A GENERALIZED UNIMODALITY

by

Richard A. Olshen and Leonard J. Savage

TECHNICAL REPORT NO. 143
April 1, 1969

PREPARED UNDER CONTRACT Nonr-225(52)
(NR-342-022) FOR
OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

A GENERALIZED UNIMODALITY

Richard A. Olshen, Stanford University,

and

Leonard J. Savage, Yale University[1]

1. Introduction.

Khintchine (1938) showed that a real random variable Z has a unimodal distribution with mode at 0 iff $Z \sim UX$ (that is, Z is distributed like UX), where U is uniform on $[0,1]$ and U and X are independent. Isii (1958, p. 173) defines a modified Stieltjes transform of a distribution function F for w complex thus.

$$I(w;F) = \int (w-t)^{-1} dF(t) .$$

Apparently unaware of Khintchine's work, he proved (pp. 179-180) that F is unimodal with mode at 0 iff there is a distribution function Φ for which $I(w;\Phi) = -w\, dI(w;F)/dw$. The equivalence of Khintchine's and Isii's results is made vivid by a proof (due to L. A. Shepp) in the next section.

This paper introduces (Section 2) a definition -- more exactly, a one parameter family of definitions -- of unimodality for random objects taking values in a finite dimensional vector space. The possibility of a more general range space is briefly mentioned, and some special attention is given to the one dimensional case and its connections with ordinary unimodality (also Section 2). Two characterizations, or alternative definitions, of α-unimodality are given (Section 3). One of these

1. The work of Savage on this paper was supported in part by the Army, Navy Air Force, and NASA under Office of Naval Research Contract Nonr-609(52) and in part by the John Simon Guggenheim Foundation.

is an extension of Khintchine's theorem to α-unimodality. The other is related to an inequality discovered by Anderson (1955, pp. 170-171) for a type of unimodality stricter than n-unimodality for an n-dimensional vector space.

In more than one dimension, the distribution of an α-unimodal vector can be completely singular, but also it can be absolutely continuous. The densities of absolutely continuous α-unimodal random vectors are characterized (Section 4). The notion of α-unimodality permits a little to be salvaged from the known disaster that sums of real, independent, unimodal random numbers need not be unimodal (Section 5).

We thank David Freedman and L.A. Shepp for their friently help and shall mention instances of it as they arise.

2. Preliminaries.

Ordinarily, a real random variable Z is said to be unimodal with a mode at x if its distribution function is convex to the left and concave to the right of x. This, of course, implies the existence of a density, except possibly at x, which can be taken to be nondecreasing to the left and nonincreasing to the right of x.

According to an easy calculation, for such a Z, $tE(f(t(Z-x))$ is nondecreasing in t for $t > 0$ for every bounded, nonnegative, Borel measurable f; and in consequence of the later Theorem 3, the condition is also sufficient for Z to be unimodal with a mode at x. This motivates a more general definition of unimodality.

Definition 1. A random variable Z with values in a vector space V of dimension n is α-unimodal (about 0) iff

$$S(t;\ \alpha,f,Z) =_{df} t^{\alpha}E(f(tZ))$$

is nondecreasing in t for $t > 0$ for every bounded, nonnegative, Borel measurable f defined on V.

Ordinary unimodality with a mode at 0 occurs when $\alpha = 1$ and $n = 1$. To insure α-unimodality, it evidently suffices to require $S(t;\ \alpha,f,Z)$ to be nondecreasing for an appropriate subclass of f's such as C^{∞} functions with compact support or even a countable, dense subset of them.

In defining α-unimodality, we have formally taken V to be a finite dimensional vector space, because that is the possibility of principle interest to us, and because some of the topics to be discussed are genuinely vectorial. However, the definition and many conclusions about it are immediately seen to extend to a much more general sort of V. Namely, let V consist of a direct product of any measurable space with the positive reals, together with one additional point, written "0". If $t > 0$, then $t \times (\theta,r)$ is to be (θ,tr); and $0 \times (\theta,r) = t \times 0 = 0$. The possibility of so generalizing V is mentioned not for generality's sake but to underline that α-unimodality is not a thoroughly vectorial, or affine, concept. This contrasts with a definition (to be mentioned again) that calls a vector valued Z unimodal iff the density of Z with respect to some translation invariant measure has convex level sets.

