Final Report of the
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March 11, 1970

Part I

TRANSMISSION CHARACTERISTICS OF SPLIT-PHASE PCM CODES

Prepared
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ABSTRACT

Pulse code modulation (PCM) telemetry utilizes a series of binary digits (ones and zeros) to describe the analog level of a sample taken from a data channel. The bi-phase-level or split-phase, PCM code utilizes the binary states "10" to represent a one and the binary states "01" to represent a zero. The terms "bi-phase-level" and "split-phase" apply to the code structure of the PCM modulating sequence and not to the particular modulation scheme used. A split-phase PCM code may be used to modulate the amplitude, phase, or frequency of a carrier signal.

The present work is concerned with the determination of certain transmission characteristics of a carrier which is amplitude-shift-keyed (ASK), phase-shift-keyed (PSK), or frequency-shift-keyed (FSK) by a split-phase PCM code. Specifically, the power spectral density is determined for each modulation scheme. All calculations assume a random bit pattern with equally likely ones and zeros.

INTRODUCTION

The bi-phase-level, or split-phase, PCM code utilizes the binary states "10" to represent a one and the binary

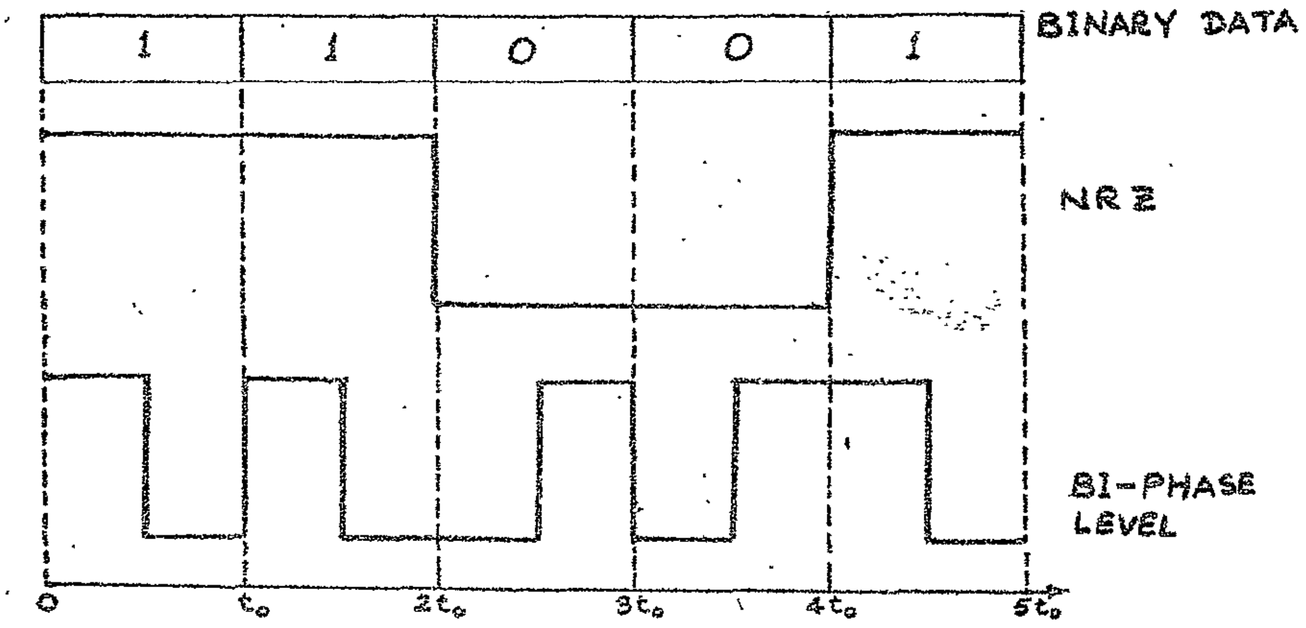

FIG 1. PCM CODE FORMATS

states "01" to represent a zero. One advantage offered by split-phase coding over other types of PCM code formats (such as NRZ and RZ) is that the transition density for a random bit pattern is higher for split-phase than for the other formats. At least one binary level transition will occur during each bit period of a split-phase code, whereas it is possible for the other code formats to have long groups of consecutive "ones" or "zeros". The greater bit transition density for the split-phase format generally allows more efficient bit synchronization (recovery of the bit rate clock frequency) to be maintained at the receiver. Figure 1 shows the non-return to zero (NRZ) and split-phase (Bi-Phase Level) PCM code formats.

A Split-phase PCM Code can be used to modulate the amplitude, phase or frequency of a carrier signal. The modulation may or may not be phase coherent. The noncoherent case is by far the simplest to analyze, because the assumption can be made that the PCM code is statistically independent of the carrier. Since, for example, modulation by the code is a multiplcative process, then this results in multiplication of autocorrelation and, therefore in convolution of power spectral densities. As the individual power spectral densities of the code are easily obtained, the determination of the power spectral density of the modulated carrier is straight forward.

If the code and the carrier are coherent, then the assumption of statistical independence is no longer valid, and multiplication of the time signals no longer result in multiplication of the autocorrelation function. Thus, determination of the coherently modulated carrier then becomes a formidable task.

The primary objective of this report is to complete this task for each type of modulation (ASK, PSK, and FSK). In addition to determination of the power spectral densities for the coherent modulation, the report presents the results for noncoherent amplitude, phase, and frequency modulation by Split-phase PCM code.

I. General

Suppose $y(t)$ is our modulated PCM signal. We assume that $y(t)$ only takes a finite number of states, $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_a$. The signals corresponding to these states are $h_1(t)$, $h_2(t)$, $h_3(t) \ldots h_a(t)$ respectively, which are defined in the interval $(0, t_o)$.

Define: p_i = The probability that state ε_i occurs in any interval.

$P_{ik}^{(n)}$ = the probability that state ε_k occurs n periods after the state ε_i occurs.

t_o = the period of the pulse

and

$$\phi(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)y(t+\tau)dt \tag{1}$$

Since $y(t)$ is bounded, the limit of (1) exists if and only if

$$\lim_{N\to\infty} \frac{1}{2Nt_o} \int_{-Nt_o}^{Nt_o} y(t)y(t+\tau)dt \qquad \text{exists.}$$

This can be proven as follows:

Let T in equation (1) be equal to Nt_o+t_1, where $o \le t_1 < t_o$,

i.e.

$$T = Nt_o+t_1 \tag{2}$$

Substituting (2) into (1), we obtain

$$\phi(\tau) = \lim_{N\to\infty} \frac{1}{2(Nt_o+t_1)} \int_{-(Nt_o+t_1)}^{(Nt_o+t_1)} y(t)y(t+\tau)dt$$

$$= \lim_{N\to\infty} \frac{1}{2Nt_o} \int_{-(Nt_o+t_1)}^{(Nt_o+t_1)} y(t)y(t+\tau)dt$$

$$= \lim_{N\to\infty} \frac{1}{2Nt_o} [\int_{-(Nt_o+t_1)}^{-(Nt_o)} y(t)y(t+\tau)dt + \int_{-Nt_o}^{Nt_o} y(t)y(t+\tau)dt$$

$$+ \int_{Nt_o}^{Nt_o+t_1} y(t)y(t+\tau)dt]$$

Since $y(t)$ is bounded, then $|y(t)\ y(t+\tau)| \leq M$ where M is a constant

$$\therefore \quad \left|\lim_{N\to\infty} \frac{1}{2Nt_o} \left[\int_{-(Nt_o+t_1)}^{-Nt_o} y(t)y(t+\tau)dt + \int_{Nt_o}^{Nt_o+t_1} y(t)y(t+\tau)dt\right]\right|$$

$$\leq \lim_{N\to\infty} \frac{1}{2Nt_o} [M\cdot t_1 + M\cdot t_1] = \lim_{N\to\infty} \frac{1}{2Nt_o} \cdot 2M\ t_1 = 0$$

Therefore

$$\phi(\tau) = \lim_{N\to\infty} \frac{1}{2Nt_o} \int_{-Nt_o}^{Nt_o} y(t)\,(y(t+\tau)\ dt \tag{3}$$

and the proof is complete.

II. Power Spectral Density

We let $\tau = nt_o + t_1$ where n is an integer and $o \leq t_1 < t_o$. Then equation (3) becomes

$$\phi(nt_o+t_1) = \lim_{N\to\infty} \frac{1}{2Nt_o} \int_{-Nt_o}^{Nt_o} y(t)y(t+nt_o+t_1)dt$$

$$= \lim_{N\to\infty} \frac{1}{2Nt_o} [\int_{o}^{t_o} y(-Nt_o+t)y(-Nt_o+nt_o+t+t_1)dt$$

$$+ \int_o^{t_o} y(-(N-1)t_o+t)y(-(N-1)t_o+nt_o+t+t_1)dt$$

$$+ \dots + \int_o^{t_o} y((N-1)t_o+t)\; y((N-1)t_o+nt_o+t+t_1)dt]$$

or

$$\phi(nt_o+t_1) = \lim_{N\to\infty} \frac{1}{2Nt_o} \sum_{m=-N}^{N-1} \int_o^{t_o} y(mt_o+t)y[(m+n)t_o+t_1+t]dt \qquad (4)$$

Also, we define

$$D_i(m) = \begin{cases} 1 & \text{if the state during the interval } [mt_o, (m+1)t_o] \\ & \text{is } \varepsilon_1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore during the interval $[mt_o, (m+1)t_o]$, $y(t)$ becomes

$$y(mt_o+t) = \sum_{i=1}^{a} D_i(m)h_i(t) \qquad (5)$$

and $y(t + nt_o + t_1)$ becomes

$$y((m+n)t_o+t_1+t) = \sum_{k=1}^{a} D_k(m+n)\; h_k\; (t+t_1) \text{ when } o<t+t_1<to \qquad (6)$$

and

$$y[(m+n)t_o+t_1+t] = \sum_{k=1}^{a} D_k(m+n+1)\; h_k\; (t_1+t-t_o) \quad t_o \leq t_1+t<2t_o \qquad (7)$$

Using (5), (6) and (7) in equation (4), we obtain

$$\phi(nt_o+t_1) = \lim_{N\to\infty} \frac{1}{2Nt_o} \sum_{m=-N}^{N-1} \sum_{i=1}^{a} \sum_{k=1}^{a} [D_i(m)D_k(m+n)\int_o^{t_o-t_1} h_i(t)h_k(t+t_1)dt + D_i(m)D_k(m+n+1) \int_{t_o-t_1}^{t_o} h_i(t)h_k(t+t_1-t_o)dt] \qquad (8)$$

Since $h_i(t) = o$ for $o>t$, $t>to$

and $h_k(t+t_1) = o$ for $t>t_o-t_1$, $t<-t_1$

Therefore $h_i(t)h_k(t+t_1) = o$ for $o>t$, $t>t_o-t_1$

and

$$\int_{0}^{t_o-t_1} h_i(t)h_k(t+t_1)dt=\int_{-\infty}^{0} h_i(t)h_k(t+t_1)dt+\int_{0}^{t_o-t_1} h_i(t)h_k(t+t_1)d$$

$$+\int_{t_o-t_1}^{\infty} h_i(t)h_k(t+t_1)dt$$

The first and third integrals are equal to 0.

$$\int_{0}^{t_o-t_1} h_i(t)h_k(t+t_1)dt = \int_{-\infty}^{\infty} h_i(t)h_k(t+t_1)dt.$$

Similarly

$$\int_{t_o-t_1}^{t_o} h_i(t)h_k(t+t_1-t_o)dt = \int_{-\infty}^{\infty} h_i(t)h_k(t+t_1-t_o)dt$$

Now equation (8) can be written as

$$\phi(nt_o+t_1) = \lim_{N\to\infty} \frac{1}{2Nt_o} \sum_{M=-N}^{N-1} \sum_{i=1}^{a} \sum_{k=1}^{a} \Big[D_i(m)D_k(m+n)\int_{-\infty}^{+\infty} h_i(t)$$

$$h_k(t+t_1)dt + D_i(m)D_k(m+n+1)\int_{-\infty}^{+\infty} h_i(t)h_k(t+t_1-t_o)dt\Big]$$

$$= \frac{1}{t_o}\sum_{i=1}^{a}\sum_{k=1}^{a}\lim_{N\to\infty}\sum_{M=-N}^{N-1}\Big[\frac{D_i(m)D_k(m+n)}{2N}\int_{-\infty}^{+\infty} h_i(t)h_k(t+t_1)dt$$

$$+\frac{1}{t_o}\sum_{i=1}^{a}\sum_{k=1}^{a}\lim_{N\to\infty}\sum_{M=-N}^{N-1}\frac{D_i(m)D_k(m+n+1)}{2N}\int_{-\infty}^{+\infty} h_i(t)h_k(t+t_1-t_o)dt\Big] \tag{9}$$

As we can see, there is a finite number of terms of the form

$$\sum_{M=-N}^{N-1} \frac{D_i(m)D_k(m+n)}{2N}$$

and these coefficients are constants as far as the variables m and n are concerned.

For n = o

$$D_i(m)D_k(m+o) = \delta_{ik}D_i(m) \tag{10}$$

where δ_{ik} is defined as

$$\delta_{ik} = \begin{cases} 1 \text{ when } i = k \\ o \text{ otherwise} \end{cases}$$

Equation (10) is true because during the interval $[mt_o, (m+1)t_o]$ only one state can happen. Therefore for $i \neq k$, if $D_i(m)=1$ then $D_k(m)$ must be equal to zero; on the other hand if $D_k(m) = 1$ then $D_i(m)$ must be equal to zero. In other words $D_i(m)D_k(m)=o$ when $i \neq k$. When $i=k$ then $D_i(m)$ times $D_i(m)$ is itself.

The next thing we want to prove is

$$\lim_{N\to\infty} \sum_{m=-N}^{N-1} \frac{D_i(m)}{2N} = P_i$$

Note that $D_i(m)$ as we have defined is a random number, which the value 1 or 0. $D_i(m)$ and $D_i(n)$ are stastically independent for $m \neq n$. P_i is the probability that $D_i(m)$ is one.

Let the variance of $D_i(m)$ be σ^2_{im}, then

$$\sigma^2_{im} = E[D^2_{im}] - \{E[D_i(m)]\}^2 = P_i \cdot 1^2 + (1-P_i) \cdot 0^2 - P_i^{\,2}$$

$$= P_i - P_i^{\,2} = (1-P_i)P_i$$

Therefore

$$\frac{\sum\limits_{m=-N}^{N-1} \sigma^2_{im}}{(2N)^2} = \frac{1}{2N} \quad \frac{2N\ P_i(1-P_i)}{2N} = \frac{P_i(1-P_i)}{2N}$$

$$\lim_{N\to\infty} \frac{P_i(1-P_i)}{2N} = 0$$

Since the random variables are independent and

$$\lim_{N\to\infty} \sum_{n=-N}^{N-1} \frac{\sigma^2_{im}}{(2N)^2} = 0, \text{ then}$$

$$\lim_{N\to\infty} \sum_{m=-N}^{N-1} \frac{D_i(m)}{2N} = E[D_i(m)] = P_i$$

with probability equal to one. Or

$$\lim_{N\to\infty} \sum_{m=-N}^{N-1} \frac{\delta_{ik} P_i(m)}{2N} = P_i \delta_{ik} \tag{11}$$

with probability equal to one. With the same reasoning we can easily prove

$$\lim_{N\to\infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n)}{2N} = P_i P_{ik}^{(n)} \qquad \text{when } n>0 \tag{12}$$

$$\lim_{N\to\infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n+1)}{2N} = P_i P_{ik}^{(n+1)}$$

and

$$\lim_{N\to\infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n)}{2N} = P_k P_{ki}^{(-n)} \qquad \text{when } n<0 \tag{13}$$

$$\lim_{N\to\infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n+1)}{2N} = P_k P_{ki}^{-(n-1)}$$

with probability equal to one.

