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ON THE INTEGRABILITY CASES OF THE EQUATION OF MOTION FOR A SATELLITE IN AN AXIALLY SYMMETRIC GRAVITATIONAL FIELD

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Mission and Systems Analysis Branch

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ABSTRACT

The projection of an axially symmetric satellite's orbit on a plane perpendicular to the rotation axis (z = const.) is given by the second-order differential equation

$$\frac{y''}{1 + y'^2} = \bar{\psi}_y - y' \bar{\psi}_x , \tag{1}$$

where the prime denotes the derivative with respect to x and $\bar{\psi}$ (x, y) is a known function. Two integrability cases of (1) have been investigated and it has been shown that for these two cases the integration of (1) can be carried out either by quadratures or reduced to a first-order differential equation. Analytical and physical properties of (1) are expressed, and it is shown that equation (1) can be derived from the classical plane eikonal equation of geometric optics.

SUMMARY

This paper discusses the integrability cases of a special equation of motion recently presented by H. Knothe* in his article "Satellites and Riemaniann Geometry". The importance of this equation (equation 1) is its close connection with the plane eikonal equation of geometric optics. The equation under discussion is given by

$$\frac{y''}{1 + y'^2} = \bar{\psi}_y - y' \bar{\psi}_x , \tag{1}$$

y(x) represents the projection of the satellite's orbit on a plane z = const., prime denotes the derivative with respect to x, and $\bar{\psi}$ (x,y) = log ψ (x,y), where ψ (x,y) is being interpreted* as the speed with which the projection of satellite moves in the two-dimensional space (x,y).

Knothe* has shown that for a given potential $\phi(u_1, u_2)$ in an axially symmetric three-dimensional Riemaniann space (u_1, u_2, u_3) there exists a function $\psi(u_1, u_2)$ by means of which the equations of motion for all point-masses with equal total energy and equal angular momentum can be reduced to a single ordinary second-order differential equation. Considering the special case of the motion of a satellite around a rotationally symmetric body in three-dimensional Euclidean space (x,y,z), Knothe* has shown that the projection of a satellite's orbit about a body of revolution on a plane orthogonal to the axis of rotation (z-axis) is a geodesic curve given by equation (1). Equation (1) can also be considered as the autonomous equation of motion of a point-mass moving in a field of force with a force function ψ (x,y) = $e^{\bar{\psi}(x,y)}$.

In this paper two integrability cases of equation (1) are considered and it is shown that for these two special cases the integration of (1) has been reduced to quadratures and to an ordinary first-order differential equation. Analytical and physical interpretations of this equation as well as its connection with the classical plane eikonal equation of Geometric Optics are also expressed.

*Celestial Mechanics 1 (1969), pp. 36-45; D. Reidel Publishing Company, Dordrecht-Holland.

ON THE INTEGRABILITY CASES OF THE EQUATION OF MOTION FOR A SATELLITE IN AN AXIALLY SYMMETRIC GRAVITATIONAL FIELD

Abohghassem Ghaffari

1. Introduction

This paper has been devoted to the investigations of the two integrability cases of the ordinary second-order differential equation

$$\frac{y''}{1 + y'^2} = \overline{\psi}_y - y' \, \overline{\psi}_x \,, \qquad (1)$$

which defines the projection (on a plane z = const.) of the motion of a satellite around an axially symmetric body in the three-dimensional Euclidean space (x, y, z). Prime denotes the derivative with respect to x.

Recently H. Knothe (ref.) showed that for a given potential $\phi(u_1, \ u_2)$ in an axially symmetric three-dimensional Riemannian space (u_1, u_2, u_3) there exists a function $\psi \ (u_1, u_2)$ by means of which the equations of motion, for all point-masses with equal total energy and equal angular momentum, can be reduced to a single ordinary second-order differential equation.

Considering the special case of the motion of a satellite around a rotationally symmetric body in three-dimensional Euclidean space (x, y, z), Knothe (ref. p. 45, formula 65) has showed that the projection of satellite's orbit on a plane z = const.

is a geodetic curve given by the equation (1), where

$$\bar{\psi} = \log \psi (x, y). \tag{2}$$

The function ψ (x, y) has been interpreted by H. Knothe (loc. cit.) as the speed with which the point-mass moves in a two-dimensional Euclidean space (x, y), i.e.,

$$\psi = \frac{d\sigma}{dt} \tag{3}$$

where σ is the arc length of the orbit's projection on the plane z = const.

Equation (1) is a special case of the general equation

$$(z_x^2 + z_y^2 + 1) y'' = z_x z_{yy} y'^3 + (2 z_x z_{xy} - z_y z_{yy}) y'^2 + (z_x^2 - 2 z_y z_{xy}) y' - z_y z_{xx} \tag{4}$$

giving the geodetic curves on a given surface

$$z = z (x, y).$$

The second-order equations (1), (4) are of the third degree in y' and there are no general rules for solving nonlinear differential equations, and skill and ingenuity are essential. The purpose of this paper is to show that there are (besides the trivial case in which the variables in $\bar{\psi}$ can be separated), at least, two cases in which the integration of equation (1) can be carried out either by quadratures or reduced to an ordinary first-order differential equation. These cases are deduced from the form and structure of the function $\overline{\psi}$ expressed in both rectangular and polar coordinates.

2. Integrability Cases

a. Equation (1) can be written

$$\frac{dy'}{1 + y'^2} = d \operatorname{arct} y' = \overline{\psi}_y \, dx - \overline{\psi}_x \, dy . \tag{5}$$

The integration of (5) would be easy if the right-hand side of (5) is also an exact total differential of a function U (x, y) such that

$$\overline{\psi}_y \, dx - \overline{\psi}_x \, dy = dU = U_x \, dx + U_y \, dy \, . \tag{6}$$

The integrability conditions of (6) leads to the Cauchy-Riemann differential equations

$$\begin{cases} U_x = \overline{\psi}_y \\ U_y = -\overline{\psi}_x , \end{cases} \tag{7}$$

provided $\overline{\psi} \in C^2$ and $U \in C^2$ in a domain D of (x, y) of the space R^2.