For one elegant proof of the result of Khintchine mentioned in the previous section due to L.A. Shepp see (Feller 1966, pp. 155-6). Another proof, which we learned largely from Shepp follows, with the "only if" implication proved first.

Assume for the moment that F is twice continuously differentiable and that F' $(=f)$ has compact support. It is easily verified that $-xf'(x)$ is nonnegative. Integration by parts shows that it is a density, and therefore that

$$G(x) = F(x) - xf(x)$$

is a distribution function. (It serves as Φ in Isii's criterion.) Let X be distributed according to G and calculate the distribution of UX thus.

$$P(UX \leq z) = \int_0^1 P(uX \leq z)du$$

$$= \int_0^1 G(z/u)du$$

$$= \int_0^1 \frac{d}{du} uF(x/u)du$$

$$= F(x) .$$

In view of the fact that limits of unimodal distributions are unimodal, a transparent approximation argument completes this half of the proof.

To see the reverse implication, let $Z \sim UX$, with U and X having the prescribed properties. Let F_n be a sequence of purely discrete distribution functions converging to the distribution function of X. Then the distribution of Z is seen as a limit of unimodal distributions.

From another point of view, the result is an immediate consequence of the integral formulation of the Krein-Milman theorem (Phelps 1966, p. 6). The set of probabilities on $[-\infty,\infty]$ whose distribution functions are convex on $(-\infty,0)$ and concave on $(0,\infty)$ is convex and weakly compact. Its extreme points are easily seen to be the closed set consisting of the uniform distributions with one endpoint at the origin together with point masses at 0, $+\infty$, and $-\infty$, whence Khintchine's theorem follows once more.

Return now to the more general exploration of α-unimodality. Taking f to be the constant 1, you see that no Z is α-unimodal for $\alpha < 0$. And taking f to be the indicator of a neighborhood of 0, you see that Z is 0-unimodal iff $Z = 0$ with probability 1. From now on, let $\alpha > 0$ be understood unless exception is explicitly made for $\alpha = 0$. An example of a random variable that is not α-unimodal for any α and cannot be translated so as to be is one whose distribution is concentrated at exactly two points.

<u>Lemma</u> 1. The set $\{\alpha\colon Z \text{ is } \alpha\text{-unimodal}\}$ is either vacuous or of the form $[\beta, \infty)$ for some $\beta \geq 0$. All these cases are possible.

Proof. Obviously for $\alpha > \alpha' \geq 0$, α'-unimodality implies α-unimodality. The converse is false, and there are random variables for every $\beta > 0$, as is shown by the following class of examples suggested by Khintchine's theorem. For $\beta > 0$ let $Z = U^{1/\beta}x$, where U is uniformly distributed on $[0,1]$ and x is a nonzero constant vector. In this case,

$$S(t; \alpha, f, Z) = t^{\alpha}E(f(tU^{1/\beta}x))$$

$$= t^{\alpha}\int_0^1 f(tu^{1/\beta}x)du = t^{\alpha-\beta}\int_0^{t^\beta} f(v^{1/\beta}x)dv$$

$$= \beta t^{\alpha-\beta}\int_0^t w^{\beta-1}f(wx)dw \ .$$

This is nondecreasing in t for all nonnegative f iff $\alpha \geq \beta$, as consideration of f with bounded carrier shows.

The topological closure of $\{\alpha: Z \text{ is } \alpha\text{-unimodal}\}$ is evident from the definition of S. ◇

For fixed β, the example ranges with x over the extreme points of the set of β-unimodal distributions, as later paragraphs will make clear.

By replacing t^α in Definition 1 by a more general nondecreasing function $\Phi(t)$ and defining Φ-unimodality in the obvious way, can the notion of α-unimodality be generalized? Not really. For X is Φ-unimodal iff ρX is Φ-unimodal for every real ρ. Assume $E(f(t\rho X))$ can be differentiated with respect to t under the integral sign; then set $\rho = 1/t$. This shows that when Φ is differentiable, Φ-unimodality amounts to α-unimodality, where

$$\alpha = \inf_t \frac{1}{t\Phi(t)} \frac{d\Phi(t)}{dt} . \tag{2.1}$$

When Φ is not differentiable, slightly more delicate reasoning shows that (2.1) still holds, with $\Phi(t)$ replaced by $\Phi(t+0)$, and $d\Phi(t)/dt$ by the lower right Dini derivate of Φ at t.

