y}{b}\right)_y\right]_y = 2\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\,\mathcal{J}_P\right]T_x$$

$$+ \left\{\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\,\mathcal{J}_P\right]_x - \frac{1}{2}\left[b\left(\frac{b_x}{b}\right)_y\right]_y\right\}T$$

$$= \left\{v\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\,\mathcal{J}_P\right] + \frac{b}{2}\left(v' - v\,\frac{b_y}{b}\right)_y\right\}_y$$

Hence,

$$b\left\{\frac{v}{b}\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\,\mathcal{J}_P\right]\right\}_y + \left[\frac{b}{2}\left(v' - v\,\frac{b_y}{b}\right)_y\right]_y = b\left\{\frac{1}{b}\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\,\mathcal{J}_P\right]\right\}_x T$$

$$- v'\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\,\mathcal{J}_P\right] - \frac{1}{2}\left[b\left(\frac{b_x}{b}\right)_y\right]_y T \qquad \text{(G2)}$$

Now

$$\frac{1}{b}\left[\frac{b}{2}\left(v' - v\,\frac{b_y}{b}\right)_y\right]_y = \frac{1}{2b}\left[v''b - b_y v' - vb\left(\frac{b_y}{b}\right)_y\right]_y$$

$$= \frac{v'''}{2} - \frac{v'}{2}\left[\frac{b_{yy}}{b} + \left(\frac{b_y}{b}\right)_y\right] - \frac{v}{2b}\left[b\left(\frac{b_y}{b}\right)_y\right]_y$$

$$= \frac{v'''}{2} - v'\left[\frac{b_{yy}}{b} - \frac{1}{2}\left(\frac{b_y}{b}\right)^2\right] - \frac{v}{2}\left[\left(\frac{b_y}{b}\right)_{yy} + \left(\frac{b_y}{b}\right)\left(\frac{b_y}{b}\right)_y\right]$$

$$= \frac{v'''}{2} + 2v'\left[\left(\frac{b_y}{2b}\right)^2 - \frac{b_{yy}}{2b}\right] + v\left[\left(\frac{b_y}{2b}\right)^2 - \frac{b_{yy}}{2b}\right]_y$$

Upon substituting this result into equation (G2) and rearranging, we obtain

$$2v'\left\{\frac{1}{b}\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\mathcal{J}_P\right] - \frac{b_{yy}}{2b} + \left(\frac{b_y}{2b}\right)^2\right\} + v\left\{\frac{1}{b}\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\mathcal{J}_P\right] - \frac{b_{yy}}{2b} + \left(\frac{b_y}{2b}\right)^2\right\}_y + \frac{v'''}{2}$$

$$= \left(\left\{\frac{1}{b}\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\mathcal{J}_P\right]\right\}_x - \frac{1}{2}\,\frac{b_y}{b}\left(\frac{b_x}{b}\right)_y - \frac{1}{2}\left(\frac{b_x}{b}\right)_{yy}\right) T$$

$$= \left(\left\{\frac{1}{b}\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\mathcal{J}_P\right]\right\}_x - \frac{1}{2}\left(\frac{b_y}{b}\right)\left(\frac{b_y}{b}\right)_x - \frac{1}{2}\left(\frac{b_y}{b}\right)_{yx}\right) T$$

$$= \left\{\frac{1}{b}\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\mathcal{J}_P\right] - \frac{b_{yy}}{2b} + \left(\frac{b_y}{2b}\right)^2\right\}_x T \qquad \text{(G3)}$$

Hence, if we define j_P by

$$j_P \equiv \frac{1}{b}\left[\left(\mathcal{J}_P\right)_x - \frac{b_x}{2b}\mathcal{J}_P\right] - \frac{b_{yy}}{2b} + \left(\frac{b_y}{2b}\right)^2 \tag{252}$$

equation (G3) becomes

$$2v' j_P + v\left(j_P\right)_y + \frac{v'''}{2} = \left(j_P\right)_x T \tag{251}$$

