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TECHNICAL MEMORANDUM

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TITLE- Differential Equations Soluble in Finite Terms of Elementary Functions

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ABSTRACT

~~It is shown that~~ the solution of a second order ordinary linear differential equation in the form $\frac{d^2y}{dx^2} + \mu(x)\frac{dy}{dx} + \omega^2(x)y = 0$ is expressible in finite terms by means of elementary functions, provided the coefficients $\mu(x)$ and $\omega^2(x)$ are related in certain specific ways. By employing the general solution in the form $y=C(x)e^{\pm iQ(x)}$, two coupled nonlinear differential equations of second order are obtained, and one is able to classify various specific relations between the coefficients via these equations. The differential equations thus obtained include those hypergeometric equations which, owing to specifically assigned values of constants in their coefficients, can be solved in terms of elementary functions.

The conditions of integrability in finite terms for the Malmstén equation $\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + \left(bx^m + \frac{c}{x^2}\right)y = 0$ are obtained immediately by the present method.

Differential equations soluble in finite terms are thus classified in a manner similar to a table of integrals so that one can recognize the variant forms of differential equations and find their solutions from the table.

This work arose in search for analytical solutions to a linearized form of the restricted three-body problem.

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955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

SUBJECT: Differential Equations Soluble in Finite Terms of Elementary Functions - Case 103-7

DATE: March 26, 1970

FROM: C. C. H. Tang

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TECHNICAL MEMORANDUM

1.0 INTRODUCTION

To define precisely the scope of the paper, we shall study only the problem of solving classes of second order ordinary linear differential equations in finite terms by means of algebraic and elementary transcendental functions. The standard form of the equation will be taken to be

$$\frac{d^2y}{dx^2} + \mu(x)\frac{dy}{dx} + \omega^2(x)y = 0, \qquad (1)$$

and it will be assumed that there is a domain in which both $\mu(x)$ and $\omega^2(x)$ are real and analytic except at a finite number of poles. The behavior of solutions in the neighborhood of singular points will be investigated briefly.

To appreciate the difficulties in solving eq. (1) with arbitrary coefficients $\mu(x)$ and $\omega^2(x)$, we quote Ince:[1] "Apart from equations with constant coefficients, and such equations as can be derived therefrom by a change of independent variable, there is no known type of linear equation of general order n which can be fully and explicitly integrated in terms of elementary functions." Accordingly for a differential equation with arbitrary coefficients, its solution has to be expressed in an infinite form, i.e., an infinite series, an infinite continued fraction, or a definite integral. Thus, most equations which arise out of problems of physics and applied mathematics have their solutions expressible only in terms of new or higher transcendental functions, such as hypergeometric functions for hypergeometric equations and Bessell functions for Bessel equations except for those special cases, e.g., when the order of the Bessel equation is an odd half-integer. In fact, Liouville[2] has shown the impossibility of integrating Bessel equations in finite terms of elementary functions except for orders of an odd half-integer.

It is true that most differential equations arising in practice can be solved by brute-force numerical integration. The method of numerical integration can yield useful numbers but not the insight that an analytic solution can supply. In addition, analytic solutions usually save enormous computing time.

The purpose of this paper is to show that the solution of eq. (1) is expressible in a general form, in finite terms of elementary functions, provided the coefficients $\mu(x)$ and $\omega^2(x)$ of eq. (1) are nonarbitrary or related in a specific fashion. A major portion of the paper will deal with the process of showing criteria under which $\mu(x)$ and $\omega^2(x)$ can be related in various specific fashions. In the process we also obtain directly the conditions of integrability in finite terms for Malmstén's equation[3]

$$\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + \left(bx^m + \frac{c}{x^2}\right) y = 0,$$

which clearly is a generalization of Bessel's equation and was investigated in 1850 by Malmstén.

Differential equations soluble in finite terms are classified in a manner similar to a table of integrals so that one can recognize the variant forms of differential equations and find their solutions from the table. This table concludes the paper.