The following lemma, whose proof is an almost immediate consequence of Definition 1, will be important in one characterization of α-unimodality.

Lemma 2. If Z is 1-dimensional, and $\Pr(Z \geq 0) = 1$, then Z is α-unimodal for positive α iff Z^α is 1-unimodal, that is, unimodal about 0.

This lemma can be given a more pat expression thus. For all nonzero α, Z^{α} is unimodal iff (signum α)$S(t; \alpha, f, Z)$ is nondecreasing in t. For negative α, this fact suggests a definition of α-unimodality at ∞, which will not be further pursued here.

Incidentally, according to the lemma together with Khintchine's theorem, the density of a one-dimensional α-unimodal random variable for $\alpha < 1$ is unbounded near 0.

3. Characterizations of α-unimodality.

Definition 2. If $Z \sim U^{1/\alpha}X$, where U and X are independent, U is uniform on [0,1], and X has its values in V, then Z is an α-star variable on V and the distribution of X is a point distribution for it.

As motivation for this terminology, consider an X concentrated at a finite number of points. The α-star variables will ultimately be identified as the α-unimodal distributions. Each α-star distribution, as such, has but one point distribution, as will also be shown. But each α-star distribution is also α'-star for all α' in $[\beta, \infty)$ for some $\beta \leq \alpha$; for each such α' there is (unless $\beta = 0$) a different point distribution.

If Z is an α-star distribution with point distribution that of X, then

$$(3.1) \qquad S(t;\ \alpha,f,Z) = t^{\alpha}E(f(tU^{1/\alpha}X))$$

$$= \int_0^1 E(f(t^{\alpha}u)^{1/\alpha}X)t^{\alpha}du$$

$$= \int_0^{t^{\alpha}} E(f(v^{1/\alpha}X))dv$$

$$= \alpha \int_0^t w^{\alpha-1}E(f(wX))dw ,$$

which is indeed nondecreasing in t for $t > 0$. The proof of the next result is now obvious.

<u>Lemma</u> 3. If Z is α-star it is also α-unimodal, and S is not only nondecreasing in t $(t > 0)$ but also absolutely continuous; and

$$(3.2) \qquad \frac{d}{dt} S(t;\ \alpha,f,Z) = \alpha t^{\alpha-1}E(f(tX))$$

almost everywhere on $(0, \infty)$. Further, if f is continuous, (3.2) holds for all positive t.

<u>Definition</u> 3. If the real function f on V has a continuous gradient and is constant off some compact set, then f is said to be smooth, or $f \in \mathcal{S}$.

According to an easy application of dominated convergence,

$$\frac{d}{dt} E(f(tZ)) = E(\frac{d}{dt} f(tZ))$$

for every f in $\mathcal{S}$ and every random variable Z on V. (What is used here is only continuity of $df(tw)/dt$ in t and its boundedness in w for each t.)

Definition 4. The not necessarily nonnegative Borel measure Φ on V is an α-partner of the random variable Z on V iff

$$(3.3) \qquad \int f(x)\, d\Phi(x) = \alpha^{-1} \frac{d}{dt} S(t;\, \alpha, f, Z)\Big|_{t=1}$$

for every f in $\mathcal{S}$.

Theorem 1. Each Z has at most one α-partner. If Z is α-unimodal, it does have an α-partner, and the α-partner is a probability measure. If Z is α-star with point distribution Φ, then Φ is also the α-partner of Z.

Proof. According to an easy approximation based on the left hand side of (3.3), any two α-partners of a random variable Z must assign the same measure to parallelepipeds, and hence they must be equal.