APPENDIX H

PROOF THAT EQUATIONS (249) AND (250) CAN ALWAYS BE SATISFIED IF $\left(j_P\right)_x = 0$

Let the domain of definition D of equations (249) and (250) be divided into two subdomains D^+ and D^- such that

$$b(x,y) \geq 0 \qquad \text{for} \quad (x,y) \in D^+$$

and

$$b(x,y) \leq 0 \qquad \text{for} \quad (x,y) \in D^-$$

We shall show that if $\left(j_P\right)_x = 0$, equations (249) and (250) always possess nonzero solutions which contain three arbitrary constants for $(x,y) \in D^+$. A similar proof will show that they also possess nonzero solutions for $(x,y) \in D^-$. We can conclude from this that these equations possess nonzero solutions in D which involve three arbitrary constants but which may have to be specified by different functions in different parts of D.

Hence, suppose that $(x,y) \in D^+$ and $\left(j_P\right)_x = 0$. Then there exists a function γ of y only such that $j_P = \gamma$.

Both equations (251) and (257) now reduce to the same equation, namely,

$$2v'\gamma + v\gamma' + \frac{1}{2} v''' = 0 \tag{H1}$$

Now the derivation of equation (251) from equations (249) and (250) shows that for every solution T of equation (250) (and this equation certainly always possesses a solution when v is any given function of y)

$$\left(\frac{\partial}{\partial x} - \frac{1}{2}\frac{b_x}{b}\right)\left[\left(v j_P\right)_y - \left(bT_y\right)_y - \left(T j_P\right)_x\right] = \sqrt{b}\,\frac{\partial}{\partial x}\,\frac{1}{\sqrt{b}}\left[\left(v j_P\right)_y - \left(bT_y\right)_y - \left(T j_P\right)_x\right]$$

$$= 2v'\gamma + v\gamma' + \frac{1}{2} v''' \tag{H2}$$

since $b \geq 0$.

Now equation (H1) shows that equation (257) has infinitely many nonzero solutions. (Recall that a third-order linear ordinary differential equation has three linearly independent solutions and that any linear combination of these solutions is also a solution.) Hence, let the function v_o of y only be any solution to equation (257).

Then equation (H2) shows that there exists a function r_T of y only such that

$$\left(v_o f_P\right)_y - \left(bT_y\right)_y - \left(T f_P\right)_x = r_T(y)\sqrt{b} \tag{H3}$$

for each solution T of the equation

$$T_x + \frac{1}{b}\,\frac{b_x}{b}\,T = \frac{1}{2}\,v_o\,\frac{b_y}{b} - \frac{1}{2}\,v_o' \tag{H4}$$

(This equation certainly has a solution T.)

It follows from definition (252) that

$$\gamma(y) = \frac{1}{\sqrt{b}}\left[\left(\frac{1}{\sqrt{b}}\, f_P\right)_x - \frac{1}{2}\left(\frac{b_y}{\sqrt{b}}\right)_y\right] \tag{H5}$$

Now let the function ρ of y only be any solution of the ordinary differential equation

$$\rho_{yy} + \gamma\rho = r_T \tag{H6}$$

and define the function $T^{(0)}$ by

$$T^{(0)} \equiv T + \frac{\rho(y)}{\sqrt{b}} \tag{H7}$$

Hence, the function $T^{(0)}$ can involve two arbitrary constants.

Substituting equation (H7) into equation (H4) shows that

$$T_x^{(0)} + \frac{1}{2}\,\frac{b_x}{b}\,T^{(0)} = \frac{1}{2}\,v_o\,\frac{b_y}{b} - \frac{1}{2}\,v_o' \tag{H8}$$

Substituting equation (H7) into equation (H3) shows that

$$\left(v_o f_P\right)_y - \left[bT^{(0)}_y\right]_y - \left[T^{(0)} f_P\right]_x = r_T(y)\sqrt{b} - \left[b\left(\frac{\rho}{\sqrt{b}}\right)_y\right]_y - \rho\left(\frac{1}{\sqrt{b}}\, f_P\right)_x \tag{H9}$$