Again we define

$$a_{ik}(n) = \begin{cases} P_i \; \delta_{ik} & \text{if } n = 0 \\ P_i \; P_{ik}^{(n)} & \text{if } n > 0 \\ P_k \; P_{ik}^{(-n)} & \text{if } n < 0 \end{cases} \tag{14}$$

and

$$R(\tau) = R(nt_o+t_1) = \frac{1}{t_o} \sum_{i,k=1}^{a} \quad a_{ik}(n) \int_{-\infty}^{+\infty} h_i(t)h_k(t+t_1)dt$$

$$+ a_{ik}(n+1) \int_{-\infty}^{+\infty} h_i(t)h_k(t+t_1-t_o)dt \tag{15}$$

then we have

$\phi(\tau) = R(\tau)$ with probability equal to one.

Since

$$\phi(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)y(t+\tau)dt, \tag{16}$$

the power spectrum $S(\omega)$ of $y(t)$ would be just the Fourier transform of $R(nt_o+t)$.

We can separate $R(\tau)$ into a periodic component and an aperiodic component. The first part corresponds to jumps in $S(\omega)$, and the second part corresponds to a part of $S(\omega)$ which is the integral of a spectral density.

Note that the probability that ε_i occurs during a period is statistically independent of the other period. Therefore $P_{ik}^{(n)} = P_k$. We can write $R(nt_o+t)$ as

$$R(nt_o+t) = \frac{1}{t_o} \sum_{i,k=1}^{a} \quad P_i \; \delta_{ik} \; \delta(n) \int_{-\infty}^{+\infty} h_i(t_1)h_k(t+t_1)dt_1$$

$$+ \sum_{m=-\infty}^{-1} P_k P_i \delta(m+n) \int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1) dt_1$$

$$+ \sum_{m=1}^{\infty} P_i P_k \delta(m-n) \int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1) dt_1$$

$$+ P_i \delta_{ik} \delta(n+1) \int_{-\infty}^{+\infty} h_i(t_1) h_k(t_1+t-t_o) dt_1 \tag{17}$$

$$+ \sum_{m=-\infty}^{-2} P_k P_i \delta(m-n) \int_{-\infty}^{+\infty} h_i(t_1) h_k(t_1+t-t_o) dt_1$$

$$+ \sum_{m=o}^{\infty} P_i P_k \delta(m-n) \int_{+\infty}^{+\infty} h_i(t_1) h_k(t_1+t-t_o) dt_1]$$

The Periodic Part

To see what the periodic part of $R(nt_o + t)$ is, the easiest way is to let $n \to \infty$. Then $\tau = nt_o + t \to \infty$ and $R(\tau) \to R_p(\tau)$ where $R_p(\tau)$ is the periodic part of $R(\tau)$.

If we do this for equation (17), we obtain

$$R_p(nt_o+t) = \frac{1}{t_o} \sum_{i,k=1}^{a} [P_i P_k \int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1) dt_1$$

$$+ P_i P_k \int_{-\infty}^{+\infty} h_i(t_1) h_k(t_1+t-t_o) dt_1]$$

$$= \sum_{i,k=1}^{a} \frac{P_i P_k}{t_o} [\int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1) dt_1 + \int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1-t_o) dt_1] \tag{18}$$

with period t_o.

The power spectrum of a periodic function with period t_o is by definition

$$G_p(\omega) = \sum_{n=-\infty}^{n=\infty} a_n\, \delta(\omega - n\omega_o) \quad \text{where } \omega_o = \frac{2\pi}{t_o} \tag{19}$$

$$a_n = \frac{1}{t_o} \int_0^{t_o} f(t) e^{-jn\omega_o t}\, dt$$

Accordingly in our case

$$a_n = \sum_{i,k=1}^{a} \frac{P_i P_k}{t_o} \Big[\frac{1}{t_o} \int_0^{t_o} \int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1) e^{-j2\pi nt/t_o}\, dt_1 dt$$

$$+ \frac{1}{t_o} \int_0^{t_o} \int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1-t_o) e^{-j2\pi nt/t_o}\, dt_1 dt\Big] \tag{20}$$

By letting $t' = t - t_o$ or $t = t' + t_o$ then when $t = o$, $t' = -t_o$, when $t = t_o$, $t' = o$, and $dt' = dt$, the second integral of equation (20) becomes

$$= \frac{1}{t_o} \int_{-t_o}^{o} \int_{-\infty}^{+\infty} h_i(t_1) h_k(t'+t_1) e^{-j2\pi n(t'+t_o)/t_o}\, dt_1 dt'$$

$$= \frac{1}{t_o} \int_{-t_o}^{o} \int_{-\infty}^{+\infty} h_i(t_1) h_k(t'+t_1) e^{-j2\pi nt'/t_o}\, dt_1 dt'$$

Changing the dummy variable t' to t, we have

$$= \frac{1}{t_o} \int_{-t_o}^{o} \int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1) e^{-j2\pi nt/t_o}\, dt_1 dt$$

Therefore

$$a_n = \sum_{i,k=1}^{a} \frac{P_i P_k}{t_o} \Big[\frac{1}{t_o} \int_0^{t_o} \int_{-\infty}^{+\infty} h_i(t_1) h_k(t+t_1) e^{-j2\pi nt/t_o}\, dt_1 dt$$

$$+ \frac{1}{t_o} \int_{-to}^{o} \int_{-\infty}^{+\infty} h_i(t_1)h_k(t+t_1)e^{-j2\pi nt/t_o} \quad dt_1dt] \tag{21}$$

or

$$a_n = \sum_{i,k=1}^{a} \frac{P_iP_k}{t_o^{\,2}} \int_{-t_o}^{t_o} \int_{-\infty}^{+\infty} h_i(t_1)h_k(t+t_1)e^{-j2\pi nt/t_o} \quad dt_1dt \tag{22}$$

Since $h_i(t_1)h_k(t+t_1) = o$ for $o>t_1$, $t_1>t_o-t$

equation (22) can be written as

$$a_n = \sum_{i,k=1}^{a} \frac{P_iP_k}{t_o^{\,2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_i(t_1)h_k(t+t_1)e^{-j2\pi nt/t_o} \quad dt_1dt \tag{23}$$

Integrating with respect to t first and then with respect to t_1, equation (23) gives

$$a_n = \sum_{i,k=1}^{a} \frac{P_iP_k}{t_o^{\,2}} \; H_i^* \left(\frac{2\pi n}{t_o}\right) H_k \left(\frac{2\pi n}{t_o}\right)$$

$$= \frac{1}{t_o^{\,2}} \left| \sum_{i=1}^{a} P_i \; H_i \left(\frac{2\pi n}{t_o}\right) \right|^2 \tag{24}$$

where $H_i(\omega)$ and $H_k(\omega)$ are the fourier transform of $h_i(t)$ and $h_k(t)$ respectively. Therefore the power spectral density of the periodic part is

$$G_p(\omega) = \sum_{n=-\infty}^{n=\infty} \frac{1}{t_o^2} \left| \sum_{i=1}^{a} P_iH_i\left(\frac{2\pi n}{t_o}\right) \right|^2 \delta\left(\omega - \frac{2\pi n}{t_o}\right) \tag{25}$$

The Nonperiodic Part

To find the power spectral density of the nonperiodic part of the autocorrelation, $R_s(nt_o+t)$, we observe that

$$R_s(nt_o+t) = R(nt_o+t) - R_p(nt_o+t)$$

$$= \frac{1}{t_o} \sum_{i,k=1}^{a} \{[a_{ik}(n) - P_iP_k] \int_{-\infty}^{\infty} h_i(t_1)h_k(t+t_1)dt_1$$

$$+ [a_{ik}(n+1) - P_iP_k] \int_{-\infty}^{+\infty} h_i(t_1)h_k(t_1+t-t_o)dt_1\} \qquad (26)$$

The integral which gives the Fourier transform may be split into intervals to give

$$S_s(\omega) = \frac{1}{t_o} \sum_{i,k=1}^{a} \sum_{n=+\infty}^{n=+\infty} \{[a_{ik}(n) - P_iP_k]e^{-j\omega nt_o} \int_{-\infty}^{+\infty} \int_{o}^{t_o} \qquad (27)$$

$$h_i(t_1)h_k(t+t_1)e^{-j\omega t}dt\, dt_1$$

$$+[a_{ik}(n+1) - P_iP_k]e^{-j\omega nt_o} \int_{-\infty}^{+\infty} \int_{o}^{t_o} h_i(t_1)h_k(t+t_1-t_o)e^{-j\omega t}dt\, dt_1\}$$

By letting $n' = n+1$ and $t = t'+t_o$ in the second integral of equation (27), we have

$$=[a_{ik}(n') - P_iP_k]e^{-j\omega(n'-1)t_o} \int_{-\infty}^{+\infty} \int_{-t_o}^{o} h_i(t_1)h_k(t'+t_1)e^{-j\omega(t'+t_o)}$$

$$dt'\, dt_1$$

$$=[a_{ik}(n') - P_iP_k]e^{-j\omega n't_o} \int_{-\infty}^{+\infty} \int_{-t_o}^{o} h_i(t_1)h_k(t'+t_1)e^{-j\omega t'}dt'dt,$$

Changing the dummy variables n' to n and t' to t we have

$$=[a_{ik}(n) - P_iP_k]e^{-j\omega nt_o} \int_{-\infty}^{+\infty} \int_{-t_o}^{o} h(t_1)h_k(t+t_1)e^{-j\omega t}dt\, dt_1$$

Therefore

$$S_s(\omega) = \frac{1}{t_o} \sum_{i,k=1}^{a} \sum_{n=-\infty}^{n=\infty} \{ [a_{ik}(n) - P_i P_k] e^{-j\omega n t_o} \int_{-\infty}^{+\infty} \int_{-t_o}^{t_o} h_i(t_1) h_k(t+t_1)$$

$$e^{-j\omega t} dt \, dt_1 \} \tag{28}$$

Using the same argument as we used in finding a_n we have

$$S_s(\omega) = \frac{1}{t_o} \sum_{i,k=1}^{a} \sum_{n=-\infty}^{\infty} [a_{ik}(n) - P_i P_k] e^{-jn\omega t_o} H_i^{*}{*}(\omega) H_k(\omega) \tag{29}$$

$$= \frac{1}{t_o} \sum_{i,k=1}^{a} \sum_{n=-\infty}^{\infty} a_{ik}(n) e^{-jn\omega t_o} H_i{*}(\omega) H_k(\omega)$$

$$- \frac{1}{t_o} \sum_{i,k=1}^{a} \sum_{n=-\infty}^{\infty} P_i P_k e^{-jn\omega t_o} H_i{*}(\omega) H_k(\omega)$$

$$= \frac{1}{t_o} \sum_{i,k=1}^{a} \{ \sum_{n=-\infty}^{-1} P_k P_{ki}^{(-n)} e^{jn\omega t_o} + P_i \delta_{ik} + \sum_{n=1}^{\infty} P_i P_{ik}^{(n)}$$

$$e^{-jn\omega t_o} \} H_i{*}(\omega) H_k(\omega)$$

$$- \sum_{i,k=1}^{a} \sum_{n=-\infty}^{\infty} P_i P_k \, \delta(\omega - \frac{2\pi n}{t_o}) \, H_i{*}(\omega) H_k(\omega)$$

The second term is a train of negative impulses located at $\omega = \dfrac{2\pi n}{t_o}$.

Therefore $S_s(\omega)$ can be written as follows

$$S_s(\omega) = \frac{1}{t_o} \sum_{i,k=1}^{a} \{ \sum_{n=-\infty}^{-1} P_k P_{ki}^{(-n)} e^{-jn\omega t_o} + P_i \delta_{ik} + \sum_{n=1}^{\infty} P_i P_{ik}^{(n)} e^{-jn\omega t_o} \}$$

$$H_i^{*}(\omega) H_k(\omega)$$

$$= \frac{1}{t_o} \sum_{i,k=1}^{a} \{ \sum_{n=1}^{\infty} P_k P_{ki}^{(n)} e^{jn\omega t_o} + P_i \delta_{ik} + \sum_{n=1}^{\infty} P_i P_{ik}^{(n)} e^{-jn\omega t_o} \}$$

$$H_i^{*}(\omega) H_k(\omega) \qquad (30)$$

$$= \frac{1}{t_o} \sum_{i,k=1}^{a} \{ P_k | P_{ki}(e^{j\omega t_o}) + P_i \delta_{ik} + P_i | P_{ik}(e^{-j\omega t_o}) \} H_i^{*}(\omega) H_k(\omega)$$

where $|P_{ki}(e^{j\omega t_o}) = \sum_{n=1}^{\infty} P_{ki}^{(n)} e^{jn\omega t_o}$

$|P_{ik}(e^{-j\omega t_o}) = \sum_{n=1}^{\infty} P_{ik}^{(n)} e^{-jn\omega t_o}$

when $\omega \neq \frac{2\pi n}{t_o}$

and

$$S_s(\omega) = \lim_{\delta \to 0} \frac{S_s(\omega+\delta) + S_s(\omega-\delta)}{2}$$

for $\omega = \frac{2\pi n}{t_o}$ for all integer values of n.

As we can see

$$P_i | P_{ik}(e^{-j\omega t_o}) = [P_k \; P_{ki}(e^{j\omega t_o})]^{*}$$

Then

$$S_s(\omega) = \frac{1}{t_o} \sum_{i=1}^{a} P_i |H_i(\omega)|^2 + \frac{2}{t_o} \mathrm{Re}[\sum_{i=1,k}^{a} P_i H_i^{*}(\omega) H_k(\omega) \, | P_{ik}(e^{-j\omega t_o})]$$

when $\omega \neq \frac{2\pi n}{t_o}$

and

$$S_s(\omega) = \lim_{\delta \to o} \frac{S_s(\omega+\delta) + S_s(\omega-\delta)}{2} \qquad (31)$$

for $\omega = \frac{2\pi n}{t_o}$.

Therefore the power spectral density is given by

$$S(\omega) = S_p(\omega) + S_s(\omega)$$

$$= \frac{1}{t_o{}^2} \left| \sum_{i=1}^{a} P_i H_i(\omega) \right|^2 \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{t_o}\right) + \frac{1}{t_o} \sum_{i=1}^{a} P_i |H_i(\omega)|^2$$

$$+ \frac{2}{t_o} \operatorname{Re} \sum_{i=1}^{a} \sum_{i=k}^{a} [P_i H_i{}^*(\omega) H_k(\omega) \{ P_{ik}(e^{-j\omega t_o})] \qquad (32)$$

A signal y(t) is called NEP (negative equally probable) process if (1) for each element $h_i(t)$ of the modulating set $\{h_i(t)\}$ of a Markov process, $-h_i(t)$ is also in the set and (2) the stationary probabilities of $h_i(t)$ and $-h_i(t)$ are equal. Also the transitional properties of $h_i(t)$ are the same as those of $-h_i(t)$; that is $P_{jk} = P_{rs}$ whenever $h_j(t) = \pm h_r(t)$ and $h_k(t) = \pm h_s(t)$. Then the first and third terms of $S(\omega)$ turn to zero and $S(\omega)$ becomes

$$S(\omega) = \frac{1}{t_o} \sum_{i=1}^{a} P_i |H_i(\omega)|^2$$

We will apply this to our problem.

III. Amplitude Modulation

An expression for a sinusoidal carrier amplitude modulated by a split-phase code is

$$e_{AM}(t) = A[1 + \beta V m(t)] \cos(\omega_c t + \phi) \qquad (33)$$

where

A - carrier peak amplitude

β - modulation index

V - voltage level of split-phase code

$m(t)$ - split-phase code switching function (± 1)

ω_c - carrier angular frequency

θ - initial phase of the carrier

Observe for $\beta V = 1$, we have the special case of "on-off" keying.

Equation (33) can be written as

$$e_{AM}(t) = [A + Bm(t)]\cos(\omega_c t + \phi)$$

where $B = A\beta V$.

The autocorrelation of $e_{AM}(t)$ is

$$R(t_1,t_1+\tau) = E\{e_{AM}(t_1)e_{AM}(t_1+\tau)\}$$

$$= E\{[A+Bm(t_1)]\cos(\omega_c t_1+\phi)\cdot[A+Bm(t_1+\tau)]\cos(\omega_c(t_1+\tau)+\phi)\}$$

$$= E\{[A^2+ABm(t_1)+ABm(t_1+\tau)+B^2m(t_1+\tau)]\cos(\omega_c t_1+\phi)\cos(\omega_c(t_1+\tau)+\phi)\}$$

$$= E\{A^2+ABm(t_1)+ABm(t_1+\tau)+B^2m(t_1)m(t_1+\tau)\}\cos(\omega_c(t_1+\tau)+\phi)\cos(\omega_c t_1+\phi)$$

since ϕ is constant in the coherent case .