Now an α-unimodal Z will be shown to have an α-partner that is a probability. To begin with, the right hand side of (3.3) defines a linear functional L on the vector space $\mathcal{S}$. According to the definition of α-unimodality, L is nonnegative, that is, $f \in \mathcal{S}$ and $f \geq 0$ together imply $L(f) \geq 0$. Moreover, $L(1) = 1$. According to Krein's extension theorem (Hewitt and Stromberg 1965, pp. 219-220), L admits extension to a nonnegative linear functional on the space of continuous functions on the one point compactification of V. Hence the Riesz representation theorem insures the existence of an α-partner that is a probability

Lemma 3 delivers the required conclusion about α-star variables. $\diamond$

In applying Theorem 1 it will be helpful to notice another version of (3.3), namely.

(3.4) $$\int f(tx)d\Phi(x) = \alpha^{-1}t^{1-\alpha} \frac{d}{dt} S(t;\ \alpha,f,Z)$$

for all smooth f and positive t.

Theorem 2. If Z is α-unimodal, it is also α-star with its α-partner Φ as its point distribution.

Proof. Let X be distributed according to Φ. Let $Z^* = U^{1/\alpha}X$ where U is independent of X and uniform on [0,1], and compare $S(t;\ \alpha,f,Z)$ with $S(t;\ \alpha,f,Z^*)$ for smooth f. According to (3.2) and (3.4)

(3.5) $$\begin{aligned}\frac{d}{dt} S(t;\ \alpha,f,Z^*) &= \alpha t^{\alpha-1}E(f(tX)) \\ &= \alpha t^{\alpha-1}\alpha^{-1}t^{1-\alpha}\frac{d}{dt} S(t;\ \alpha,f,Z) \\ &= \frac{d}{dt}\ S(t;\ \alpha,f,Z)\ .\end{aligned}$$

Therefore, for smooth f,

$$E(f(Z^*)) = S(1;\ \alpha,f,Z^*) = S(1;\ \alpha,f,Z) = E(f(Z))\ .$$

And so $Z \sim Z^*$. ◇

Corollary 1. If Z is α-unimodal for some α, then $E(f(tZ))$ is absolutely continuous in t for every bounded, real, Borel measurable f. If f is also continuous, then $E(f(tZ))$ is continuously differentiable for $t > 0$. If f is nonnegative, then

(3.6) $$\alpha E(f(tZ)) + t \frac{d}{dt} E(f(tZ)) \geq 0$$

whenever the derivative exists. And (3.6) for nonnegative, smooth f characterizes α-unimodality.

Proof. Immediate from Theorem 2, Lemma 3, and Definition 1. ◇

Constant Z shows how important unimodality is for the differentiability conclusion.

In Khintchine's original work, something close to Corollary 1 was the basic tool in proving his special case of Theorem 2. See (Gnedenko and Kolmogorov 1954, Section 32 as corrected by K. L.Chung in Appendix II).

An α-star vector $Z = U^{1/\alpha}X$ is also β-star if $\beta > \alpha$. As such, the point distribution Φ_β is now easily seen to be characterized by the following calculation, applicable to smooth f.

$$\int f(x)d\Phi_\beta(x) = \beta^{-1} \frac{d}{dt} t^\beta E(f(tZ))\Big|_{t=1}$$

$$= \frac{\alpha}{\beta} \frac{d}{dt} t^{\beta-\alpha} \int_0^t w^{\alpha-1} E(f(wX))dw\Big|_{t=1}$$

$$= \frac{\alpha}{\beta} E(f(X)) + (1 - \frac{\alpha}{\beta})E(f(Z)) .$$

So Φ_β is the weighted mixture of the distributions of X and Z with weights α/β and $1-(\alpha/\beta)$. When is $\Phi_\beta = \Phi$? Exactly when X and $U^{1/\alpha}X$ have the same distribution. But, as is rather easily seen that occurs iff $X = 0$ with probability 1.

The next corollary, however obvious, is historically important. It extends the most popular version of Khintchine's theorem (see Gnedenko and Kolmogorov 1954, p. 160 and Feller 1966, p. 501, which requires an obvious correction).

<u>Corollary</u> 2. The characteristic function φ is that of an α-unimodal random vector iff it is of the form

$$\varphi(\underset{\sim}{\delta}) = \int_0^1 \psi(u^{1/\alpha}\underset{\sim}{\delta})du$$

$$= \alpha\int_0^1 v^{\alpha-1}\,\psi(v\underset{\sim}{\delta})dv$$

where ψ is also a characteristic function.

