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Technical Report 32-1477

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Preface

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Abstract

The increase in the pointing accuracy requirements of space vehicles has made it mandatory to take into account the structural flexibilities in the transfer function relating control torques to attitude angles. This report provides a finite element formulation of the governing equations of the perturbations and small vibrations of flexible structures undergoing arbitrary translations or rotational motion, or both. The structure may or may not contain dampers, or rotating parts on flexible or rigid mounts. Having obtained the linearized governing equations, an approximate but practical method is described for obtaining the transfer function relating control torques to attitude angles.

On Small Vibrations and Perturbations of Flexible Bodies Undergoing Arbitrary Nominal Motion

I. Introduction

The need for inclusion of the structural flexibilities in the "dynamics block" of the attitude control system of spacecrafts has been given in Ref. 1. A summary of the state of the art, and a discrete formulation with a method of obtaining the transfer function relating control torques to attitude angles are given in Refs. 2 and 3. Since the linearized equations of perturbations are obtained from those of the general motion with not all the nonlinearities, they are not complete. Moreover, the approximate numerical method suggested in this reference for obtaining the transfer function rapidly becomes impractical with the increased number of degrees of freedom.

In this work, the linearized governing equations of small vibrations and perturbations of flexible bodies undergoing arbitrary motion are obtained in discrete form by means of a finite element technique without first deriving the governing equations of the arbitrary motion. The governing equations thus obtained include those of Ref. 3 as a special case. Also an approximate numerical method is given for obtaining the transfer function relating control torques to attitude angles. This method is a practical one, since it preserves the bandedness of the coefficient matrices. In what follows, only the repeated latin subscripts i, j, p, and q imply summation over the range.

The flexible body is considered as the assembly of $(n+1)$ subbodies interconnected by elastic springs. Let V denote the total material volume of the body, and v_i the material volume associated with the ith subbody, such that

$$V = \sum_{i=0}^{n} v_i \tag{1a}$$

Let ν denote unit mass, and m_i the mass of the ith subbody. Then

$$m_i = \int_{v_i} \nu \, dv \qquad (i = 0,1,\cdots,n) \tag{1b}$$

Let $\mathbf{e}_i^0$ denote the position vector of any particle in the ith subbody, relative to its mass center before perturbations. Therefore

$$\int_{v_i} \nu \, \mathbf{e}^0 \, dv = \mathbf{0} \qquad (i = 0,1,\cdots,n) \tag{1c}$$

where the index in the integral bounds is also the index of the indexable quantities of the integrand. Denoting the position vector of any particle in the subbody with $\boldsymbol{\rho}_i^0$ and the position vector of the mass center of this subbody with $\mathbf{R}_i^0$, relative to the mass center of the whole $(n+1)$ body system before perturbations, we may write

$$\boldsymbol{\rho}_i^0 = \mathbf{R}_i^0 + \mathbf{e}_i^0 \tag{1d}$$

and

$$\int_{v_i} \nu \, \boldsymbol{\rho}_i^0 \, dv = \mathbf{0} \tag{1e}$$

where the repeated index indicates summation over the range of zero to n. With the help of (1*d*), (1*b*), and (1*c*), Eq. (1*e*) reduces to

$$\mathbf{R}_i^0 \, m_i = \mathbf{0} \tag{1f}$$

where the repeated index shows summation over the range of zero to n. Let $\mathbf{p}_i^0$ denote the position vector of a particle in the ith subbody, and $\mathbf{X}$ denote the position vector of the mass center of the whole system, relative to an inertially fixed coordinate system, such that

$$\mathbf{p}_i^0 = \mathbf{X} + \boldsymbol{\rho}_i^0 \tag{1g}$$

or, by the use of (1*d*),

$$\mathbf{p}_i^0 = \mathbf{X} + \mathbf{R}_i^0 + \mathbf{e}_i^0 \tag{1h}$$

Let $\mathbf{c}$ and $\boldsymbol{\theta}$ denote the perturbation translation and the perturbation rotation, respectively, of the whole system, and $\mathbf{u}_i$ and $\boldsymbol{\beta}_i$ denote the perturbation translation and the perturbation rotation of the ith subbody relative to the zeroth subbody, respectively. Let $\mathbf{e}_i$, $\mathbf{R}_i$, $\boldsymbol{\rho}_i$, and $\mathbf{p}_i$ denote the quantities in the perturbed state corresponding to $\mathbf{e}_i^0$, $\mathbf{R}_i^0$, $\boldsymbol{\rho}_i^0$, and $\mathbf{p}_i^0$, respectively. With the help of Fig. 1, the linearized relationships between these quantities may be written as

$$\mathbf{e}_i = \mathbf{e}_i^0 - \mathbf{e}_i^0 \times \boldsymbol{\theta} - (\mathbf{e}^0 \times \boldsymbol{\beta})_i \tag{2a}$$

$$\mathbf{R}_i = \mathbf{R}_i^0 + \mathbf{c} - \mathbf{R}_i^0 \times \boldsymbol{\theta} + \mathbf{u}_i \tag{2b}$$

$$\boldsymbol{\rho}_i = \boldsymbol{\rho}_i^0 + \mathbf{c} - (\mathbf{R}_i^0 + \mathbf{e}_i^0) \times \boldsymbol{\theta} - (\mathbf{e}^0 \times \boldsymbol{\beta})_i + \mathbf{u}_i \tag{2c}$$

$$\mathbf{p}_i = \mathbf{p}_i^0 + \mathbf{c} - (\mathbf{R}_i^0 + \mathbf{e}_i^0) \times \boldsymbol{\theta} - (\mathbf{e}^0 \times \boldsymbol{\beta})_i + \mathbf{u}_i \tag{2d}$$

If we assume that the system is subjected to disturbance torques but not to disturbance forces, the position vector

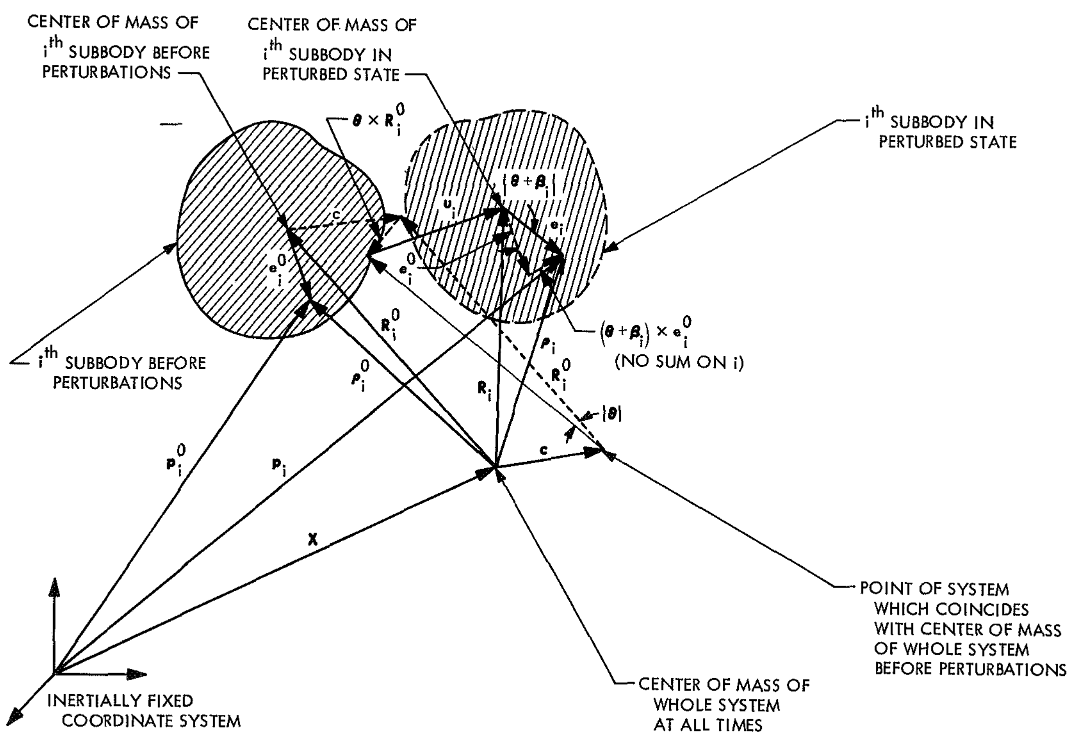

Fig. 1. Sketch for the definitions of various vectors

$\mathbf{X}$ always defines the trajectory of the mass center of the whole system at all times, and $\mathbf{c}$ defines the location of the centroid before perturbations, relative to the point defined by $\mathbf{X}$. The first objective of this work is to obtain the governing equations on $\mathbf{c}$, $\boldsymbol{\theta}$, $\mathbf{u}_i$, and $\boldsymbol{\beta}_i$, $i = 1,2,\cdots, n$.

II. Governing Equations

In this section the governing equations are derived in vector notation. The four groups of equations are a set of compatibility equations, n number of force equilibrium equations of subbodies, n number of moment equilibrium equations of subbodies, and a moment equilibrium equation of the whole system.

A. Compatibility Equation

Assuming that the boundaries of v_i, $i = 0,1,\cdots, n$ do not change appreciably during the perturbations, we may write

$$\int_{v_i} \nu\, \boldsymbol{\rho}_i \, dv = 0 \qquad (i = 0,1,\cdots, n) \tag{3a}$$

With the help of (2c), (1e), (1f), (1c), and (1a), Eq. (3a) may be reduced to

$$\left(\sum_{i=0}^{n} m_i\right)\mathbf{c} + m_i\mathbf{u}_i = 0 \tag{3b}$$

or, by defining μ_i as

$$\mu_i = \frac{m_i}{\sum\limits_{i=0}^{n} m_i} \tag{3c}$$

(3b) may be rewritten as

$$\mathbf{c} + \mu_i \mathbf{u}_i = 0 \tag{3d}$$

where the repeated index means summation over the range of 1 to n. Equation (3d) may be considered as a geometric compatibility equation.