Since the $E\{m(t_1)\} = E\{m(t_1+\tau)\} = o$, then we have

$$R(t_1,t_1+\tau) = \{A^2+E[B^2m(t_1)m(t_1+\tau)]\}\cos(\omega_c t_1+\phi)\cos(\omega_c(t_1+\tau)+\phi)$$

$$= A\cos(\omega_c t_1+\phi)A\cos(\omega_c(t_1+\tau)+\phi)+E\{Bm(t_1)\cos(\omega_c t_1+\phi)$$

$$Bm(t_1+\tau)\cos(\omega_c(t_1+\tau)\} \qquad (34)$$

From the equation (34) we can see that the autocorrelation

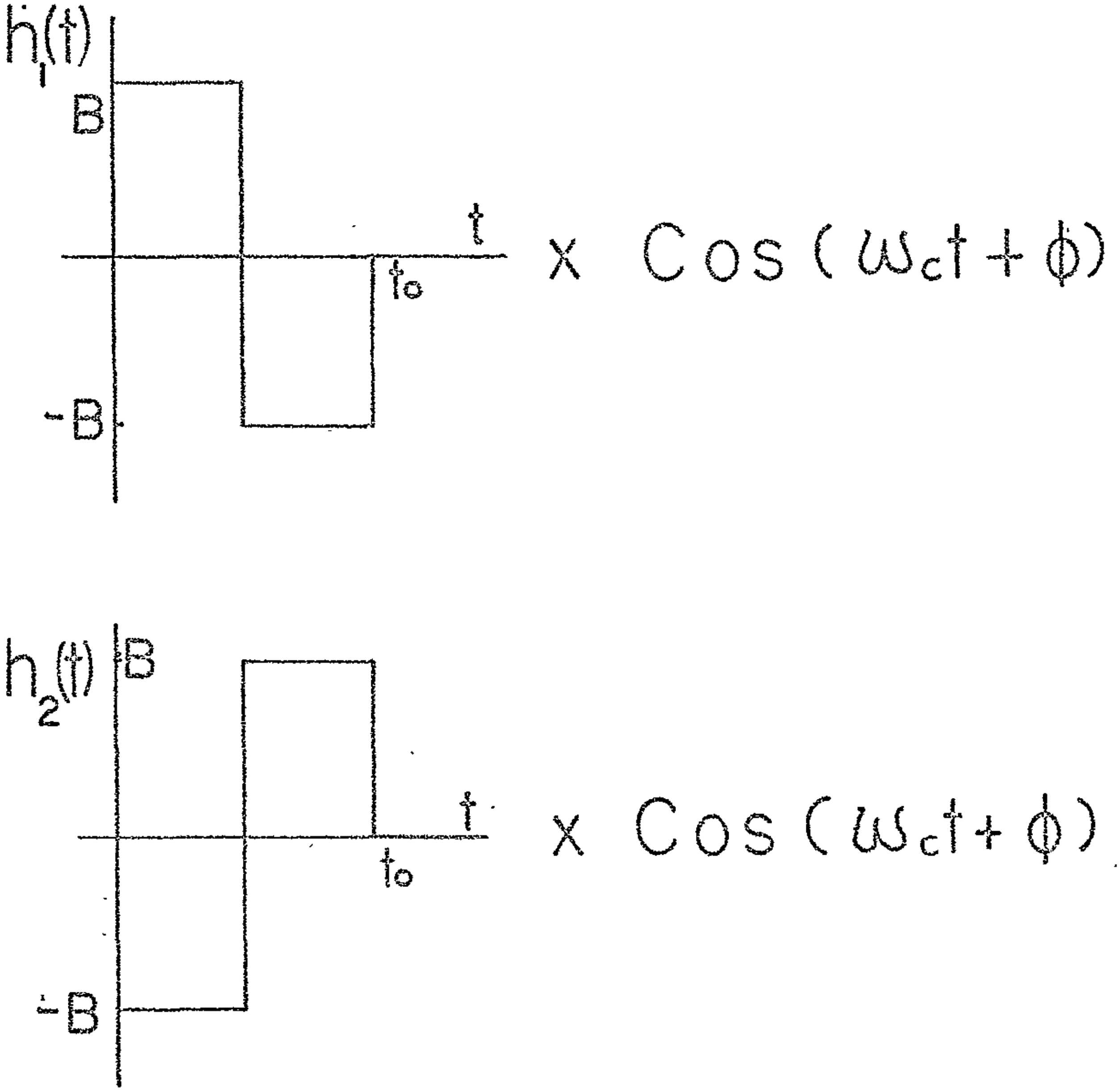

Fig. 2: The Two States

of $e_{AM}(t)$ is equal to the sum of the autocorrelation of the deterministic signal $A \cos(\omega_c t + \phi)$ and the autocorrelation of the random signal $Bm(t) \cos(\omega_c t+\phi)$.

Since

$$F[f_1(t)+f_2(t)] = F[f_1(t)]+F[f_2(t)].$$

(where F denotes the Fourier Transform), the power spectral density of $e_{AM}(t)$ is equal to the sum of the power spectral density of $A \cos(\omega_c t + \phi)$ and the power spectral density of $Bm(t) \cos (\omega_c t + \phi)$.

It is well known that the power spectral density of the deterministic signal $A \cos (\omega_c t + \phi)$, is equal to

$$\frac{A^2}{4} \delta(\omega_c - \omega) + \frac{A^2}{4} \delta(\omega_c + \omega)$$

or

$$\frac{A^2}{4} \delta(\omega - \omega_c) + \frac{A^2}{4} \delta(\omega + \omega_c).$$

The autocorrelation of the random signal $Bm(t) \cos(\omega_c t+\phi)$ is the main thing we have to find in order to find the power spectrum of $e_{AM}(t)$.

During the interval $o<t<t_o$, $Bm(t) \cos (\omega_c t+\phi)$ has only two possible states which are as shown in Figure 2.

and $h_1(t) = -h_2(t)$.

We can see that $h_1(t)$ and $h_2(t)$ have the following properties

$$p_1 = p_2 = \frac{1}{2}\ ,\ p_{11}^{(n)} = p_{12}^{(n)} = p_{21}^{(n)} = p_{22}^{(n)} = \frac{1}{2}$$

Therefore the random signal $Bm(t)\cos(\omega_c t+\phi)$ is a NEP [2] process, and therefore the first term and the third term of the general form of $S(\omega)$ of equation (31) are equal to zero. The power spectrum of $Bm\cos(\omega_c t+\phi)$ is simply given by equation (32) Clearly $H_1(\omega) = -H_2(\omega)$, where $H_1(\omega)$ and $H_2(\omega)$ are the Fourier Transforms of $h_1(t)$ and $h_2(t)$ respectively.

Therefore

$$S(\omega) = \frac{1}{t_o} \sum_{i=1}^{2} p_i |H_i(\omega)|^2 = \frac{1}{t_o}[\frac{1}{2}|H_1(\omega)|^2 + \frac{1}{2}|H_2(\omega)|^2]$$

or

$$S(\omega) = \frac{1}{t_o} |H_1(\omega)|^2 \tag{35}$$

<u>Calculation of $H_1(\omega)$</u>

$$H_1(\omega) = \int_0^{\frac{t_o}{2}} B\cos(\omega_c t+\phi)e^{-j\omega t}dt - \int_{t_o/2}^{t_o} B\cos(\omega_c t+\phi)e^{-j\omega t}dt$$

$$= \int_0^{\frac{t_o}{2}} B\,\frac{e^{j(\omega_c t+\phi)}+e^{-j(\omega_c t+\phi)}}{2}\,e^{-j\omega t}dt - \int_{t/2}^{t_o} B\,\frac{e^{j(\omega_c t+\phi)}+e^{-j(\omega_c t+\phi)}}{2}\,e^{-j\omega t}dt$$

$$= \frac{Be^{j\phi}}{2} \cdot \frac{e^{j(\omega_c-\omega)t}}{j(\omega_c-\omega)} \Bigg|_{o}^{t_o/2} + \frac{Be^{-j\phi}}{2} \cdot \frac{e^{-j(\omega_c+\omega)t}}{-j(\omega_c+\omega)} \Bigg|_{o}^{t_o/2}$$

$$- \frac{Be^{j\phi}}{2} \frac{e^{j(\omega_c-\omega)t}}{j(\omega_c-\omega)} \Bigg|_{t_o/2}^{t_o} - \frac{Be^{-j\phi}}{2} \cdot \frac{e^{-j(\omega_c+\omega)t}}{-j(\omega_c+\omega)} \Bigg|_{t_o/2}^{t_o}$$

$$= \frac{Be^{j\phi}}{2} \frac{e^{j(\omega_c-\omega)t_o/2}}{j(\omega_c-\omega)} - \frac{Be^{j\phi}}{2} \frac{1}{j(\omega_c-\omega)} - \frac{e^{-j(\omega_c+\omega)t_o/2}}{j(\omega_c+\omega)}$$

$$+ \frac{Be^{-j\phi}}{2j(\omega_c+\omega)} - \frac{Be^{j\phi}e^{j(\omega_c-\omega)t_o}}{2j(\omega_c-\omega)} + \frac{Be^{j\phi}e^{j(\omega_c-\omega)t_o/2}}{2j(\omega_c-\omega)} + \frac{Be^{-j\phi}e^{-j(\omega_c+\omega)t_o}}{2j(\omega_c+\omega)}$$

$$- \frac{Be^{-j\phi}e^{-j(\omega_c+\omega)t_o/2}}{2j(\omega_c+\omega)}$$

$$= \frac{Be^{-j\phi}}{2j(\omega_c+\omega)} - \frac{Be^{j\phi}}{2j(\omega_c-\omega)} + \frac{Be^{j\phi}e^{j(\omega_c-\omega)t_o/2}}{j(\omega_c-\omega)} - \frac{Be^{-j\phi}e^{-j(\omega_c+\omega)t_o/2}}{j(\omega_c+\omega)}$$

$$+ \frac{Be^{-j\phi}e^{-j(\omega_c+\omega)t_o}}{2j(\omega_c+\omega)} - \frac{Be^{j\phi}e^{j(\omega_c-\omega)t_o}}{2j(\omega_c-\omega)}$$

$$= \frac{-jB(\omega_c-\omega)e^{-j\phi}+jB(\omega_c+\omega)e^{j\phi}}{2(\omega_c^2-\omega^2)}$$

$$+ \frac{-jBe^{j\phi}(\omega_c+\omega)e^{j(\omega_c-\omega)t_o/2}+jBe^{-j\phi}e^{-j(\omega_c+\omega)t_o/2}(\omega_c-\omega)}{(\omega_c^2-\omega)^2}$$

$$+ \frac{-jBe^{-j\phi}(\omega_c-\omega)e^{-j(\omega_c+\omega)t_o}+jBe^{j\phi}e^{j(\omega_c-\omega)t_o}(\omega_c+\omega)}{2(\omega_c^2-\omega^2)}$$

For $\omega_c = \frac{2n\pi}{t_o}$ when n is an integer, then

$$H_1(\omega) = \frac{jB\omega_c(e^{j\phi}-e^{-j\phi})+jB\omega_c(e^{j\phi}+e^{-j\phi})}{2(\omega_c^{2}-\omega^{2})}$$

$$+ \frac{-jBe^{j\phi}(\omega_c+\omega)e^{j(\omega_c-\omega)t_o/2}+jBe^{-j\phi}e^{j(\omega_c-\omega)t_o/2}(\omega_c-\omega)}{2(\omega_c^{2}-\omega^{2})}$$

$$+ \frac{-jBe^{-j\phi}(\omega_c-\omega)e^{j(\omega_c-\omega)t_o}+jBe^{j\phi}d^{j(\omega_c-\omega)t_o}(\omega_c+\omega)}{2(\omega_c^{2}-\omega^{2})}$$

$$= \frac{-Be\omega_c\sin\phi+jB\omega\cos\phi}{\omega_c^{2}-\omega^{2}}$$

$$+ \frac{-Be^{j(\omega_c-\omega)t_o/2}[-j(\omega_c+\omega)e^{j\phi}+j(\omega_c-\omega)e^{-j\phi}]}{\omega_c^{2}-\omega^{2}}$$

$$+ \frac{Be^{j(\omega_c-\omega)t_o/2}[-j(\omega_c+\omega)e^{j\phi}+j(\omega_c-\omega)e^{-j\phi}]}{2(\omega_c^{2}-\omega^{2})}$$

$$= \frac{-B\omega_c\sin\phi+jB\omega\cos}{\omega_c^{2}-\omega^{2}} - \frac{-B\omega_c\sin\phi+jB\omega\cos\phi}{\omega_c^{2}-\omega^{2}} \cdot 2e^{j(\omega_c-\omega)t_o/2}$$

$$+ \frac{-B\omega_c\sin\phi+jB\omega\cos\phi}{\omega_c^{2}-\omega^{2}}\, e^{j(\omega_c-\omega)t_o}$$

$$= \frac{-B\omega_c\sin\phi+jB\omega\cos\phi}{\omega_c^{2}-\omega^{2}}\,[1-2e^{j(\omega_c-\omega)t_o/2} + e^{j(\omega_c-\omega)t_o}]$$

$$= \frac{-B\omega_c\sin\phi+jB\omega\cos\phi}{\omega_c^{2}-\omega^{2}}\,(1-e^{j(\omega_c-\omega)t_o/2})^{2}$$

$$= \frac{-B\omega_c\sin\phi+jB\omega\cos\phi}{\omega_c^{2}-\omega^{2}}\,[(\frac{e^{-j(\omega_i-\omega)t_o/4}-e^{j(\omega_i-\omega)t_o/4}}{2j})\ 2je^{j(\omega_i-\omega)t_o/4}]$$

$$= \frac{-B\omega_c\sin\phi+jB\omega\cos\phi}{\omega_c^{2}-\omega^{2}} \cdot (-4)\cdot(\sin\frac{(\omega_c-\omega)t_o}{4})^{2} \cdot e^{j(\omega_c-\omega)t_o/2}$$

$$= \frac{-B\omega_c \sin\phi + jB\omega\cos\phi}{\omega^2-\omega_c^2} \cdot 4 \cdot \left[\sin\left(\frac{\omega_c-\omega}{4}\right)t_o\right]^2 e^{j(\omega_c-\omega)t_o/2}$$

The Power Spectral Density of the Coherent AM Signal

By substitution in equation (35), we have the power spectral density for the coherent case, i.e.

$$S_c(\omega) = \frac{1}{t_o} \frac{B^2\omega_c^2\sin^2\phi + B^2\omega^2\cos^2\phi}{(\omega^2-\omega_c^2)^2} \cdot 16 \cdot (\sin(\omega-\omega_c)t_o/4)^4$$

$$= \frac{1}{t_o} \frac{B^2\omega_c^2\sin^2\phi + B^2\omega^2\cos^2\phi}{(\omega+\omega_c)^2\ (\omega-\omega_c)^2} \cdot 16 \cdot (\sin(\omega-\omega_c)t_o/4)^4$$

$$= B^2 t_o \left[\frac{(\sin(\omega-\omega_c)t_o/4)^2}{\frac{(\omega-\omega_c)t_o}{4}}\right]^2 \left[\frac{\omega_c^2\sin^2\phi + \omega^2\cos^2\phi}{(\omega+\omega_c)^2}\right]$$

$$= B^2 t_o \left[\frac{(\sin(\omega-\omega_c)t_o/4)^2}{\frac{(\omega-\omega_c)t_o}{4}}\right]^2 \left[\frac{\sin^2\phi + (\frac{\omega}{\omega_c})^2\cos^2\phi}{(1+\frac{\omega}{\omega_c})^2}\right]$$