Using equation (H5) to eliminate $\left(\mathscr{I}_P/\sqrt{b}\right)_x$ in equation (H9) gives

$$\left(v_o\mathscr{I}_P\right)_y - \left[bT^{(0)}_y\right]_y - \left[T^{(0)}\mathscr{I}_P\right]_x = \sqrt{b}\,(r_T - \rho\gamma) - \left[b\left(\frac{\rho}{\sqrt{b}}\right)_y\right]_y - \frac{1}{2}\left(\frac{b_y}{\sqrt{b}}\right)_y \rho$$

$$= \sqrt{b}\,(r_T - \rho\gamma) - \left(\sqrt{b}\,\rho_y - \frac{1}{2}\,\frac{b_y}{\sqrt{b}}\,\rho\right)_y - \frac{1}{2}\,\rho\left(\frac{b_y}{\sqrt{b}}\right)_y$$

$$= \sqrt{b}\,(r_T - \rho\gamma - \rho_{yy})$$

Hence, equation (H6) now shows that

$$\left(v_o\mathscr{I}_P\right)_y - \left[bT^{(0)}_y\right]_y = \left[T^{(0)}\mathscr{I}_P\right]_x \tag{H10}$$

Thus, we have shown that, if $\left(\mathscr{I}_P\right)_x = 0$, then for each solution v_o of the ordinary differential equation (257) there exists a function $T^{(0)}$ (containing two arbitrary constants) such that v_o and $T^{(0)}$ satisfy equations (249) and (250). Notice that this proof was constructive and can therefore be used in practice to find the function $T^{(0)}$ once a solution $v^{(0)}$ to differential equation (257) is known.

APPENDIX I

SOLUTIONS TO ELLIPTIC LIOUVILLE'S EQUATION

The elliptic form of Liouville's equation is

$$\frac{\partial}{\partial x}\left(\frac{1}{U}\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{U}\frac{\partial U}{\partial y}\right) = c_o U \tag{I1}$$

where c_o is a constant. Let

$$\chi_1 = h_1 + ig_1$$

be any nonconstant analytic function of the complex variable

$$z = x + iy$$

and let H be a nonzero function of h_1 and g_1. Put

$$U = H\left|\frac{d\chi_1}{dz}\right|^2 \tag{I2}$$

Then since $\ln\left|d\chi_1/dz\right|$ is a harmonic function,

$$\frac{\partial}{\partial x}\left(\frac{1}{U}\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{U}\frac{\partial U}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{H}\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{H}\frac{\partial H}{\partial y}\right) = \left|\frac{dx_1}{dz}\right|^2\left[\frac{\partial}{\partial h_1}\left(\frac{1}{H}\frac{\partial H}{\partial h_1}\right) + \frac{\partial}{\partial g_1}\left(\frac{1}{H}\frac{\partial H}{\partial g_1}\right)\right]$$

Hence, equation (I1) becomes

$$\frac{\partial}{\partial h_1}\left(\frac{1}{H}\frac{\partial H}{\partial h_1}\right) + \frac{\partial}{\partial g_1}\left(\frac{1}{H}\frac{\partial H}{\partial g_1}\right) = c_o H \tag{I3}$$

If we put $H = F$, where F is a function of h_1 only, then equation (I3) becomes

$$\frac{F''}{F} - \frac{F'^2}{F^2} = c_o F \tag{I4}$$

Hence, upon setting $F' = P$, we find

$$F'' = P \frac{dP}{dF}$$

and equation (I4) becomes

$$\frac{1}{2} \frac{1}{F} \frac{d(P^2)}{dF} - \frac{P^2}{F^2} = c_o F$$

or

$$(F')^2 = P^2 = F^2(2c_o F + c_1)$$

where c_1 is a constant of integration.