The infinite number of functions U (x, y) are given by the two following quadratures

$$U(x, y) = \int_{x_0}^{x} \overline{\psi}_y (x, y) \, dx - \int_{y_0}^{y} \overline{\psi}_x (x_0, y) \, dy + \alpha \tag{8}$$

where x_0 and y_0 are arbitrary numerical values of x and y respectively, and α is an arbitrary constant. The particular function U corresponding to $\alpha = 0$ is

$$U(x, y) = \int_{x_0}^{x} \overline{\psi}_y (x, y)\, dx - \int_{y_0}^{y} \overline{\psi}_x (x_0, y)\, dy, \tag{9}$$

and the integration of equation (5) gives

$$\operatorname{arctg} y' = U(x, y) + C, \qquad y' = \tan (U + C) = \tan \omega$$

or

$$U + C = \omega \tag{10}$$

where the angle ω has been defined by H. Knothe (ref.) such that the tangent vector of the trajectory x(t), y(t) has the components $\cos \omega$ and $\sin \omega$ respectively, and C is an integration constant.

Analytic and physical interpretations.

The Cauchy-Riemann conditions (7) state that there exists an analytic function W of the complex variable $z = x + iy$ such that

$$W(x + iy) = U(x, y) + i\overline{\psi}(x, y), \tag{11}$$

where the two harmonic functions U and $\overline{\psi}$ satisfy Laplace equations

$$\Delta U = 0, \qquad \Delta \overline{\psi} = 0. \tag{12}$$

From the Cauchy-Reimann conditions (7) one can also deduce that

$$U_x^2 + U_y^2 = \overline{\psi}_x^2 + \overline{\psi}_y^2 \tag{13}$$

or the expression

$$z_x^2 + z_y^2 \tag{14}$$

is an invariant (in terms of x and y) on either surfaces

$$\begin{cases} z = U(x, y) \\ z = \overline{\psi}(x, y) . \end{cases} \tag{15}$$

There is a close relation between equation (1) and the classical plane eikonal equation of geometric optics

$$(\nabla z)^2 = z_x^2 + z_y^2 = n^2 (x, y) \tag{16}$$

where n is a function of x and y. In fact, equation (1) can be derived, for a special form of n (x, y), from equation (16). The form of equation (16) suggests to write

$$n (x, y) = e^{\overline{\psi}} , \tag{17}$$

and setting

$$\begin{cases} z_x = e^{\overline{\psi}} \cos \omega \\ z_y = e^{\overline{\psi}} \sin \omega, \end{cases} \tag{18}$$

one finds that ω_x and ω_y satisfy the first-order partial differential equation

$$\cos \omega \cdot \omega_x + \sin \omega \cdot \omega_y = \overline{\psi}_y \cos \omega - \overline{\psi}_x \sin \omega \tag{19}$$

with the corresponding characteristic differential equations

$$dx = \frac{dy}{\operatorname{tg} \omega} = \frac{d\omega}{\overline{\psi}_y - \overline{\psi}_x \operatorname{tg} \omega} \tag{20}$$

which give

$$\frac{dy}{dx} = y' = \operatorname{tg} \omega, \text{ and } \frac{d\omega}{dx} = \omega' = \overline{\psi}_y - \overline{\psi}_x \operatorname{tg} \omega . \tag{21}$$

Taking the derivative of the first equation of (21) with respect to x and considering the second equation of (21) we get exactly the equation (1).

The case n = const. corresponds to the developable conoid surfaces whereas n as a function of x and y corresponds to the plane wave fronts of light in an inhomogeneous medium with a variable index of refraction n (x, y).

If n is a function of z alone, equation (16) can be integrated by classical methods and represents the well-known parallel (moulding) surfaces.

Equation (1) can also be considered as the plane autonomous equation of motion of a point-mass moving in a field of force with a force function $\psi(x,y) = e^{\bar{\psi}(x,y)}$.

b. If $\bar{\psi}(x,y)$ is an homogeneous function of degree zero, that is to say if

$$\bar{\psi}(x, y) = \bar{\psi}\left(1, \frac{y}{x}\right) \to f\left(\frac{y}{x}\right) = f(u) \tag{22}$$

where

$$u = \frac{y}{x}, \tag{23}$$

then the equation (1) will be reduced to an ordinary first-order differential equation. The new variables are x and u.

In fact, equation (1) becomes

$$\frac{x y''}{1 + y'^2} = f'(u)\,(1 + u y'), \tag{24}$$

and the substitution (23) gives

$$\begin{cases} y' = u' x + u = t + u \\ y'' = u'' x + 2 u' = u' \left(\dfrac{dt}{dx} + 1\right) \end{cases} \tag{25}$$

and equation (24) becomes

$$t\left(\frac{dt}{du} + 1\right) = f'(u)\,[1 + u\,(t + u)]\,[1 + (t + u)^2] \tag{26}$$

which is of the first order in u and t.

If, in polar coordinates, $\overline{\psi}$ is a function of the argument

$$\theta = \mathrm{arctg}\, \frac{y}{x}$$

alone, the substitution (23) leads to the same equation (26).

It may exist, besides the two above mentioned cases, other cases in which the integration of (1) can be carried out by quadratures or reduced to an ordinary first-order differential equation.

REFERENCE

H. Knothe, Satellites and Riemmanian Geometry. Celestial Mechanics 1 (1969), 36-45, D. Reidel Publishing Company, Dordrecht-Holland.