Therefore the power spectrum of $e_{AM}(t)$ is

$$S_c(\omega) = \frac{A^2}{4}\delta(\omega_c-\omega) + \frac{A^2}{4}\delta(\omega_c+\omega) + B^2 t_o \left[\frac{(\sin(\omega-\omega_c)t_o/4)^2}{\frac{(\omega-\omega_c)t_o}{4}}\right]^2 \left[\frac{\sin^2\phi + (\frac{\omega}{\omega_c})^2\cos^2\phi}{(1+\frac{\omega}{\omega_c})^2}\right] \qquad (36)$$

Equation (36) is plotted in Figure 3 for the values of $\phi = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

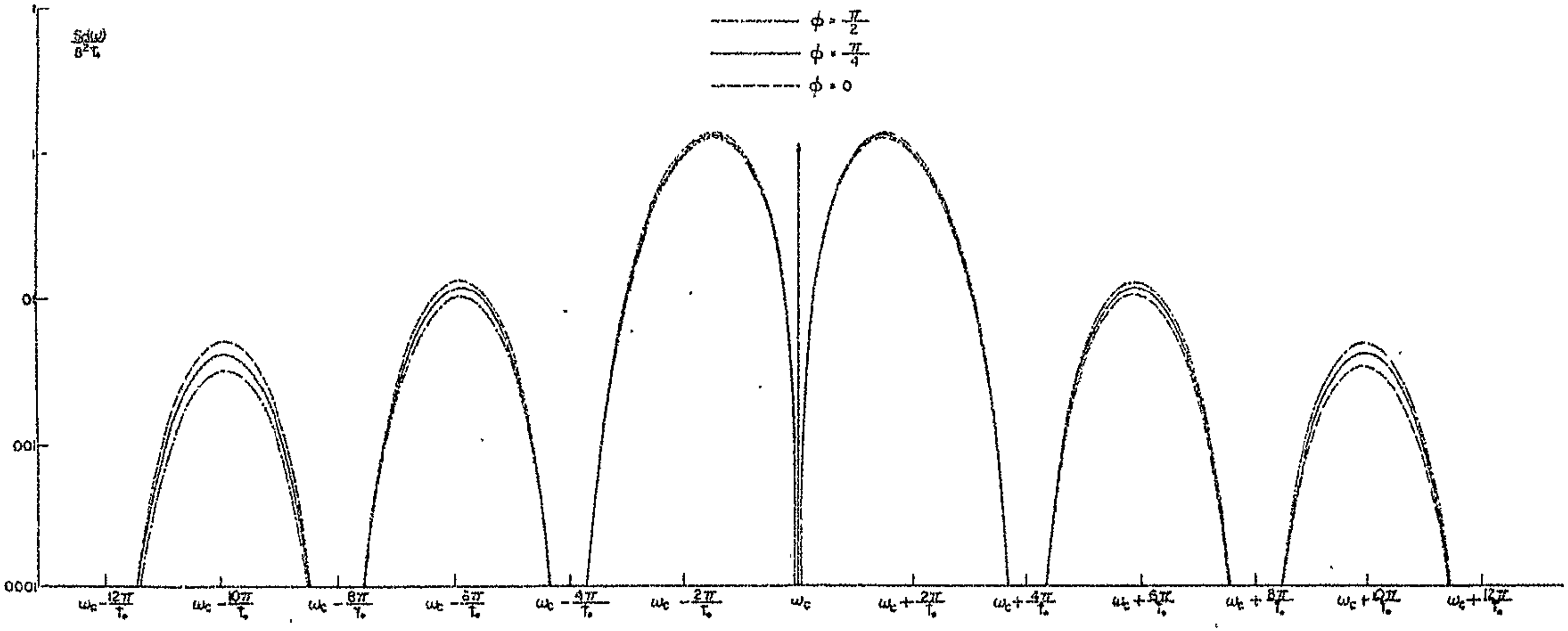

Figure 3. Power Spectral Density of Amplitude Modulated Signal

$S_c(\omega)$ can be written as

$$S_c(\omega) = S_{env}(\omega) \sin^4(\omega-\omega_c) \frac{t_o}{4}$$

where

$$S_{env}(\omega) = \frac{16\ B^2}{t_o} \frac{\omega_c^{\ 2} \sin^2\phi + \omega^2 \cos^2\phi}{(\omega_c^{\ 2} - \omega^2)^2} \tag{37}$$

Inspection of Eq. (37) reveals the following:

A. If ϕ is zero or a multiple of π, the envelope reduces to

$$S_{env}(\omega) = \frac{16B^2}{t_o} \frac{\omega^2}{(\omega_c^{\ 2} - \omega^2)^2}$$

and, for large ω,

$$S_{env}(\omega) \simeq \frac{K}{\omega^2}$$

Thus for large ω the power spectral density falls off at 6dB/octave. This corresponds to the case of coherent modulation of the carrier, with bit transitions occuring at the peaks of the carrier. Intuitively, it would be expected that this is the maximum bandwidth case (see Figure 4).

B. If $\phi = K \frac{\pi}{2}$ where K is an odd integer, then

$$S_{env}(\omega) = \frac{16\ B^2}{t_o} \frac{\omega_c^{\ 2}}{(\omega_c^{\ 2} - \omega^2)^2}$$

and for large ω

$$S_{env}(\omega) \simeq \frac{1}{\omega^4}$$

Thus for large ω, the power spectral density falls off at 12 dB/octave. This corresponds to the case of coherent modulation of the carrier, with bit transitions occuring at the zero crossings of the carrier. Intuitively, it would be expected that this is the minimum bandwidth case, because the modulated signal is never discontinuous.

C. If ϕ is equal to $\frac{\pi}{4}$

$$S_{env}(\omega) = \frac{8B^2}{t_o} \quad \frac{\omega_c^{\ 2} + \omega^2}{(\omega^2-\omega_c^{\ 2})^2}$$

and for large ω, the power spectral density falls off at approximately 6 dB/octave. Intuitively, this would be expected to be an average bandwidth case. See Figure 4 .

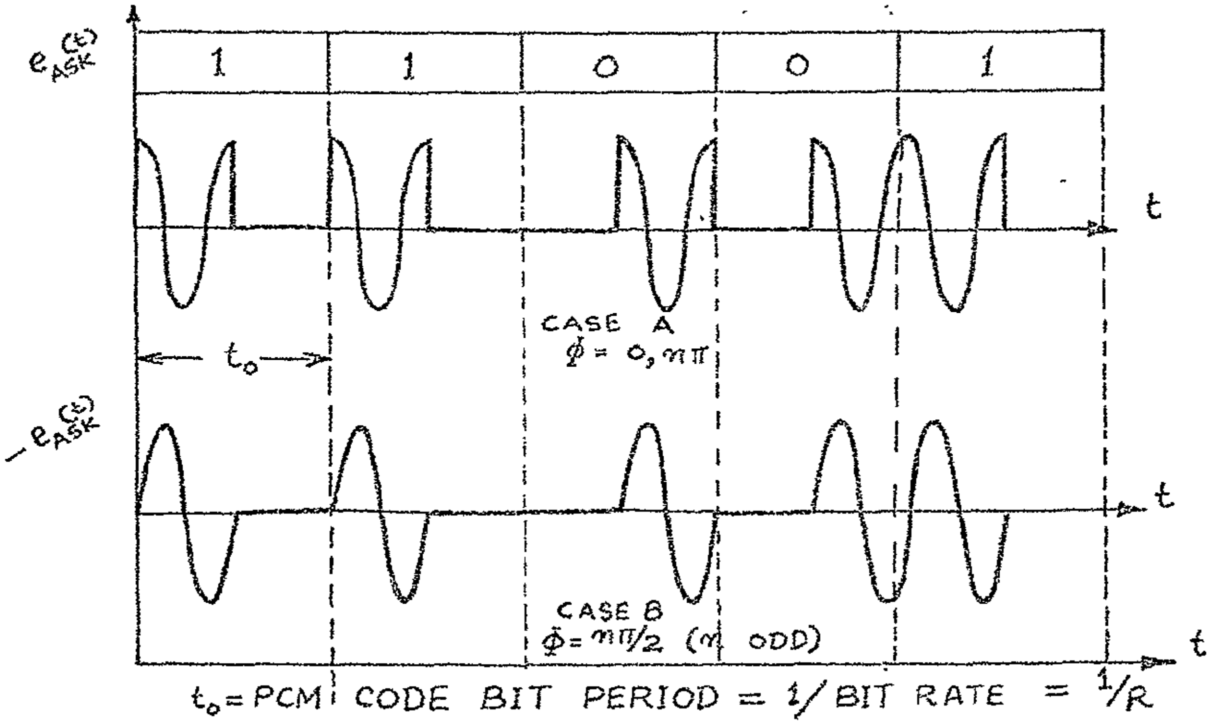

FIG. 4 – ON-AND-OFF KEYING BY A SPLIT PHASE CODE

The Power Spectral Density of the Noncoherent AM Signal

The autocorrelation function of the AM signal of equation (33) is given by

$$R_{AM}(\tau) = E\ \{e_{AM}(t_1)\ e_{AM}(t_1+\tau)\} \qquad (38)$$

where t_1 and $t_1+\tau$ are the times at which the members of the ensemble are sampled.

Equation (38) may be further expressed as

$$R_{AM}(\tau) = E\ \{[A+Bm(t_1)]\cos(\omega_c t_1+\phi)\,[A+Bm(t_1+\tau)]\cos(\omega_c(t_1+\tau)+\phi)\}$$

or

$$R_{AM}(\tau) = E\ \{[A+Bm(t_1)]\,[A+Bm(t_1+\tau)]\cos(\omega_c t_1+\phi)\cos(\omega_c(t_1+\tau)+\phi)\}$$

Since the modulation process is noncoherent, the PCM signal and the carrier may be assumed to be statistically independent[4]. Since the expected value of the product of two statistically independent random variables is equal to the product of their expected values, then

$$R_{AM}(\tau) = E\ \{[A+Bm(t_1)]\,[A+Bm(t_1+\tau)]\}E[\cos(\omega_c t_1+\tau)\cos(\omega_c(t_1+\tau)+\phi)]$$

But (39)

$$\cos(\omega_c t_1+\phi)\cos(\omega_c t_1+\omega_c\tau+\phi) = \frac{1}{2}\cos(\omega_c\tau)+\frac{1}{2}\cos(2\omega_c t_1+\omega_c\tau+2\phi)$$

Then equation (39) becomes

$$R_{AM}(\tau) = E\{A^2+ABm(t_1)+ABm(t_1+\tau)+B^2m(t_1)m(t_1+\tau)\}$$

$$\cdot E\{\frac{1}{2}\cos\omega_c\tau+\frac{1}{2}\cos(2\omega_c t_1+\omega_c\tau+2\phi\}$$

$$=[A^2+AB\ E\{m(t_1)\}+AB\ E\{m(t_1+\tau)\}+B^2E\{m(t_1)m(t_1+\tau)\}]$$

$$\cdot\ [\frac{1}{2}E(\cos\omega_c\tau)+\frac{1}{2}E(\cos(2\omega_c t_1+\omega_c\tau+2\phi)] \qquad (40)$$

If ϕ_c is assumed to be a random variable, uniformly distributed over the range 0 to 2π, then

$$E[\cos(2\omega_c t_1+\omega_c\tau+2\phi)] = 0$$

It can also be noted that the assumption of a random PCM code with equally-likely ones and zeros results in

$$E[m(t_1)] = E[m(t_1+\tau)] = 0$$

Also since $\omega_c\tau$ is constant for a given value of τ, then

$$E[\cos(\omega_c\tau)] = \cos(\omega_c\tau)$$

Therefore equation (40) becomes

$$R_{AM}(\tau) = \frac{A^2}{2}\cos\omega_c\tau+\frac{B^2}{2}\,E\{m(t_1)m(t_1+\tau)\}\cos\omega_c\tau$$

or

$$R_{AM}(\tau) = \frac{A^2}{2}\cos\omega_c\tau+\frac{A^2V^2\beta^2}{2}\,E\{m(t_1)m(t_1+\tau)]\}\cos\omega_c\tau \qquad (41)$$

Several observations can be made regarding equation (41). First, the term $\frac{A^2}{2}\cos(\omega_c\tau)$ is recognized as being the autocorrelation function of the carrier, $A\cos(\omega_c t+\phi)$. Second, the term $V^2\ E[m(t_1)m(t_1+\tau)]$ is recognized as being an expression for the autocorrelation function of the binary sequence (split-phase PCM code) under consideration. Thus,

$$R_{AM}(\tau) = R_{CARRIER}(\tau)+\beta^2 R_{Bi\phi-L}(\tau)R_{CARRIER}(\tau) \qquad (42)$$

And since

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \qquad (43)$$

and multiplication of autocorrelation functions results in convolution of power spectra, we have

$$S_{AM}(\omega) = S_{CARRIER}(\omega)+\beta^2[S_{Bi\phi-L}(\omega)*S_{CARRIER}(\omega)] \qquad (44)$$

The autocorrelation function for the splitphase code has been found to be as shown in Figure 5[3] with the corresponding spectral density,

$$S_{Bi\phi-L}(\omega) = \frac{V^2 t_o}{2\pi} \quad \left[\frac{\sin^4 \frac{t_o}{4}}{\left(\frac{\omega t_o}{4}\right)^2}\right] \qquad (45)$$

Clearly $S_{CARRIER}(\omega) = \frac{A^2}{4}[\delta(\omega+\omega_c)+\delta(\omega-\omega_c)]$

The second term of equation (44) can be calculated easily to be

$$\beta^2[S_{Bi\phi-L}(\omega)*S_{CARRIER}(\omega)]$$

$$= \frac{A^2V^2\beta^2t_o}{8\pi}\left[\frac{\sin^4 \frac{(\omega+\omega_c)t_o}{4}}{\left[\frac{(\omega+\omega_c)t_o}{4}\right]^2} + \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left[\frac{(\omega-\omega_c)t_o}{4}\right]^2}\right]$$

Therefore, $S_{AM}(\omega) = \frac{A^2}{4}[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$

$$+ \frac{A^2V^2\beta^2t_o}{8\pi}\left[\frac{\sin^4 \frac{(\omega+\omega_c)t_o}{4}}{\left[\frac{(\omega-\omega_c)t_o}{4}\right]^2} + \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left[\frac{(\omega-\omega_c)t_o}{4}\right]^2}\right] \qquad (46)$$

As indicated in Figure 6 this expression clearly consists of discrete carrier components plus sidebands