2.0 FORMULATION

In an attempt to obtain the general solution of eq. (1) we assume that it takes the following general form:[4]

$$y = C(x)e^{\pm iQ(x)}, \tag{2}$$

where C(x) and Q(x) are functions to be determined in terms of the coefficients $\mu(x)$ and $\omega^2(x)$ of eq. (1). For simplicity we assume that both $\mu(x)$ and $\omega^2(x)$ are either real or imaginary. Substituting eq. (2) in eq. (1), we obtain

$$Q'' + \mu Q' + 2\frac{C'}{C}\,Q' = 0, \tag{3}$$

and

$$\left(\frac{C'}{C}\right)' + \left(\frac{C'}{C}\right)^2 + \mu\frac{C'}{C} + \omega^2 - (Q')^2 = 0, \tag{4}$$

where single prime and double primes indicate first and second derivatives, respectively, with respect to x. Equations (3) and (4) are coupled nonlinear differential equations of second order. At a first glance, we might think that the solution in the form of equation (2) may represent a step backward in solving equation (1). Fortunately equation (3) can be neatly integrated for C in terms of Q' and μ, and its solution is:

$$C = \frac{C_o}{\sqrt{Q'}} e^{-1/2 \int_{x_o}^{x} \mu dx} , \tag{5}$$

where C_o is a constant of integration. The formal solution of eq. (1), therefore, is

$$y = \frac{C_o}{\sqrt{Q'}} e^{\pm iQ} e^{-1/2 \int_{x_o}^{x} \mu dx} , \tag{6}$$

where Q' is the solution of the following nonlinear differential equation obtained by substituting eq. (3) into eq. (4):

$$\frac{1}{2}\left[\left(\frac{Q''}{Q'}\right)' - \frac{1}{2}\left(\frac{Q''}{Q'}\right)^2 + 2(Q')^2\right] = \omega^2 - \frac{\mu'}{2} - \frac{\mu^2}{4} . \tag{7}$$

Through the transformation

$$Q' = \pm \frac{i}{2} \frac{z'}{z}, \tag{8}$$

eq. (7) can be converted into

$$\frac{1}{2}\left[\left(\frac{z''}{z'}\right)' - \frac{1}{2}\left(\frac{z''}{z'}\right)^2\right] = \left(\omega^2 - \frac{\mu'}{2} - \frac{\mu^2}{4}\right). \tag{9}$$

Equation (9) can be recognized as a nonlinear differential equation in Riccati form. Substituting eq. (8) into eq. (6), we can also obtain the formal solution of eq. (1) in terms of z and z' as

$$y = (z')^{-1/2} \left[C_1 + C_2 z\right] e^{-1/2 \int \mu dx} \tag{10}$$

where z' is the solution of eq. (9) and C_1 and C_2 are arbitrary constants.

Our task of solving the second order ordinary linear differential equation (1) has been converted into that of solving the nonlinear differential equation (7) or (9). The formidable task of solving these equations in finite terms for arbitrarily prescribed $\mu(x)$ and $\omega^2(x)$ may not appear to be too helpful in general, since it is clear that only a limited number of classes of $\mu(x)$ and $\omega^2(x)$ will render eq. (7) or (9) soluble in finite terms. In fact, the only cases in which a class of Riccati equation is integrable in finite terms are the classical cases discovered by Daniel Bernoulli.[5] Examples to solve eq. (9) in finite terms for "arbitrarily" prescribed $\mu(x)$ and $\omega^2(x)$ will be shown later. Equation (9), however, can be solved approximately in finite terms for arbitrarily prescribed $\mu(x)$ and $\omega^2(x)$, and we shall show this briefly later to avoid a lengthy diversion. For the moment, we will concentrate on finding solutions of either eq. (7) or (9) in finite terms, when the $\mu(x)$ and $\omega^2(x)$ are related in a specific fashion. The choice of using eq. (7) or (9) depends on the specific relation between $\mu(x)$ and $\omega^2(x)$ in the differential equation to be solved.

In an effort to show the significance of eq. (7), we show that the following five equations all have the same Q equation (i.e., eq. (7)):

$$y'' + \frac{(2n+1)}{x} y' + \xi^2 y = 0$$

$$y'' + \frac{1}{x}y' + \left(\xi^2 - \frac{n^2}{x^2}\right) y = 0$$

$$y'' + \frac{2}{x}y' + \left(\xi^2 - \frac{n^2-\frac{1}{4}}{x^2}\right) y = 0$$

$$y'' \pm i2\xi y' - \frac{n^2-\frac{1}{4}}{x^2} y = 0$$

$$y'' + \left(\xi^2 - \frac{n^2-\frac{1}{4}}{x^2}\right) y = 0$$

These five equations are transformed equations of one another by a change of dependent variable, and their solutions all depend on the solution of the invariant Q equation

$$\left(\frac{Q''}{Q'}\right)' - \frac{1}{2}\left(\frac{Q''}{Q'}\right)^2 + 2(Q')^2 = 2\left(\xi^2 - \frac{n^2-\frac{1}{4}}{x^2}\right) .$$