This shows that

$$\pm \int \frac{dF}{F\sqrt{2c_o F + c_1}} = h_1 + c_2$$

Thus,

$$h_1 + c_2 = \begin{cases} \pm \dfrac{2}{\sqrt{c_1}} \tanh^{-1} \sqrt{\dfrac{2c_o F + c_1}{c_1}} & \text{if} \quad c_1 \neq 0 \\[2ex] \pm \sqrt{\dfrac{2}{c_o}} \dfrac{1}{\sqrt{F}} & \text{if} \quad c_1 = 0 \end{cases}$$

Hence,

$$F = \begin{cases} -\dfrac{2}{c_o} \left(\dfrac{\sqrt{c_1}}{2}\right)^2 \dfrac{1}{\cosh^2\left[\dfrac{\sqrt{c_1}}{2}(h_1 + c_2)\right]} & \text{if} \quad c_1 \neq 0 \\[2ex] \dfrac{2}{c_o(h_1 + c_2)^2} & \text{if} \quad c_1 = 0 \end{cases}$$

The solution (I2) of equation (I1) is therefore

$$U = \begin{cases} -\dfrac{2}{c_o} \dfrac{\left(\dfrac{\sqrt{c_1}}{2}\right)^2 \left|\dfrac{d\chi_1}{dz}\right|^2}{\cosh^2\left[\dfrac{\sqrt{c_1}}{2}(h_1 + c_2)\right]} & \text{if } c_1 \neq 0 \\[2em] \dfrac{2}{c_o} \dfrac{\left|\dfrac{d\chi_1}{dz}\right|^2}{(h_1 + c_2)^2} & \text{if } c_1 = 0 \end{cases} \tag{I5}$$

Now put $c_3 = |c_1|$ and define the harmonic conjugate functions h and g by

$$h = \begin{cases} \dfrac{\sqrt{c_3}}{2}(h_1 + c_2) & \text{if } c_1 \neq 0 \\[1em] h_1 + c_2 & \text{if } c_1 = 0 \end{cases}$$

$$g = \begin{cases} \dfrac{\sqrt{c_3}}{2} g_1 & \text{if } c_1 \neq 0 \\[1em] g_1 & \text{if } c_1 = 0 \end{cases}$$

If we define the analytic function χ of the complex variable z by

$$\chi = h + ig$$

the solution (I5) becomes

$$U = \begin{cases} -\dfrac{2}{c_o} \dfrac{\left|\dfrac{d\chi}{dz}\right|^2}{\cosh^2 h} & \text{if } c_1 > 0 \\[2ex] \dfrac{2}{c_o} \dfrac{\left|\dfrac{d\chi}{dz}\right|^2}{h^2} & \text{if } c_1 = 0 \\[2ex] \dfrac{2}{c_o} \dfrac{\left|\dfrac{d\chi}{dz}\right|^2}{\cos^2 h} & \text{if } c_1 < 0 \end{cases} \tag{I6}$$

for any nonconstant analytic function χ. Actually, since the analytic function χ is quite arbitrary, the expressions on the second and third lines of (I6) are the same, for if we put $\chi = 1/i \ln \chi_2$, $h_2 = \mathcal{R}e\chi_2$ where χ_2 is any analytic function. Then,

$$\chi_2 = e^{i\chi}$$

$$\frac{d\chi_2}{dz} = ie^{i\chi} \frac{d\chi}{dz}$$

$$\left|\frac{d\chi_2}{dz}\right|^2 = e^{i(\chi-\chi^*)} \left|\frac{d\chi}{dz}\right|^2$$

$$\cos^2 h = \left(\frac{e^{ih} + e^{-ih}}{2}\right)^2 = \left\{\frac{e^{i[(\chi+\chi^*)/2]} + e^{-i[(\chi+\chi^*)/2]}}{2}\right\}^2$$

$$= e^{i(\chi^*-\chi)} \left(\frac{e^{i\chi} + e^{-i\chi^*}}{2}\right)^2$$

$$= e^{i(\chi^*-\chi)} \left(\frac{\chi_2 + \chi_2^*}{2}\right)^2$$

$$= e^{i(\chi^*-\chi)} (h_2)^2$$

Hence,

$$\frac{\left|\frac{d\chi_2}{dz}\right|^2}{(h_2)^2} = \frac{\left|\frac{d\chi}{dz}\right|^2}{\cos^2 h}$$