The behavior of S for f's that decrease along rays leads to a characterization of α-unimodality that generalizes the usual definition of unimodality in one dimension. The basic example of such an f is the indicator of a "star shaped" set.

<u>Definition</u> 5. A real-valued function f on V is <u>star down</u> iff, for each x in V, f(tx) is nonincreasing in t in $[0, \infty)$, and f is bounded and nonegative.

For an α-unimodal Z and a star down f, consider $Q(t) = E(f(tZ))$ for $t \geq 0$. Evidently Q is nonincreasing. But $t^\alpha Q = S$ is nondecreasing. Therefore, except for the trivial possibility that $Q \equiv 0$ for $t > 0$,

$$\epsilon^\alpha \leq Q(t)/Q(t\epsilon) \leq 1$$

for all t in $[0,\infty)$ and all ϵ in $[0,1]$. Or, for $0 \leq \delta < t$,

$$0 \leq Q(t-\delta) - Q(t)$$

$$= Q(t)\{[Q(\tfrac{t-\delta}{t})t)/Q(t)] - 1\}$$

$$\leq Q(t)\{(1 - \tfrac{\delta}{t})^{-\alpha} - 1\}$$

$$= Q(t)\,\frac{\alpha\delta}{t} + Q(t)o((\tfrac{\delta}{t})^2) .$$

In particular, the above argument shows without recourse to Theorem 2 that Q and S are absolutely continuous.

Since $Q(t\epsilon) - Q(t) = E\{f(t\epsilon Z) - f(tZ)\}$, $t^{\alpha}\{Q(t\epsilon) - Q(t)\}$ is nondecreasing in t for fixed ϵ in $[0,1]$. Let $K(s) = Q(s^{-1/\alpha})$, so that $Q(t) = K(t^{-\alpha})$. K is continuous -- in fact absolutely continuous -- and

$$\text{(3.7)} \qquad \frac{K(bs) - K(s)}{s}$$

is nonincreasing in s for fixed b in $[1,\infty)$. It will be argued now, in more than one way, that K is concave.

Since K is absolutely continuous, it is concave if its derivative K' is nonincreasing where defined. But this it is according to the monotony of (3.7), or the <u>preconcavity</u> of K, as we shall momentarily call it. For if K is differentiable at s, then

$$\frac{K(bs) - K(s)}{s} = (b-1)K'(s) + o(b-1) .$$

Concavity quite easily implies preconcavity, and it would be somewhat interesting to know to what extent the converse is true. Certainly, absolute continuity is an unnecessarily strong supplementary condition.

Perhaps none at all is needed, and measurability seems very likely to be adequate. We have not resolved these conjectures but can point out that preconcavity, even much weakened, implies concavity if K is continuous. To see this, specialize preconcavity to conclude that, for $b > 1$,

$$K(bs) - K(s) \geq \frac{1}{b} \{K(b^2 s) - K(bs)\} .$$

Therefore,

$$K(bs) \geq \frac{K(b^2 s) + bK(s)}{1+b} .$$

That is, there is a point on each chord of K, in fact that geometric mean of the endpoints of its base, where the chord does not exceed the function. For a continuous function, this is known to imply concavity (Hardy, Littlewood, and Polya 1934, p. 73). As David Freedman remarked to us, a preconvex K that is monotone -- as is the K in our particular application -- is automatically continuous.

Several proofs have now been given for the next theorem.

Theorem 3. If f is star down and Z is α-unimodal, then $E(f(s^{-1/\alpha}Z))$ is concave in s.

As was mentioned in Section 1, Theorem 3 is related to an inequality of Anderson (1955, p. 172). He defines Z in n dimensions to be unimodal iff Z has a density h that is nonincreasing along rays from the origin and has convex contours. So, as is easily verified, his unimodality implies n-unimodality but not conversely. Now let $h(u) = h(-u)$ for all u, and let N be a convex neighborhood of the origin for which $u \in N$ iff $-u \in N$. Anderson proves that as N slides out a ray from the origin, the probability of N decreases. This conclusion is in general false for n-unimodality, yet his and our conclusions are similar. Both refer to diminution of probability as certain sets are removed from the origin, in his case as spheres are translated and in ours as shells are dilated.