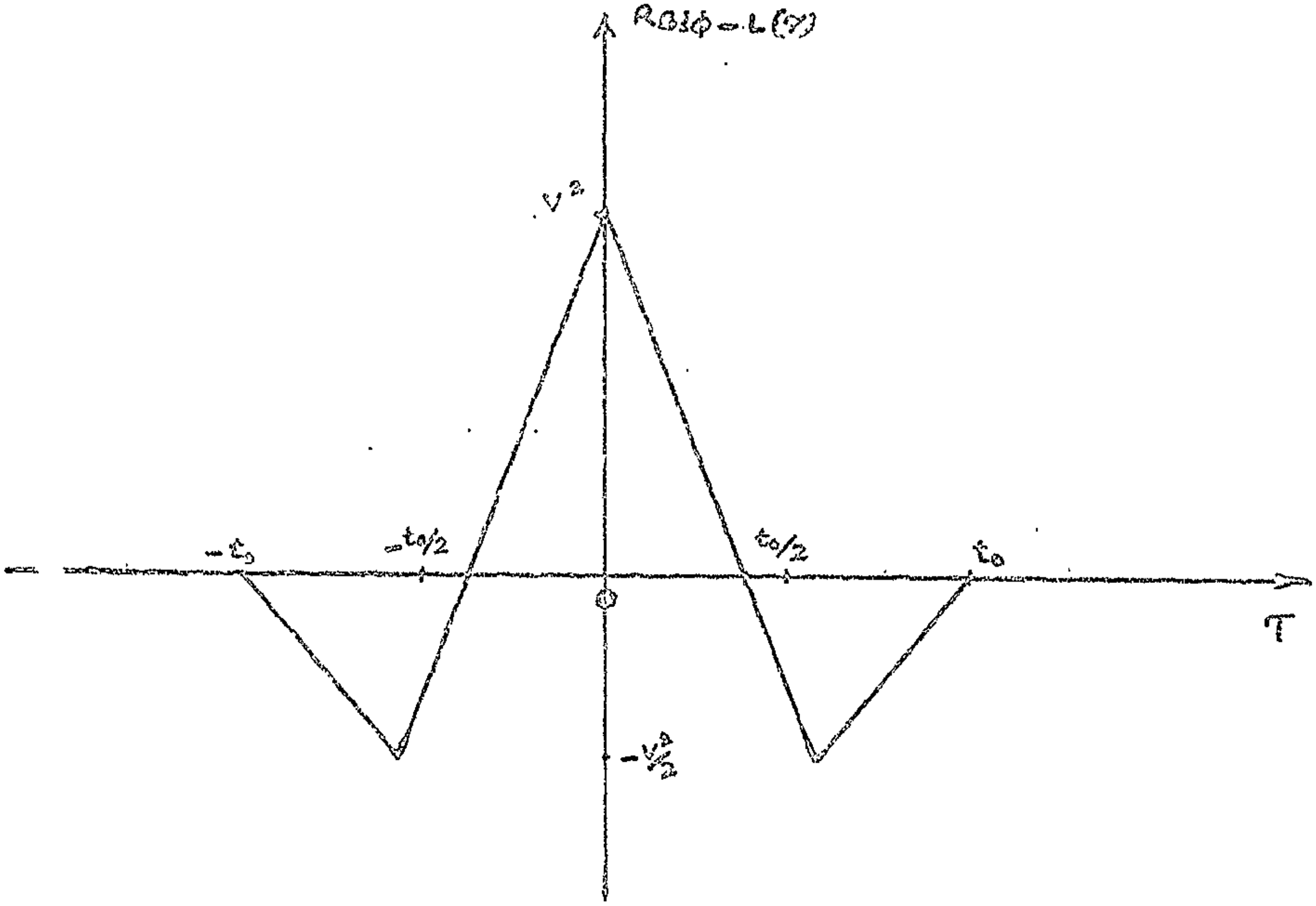

FIG.5 AUTOCORRELATION OF SPLIT-PHASE CODE

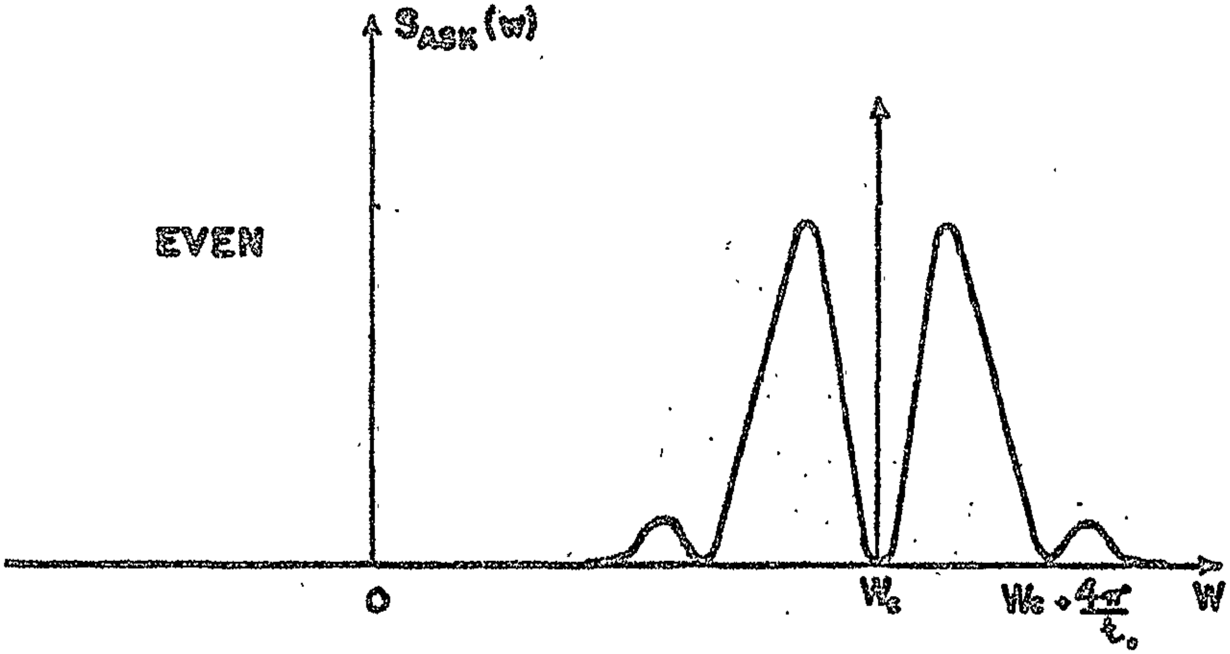

FIG. 6 – POWER SPECTRAL DENSITY OF NON-COHERENT ASK SIGNAL.

resulting from the split-phase baseband spectrum being translated to appear about plus and minus the carrier frequency. Maximum sideband power occurs for $V\beta = 1$, and as noted previously this corresponds to "on-off" keying of the carrier by the split-phase code.

If the carrier frequency, ω_c, is related to the frequency of the modulating sequence, equation (16) can be reduced to a single term. Specifically, if

$\omega_c = \frac{2n\pi}{t_o}$, n is integer, then

$$
\begin{aligned}
\sin^4\left(\frac{(\omega+\omega_c)t_o}{4}\right) &= \sin^4\left(\frac{\omega-\omega_c}{4}\,t_o + \frac{\omega_c t_o}{2}\right) \\
&= \sin^4\left(\frac{\omega-\omega_c}{4}\,t_o + n\pi\right) \\
&= \sin^4\left(\frac{\omega-\omega_c}{4}\,t_c\right) \\
&= \sin^4\left(\frac{\omega-\omega_c}{4}\,t_o\right)
\end{aligned}
$$

Thus

$$
S_{ps}(\omega) = \frac{A^2}{4}\,\delta(\omega+\omega_c) + \frac{A^2}{4}\,\delta(\omega-\omega_c) + \frac{A^2 t_o}{4}\left[\frac{\sin^4\frac{\omega-\omega_c}{4}\,t_o}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2} + \frac{\sin^4\frac{\omega-\omega_c}{4}\,t_o}{\left(\frac{(\omega+\omega_c)t_o}{4}\right)^2}\right]
$$

$$
= \frac{A^2}{4}\,\delta(\omega+\omega_c) + \frac{A^2}{4}\,\delta(\omega-\omega_c) + \frac{A^2 t_o}{4}\;\frac{\sin^4\frac{\omega-\omega_c}{4}\,t_o}{\left[\frac{(\omega-\omega_c)t_o}{4}\right]^2}\left[1 + \frac{\left(\frac{\omega-\omega_c}{2}\,t_o\right)^2}{\left(\frac{\omega+\omega_c}{2}\,t_o\right)^2}\right]
$$

Assume $\omega_c = \frac{2n\pi}{t_o}$

$$S_{nc}(\omega) = \frac{A^2}{4}\delta(\omega+\omega_c) + \frac{A^2}{4}\delta(\omega-\omega_c) + \frac{B^2 t_o}{4} \cdot \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2} \left[1 + \frac{(\omega-\omega_c)^2}{(\omega+\omega_c)^2}\right]$$

$$= \frac{A^2}{4}\delta(\omega+\omega_c) + \frac{A^2}{4}\delta(\omega-\omega_c) + \frac{B^2 t_o}{4} \cdot \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2} \cdot \left[\frac{2\omega^2+2\omega_c^2}{(\omega+\omega_c)^2}\right]$$

$$= \frac{A^2}{4}\delta(\omega+\omega_c) + \frac{A^2}{4}(\omega-\omega_c) + \frac{B^2 t_o}{2} \cdot \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2} \cdot \left[\frac{\omega^2+\omega_c^2}{(\omega+\omega_c)^2}\right]$$

or

$$S_{nc}(\omega) = \frac{A^2}{4}\delta(\omega+\omega_c) + \frac{A^2}{4}(\omega-\omega_c) + \frac{B^2 t_o}{2} \cdot \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2} \cdot \left[\frac{1+\left(\frac{\omega}{\omega_c}\right)^2}{\left(1+\frac{\omega}{\omega_c}\right)^2}\right] \quad (47)$$

The easiest way to find the power spectrum of the noncoherent case is by finding the expected value of the power spectrum of the coherent case with respect to the random number ϕ, that is

$$S_{nc}(\omega) = E[S_c(\omega)] = E\left\{\frac{A^2}{4}\delta(\omega-\omega_c) + \frac{A^2}{4}\delta(\omega+\omega_c)\right.$$

$$\left. + B^2 t_o \left[\frac{\sin^2 (\omega-\omega_c)t_o/4}{\frac{(\omega-\omega_c)t_o}{4}}\right]^2 E\left[\frac{\sin^2\phi + \left(\frac{\omega}{\omega_c}\right)^2\cos^2\phi}{\left(1+\frac{\omega}{\omega_c}\right)^2}\right]\right\}$$

$$= \frac{A^2}{4}\delta(\omega-\omega_c) + \frac{A^2}{4}\delta(\omega+\omega_c) + B^2 t_o \left[\frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2}\right]\left[\frac{\frac{1}{2} + \frac{1}{2}\left(\frac{\omega}{\omega_c}\right)^2}{\left(1+\frac{\omega}{\omega_c}\right)^2}\right]$$

$$= \frac{A^2}{4}\delta(\omega-\omega_c)+\frac{A^2}{4}\delta(\omega+\omega_c)+\frac{B^2 t_o}{2}\left[\frac{\sin^4\frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2}\right]\cdot\left[\frac{1+\left(\frac{\omega}{\omega_c}\right)^2}{\left(1+\frac{\omega}{\omega_c}\right)^2}\right] \quad (48)$$

which agrees with equation (45).

IV. Phase-Shift Keying

The expression for a sinusoidal carrier which is phase-shift-keyed (PSK) by a Split-Phase Code is

$$e_{PSK}(t) = A\cos[\omega_c t+\phi +\beta V m(t)] \quad (49)$$

when A, ϕ , m(t), V are as defined before and β is the modulation index.

Equation (49) can be expanded to give

$$e_{PSK}(t) = A\cos(\omega_c t+\phi)\cos V\beta - A\,m(t)\sin(\omega_c t+\phi)\sin V\beta \quad (50)$$

Let $B = A\cos V\beta$ and

$C = A\sin V\beta$

Then equation (50) becomes

$$e_{PSK}(t) = B\cos(\omega_c t+\phi) - Cm(t)\sin(\omega_c t+\phi) \quad (51)$$

The autocorrelation of this signal is

$$R(t,t+\tau) = E\{e_{PSK}(t)e_{PSK}(t+\tau)\}$$

$$= E\{[B\cos(\omega_c t+\phi) - Cm(t)\sin(\omega_c t+\phi)].$$

$$[B\cos(\omega_c(t+\tau)+\phi) - Cm(t+\tau)\sin(\omega_c(t+\tau)+\phi)]\}$$

$$= B^2\cos(\omega_c t+\phi)\cos(\omega_c(t+\tau)+\phi) +$$

$$E\{Cm(t)\sin(\omega_c t+\phi)Cm(t+\tau)\sin(\omega_c(t+\tau)+\phi)\} \quad (52)$$

The Power Spectral Density of the Coherent Case

The power spectral density of $e_{PSK}(t)$ is the power spectral density of the deterministic signal $B\cos(\omega_c t+\phi)$ and the power spectral density of the random signal $Cm(t)\sin(\omega_c t+\phi)$.

The power spectral density of $B\cos(\omega_c t+\phi)$ is

$$\frac{B^2}{4}\delta(\omega-\omega_c)+\frac{B^2}{4}\delta(\omega+\omega_c)$$

We can find the power spectrum of the random signal $Cm(t)\sin(\omega_c t+\phi)$ as follows:

Let $\phi = \phi_1 + \frac{\pi}{2}$

Then $Cm(t)\sin(\omega_c t+\phi) = Cm(t)\sin(\omega_c t+\phi_1+\frac{\pi}{2}) = Cm(t)\cos(\omega_c t+\phi_1)$

The power spectrum of $Cm(t)\cos(\omega_c t+\phi_1)$ has been found before for the case of amplitude modulation to be

$$C^2 t_o \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2} \left[\frac{\sin^2\phi_1+\left(\frac{\omega}{\omega_c}\right)^2\cos^2\phi_1}{\left(1+\frac{\omega}{\omega_c}\right)^2}\right]$$

and for $\phi_1 = \phi - \frac{\pi}{2}$, it becomes

$$C^2 t_o \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2} \left[\frac{\cos^2\phi+\left(\frac{\omega}{\omega_c}\right)^2\sin^2\phi}{\left(1+\frac{\omega}{\omega_c}\right)^2}\right]$$

The power spectral density of the phase-shift keyed signal by split-phase is

$$S_{PSK-c}(\omega) = \frac{B^2}{4}\delta(\omega-\omega_c) + \frac{B^2}{4}\delta(\omega+\omega_c)$$

$$+ C^2 t_o \frac{\sin^4 \frac{(\omega-\omega_c)}{4} t_o}{\frac{(\omega-\omega_c)t_o{}^2}{4}} \left[\frac{\cos^2\phi + (\frac{\omega}{\omega_c})^2 \sin^2\phi}{(1+\frac{\omega}{\omega_c})^2}\right]$$

which can be written as

$$S_{PSK-C}(\omega) = \frac{A^2\cos^2 V\beta}{4} \left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right] .$$

$$+ t_o A^2 \sin^2 V\beta \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{\omega-\omega_c}{4} t_o\right)^2} \cdot \left[\frac{\cos^2\phi + (\frac{\omega}{\omega_c})^2 \sin^2\phi}{(1 + \frac{\omega}{\omega_c})^2}\right] \quad (53)$$

The power spectral density of the phase-modulated signal of equation (53) resembles the power spectral density of the amplitude-modulated signal of equation (36) except as the sideband power is maximized, the carrier components tend to vanish, and indeed at $\beta V = K\frac{\pi}{2}$, K = odd integer, the carrier does vanish and the sideband power is a maximum.

The Power Spectral Density of the Noncoherent Case

For finding the power spectral density of the noncoherent case, we take the expected value of the power spectral density of the coherent case and consider the phase ϕ to be a random variable uniformly distributed between 0 and 2π. Thus we find that

$$S_{PSK-nc} = \frac{A^2}{4} \cos^2 V\beta \left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right]$$

$$+\frac{t_o A^2}{2} \quad \sin V\beta \quad \frac{\sin^4 \frac{(\omega-\omega_c)t_o}{4}}{\left(\frac{(\omega-\omega_c)t_o}{4}\right)^2} \left[\frac{1+(\frac{\omega}{\omega_c})^2}{(1+\frac{\omega}{\omega_c})^2}\right] \tag{54}$$

Similar conclusions can be made regarding the noncoherent case as for the coherent case.