Notice that when the solution U is given by the first line of equation (I6)

$$Uc_o \leq 0$$

and when the solution U is given by the second (or third) line of equation (I6)

$$Uc_o \geq 0$$

Thus, these forms of the solution cannot be transformed into one another simply by changing the arbitrary analytic function χ, and they therefore represent totally different expressions for the solution. Hence,

$$U = \begin{cases} -\dfrac{2}{c_o} \dfrac{\left|\frac{d\chi}{dz}\right|^2}{\cosh^2 h} & \text{if } Uc_o \leq 0 \\[2ex] \dfrac{2}{c_o} \dfrac{\left|\frac{d\chi}{dz}\right|^2}{\cos^2 h} & \text{if } Uc_o \geq 0 \end{cases} \tag{I7}$$

for any analytic function χ.

We shall give an alternate derivation of equation (I7) and at the same time demonstrate that this is the most general real solution to equation (I1). It is shown in reference 3 (p. 194) that the general solution to equation (I1) is

$$U = \frac{8}{c_o} \frac{\frac{d\chi}{dz} \frac{d\Lambda}{dz*}}{(\chi + \Lambda)^2} \tag{I8}$$

where $\chi = h + g$ is any nonconstant analytic function of the complex variable z and Λ

is any analytic function of the complex conjugate variable $z^* = \chi - iy$. This solution is not real valued, however, unless the functions χ and Λ are related. In order to find this relation, notice that for any function Λ there exists an analytic function Γ of the complex conjugate variable χ^* such that

$$\Gamma\left[\chi^*(z^*)\right] = \Gamma(z^*)$$

Hence, equation (I8) becomes

$$U = \frac{8}{c_o} \frac{\frac{d\chi}{dz}\frac{d\chi^*}{dz^*}\Gamma'}{(\chi + \Gamma)^2} = \frac{\left|\frac{d\chi}{dz}\right|^2 \Gamma'}{(\chi + \Gamma)^2} \tag{I9}$$

where

$$\Gamma' \equiv \frac{d\Gamma}{d\chi^*} \neq 0$$

Now U is real if, and only if,

$$\frac{\Gamma'}{(\chi + \Gamma)^2} = \frac{(\Gamma')^*}{(\chi^* + \Gamma^*)^2} \tag{I10}$$

or

$$\frac{\chi + \Gamma}{(\Gamma')^{1/2}} = \frac{\chi^* + \Gamma^*}{(\Gamma^{*\prime})^{1/2}} \tag{I11}$$

Now this equation can be extend to complex values of h and g, and the variables χ and χ^* are then independent. Recalling that Γ^* and $(\Gamma')^*$ are functions of χ only, we find, upon differentiating equation (I-11) with respect to χ^*, that

$$\left[\frac{\Gamma}{(\Gamma')^{1/2}}\right]' + \chi\left[\frac{1}{(\Gamma')^{1/2}}\right]' = \frac{1}{(\Gamma^{*\prime})^{1/2}}$$

Upon differentiating again with respect to χ^* we obtain

$$\left[\frac{\Gamma}{(\Gamma')^{1/2}}\right]'' + \chi\left(\frac{1}{\sqrt{\Gamma'}}\right)'' = 0$$

