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MODIFICATIONS TO FOURIER'S LAW OF HEAT CONDUCTION THROUGH THE USE OF A SIMPLE COLLISION MODEL

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SUMMARY

The exact first-order solution of an approximate kinetic equation is obtained for the problem of viscous gases flowing through circular pipes with constant temperature gradients. When expressed in terms of transport coefficients, the resulting flux-force relations are assumed to be correct since they agree in the proper limit with the Grad 13-moment approximation. However, the complete flux-force relations have important contributions from at least the first 22 moments; in particular, terms of convection magnitude are added to both the axial and radial components of the 13-moment equations for thermal conduction. These terms, which are linearly proportional to the magnitude of the flow velocity (instead of the square), significantly alter the conventional Navier-Stokes description of heat transfer in the problem considered. They also affect the evaluation of the pressure tensor.

INTRODUCTION

Calculations of heat transfer in flowing gases are normally based upon the conventional theory of Navier and Stokes (ref. 1), in which Newton's law of viscosity and Fourier's law of thermal conduction are employed in the macroscopic equations of change. Despite the fact that the validity of such laws has been demonstrated in many practical applications and also on the basis of the second approximation to the kinetic theory of Chapman and Enskog (ref. 2), their general applicability to all near-equilibrium systems composed of ordinary monatomic gases is not guaranteed. The purpose of the present research to to investigate the accuracy with which the conventional theory describes a particular example of viscous pipe flow.

Two generalizations of the simple Newton and Fourier laws are well known – namely, Grad's 13-moment addition of coupling terms (ref. 1, p. 495) to the heat flux and traceless pressure tensor. However, since the coupling terms can be obtained from the Chapman-Enskog third approximation, the assertion is frequently made that they are of little importance unless the gas properties vary appreciably in the distance of a mean free path. This argument implicitly assumes the mean free path to be a valid perturbation parameter and the divergence of the traceless pressure tensor to be negligible compared with the pressure gradient, even though simple examples can be cited to the contrary.

One such example is the steady-state, pressure-driven, viscous flow of a gas through a pipe. The pressure gradient is balanced in this problem by the divergence of the traceless pressure tensor, which therefore cannot be regarded as a third-approximation force and should not be neglected in the velocity distribution function. In addition, the 13-moment coupling contribution to the conductive heat flux is directly proportional to the divergence of the traceless pressure tensor and is just as important as Fourier's term. Although neither contribution is significant when compared with heat transfer by convection, the question naturally arises as to whether higher Chapman-Enskog or higher Grad moment approximations yield conductive terms of much larger magnitude. The answer to this question is the primary objective of the present research.

Since higher approximations to either the Chapman-Enskog or the Grad solutions of the Boltzmann integro-differential equation are very difficult to obtain, a common practice (see ref. 3, for example) is the substitution of simple collision models for the complete Boltzmann collision integrals in order to reduce the kinetic equations to differential form. Perhaps the most frequently used model is that of Krook (ref. 3), which assumes the collision integrals to be directly proportional to the difference between the equilibrium and nonequilibrium velocity distribution functions. This model has several attractive features, but it does not distinguish between the effective collision frequencies for different transport phenomena; hence, the Krook predictions of quantities like the Prandtl number contain substantial errors. The model developed in the present research for obtaining a differential form of the kinetic equation avoids such difficulties.

The first-order (that is, second-approximation) form of the resulting kinetic equation is the fundamental relation used in the present study. It is obtained from the approximate model equation for viscous pipe flow by deleting all terms corresponding to squares and higher powers of the mean gas velocity and its driving force. This procedure is less restrictive than Chapman and Enskog's second approximation because spatial derivatives of the first-order perturbation function are not arbitrarily neglected in the first-order kinetic equation. Accordingly, the aforementioned 13-moment additions to the traceless pressure tensor and the heat flux, as well as other phenomena of equal or greater importance, are included automatically without reference to higher approximations or orders.

The utility of the approximate first-order kinetic equation rests in the fact that exact solutions can be found for many problems – including the one of viscous gases flowing through circular pipes with constant temperature gradients. With the reasonable assumption that errors inherent in the collision model affect the values of effective collision frequencies but not their ratios, the velocity moments of the exact solution are expected to give accurate flux-force relations if precise transport coefficients are used in the final expressions. Some support for this speculation is provided by the precise reduction to Grad's formulas in the 13-moment limit and by the correct prediction of the Prandtl number. However, the complete flux-force relations have important contributions from at least the first 22 moments; in particular, contributions of the same magnitude as heat convection are added to both the axial and radial components of the 13-moment equations for thermal conduction. These terms, which are proportional to the magnitude of the flow velocity (instead of the magnitude squared), significantly alter the conventional Navier-Stokes description of heat transfer in the problem considered. They also affect the evaluation of the pressure tensor.

Previous efforts to obtain exact solutions of linearized Boltzmann equations have been made by Wang Chang and Uhlenbeck for the problems of heat transport between parallel plates (ref. 4), Couette flow (ref. 5), and flow near a surface (ref. 6). However, these solutions are not given in the closed form of the present research but appear instead as formal expansions in the complete set of spherical harmonics. Only the first few coefficients in these expansions are derived. Truesdell (ref. 7, see pp. 74-83) exactly solved a problem of simple shear in an infinite medium by considering the complete set of macroscopic equations of change and making assumptions about the behavior of the higher moments. These assumptions do not apply to the problem treated herein.

SYMBOLS

a	radius of circular pipe
A_i	ratio of collision integrals defined by equation (21)
b	impact parameter
$\vec{c}$	particle velocity relative to pipe
c_p	specific heat per particle at constant pressure
e_{ij},e_{ij}^*	combinations of velocity or heat-flux derivatives (see eqs. (87) and (88))
f	velocity distribution function
$f^{(o)}$	Maxwellian distribution function relative to mean gas flow

$g; g_1, \ldots, g_9$	trial velocity functions
i,j,k	indices; also used to denote vector and tensor components
$\hat{i},\hat{j},\hat{k}$	unit vectors along x-, y-, and z-axes
k	Boltzmann's constant
l	mean free path
m	particle mass
n	particle number density
p, p^o	local and equilibrium pressures, respectively
$\overset{o}{\overleftrightarrow{P}}$	traceless pressure tensor
N_{Pr}	Prandtl number
$\vec{q}, \vec{q}_v$	conductive and convective heat flux, respectively
q_r	radial component of $\vec{q}$
$\vec{Q}$	total heat flux
Q_r	radial component of $\vec{Q}$
$\vec{r}$	radial vector $\hat{i}x + \hat{j}y$; magnitude r is radial distance from pipe axis
R	rate of heat transfer per unit area to pipe
s	entropy density
$s^{(o)}$	equilibrium entropy density
$\dot{s}_c$	entropy source strength (collisional production rate of entropy density)

t	time
T, T^o	local and equilibrium temperatures, respectively
$\vec{u}$	reduced particle velocity relative to mean gas flow (see eq. (17))
$\overleftrightarrow{U}$	unit tensor
$\vec{v}$	local flow velocity
$\bar{v}$	mean particle speed at equilibrium
v_o	magnitude of flow velocity at axis of pipe (r = 0)
W	scalar defined by equation (50)
x,y,z	Cartesian coordinates; also used to denote vector and tensor components
x_i, x_j, x_k	same as x, y, and z, respectively
$\vec{X}$	body force per unit mass
α	flow parameter defined by equation (35)
$\vec{\beta}$	reduced flow velocity defined by equation (33)
$\vec{\gamma}$	reduced particle velocity relative to mean gas flow; see equation (27)
δ_{ij}	Kronecker delta function
ϵ	azimuthal angle in particle collisions
η	viscosity
θ	angle between z-axis and particle velocity before collision
λ	thermal conductivity
ρ	mass density

τ collision time

ϕ first-order (or second-approximation) perturbation function

Φ, Φ^* dissipation functions

χ scattering or deflection angle

Special notations:

$\left(\frac{\partial f}{\partial t}\right)_c$ collisional time rate of change of f

$\frac{D}{Dt}$ substantial derivative – that is, $\frac{\partial}{\partial t} + \vec{v} \cdot \nabla$

$\langle \ \rangle$ velocity average using f

$\langle \ \rangle_o$ velocity average using $f^{(o)}$

Primed symbols in collision integrals refer to quantities after a collision; unprimed symbols to quantities before a collision. Magnitudes of vector quantities are denoted by the same symbol without the arrow.

THE COLLISION MODEL

The purpose of this section is to develop a modification of the Krook collision model which can serve as the basis of the approximate kinetic equation employed in the present research. In particular, the real gas is replaced with a hypothetical system having the same gross collision properties but which simplifies the kinetic theory to permit exact solvability for the velocity distribution function. The reliability of this procedure is tested in subsequent sections by calculations of the Prandtl number and comparisons of flux-force relations with Grad's 13-moment results.

A convenient hypothetical gas system for reducing the kinetic equation to a solvable differential form is constructed as follows: Introduce a fictitious second gas of infinitely massive particles at rest relative to the mean velocity of the real gas at every point; assume collisions between real-gas particles to have negligible macroscopic effects compared with real-gas – fictitious-gas interactions. The collision integrals in the corresponding Boltzmann kinetic equation are especially simple and resemble in many respects the Lorentz collision integrals employed for certain types of plasma. (See refs. 8 to 11.) They comprise the complete collision model used in the present research.

The utility of the hypothetical gas depends upon its macroscopic correspondence to the real gas alone. Although the detailed collision processes and the values of individual transport coefficients are different in the two systems, the same general transport phenomena must occur in both. Undesirable phenomena such as binary and thermal diffusion of the real gas relative to the fictitious gas are prohibited by the restriction to the same local flow velocity. Hence, the substitution of the values of real-gas transport coefficients into the final exact macroscopic equations derived for the hypothetical system should yield reliable expressions for the real gas alone; in particular, valuable information should be obtained on the nature and importance of previously neglected contributions to flux-force relations from higher Grad moment and Chapman-Enskog approximations. The one basic assumption in this statement is that the collision model affects only the values of the effective collision frequencies and not the relative importance of transport terms, a critical test being the accurate prediction of the Prandtl number. Such numbers (that is, ratios of collision frequencies) are generally constant over a large range of mass differences between constituent species.