V. Frequency-Shift Keying

From equations (25) and (29) we can write the general formula of the power spectral density as

$$S(\omega) = \frac{1}{t_o^2} \left|\sum_{i=1}^{a} P_i H_i(\omega)\right|^2 \sum_{n=-\infty}^{\infty} \delta\left(\omega-\frac{2\pi n}{t_o}\right)$$

$$+\frac{1}{t_o} \sum_{i,k=1}^{a} \sum_{n=-\infty}^{\infty} [a_{ik}(n) - P_i P_k] e^{j\omega n t_o} H_i^*(\omega) H_k(\omega) \tag{55}$$

where $a_{ik}(n) = \begin{cases} P_i \delta_{ik} & \text{if } n = o \\ P_i P_{ik}^{(n)} & \text{if } n > o \\ P_k P_{ki}^{(-n)} & \text{if } n < o \end{cases}$

In our present case, we have only two states and with equal probability, therefore $P_1 = P_2 = \frac{1}{2}$. Moreover in every period we assume they are statistically independent, consequently

$$a_{ik}(n) = \begin{cases} P_i \delta_{ik} = \frac{1}{2}\delta_{ik} & \text{if } n = o \\ P_i P_k = \frac{1}{4} & \text{if } n > o \\ P_k P_i = \frac{1}{4} & \text{if } n < o \end{cases}$$

i.e.

$$a_{ik}(n) = \begin{cases} \frac{1}{2}\delta_{ik} & \text{if } n = o \\ \frac{1}{4} & \text{if } n \neq o \end{cases} \tag{56}$$

We substitute $a_{ik}(n)$ of equation (56) into $S(\omega)$ of equation (55) we immediately get

$$S(\omega) = \frac{1}{4t_o^2} |H_1(\omega) + H_2(\omega)|^2 \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{t_o}) + \frac{1}{t_o} \sum_{i,k=1}^{2} [\frac{1}{2}\delta_{ik} - \frac{1}{4}] H_i^*(\omega) H_k(\omega)$$

$$= \frac{1}{4t_o^2} |H_1(\omega) + H_2(\omega)|^2 \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{t_o}) + \frac{1}{t_o} [H_1^*(\omega)\ H_1(\omega) + H_2(\omega)\ H_2^*(\omega)] - \frac{1}{4t_o} \sum_{i,k=1}^{2} H_i^*(\omega) H_k(\omega)$$

$$= \frac{1}{4t_o^2} |H_1(\omega) + H_2(\omega)|^2 \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{t_o}) + \frac{1}{4t_o} |H_1(\omega) - H_2(\omega)|^2 \qquad (57)$$

Now for frequency-shift keying (FSK)

$$h_1(t) = \begin{cases} A\cos(\omega_1 t + \phi_1) & o \le t < \frac{t_o}{2} \\ A\cos[\omega_2 (t - \frac{t_o}{2}) + \phi_2] & \frac{t_o}{2} \le t < t_o \end{cases}$$

and

$$h_2(t) = \begin{cases} A\cos(\omega_2 t + \phi_2) & o \le t < \frac{t_o}{2} \\ A\cos[\omega_1 (t - \frac{t_o}{2}) + \phi_1] & \frac{t_o}{2} \le t < t_o \end{cases}$$

where ω_1 and ω_2 are the carrier frequencies. We have

$$H_1(\omega) = \int_0^{t_o} h_1(t)e^{-j\omega t}dt$$

$$= \int_0^{\frac{t_o}{2}} A\cos(\omega_1 t+\phi_1)e^{-j\omega t}dt+\int_{\frac{t_o}{2}}^{t_o} A\cos[\omega_2(t-\frac{T}{2})+\phi_2]e^{-j\omega t}dt$$

The first integral is

$$\int_0^{\frac{t_o}{2}} A\cos(\omega_1 t+\phi_1)e^{-j\omega t}dt$$

$$= \int_0^{\frac{t_o}{2}} \frac{A}{2}[e^{j(\omega_1-\omega)t+j\phi_1} + e^{-j(\omega_1+\omega)t-j\phi_1}]dt$$

$$= \frac{A}{2}e^{j\phi_1}\frac{e^{j(\omega_1-\omega)t}}{j(\omega_1-\omega)}\Bigg|_0^{\frac{t_o}{2}} + \frac{A}{2}e^{-j\phi_1}\frac{e^{-j(\omega_1+\omega)t}}{-j(\omega_1+\omega)}\Bigg|_0^{\frac{t_o}{2}}$$

$$= \frac{A}{2}e^{j\phi_1}[\frac{e^{j(\omega_1-\omega)\frac{t_o}{2}}}{j(\omega_1-\omega)} - \frac{1}{j(\omega_1-\omega)}]+\frac{A}{2}e^{-j\phi_1}[\frac{e^{-j(\omega_1+\omega)\frac{t_o}{2}}}{-j(\omega_1+\omega)} - \frac{1}{-j(\omega_1+}$$

For simplicity, we assume $\omega_1 = \frac{4n\pi}{t_o}$, $\omega_2 = \frac{4m\pi}{t_o}$ where n, m are integers.

$$= \frac{A}{2}e^{j(\omega_1-\omega)\frac{t_o}{2}}[\frac{e^{j\phi_1}}{j(\omega_1-\omega)} - \frac{e^{-j\phi_1}}{j(\omega_1+\omega)}] - \frac{A}{2}[\frac{e^{j\phi_1}}{j(\omega_1-\omega)} - \frac{e^{-j\phi_1}}{j(\omega_1+\omega)}]$$

$$= \frac{A}{2}\ [e^{j(\omega_1-\omega)\frac{t_o}{2}} - 1]\ [\frac{(\omega_1+\omega)e^{j\phi_1} - (\omega_1-\omega)e^{-j\phi_1}}{j(\omega_1-\omega)\ \ (\omega_1+\omega)}]$$

$$= \frac{A}{2}\ e^{j(\omega_1-\omega)\frac{t_o}{4}}\ [e^{j(\omega_1-\omega)\frac{t_o}{4}} - e^{-j(\omega_1-\omega)\frac{t_o}{4}}]\ [\frac{\omega_1(e^{j\phi_1}-e^{-j\phi_1})}{j(\omega_1-\omega)\cdot}$$

$$\frac{+\omega(e^{j\phi_1}+e^{j\phi_1})}{(\omega_1+\omega)}\ ]$$

$$= A\ e^{j(\omega_1-\omega)\frac{t_o}{4}} \cdot \sin(\omega_1-\omega)\frac{t_o}{4} \cdot \frac{2j\omega_1\sin\phi_1 + 2\omega\cos\phi_1}{(\omega_1{}^2-\omega^2)}$$

The second integral is

$$\int_{\frac{t_o}{2}}^{t_o} A\ \cos\ [\omega_2(t-\frac{t_o}{2})+\phi_2]\ e^{-j\omega t}dt$$

$$= e^{-j\omega\frac{t_o}{2}} \int_{0}^{\frac{t_o}{2}} A\ \cos(\omega_2t+\omega_2)e^{-j\omega t}dt$$

$$= Ae^{-j\omega\frac{t_o}{2}} \cdot e^{j(\omega_2-\omega)\frac{t_o}{4}}\ \sin(\omega_2-\omega)\frac{t_o}{4} \cdot \frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{\omega_2{}^2-\omega^2}$$

$$= Ae^{+j(\omega_2-\omega)3\frac{t_o}{4}}\ \sin(\omega_2-\omega)\frac{t_o}{4} \cdot \frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{\omega_2{}^2-\omega^2}$$

$$(\text{since } e^{j\omega_2\frac{t_o}{2}} = 1)$$

Therefore

$$\begin{aligned} H_1(\omega) &= Ae^{j(\omega_1-\omega)t_o/4}\sin(\omega_1-\omega)t_o/4\cdot\frac{2j\omega_1\sin\phi_1+2\omega\cos\phi_1}{\omega_1^2-\omega^2} \\ &\quad +Ae^{j(\omega_2-\omega)3t_o/4}\sin(\omega_2-\omega)t_o/4\cdot\frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{\omega_2^2-\omega^2} \end{aligned} \tag{58}$$

We can see that $H(\omega)$ is simply.s simply:

$$\begin{aligned} H_2(\omega) &= Ae^{j(\omega_2-\omega)t_o/4}\sin(\omega_2-\omega)t_o/4\cdot\frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{\omega_2^2-\omega^2} \\ &\quad +Ae^{j(\omega_1-\omega)3t_o/4}\sin(\omega_1-\omega)t_o/4\cdot\frac{2j\omega_1\sin\phi_1+2\omega\cos\phi_1}{\omega_1^2-\omega^2} \end{aligned} \tag{59}$$

Thus

$$\begin{aligned} H_1(\omega)+H_2(\omega) &= A[e^{j(\omega_1-\omega)t_o/4}+e^{j(\omega_1-\omega)3t_o/4}]\sin(\omega_1-\omega)t_o/4 \\ &\qquad\cdot\frac{2j\omega_1\sin\phi_1+2\omega\cos\phi_1}{\omega_1^2-\omega^2} \\ &\quad + A[e^{j(\omega_2-\omega)t_o/4}+e^{j(\omega_2-\omega)3t_o/4}]\sin(\omega_2-\omega)t_o/4\cdot\frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{\omega_2^2-\omega^2} \\ &= 2Ae^{j(\omega_1-\omega)t_o/2}\cos(\omega_1-\omega)t_o/4\ \sin(\omega_1-\omega)t_o/4\cdot\frac{2j\omega_1\sin\phi_1+2\omega\cos\phi_1}{\omega_1^2-\omega^2} \\ &\quad + 2Ae^{j(\omega_2-\omega)t_o/2}\cos(\omega_2-\omega)t_o/4\ \sin(\omega_2-\omega)t_o/4\cdot\frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{\omega_2^2-\omega^2} \end{aligned} \tag{60}$$

Since $\sin 2X = 2\sin X\cos X$, equation (60) becomes

$$H_1(\omega)+H_2(\omega)=Ae^{j(\omega_1-\omega)t_o/2}\sin(\omega_1-\omega)t_o/2\cdot\frac{2j\omega_1\sin\phi_1+2\omega\cos\phi_1}{\omega_1^2-\omega^2}$$

$$+\ Ae^{j(\omega_2-\omega)t_o/2}\sin(\omega_2-\omega)t_o/2\cdot\frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{\omega_2^2-\omega^2}$$

$$=\ A[\cos(\omega_1-\omega)t_o/2+j\sin(\omega_1-\omega)t_o/2]\sin(\omega_1-\omega)t_o/2$$

$$\cdot\ \frac{2j\omega_1\sin\phi_1+2\omega\cos\phi_1}{\omega_1^2-\omega^2}$$

$$+\ A[\cos(\omega_2-\omega)t_o/2+j\sin(\omega_2-\omega)t_o/2]\sin(\omega_2-\omega)t_o/2$$

$$\cdot\ \frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{\omega_2^2-\omega^2}$$

$$=\ A\ \frac{2\omega\cos(\omega_1-\omega)t_o/2\sin(\omega_1-\omega)t_o/2\ \cos\phi_1}{\omega_1^2-\omega^2}$$

$$-\ A\ \frac{2\omega_1\sin(\omega_1-\omega)t_o/2\sin(\omega_1-\omega)t_o/2\sin\phi_1}{\omega_1^2-\omega^2}$$

$$+\ A\ \frac{2\omega\cos(\omega_2-\omega)t_o/2\sin(\omega_2-\omega)t_o/2\cos\phi_2}{\omega_1^2-\omega^2}$$

$$-\ A\ \frac{2\omega_2\sin(\omega_2-\omega)t_o/2\sin(\omega_2-\omega)t_o/2\sin\phi_2}{\omega_2^2-\omega^2}$$

$$+ j[A\frac{2\omega\sin(\omega_1-\omega)t_o/2\sin(\omega_1-\omega)t_o/2\cos\phi_1}{\omega_1^2-\omega^2}$$

$$+ A\frac{2\omega_1\cos(\omega_1-\omega)t_o/2\sin(\omega_1-\omega)t_o/2\sin\phi_1}{\omega_1^2-\omega^2}$$

$$+ A\frac{2\omega\sin(\omega_2-\omega)t_o/2\sin(\omega_2-\omega)t_o\cos\phi_2}{\omega_2^2-\omega^2}$$

$$+ A\frac{2\omega_2\cos(\omega_2-\omega)t_o/2\sin(\omega_2-\omega)t_o/2\sin\phi_2}{\omega_2^2-\omega^2}]$$

continuing the trigonometric substitutions, we have

$$H_1(\omega)+H_2(\omega)=A[\frac{\omega\sin(\omega_1-\omega)t_o\cos\phi_1}{\omega_1^2-\omega^2} - \frac{\omega_1(1-\cos(\omega_1-\omega)t_o)\sin\phi_1}{\omega_1^2-\omega^2}$$

$$+ \frac{\omega\sin(\omega_2-\omega)t_o\cos\phi_2}{\omega_2^2-\omega^2} - \frac{\omega_2(1-\cos(\omega_2-\omega)t_o)\sin\phi_2}{\omega_2^2-\omega^2}$$

$$+jA[\frac{\omega[1-\cos(\omega_1-\omega)t_o]\cos\phi_1}{\omega_1^2-\omega^2} + \frac{\omega_1\sin(\omega_1-\omega)t_o\sin\phi_1}{\omega_1^2-\omega^2}$$

$$+ \frac{\omega[1-\cos(\omega_2-\omega)t_o]\cos\phi_2}{\omega_2^2-\omega^2} + \frac{2\omega_2\sin(\omega_2-\omega)t_o\sin\phi_2}{\omega_2^2-\omega^2}]$$

or

$$H_1(\omega)+H_2(\omega)=A\frac{\omega\sin(\omega_1-\omega)t_o\cos\phi_1+\omega_1\cos(\omega_1-\omega)t_o\sin\phi_1-\sin\phi_1}{\omega_1^2-\omega^2}$$

$$+ A\frac{\omega\sin(\omega_2-\omega)t_o\cos\phi_2+\omega_2\cos(\omega_2-\omega)t_o\sin\phi_2-\omega_2\sin\phi_2}{\omega_1^2-\omega^2}$$

$$+ jA[\frac{\omega_1\sin(\omega_1-\omega)t_o\sin\omega_1-\omega\cos(\omega_1-\omega)t_o\cos\phi_1+\omega\cos\phi_1}{\omega_1^2-\omega^2}$$

$$+ \frac{\omega\sin(\omega_2-\omega)t_o\sin\phi_2-\omega\cos(\omega_2-\omega)t_o\cos\phi_2+\omega\cos\phi_2}{\omega_2^2-\omega^2}]$$

But $|H_1(\omega)+H_2(\omega)|^2 = [H_1(\omega)+H_2(\omega)][H_1(\omega)+H_2(\omega)]^*$

$= [\text{Real Part}]^2 + [\text{Imaginary Part}]^2$

Thus

$$|H_1(\omega)+H_2(\omega)|^2/A^2=[\frac{\omega\sin(\omega_1-\omega)t_o\cos\phi_1+\omega_1\cos(\omega_1-\omega)t_o\sin\phi_1-\omega_1\sin\phi_1}{\omega_1^2-\omega^2}$$

$$+ \frac{\omega\sin(\omega_2-\omega)t_o\cos\phi_2+\omega_2\cos(\omega_2-\omega)t_o\sin\phi_2-\omega_2\sin\phi_2}{\omega_2^2-\omega^2}]^2$$

$$+[\frac{\omega_1\sin(\omega_1-\omega)t_o\sin\phi_1-\omega\cos(\omega_1-\omega)t_o\cos\phi_1+\omega\cos\phi_1}{\omega_1^2-\omega^2}$$

$$+ \frac{\omega_2\sin(\omega_2-\omega)t_o\sin\phi_2-\omega\cos(\omega_2-\omega)t_o\cos\phi_2+\omega\cos\phi_2}{\omega_2^2-\omega^2}]^2$$

$$= \frac{\sin^2(\omega_1-\omega)t_o[\omega^2\cos^2\phi_1+\omega_1^2\sin^2\phi_1]+\cos^2(\omega_1-\omega)t_o[\omega_1^2\sin^2\phi_1+\omega^2\cos^2\phi_1]}{(\omega_1^2-\omega^2)^2}$$

$$\frac{+\omega_1^2\sin^2\phi_1+\omega^2\cos^2\phi_1-2\cos(\omega_1-\omega)t_o[\omega_1^2\sin^2\phi_1+\omega^2\cos^2\phi_1]}{1}$$

$$+\frac{\sin^2(\omega_2-\omega)t_o[\omega^2\cos^2\phi_2+\omega_2^2\sin^2\phi_2]+\cos^2(\omega_2-\omega)t_o[\omega_2^2\sin^2\phi_2+\omega^2\cos^2\phi_2]}{(\omega_2^2-\omega^2)^2}$$

$$\frac{+\omega_2^2\sin^2\phi_2+\omega^2\cos^2\phi_2-2\cos(\omega_2-\omega)t_o[\omega_2^2\sin^2\phi_2+\omega^2\cos^2\phi_2]}{1}$$

$$+\frac{\omega^2\sin(\omega_1-\omega)t_o\sin(\omega_2-\omega)t_o\cos\phi_1\cos\phi_2+\omega\omega_2\sin(\omega_1-\omega)t_o\cos(\omega_2-\omega)t_o}{(\omega_1^2-\omega^2)\quad(\omega_2^2-\omega^2)}$$