B. Force Equilibrium of Subbodies

Let $\mathbf{f}^0_i$ denote the forces acting at the mass center of the ith subbody before perturbations. Let $\mathbf{F}_i$ denote forces created by the perturbations at the mass center of the ith subbody. The force equilibrium equations of the subbodies in the unperturbed state may be written as

$$\int_{v_i} \nu\, \ddot{\mathbf{p}}^0 \, dv + \mathbf{f}^0_i = 0 \qquad (i = 1,2,\cdots, n) \tag{4a}$$

and the force equilibrium equations in the perturbed state are

$$\int_{v_i} \nu\, \ddot{\mathbf{p}} \, dv + \mathbf{f}^0_i - \mathbf{f}^0_i \times \boldsymbol{\theta} - (\mathbf{f}^0 \times \boldsymbol{\beta})_i + \mathbf{F}_i = 0 \qquad (i = 1,2,\cdots, n) \tag{4b}$$

Substracting (4a) from (4b), one obtains

$$\int_{v_i} \nu(\ddot{\mathbf{p}} - \ddot{\mathbf{p}}^0) \, dv + \mathbf{F}_i - \mathbf{f}^0_i \times \boldsymbol{\theta} - (\mathbf{f}^0 \times \boldsymbol{\beta})_i = 0 \qquad (i = 1,\cdots, n) \tag{4c}$$

Substituting $\mathbf{p}_i$ from (2d) and using (1b) and (1c), after cancellation one finally obtains

$$(m\ddot{\mathbf{u}})_i - (m\ddot{\mathbf{R}}^0)_i \times \boldsymbol{\theta} - 2(m\dot{\mathbf{R}}^0)_i \times \dot{\boldsymbol{\theta}} - (m\mathbf{R}^0)_i \times \ddot{\boldsymbol{\theta}} + m_i\ddot{\mathbf{c}} + \mathbf{F}_i - (\mathbf{f}^0 \times \boldsymbol{\beta})_i - \mathbf{f}^0_i \times \boldsymbol{\theta} = 0 \qquad (i = 1,\cdots, n) \tag{4d}$$

which are the force equilibrium equations of the subbodies. In Eqs. (4), dots over symbols indicate differentiation with respect to time. If a parenthetical quantity carries a subscript, this implies that all the indexable quantities within the parentheses carry the same subscript. These conventions are used throughout this work.

C. Moment Equilibrium of Subbodies

Let $\mathbf{f}_i^{0\prime}$ denote the moments acting at the mass center of the ith subbody before perturbations, and $\mathbf{F}_i'$ those moments created by the perturbations. The moment equilibrium of the subbodies in the unperturbed state may be written as

$$\int_{v_i} \nu\, \mathbf{e}^0 \times \ddot{\mathbf{e}}^0 dv + \mathbf{f}_i^{0\prime} = 0 \qquad (i = 1,2, \cdots, n) \tag{5a}$$

and the moment equilibrium equations in the perturbed state are

$$\int_{v_i} \nu\, \mathbf{e} \times \ddot{\mathbf{e}}\, dv + \mathbf{f}_i^{0\prime} - \mathbf{f}^{0\prime} \times \boldsymbol{\theta} - (\mathbf{f}^{0\prime} \times \boldsymbol{\beta})_i + \mathbf{F}_i' = 0 \qquad (i = 1, \cdots, n) \tag{5b}$$

Subtracting (5a) from (5b), one obtains

$$\int_{v_i} \nu(\mathbf{e} \times \ddot{\mathbf{e}} - \mathbf{e}^0 \times \ddot{\mathbf{e}}^0)\, dv + \mathbf{F}_i' - \mathbf{f}_i^{0\prime} \times \boldsymbol{\theta} - (\mathbf{f}^{0\prime} \times \boldsymbol{\beta})_i = 0 \qquad (i = 1,2, \cdots, n) \tag{5c}$$

Substituting $\mathbf{e}_i$ from (2a), and rearranging, one finally obtains

$$\int_{v_i} \nu[\ddot{\mathbf{e}}^0 \times (\mathbf{e}^0 \times \boldsymbol{\beta}) - \mathbf{e}^0 \times (\ddot{\mathbf{e}}^0 \times \boldsymbol{\beta}) - 2\mathbf{e}^0 \times (\dot{\mathbf{e}}^0 \times \dot{\boldsymbol{\beta}}) - \mathbf{e}^0 \times (\mathbf{e}^0 \times \ddot{\boldsymbol{\beta}}) + \ddot{\mathbf{e}}^0 \times (\mathbf{e}^0 \times \boldsymbol{\theta}) - \mathbf{e}^0 \times (\ddot{\mathbf{e}}^0 \times \boldsymbol{\theta})$$

$$- 2\mathbf{e}^0 \times (\dot{\mathbf{e}}^0 \times \dot{\boldsymbol{\theta}}) - \mathbf{e}^0 \times (\mathbf{e}^0 \times \ddot{\boldsymbol{\theta}})]\, dv + \mathbf{F}_i' - (\mathbf{f}^{0\prime} \times \boldsymbol{\beta})_i - \mathbf{f}_i^{0\prime} \times \boldsymbol{\theta} = 0 \qquad (i = 1, \cdots, n) \tag{5d}$$

which are the moment equilibrium equations of the subbodies.

D. Moment Equilibrium of Whole System

Let $\mathbf{t}^0$ denote the torques acting at the mass center of the whole system before perturbations, and let $\mathbf{T}$ denote those causing the perturbed motion. The overall moment equilibrium of the system in the unperturbed state may be written as

$$\int_V \nu \boldsymbol{\rho}_i^0 \times \ddot{\boldsymbol{\rho}}_i^0\, dv + \mathbf{t}^0 = 0 \tag{6a}$$

and the moment equilibrium equation in the perturbed state is

$$\int_V \nu \boldsymbol{\rho}_i \times \ddot{\boldsymbol{\rho}}_i\, dv + \mathbf{t}^0 - \mathbf{t}^0 \times \boldsymbol{\theta} = \mathbf{T} \tag{6b}$$

Subtracting (6a) from (6b), one obtains

$$\int_V \nu(\boldsymbol{\rho}_i \times \ddot{\boldsymbol{\rho}}_i - \boldsymbol{\rho}_i^0 \times \ddot{\boldsymbol{\rho}}_i^0)\, dv - \mathbf{T} - \mathbf{t}^0 \times \boldsymbol{\theta} = 0 \tag{6c}$$

Substituting $\boldsymbol{\rho}_i$ from (2c), and after the cancellation of $\boldsymbol{\rho}_i^0 \times \ddot{\boldsymbol{\rho}}_i^0$ terms and linearizing, using (1d), (1a), (1b), (1c), (1e), and (1f), one finally obtains

$$m_i (\mathbf{R}^0 \times \ddot{\mathbf{u}})_i - m_i (\ddot{\mathbf{R}}^0 \times \mathbf{u})_i + m_i [\ddot{\mathbf{R}}^0 \times (\mathbf{R}^0 \times \boldsymbol{\theta}) - \mathbf{R}^0 \times (\ddot{\mathbf{R}}^0 \times \boldsymbol{\theta} + 2\dot{\mathbf{R}}^0 \times \dot{\boldsymbol{\theta}} + \mathbf{R}^0 \times \ddot{\boldsymbol{\theta}})]_i$$

$$+ \int_{v_i} \nu [\ddot{\mathbf{e}}^0 \times (\mathbf{e}^0 \times \boldsymbol{\beta}) - \mathbf{e}^0 \times (\ddot{\mathbf{e}}^0 \times \boldsymbol{\beta} + 2\dot{\mathbf{e}}^0 \times \dot{\boldsymbol{\beta}} + \mathbf{e}^0 \times \ddot{\boldsymbol{\beta}}) + \ddot{\mathbf{e}}^0 \times (\mathbf{e}^0 \times \boldsymbol{\theta}) - \mathbf{e}^0 \times (\ddot{\mathbf{e}}^0 \times \boldsymbol{\theta} + 2\dot{\mathbf{e}}^0 \times \dot{\boldsymbol{\theta}}$$

$$+ \mathbf{e}^0 \times \ddot{\boldsymbol{\theta}})]_i \, dv - \mathbf{T} - \mathbf{t}^0 \times \boldsymbol{\theta} = 0 \tag{6d}$$

which is the moment equilibrium equation of the whole system.

Equations (3d), (4d), (5d), and (6d) are the vectorial equations from which unknown quantities $\mathbf{c}$, $\boldsymbol{\theta}$, $\mathbf{u}_i$ and $\boldsymbol{\beta}_i$, $i = 1, 2, \cdots, n$ can be obtained with the knowledge of initial conditions.

III. Governing Equations in Cartesian Coordinates

Let α, τ_i, and γ_i denote right-hand cartesian coordinate systems attached to and moving with the center of mass of the unperturbed system, the center of mass of the ith subbody, and the ith subbody, respectively. Let $[\alpha]$, $[\tau]_i$, and $[\gamma]_i$ denote 3×3 matrices where the columns are the direction cosines of the axes of similarly named coordinate systems, such that $[\alpha]$ contains the direction cosines of the α-coordinate axes in the inertially fixed coordinate system (see Fig. 1), $[\tau]_i$ contains the direction cosines of the τ_i-coordinate axes in the α-coordinate system, and $[\gamma]_i$ contains the direction cosines of the γ_i-coordinate axes in the τ_i-coordinate system. Let $\{u\}_i$, $\{\beta\}_i$ denote the descriptions of $\mathbf{u}_i$, $\boldsymbol{\beta}_i$ in the τ_i-coordinate system, and let $\{\theta\}$, $\{c\}$ denote the descriptions of $\boldsymbol{\theta}$, $\mathbf{c}$ in the α-coordinate system. Let $\{R_\alpha\}_i$ and $\{e_\alpha\}_i$ denote the descriptions of $\mathbf{R}_i^0$ and $\mathbf{e}_i^0$ in the α-coordinate system, and $\{R_\tau\}_i$ and $\{e_\tau\}_i$ those in the τ_i-coordinate system. With these definitions, Eqs. (3d), (4d), (5d), and (6d) may be expressed in α- and τ_i-coordinate systems as in Table 1 where a tilde has the meaning of

$$\mathbf{a} \times \mathbf{b} = [\tilde{a}]\{b\} = -[\tilde{b}]\{a\} \tag{7a}$$

where $\{a\}$ and $\{b\}$ are the descriptions of $\mathbf{a}$ and $\mathbf{b}$ in a right-hand cartesian coordinate system. If a_1, a_2, and a_3 are the components of $\mathbf{a}$ in this coordinate system, it may be observed that

$$[\tilde{a}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{7b}$$

and

$$[\tilde{a}] = -[\tilde{a}]^T \tag{7c}$$

where superscript T indicates transposition. Moreover, if $\{a_\alpha\}$ is the description of $\mathbf{a}$ in the α-coordinate system, and $\{a_\tau\}$ is the description of $\mathbf{a}$ in the τ-coordinate system, then

$$\{a_\alpha\} = [\tau]\{a_\tau\} \tag{7d}$$

and

$$\widetilde{[\tau]\{a_\tau\}} = [\tau][\tilde{a}_\tau][\tau]^T \tag{7e}$$

The identities given by Eqs. (7) are all used in obtaining the governing equations in Table 1 from Eqs. (3d), (4d), (5d), and (6d).