A simple but informative preliminary illustration of the mathematical character of the collision model follows from the application of temperature and pressure gradients to the hypothetical gas in such a way as to cause no average flow with respect to the laboratory frame of reference. With the concept of order as defined in the "Introduction" (that is, with the first-order perturbation function ϕ being proportional in this case to the applied pressure gradient), the corresponding first-order kinetic equation for real-gas particles can be written in the following manner for steady-state conditions and no body forces:

$$\vec{c} \cdot \nabla f = \vec{c} \cdot \nabla f^{(o)} + f^{(o)} \vec{c} \cdot \nabla \phi = \left(\frac{\partial f}{\partial t}\right)_c \tag{1}$$

where $\vec{c}$ is the particle velocity, $\left(\frac{\partial f}{\partial t}\right)_c$ is the collisional time rate of change of the velocity distribution function f, and

$$f = f^{(o)}(1 + \phi) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mc^2}{2kT}\right)(1 + \phi) \tag{2}$$

If the temperature and pressure are assumed to vary only in the z-direction and to have no second or higher spatial derivatives, the gradient of ϕ can be deleted from equation (1) to yield

$$c_z \frac{\partial f^{(o)}}{\partial z} = \frac{c_z f^{(o)}}{T}\left[\left(\frac{mc^2}{2kT} - \frac{5}{2}\right)\frac{\partial T}{\partial z} + \frac{1}{nk}\frac{\partial p}{\partial z}\right] = \left(\frac{\partial f}{\partial t}\right)_c \tag{3}$$

with the aid of the ideal gas law $p = nkT$. Finally, the collision term becomes

$$\left(\frac{\partial f}{\partial t}\right)_c = -\int \left[ff(\text{fictitious}) - f'f'(\text{fictitious})\right]\left|\vec{c} - \vec{c}(\text{fictitious})\right| b\ db\ d\epsilon\ d\vec{c}(\text{fictitious})$$

$$= -\int (f - f')f(\text{fictitious})cb\ db\ d\epsilon\ d\vec{c}(\text{fictitious})$$

$$= -n\int (f - f')cb\ db\ d\epsilon = -ncf^{(o)}\int (\phi - \phi')b\ db\ d\epsilon \tag{4}$$

according to the properties previously outlined for the fictitious gas and its interactions with particles of the real species. Both the real and the fictitious species are assumed to have the same number density n in order to facilitate the ultimate substitution of the real system for the hypothetical one.

A convenient trial solution of the resulting kinetic equation

$$\frac{c_z}{T}\left[\left(\frac{mc^2}{2kT} - \frac{5}{2}\right)\frac{\partial T}{\partial z} + \frac{1}{nk}\frac{\partial p}{\partial z}\right] = -nc\int (\phi - \phi')b\ db\ d\epsilon \tag{5}$$

is

$$\phi = g(c)\ c_z \tag{6}$$

where g is an unknown function of the magnitude of $\vec{c}$ and thus is not changed by collisions with the fixed scattering centers of the infinitely massive fictitious gas. Accordingly, equation (5) becomes

$$\frac{c_z}{T}\left[\left(\frac{mc^2}{2kT} - \frac{5}{2}\right)\frac{\partial T}{\partial z} + \frac{1}{nk}\frac{\partial p}{\partial z}\right] = -ncg\int \left(c_z - c_z'\right)b\ db\ d\epsilon \tag{7}$$

As shown in references 8 to 11, the Lorentz-like collision integral in equation (7) satisfies

$$\int \left(c_z - c_z'\right)b\ db\ d\epsilon = \frac{c_z}{n\tau c} \tag{8}$$

if the interaction potential corresponds to Maxwellian particles – that is, if the force between particles varies as the inverse fifth power of the separation. This requirement is a necessary condition (ref. 8) for the Krook model to describe precisely the collision dynamics of the hypothetical system, but it does not diminish the scope of the present research because again only the values of transport coefficients are ultimately affected. The collision time τ in equation (8) is appropriate for the relative diffusion of the two species if such diffusion were permitted to occur.

The substitution of equation (8) into equation (7) yields

$$\frac{c_z}{T}\left[\left(\frac{mc^2}{2kT} - \frac{5}{2}\right)\frac{\partial T}{\partial z} + \frac{1}{nk}\frac{\partial p}{\partial z}\right] = -\frac{gc_z}{\tau} \tag{9}$$

Thus,

$$\phi = -\frac{\tau}{T}\left[\left(\frac{mc^2}{2kT} - \frac{5}{2}\right)\frac{\partial T}{\partial z} + \frac{1}{nk}\frac{\partial p}{\partial z}\right]c_z \tag{10}$$

from equation (6). In addition,

$$\left(\frac{\partial f}{\partial t}\right)_c = -\frac{1}{\tau}f^{(o)}\phi = -\frac{1}{\tau}\left(f - f^{(o)}\right) \tag{11}$$

is obtained from equations (4), (8), and (10) and identifies the present method with the Krook collision model in this problem.

Krook's value of the thermal conductivity is computed from equations (2) and (10) and the use of Fourier's law of thermal conduction in the following manner:

$$\begin{aligned} q_z &= \frac{nm}{2}\langle c^2 c_z\rangle = \frac{m}{2}\int c^2 c_z f \, d\vec{c} \\ &= -\frac{5kp\tau}{2m}\left(\frac{\partial T}{\partial z} + \frac{1}{nk}\frac{\partial p}{\partial z}\right) = -\lambda\frac{\partial T}{\partial z} - \frac{5p\tau}{2nm}\frac{\partial p}{\partial z} \end{aligned} \tag{12}$$

where

$$\lambda = \frac{5kp\tau}{2m} \tag{13}$$

Equation (13) differs from the standard expression (ref. 1) by a factor of $2^{-1/2}$. Although an exact conformity can be obtained with a redefinition of τ, this redefinition is not essential for present purposes.

The complete set of collision integrals comprising the definition of the collision model in problems of steady-state viscous flow driven by an applied pressure gradient requires a consideration of the following first-order kinetic equation:

$$\vec{c} \cdot \nabla f^{(o)} + f^{(o)} \vec{c} \cdot \nabla\phi = -n|\vec{c}-\vec{v}| f^{(o)} \int (\phi - \phi')b \, db \, d\epsilon \tag{14}$$

Precise details clearly depend on the exact nature of ϕ and are deferred in part to the next section; however, a significant amount of information can be obtained from the Grad 13-moment velocity distribution function (ref. 9)

$$f = f^{(o)}(1 + \phi) = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-u^2}\left[1 + \frac{4}{5}\left(u^2 - \frac{5}{2}\right)\langle u^2\vec{u}\rangle \cdot \vec{u} + \frac{1}{p}\overset{o}{\overleftrightarrow{P}}:\vec{u}\,\vec{u}\right] \tag{15}$$

where $\overset{o}{\overleftrightarrow{P}}$ is the traceless pressure tensor defined by

$$\overset{o}{\overleftrightarrow{P}} = 2p\left(\langle\vec{u}\,\vec{u}\rangle - \frac{1}{3}\langle u^2\rangle\overleftrightarrow{U}\right) \tag{16}$$

and

$$\vec{u} = \left(\frac{m}{2kT}\right)^{1/2}(\vec{c} - \vec{v}) \tag{17}$$

is a dimensionless particle velocity relative to the mean gas motion.

In particular, the integral

$$\int \left(\vec{u}\,\vec{u} - \vec{u}'\,\vec{u}'\right) b \, db \, d\epsilon$$

along with equation (8) written in the form

$$\int\left(\vec{u} - \vec{u}'\right) b \, db \, d\epsilon = \frac{1}{n\tau u}\left(\frac{m}{2kT}\right)^{1/2} \vec{u} \tag{18}$$

are important in viscous flow. This set is the complete model only if the 13-moment approximation is used.

The evaluation of the new collision integral is very simple in the case of the hypothetical real-gas—fictitious-gas system. Since u_z' is related to u_z by the expression (ref. 8)

$$u_z' = u_z \cos\chi - u \sin\chi \sin\theta \sin\epsilon \tag{19}$$

there results

$$\int\left(u_z^2 - u_z'^2\right)b\ db\ d\epsilon = \int\left\{\left[u_z^2 + \left(u_z^2 - u^2\right)\sin^2\epsilon\right]\left(1 - \cos^2\chi\right) + 2uu_z \sin\chi\cos\chi\sin\theta\sin\epsilon\right\}b\ db\ d\epsilon$$

$$= \pi\left(3u_z^2 - u^2\right)\int_0^\infty\left(1 - \cos^2\chi\right)b\ db$$

$$= \pi A_2\left(3u_z^2 - u^2\right)\int_0^\infty (1 - \cos\chi)b\ db$$

$$= \frac{A_2}{2u_z}\left(3u_z^2 - u^2\right)\int\left(u_z - u_z'\right)b\ db\ d\epsilon$$

$$= \frac{A_2}{2n\tau u}\left(\frac{m}{2kT}\right)^{1/2}\left(3u_z^2 - u^2\right) \tag{20}$$

with the aid of equation (18) and the definition

$$A_i = \frac{\int_0^\infty\left(1 - \cos^i\chi\right)b\ db}{\int_0^\infty (1 - \cos\chi)b\ db} \tag{21}$$

The remaining tensor elements are derived in similar fashion to yield

$$\int\left(\vec{u}\,\vec{u} - \vec{u}'\vec{u}'\right)b\ db\ d\epsilon = \frac{A_2}{2n\tau u}\left(\frac{m}{2kT}\right)^{1/2}\left(3\vec{u}\,\vec{u} - u^2\,\overleftrightarrow{U}\right) \tag{22}$$

A preliminary evaluation of the Prandtl number is made by comparing equation (18) with the xz-element of equation (22) for a viscous flow in the z-direction. The principal difference between the two collision integrals lies in the values of the coefficients; hence, the effective collision time for the rate of change of $u_x u_z$ appears to be $2/\left(3A_2\right)$ times that of $\vec{u}$. Also, since these integrals are the collision integrals associated with the coefficients of thermal conductivity and viscosity, the same ratio $2/\left(3A_2\right)$ is expected to be the value of the Prandtl number for the hypothetical real-gas—fictitious-gas combination. The exact solution of equation (14) merely confirms this prediction and, together with the computation of $\overset{o}{P}_{xz}$, is deferred to a subsequent section.

Meanwhile, a predicted Prandtl number of $2/(3A_2)$ is observed to be very close to the Chapman-Enskog value of 2/3 because the calculated value of A_2 is 1.034 for Maxwellian particles (ref. 9). Since the Chapman-Enskog result refers to the real-gas system and thus involves the rigorous Boltzmann integrals for collisions between real-gas particles, an important justification is obtained for the basic assumption of the present research – namely, that the collision model does not affect the relative importance of transport terms. The Krook model, on the other hand, yields

$$\int \left(u_x u_z - u_x' u_z'\right) b \, db \, d\epsilon = \frac{1}{n\tau u}\left(\frac{m}{2kT}\right)^{1/2} u_x u_z \tag{23}$$

from equations (11) and (15) and therefore corresponds to a Prandtl number of unity.