In passing it is interesting to note that the exponential factor in eq. (5) is exactly the standard transformation used in reducing eq. (1) into the normal form

$$Y'' + F(x)Y = 0 \quad , \tag{11}$$

where

$$Y = ye^{\frac{1}{2}\int \mu dx} = \frac{C_o}{\sqrt{Q'}} e^{\pm iQ} , \tag{12}$$

and

$$F(x) = \omega^2 - \frac{\mu'}{2} - \frac{\mu^2}{4} . \tag{13}$$

Note the important fact that eq. (13) is identical to eq. (7). It is well-known that eq. (11) can be transformed into the canonical Riccati form

$$Z' + Z^2 = -\left(\omega^2 - \frac{\mu'}{2} - \frac{\mu^2}{4}\right) \quad , \tag{14}$$

through the transformation

$$Y = e^{\int Z dx} \quad . \tag{14a}$$

The identity between equations (9) and (14) can be established by noting

$$Z = -\frac{1}{2}\frac{z''}{z'} \quad . \tag{14b}$$

Accordingly it is not surprising that eq. (7) can be transformed into the Riccati form of eq. (9).

It is also important to note the similarity between the solution of eq. (11) in the form of eq. (12) and that in the form of the WKB method of approximation. A comparison study between the two forms of solutions should be illuminating but will not be carried out here.

3.0 DIFFERENTIAL EQUATIONS SOLUBLE IN FINITE TERMS OF ELEMENTARY FUNCTIONS

Under the constraint that the coefficients $\mu(x)$ and $\omega^2(x)$ of eq. (1) can be related in a specific fashion, we shall first obtain constraining relations between $\mu(x)$ and $\omega^2(x)$ from either eq. (7) or eq. (9) by simple assumptions and then solve the resulting nonlinear differential equation. In this way both the specific form of the differential equation and its solution can be obtained simultaneously. We start from the simplest case of both $\mu(x)$ and $\omega^2(x)$ being constant.

(I) Let both $\mu(x)$ and $\omega^2(x)$ be constant:

(I) (A) For $\omega_c^2 - \frac{\mu_c^2}{4}$ = constant $\xi^2 \neq 0$:

In this case it is simpler to use eq. (9) than eq. (7), and eq. (9) becomes

$$\left(\frac{z''}{z'}\right)' - \frac{1}{2}\left(\frac{z''}{z'}\right)^2 = 2\left(\omega_c^2 - \frac{\mu_c^2}{4}\right) = 2\xi^2 \tag{15}$$

By inspection we obtain a particular solution of eq. (15) as

$$\frac{z''}{z'} = \pm i2\xi \tag{16}$$

Euler[6] has shown that if a particular solution of a Riccati equation is known the general solution can be effected by two quadratures. By his method the general solution of eq. (15) can be shown to be

$$\frac{z''}{z'} = -i2\xi \, \frac{e^{i\xi x} - ke^{-i\xi x}}{e^{i\xi x} + ke^{-i\xi x}} , \tag{16a}$$

where k is an arbitrary constant.

Equation (16a) reduces to eq. (16)

$$\frac{z''}{z'} = -i2\xi \quad \text{for } k=0; \qquad \frac{z''}{z'} = i2\xi \quad \text{for } k = \pm\infty$$

Equation (16a) becomes

$$\frac{z''}{z'} = 2\xi \tan\ (\xi x + \phi) \qquad \text{for } k>0$$

$$\text{or } -2\xi \cot\ (\xi x + \phi) \qquad \text{for } k<0\ , \tag{16b}$$

where ϕ is the phase constant and is equal to zero for $k = \pm 1$. Integrating either the particular solution [eq. (16)] or the general solution [eq. (16a)] to obtain z' and z, we have from eq. (10)

$$y = e^{-\frac{1}{2}\mu_c x}\left[C_1 e^{i\sqrt{\omega_c^2 - \frac{1}{4}\mu_c^2}\ x} + C_2 e^{-i\sqrt{\omega_c^2 - \frac{1}{4}\mu_c^2}\ x}\right] \tag{17}$$

as the well-known general solution of the differential equation with constant coefficients:

$$y'' + \mu_c y' + \omega_c^2 y = 0\ . \tag{17a}$$

The normal form of equation (17a) becomes:

$$Y'' + \left(\omega_c^2 - \frac{1}{4}\mu_c^2\right) Y = 0\ . \tag{17b}$$

(I) (B) For $\omega_c^2 - \frac{\mu_c^2}{4} = 0$:

In this case the general solution of eq. (15) is:

$$\frac{z''}{z'} = \frac{k}{1 - \frac{1}{2}kx}\ . \tag{18}$$

Equation (18) reduces to the particular solution $\frac{z''}{z'} = 0$ for $k = 0$. The solution of eq. (17a) then is, from eq. (10),

$$y = \left(1 - \frac{1}{2}kx\right)\left[C_1 + C_2\frac{2}{k}\left(1 - \frac{1}{2}kx\right)^{-1}\right]e^{-\frac{1}{2}\mu}c^x = \left(C_3 + C_4x\right)e^{-\frac{1}{2}\mu}c^x \quad (19)$$

(II) Let $\omega^2 = \mu'$ or 0: (20)

Equation (9) then becomes

$$\left(\frac{z''}{z'}\right)' - \frac{1}{2}\left(\frac{z''}{z'}\right)^2 = \mu' - \frac{1}{2}\mu^2 \text{ or } -\mu' - \frac{1}{2}\mu^2 \quad (21)$$

By inspection we find that a particular solution of eq. (21) is μ or $-\mu$ and the general solution is:

$$\frac{z''}{z'} = \mu + \frac{ke^{\int\mu dx}}{1 - \frac{k}{2}\int e^{\int\mu dx}\,dx}, \text{or } -\mu + \frac{ke^{-\int\mu dx}}{1 - \frac{k}{2}\int e^{-\int\mu dx}dx}. \quad (22)$$

When k = 0, eq. (22) reduces to the particular solution. Integrating $\frac{z''}{z'} = \mu$ or eq. (22), we obtain from eq. (10)

$$y = e^{-\int\mu dx}\left(C_1 + C_2\int e^{\int\mu dx}\,dx\right) \text{ or} \quad (23)$$

$$y = C_3 + C_4\int e^{-\int\mu dx}\,dx$$

as the general solution of

$$y'' + \mu y' + \mu' y = 0 \quad \text{or } y'' + \mu y' = 0. \quad (24)$$

The normal form of eq. (24) becomes

$$Y'' + \frac{1}{2}\left(\mu' - \frac{1}{2}\mu^2\right)Y = 0 \text{ or } Y'' - \frac{1}{2}(\mu' + \frac{1}{2}\mu^2)\ Y = 0 \quad (25)$$

Note that eq. (23) can be obtained by two simple quadratures of eq. (24)

(III) Let $\omega^2 = (Q')^2 + \mu'$: (26)

In this case we shall use eq. (7) instead of eq. (9), and eq. (7) becomes

$$\left(\frac{Q''}{Q'}\right)' - \frac{1}{2}\left(\frac{Q''}{Q'}\right)^2 = \mu' - \frac{1}{2}\mu^2 \quad , \tag{27}$$

Equation (27) is identical to eq. (21) and its solution is shown in eq. (22). Integrating eq. (22), we obtain

$$Q' = Ke^{\int\mu dx}\left(1 - \frac{k}{2}\int e^{\int\mu dx}\, dx\right)^{-2} \tag{28}$$

$$Q = \frac{2K}{k}\left(1 - \frac{k}{2}\int e^{\int\mu dx} dx\right)^{-1} + K_1 \quad , \quad \text{when } k \neq 0. \tag{29}$$

$$= K\int e^{\int\mu dx} dx \quad , \qquad \text{when} \quad k = 0 \quad . \tag{29a}$$

Equation (1) then takes the form

$$y'' + \mu y' + \left[K^2 e^{2\int\mu dx}\left(1 - \frac{k}{2}\int e^{\int\mu dx} dx\right)^{-4} + \mu'\right] y=0 \tag{30}$$

and its solution is, from eq. (6),

$$y = \left(1 - \frac{k}{2}\int e^{\int\mu dx} dx\right) e^{-\int\mu dx}\, e^{\pm i\frac{2K}{k}\left(1-\frac{k}{2}\int e^{\int\mu dx} dx\right)^{-1}} \tag{30a}$$