IV. Special Cases of Steady Motion

One can specialize the formulation given in Table 1 by assuming steady rotational motion. Let $\boldsymbol{\omega}$ denote the rotation vector of the whole system relative to the fixed coordinate system, $\boldsymbol{\Omega}_i$ the rotation vector of the τ_i-coordinate system relative to the α-coordinate system, and $\mathbf{G}_i$ the rotation vector of ith subbody itself relative to the τ_i-coordinate system. Let $\{\omega_0\}$, $\{\omega_\alpha\}$, and $\{\omega_\tau\}_i$ denote the descriptions of $\boldsymbol{\omega}$ vector in the fixed, α-, and τ_i-coordinate systems, respectively. Let $\{\Omega_0\}_i$, $\{\Omega_\alpha\}_i$ and

Table 1. The governing equations of perturbations in cartesian coordinates

$$
\left[\begin{array}{c:c:c:c}
A^{uu}_{ij} & 0 & A^{uo}_{i} & A^{uc}_{i} \\ \hdashline
0 & A^{\beta\beta}_{ij} & A^{\beta o}_{i} & 0 \\ \hdashline
A^{ou}_{j} & A^{o\beta}_{j} & A^{oo} & 0 \\ \hdashline
0 & 0 & 0 & 0
\end{array}\right]
\left\{\begin{array}{c}
\ddot{u}_j \\ \hdashline \ddot{\beta}_j \\ \hdashline \ddot{\theta} \\ \hdashline \ddot{c}
\end{array}\right\}
+2
\left[\begin{array}{c:c:c:c}
b^{uu}_{ij} & 0 & b^{uo}_{i} & b^{uc}_{i} \\ \hdashline
0 & b^{\beta\beta}_{ij} & b^{\beta o}_{i} & 0 \\ \hdashline
b^{ou}_{j} & b^{o\beta}_{j} & b^{oo} & 0 \\ \hdashline
0 & 0 & 0 & 0
\end{array}\right]
\left\{\begin{array}{c}
\dot{u}_j \\ \hdashline \dot{\beta}_j \\ \hdashline \dot{\theta} \\ \hdashline \dot{c}
\end{array}\right\}
+
\left[\begin{array}{c:c:c:c}
d^{uu}_{ij} & 0 & d^{uo}_{i} & d^{uc}_{i} \\ \hdashline
0 & d^{\beta\beta}_{ij} & d^{\beta o}_{i} & 0 \\ \hdashline
d^{ou}_{j} & d^{o\beta}_{j} & d^{oo} & 0 \\ \hdashline
d^{cu}_{j} & 0 & 0 & I
\end{array}\right]
\left\{\begin{array}{c}
u_j \\ \hdashline \beta_j \\ \hdashline \theta \\ \hdashline c
\end{array}\right\}
=-
\left\{\begin{array}{c}
F_i \\ \hdashline F'_i \\ \hdashline 0 \\ \hdashline 0
\end{array}\right\}
+
\left\{\begin{array}{c}
\mathbf{f}^{o}_{i}\times\boldsymbol{\theta}+(\mathbf{f}^{o}\times\boldsymbol{\beta})_i \\ \hdashline
\mathbf{f}'^{o}_{i}\times\boldsymbol{\theta}+(\mathbf{f}^{o\prime}\times\boldsymbol{\beta})_i \\ \hdashline
\mathbf{t}^{o}\times\boldsymbol{\theta} \\ \hdashline
0
\end{array}\right\}
+
\left\{\begin{array}{c}
0 \\ \hdashline 0 \\ \hdashline T \\ \hdashline 0
\end{array}\right\}
$$

$$[A^{uu}_{ij}] = m_i[I], i=j,$$

$$[A^{uu}_{ij}] = [0], i\neq j,$$

$$[A^{uo}_{i}] = (m[\widetilde{R}_\alpha][\tau])^T_i,$$

$$[A^{ou}_{j}] = (m[\widetilde{R}_\alpha][\tau])_j,$$

$$[A^{\beta\beta}_{ij}] = \int_{v_i}\nu[\widetilde{e}_\tau][\widetilde{e}^T_\tau]dv = [J_\tau]_i, i=j,$$

$$[A^{\beta\beta}_{ij}] = [0], i\neq j,$$

$$[A^{\beta o}_{i}] = \int_{v_i}\nu[\widetilde{e}_\tau][\widetilde{e}^T_\tau][\tau]^T dv = ([\tau][J_\tau])^T_i,$$

$$[A^{o\beta}_{j}] = ([\tau][J_\tau])_j,$$

$$[A^{oo}] = m_i([\widetilde{R}_\alpha][\widetilde{R}^T_\alpha])_i + ([\tau][J_\tau])_i[\tau]^T_i = [J],$$

$$[A^{uc}_{i}] = (m[\tau])^T_i,$$

$$[b^{uu}_{ij}] = (m([\alpha][\tau])^T(\overset{\cdot}{\overbrace{[\alpha][\tau]}}))_i, i=j,$$

$$[b^{uu}_{ij}] = [0], i\neq j,$$

$$[b^{uo}_{i}] = -(m[\widetilde{R}_\alpha][\alpha]^T[\dot{\alpha}][\tau])^T_i,$$

$$[b^{ou}_{j}] = (m[\widetilde{R}_\alpha][\alpha]^T(\overset{\cdot}{\overbrace{[\alpha][\tau]}}))_j,$$

$$[b^{\beta\beta}_{ij}] = \int_{v_i}\nu[\widetilde{e}_\tau]([\alpha][\tau])^T(\overset{\cdot}{\overbrace{[\alpha][\tau][\widetilde{e}^T_\tau]}})dv, i=j,$$

$$[b^{\beta\beta}_{ij}] = [0], i\neq j,$$

$$[b^{\beta o}_{i}] = \int_{v_i}\nu[\widetilde{e}_\tau]([\alpha][\tau])^T(\overset{\cdot}{\overbrace{[\alpha](\widetilde{[\tau]\{e_\tau\}})^T}})dv,$$

$$[b^{o\beta}_{j}] = \int_{v_j}\nu\widetilde{[\tau]\{e_\tau\}}([\alpha][\tau])^T(\overset{\cdot}{\overbrace{[\alpha][\tau][\widetilde{e}^T_\tau]}})dv,$$

$$[b^{oo}] = m_i([\widetilde{R}_\alpha][\alpha]^T[\dot{\alpha}][\widetilde{R}^T_\alpha])_i + [\tau]_i\int_{v_i}\nu[\widetilde{e}_\tau]([\alpha][\tau])^T(\overset{\cdot}{\overbrace{[\alpha](\widetilde{[\tau]\{e_\tau\}})^T}})dv,$$

$$[b^{uc}_{i}] = (m([\alpha][\tau])^T[\dot{\alpha}])_i,$$

$$[d^{uu}_{ij}] = (m([\alpha][\tau])^T(\overset{\cdot\cdot}{\overbrace{[\alpha][\tau]}}))_i, i=j,$$

$$[d^{uu}_{ij}] = [0], i\neq j,$$

$$[d^{uo}_{i}] = (m[\tau]^T[\alpha]^T[\ddot{\alpha}][\widetilde{R}^T_\alpha])_i,$$

$$[d^{ou}_{j}] = (m[\widetilde{R}_\alpha][\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha][\tau]}}) - m[\alpha]^T(\widetilde{[\ddot{\alpha}]\{R_\alpha\}})[\alpha][\tau])_j,$$

$$[d^{\beta\beta}_{ij}] = \int_{v_i}\nu([\tau]^T[\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha][\tau]\{e_\tau\}}})[\alpha][\tau][\widetilde{e}_\tau] + [\widetilde{e}_\tau][\tau]^T[\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha][\tau][\widetilde{e}^T_\tau]}}))dv, i=j,$$

$$[d^{\beta\beta}_{ij}] = [0], i\neq j,$$

$$[d^{\beta o}_{i}] = \int_{v_i}\nu([\tau]^T[\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha][\tau]\{e_\tau\}}})[\alpha](\widetilde{[\tau]\{e_\tau\}}) + [\widetilde{e}_\tau][\tau]^T[\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha]\widetilde{[\tau]\{e_\tau\}}}})^T dv,$$

$$[d^{o\beta}_{j}] = \int_{v_i}\nu([\tau][\widetilde{e}_\tau][\tau]^T[\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha][\tau][\widetilde{e}^T_\tau]}}) - [\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha][\tau]\{e_\tau\}}})[\alpha][\tau][\widetilde{e}^T_\tau])dv,$$

$$[d^{uc}_{i}] = (m([\alpha][\tau])^T[\ddot{\alpha}])_i,$$

$$[d^{cu}_{j}] = (\mu[\tau])_j,$$

$$[d^{oo}] = m_i([\widetilde{R}_\alpha][\alpha][\ddot{\alpha}][\widetilde{R}^T_\alpha])_i - m_i([\alpha]^T(\widetilde{[\ddot{\alpha}]\{R_\alpha\}})[\alpha][\widetilde{R}^T_\alpha])_i + [\tau]_i\int_{v_i}\nu[\widetilde{e}_\tau][\tau]^T[\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha](\widetilde{[\tau]\{e_\tau\}})^T}})dv - \left(\int_{v_i}\nu[\alpha]^T(\overset{\cdot\cdot}{\overbrace{[\alpha][\tau]\{e_\tau\}}})[\alpha][\tau][\widetilde{e}_\tau]dv\right)[\tau]^T_i$$

$[I]$ = Unit matrix of order three. $[J_\tau]_i = \int_{v_i}\nu[\widetilde{e}_\tau][\widetilde{e}^T_\tau]dv$ = Moment-of-inertia dyadic of ith subbody in τ_i-coordinate system.

$[J]$ = $m_i([\widetilde{R}_\alpha][\widetilde{R}^T_\alpha])_i + ([\tau][J_\tau])_i[\tau]^T_i$ = Moment-of-inertia dyadic of whole structure in α-coordinate system.

Repeated index i implies summation over the range of zero to n.