The only remaining collision integral associated with the exact solution of equation (14) is shown in the next section to be

$$\int \left(u_z^3 - u_z'^3\right) b \, db \, d\epsilon = \frac{1}{2n\tau u}\left(\frac{m}{2kT}\right)^{1/2}\left[\left(5A_3 - 3\right)u_z^3 - 3\left(A_3 - 1\right)u^2 u_z\right] \tag{24}$$

the evalutation of which is analogous to the treatment used in equations (19) and (20).

Equations (18), (22), and (24), together with the statement that u equals u', thus contain all the information necessary for a complete description of the collision model employed in the present research. The modifications to the Krook model are systematically found from exact evaluations of Lorentz-like collision integrals and correspond to the introduction of different effective collision times for the transport of different macroscopic gas properties. A convenient summary of these modifications is provided in the 13-moment approximation by the following expressions for the Krook collision model and the present collision model, respectively:

$$\left(\frac{\partial f}{\partial t}\right)_c = -\frac{1}{\tau}\left(f - f^{(o)}\right) \tag{25}$$

and

$$\left(\frac{\partial f}{\partial t}\right)_c = -\frac{1}{\tau}\left(f - f^{(o)}\right) - \frac{1}{2p\tau} f^{(o)} \overset{o}{\overleftrightarrow{P}} : \left[A_2\left(3\vec{u}\,\vec{u} - u^2 \overleftrightarrow{U}\right) - 2\vec{u}\,\vec{u}\right] \tag{26}$$

Equation (26) is derived from equations (15), (18), and (22).

THE EXACT FIRST-ORDER SOLUTION

A convenient form of the first-order kinetic equation for viscous pipe flow is obtained from equation (14) by introducing the equilibrium temperature T^o into the dimensionless particle velocity

$$\vec{\gamma} = \left(\frac{m}{2kT^o}\right)^{1/2}\left(\vec{c} - \vec{v}\right) \tag{27}$$

and requiring the perturbation function ϕ to satisfy

$$f = n\left(\frac{m}{2\pi kT^o}\right)^{3/2} e^{-\gamma^2} (1 + \phi) \tag{28}$$

This procedure permits the spatial dependence of T for a given velocity profile to be determined from the solution of the kinetic equation instead of being regarded as input data.

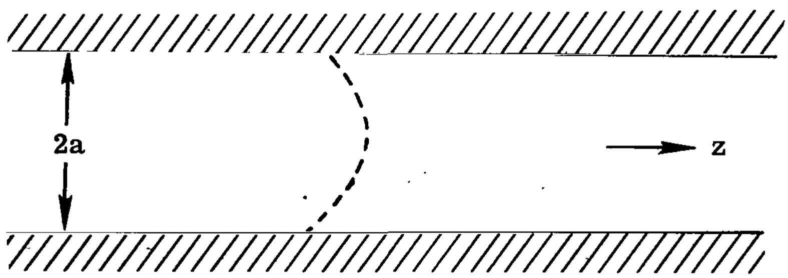
The number density n can be treated in similar fashion, but such treatment is of no advantage in the present problem because n is independent of the axial coordinate z. (See sketch (a).) This result is required by the equation of continuity

$$\frac{\partial n}{\partial t} + \nabla \cdot \left(n\vec{v}\right) = 0 \tag{29}$$

if the system is in a steady state and if

$$\vec{v} = \hat{k}v(r) \tag{30}$$

where r is the radial distance from the pipe axis. Moreover, if the induced radial variations of n and T are relegated to second order (that is, proportional to the square of the flow velocity as partly implied by the energy equation and Fourier's law), the number density is constant in equation (28).



Sketch (a).- Diagram of viscous pipe flow. The dashed curve represents the velocity profile.

These restrictions to a steady-state condition, equation (30), and constant n through first order, together with the assumptions of no body forces (for example, gravitation) and a parabolic velocity profile

$$v = v_o\left(1 - \frac{r^2}{a^2}\right) \tag{31}$$

where v_o is the value of v on the axis and a is the pipe radius, constitute a complete set of input data for the solution of equation (14) by equation (28). Whether such conditions are possible in principle is left for the kinetic theory to answer (by permitting or rejecting a solution), as are the predictions of $T - T^o$, transport coefficients, and flux-force relations. Velocity and temperature jumps at the walls are ignored in this treatment, but can be inserted directly into the final macroscopic expressions.

The substitution of equations (27) and (28) into equation (14) yields

$$\frac{1}{f^{(o)}}\left(\gamma_x \frac{\partial f^{(o)}}{\partial x} + \gamma_y \frac{\partial f^{(o)}}{\partial y}\right) + \vec{\gamma} \cdot \nabla\phi = 2\left(\gamma_x \frac{\partial \beta}{\partial x} + \gamma_y \frac{\partial \beta}{\partial y}\right)\gamma_z + \vec{\gamma} \cdot \nabla\phi$$

$$= -n\gamma \int (\phi - \phi')b \, db \, d\epsilon \tag{32}$$

if terms containing v^2 are neglected and if β is the magnitude of the dimensionless flow velocity defined by

$$\vec{\beta} = \left(\frac{m}{2kT^o}\right)^{1/2} \vec{v} \tag{33}$$

Equation (32) can be written in the more convenient form

$$2\alpha\left(x\gamma_x + y\gamma_y\right)\gamma_z + \vec{\gamma} \cdot \nabla\phi = -n\gamma \int (\phi - \phi')b \, db \, d\epsilon \tag{34}$$

by using the following combination of equations (31) and (33):

$$\frac{\partial \beta}{\partial x} = -\frac{2v_o}{a^2}\left(\frac{m}{2kT^o}\right)^{1/2} x = \alpha x \tag{35}$$

Since the collision integrals of equations (18), (22), and (24) can be adapted to the present notation merely by substituting T^o for T and $\vec{\gamma}$ for $\vec{u}$, a simple inspection of these expressions and their role in balancing the left-hand side of equation (34) yields

$$\phi = \left(g_1 + r^2 g_2\right)\gamma_z + \left(x\gamma_x + y\gamma_y\right) g_3\gamma_z + g_4\gamma_z^3 + z\left[r^2 g_5 + \left(x\gamma_x + y\gamma_y\right) g_6 + g_7\gamma_z^2 + zg_8\gamma_z + z^2 g_9\right] \tag{36}$$

as a trial solution. The unknown spatially independent factors g_i are functions only of the magnitude of $\vec{\gamma}$ and thus are not affected by collisions according to the principles of energy and momentum conservation in the present model.

The substitution of equation (36) into equation (34) then gives

$$\tau\left(\frac{2kT^o}{m}\right)^{1/2}\left[2\left(\alpha + g_2\right)\left(x\gamma_x + y\gamma_y\right)\gamma_z + \left(\gamma^2 - \gamma_z^2\right) g_3\gamma_z + 2z\left(x\gamma_x + y\gamma_y\right) g_5 + r^2 g_5\gamma_z + z\left(\gamma^2 - \gamma_z^2\right) g_6 + \left(x\gamma_x + y\gamma_y\right) g_6\gamma_z + g_7\gamma_z^3 + 2zg_8\gamma_z^2 + 3z^2 g_9\gamma_z\right]$$

$$= -\left(g_1 + r^2 g_2\right)\gamma_z - \frac{3A_2 g_3}{2}\left(x\gamma_x + y\gamma_y\right)\gamma_z - \frac{g_4}{2}\left[\left(5A_3 - 3\right)\gamma_z^3 - 3\left(A_3 - 1\right)\gamma^2\gamma_z\right] - z\left(x\gamma_x + y\gamma_y\right) g_6 - \frac{zA_2 g_7}{2}\left(3\gamma_z^2 - \gamma^2\right) - z^2 g_8\gamma_z \tag{37}$$

so that

$$\phi = \frac{8p^o\tau^2\gamma_z}{3A_2\left(5A_3 - 3\right)nm}\left\{2\alpha\left(A_3\gamma^2 - \gamma_z^2\right) - \left[\left(5A_3 - 9\right)\gamma^2 + 10\gamma_z^2\right] g_2\right\} + \frac{4\tau z\gamma_z^2}{A_2}\left(\frac{2kT^o}{m}\right)^{1/2} g_2 - \frac{4\tau}{3A_2}\left(\frac{2kT^o}{m}\right)^{1/2}\left(\alpha + 2g_2\right)\left(x\gamma_x + y\gamma_y\right)\gamma_z + \left(r^2 - 2z^2\right) g_2\gamma_z + 2zg_2\left(x\gamma_x + y\gamma_y\right) - \frac{z}{3\tau}\left(\frac{m}{2kT^o}\right)^{1/2}\left(3r^2 - 2z^2\right) g_2 \tag{38}$$

from the equating of coefficients of like powers of the independent phase-space variables in equation (37) and the use again of equation (36).

Equations (28) and (38) represent the exact first-order (that is, linear v) solution of the viscous-flow problem in question – subject, of course, to the limitations of the approximate collision model. This solution is more complete than that of Chapman and

Enskog at the same level and for the same model because spatial derivatives of ϕ are retained on the left-hand side of the first-order kinetic equation.

Conditions on the unknown function g_2 are obtained by taking various velocity moments of the distribution function. For example, the relations

$$\int_0^\infty e^{-\gamma^2}\gamma^2 g_2 \, d\gamma = 0 \tag{39}$$

and

$$\int_0^\infty e^{-\gamma^2}\gamma^4 g_2 \, d\gamma = 0 \tag{40}$$

follow from the requirement that the entire contribution to n should come from the Maxwellian contribution to equation (28). These integrals further demand zero values for $\langle\gamma_x\rangle$, $\langle\gamma_y\rangle$, and $\partial T/\partial r$ in this problem.

The consistency of the problem constraints is thus assured if g_2 also satisfies the requirement

$$\begin{aligned}\langle\gamma_z\rangle &= \frac{1}{n}\int \gamma_z f \, d\vec{c} = \frac{32p^o\tau^2}{45\pi^{1/2} A_2 nm}\int_0^\infty e^{-\gamma^2}\gamma^6\left(2\alpha - 5g_2\right)d\gamma \\ &= \frac{32p^o\tau^2}{9\pi^{1/2} A_2 nm}\left(\frac{3\pi^{1/2}\alpha}{8} - \int_0^\infty e^{-\gamma^2}\gamma^6 g_2 \, d\gamma\right) = 0\end{aligned} \tag{41}$$

that is, if

$$\int_0^\infty e^{-\gamma^2}\gamma^6 g_2 \, d\gamma = \frac{3\pi^{1/2}\alpha}{8} \tag{42}$$

Even though equations (39), (40), and (42) do not determine g_2 uniquely, they are sufficient to describe completely the usual macroscopic properties of the gas. One illustration is the local temperature defined by

$$\begin{aligned}T = \frac{2T^o}{3}\langle\gamma^2\rangle &= T^o\left[1 + \frac{32\tau z}{9\pi^{1/2} A_2}\left(\frac{2kT^o}{m}\right)^{1/2}\int_0^\infty e^{-\gamma^2}\gamma^6 g_2 \, d\gamma\right] \\ &= T^o\left(1 + \frac{z}{T^o}\frac{\partial T}{\partial z}\right)\end{aligned} \tag{43}$$

so that

$$\frac{\partial T}{\partial z} = \frac{32T^{o}\tau}{9\pi^{1/2}A_2}\left(\frac{2kT^{o}}{m}\right)^{1/2}\int_0^\infty e^{-\gamma^2}\gamma^6 g_2\, d\gamma$$

$$= \frac{4T^{o}\tau\alpha}{3A_2}\left(\frac{2kT^{o}}{m}\right)^{1/2} = -\frac{8T^{o}\tau v_o}{3A_2a^2} \tag{44}$$

from equations (35), (42), and (43).