$$\frac{\cdot\cos\phi_1\sin\phi_2-\omega\omega_2\sin(\omega_1-\omega)t_o\cos\phi_1\sin\phi_2+\omega\omega_1\sin(\omega_2-\omega)t_o\cos(\omega_1-\omega)t_o}{1}$$

$$\frac{\cdot\sin\phi_1\cos\phi_2+\omega_1\omega_2\cos(\omega_1-\omega)t_o\cos(\omega_2-\omega)t_o\sin\phi_1\sin\phi_2-\omega_1\omega_2\cos(\omega_1-\omega)}{1}$$

$$\frac{\cdot t_o\sin\phi_1\sin\phi_2-\omega_1\omega\sin\phi_1\cos\phi_2\sin(\omega_2-\omega)t_o-\omega_1\omega_2\sin\phi_1\sin\phi_2\cos(\omega_2-\omega)t_o}{1}$$

$$\frac{+\omega_1\omega_2\sin\phi_1\sin\phi_2+\omega_1\omega_2\sin(\omega_1-\omega)t_o\sin(\omega_2-\omega)t_o\sin\phi_1\sin\phi_2}{1}$$

$$\frac{-\omega_1\omega\sin(\omega_1-\omega)t_o\cos(\omega_2-\omega)t_o\sin\phi_1\cos\phi_2+\omega\omega_1\sin\phi_1\cos\phi_2\sin(\omega_1-\omega)t_o}{1}$$

$$\frac{-\omega\omega_2\sin(\omega_2-\omega)t_o\cos(\omega_1-\omega)t_o\cos\phi_1\sin\phi_2+\omega^2\cos(\omega_1-\omega)t_o\cos(\omega_2-\omega)t_o}{1}$$

$$\frac{\cdot\cos\phi_1\cos\phi_2}{1}$$

$$=\frac{4\sin^2(\frac{\omega_1-\omega}{2})t_o[\omega_1^2\sin^2\phi_1+\omega^2\cos^2\phi_1]}{(\omega_1^2-\omega^2)^2}$$

$$+\frac{4\sin^2\left(\frac{\omega_2-\omega}{2}\right)t_o\,[\omega_2^2\sin^2\phi_2+\omega^2\cos^2\phi_2]}{(\omega_2^2-\omega^2)^2}$$

$$+\frac{1}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}\,[\omega^2\cos\phi_1\cos\phi_2\{1+\cos(\omega_2-\omega_1)t_o$$

$$-2\cos\frac{(\omega_2-\omega_1)t_o}{2}\cos\frac{(\omega_1+\omega_2-2\omega)t_o}{2}\}$$

$$+\omega_1\omega_2\sin\phi_1\sin\phi_2\{1+\cos(\omega_2-\omega_1)t_o-2\cos\frac{(\omega_2-\omega_1)t_o}{2}\cos\frac{(\omega_1+\omega_2-2\omega)t_o}{2}\}$$

$$+\omega\omega_2\cos\phi_1\sin\phi_2\{\sin(\omega_1-\omega_2)t_o+2\cos\left(\frac{\omega_1+\omega_2-2\omega}{2}\right)t_o\sin\left(\frac{\omega_2-\omega_1}{2}\right)t_o\}$$

$$+\omega\omega_1\sin\phi_1\cos\phi_2\{\sin(\omega_2-\omega_1)t_o+2\cos\left(\frac{\omega_1+\omega_2-2}{2}\right)t_o\sin\left(\frac{\omega_1-\omega_2}{2}\right)t_o\}]$$

Since $\omega_1 = \frac{4n\pi}{t_o}$, $\omega_2 = \frac{4m\pi}{t_o}$ n and m are positive integers then

$\omega_1-\omega_2 = \frac{4\pi}{t_o}(n-m) = \frac{4\pi k}{t_o}$ where $k = n-m$ and

$\cos(\omega_1-\omega_2)t_o = \cos\frac{4\pi k}{t_o}\cdot t_o = \cos 4k\pi = 1$

$\sin(\omega_1-\omega_2)t_o = \sin\frac{4k\pi}{t_o}\cdot t_o = \sin 4k\pi = o$

$\sin(\omega_1-\omega_2)t_o = \sin\frac{2k\pi}{t_o}\cdot t_o = \sin 2k\pi = o$

$\cos\frac{1}{2}(\omega_1-\omega_2)t_o = \cos\frac{2k\pi}{t_o}\cdot t_o = \cos 2k\pi = 1$

Accordingly

$$|H_1(\omega)+H_2(\omega)|^2 = \frac{\sin^2(\omega_1-\omega)\frac{t_o}{2}\,[\omega_1^2\sin^2\phi_1+\omega^2\cos^2\phi_1]}{(\omega_1^2-\omega^2)^2}$$

$$+\frac{4\sin^2(\omega_2-\omega)\frac{t_o}{2}\,[\omega_2^2\sin^2\phi_2+\omega^2\cos^2\phi_2]}{(\omega_2^2-\omega^2)^2}$$

$$+\frac{2\omega^2\cos\phi_1\cos\phi_2+2\omega_1\omega_2\sin\phi_1\sin\phi_2-2\omega^2\cos\phi_1\cos\phi_2\cos[\omega-\frac{1}{2}(\omega_1+\omega_2)]t_o}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}$$

$$\frac{-2\omega_1\omega_2\sin\phi_1\sin\phi_2\cos[\omega-\frac{1}{2}(\omega_1+\omega_2)]\,t_o/2}{}$$

$$= t_o^2\left[\frac{\sin(\omega_1-\omega)t_o/2}{(\omega_1-\omega)t_o/2}\right]^2 \frac{\sin^2\phi_1+(\frac{\omega}{\omega_1})^2\cos^2\phi_1}{(1+\frac{\omega}{\omega_1})^2}$$

$$+\ t_o^2\left[\frac{\sin(\omega_2-\omega)t_o/2}{(\omega_2-\omega)t_o/2}\right]^2 \frac{\sin^2\phi_2+(\frac{\omega}{\omega_2})^2\cos^2\phi_2}{(1+\frac{\omega}{\omega_2})^2}$$

$$+\ 2\frac{\omega^2\cos\phi_1\cos\phi_2\{1-\cos[\omega-\frac{1}{2}(\omega_1+\omega_2)]\,t_o/2\}+\omega_1\omega_2\sin\phi_1\sin\phi_2}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}$$

$$\cdot\{\cos[\omega-\frac{1}{2}(\omega_1+\omega_2)]\,t_o/2\}$$

$$= t_o^2\left[\frac{\sin(\omega_1-\omega)t_o/2}{(\omega_1-\omega)t_o/2}\right]^2 \left[\frac{\sin^2\phi_1+(\frac{\omega}{\omega_1})^2\cos^2\phi_1}{(1+\frac{\omega}{\omega_1})^2}\right]$$

$$+ t_o^2 [\frac{\sin(\omega_2-\omega)t_o/2}{(\omega_2-\omega)t_o/2}]^2 \quad [\frac{\sin^2\phi_2+(\frac{\omega}{\omega_2})^2\cos^2\phi_2}{(1+\frac{\omega}{\omega_2})^2}\}$$

$$+ \frac{\{\omega^2\cos\phi_1\cos\phi_2+\omega_1\omega_2\sin\phi_1\sin\phi_2\}\{2\sin^2[\omega\pm\frac{1}{2}(\omega_1+\omega_2)]t_o/4}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)} \quad (61)$$

From equation (58) and (59) we have

$[H_1\ (\omega)-H_2(\omega)]/A$

$$= [e^{j(\omega_1-\omega)t_o/4} - e^{j(\omega_1-\omega)3t_o/4}]\sin(\omega_1-\omega)t_o/4 \cdot \frac{2j\omega_1\sin\phi_1+2\omega\cos\phi_1}{(\omega_2^2-\omega^2)}$$

$$- [e^{j(\omega_2-\omega)t_o/4} - e^{j(\omega_2-\omega)3t_o/4}]\sin(\omega_2-\omega)t_o/4 \cdot \frac{2j\omega_2\sin\phi_2+2\omega\cos\phi_2}{(\omega_2^2-\omega^2)}$$

$$= e^{j(\omega_1-\omega)t_o/2}\sin^2(\omega_1-\omega)t_o/4 \cdot \frac{4\omega_1\sin\phi_1-4j\omega\cos\phi_1}{\omega_1^2-\omega}$$

$$- e^{j(\omega_2-\omega)t_o/2}\sin^2(\omega_2-\omega)t_o/4 \cdot \frac{4\omega_2\sin\phi_2-4j\omega\cos\phi_2}{\omega_2^2-\omega^2} \quad (62)$$

Following the same procedure used to obtain equation (61), we get

$|H_1(\omega) - H_2(\omega)|^2/A^2$

$$= \sin^4(\omega_1-\omega)t_o/4 \cdot \frac{16\omega_1^2\sin^2\phi_1+16\omega^2\cos^2\phi_1}{(\omega_1^2-\omega^2)^2}$$

$$+\sin^4(\omega_2-\omega)t_o/4\cdot\frac{16\omega_1^2\sin^2\phi_2+16\omega^2\cos^2\phi_2}{(\omega_2^2-\omega^2)^2}$$

$$+2\left[\frac{\cos(\omega_1-\omega)t_o/2\cdot 4\omega_1\sin\phi_1\sin^2(\omega_1-\omega)t_o/4}{\omega_1^2-\omega^2}\right.$$

$$+\frac{\sin(\omega_1-\omega)t_o/2\cdot\sin^2(\omega_1-\omega)t_o/4\cdot 4\omega\cos\phi_1}{\omega_1^2-\omega^2}$$

$$\cdot\left[\frac{-\cos(\omega_2-\omega)t_o/2\sin^2(\omega_2-\omega)t_o 4\cdot 4\omega_2\sin\phi_2}{\omega_1^2-\omega^2}\right.$$

$$\left.-\frac{\sin(\omega_2-\omega)t_o/2\cdot\sin^2(\omega_2-\omega)t_o/4\cdot 4\omega\cos\phi_2}{\omega_2^2-\omega^2}\right]$$

$$+2\left[\frac{\sin(\omega_1-\omega)t_o/2\cdot 4\omega_1\sin\phi_1\sin^2(\omega_1-\omega)t_o/4}{\omega_1^2-\omega^2}\right.$$

$$-\frac{\cos(\omega_1-\omega)t_o/2\cdot\sin^2(\omega_1-\omega)t_o/4\cdot 4\omega\cos\phi_2}{\omega_1^2-\omega^2}$$

$$\cdot\left[\frac{-\sin(\omega_2-\omega)\frac{t_o}{2}\sin^2(\omega_2-\omega)\frac{t_o}{4}\cdot 4\omega_2\sin\phi_2}{\omega_2^2-\omega^2}\right.$$

$$\left.+\frac{\cos(\omega_2-\omega)\frac{t_o}{4}\cdot 4\omega\cos\phi_2}{\omega_2^2-\omega^2}\right]$$

Continuing simplification, we obtain

$$\frac{|H_1(\omega)-H_2(\omega)|^2}{A^2}$$

$$= t_o^{\,2}\left[\frac{\sin^2(\omega_2-\omega)\frac{t_o}{4}}{(\omega_2-\omega)\frac{t_o}{4}}\right]^2 \cdot \left[\frac{\sin^2 {}_1+(\frac{\omega}{\omega_1})^2\cos^2\phi_1}{(1+\frac{\omega}{\omega_1})^2}\right]$$

$$+ t_o^{\,2}\left[\frac{\sin^2(\omega_2-\omega)\frac{t_o}{4}}{(\omega_2-\omega)\frac{t_o}{4}}\right]^2 \cdot \left[\frac{\sin^2\phi_2+(\frac{\omega}{\omega_2})^2\cos^2\phi_2}{(1+\frac{\omega}{\omega_2})^2}\right]$$

$$-2t_o^{\,2} \quad \frac{\sin^2(\omega_1-\omega)\frac{t_o}{4}}{(\omega_1-\omega)\frac{t_o}{4}} \cdot \frac{\sin^2(\omega_2-\omega)\frac{t_o}{4}}{(\omega_2-\omega)\frac{t_o}{4}} \cdot \frac{\sin\phi_1\cos(\omega_1-\omega)\frac{t_o}{2}}{(1+\frac{\omega}{\omega_1})(1+\frac{\omega}{\omega_2})}$$

$$\frac{+\frac{\omega}{\omega_1}\cos\phi_1\sin(\omega_1-\omega)\frac{t_o}{2}\,)}{1} \cdot \frac{[\sin\phi_2\cos(\omega_2-\omega)\frac{t_o}{2}+\frac{\omega}{\omega_2}\cos\phi_2\sin(\omega_2-\omega)\frac{t_o}{2}}{1}$$

$$-2t_o^{\,2} \quad \frac{\sin^2(\omega_1-\omega)\frac{t_o}{4}}{(\omega_1-\omega)\frac{t_o}{4}} \cdot \frac{\sin^2(\omega_2-\omega)\frac{t_o}{4}}{(\omega_2-\omega)\frac{t_o}{4}}$$

$$\cdot \frac{[\sin\phi_1\sin(\omega_1-\omega)\frac{t_o}{2} - \frac{\omega}{\omega_1}\cos\phi_1\cos(\omega_1-\omega)\frac{t_o}{2}]\,[\sin\phi_2\cos(\omega_2-\omega)t_o/2}{(1+\frac{\omega}{\omega_1})\quad(1+\frac{\omega}{\omega_2})}$$

$$\frac{-\frac{\omega}{\omega_2}\cos\phi_2\cos(\omega_2-\omega)t_o/2]}{1} \qquad (63)$$

The first term of equation (57) after substituting equation (61) becomes

$$\frac{1}{4t_o^2}|H_1(\omega)+H_2(\omega)|^2 \cdot \sum_{L=-\infty}^{\infty} \delta(\omega-\frac{2\pi L}{t_o})$$

$$= \frac{1}{4t_o^2} \cdot A^2\{t_o^2 {\frac{\sin(\omega_1-\omega)t_o/2}{(\omega_1-\omega)t_o/2}}^2 \cdot \frac{\sin^2\phi_1+(\frac{\omega}{\omega_1})^2\cos^2\phi_1}{(1+\frac{\omega}{\omega_1})^2}$$

$$+ t_o^2 \frac{\sin(\omega_2-\omega)t_o/2}{(\omega_2-\omega)t_o/2} \frac{\sin^2\phi_2+(\frac{\omega}{\omega_2})^2\cos^2\phi_2}{(1+\frac{\omega}{\omega_2})^2}$$

$$+ \frac{2\{\sin^2[\omega-\frac{1}{2}(\omega_1+\omega_2)]t_o/4\}\{\omega^2\cos\phi_1\cos\phi_2+\omega_1\omega_2\sin\phi_1\sin\phi_2\}}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}$$

$$\sum_{L=-\infty}^{\infty} \delta(\omega-\frac{2\pi L}{t_o}) \qquad (64)$$

Since $\omega_1 = \frac{4\pi n}{t_o}$ and $\omega_2 = \frac{4\pi m}{t_o}$ and $\omega_1 \neq \omega_2$, then the above will result in

$$\frac{1}{4t_o^2}|H_1(\omega)+H_2(\omega)|^2 = \frac{A^2}{16}\delta(\omega-\omega_1)+\frac{A^2}{16}\delta(\omega+\omega_1)+\frac{A^2}{16}\delta(\omega-\omega_2)+\frac{A^2}{16}\delta(\omega+\omega_2) \qquad (65)$$

This is because

$$[\frac{\sin(\omega_1-\omega)t_o/2}{(\omega_1-\omega)t_o/2}]^2 \cdot \frac{\sin^2\phi_1+(\frac{\omega}{\omega_1})^2\cos^2\phi_1}{(1+\frac{\omega}{\omega_1})^2} \sum_{L=-\infty}^{\infty} \delta(\omega-\frac{2\pi L}{t_o})$$

$$= \begin{cases} 0 & \text{when } \omega \neq \pm\omega_1 \\ \frac{1}{4} & \text{when } \omega = \pm\omega_1 \end{cases}$$

and

$$\frac{\sin(\omega_2-\omega)t_o/2}{(\omega_1-\omega)t_o/2}\quad\frac{\sin^2\phi_1+(\frac{\omega}{\omega_2})^2\cos^2\phi_1}{(1+\frac{\omega}{\omega_2})^2}\quad\sum_{L=-\infty}^{\infty}\delta(\omega-\frac{2\pi L}{t_o})$$

$$=\begin{cases}0 & \text{when } \omega\neq\pm\omega_2\\ \frac{1}{4} & \text{when } \omega=\pm\omega_2\end{cases}$$

and the third term of equation (64) is zero.