$\{\Omega_\tau\}_i$, and $\{G_0\}_i$, $\{G_\alpha\}_i$, and $\{G_\tau\}_i$ denote the descriptions of $\mathbf{\Omega}_i$ and $\mathbf{G}_i$ vectors in the coordinate systems implied by their subscripts (subscript zero is for the inertially fixed coordinate system). With these definitions, one may write that

$$\{\omega_0\} = [\alpha]\{\omega_\alpha\} = [\alpha]([\tau]\{\omega_\tau\})_i \tag{8a}$$

$$\{\Omega_0\}_i = [\alpha]\{\Omega_\alpha\}_i = [\alpha]([\tau]\{\Omega_\tau\})_i \tag{8b}$$

$$\{G_0\}_i = [\alpha]\{G_\alpha\}_i = [\alpha]([\tau]\{G_\tau\})_i \tag{8c}$$

and

$$[\dot{\alpha}] = [\tilde{\omega}_0][\alpha] \tag{8d}$$

$$[\dot{\tau}]_i = ([\tilde{\Omega}_\alpha][\tau])_i \tag{8e}$$

$$[\dot{\gamma}]_i = ([\tilde{G}_\tau][\gamma])_i \tag{8f}$$

For steady rotational motion about the principal axes of moment of inertia

$$\dot{\boldsymbol{\omega}} = \dot{\mathbf{\Omega}}_i = \dot{\mathbf{G}}_i = 0 \qquad (i = 0, 1, \cdots, n) \tag{8g}$$

which means that in the time derivatives of (8*d*), (8*e*), and (8*f*)

$$[\ddot{\alpha}] = [\tilde{\omega}_0][\tilde{\omega}_0][\alpha] + [\dot{\tilde{\omega}}_0][\alpha]$$

$$[\ddot{\tau}]_i = ([\tilde{\Omega}_\alpha][\tilde{\Omega}_\alpha][\tau] + [\dot{\tilde{\Omega}}_\alpha][\tau])_i$$

$$[\ddot{\gamma}]_i = ([\tilde{G}_\tau][\tilde{G}_\tau][\gamma] + [\dot{\tilde{G}}_\tau][\gamma])_i$$

the second terms on the right are zero; therefore,

$$[\ddot{\alpha}] = [\tilde{\omega}_0][\tilde{\omega}_0][\alpha] \tag{8h}$$

$$[\ddot{\tau}]_i = ([\tilde{\Omega}_\alpha][\tilde{\Omega}_\alpha][\alpha])_i \tag{8i}$$

$$[\ddot{\gamma}]_i = ([\tilde{G}_\tau][\tilde{G}_\tau][\gamma])_i \tag{8j}$$

From the definitions of $[\alpha]$ and $[\tau]_i$, it follows that

$$[\alpha]^{-1} = [\alpha]^T \tag{8k}$$

$$[\tau]_i^{-1} = [\tau]_i^T \tag{8l}$$

With the help of Eqs. (8), one can specialize the general formulation given in Table 1 for the steady state rotational motion, and obtain the formulation in Table 2. The formulation in Table 2 may be further specialized for the following cases.

A. Nonrotating System With No Rotating Parts

If the system is not rotating and it has no rotating parts, one observes that

$$\boldsymbol{\omega} = \mathbf{\Omega}_i = \mathbf{G}_i = 0 \qquad (i = 0, 1, \cdots, n) \tag{9a}$$

This implies that

$$\{\omega_\alpha\} = \{\omega_\tau\}_i = \{\Omega_\alpha\}_i = \{\Omega_\tau\}_i = \{G_\tau\}_i = 0 \qquad (i = 0, 1, \cdots, n) \tag{9b}$$

By using Eqs. (9*b*) in Table 2, the governing equations of this case may be obtained as in the next table (see Table 3). This case is dealt with in Ref. 4.

B. Rotating System With No Rotating Parts

In this case

$$\mathbf{\Omega}_i = \mathbf{G}_i = 0 \qquad (i = 0, 1, \cdots, n) \tag{10a}$$

but $\boldsymbol{\omega} \neq 0$. From (10*a*), it follows that

$$\{\Omega_\alpha\}_i = \{\Omega_\tau\}_i = \{G_\tau\}_i = 0 \qquad (i = 0, 1, \cdots, n) \tag{10b}$$

By using Eqs. (10*b*) in Table 2, the governing equations of rotating system with no rotating parts may be obtained as in Table 4. This case is dealt with in Ref. 3.

C. Rotating System With Rotating Parts Where $\mathbf{\Omega}_i = 0$

In this case $\boldsymbol{\omega} \neq 0$, and $\mathbf{G}_i \neq 0$, but

$$\mathbf{\Omega}_i = 0 \qquad (i = 0, 1, \cdots, n) \tag{11a}$$

which implies that

$$\{\Omega_\alpha\}_i = \{\Omega_\tau\}_i = 0 \qquad (i = 0, 1, \cdots, n) \tag{11b}$$

Using Eqs. (11*b*) in Table 2, one obtains the formulation given in a later table (see Table 5).

Table 2. The governing equations of perturbations in cartesian coordinates for steady nominal rotational motion

$$
\underbrace{\begin{bmatrix} A_{ij}^{uu} & 0 & A_i^{u\theta} & A_i^{uc} \\ 0 & A_{ij}^{\beta\beta} & A_i^{\beta\theta} & 0 \\ A_j^{\theta u} & A_j^{\theta\beta} & A^{\theta\theta} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u}_j \\ \ddot{\beta}_j \\ \ddot{\theta} \\ \ddot{c} \end{Bmatrix}}_{\text{INERTIAL}} + 2 \underbrace{\begin{bmatrix} b_{ij}^{uu} & 0 & b_i^{u\theta} & b_i^{uc} \\ 0 & b_{ij}^{\beta\beta} & b_i^{\beta\theta} & 0 \\ b_j^{\theta u} & b_j^{\theta\beta} & b^{\theta\theta} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_j \\ \dot{\beta}_j \\ \dot{\theta} \\ \dot{c} \end{Bmatrix}}_{\text{CORIOLIS}} + \underbrace{\begin{bmatrix} d_{ij}^{uu} & 0 & d_i^{u\theta} & d_i^{uc} \\ 0 & d_{ij}^{\beta\beta} & d_i^{\beta\theta} & 0 \\ d_j^{\theta u} & d_j^{\theta\beta} & d^{\theta\theta} & 0 \\ d_j^{cu} & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} u_j \\ \beta_j \\ \theta \\ c \end{Bmatrix}}_{\text{CENTRIFUGAL}} = -\underbrace{\begin{Bmatrix} F_i \\ F_i' \\ 0 \\ 0 \end{Bmatrix}}_{\text{ELASTIC}} + \underbrace{\begin{Bmatrix} \mathbf{f}_i^{0\prime} \times \boldsymbol{\theta} + (\mathbf{f}^0 \times \boldsymbol{\beta})_i \\ \mathbf{f}_i^{0\prime} \times \boldsymbol{\theta} + (\mathbf{f}^{0\prime} \times \boldsymbol{\beta})_i \\ 0 \\ 0 \end{Bmatrix}}_{\text{INITIAL STRESS}} + \underbrace{\begin{Bmatrix} 0 \\ 0 \\ T \\ 0 \end{Bmatrix}}_{\text{CONTROL}}
$$

$[A_{ij}^{uu}] = m_i[I], i = j,$

$[A_{ij}^{uu}] = [0], i \neq j,$

$[A_i^{u\theta}] = (m[\widetilde{R}_\alpha][\tau])_i^T,$

$[A_j^{\theta u}] = (m[\widetilde{R}_\alpha][\tau])_j,$

$[A_{ij}^{\beta\beta}] = [J_\tau]_i, i = j,$

$[A_{ij}^{\beta\beta}] = [0], i \neq j,$

$[A_i^{\beta\theta}] = ([J_\tau][\tau]^T)_i,$

$[A_j^{\theta\beta}] = ([\tau][J_\tau])_j,$

$[A^{\theta\theta}] = [J],$

$[A_i^{uc}] = (m[\tau])_i^T,$

$[b_{ij}^{uu}] = (m([\widetilde{\omega}_\tau] + [\widetilde{\Omega}_\tau]))_i, i = j,$

$[b_{ij}^{uu}] = [0], i \neq j,$

$[b_i^{u\theta}] = -(m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\tau])_i^T,$

$[b_j^{\theta u}] = (m[\widetilde{R}_\alpha]([\widetilde{\omega}_\alpha] + [\widetilde{\Omega}_\alpha])[\tau])_j,$

$[b_{ij}^{\beta\beta}] = [L_\tau]_i, i = j,$

$[b_{ij}^{\beta\beta}] = [0], i \neq j,$

$[b_i^{\beta\theta}] = (([L_\tau] + [J_\tau][\widetilde{\Omega}_\tau])[\tau]^T)_i,$

$[b_j^{\theta\beta}] = ([\tau][L_\tau])_j,$

$[b^{\theta\theta}] = [L],$

$[b_i^{uc}] = (m[\widetilde{\omega}_\tau][\tau]^T)_i,$

$[d_{ij}^{uu}] = (m([\widetilde{\omega}_\tau] + [\widetilde{\Omega}_\tau])^2)_i, i = j,$

$[d_{ij}^{uu}] = [0], i \neq j,$

$[d_i^{u\theta}] = (m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha]^2[\tau])_i^T,$

$[d_j^{\theta u}] = (m([\widetilde{R}_\alpha]([\widetilde{\omega}_\alpha] + [\widetilde{\Omega}_\alpha])^2 - (\widetilde{[\widetilde{\omega}_\alpha]^2 \{R_\alpha\}}))[\tau])_j,$

$[d_{ij}^{\beta\beta}] = [N_\tau]_i, i = j,$

$[d_{ij}^{\beta\beta}] = [0], i \neq j,$

$[d^{\theta\theta}] = [N], \qquad [d_j^{cu}] = (\mu[\tau])_j,$

$[d_i^{uc}] = (m[\widetilde{\omega}_\tau]^2[\tau]^T)_i$

$[I] \quad$ = Unit matrix of order three. $[J_\tau]_i = \int_{v_i} \nu[\widetilde{e}_\tau][\widetilde{e}_\tau^T]dv$ = Moment-of-inertia dyadic of ith subbody in τ_i-coordinate system.

$[J] \quad = m_i([\widetilde{R}_\alpha][\widetilde{R}_\alpha^T])_i + ([\tau][J_\tau])_i[\tau]_i^T$ = Moment-of-inertia dyadic of whole structure in α-coordinate system.