Equations (43) and (44) can be employed in the following manner to express equations (28) and (38) in a more convenient form:

$$f^{(o)} = n\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-u^2} = n\left(\frac{m}{2\pi kT^{o}}\right)^{3/2}\left(\frac{T^{o}}{T}\right)^{3/2}\exp\left[-\frac{m}{2kT^{o}}\left(\frac{T^{o}}{T}\right)\left(c^2 - 2vc_z + v^2\right)\right]$$

$$\approx n\left(\frac{m}{2\pi kT^{o}}\right)^{3/2}\left(\frac{T^{o}}{T}\right)^{3/2}\exp\left[-\frac{m}{2kT^{o}}\left(\vec{c} - \vec{v}\right)^2 - \frac{8\tau v_o z}{3A_2a^2}u^2\right]$$

$$\approx n\left(\frac{m}{2\pi kT^{o}}\right)^{3/2}e^{-\gamma^2}\left(1 + \frac{4\tau v_o z}{A_2a^2}\right)\left(1 - \frac{8\tau v_o z}{3A_2a^2}u^2\right)$$

$$\approx n\left(\frac{m}{2\pi kT^{o}}\right)^{3/2}e^{-\gamma^2}\left[1 - \frac{4\tau v_o z}{3A_2a^2}\left(2u^2 - 3\right)\right] \tag{45}$$

Accordingly, the complete first-order solution is written as

$$f = f^{(o)}(1 + \phi) \tag{46}$$

where $f^{(o)}$ is defined now by the first equality in equation (45) and the new ϕ satisfies the expression

$$\phi = \frac{8\tau v_o}{3A_2a^2}\left[z\left(u^2 - \frac{3}{2}\right) - \frac{2\tau}{5A_3 - 3}\left(\frac{2kT}{m}\right)^{1/2}\left(A_3u^2 - u_z^2\right)u_z + \left(xu_x + yu_y\right)u_z\right] - \frac{8p\tau^2u_z}{3A_2\left(5A_3 - 3\right)nm}\left[\left(5A_3 - 9\right)u^2\right.$$
$$\left. + 10u_z^2\right]g_2 + \frac{4\tau zu_z^2}{A_2}\left(\frac{2kT}{m}\right)^{1/2}g_2 - \frac{8\tau}{3A_2}\left(\frac{2kT}{m}\right)^{1/2}\left(xu_x + yu_y\right)g_2u_z + \left(r^2 - 2z^2\right)g_2u_z + 2zg_2\left(xu_x + yu_y\right)$$
$$- \frac{z}{3\tau}\left(\frac{m}{2kT}\right)^{1/2}\left(3r^2 - 2z^2\right)g_2 \tag{47}$$

This formulation of the velocity distribution function clearly shows an important distinction between the Chapman-Enskog first-order perturbation function and that of equation (47). Whereas the Chapman-Enskog method has the primitive property of relegating the entire spatial dependence to the first five moments n, T, and $\vec{v}$, numerous terms in equation (47) involve x, y, and z explicitly. Hence, many higher moments must be introduced and permitted their own spatial variations before the first-order distribution function is completely described by local values of velocity moments. The remainder of the present section is concerned with several steps toward such a representation.

The next higher vector moment is the conductive heat flux given by

$$\vec{q} \equiv \frac{nm}{2}\left\langle (\vec{c} - \vec{v})^2 (\vec{c} - \vec{v}) \right\rangle = p\left(\frac{2kT}{m}\right)^{1/2} \left\langle u^2 \vec{u} \right\rangle = \frac{p}{n}\left(\frac{2kT}{m}\right)^{1/2} \int u^2 \vec{u} f \, d\vec{c}$$

$$= \frac{4p}{3}\left(\frac{2kT}{\pi m}\right)^{1/2}\left[2z\vec{r} + \hat{k}\left(r^2 - 2z^2\right)\right]\int_0^\infty e^{-u^2} u^6 g_2 \, du$$

$$- \hat{k}\,\frac{28p^2\tau^2 v_o}{3A_2 nma^2}\left[1 + \frac{8a^2}{21v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2} u^8 g_2 \, du\right]$$

$$= -\frac{pv_o}{a^2}\left[2z\vec{r} + \hat{k}\left(r^2 - 2z^2 + \frac{28p\tau^2}{3A_2 nm}\right)\right] - \hat{k}\,\frac{32p^2\tau^2}{9A_2 nm}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2} u^8 g_2 \, du \qquad (48)$$

Besides the direct use in this expression of equation (47) to evaluate the original integral, equations (42) and (44) are employed in the subsequent simplifications.

If $\vec{q}$ and its spatial derivatives from equation (48) and the spatial derivatives of v from equation (31) are substituted for certain combinations of terms in equation (47), there results

$$\phi = -\frac{20\tau}{27A_2}\left\{\left[\frac{\partial v}{\partial x} + \frac{2}{5p}\left(\frac{\partial q_x}{\partial z} + \frac{\partial q_z}{\partial x}\right)\right]u_x + \left[\frac{\partial v}{\partial y} + \frac{2}{5p}\left(\frac{\partial q_y}{\partial z} + \frac{\partial q_z}{\partial y}\right)\right]u_y\right\}\left[1 - \frac{a^2}{v_o}\left(\frac{2kT}{m}\right)^{1/2} g_2\right]u_z$$

$$+ \frac{4\tau}{15A_2 p}\left(u^2 - 3u_z^2\right)\frac{\partial q_z}{\partial z} + \frac{2\tau}{5A_2 p}\left\{u^2 - \frac{5}{2} + 2u_z^2\left[1 + \frac{5a^2}{4v_o}\left(\frac{2kT}{m}\right)^{1/2} g_2\right]\right\}\frac{\partial q_z}{\partial z}$$

$$+ \frac{4}{5p}\left(\frac{m}{2kT}\right)^{1/2}\left(u^2 - \frac{5}{2}\right)\vec{q}\cdot\vec{u} + \frac{16p\tau^2 v_o u_z}{15A_2(5A_3 - 3)nma^2}\left(\frac{m}{2kT}\right)^{1/2}\left\{4(5A_3 - 3)\left(u^2 - \frac{35}{8}\right)\right.$$

(Equation continued on next page)

$$+ \left[\left(5A_3 - 9\right)u^2 + 10u_z^2\right]\left[1 - \frac{5a^2}{2v_o}\left(\frac{2kT}{m}\right)^{1/2} g_2\right]$$

$$+ \frac{8\left(5A_3 - 3\right)a^2}{3v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\left(u^2 - \frac{5}{2}\right)\int_0^\infty e^{-u^2}u^8 g_2 \, du\Bigg\}$$

$$+ \frac{4v_o}{5a^2}\left(\frac{m}{2kT}\right)^{1/2}\left[2z\left(xu_x + yu_y\right) + \left(r^2 - 2z^2\right)u_z\right]\left[u^2 - \frac{5}{2} + \frac{5a^2}{4v_o}\left(\frac{2kT}{m}\right)^{1/2} g_2\right]$$

$$- \frac{z}{3\tau}\left(\frac{m}{2kT}\right)^{1/2}\left(3r^2 - 2z^2\right)g_2 \tag{49}$$

as a partially complete moment representation of the perturbation function.

Another step toward the complete moment representation of the first-order distribution function is suggested by the last term in equation (49). Since the components of $\vec{u}$ do not appear in this term, it is convenient to introduce the scalar W according to the definition

$$W = \langle u^4 \rangle - \langle u^4 \rangle_o = \frac{1}{n}\int u^4\left(f - f^{(o)}\right)d\vec{c} = \frac{1}{n}\int u^4 f^{(o)}\phi \, d\vec{c} \tag{50}$$

This definition makes W the next higher scalar moment beyond $\langle u^o \rangle$ corresponding to the number density n and $\langle u^2 \rangle$ corresponding to the temperature T; hence, the mathematical sequence is well defined even though the moments higher than $\langle u^2 \rangle$ are not familiar physical entities.

The evaluation of W by the use of equation (49) in equation (50) and the aid of equations (44) and (48) yields

$$W = \frac{5\tau}{A_2 p}\left[1 + \frac{4a^2}{15v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2}u^8 g_2 \, du\right]\frac{\partial q_z}{\partial z}$$

$$- \frac{2nmz}{3p\tau}\left(\frac{2kT}{\pi m}\right)^{1/2}\left(3r^2 - 2z^2\right)\int_0^\infty e^{-u^2}u^6 g_2 \, du$$

$$= \frac{v_o z}{a^2}\left\{\frac{20\tau}{A_2}\left[1 + \frac{4a^2}{15v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2}u^8 g_2 \, du\right] + \frac{nm}{2p\tau}\left(3r^2 - 2z^2\right)\right\} \tag{51}$$