In view of Lemma 2, the condition of Theorem 3 characterizes α-unimodality in case V is 1-dimensional, and in the particular case $\alpha = 1$ it is comforting to realize the equivalence of Khintchine's definition and ours. This characterization extends to arbitrary V. In preparation for proving that, we present an argument, which incidentally leads to a new proof of Theorem 2, due to David Freedman.

We assume for clarity that $\Pr(Z = 0) = 0$; the details thus omitted would be routine to supply. Introduce an arbitrary Euclidean norm $\| \|$ on V and represent Z as DL, where $D = Z/\|Z\|$ is the "direction" of Z and $L = \|Z\|$ is its length.

Theorem 4. Z is α-unimodal iff L given D is effectively 1-dimensional α-unimodal with probability 1, that is, for $s < t$ and nonnegative g, defined on $[0, \infty)$,

$$s^{\alpha}E(g(sL)|D) \leq t^{\alpha}E(g(tL)|D) \tag{3.8}$$

with probability 1. And it is enough that (3.8) hold for all smooth g.

Proof. The "if" implication. For a smooth f, use a regular conditional probability given D to compute thus.

$$E(t^\alpha f(tZ) - s^\alpha f(sZ))$$

$$= E\{E(t^\alpha f(DtL) - s^\alpha f(DsL) \mid D)\} > 0 \ .$$

The "only if" implication. Suppose (3.8) fails on a set of positive measure with indicator $h(D)$. Let $f(Z) = h(D)g(L)$.

$$E(t^\alpha f(tZ) - s^\alpha f(sZ))$$

$$= E\{h(D)E(t^\alpha g(tL) - s^\alpha g(sL) \mid D)\}$$

$$< 0 \ . \qquad \diamond$$

By means of Theorem 4, the assertion in Theorem 2 that an α-unimodal Z is necessarily α-star can be reduced to Khintchine's theorem about 1-unimodality on the line, of which it is of course a generalization. In the first place, calculating with a regular conditional probability given D, (3.8) implies that, for almost all D, $t^\alpha E(g(tL) \mid D)$ is nondecreasing for all smooth g. That implies that L is 1-unimodal for almost all D. So according to Khintchine's theorem, L given D is almost surely distributed like UR where U and R are independent and the distribution of R is a function of D. A distribution for X is well defined by the conditions that $X/\|X\| \sim D$ and $\|X\|$ given D is distributed like R given D. This variable does what is required.

Now the converse of Theorem 3 will be established, leaving to the reader the slight additional complication that arises if $\Pr(Z = 0)$ is not assumed to be 0.

Theorem 5. If $E(f(t^{-1/\alpha}Z))$ is concave for each star-down f, then Z is α-unimodal.

Proof. Let g_x be the indicator of $[0,x]$ and h any function of D. Then f, where $f(Z) = h(D)g_x(L)$, is star down. The concavity of $E(f(t^{-1/\alpha}Z)) = E(h(D)\ Pr(t^{-1/\alpha}L \leq x|D))$ shows, for the regular version, that the distribution function of L^α is concave, that is unimodal at 0, for almost all D. According to Lemma 2, these conditional distributions are therefore α-unimodal. Theorem 4 now applies to show that Z is α-unimodal. ◇

4. Densities.

In 1 dimension, an α-unimodal variable has a density except possibly at 0, as previous discussion has made clear. But in higher dimensions, the existence of a density is atypical in that many point distributions -- including all finite ones -- result in singular α-unimodal distributions. Suppose, though, that Z is α-unimodal and does have a density ρ, what can be said about ρ ?

Theorem 6. A probability density ρ on V is that of an α-unimodal Z iff for all s, t with $0 < s < t$

$$(4.1) \qquad s^{n-\alpha}\rho(sz) \geq t^{n-\alpha}\rho(tz)$$

for almost all z with respect to a Lebesgue measure.