$[L_\tau]_i = \int_{v_i} \nu([\widetilde{e}_\tau]([\widetilde{\omega}_\tau] + [\widetilde{\Omega}_\tau])[\widetilde{e}_\tau^T] + [\widetilde{e}_\tau](\widetilde{[\widetilde{G}_\tau]\{e_\tau\}})^T)dv, \qquad [L] = m_i([\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\widetilde{R}_\alpha^T])_i + [\tau]_i([L_\tau][\tau]^T)_i.$

$[N_\tau]_i = \int_{v_i} \nu([\widetilde{e}_\tau]([\widetilde{\omega}_\tau] + [\widetilde{\Omega}_\tau])^2[\widetilde{e}_\tau^T] - (\widetilde{([\widetilde{\omega}_\tau] + [\widetilde{\Omega}_\tau])^2\{e_\tau\}}\,[\widetilde{e}_\tau^T] + [\widetilde{e}_\tau](\widetilde{[\widetilde{G}_\tau]^2\{e_\tau\}})^T - (\widetilde{[\widetilde{G}_\tau]^2\{e_\tau\}})[\widetilde{e}_\tau^T])dv$

$\quad + 2\int_{v_i} \nu([\widetilde{e}_\tau]([\widetilde{\omega}_\tau] + [\widetilde{\Omega}_\tau])(\widetilde{[\widetilde{G}_\tau]\{e_\tau\}})^T - (\widetilde{([\widetilde{\omega}_\tau] + [\widetilde{\Omega}_\tau])[\widetilde{G}_\tau]\{e_\tau\}})[\widetilde{e}_\tau^T])dv.$

$[N] \quad = m_i([\widetilde{R}_\alpha][\widetilde{\omega}_\alpha]^2[\widetilde{R}_\alpha^T])_i - m_i(\widetilde{[\widetilde{\omega}_\alpha]^2\{R_\alpha\}}[\widetilde{R}_\alpha^T])_i + [\tau]_i([N_\tau][\tau]^T)_i.$

Repeated index i implies summation over the range of zero to n.

V. Matrix of Transfer Functions Relating Control Torques to Attitude Angles

In this section only the governing equations given in Tables 3, 4, 5 are considered. These equations are for the small perturbations from the nominal motion. The perturbations are caused by the control torque $\{T\}$. The angles $\{\theta\}$ represent the errors in the attitude angles. The second order system in Table 5 contains state variables $\{c\}$, $\{\theta\}$, $\{u\}_i$, and $\{\beta\}_i$, $i = 1, 2, \cdots, n$, from which $\{c\}$, $\{u\}_i$, and $\{\beta\}_i$, $i = 1, 2, \cdots, n$ are to be eliminated. The required matrix of transfer functions can be obtained from the remaining equations with relative ease.

The first vector in the right hand side of the equations in Table 5 represents the elastic and damping forces caused by the perturbation deformations, the second vector in the right hand side represents the effect of initial stresses which may exist before perturbations, and the last vector in the right hand side represents the control torques which are assumed to be acting at the center of mass of the system before perturbations.

Let the stiffness matrix (Ref. 5), the geometric matrix (Ref. 6), and the general damping matrix (Ref. 7) associated with $\{u\}_i$ and $\{\beta\}_i$ directions be denoted by

$$\left[\begin{array}{c|c} K_{ij}^{uu} & K_{ij}^{u\beta} \\ \hline K_{ij}^{\beta u} & K_{ij}^{\beta\beta} \end{array}\right], \qquad \left[\begin{array}{c|c} K_{0ij}^{uu} & K_{0ij}^{u\beta} \\ \hline K_{0ij}^{\beta u} & K_{0ij}^{\beta\beta} \end{array}\right], \qquad \left[\begin{array}{c|c} c_{ij}^{uu} & c_{ij}^{u\beta} \\ \hline c_{ij}^{\beta u} & c_{ij}^{\beta\beta} \end{array}\right]$$

respectively. With these, one may write

$$\left\{\begin{array}{c} \mathbf{F}_i - \mathbf{f}_i^0 \times \boldsymbol{\theta} - (\mathbf{f}^0 \times \boldsymbol{\beta})_i \\ \hline \mathbf{F}'_i - \mathbf{f}_i^{0\prime} \times \boldsymbol{\theta} - (\mathbf{f}^{0\prime} \times \boldsymbol{\beta})_i \end{array}\right\} = \left[\begin{array}{c|c} K_{ij}^{uu} + K_{0ij}^{uu} & K_{ij}^{u\beta} + K_{0ij}^{u\beta} \\ \hline K_{ij}^{\beta u} + K_{0ij}^{\beta u} & K_{ij}^{\beta\beta} + K_{0ij}^{\beta\beta} \end{array}\right] \left\{\begin{array}{c} u_j \\ \hline \beta_j \end{array}\right\} + \left[\begin{array}{c|c} c_{ij}^{uu} & c_{ij}^{u\beta} \\ \hline c_{ij}^{\beta u} & c_{ij}^{\beta\beta} \end{array}\right] \left\{\begin{array}{c} \dot{u}_j \\ \hline \dot{\beta}_j \end{array}\right\} \tag{12a}$$

Substituting these into the governing equations in Table 5, one obtains

$$\left[\begin{array}{c|c|c|c} A_{ij}^{uu} & 0 & A_i^{u\theta} & A_i^{uc} \\ \hline 0 & A_{ij}^{\beta\beta} & A_i^{\beta\theta} & 0 \\ \hline A_j^{\theta u} & A_j^{\theta\beta} & A^{\theta\theta} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \left\{\begin{array}{c} \ddot{u}_j \\ \hline \ddot{\beta}_j \\ \hline \ddot{\theta} \\ \hline \ddot{c} \end{array}\right\} + \left[\begin{array}{c|c|c|c} B_{ij}^{uu} & B_{ij}^{u\beta} & B_i^{u\theta} & B_i^{uc} \\ \hline B_{ij}^{\beta u} & B_{ij}^{\beta\beta} & B_i^{\beta\theta} & 0 \\ \hline B_j^{\theta u} & B_j^{\theta\beta} & B^{\theta\theta} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \left\{\begin{array}{c} \dot{u}_j \\ \hline \dot{\beta}_j \\ \hline \dot{\theta} \\ \hline \dot{c} \end{array}\right\} + \left[\begin{array}{c|c|c|c} C_{ij}^{uu} & C_{ij}^{u\beta} & C_i^{u\theta} & C_i^{uc} \\ \hline C_{ij}^{\beta u} & C_{ij}^{\beta\beta} & C_i^{\beta\theta} & 0 \\ \hline C_j^{\theta u} & C_j^{\theta\beta} & C^{\theta\theta} & 0 \\ \hline C_j^{cu} & 0 & 0 & I \end{array}\right] \left\{\begin{array}{c} u_j \\ \hline \beta_j \\ \hline \theta \\ \hline c \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \\ \hline T \\ \hline 0 \end{array}\right\} \tag{12b}$$

where

$$
\begin{bmatrix}
B^{uu}_{ij} & B^{u\beta}_{ij} & B^{u\theta}_{i} & B^{uc}_{i} \\
B^{\beta u}_{ij} & B^{\beta\beta}_{ij} & B^{\beta\theta}_{i} & 0 \\
B^{\theta u}_{j} & B^{\theta\beta}_{j} & B^{\theta\theta} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
= 2
\begin{bmatrix}
b^{uu}_{ij} & 0 & b^{u\theta}_{i} & b^{uc}_{i} \\
0 & b^{\beta\beta}_{ij} & b^{\beta\theta}_{i} & 0 \\
b^{\theta u}_{j} & b^{\theta u}_{j} & b^{\theta\theta} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
+
\begin{bmatrix}
c^{uu}_{ij} & c^{u\beta}_{ij} & 0 & 0 \\
c^{\beta u}_{ij} & c^{\beta\beta}_{ij} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\tag{12c}
$$

and

$$
\begin{bmatrix}
C^{uu}_{ij} & C^{u\beta}_{ij} & C^{u\theta}_{i} & C^{uc}_{i} \\
C^{\beta u}_{ij} & C^{\beta\beta}_{ij} & C^{\beta\theta}_{i} & 0 \\
C^{\theta u}_{j} & C^{\theta\beta}_{j} & C^{\theta\theta} & 0 \\
C^{cu}_{j} & 0 & 0 & I
\end{bmatrix}
=
\begin{bmatrix}
d^{uu}_{ij} & 0 & d^{u\theta}_{i} & d^{uc}_{i} \\
0 & d^{\beta\beta}_{ij} & d^{\beta\theta}_{i} & 0 \\
d^{\theta u}_{j} & d^{\theta\beta}_{j} & d^{\theta\theta} & 0 \\
d^{cu}_{j} & 0 & 0 & I
\end{bmatrix}
+
\begin{bmatrix}
K^{uu}_{ij} + K^{uu}_{0ij} & K^{u\beta}_{ij} + K^{u\beta}_{0ij} & 0 & 0 \\
K^{\beta u}_{ij} + K^{\beta u}_{0ij} & K^{\beta\beta}_{ij} + K^{\beta\beta}_{0ij} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\tag{12d}
$$

and all others are as defined in Table 5. Equations (12*b*) may be put in a more useful form by reordering $\{u\}_i$ and $\{\beta\}_i$ subvectors as in $\{(u,\beta)_i\} = \{q_i\}$:

$$
\begin{bmatrix}
A^{qq}_{ij} & A^{q\theta}_{i} & A^{qc}_{i} \\
A^{\theta q}_{j} & A^{\theta\theta} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix} \ddot{q}_j \\ \ddot{\theta} \\ \ddot{c} \end{Bmatrix}
+
\begin{bmatrix}
B^{qq}_{ij} & B^{q\theta}_{i} & B^{qc}_{i} \\
B^{\theta q}_{j} & B^{\theta\theta} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix} \dot{q}_j \\ \dot{\theta} \\ \dot{c} \end{Bmatrix}
+
\begin{bmatrix}
C^{qq}_{ij} & C^{q\theta}_{i} & C^{qc}_{i} \\
C^{\theta q}_{j} & C^{\theta\theta} & 0 \\
C^{cq}_{j} & 0 & I
\end{bmatrix}
\begin{Bmatrix} q_j \\ \theta \\ c \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ T \\ 0 \end{Bmatrix}
\tag{12e}
$$

where

$$[B^{qq}_{ij}] = 2\,[b^{qq}_{ij}] + [c^{qq}_{ij}] \tag{12f}$$

and

$$[C^{qq}_{ij}] = [d^{qq}_{ij}] + [K^{qq}_{ij}] + [K^{qq}_{0ij}] \tag{12g}$$

Matrices $[A^{qq}_{ij}]$, $[b^{qq}_{ij}]$, and $[d^{qq}_{ij}]$ are always at the most tridiagonal, and matrices $[c^{qq}_{ij}]$, $[K^{qq}_{0ij}]$, and $[K^{qq}_{ij}]$ are usually banded. Matrices $[A^{qq}_{ij}]$ and $[K^{qq}_{ij}]$ are always symmetric and positive definite. In the absence of damping, matrix $[c^{qq}_{ij}]$ is a zero matrix. Matrix $[K^{qq}_{0ij}]$ may or may not be symmetrical; however, it is always zero in the absence of initial stresses. If there are no rotating subbodies, i.e., if $\mathbf{G}_i = 0$ for all i, then $[b^{qq}_{ij}]$ is skew-symmetric, and $[d^{qq}_{ij}]$ can be assumed symmetric by ignoring the skew-symmetric part, as explained in the next paragraph.