This expression and

$$\nabla W = \frac{3nmv_o z}{p\tau a^2}\vec{r} + \hat{k}\,\frac{v_o}{a^2}\left\{\frac{20\tau}{A_2}\left[1 + \frac{4a^2}{15v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2}u^8 g_2\,du\right] + \frac{3nm}{2p\tau}\left(r^2 - 2z^2\right)\right\} \tag{52}$$

combine with equation (49) to give

$$\begin{aligned}
\phi = &-\frac{20\tau}{27A_2}\left\{\left[\frac{\partial v}{\partial x} + \frac{2}{5p}\left(\frac{\partial q_x}{\partial z} + \frac{\partial q_z}{\partial x}\right)\right]u_x + \left[\frac{\partial v}{\partial y} + \frac{2}{5p}\left(\frac{\partial q_y}{\partial z} + \frac{\partial q_z}{\partial y}\right)\right]u_y\right\}\left[1 - \frac{a^2}{v_o}\left(\frac{2kT}{m}\right)^{1/2} g_2\right]u_z \\
&+ \frac{4\tau}{15A_2 p}\left(u^2 - 3u_z^2\right)\frac{\partial q_z}{\partial z} + \frac{4}{5p}\left(\frac{m}{2kT}\right)^{1/2}\left(u^2 - \frac{5}{2}\right)\vec{q}\cdot\vec{u} - \frac{2pa^2 W}{3nmv_o}\left(\frac{m}{2kT}\right)^{1/2} g_2 \\
&+ \frac{8p\tau}{15nm}\left(\frac{m}{2kT}\right)^{1/2}\left(u^2 - \frac{7}{2}\right)\vec{u}\cdot\nabla W \\
&+ \frac{8p\tau}{15nm}\left(\frac{m}{2kT}\right)^{1/2}\left[1 + \frac{5a^2}{4v_o}\left(\frac{2kT}{m}\right)^{1/2} g_2\right]\left[\vec{u}\cdot\nabla W + \frac{3nmu_z^2}{2A_2p^2}\left(\frac{2kT}{m}\right)^{1/2}\frac{\partial q_z}{\partial z}\right] \\
&+ \frac{2\tau}{5A_2 p}\left[\frac{\partial q_z}{\partial z} - \frac{4p\tau v_o}{a^2}\left(\frac{2kT}{m}\right)^{1/2}u_z\right]\left\{u^2 - \frac{5}{2}\right. \\
&\left. + \frac{25pa^2 g_2}{3nmv_o}\left(\frac{m}{2kT}\right)^{1/2}\left[1 + \frac{4a^2}{15v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2}u^8 g_2\,du\right]\right\} \\
&- \frac{16p\tau^2 v_o u_z}{15A_2\left(5A_3 - 3\right)nma^2}\left(\frac{m}{2kT}\right)^{1/2}\left\{3\left(5A_3 - 3\right)u^2\right. \\
&\left. - \left[\left(5A_3 - 9\right)u^2 + 10u_z^2\right]\left[1 - \frac{5a^2}{2v_o}\left(\frac{2kT}{m}\right)^{1/2} g_2\right]\right\}
\end{aligned} \tag{53}$$

Despite the fact that the explicit spatial dependence has been removed in equation (53), that expression is still somewhat complicated because the function g_2 is unknown. The conditions given in equations (39), (40), and (42) do not determine g_2 uniquely; thus, the present problem is not completely defined by the restrictions to a steady state, constant number density, and parabolic velocity profile. Fortunately, however, the choice of g_2 has almost no practical significance because of the following summary of the effects of different g_2 selections on the usual 13 moments: none at all on n, T, $\vec{v}$, and $\overset{\circ}{\overleftrightarrow{P}}$ if equations (39), (40), and (42) hold; almost none on $\vec{q}$ as is shown in a subsequent section.

The simplest form of g_2 which is consistent with equations (39), (40), and (42) is

$$g_2 = -\frac{2v_o}{5a^2}\left(\frac{m}{2kT}\right)^{1/2}\left(u^4 - 5u^2 + \frac{15}{4}\right) \tag{54}$$

Accordingly, since

$$\int_0^\infty e^{-u^2}u^8 g_2 \, du = -\frac{63v_o}{8a^2}\left(\frac{\pi m}{2kT}\right)^{1/2} \tag{55}$$

equation (53) becomes

$$\begin{aligned}
\phi = &-\frac{8\tau u_z}{27A_2}\left(u^4 - 5u^2 + \frac{25}{4}\right)\left\{\left[\frac{\partial v}{\partial x} + \frac{2}{5p}\left(\frac{\partial q_x}{\partial z} + \frac{\partial q_z}{\partial x}\right)\right]u_x + \left[\frac{\partial v}{\partial y} + \frac{2}{5p}\left(\frac{\partial q_y}{\partial z} + \frac{\partial q_z}{\partial y}\right)\right]u_y\right\} \\
&+ \frac{4}{5p}\left(\frac{m}{2kT}\right)^{1/2}\left(u^2 - \frac{5}{2}\right)\vec{q}\cdot\vec{u} + \frac{2W}{15}\left(u^4 - 5u^2 + \frac{15}{4}\right) \\
&- \frac{4p\tau}{15nm}\left(\frac{m}{2kT}\right)^{1/2}\left(u^4 - 7u^2 + \frac{35}{4}\right)\vec{u}\cdot\nabla W \\
&+ \frac{\tau}{15A_2 p}\left[11u^4 - 45u^2 + \frac{105}{4} - 6\left(u^4 - 5u^2 + \frac{15}{4}\right)u_z^2\right]\frac{\partial q_z}{\partial z} \\
&- \frac{4\tau^2 v_o u_z}{15A_2(5A_3 - 3)a^2}\left(\frac{2kT}{m}\right)^{1/2}\left\{(5A_3 - 3)\left(11u^4 - 43u^2 + \frac{105}{4}\right)\right. \\
&\left. - 2\left[(5A_3 - 9)u^2 + 10u_z^2\right]\left(u^4 - 5u^2 + \frac{19}{4}\right)\right\}
\end{aligned} \tag{56}$$

THE TRACELESS PRESSURE TENSOR

The traceless pressure tensor defined in equation (16) is computed by equations (44), (48), and (53) to satisfy

$$\overset{o}{\overleftrightarrow{P}} = -\frac{10p\tau}{27A_2}\left\{\left(\hat{i}\hat{k} + \hat{k}\hat{i}\right)\left[\frac{\partial v}{\partial x} + \frac{2}{5p}\left(\frac{\partial q_x}{\partial z} + \frac{\partial q_z}{\partial x}\right)\right]\right.$$

(Equation continued on next page)

$$+\left(\hat{j}\hat{k}+\hat{k}\hat{j}\right)\left[\frac{\partial v}{\partial y}+\frac{2}{5p}\left(\frac{\partial q_y}{\partial z}+\frac{\partial q_z}{\partial y}\right)\right]\Bigg\}\left[1-\frac{16a^2}{15v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2}u^6 g_2\,du\right]$$

$$+\frac{2\tau}{3A_2}\Bigg\{\left(\hat{i}\hat{i}+\hat{j}\hat{j}\right)\left[1+\frac{4a^2}{5v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2}u^6 g_2\,du\right]$$

$$+\hat{k}\hat{k}\left[1+\frac{12a^2}{5v_o}\left(\frac{2kT}{\pi m}\right)^{1/2}\int_0^\infty e^{-u^2}u^6 g_2\,du\right]\Bigg\}\frac{\partial q_z}{\partial z}$$

$$=-\frac{2p\tau}{3A_2}\Bigg\{\left(\hat{i}\hat{k}+\hat{k}\hat{i}\right)\left[\frac{\partial v}{\partial x}+\frac{2}{5p}\left(\frac{\partial q_x}{\partial z}+\frac{\partial q_z}{\partial x}\right)\right]+\left(\hat{j}\hat{k}+\hat{k}\hat{j}\right)\left[\frac{\partial v}{\partial y}+\frac{2}{5p}\left(\frac{\partial q_y}{\partial z}+\frac{\partial q_z}{\partial y}\right)\right]\Bigg\}$$

$$-\frac{4\tau}{15A_2}\left[\hat{i}\hat{i}\left(2\frac{\partial q_x}{\partial x}-\frac{2}{3}\nabla\cdot\vec{q}\right)+\hat{j}\hat{j}\left(2\frac{\partial q_y}{\partial y}-\frac{2}{3}\nabla\cdot\vec{q}\right)+\hat{k}\hat{k}\left(2\frac{\partial q_z}{\partial z}-\frac{2}{3}\nabla\cdot\vec{q}\right)\right] \tag{57}$$

which is identical with the Grad 13-moment result (ref. 1, p. 495)

$$\overset{o}{P}_{ij}=-\eta\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\vec{v}\right)-\frac{2\eta}{5p}\left(\frac{\partial q_i}{\partial x_j}+\frac{\partial q_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\vec{q}\right) \tag{58}$$

if the viscosity η is given by

$$\eta=\frac{2p\tau}{3A_2} \tag{59}$$

and if proper account is taken of the characteristics of $\vec{v}$ in the present problem.

This value of the viscosity combines with the thermal conductivity of equation (13) to yield the correct Prandtl number

$$N_{Pr}=\frac{c_p\eta}{\lambda}=\frac{5k\eta}{2m\lambda}=\frac{2}{3A_2}\approx\frac{2}{3} \tag{60}$$

thereby at least one principal improvement of the present collision model over that of Krook is confirmed. Equations (57), (58), and (60) thus provide some support for the hypothesis that precise flux-force relations and ratios of transport coefficients for real gases can be obtained from the kinetic theory of the real-gas—fictitious-gas combination.

Although the present formulation of the traceless pressure tensor is the same as the 13-moment approximation, the evaluations of the tensor elements are different because of different results for the spatial derivatives of $\vec{q}$. This aspect is examined in some detail in the next section, where a more convenient representation of $\vec{q}$ is deduced.

CONDUCTIVE HEAT FLUX

An expression for the conductive heat flux $\vec{q}$ has already been derived in equation (48), the combination of which with equations (13), (44), and (52) gives

$$\vec{q} = \hat{k}\,\frac{4p^2\tau^2 v_o}{A_2 nma^2} - \frac{2p^2\tau}{3nm}\,\nabla W = -\hat{k}\,\frac{3\lambda}{5}\,\frac{\partial T}{\partial z} - \frac{4\lambda T}{15}\,\nabla W$$

$$= -\hat{k}\lambda\,\frac{\partial T}{\partial z} + \hat{k}\,\frac{2\lambda}{5}\,\frac{\partial T}{\partial z} - \frac{4\lambda T}{15}\,\nabla W \tag{61}$$

Since the divergence of $\overset{\leftrightarrow}{\overset{o}{P}}$ satisfies

$$\nabla \cdot \overset{\leftrightarrow}{\overset{o}{P}} = \hat{k}\left(\frac{\partial \overset{o}{P}_{xz}}{\partial x} + \frac{\partial \overset{o}{P}_{yz}}{\partial y} + \frac{\partial \overset{o}{P}_{zz}}{\partial z}\right) = -\hat{k}nk\,\frac{\partial T}{\partial z} = -nk\nabla T \tag{62}$$

from equations (31), (44), (48), (58), and (59), equation (61) becomes

$$\vec{q} = -\lambda\nabla T - \frac{2\lambda}{5nk}\,\nabla \cdot \overset{\leftrightarrow}{\overset{o}{P}} - \frac{4\lambda T}{15}\,\nabla W \tag{63}$$