Now if $\left[1/(\Gamma')^{1/2}\right]'' \neq 0$, this equation would show, upon division by $\left[1/(\Gamma')^{1/2}\right]''$, that χ was equal to a constant. Since χ is a nonconstant function, we conclude that

$$\left[\frac{1}{(\Gamma')^{1/2}}\right]'' = 0$$

Upon integrating this equation, we find that

$$\Gamma = \begin{cases} \dfrac{1}{c_1\chi^* + c_2} + c_3 & \text{if } c_1 \neq 0 \\ \\ c_4\chi^* + c_5 & \text{if } c_1 = 0 \end{cases} \tag{I12}$$

where c_1 to c_5 are constants of integration. Upon substituting these results into equation (I11) we find that

$$\frac{(c_3 + \chi)(c_1\chi^* + c_2) + 1}{\sqrt{c_1}} = \frac{(c_3^* + \chi^*)(c_1^*\chi + c_2^*) + 1}{\sqrt{c_1^*}} \tag{I13}$$

and

$$\frac{\chi + c_4\chi^* + c_5}{\sqrt{c_4}} = \frac{\chi^* + c_4^*\chi + c_5^*}{\sqrt{c_4^*}} \tag{I14}$$

Differentiating equation (I13) with respect to χ shown that

$$\frac{c_1\chi^* + c_2}{\sqrt{c_1}} = \frac{c_1^*(\chi^* + c_3^*)}{\sqrt{c_1^*}}$$

Hence, c_1 is real and $c_2 = c_1c_3^*$.

Equation (I14) shows that there exists a real constant c_6 such that

$$c_4 = e^{-i2c_6}$$

and a real constant c_7 such that

$$c_5 = e^{-ic_6} c_7$$

Hence, equation (I12) becomes

$$\Gamma = \begin{cases} c_3 + \dfrac{1}{c_1(\chi + c_3)^*} & \text{if } c_1 \neq 0 \\ \\ e^{-i2c_6}\chi^* + 2c_7 e^{-2ic_6} & \text{if } c_1 = 0 \end{cases} \tag{I15}$$

where c_1, c_6, and c_7 are real.

Upon substituting equation (I15) into equation (I9), we obtain

$$U = \begin{cases} -\dfrac{8c_1}{c_o} \dfrac{\left|\dfrac{d\chi}{dz}\right|^2}{\left(c_1|\chi + c_3|^2 + 1\right)^2} & \text{if } c_1 \neq 0 \\ \\ \dfrac{8}{c_o} \dfrac{\left|\dfrac{d\chi}{dz}\right|^2}{\left[\left(e^{ic_6}\chi + c_7\right) + \left(e^{ic_6}\chi + c_7\right)^*\right]^2} & \text{if } c_1 = 0 \end{cases}$$

Hence, upon defining the analytic function $\chi_1 = h_1 + ig_1$ by

$$\chi_1 = \begin{cases} |c_1|(\chi + c_3) & \text{if } c_1 \neq 0 \\ \\ e^{ic_6}\chi + c_7 & \text{if } c_1 = 0 \end{cases}$$

we obtain

$$U = \begin{cases} -\dfrac{8}{c_o} \dfrac{\left|\dfrac{d\chi_1}{dz}\right|^2}{\left(|\chi_1|^2 + 1\right)^2} & \text{if } c_1 > 0 \\[2ex] \dfrac{8}{c_o} \dfrac{\left|\dfrac{d\chi_1}{dz}\right|}{\left(1 - |\chi_1|^2\right)^2} & \text{if } c_1 < 0 \\[2ex] \dfrac{2}{c_o} \dfrac{\left|\dfrac{d\chi_1}{dz}\right|}{(h_1)^2} & \text{if } c_1 = 0 \end{cases} \tag{I16}$$

Notice that, for the first form of the solution (I16), $Uc_o \leq 0$; and for the second two forms, $Uc_o \geq 0$. In fact, these latter two forms are equivalent since they can be transformed into one another by the conformal transformation

$$\chi_2 = \frac{1 - \chi_1}{1 + \chi_2}$$

Hence,

$$U = \begin{cases} -\dfrac{8}{c_o} \dfrac{\left|\dfrac{d\chi_1}{dz}\right|^2}{\left(|\chi_1|^2 + 1\right)^2} & \text{if } Uc_o \leq 0 \\[2ex] \dfrac{2}{c_o} \dfrac{\left|\dfrac{d\chi_1}{dz}\right|^2}{(h_1)^2} & \text{if } Uc_o \geq 0, \end{cases} \tag{I17}$$