Adding equation (63) to (65) the expression for the power spectral density of the FSK signal is

$$S_{FSK-C}(\omega) = \frac{A^2}{16}\delta(\omega-\omega_1)+\frac{A^2}{16}\delta(\omega+\omega_1)+\frac{A^2}{16}\delta(\omega-\omega_2)+\frac{A^2}{16}\delta(\omega+\omega_2)$$

$$+\frac{A^2t_o}{4}\left[\frac{\sin^2(\omega_1-\omega)\frac{t_o}{4}}{(\omega_1-\omega)t_o/4}\right]^2\cdot\left[\frac{\sin^2\phi_1+(\frac{\omega}{\omega_1})^2\cos^2\phi_1}{(1+\frac{\omega}{\omega_1})^2}\right]$$

$$+\frac{A^2t_o}{4}\left[\frac{\sin^2(\omega_2-\omega)t_o/4}{(\omega_2-\omega)t_o/4}\right]^2\left[\frac{\sin^2\phi_2+(\frac{\omega}{\omega_2})^2\cos^2\phi_2}{(1+\frac{\omega}{\omega_2})^2}\right]$$

$$-\frac{A^2t_o}{2}\;\frac{\sin^2(\omega_1-\omega)t_o/4}{(\omega_1-\omega)t_o/4}\cdot\frac{\sin^2(\omega_2-\omega)t_o/4}{(\omega_2-\omega)t_o/4}\cdot\frac{[\sin\phi_1\cos(\omega_1-\omega)t_o/2}{1}$$

$$\underline{+\frac{\omega}{\omega_1}\cos\phi_1\cdot\sin(\omega_1-\omega)t_o/2]\cdot[\sin\phi_2\cos(\omega_2-\omega)t_o/2+\frac{\omega}{\omega_2}\cos\phi_2\sin(\omega_2-\omega)t_o/}$$

$$
-\frac{A^2 t_o}{2}\cdot\frac{\sin^2(\omega_1-\omega)t_o/4}{(\omega_1-\omega)t_o/4}\cdot\frac{\sin^2(\omega_2-\omega)t_o/4}{(\omega_2-\omega)t_o/4}
$$

$$
\cdot\frac{[\sin\phi_1\sin(\omega_1-\omega)t_o/2-\frac{\omega}{\omega_1}\cos\phi_1\cos(\omega_1-\omega)t_o/2]}{(1+\frac{\omega}{\omega_1})(1+\frac{\omega}{\omega_2})}
$$

$$
\cdot\frac{[\sin\phi_2\cos(\omega_2-\omega)t_o/2-\frac{\omega}{\omega_2}\cos\phi_2\cos(\omega_2-\omega)t_o/2]}{1} \qquad (66)
$$

Assuming ϕ_1, ϕ_2 to be two independent random variables, each uniformly distributed between 0 and 2π, and taking the expected value of $S_{FSK-C}(\omega)$ with respect to ϕ_1 and ϕ_2, we get the power spectrum of the noncoherent case. This can be done easily by inspection to obtain

$$
S_{FSK-n}(\omega) = \frac{A^2}{16}\delta(\omega-\omega_1)+\frac{A^2}{16}\delta(\omega+\omega_1)+\frac{A^2}{16}\delta(\omega-\omega_2)+\frac{A^2}{16}\delta(\omega+\omega_2)
$$

$$
+\frac{A^2 t_o}{8}\left[\frac{\sin^2(\omega_1-\omega)t_o/4}{(\omega_1-\omega)t_o/4}\right]^2\left[\frac{\overset{1+(\frac{\omega}{\omega})^2}{1}}{(1+\frac{\omega}{\omega_2})^2}\right]
$$

$$
+\frac{A^2 t_o}{8}\left[\frac{\sin^2(\omega_2-\omega)t_o/4}{(\omega_2-\omega)t_o/4}\right]^2\left[\frac{1+(\frac{\omega}{\omega_2})^2}{(1+\frac{\omega}{\omega_2})^2}\right] \qquad (67)
$$

2

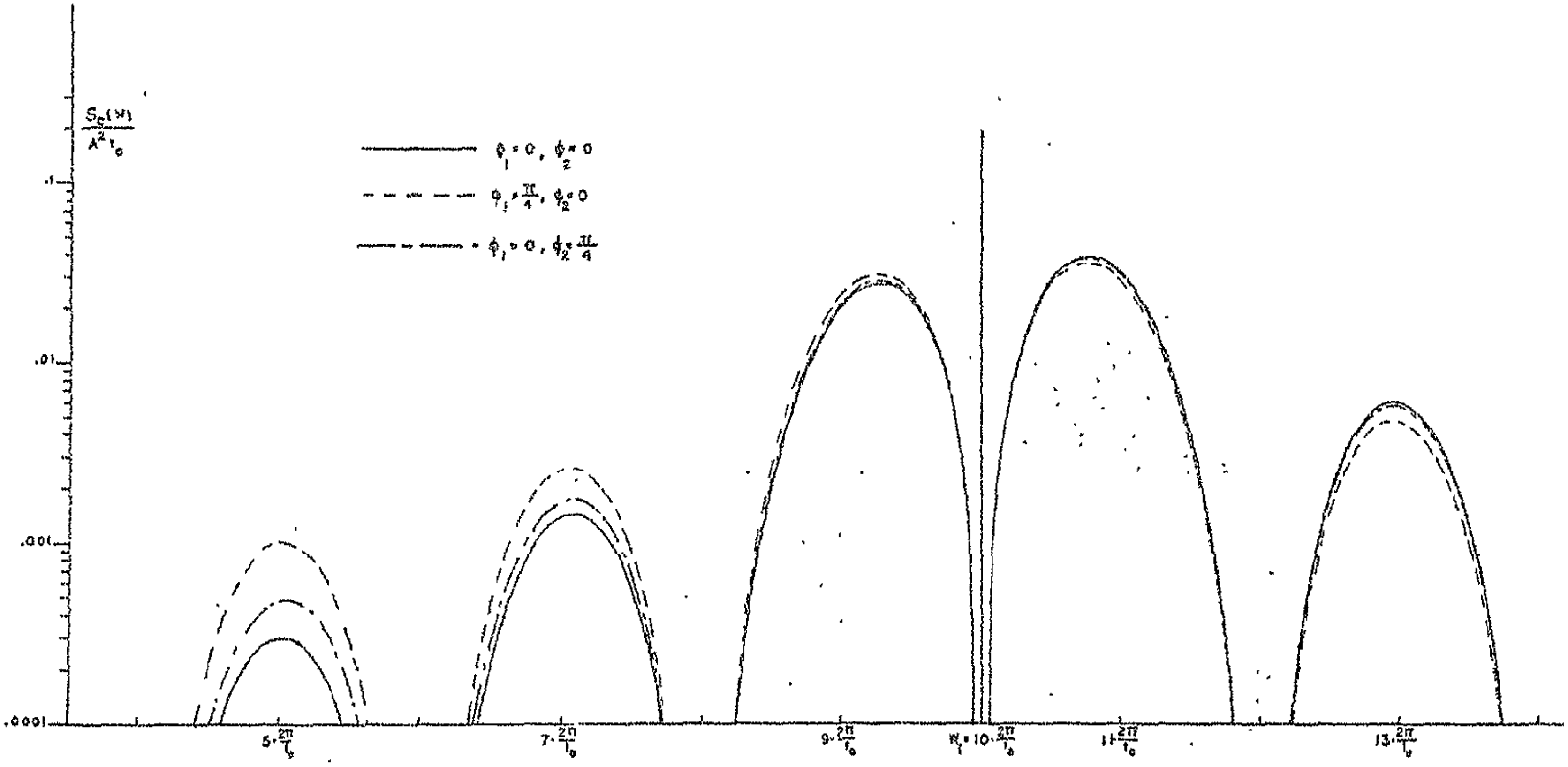

FIG. 7 POWER SPECTRAL DENSITY OF COHERENT FSK SIGNAL.

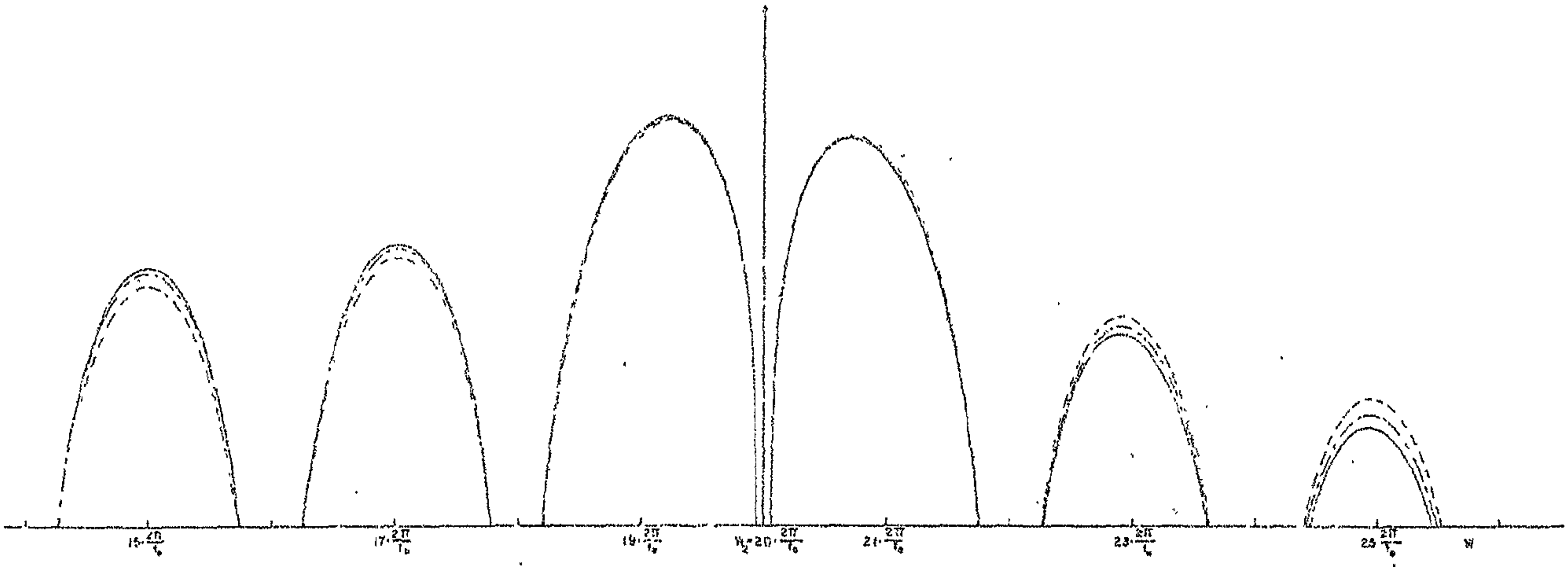

FIG. 7 POWER SPECTRAL DENSITY OF COHERENT FSK SIGNAL.

The power spectral density of the FSK signal of equation (66) is shown on Figure 7, for $\omega_2 = 2\omega_1$ and for the following cases:

(a) $\phi_1 = 0$, $\phi_2 = 0$

(b) $\phi_1 = \frac{\pi}{4}$, $\phi_2 = 0$

(c) $\phi_1 = 0$, $\phi_2 = \frac{\pi}{4}$.

The FSK signal corresponding to case (a) will have the smallest bandwidth and this should be expected because there are no discontinuities in the signal. The maximum bandwidth will correspond to the case where $\phi_1 = 0$, and $\phi_2 = 180$ or visa versa. Cases (b) and (c) will correspond to the average bandwidth.

The power spectral density for the noncoherent FSK signal of equation (67) is plotted in Figure 8.

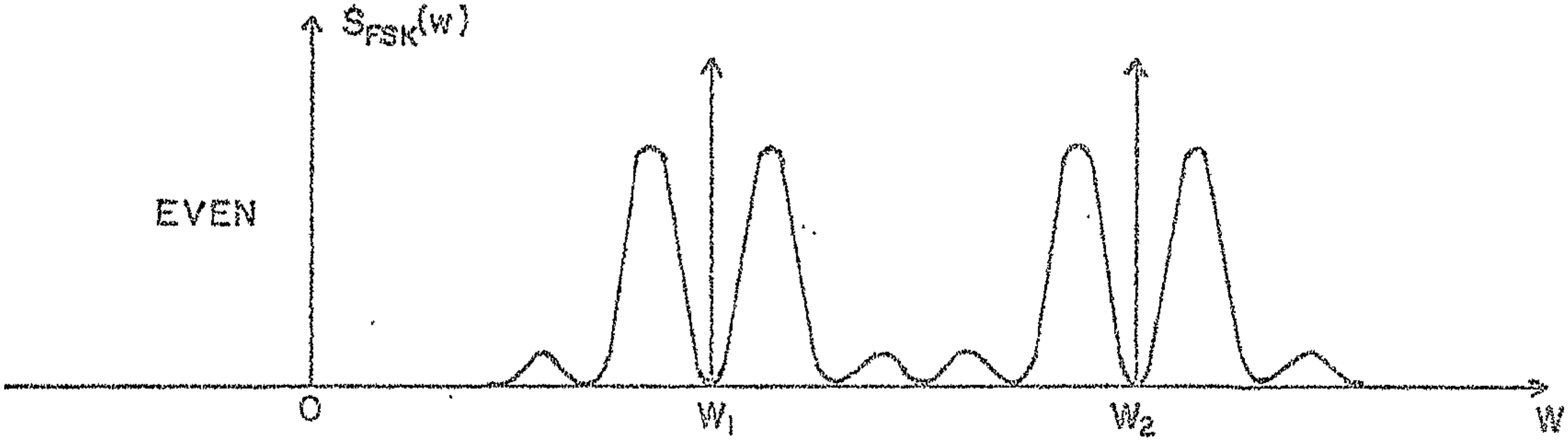

FIG.8. POWER SPECTRAL DENSITY OF NON-COHERENT FSK SIGNAL.

VI. SUMMARY AND CONCLUSIONS

This report presents the power spectral density calculations for a carrier which is modulated in amplitude, phase or frequency by a split-phase PCM code. Such modulation can, in general, be either phase coherent or phase incoherent; both cases have been investigated in detail.

Equation (36) gives the expression for the power spectral density for the coherent case of amplitude modulation. It was shown that when the transitions occur at the peaks of the carrier, maximum bandwidth is required for transmitting the signal. This corresponds to Φ equal to zero or multiple of π. Minimum bandwidth is required when Φ is an odd multiple of $\frac{\pi}{2}$.

Equation (48) gives the expression of the power spectral density for the noncoherent case of amplitude modulation.

The calculation of the power spectral density of the phase-shift Keyed (PSK) by a split-phase has been summarized on pages 37 through 40. Equations (53) and (54) give the expressions for the power spectral density for the coherent case and the noncoherent case, respectively. It should be observed that the sideband power can be maximized by letting βV equal to $K \frac{\pi}{2}$, where K is an odd integer, β is the modulating index and V is the amplitude of the split-phase code. Also, maximum bandwidth corresponds to the case when Φ is an odd multiple of $\frac{\pi}{2}$, and minimum bandwidth corresponds to the case when Φ is a multiple of π.

The power spectral density of the FSK signal has been calculated on pages 40 through 60. Equation (66) gives the expression for the power spectral density of the coherent case of FSK signal and equation (67) gives the expression for the noncoherent case. It should be observed that minimum bandwidth will correspond to the case when Φ_1 and Φ_2 equal to $n\pi$, where n is zero or an even integer. Maximum bandwidth will result when $\Phi_1 = n\pi$ and $\Phi_2 = m\pi$ where n is zero or an even integer and m is an odd integer.

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