A convenient comparison with the Grad 13-moment result is obtained by taking the $mc^2\vec{c}$-moment of equation (1) and using either equation (15) or equation (56) in the present collision model. The result is

$$\nabla \cdot \left(nm\left\langle c^2\vec{c}\vec{c}\right\rangle\right) = \nabla \cdot \left[\frac{4nk^2T^2}{m}\left(\left\langle u^2\vec{u}\vec{u}\right\rangle + \vec{\beta}\left\langle u^2\vec{u}\right\rangle + \left\langle u^2\vec{u}\right\rangle\vec{\beta} + 2\vec{\beta}\cdot\left\langle \vec{u}\vec{u}\vec{u}\right\rangle\right)\right]$$

$$= \frac{4nk^2}{m}\,\nabla \cdot \left(T^2\left\langle u^2\vec{u}\vec{u}\right\rangle\right) = \hat{k}\,\frac{8nk^2T}{m}\left\langle u^2u_z^2\right\rangle_o\,\frac{\partial T}{\partial z} + \frac{4p^2}{nm}\,\nabla \cdot \left\langle u^2\vec{u}\vec{u}\right\rangle$$

$$= \frac{10nk^2T}{m}\,\nabla T + \frac{4p^2}{nm}\,\nabla \cdot \left\langle u^2\vec{u}\vec{u}\right\rangle = m\int c^2\vec{c}\left(\frac{\partial f}{\partial t}\right)_c d\vec{c}$$

$$= -\frac{4p^2}{\pi^{3/2}m}\int e^{-u^2}u^3\vec{u}\left(\phi - \phi'\right)b\,db\,d\epsilon\,d\vec{u} = -\frac{2}{\tau}\,\vec{q} \tag{64}$$

Second-order terms are neglected in the simplifications of $\nabla \cdot \left(nm\left\langle c^2\vec{c}\vec{c}\right\rangle\right)$, whereas equations (4) and (18) are employed in the evaluation of the collision integral.

Equation (64) yields

$$\vec{q} = -\frac{2p^2\tau}{nm}\left[\frac{5}{2T}\nabla T + \nabla \cdot \left\langle u^2\vec{u}\vec{u}\right\rangle_o + \nabla \cdot \left(\left\langle u^2\vec{u}\vec{u}\right\rangle - \left\langle u^2\vec{u}\vec{u}\right\rangle_o\right)\right]$$

$$= -2\lambda\nabla T - \frac{4\lambda T}{5}\nabla \cdot \left(\left\langle u^2\vec{u}\vec{u}\right\rangle - \left\langle u^2\vec{u}\vec{u}\right\rangle_o\right) \tag{65}$$

with the aid of equation (13). Unlike the collision integral in equation (64), the value of which is independent of whether equation (15) or equation (56) is used, the evaluations of $\left\langle u^2\vec{u}\vec{u}\right\rangle - \left\langle u^2\vec{u}\vec{u}\right\rangle_o$ are

$$\left\langle u^2\vec{u}\vec{u}\right\rangle - \left\langle u^2\vec{u}\vec{u}\right\rangle_o = \frac{7}{4p}\overset{o}{\overleftrightarrow{P}} \tag{66}$$

and

$$\left\langle u^2\vec{u}\vec{u}\right\rangle - \left\langle u^2\vec{u}\vec{u}\right\rangle_o = \frac{7}{36p}\left[17\overset{o}{\overleftrightarrow{P}} + 10\left(\hat{i}\hat{i}\overset{o}{P}_{xx} + \hat{j}\hat{j}\overset{o}{P}_{yy} + \hat{k}\hat{k}\overset{o}{P}_{zz}\right)\right] + \frac{W}{3}\overleftrightarrow{U} \tag{67}$$

respectively, for the Grad 13-moment and the present velocity distribution functions.

Accordingly, the two determinations of the conductive heat flux are given by

$$\vec{q}(13\text{ moment}) = -\lambda\nabla T - \frac{2\lambda T}{5p}\nabla \cdot \overset{o}{\overleftrightarrow{P}} \tag{68}$$

and

$$\vec{q}(\text{exact first order}) = -\lambda\nabla T - \frac{2\lambda T}{5p}\nabla \cdot \overset{o}{\overleftrightarrow{P}} - \frac{4\lambda T}{15}\nabla W \tag{69}$$

from equations (44), (48), (58), (62), and (65) to (67). Equation (68) is also identical with the Grad 13-moment result quoted in reference 1 (p. 495; note, however, the typographical error).

The fact that equation (69) reduces to equation (68) in the 13-moment limit of $W = 0$ completes the theoretical support for the hypothesis that precise flux-force relations for real gases can be obtained from the kinetic theory of the real-gas—fictitious-gas combination. Of the 13 physically important moments n, $\overleftrightarrow{P}$, $\vec{q}$, and

$$\vec{v} = \hat{k}v_o\left(1 - \frac{r^2}{a^2}\right) = -\frac{a^2}{4\eta}\left(1 - \frac{r^2}{a^2}\right)\nabla p \tag{70}$$

from equations (31) and (44), the formal results of the present research differ from those of the Grad 13-moment approximation only in the expression for $\vec{q}$; this one difference stems not from the collision model, but from the inaccurate 13-moment determination of the quantity $\langle u^2\vec{u}\vec{u}\rangle$ in equation (65).

Consequently, the six moments corresponding to $\langle u^2\vec{u}\vec{u}\rangle$ and three more to cover the $\vec{u} \cdot \nabla W$ term in equation (56) are minimal additions to the 13-moment distribution function in order to provide an adequate first-order description of the present problem. The flux-force relations of equations (58), (69), and (70) are thus equivalent to a Grad 22-moment approximation or better.

This discussion would be trivial, of course, if the last term on the right side of equation (69) were negligible compared with the first two. Actually, the reverse is true because of the following combinations (correct through first order) of equations (13), (52), (55), (59), (62), (69), and (70):

$$\nabla W = \frac{15pv_o}{4\lambda Ta^2}\left[2z\vec{r} + \hat{k}\left(r^2 - 2z^2\right)\right] - \frac{33}{4p}\nabla \cdot \overset{o}{\overleftrightarrow{P}} \tag{71}$$

and

$$\begin{aligned}\vec{q} &= -\lambda\nabla T + \frac{9\lambda T}{5p}\nabla \cdot \overset{o}{\overleftrightarrow{P}} - \frac{pv_o}{a^2}\left[2z\vec{r} + \hat{k}\left(r^2 - 2z^2\right)\right] \\ &= -\frac{pv_o}{a^2}\left[2z\vec{r} + \hat{k}\left(r^2 - 2z^2 - \frac{56\eta\lambda T}{5p^2}\right)\right] \approx -\frac{pv_o}{a^2}\left[2z\vec{r} + \hat{k}\left(r^2 - 2z^2\right)\right]\end{aligned} \tag{72}$$

Not only does the ∇W contribution to $\vec{q}$ add a term of ordinary conduction magnitude (that is, $\lambda\nabla T$), it also adds much larger terms of convection magnitude which are independent of the choice of g_2 in the perturbation function. An interesting curiosity of equation (72) with regard to the second law of thermodynamics is given in the appendix.

Since the convective heat flux is deduced from equation (70) to be

$$\vec{q}_v = \frac{5p}{2}\vec{v} = \frac{5pv_o\left(a^2 - r^2\right)}{2a^2}\hat{k} \tag{73}$$

the total heat flux $\vec{Q}$ can be written as

$$\vec{Q} = \vec{q} + \vec{q}_v \approx -\frac{pv_o}{2a^2}\left[4z\vec{r} - \hat{k}\left(5a^2 - 7r^2 + 4z^2\right)\right] \tag{74}$$

Consequently, the ratio of the maximum r-component to the maximum z-component of $\vec{Q}$ is

$$\frac{Q_r(r = a)}{Q_z(r = 0)} = -\frac{4az}{5a^2 + 4z^2} \tag{75}$$

Except at $z = 0$, which corresponds to $T = T^o$ from equation (43), the picture of heat transfer obtained from equations (72) to (75) is entirely different from the 13-moment result of $\vec{Q} \approx \vec{q}_v$. This difference is especially noticeable in the first-order expressions

$$Q_r(13 \text{ moment}) = 0 \tag{76}$$

and

$$Q_r(\text{exact solution}) = -\frac{2pv_o rz}{a^2} = \frac{p^2\left(T - T^o\right)r}{2\eta T} \tag{77}$$

from equations (43), (70), and (74). Whereas the 13-moment approximation predicts no heat transfer to the walls of convection magnitude (that is, first order or proportional to v), the present solution predicts an outflow of energy in the heated regions of the gas and an inflow of energy in the cooled regions – even though T (first order) does not vary with r.

It is important also to realize that the prediction of a first-order radial heat transfer has nothing to do with the usual macroscopic energy equation, which is concerned with the divergence of $\vec{q}$ and yields a second-order or v^2-contribution to q_r. Equation (72) states that the spatial variations of q_r and q_z are such as to make the divergence of $\vec{q}$ vanish in first order; hence, the phenomenon discovered here is indeed new in the sense that it cannot be predicted for this specific problem by either the Navier-Stokes theory or the Grad 13-moment approximation. The latter method modifies only the negligible contributions to $\vec{Q}$.

As mentioned previously, equation (72) also has important effects on the values of the traceless pressure tensor elements in equation (58). Whereas the 13-moment approximation to $\overset{\leftrightarrow}{\overset{o}{P}}$ satisfies the Navier-Stokes expression

$$\begin{aligned}\overset{\leftrightarrow}{\overset{o}{P}} &= -\eta\left[\left(\hat{i}\hat{k} + \hat{k}\hat{i}\right)\frac{\partial v}{\partial x} + \left(\hat{j}\hat{k} + \hat{k}\hat{j}\right)\frac{\partial v}{\partial y}\right] \\ &= \frac{2\eta v_o}{a^2}\left[\left(\hat{i}\hat{k} + \hat{k}\hat{i}\right)x + \left(\hat{j}\hat{k} + \hat{k}\hat{j}\right)y\right]\end{aligned} \tag{78}$$

in the present problem because of the vanishing of the spatial derivatives of $\vec{q}$ (13 moment) to first order, the exact solution gives

$$\overset{o}{\overleftrightarrow{P}} = \frac{2\eta v_o}{a^2}\left[\left(\hat{i}\hat{k} + \hat{k}\hat{i}\right)x + \left(\hat{j}\hat{k} + \hat{k}\hat{j}\right)y\right] - \frac{2\eta}{5p}\left[\left(\hat{i}\hat{k} + \hat{k}\hat{i}\right)\left(\frac{\partial q_x}{\partial z} + \frac{\partial q_z}{\partial x}\right)\right.$$

$$\left. + \left(\hat{j}\hat{k} + \hat{k}\hat{j}\right)\left(\frac{\partial q_y}{\partial z} + \frac{\partial q_z}{\partial y}\right) + 2\hat{i}\hat{i}\,\frac{\partial q_x}{\partial x} + 2\hat{j}\hat{j}\,\frac{\partial q_y}{\partial y} + 2\hat{k}\hat{k}\,\frac{\partial q_z}{\partial z}\right]$$

$$= \frac{2\eta v_o}{5a^2}\left[9\left(\hat{i}\hat{k} + \hat{k}\hat{i}\right)x + 9\left(\hat{j}\hat{k} + \hat{k}\hat{j}\right)y + 4\left(\overleftrightarrow{U} - 3\hat{k}\hat{k}\right)z\right] \tag{79}$$