We have already shown that the second form of the solution (I17) is the same as the second form of the solution (I7). We shall now show that the first form of the solution (I17) (which is the form given in ref. 13) is the same as the first form of the solution (I7). To this end put

$$\chi = \ln \chi_1$$

where $\chi = h + ig$, and obtain

$$\frac{4}{\left(|\chi_1|^2+1\right)^2} = \frac{4}{\left(e^{\chi}e^{\chi^*}+1\right)^2} = \frac{4}{e^{\chi+\chi^*}\left\{e^{(\chi+\chi^*)/2}+e^{-[(\chi+\chi^*)/2]}\right\}^2} = \frac{e^{-(\chi+\chi^*)}}{\cosh^2 h}$$

$$\left|\frac{d\chi_1}{dz}\right|^2 = e^{\chi+\chi^*}\left|\frac{d\chi}{dz}\right|^2$$

which proves the assertion.

Hence, we conclude that the most general real solution to equation (I1) is given by equation (I7), with $\chi = h + ig$ an arbitrary analytic function of z.

There are many different expressions which can be given for the solution (I7). These can be obtained by varying the arbitrary function χ. For our purposes, the most useful of these, which is an alternate for the second form of equation (I7), is the one which is analogous to that obtained in the hyperbolic case. To obtain this expression, it is more convenient to start with the form of the second part of equation (I7) given in the center of equation (I6), which is

$$U = \frac{2}{c_o}\,\frac{\left|\frac{d\chi}{dz}\right|^2}{h^2} \tag{I18}$$

or, equivalently,

$$U = \frac{8}{c_o}\,\frac{\left|\frac{d\chi}{dz}\right|^2}{(\chi+\chi^*)^2} \tag{I19}$$

We now define the analytic function $W = u + iv$ of the complex variable z by

$$W(z) \equiv 2\int_{i\chi(z)}^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}}\,dt \tag{I20}$$

where g_2 and g_3 are real constants and the path of integration may be any curve which does not pass through a zero of $4t^3 - g_2 t - g_3$. Then (ref. 12)

$$i\chi = \wp\left(\frac{1}{2} W; g_2, g_3\right)$$

Now it is shown in reference 12 that $\wp(Z)$ is real when Z is real. This shows that

$$\left[\wp(Z)\right]^* = \wp(Z^*)$$

Hence,

$$-i\chi^* = \wp\left(\frac{1}{2} W^*; g_2, g_3\right)$$

Upon using these results in equation (I19), we find

$$U = -\frac{2}{c_o}\left|\frac{dW}{dz}\right|^2 \frac{\wp'\left(\frac{1}{2} W\right)\wp'\left(\frac{1}{2} W^*\right)}{\left[\wp\left(\frac{1}{2} W\right) - \wp\left(\frac{1}{2} W^*\right)\right]^2}$$

The results of example 1 on page 456 of reference 12 now show that

$$U = \frac{2}{c_o}\left[\wp\left(\frac{1}{2} W + \frac{1}{2} W^*\right) - \wp\left(\frac{1}{2} W - \frac{1}{2} W^*\right)\right]\left|\frac{dW}{dz}\right|^2$$

$$= \frac{2}{c_o}\left[\wp(u; g_2, g_3) - \wp(iv; g_2, g_3)\right]\left|\frac{dW}{dz}\right|^2$$

But example 2 on page 439 of reference 12 shows that

$$\wp(iv; g_2, g_3) = -\wp(v; g_2, -g_3)$$

Hence,

$$U = \frac{2}{c_o}\left[\wp(u; g_2, g_3) + \wp(v; g_2, -g_3)\right]\left|\frac{dW}{dz}\right|^2 \tag{I21}$$

This form of the solution to equation (I1) could also have been obtained by choosing the function H to be of the form

$$H(h_1, g_1) = F(h_1) + G(g_1)$$

It is not hard to show by using the results on page 453 of reference 12 that, if we had defined the function W by

$$W(z) \equiv 2 \int_{\chi_o}^{+i\chi(z)} \frac{1}{\sqrt{a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4}}\, dt \tag{I22}$$

(where a_0 to a_4 are any real constants) instead of by equation (I20), then there would still exist real constants g_2 and g_3 such that equation (I21) holds.