Table 3. The governing equations of perturbations in cartesian coordinates for nonrotating system with no rotating parts

$$
\left[\begin{array}{c:c:c:c} A^{uu}_{ij} & 0 & A^{uo}_{i} & A^{uc}_{i} \\ \hdashline 0 & A^{\beta\beta}_{ij} & A^{\beta o} & 0 \\ \hdashline A^{ou}_{j} & A^{o\beta}_{j} & A^{oo} & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right]
\left\{\begin{array}{c} \ddot{u}_j \\ \hdashline \ddot{\beta}_j \\ \hdashline \ddot{\theta} \\ \hdashline \ddot{c} \end{array}\right\}
+2\left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right]
\left\{\begin{array}{c} \dot{u}_j \\ \hdashline \dot{\beta}_j \\ \hdashline \dot{\theta} \\ \hdashline \dot{c} \end{array}\right\}
+\left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline [\mu\tau]_j & 0 & 0 & I \end{array}\right]
\left\{\begin{array}{c} u_j \\ \hdashline \beta_j \\ \hdashline \theta \\ \hdashline c \end{array}\right\}
= -\left\{\begin{array}{c} F_i \\ \hdashline F'_i \\ \hdashline 0 \\ \hdashline 0 \end{array}\right\}
+\left[\begin{array}{c} \mathbf{f}^o \times \boldsymbol{\theta} + (\mathbf{f}^o \times \boldsymbol{\beta})_i \\ \hdashline \mathbf{f}^{o\prime} \times \boldsymbol{\theta} + (\mathbf{f}^{o\prime} \times \boldsymbol{\beta})_i \\ \hdashline 0 \\ \hdashline 0 \end{array}\right]
+\left\{\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline T \\ \hdashline 0 \end{array}\right\}
$$

$$[A^{uu}_{ij}] = m_i[I], i = j$$

$$[A^{uu}_{ij}] = [0], i \neq j$$

$$[A^{uo}_{i}] = (m[\widetilde{R}_\alpha][\tau])^T_i$$

$$[A^{ou}_{j}] = (m[\widetilde{R}_\alpha][\tau])_j$$

$$[A^{\beta\beta}_{ij}] = [J_\tau]_i, i = j$$

$$[A^{\beta\beta}_{ij}] = [0], i \neq j$$

$$[A^{\beta o}_{i}] = ([\tau][J_\tau])^T_i$$

$$[A^{o\beta}_{j}] = ([\tau][J_\tau])_j$$

$$[A^{oo}] = [J]$$

$$[A^{uc}_{i}] = (m[\tau])^T_i$$

$[I]$ = Unit matrix of order three.

$[J_\tau]_i = \int_{\tau_i} \nu[\widetilde{e}_\tau][\widetilde{e}^T_\tau]dv$ = Moment-of-inertia dyadic of ith subbody in τ_i-coordinate system.

$[J] = m_i([\widetilde{R}_\alpha][\widetilde{R}^T_\alpha])_i + ([\tau][J_\tau])_i[\tau]^T_i$ = Moment-of-inertia dyadic of whole structure in α-coordinate system.

Repeated index i implies summation over the range of zero to n.

Table 4. The governing equations of perturbations in cartesian coordinates for rotating system with no rotating parts

$$
\left[\begin{array}{c:c:c:c}
A_{ij}^{uu} & 0 & A_i^{u\theta} & A_i^{uc} \\ \hdashline
0 & A_{ij}^{\beta\beta} & A_i^{\beta\theta} & 0 \\ \hdashline
A_j^{\theta u} & A_j^{\theta\beta} & A^{\theta\theta} & 0 \\ \hdashline
0 & 0 & 0 & 0
\end{array}\right]
\left\{\begin{array}{c}
\ddot{u}_j \\ \hdashline \ddot{\beta}_j \\ \hdashline \ddot{\theta} \\ \hdashline \ddot{c}
\end{array}\right\}
+2
\left[\begin{array}{c:c:c:c}
b_{ij}^{uu} & 0 & b_i^{u\theta} & b_i^{uc} \\ \hdashline
0 & b_{ij}^{\beta\beta} & b_i^{\beta\theta} & 0 \\ \hdashline
b_j^{\theta u} & b_j^{\theta\beta} & b^{\theta\theta} & 0 \\ \hdashline
0 & 0 & 0 & 0
\end{array}\right]
\left\{\begin{array}{c}
\dot{u}_j \\ \hdashline \dot{\beta}_j \\ \hdashline \dot{\theta} \\ \hdashline \dot{c}
\end{array}\right\}
+
\left[\begin{array}{c:c:c:c}
d_{ij}^{uu} & 0 & d_i^{u\theta} & d_i^{uc} \\ \hdashline
0 & d_{ij}^{\beta\beta} & d_i^{\beta\theta} & 0 \\ \hdashline
d_j^{\theta u} & d_j^{\theta\beta} & d^{\theta\theta} & 0 \\ \hdashline
d_j^{cu} & 0 & 0 & \mathrm{I}
\end{array}\right]
\left\{\begin{array}{c}
u_j \\ \hdashline \beta_j \\ \hdashline \theta \\ \hdashline c
\end{array}\right\}
=-
\left\{\begin{array}{c}
F_i \\ \hdashline F'_i \\ \hdashline 0 \\ \hdashline 0
\end{array}\right\}
+
\left[\begin{array}{c}
\mathbf{f}_i^0 \times \boldsymbol{\theta} + (\mathbf{f}^0 \times \boldsymbol{\beta})_i \\ \hdashline
\mathbf{f}_i^{0\prime} \times \boldsymbol{\theta} + (\mathbf{f}^{0\prime} \times \boldsymbol{\beta})_i \\ \hdashline
0 \\ \hdashline 0
\end{array}\right]
+
\left\{\begin{array}{c}
0 \\ \hdashline 0 \\ \hdashline T \\ \hdashline 0
\end{array}\right\}
$$

$[A_{ij}^{uu}] = m_i[I], i = j,$

$[A_{ij}^{uu}] = [0], i \neq j,$

$[A_i^{u\theta}] = (m[\widetilde{R}_\alpha][\tau])_i^T,$

$[A_j^{\theta u}] = (m[\widetilde{R}_\alpha][\tau])_j,$

$[A_{ij}^{\beta\beta}] = [J_\tau]_i, i = j,$

$[A_{ij}^{\beta\beta}] = [0], i \neq j,$

$[A_i^{\beta\theta}] = ([\tau][J_\tau])_i^T,$

$[A_j^{\theta\beta}] = ([\tau][J_\tau])_j,$

$[A^{\theta\theta}] = [J],$

$[A_i^{uc}] = (m[\tau])_i^T,$

$[b_{ij}^{uu}] = (m[\widetilde{\omega}_\tau])_i, i = j,$

$[b_{ij}^{uu}] = [0], i \neq j,$

$[b_i^{u\theta}] = -(m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\tau])_i^T,$

$[b^{\theta u}] = (m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\tau])_j,$

$[b_{ij}^{\beta\beta}] = [L_\tau]_i, i = j,$

$[b_{ij}^{\beta\beta}] = [0], i \neq j,$

$[b_i^{\beta\theta}] = -([\tau][L_\tau])_i^T,$

$[b_j^{\theta\beta}] = ([\tau][L_\tau])_j,$

$[b^{\theta\theta}] = [L],$

$[b_i^{uc}] = (m[\widetilde{\omega}_\tau][\tau]^T)_i,$

$[d_{ij}^{uu}] = (m[\widetilde{\omega}_\tau]^2)_i, i = j,$

$[d_{ij}^{uu}] = [0], i \neq j,$

$[d_i^{u\theta}] = (m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha]^2[\tau])_i^T,$

$[d_j^{\theta u}] = (m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha]^2[\tau])_j,$

$[d_{ij}^{\beta\beta}] = [N_\tau]_i, i = j,$

$[d_{ij}^{\beta\beta}] = [0], i \neq j,$

$[d^{\theta\theta}] = [N],$ $\qquad [d_j^{cu}] = (\mu[\tau])_j,$

$[d_i^{uc}] = (m[\widetilde{\omega}_\tau]^2[\tau]^T)_i$

$[I]$ = Unit matrix of order three.

$[J_\tau]_i = \int_{r_i} \nu[\widetilde{e}_\tau][\widetilde{e}_\tau^T]\,dv$ = Moment-of-inertia dyadic of ith subbody in τ_i-coordinate system.

$[J] = m_i([\widetilde{R}_\alpha][\widetilde{R}_\alpha^T])_i + ([\tau][J_\tau])_i[\tau]_i^T$ = Moment of inertia of whole structure in α-coordinate system.

$[L_\tau] = \int_{r_i} \nu[\widetilde{e}_\tau][\widetilde{\omega}_\tau][\widetilde{e}_\tau^T]\,dv,$

$[L] = m_i([\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\widetilde{R}_\alpha^T])_i + ([\tau][L_\tau])_i[\tau]_i^T.$

$[N_\tau]_i = \int_{r_i} \nu([\widetilde{e}_\tau][\widetilde{\omega}_\tau]^2[\widetilde{e}_\tau^T] - (\widetilde{[\widetilde{\omega}_\tau]^2\{e_\tau\}})[\widetilde{e}_\tau^T])\,dv.$

$[N] = m_i([\widetilde{R}_\alpha][\widetilde{\omega}_\alpha]^2[\widetilde{R}_\alpha^T])_i - m_i(\widetilde{[\widetilde{\omega}_\alpha]^2\{R_\alpha\}}[\widetilde{R}^T])_i + ([\tau][N_\tau])_i[\tau]_i^T.$

Repeated index i implies summation over the range of zero to n.