Accordingly, major changes are made in Newton's law of viscosity as well as in Fourier's law of thermal conduction; in particular, numerical changes occur in the off-diagonal elements of $\overset{o}{\overleftrightarrow{P}}$ and nonzero diagonal elements are introduced. However, no change appears in the important relation

$$\nabla \cdot \overset{o}{\overleftrightarrow{P}} = \hat{k}\,\frac{4\eta v_o}{a^2} = -nk\nabla T \tag{80}$$

which is fixed by the macroscopic equation of motion

$$\rho\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right)\vec{v} - \rho\vec{X} + \nabla p + \nabla \cdot \overset{o}{\overleftrightarrow{P}} = \nabla p + \nabla \cdot \overset{o}{\overleftrightarrow{P}} = 0 \tag{81}$$

THE ENERGY EQUATION

The final computations of the present research concern the energy equation, which is obtained as follows by taking the mc^2-moment of the kinetic equation and using the equations of continuity and motion:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(nm\langle c^2\rangle\right) - 2nm\vec{X} \cdot \vec{v} + \nabla \cdot \left(nm\langle c^2\vec{c}\rangle\right) \\
&= \frac{\partial}{\partial t}\left[nm\left\langle\left(\vec{c} - \vec{v}\right)^2\right\rangle\right] + \frac{\partial}{\partial t}\left(nmv^2\right) - 2nm\vec{X} \cdot \vec{v} + \nabla \cdot \left[nm\left\langle\left(\vec{c} - \vec{v}\right)^2\left(\vec{c} - \vec{v}\right) + \left(\vec{c} - \vec{v}\right)^2\vec{v} + 2\vec{v} \cdot \left(\vec{c} - \vec{v}\right)\left(\vec{c} - \vec{v}\right) + v^2\vec{v}\right\rangle\right] \\
&= 3\,\frac{\partial p}{\partial t} + 2\rho\vec{v} \cdot \left(\frac{D\vec{v}}{Dt} - \vec{v} \cdot \nabla\vec{v}\right) - v^2\nabla \cdot \left(\rho\vec{v}\right) - 2\rho\vec{v} \cdot \vec{X} + 2\nabla \cdot \vec{q} + 3p\nabla \cdot \vec{v} + 3\vec{v} \cdot \nabla p + 2\nabla \cdot \left[\vec{v} \cdot \left(\overset{o}{\overleftrightarrow{P}} + p\overleftrightarrow{U}\right)\right] + v^2\nabla \cdot \left(\rho\vec{v}\right) + 2\rho\vec{v} \cdot \left(\vec{v} \cdot \nabla\vec{v}\right) \\
&= 2\rho c_p\,\frac{DT}{Dt} - 2\,\frac{Dp}{Dt} + 2\nabla \cdot \vec{q} - 2\left[\vec{v} \cdot \left(\nabla \cdot \overset{o}{\overleftrightarrow{P}}\right) - \nabla \cdot \left(\vec{v} \cdot \overset{o}{\overleftrightarrow{P}}\right)\right] \\
&= 0
\end{aligned} \tag{82}$$

Thus

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} - \nabla \cdot \vec{q} + \Phi + \Phi^* \tag{83}$$

where Φ and Φ^* are dissipation functions satisfying

$$\Phi + \Phi^* = \vec{v} \cdot \left(\nabla \cdot \overset{\text{o}}{\overrightarrow{P}}\right) - \nabla \cdot \left(\vec{v} \cdot \overset{\text{o}}{\overrightarrow{P}}\right) \tag{84}$$

$$\Phi = \eta \left\{ e_{12}^{\;2} + e_{13}^{\;2} + e_{23}^{\;2} + \frac{1}{6}\left[\left(e_{11} - e_{22}\right)^2 + \left(e_{11} - e_{33}\right)^2 + \left(e_{22} - e_{33}\right)^2\right]\right\} \tag{85}$$

$$\begin{aligned}\Phi^* = \frac{2\eta}{5p} \Bigg\{ & e_{12}e_{12}^{\;*} + e_{13}e_{13}^{\;*} + e_{23}e_{23}^{\;*} + \frac{1}{6}\Big[\left(e_{11} - e_{22}\right)\left(e_{11}^{\;*} - e_{22}^{\;*}\right) \\ & + \left(e_{11} - e_{33}\right)\left(e_{11}^{\;*} - e_{33}^{\;*}\right) + \left(e_{22} - e_{33}\right)\left(e_{22}^{\;*} - e_{33}^{\;*}\right)\Big]\Bigg\}\end{aligned} \tag{86}$$

$$e_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \tag{87}$$

and

$$e_{ij}^{\;*} = \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \tag{88}$$

Third-order terms proportional to v^3 and containing spatial derivatives of η are neglected in equations (85) and (86).

Except for the dissipation function Φ^*, which derives from the heat-flux coupling terms in equation (58), equation (83) is the Navier-Stokes energy equation. The values of Φ and Φ^* for the two distribution functions are obtained from equations (68), (70), (72), and (85) to (88) to be

$$\Phi(\text{13 moment}) = \Phi(\text{exact solution}) = \eta\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] = \frac{4\eta v_o^{\;2} r^2}{a^4} \tag{89}$$

$$\Phi^*(\text{13 moment}) = 0 \tag{90}$$

and

$$\Phi^*(\text{exact solution}) = \frac{16\eta v_o^2 r^2}{5a^4} \tag{91}$$

through second order.

The resulting energy equations for the present problem are for the 13-moment solution:

$$\nabla \cdot \vec{q} = -\frac{3}{2}\vec{v} \cdot \nabla p + \Phi + \Phi^* = \frac{6\eta v_o^2}{a^4}\left(a^2 - \frac{1}{3}r^2\right) \tag{92}$$

and for the exact solution:

$$\nabla \cdot \vec{q} = -\frac{3}{2}\vec{v} \cdot \nabla p + \Phi + \Phi^* = \frac{6\eta v_o^2}{a^4}\left(a^2 + \frac{1}{5}r^2\right) \tag{93}$$

Equation (70) is used in expressing ∇p in terms of $\hat{k}v_o$.

As in the evaluation of the traceless pressure tensor elements, and even though the formal expression of the energy equation is the same as the 13-moment approximation, the new result for $\vec{q}$ given in equation (72) has pronounced effects on the values of Φ^* and the divergence of $\vec{q}$. Equation (72) and the solutions of equations (92) and (93) imply not only the substantial changes discussed previously with regard to the Navier-Stokes picture of first-order heat flux, but also significant alterations in the v^2-dependent contribution to q_r.

The rate of heat transfer per unit area to the walls of the pipe is defined by

$$R = \frac{1}{2\pi a\ \Delta z}\left\{2\pi a Q_r(\text{1st order; } r = a)\Delta z + \int \nabla \cdot \vec{Q}(\text{2d order})dx\ dy\ dz\right.$$

$$\left. - 2\pi\int_0^a \left[Q_z(\text{2d order; } z + \Delta z) - Q_z(\text{2d order; } z)\right] r\ dr\right\}$$

$$\approx Q_r(\text{1st order; } r = a) + \frac{1}{a}\int_0^a r\nabla \cdot \vec{q}\ dr \tag{94}$$

where Δz is a small increment of axial length. Two assumptions are made in the simplifications of equation (94): (1) the convective contribution to $\nabla \cdot \vec{Q}$ (2d order) is negligible or zero, and (2) changes in Q_z (2d order) along the z-axis are third order.

Both are reasonable in view of the fact that the flow velocity has no radial component and the likelihood that the nth derivative with respect to z corresponds to a term of nth order.

Equations (43), (70), (74), and (92) to (94) thus yield

$$R(13\text{ moment}) = \frac{5\eta v_o^2}{2a} \tag{95}$$

and

$$R(\text{exact solution}) = -\frac{2pv_o z}{a} + \frac{33\eta v_o^2}{10a} = \frac{ap^2\left(T - T^o\right)}{2\eta T} + \frac{33\eta v_o^2}{10a} \tag{96}$$

COMPARISONS WITH CHAPMAN-ENSKOG THEORY

Correlations of the preceding results with the kinetic theory of Chapman and Enskog (ref. 2) are somewhat ambiguous because the use of τ (or the mean free path) as a perturbation parameter is not valid in the present problem. The reasons are obvious both physically and from the exact first-order solution in equation (47). In particular, the approach of τ (and thus the viscous forces) to zero is inconsistent with the steady-state condition unless the applied temperature gradient also vanishes, for otherwise an unbalanced driving force would exist and cause the gas to be constantly accelerated through the pipe. The same conclusion is reached by analyzing equation (47), the last three terms of which are proportional to τ^{-1}, τ^{-1}, and τ^{-2} because of the τ^{-1} dependence of g_2 from equations (54), (59), and (70). Hence, the fundamental perturbation principle of Chapman and Enskog does not apply to this case.

Comparisons can be obtained, however, without utilizing the explicit and implicit appearances of τ in equation (47); for example, one can simply ignore the perturbation principle and look directly at the Chapman-Enskog perturbation equations for a basis of correlation. A particularly convenient and systematic scheme is established as follows: first assume that no term in equation (47) occurs below the Chapman-Enskog second approximation because of the factor v_o or g_2; next assume that $\nabla\phi$ fails to appear in the second-approximation kinetic equation because ϕ (second approximation) does not contain x, y, or z explicitly; then assume that second spatial derivatives of ϕ are absent from the third-approximation kinetic equation because ϕ (third approximation) contains no explicit x^2, y^2, z^2, xy, xz, or yz factors and so forth. Each power of x, y, or z appearing explicitly in equation (47) thus represents a raising of the Chapman-Enskog level of approximation by unity in order to include the corresponding term. As explained previously, the occurrence of such higher approximations in the present first-order solution derives from the use of a more complete first-order (or second-approximation) kinetic equation which includes all terms linearly proportional to the viscous flow velocity.

The assignment of a level of approximation to a particular form of a flux-force relation must also involve the selection of the pertinent terms in equation (47). For example, the ∇W-contribution to equation (69) derives from the $\left(r^2 - 2z^2\right)g_2 u_z$ and $2zg_2\left(xu_x + yu_y\right)$ terms in equation (47); accordingly, that expression for the conductive heat flux is assigned to the fourth Chapman-Enskog approximation. A complete summary is given in table I.