APPENDIX J

SYMBOLS

A	coefficient of first derivative in general partial differential equation
a	coefficient of first derivative in canonical partial differential equation
$\tilde{a}$	transformed coefficient of first derivative in canonical partial differential equation
B	coefficient of first derivative in general partial differential equation
b	coefficient of first derivative in canonical partial differential equation
$\tilde{b}$	transformed coefficient of first derivative in canonical partial differential equation
C	coefficient of linear term in general partial differential equation
c	coefficient of linear term in canonical partial differential equation
$\tilde{c}$	transformed coefficient of linear term in canonical partial differential equation
c_o	constant
D	domain of definition of differential equation
d_r	functions of ξ for $r = 1, 2, 3, \ldots$
e_r	functions of η for $r = 1, 2, 3, \ldots$
F	function of σ
f	function of ξ and η
G	function of τ
g	$\mathcal{I}m\,\chi$
g_r	constants, $r = 1, 2$
H	function of η only in trial solution
h	$\mathcal{R}e\,\chi$
I	special notation for canonical invariant used in appendix A
i	$\sqrt{-1}$
$\mathcal{J}$	canonical invariant
$\mathscr{J}$	canonical invariant
j	index in operator $L^{(j)}$ (can take on values -1, 1, 0)

$\mathcal{J}$	quantity defined in terms of canonical invariant
K	quantity defined in terms of canonical invariant
k	quantity defined in terms of canonical invariant
k_r	quantity defined in terms of canonical invariant for r for $r = 1, 2$
$L^{(j)}$	linear operator defined by eq. (54)
M	constant
p_r	function of u for $r = 1, 2$
$\wp$	Weierstrass $\wp$-function
q_r	function of v for $r = 1, 2$
S	function of σ, function of complex variable z
$s^{(r)}$	coefficients in ordinary differential equations for $r = 0, 1, 2$
T	function of τ
t	variable of integration
$t^{(r)}$	coefficients in ordinary differential equations for $r = 0, 1, 2$
U	dependent variable
u	transformed independent variable
V	transformed dependent variable
v	transformed independent variable
W	analytic function of the complex variable z
w	function of x and y defined in terms of u and v by eq. (193)
x	independent variable in canonical partial differential equation
$Y^{(r)}$	coefficients in ordinary differential equation for $r = 0, 1, 3$
y	independent variable in canonical partial differential equation
z	complex variable, $x + iy$
α	coefficient of second derivative in general partial differential equation
β	coefficient of mixed derivative in general partial differential equation
γ	coefficient of second derivative in general partial differential equation - general purpose function of σ
η	independent variable in general partial differential equation
θ	$\int a\, dx$

λ function for transforming dependent variable - general purpose function of τ

μ special symbol used in appendix A

ξ independent variable in general partial differential equation

Ξ function of ξ in trial solution

π_r constants for $r = 1, 2, 3$

σ $x + y$

τ $x - y$

Φ function u

φ function connecting ξ with x and y

χ function of the complex variable z

Ψ function of v

ψ function connecting η with x and y

Ω defined by $\mathcal{J}_P = \partial\Omega/\partial x$

$\Omega^{(r)}$ quantities depending on invariant $\mathcal{J}$ for $r = 1, 2$

ω function used in transforming dependent variable

Subscripts:

E elliptic

H hyperbolic

P parabolic

Superscripts:

* complex conjugate or quantity which reduces to complex conjugate when independent variables x and y are real

' denotes differentiation with respect to argument (prime)

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