Table 5. The governing equations of perturbations in cartesian coordinates for rotating system with rotating parts where $\Omega_i = 0$

$$
\left[\begin{array}{c|c|c|c} A_{ij}^{uu} & 0 & A_i^{u\theta} & A_i^{uc} \\ \hline 0 & A_{ij}^{\beta\beta} & A_i^{\beta\theta} & 0 \\ \hline A_j^{\theta u} & A_j^{\theta\beta} & A^{\theta\theta} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]
\left\{\begin{array}{c} \ddot{u}_j \\ \hline \ddot{\beta}_j \\ \hline \ddot{\theta} \\ \hline \ddot{c} \end{array}\right\}
+ 2 \left[\begin{array}{c|c|c|c} b_{ij}^{uu} & 0 & b_i^{u\theta} & b_i^{uc} \\ \hline 0 & b_{ij}^{\beta\beta} & b_i^{\beta\theta} & 0 \\ \hline b_j^{\theta u} & b_j^{\theta\beta} & b^{\theta\theta} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]
\left\{\begin{array}{c} \dot{u}_j \\ \hline \dot{\beta}_j \\ \hline \dot{\theta} \\ \hline \dot{c} \end{array}\right\}
+ \left[\begin{array}{c|c|c|c} d_{ij}^{uu} & 0 & d_i^{u\theta} & d_i^{uc} \\ \hline 0 & d_{ij}^{\beta\beta} & d_i^{\beta\theta} & 0 \\ \hline d_j^{\theta u} & d_j^{\theta\beta} & d^{\theta\theta} & 0 \\ \hline d_j^{cu} & 0 & 0 & I \end{array}\right]
\left\{\begin{array}{c} u_j \\ \hline \beta_j \\ \hline \theta \\ \hline c \end{array}\right\}
= - \left\{\begin{array}{c} F_i \\ \hline F_i' \\ \hline 0 \\ \hline 0 \end{array}\right\}
+ \left\{\begin{array}{c} \mathbf{f}^{o} \times \boldsymbol{\theta} + (\mathbf{f}^{o} \times \boldsymbol{\beta})_i \\ \hline \mathbf{f}^{o\prime} \times \boldsymbol{\theta} + (\mathbf{f}^{o\prime} \times \boldsymbol{\beta})_i \\ \hline 0 \\ \hline 0 \end{array}\right\}
+ \left\{\begin{array}{c} 0 \\ \hline 0 \\ \hline T \\ \hline 0 \end{array}\right\}
$$

$[A_{ij}^{uu}] = m_i[I], i = j,$	$[b_{ij}^{uu}] = (m[\widetilde{\omega}_\tau])_i, i = j,$	$[d_{ij}^{uu}] = (m[\widetilde{\omega}_\tau]^2)_i, i = j,$
$[A_{ij}^{uu}] = [0], i \neq j,$	$[b_{ij}^{uu}] = [0], i \neq j,$	$[d_{ij}^{uu}] = [0], i \neq j,$
$[A_i^{u\theta}] = (m[\widetilde{R}_\alpha][\tau])_i^T,$	$[b_i^{u\theta}] = -(m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\tau])_i^T,$	$[d_i^{u\theta}] = (m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha]^2[\tau])_i^T,$
$[A_j^{\theta u}] = (m[\widetilde{R}_\alpha][\tau])_j,$	$[b_j^{\theta u}] = (m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\tau])_j,$	$[d_j^{\theta u}] = (m[\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\tau] - m[\widetilde{R}_\alpha](\widetilde{[\widetilde{\omega}_\alpha]\{R_\alpha\}})[\tau])_j,$
$[A_{ij}^{\beta\beta}] = [J_\tau]_i, i = j,$	$[b_{ij}^{\beta\beta}] = [L_\tau]_i, i = j,$	$[d_{ij}^{\beta\beta}] = [N_\tau]_i, i = j,$
$[A_{ij}^{\beta\beta}] = [0], i \neq j,$	$[b_{ij}^{\beta\beta}] = [0], i \neq j,$	$[d_{ij}^{\beta\beta}] = [0], i \neq j,$
$[A_i^{\beta\theta}] = ([\tau][J_\tau])_i^T,$	$[b_i^{\beta\theta}] = ([L_\tau][\tau]^T)_i,$	$[d^{\theta\theta}] = [N], \quad [d_j^{cu}] = (\mu[\tau])_j$
$[A_j^{\theta\beta}] = ([\tau][J_\tau])_j,$	$[b_j^{\theta\beta}] = ([\tau][L_\tau])_j$	$[d_i^{uc}] = (m[\widetilde{\omega}_\tau]^2[\tau]^T)_i$
$[A^{\theta\theta}] = [J],$	$[b^{\theta\theta}] = [L],$	
$[A_i^{uc}] = (m[\tau])_i^T,$	$[b_i^{uc}] = (m[\widetilde{\omega}_\tau][\tau]^T)_i,$	

$[I]$ = Unit matrix of order three.

$[J_\tau]_i = \int_{v_i} \nu[\widetilde{e}_\tau][e_\tau^T]dv$ = Moment-of-inertia dyadic of ith subbody in τ_i-coordinate system.

$[J] = m_i([\widetilde{R}_\alpha][\widetilde{R}_\alpha^T])_i + ([\tau][J_\tau])_i[\tau]_i^T$ = Moment-of-inertia dyadic of whole structure in α-coordinate system.

$[L_\tau]_i = \int_{v_i} \nu([\widetilde{e}_\tau][\widetilde{\omega}_\tau][\widetilde{e}_\tau^T] + [\widetilde{e}_\tau](\widetilde{[\widetilde{G}_\tau]\{e_\tau\}})^T)dv,$

$[L] = m_i([\widetilde{R}_\alpha][\widetilde{\omega}_\alpha][\widetilde{R}_\alpha^T])_i + ([\tau][L_\tau])_i[\tau]_i^T.$

$[N_\tau]_i = \int_{v_i} \nu([\widetilde{e}_\tau][\widetilde{\omega}_\tau]^2[\widetilde{e}_\tau^T] - (\widetilde{[\widetilde{\omega}_\tau]^2\{e_\tau\}})[\widetilde{e}_\tau^T] + [\widetilde{e}_\tau](\widetilde{[\widetilde{G}_\tau]^2\{e_\tau\}})^T - (\widetilde{[\widetilde{G}_\tau]^2\{e_\tau\}})[\widetilde{e}_\tau^T])\,dv.$

$\qquad + 2\int_{v_i} \nu([\widetilde{e}_\tau][\widetilde{\omega}_\tau](\widetilde{[\widetilde{G}_\tau]\{e_\tau\}})^T - (\widetilde{[\widetilde{\omega}_\tau][\widetilde{G}_\tau]\{e_\tau\}})[\widetilde{e}_\tau^T])dv.$

$[N] = m_i([\widetilde{R}_\alpha][\widetilde{\omega}_\alpha]^2[\widetilde{R}_\alpha^T])_i - m_i(\widetilde{[\widetilde{\omega}_\alpha]^2\{R_\alpha\}}[\widetilde{R}_\alpha^T])_i + ([\tau][N_\tau])_i[\tau]_i^T.$

Repeated index i implies summation over the range of zero to n.

It may be observed from Table 4 that one of the generating submatrices of $[d_{ij}^{qq}]$, that is, $[N_T]_i$ is not symmetrical; however, it may be expressed as the sum of a symmetric and a skew-symmetric matrix:

$$[N_T] = [n_T] + [n_T'] \tag{13a}$$

where the symmetric matrix may be expressed as

$$[n_T]_i = \int_{v_i} \nu \left[[\tilde{\omega}_T^T]\{e_T\}[e^T][\tilde{\omega}_T] - [\omega_T]\{e_T\}[e_T]\{\omega_T\}[I] + \frac{1}{2}(\{e_T\}[e_T]\{\omega_T\}[\omega_T] + \{\omega_T\}[\omega_T]\{e_T\}[e_T]) \right] dv \tag{13b}$$

and the skew-symmetric matrix is

$$[n_T']_i = \frac{1}{2}\int_{v_i} \nu(\{e_T\}[e_T]\{\omega_T\}[\omega_T] - \{\omega_T\}[\omega_T]\{e_T\}[e_T])\, dv \tag{13c}$$

Observing that the norm of $[n_T']_i$ is at least one order of magnitude smaller than that of $[n_T]_i$ for small rotation rates, the approximation of

$$[N_T]_i \approx [n_T]_i \tag{13d}$$

may sometimes be justifiable.

Note that the state variables $\{c\}$ can be easily eliminated from the differential equations set of (12e) at the expense of losing the bandedness of matrices $[A_{ij}^{qq}]$, $[B_{ij}^{qq}]$, and $[C_{ij}^{qq}]$, since the elimination process amounts to replacing all the zero entries of these matrices with nonzero quantities (see Ref. 3). In this work, the elimination of $\{c\}$ is performed after the generation of the eigenvectors of the homogeneous system

$$[A_{ij}^{qq}]\{\ddot{q}_j\} + [B_{ij}^{qq}]\{\dot{q}_j\} + [C_{ij}^{qq}]\{q_j\} = \{0\} \tag{13e}$$

which is associated with the first group of equations in (12e). This way of elimination is justified since $\{c\}$ represents the shift in the center of mass of the whole system because of perturbed deformations (see Eq. 3d).