Proof. The "if" part. Evident, because

$$(4.2) \qquad t^{\alpha}E(f(tZ)) = t^{\alpha}\int f(tz)\ \rho(z)dz$$

$$= t^{\alpha-n}\int f(w)\ \rho(w/t)dw\ .$$

The "only if" part. Let f be the indicator of the set in V where (4.1) fails and show by applying (4.2) to f that Z is not α-unimodal. $\diamond$

For any ρ there is a smoothed version ρ^* such that, for any s and t for which (4.1) holds almost everywhere for ρ it holds everywhere for ρ^*. Indeed consider,

$$\rho^*(w) = \limsup_k \frac{\int_{\|kz\| \leq 1} \rho(w-z)dz}{\int_{\|kz\| \leq 1} dz} . \tag{4.3}$$

According to a known fact (Saks 1964, p. 118), the lim sup in (4.3) is almost everywhere actually a limit and equal to $\rho(w)$. Thus ρ^* is a version of ρ. If ρ satisfies (4.1) almost everywhere, then clearly ρ^* satisfies it everywhere. Also ρ^*, being a lim sup of continuous functions, is Borel measurable. Finally, if wherever ρ^* is infinite, except at $z = 0$, it is changed to 0 all of the properties mentioned will persist.

Corollary 3. The random vector Z is a α-unimodal with an absolutely continuous distribution iff there is a Borel measurable version ρ^* of its density for which $s^{n-\alpha} \rho^*(sz)$ is nonincreasing in s for each fixed $z \in V$, $z \neq 0$.

If σ is a probability density on V, the corresponding α-unimodal distribution with σ as the density of its point distribution has density ρ, where

$$\rho(z) = \int_0^1 u^{-n/\alpha} \sigma(u^{-1/\alpha}z)du = \alpha \int_1^\infty v^{n-\alpha-1} \sigma(vz)dv .$$

In particular,

$$\rho(tz) = \alpha \int_1^{\infty} v^{n-\alpha-1} \sigma(tvz) dv$$

$$= \alpha t^{\alpha-n} \int_t^{\infty} w^{n-\alpha-1} \sigma(wz) dw \ .$$

So, at least if σ is continuous,

$$\frac{d}{dt} t^{n-\alpha} \rho(tz) = -\alpha t^{n-\alpha-1} \sigma(tz) \ , \tag{4.4}$$

except possibly at $z = 0$. In 1 dimension, if σ is continuous then ρ is continuously differentiable, except possibly at 0. In 2 or more dimensions, even if σ is continuous, ρ might have directional derivatives only along the ray determined by z. Of course, under sufficient regularity, (4.4) becomes

$$-\alpha\sigma(z) = (n - \alpha)\rho(z) + z \cdot \nabla\rho(z) \ ,$$

a 1-dimensional special case of which has already been mentioned.

5. Sums.

The convolution of symmetric 1-dimensional, 1-unimodal distributions is also 1-unimodal (Wintner 1938, p. 30). The conclusion is not true in general if the assumption of symmetry is dropped; see Chung's Appendix II of (Gnedenko and Kolmogorov, 1954), also (Feller 1966, p. 164) and (Ibragimov 1956, p. 255). Two facts bear on possible extensions of the first result. First, since Anderson's conclusion fails for n-dimensional, absolutely continuous, symmetric, n-unimodal variables, Wintner's result does not extend to n-dimensional, n-unimodal variables. Second, the uniform distribution on the unit square in 2 dimensions can be viewed as the convolution of a pair of symmetric, 2-dimensional, 1-unimodal distributions, but this distribution is, as Corollary 3 shows, only 2-unimodal and cannot be translated so as to be α-unimodal for any $\alpha < 2$. The next, and final, theorem offers some solace for the disappointment of Chung's discovery.

<u>Theorem</u> 7. If Z and Z' are independent and are α and α' unimodal in V and V', then (Z,Z') is $\alpha + \alpha'$ unimodal in $V \times V'$. If further $V = V'$, then $Z + Z'$ is $(\alpha + \alpha')$-unimodal. No lower index of unimodality can be asserted, even for a new origin.