Two exceptions to the correlation scheme are noted in table I, each of which results from inconsistent Chapman-Enskog assumptions. Equation (47) indicates that the first appearance of a nonzero traceless pressure tensor should occur in the third approximation, whereas Chapman and Enskog credit Newton's law of viscosity to the second approximation; in addition, equation (47) clearly shows that the coupling terms corresponding to the Grad 13-moment and Burnett formulations cannot be a priori separated from Newton's contributions. Although spatial derivatives of the viscous flow velocity result from spatial derivatives of Chapman and Enskog's Maxwellian distribution function in the second-approximation kinetic equation, they should be deleted after the fact as being inconsistent with that equation. The reason is obvious: Since the resulting ϕ (second approximation) must contain a term proportional to $\partial v/\partial x$, which, in turn, is proportional to x itself in the case of a parabolic velocity profile, the neglect of $\partial\phi/\partial x$ in the second-approximation kinetic equation is inconsistent with the assignment of Newton's law of viscosity to that

TABLE I.- SUMMARY OF MACROSCOPIC RELATIONS

Number of moments in Grad approximation	Level of Chapman-Enskog approximation	Pressure tensor element, $\overset{0}{P}_{ij}$	Heat flux, $\vec{q}$	Energy equation
5	First	0	0	------------------------
----	Second	$-\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\vec{v}\right)$	$-\lambda\nabla T$	$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} - \nabla\cdot\vec{q} + \Phi$
13	Third	$-\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\vec{v}\right) - \frac{2\eta}{5p}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\vec{q}\right)$	$-\lambda\nabla T - \frac{2\lambda T}{5p}\nabla\cdot\overset{0}{P}$	$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} - \nabla\cdot\vec{q} + \Phi + \Phi^*$
*≧22	Fourth	$-\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\vec{v}\right) - \frac{2\eta}{5p}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\vec{q}\right)$	$-\lambda\nabla T - \frac{2\lambda T}{5p}\nabla\cdot\overset{0}{P} - \frac{4\lambda T}{15}\nabla W$	$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} - \nabla\cdot\vec{q} + \Phi + \Phi^*$

*The value of the ∇W term in the equation for $\vec{q}$ depends upon the number of moments employed. At least 22 moments are required for this term to appear.

level. Either the first spatial derivatives of ϕ should be included in the second-approximation kinetic equation, in which case the complete 13-moment formula for $\overset{o}{\overleftrightarrow{P}}$ would be second approximation, or no contribution to $\overset{o}{\overleftrightarrow{P}}$ should occur below the third approximation.

A similar argument applies to Fourier's law of thermal conduction in Chapman and Enskog's second approximation. Equation (47) again implies that the 13-moment coupling term cannot be a priori separated from Fourier's law, both contributions being assigned to the second approximation by the present correlation scheme. Chapman and Enskog's neglect of the divergence of the traceless pressure tensor in the second approximation is therefore inconsistent in the present problem.

Not only are the arguments similar, but the two Chapman-Enskog inconsistencies with respect to $\overset{o}{\overleftrightarrow{P}}$ and $\vec{q}$ are, in fact, related to each other. If the gradient of ϕ (and thus the gradient of the spatial derivatives of the viscous flow velocity) had been included in the second-approximation kinetic equation in order to be consistent with the retention of $\partial v/\partial x$, etc., the divergence of $\overset{o}{\overleftrightarrow{P}}$ would have occurred automatically in the second approximation to $\vec{q}$.

Completely consistent treatments, whether based on the present correlation scheme or something different, always predict more complicated expressions than those of Newton and Fourier for the present problem. This conclusion further explains the absence in table I of a direct correspondence between the Chapman-Enskog second approximation and one of the Grad N-moment variety, the latter method being free of basic inconsistencies. Only the use of insufficient moments seems to affect the accuracy of the general Grad approximation for neutral gases, although other difficulties arise in certain plasma applications. (See ref. 11.)

CONCLUDING REMARKS

Exact first-order solutions of an approximate kinetic equation have been found for the problem of viscous gases flowing through circular pipes with constant temperature gradients. The assumption that errors inherent in the collision model affect only the values of transport coefficients and not their ratios or the formal flux-force relations is supported in two ways: (1) the expressions for all 13 physically important moments reduce in the proper limit to those of the Grad 13-moment approximation, and (2) the ratio of the calculated viscosity and thermal conductivity agrees with Chapman and Enskog's result.

A significant new contribution to the conductive heat flux was discovered which has the same order of magnitude as heat convection and substantially changes the Navier-Stokes description of energy transfer in the problem considered; in addition, the evaluations of the traceless pressure tensor and the dissipation terms in the macroscopic energy equation are affected. The description of these effects requires at least the fourth Chapman-Enskog approximation or the use of 22 moments in the Grad distribution function.

Although the problem considered in the present research is possible in principle, experimental difficulties may prevent a direct laboratory confirmation of the results. For one thing, a two-dimensional control of the temperature variations within the pipe and its environment would have to be maintained in order to guarantee a constant axial temperature gradient and also a steady radial flow of heat into or out of the gas.

In any event, the principal value of the research probably lies more in the method introduced than in the actual problem solved. The extension of the collision model to more complicated integrals is precisely defined, so that the only question is whether the resulting differential equations can be solved for given situations. Since the technique of solution is fairly straightforward, no great difficulty is anticipated – even for time-dependent systems. The most important precaution is that the first-order kinetic equation be complete in the first power of some small physical parameter (the viscous flow velocity in the present problem), for that is the device by which significant contributions from higher Chapman-Enskog or Grad moment approximations are included with minimum effort. This saving of effort is generally very substantial.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., April 8, 1970.

APPENDIX

SECOND LAW OF THERMODYNAMICS

The purpose of this appendix is to discuss an interesting curiosity of equation (72) with regard to the second law of thermodynamics. Since the dominant contribution to the axial component of $\vec{q}$ is given by

$$q_z = -\frac{pv_o}{a^2}\left(r^2 - 2z^2\right) \tag{A1}$$

and since v_o is positive if the temperature decreases with increasing z, conductive heat flows from cooler to warmer regions in the vicinity of $z = 0$. This result directly contradicts one of the simplest and most popular interpretations of the second law.

An explanation of this paradox is obtained from the following more basic and correct formulation of the second law: The entropy source strength (that is, the rate of entropy production per unit volume) must be nonnegative at every point. The statistical expression for the entropy source strength is deduced from the local entropy density

$$s = s^{(o)} - k\int f \log_e \frac{f}{f^{(o)}}\, d\vec{c} \tag{A2}$$

to be

$$\dot{s}_c = -k\int \left[1 + \log_e(1+\phi)\right]\left(\frac{\partial f}{\partial t}\right)_c d\vec{c} = -k\int \log_e(1+\phi)\left(\frac{\partial f}{\partial t}\right)_c d\vec{c} \approx -k\int \phi\left(\frac{\partial f}{\partial t}\right)_c d\vec{c} \tag{A3}$$

If the left-hand side of equation (1) is substituted for $(\partial f/\partial t)_c$ in equation (A3), the result can be written

$$\begin{aligned}
\dot{s}_c &= -k\int \phi\vec{c}\cdot\left(\nabla f^{(o)} + f^{(o)}\nabla\phi\right) d\vec{c} \\
&= -k\int f^{(o)}\phi\left\{\vec{c}\cdot\left[\left(u^2 - \frac{5}{2}\right)\frac{\nabla T}{T} + \frac{\nabla p}{p} + \nabla\phi\right] + 2\,\frac{dv}{dr}\,u_r u_z\right\} d\vec{c} \\
&= -\frac{1}{T^2}\vec{q}\cdot\nabla T - \frac{1}{T}\overset{o}{P}_{rz}\frac{dv}{dr} - \frac{nk}{\pi}\left(\frac{2kT}{\pi m}\right)^{1/2}\int e^{-u^2}\phi\vec{u}\cdot\nabla\phi\, d\vec{u} \\
&= -\frac{1}{T^2}\vec{q}\cdot\nabla T + \frac{9\eta}{5T}\left(\frac{dv}{dr}\right)^2 - \frac{nk}{\pi}\left(\frac{2kT}{\pi m}\right)^{1/2}\int e^{-u^2}\phi\vec{u}\cdot\nabla\phi\, d\vec{u}
\end{aligned} \tag{A4}$$

with the aid of equations (16), (48), (70), and (79). Contributions of order v^3 and higher are neglected. Since the last term of equation (A4) never appears in the second Chapman-Enskog approximation or Navier-Stokes theory, the requirement that $\dot{s}_c$ be nonnegative usually restricts $\vec{q}$ to the direction opposite of ∇T.

In the present problem, however, the last term of equation (A4) is the most important part of $\dot{s}_c$. This conclusion is reached by first introducing the mean free path $l = \bar{v}\tau$, where

$$\bar{v} = 2\left(\frac{2kT}{\pi m}\right)^{1/2} \tag{A5}$$

and using equations (44), (59), (70), and (A1) to rewrite equation (A4) in the convenient form

$$\frac{\tau\dot{s}_c}{nk} \approx \frac{16\left(2r^2 + 5z^2\right)l^2}{15A_2a^4}\left(\frac{v_o}{\bar{v}}\right)^2 - \frac{l}{2\pi}\int e^{-u^2}\phi\vec{u}\cdot\nabla\phi\, d\vec{u} \tag{A6}$$

Since the function g_2 and the constant α are both proportional to τ^{-1} by equations (42), (44), (59), and (70), the largest contributions to ϕ in terms of small values of l/a are the last three terms in equation (47). Accordingly,

$$\phi \approx \left(r^2 - 2z^2\right)g_2u_z + 2rzg_2u_r - \frac{z}{3\tau}\left(\frac{m}{2kT}\right)^{1/2}\left(3r^2 - 2z^2\right)g_2 \tag{A7}$$

and

$$\vec{u}\cdot\nabla\phi \approx 4rg_2u_ru_z + 2zg_2\left(u^2 - 3u_z^2\right) - \frac{1}{\tau}\left(\frac{m}{2kT}\right)^{1/2}g_2\left[2rzu_r + \left(r^2 - 2z^2\right)u_z\right] \tag{A8}$$

so that

$$\frac{\tau\dot{s}_c}{nk} \approx \frac{4a^4}{3\pi^{1/2}}\int_0^\infty e^{-u^2}u^4g_2^2\, du = 0\left(v_o^2/\bar{v}^2\right) \tag{A9}$$

at the position $r = a$, $z = 0$ where the previously mentioned paradox with the second law is most pronounced.

No disagreement with the basic formulation of the second law of thermodynamics occurs, nor can a violation ever exist when the solution of the kinetic equation is exact in the present sense.

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