To describe the procedure adopted in this work, one may rewrite Eqs. (12e) as

$$\left[\begin{array}{c|c} B_{ij}^{qq} & A_{ij}^{qq} \\ \hline -A_{ij}^{qq} & 0 \end{array}\right] \left\{\begin{array}{c} \dot{q}_j \\ \hline \ddot{q}_j \end{array}\right\} + \left[\begin{array}{c|c} C_{ij}^{qq} & 0 \\ \hline 0 & A_{ij}^{qq} \end{array}\right] \left\{\begin{array}{c} q_j \\ \hline \dot{q}_j \end{array}\right\} = -\left(\left[\begin{array}{c|c} B_i^{qc} & A_i^{qc} \\ \hline 0 & 0 \end{array}\right] \left\{\begin{array}{c} \dot{c} \\ \hline \ddot{c} \end{array}\right\} + \left[\begin{array}{c|c} C_i^{qc} & 0 \\ \hline 0 & 0 \end{array}\right] \left\{\begin{array}{c} c \\ \hline \dot{c} \end{array}\right\} \right) - \left(\left[\begin{array}{c} A^{q\theta}\{\ddot{\theta}\} \\ \hline 0 \end{array}\right] + \left[\begin{array}{c} B^{q\theta}\{\dot{\theta}\} \\ \hline 0 \end{array}\right] + \left[\begin{array}{c} C^{q\theta}\{\theta\} \\ \hline 0 \end{array}\right] \right) \tag{14a}$$

$$[A^{\theta\theta}]\{\ddot{\theta}\} + [B^{\theta\theta}]\{\dot{\theta}\} + [C^{\theta\theta}]\{\theta\} + [B_j^{\theta q} \mid A_j^{\theta q}] \left\{\begin{array}{c} \dot{q}_j \\ \hline \ddot{q}_j \end{array}\right\} + [C^{\theta q} \mid 0] \left\{\begin{array}{c} q_j \\ \hline \dot{q}_j \end{array}\right\} = \{T\} \tag{14b}$$

$$\left\{\begin{array}{c} c \\ \hline \dot{c} \end{array}\right\} = -\left[\begin{array}{c|c} C_j^{cq} & 0 \\ \hline 0 & C_j^{cq} \end{array}\right] \left\{\begin{array}{c} q_j \\ \hline \dot{q}_j \end{array}\right\} \tag{14c}$$

Reordering these equations by $\{(q,\dot{q})_j\} = \{Q_j\}$ and $\{(c,\dot{c})\} = \{\xi\}$, one may write

$$[B_{ij}^{QQ}]\{\dot{Q}_j\} + [C_{ij}^{QQ}]\{Q_j\} = -([B_i^{Q\xi}]\{\dot{\xi}\} + [C_i^{Q\xi}]\{\xi\}) - ([A_i^{Qo}]\{\ddot{\theta}\} + [B_i^{Qo}]\{\dot{\theta}\} + [C_i^{Qo}]\{\theta\}) \tag{14d}$$

$$[A^{oo}]\{\ddot{\theta}\} + [B^{oo}]\{\dot{\theta}\} + [C^{oo}]\{\theta\} + [B_j^{oQ}]\{\dot{Q}_j\} + [C_j^{oQ}]\{Q_j\} = \{T\} \tag{14e}$$

$$\{\xi\} = -[C_j^{\xi Q}]\{Q_j\} \tag{14f}$$

where matrices $[A_{ij}^{QQ}]$ and $[B_{ij}^{QQ}]$ are banded similar to $[A_{ij}^{qq}]$ and $[B_{ij}^{qq}]$ of Eq. (13e) but their order is twice as much. Consider the homogeneous problem

$$[B_{ij}^{QQ}]\{\dot{Q}_j\} + [C_{ij}^{QQ}]\{Q_j\} = \{0\} \tag{14g}$$

and its adjoint

$$[B_{ij}^{QQ}]^T\{\dot{Q}'_j\} + [C_{ij}^{QQ}]^T\{Q'_j\} = \{0\} \tag{14h}$$

These homogeneous problems do have the same eigenvalues. The eigenvectors of Eq. (14g) are orthogonal to those of (14h) with respect to $[B_{ij}^{QQ}]$ and $[C_{ij}^{QQ}]$. Let $[\Phi_{jp}]$ and $[\Phi'_{jq}]$ denote the r eigenvectors of (14g) and (14h), respectively, such that

$$\{Q_j\} = [\Phi_{jp}]\{y_p\} \tag{15a}$$

is an acceptable approximation. Note that usually $r \ll n$. Substituting $\{Q_j\}$ from (15a) into (14d), (14e), and (14f), and premultiplying both sides of (14d) with $[\Phi'_{jq}]^T$, one obtains

$$[B_{qp}^{yy}]\{\dot{y}_p\} + [C_{qp}^{yy}]\{y_p\} = -([B_q^{y\xi}]\{\dot{\xi}\} + [C_q^{y\xi}]\{\xi\}) - ([A_q^{yo}]\{\ddot{\theta}\} + [B_q^{yo}]\{\dot{\theta}\} + [C_q^{yo}]\{\theta\} \tag{15b}$$

$$[A^{oo}]\{\ddot{\theta}\} + [B^{oo}]\{\dot{\theta}\} + [C^{oo}]\{\theta\} + [B_p^{oy}]\{\dot{y}_p\} + [C_p^{oy}]\{y_p\} = \{T\} \tag{15c}$$

$$\{\xi\} = -[C_p^{\xi y}]\{y_p\} \tag{15d}$$

Note that $[B_{qp}^{yy}]$ and $[C_{qp}^{yy}]$ are diagonal matrices of order r (indices p and q run from 1 to r, as differentiated from indices i and j, which run from 1 to n). Now, one may substitute $\{\xi\}$ from (15d) into (15b) to eliminate $\{\xi\}$ from the equations

$$([B_{qp}^{yy}] - [B_q^{y\xi}][C_p^{\xi y}])\{\dot{y}_p\} + ([C_{qp}^{yy}] - [C_q^{y\xi}][C_p^{\xi y}])\{y_p\} = -([A_q^{yo}]\{\ddot{\theta}\} + [B_q^{yo}]\{\dot{\theta}\} + [C_q^{yo}]\{\theta\}) \tag{15e}$$

Equations (15c) and (15e) are equivalent to those of (12e) with the approximation of (15a). In order to eliminate variables $\{y_p\}$ from the set of equations, consider the homogeneous part of (15e)

$$([B_{qp}^{yy}] - [B_q^{y\xi}][C_p^{\xi y}])\{\dot{y}_p\} + ([C_{qp}^{yy}] - [C_q^{y\xi}][C_p^{\xi y}])\{y_p\} = \{0\} \tag{15f}$$

and its adjoint

$$([B^{yy}] - [B^{y\xi}][C^{\xi y}])^T\{\dot{y}'\} + ([C^{yy}] - [C^{y\xi}][C^{\xi y}])^T\{y'\} = \{0\} \tag{15g}$$

These homogeneous problems do have the same eigenvalues, but in general have different eigenvectors. Let $[\psi_{pq}]$ and $[\psi'_{pq}]$ denote the eigenvectors of (15*f*) and (15*g*), respectively. One may use the transformation

$$\{y_p\} = [\psi_{pq}]\{z_q\} \tag{16a}$$

in (15*e*) and (15*c*) and premultiply both sides of (15*e*) by $[\psi'_{pq}]^T$ to obtain

$$[B^{zz}_{pq}]\{\dot{z}_q\} + [C^{zz}_{pq}]\{z_q\} = -([A^{z\theta}_p]\{\ddot{\theta}\} + [B^{z\theta}_p]\{\dot{\theta}\} + [C^{z\theta}_p]\{\theta\}) \tag{16b}$$

and

$$[A^{\theta\theta}]\{\ddot{\theta}\} + [B^{\theta\theta}]\{\dot{\theta}\} + [C^{\theta\theta}]\{\theta\} + [B^{\theta z}_q]\{\dot{z}_q\} + [C^{\theta z}_q]\{z_q\} = \{T\} \tag{16c}$$

where $[B^{zz}_{pq}]$ and $[C^{zz}_{pq}]$ are diagonal matrices of order r.

In order to eliminate variables $\{z_q\}$ from equations (16*a*) and (16*b*), one may use Laplace transforms. This transformation is also convenient for expressing the transfer functions. Let $\{z_q(s)\}$, $\{\theta(s)\}$, and $\{T(s)\}$ denote the Laplace transforms of $\{z_q\}$, $\{\theta\}$, and $\{T\}$, respectively. Let s denote the complex frequency. From the Laplace transform of Eq. (16*b*) one may write

$$\{z_q(s)\} = -\left[s[B^{zz}_{pq}] + [C^{zz}_{pq}]\right]^{-1}\left[s^2[A^{z\theta}_p] + s[B^{z\theta}_p] + [C^{z\theta}_p]\right]\{\theta(s)\} \tag{17a}$$

assuming that the inverse exists. This assumption is possible as long as the eigenvectors used previously are linearly independent. The inverse matrix is a diagonal matrix of order r. Substituting $\{z_q(s)\}$ from (17*a*) into the Laplace transform of (16*c*), and inverting the coefficient of $\{\theta(s)\}$, one obtains

$$\{\theta(s)\} = [G(s)]\{T(s)\} \tag{17b}$$

where

$$[G(s)] = \left[s^2[A^{\theta\theta}] + s[B^{\theta\theta}] + [C^{\theta\theta}] - [s[B^{\theta z}_q] + [C^{\theta z}_q]][s[B^{zz}_{pq}] + [C^{zz}_{pq}]]^{-1}[s^2[A^{z\theta}_p] + s[B^{z\theta}_p] + [C^{z\theta}_p]]\right]^{-1} \tag{17c}$$

which is the required matrix of transfer functions.

The success of the procedure described above depends on the acceptability of the approximation (15*a*), and the computation of r eigenvectors of (14g), (14*h*), and all the eigenvectors of (15*f*) and (15*g*), and the linear independence of these eigenvectors. If the matrices in equations (14g) and (15*f*) are symmetric, the eigenvectors of the adjoint problems are the same as those of (14g) and (15*f*). When $[C^{qq}_{ij}]$ is symmetric and positive definite and $[B^{qq}_{ij}]$ is skew-symmetric, the eigenvectors of (14g) may be obtained by an extended Sturm sequence method for Hermitian matrices (see Ref. 8). However, in general, matrices $[B^{qq}_{ij}]$ and $[C^{qq}_{ij}]$ may not have these favorable properties, in which event, for the economical solution of the eigenvectors of (14g) and (14*h*), an efficient algorithm which takes into account the bandedness of the coefficient matrices is needed. The eigenvalue problems associated with (15*f*) and (15*g*) are relatively easy, since these systems are only of order r whereas the ones associated with (14g) and (14*h*) are of order $12n$ ($r \ll n$).

VI. Summary

In this work, the governing equations of small vibrations and perturbations of structures undergoing arbitrary motion are obtained for the following cases:

(1) Independent of a coordinate system, in vector notation, in Eqs. (3*d*), (4*d*), (5*d*), and (6*d*).

(2) In cartesian coordinates in Table 1.

(3) In cartesian coordinates for steady nominal motion in Table 2.

(4) In cartesian coordinates for nonrotating system with no rotating parts in Table 3.

(5) In cartesian coordinates for nominally rotating system with no rotating parts in Table 4.

(6) In cartesian coordinates for nominally rotating system with nominally rotating parts in Table 5.

A method is described for the elimination of undesired state variables with the objective of obtaining the matrix of transfer functions relating control torques to attitude angles. This method takes into account the bandedness of the coefficient matrices.

The formulations given in this work may be used in analyzing the vibrations of rotating or nonrotating flexible structures with or without rotating parts, as well as in obtaining the open loop transfer functions related with the "dynamic blocks" of control systems of spin-stabilized, dual-spin-stabilized satellites, and those spacecraft which use reaction wheels or gas jets for attitude control.

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