Proof. Regard Z and Z' without loss as star variables and compute thus.

$$E(f(tZ,\ tZ')) = \int_0^1\int_0^1 E(f(tu^{1/\alpha}X_1,\ tv^{1/\alpha'}X_2))dudv$$

$$= t^{-(\alpha+\alpha')}\int_0^{t^\alpha}\int_0^{t^{\alpha'}} E(f(u^{1/\alpha}X_1,\ v^{1/\alpha'}X_2))dudv\ .$$

The assertion about the sum follows from specializing f to a function of $Z + Z'$. Unimprovability in 2 and therefore higher dimensions has been discussed; the result in 1 dimension is easily seen by varying the two point support of the point distribution of Chung's example. ◇

It is well known that in 1 dimension the sum of two independent, 1-unimodal, random numbers, one of which has a symmetric distribution, can fail to be 1-unimodal for any origin. Perhaps, however, in this context the index of Theorem 7 can be improved. We have not explored that question.

References

Anderson, T.W. (1955) The integral of a symmetric unimodal function. Proc. Amer. Math. Soc. 6, 170-176.

Blackwell, David (1955) On a class of probability spaces. Proc. Third Berkeley Symp. Math. Statist. Prob. 2, 1-6. U. of Calif. Press.

Feller, William (1966) An Introduction to Probability Theory and its Applications Vol. II. Wiley, New York

Gnedenko, B.V. and Kolmogorov, A.N. (1954) Limit Distributions for Sums of Independent Random Variables (translated and annotated by K. L. Chung and with an Appendix by J. L. Doob). Addison-Wesley, Cambridge, Mass.

Hardy, G.H., Littlewood, J.E., and Polya, G. (1934) Inequalities. Cambridge at the U. Press.

Hewitt, Edwin and Stromberg, Karl (1965) Real and Abstract Analysis. Springer, New York.

Ibragimov, I.A. (1956) On the composition of unimodal distributions. Theory Prob. Appl. (Translation) 1, 255-260.

Isii, K. (1958) Note on a characterization of unimodal distributions. Ann. Inst. Stat. Math. IX, 173-184.

Khintchine, A. Ya. (1938) On unimodal distributions (in Russian). Izv. Nauchno-Issled. Inst. Mat. Meh. Tomsk. Gos. Univ. 2, 1-7.

Phelps, Robert R. (1966) Lectures on Choquet's Theorem. Van Nostrand, Princeton.

Saks, S. (1964) Theory of the Integral 2nd Revised Ed. (translated by L.C. Young). Dover, New York.

Wintner, Aurel (1938) Asymptotic Distributions and Infinite Convolutions. Edwards Brothers, Ann Arbor.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Statistics Department Stanford University Stanford, California	2b. GROUP

3. REPORT TITLE

A GENERALIZED UNIMODALITY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

TECHNICAL REPORT

5. AUTHOR(S) (Last name, first name, initial)

OLSHEN, RICHARD A. and SAVAGE, LEONARD J.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
April 1, 1969	21	1f

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
Nonr 225(52)	TECHNICAL REPORT NO. 143
b. PROJECT NO. NR 342-022 c. d.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Logistics and Mathematical Sciences Branch Office of Naval Research Washington, D.C.

13. ABSTRACT

This paper introduces (Section 2) a definition -- more exactly, a one parameter family of definitions -- of unimodality for random objects taking values in a finite dimensional vector space. The possibility of a more general range space is briefly mentioned, and some special attention is given to the one dimensional case and its connections with ordinary unimodality (also Section 2). Two characterizations, or alternative definitions, of α-unimodality are given (Section 3). One of these is an extension of Khintchine's theorem to α-unimodality. The other is related to an inequality discovered by Anderson for a type of unimodality stricter than n-unimodality for an n-dimensional vector space.

In more than one dimension, the distribution of an α-unimodal vector can be completely singular, but also it can be absolutely continuous. The densities of absolutely continuous α-unimodal random vectors are characterized (Section 4). The notion of α-unimodality permits a little to be salvaged from the known disaster that sums of real, independent, unimodal random numbers need not be unimodal (Section 5).

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
α-unimodality						
Star variable						
Point distribution						
Star down						
Preconcavity						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
__________________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
__________________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
__________________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.