The normal form of eq. (30) becomes, from eq. (11),

$$Y'' + \left[K^2 e^{2\int\mu dx}\left(1-\frac{k}{2}\int e^{\int\mu dx} dx\right)^{-4} + \frac{\mu'}{2} - \frac{\mu^2}{4}\right] Y=0 \tag{30b}$$

The solution of eq. (30b) can be obtained from eq. (12). Since k is an arbitrary constant we first simplify the above equation by letting k = 0:

$$y'' + \mu y' + \left(K^2 e^{2\int \mu dx} + \mu'\right) y = 0 \tag{31}$$

$$y = e^{-\int \mu dx} e^{\pm iK \int e^{\int \mu dx} dx} \tag{31a}$$

$$Y'' + \left(K^2 e^{2\int \mu dx} + \frac{\mu'}{2} - \frac{\mu^2}{4}\right) Y = 0 \tag{31b}$$

Letting the functional dependence of μ on x assume various forms, we obtain the following cases:

(III) (A) (1) For μ = constant α :

$$y'' + \alpha y' + \beta^2 e^{2\alpha x} y = 0 \tag{32}$$

$$y = \alpha e^{-\alpha x} e^{\pm i\frac{\beta}{\alpha} e^{\alpha x}} \tag{32a}$$

$$Y'' + \left(\beta^2 e^{2\alpha x} - \frac{\alpha^2}{4}\right) Y = 0 \tag{32b}$$

(III) (A) (2) For $\mu = \alpha x$:

$$y'' + \alpha x y' + \left(\beta^2 e^{\alpha x^2} + \alpha\right) y = 0 \tag{33}$$

$$y = \exp\left(-\frac{\alpha}{2}x^2 \pm i\beta\int \exp\frac{\alpha}{2}x^2\,dx\right) \tag{33a}$$

$$Y'' + \left(\beta^2 e^{\alpha x^2} - \frac{\alpha^2 x^2}{4} + \frac{\alpha}{2}\right) y = 0 \tag{33b}$$

(III) (A) (3) For $\mu = \frac{\alpha_1}{\alpha_2 + x}$, $\alpha_1 \neq 1$:

$$y'' + \frac{\alpha_1}{\alpha_2+x}y' + \left[\beta^2\left(\alpha_2 + x\right)^{2\alpha_1} - \frac{\alpha_1}{(\alpha_2+x)^2}\right] y = 0 \tag{34}$$

$$y = (\alpha_2+x)^{-\alpha_1} e^{\pm i\frac{\beta}{\alpha_1+1}(\alpha_2+x)^{\alpha_1+1}} \tag{34a}$$

$$Y'' + \left[\beta^2(\alpha_2+x)^{2\alpha_1} - \frac{\alpha_1(\alpha_1+2)}{4(\alpha_2+x)^2}\right] Y = 0. \tag{34b}$$

It is important to note that we can identify eq. (34) with the transformed Bessel equation in the form[7]

$$y'' + \frac{1}{x}y' + 4\left(x^2 - \frac{n^2}{x^2}\right) y = 0 \quad , \quad y = Z_n(x^2) \quad . \tag{35}$$

Clearly eq. (34) and eq. (35) become identical, when

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \beta = \pm 2, \text{ and } n = \pm\frac{1}{2} \quad .$$

Accordingly, the solution of Bessel equation can be expressed by means of elementary functions in finite terms when its order is $\pm\frac{1}{2}$ and has the form, from eq. (34a),

$$y = \frac{1}{x}\left(C_1 e^{ix^2} + C_2 e^{-ix^2}\right) = Z_{\pm\frac{1}{2}}(x^2) \quad . \tag{36}$$

It is also worthwhile to note the similarity between eq. (34) for $\alpha_2=0$ and Malmstén's differential equation[3]

$$y'' + \frac{a}{x} y' + \left(bx^m + \frac{c}{x^2}\right) y = 0 \quad , \tag{37}$$

for which Malmstén investigated conditions of integrability in finite terms. We shall return to this equation later.

(III) (A) (4) For $\mu = \frac{a}{x-a_o} + \frac{b}{x-b_o} + \frac{c}{x-c_o}$:

$$y'' + \left(\frac{a}{x-a_o} + \frac{b}{x-b_o} + \frac{c}{x-c_o}\right) y' +$$

$$\left[\beta^2 (x-a_o)^{2a}(x-b_o)^{2b}(x-c_o)^{2c} - \frac{a}{(x-a_o)^2} - \frac{b}{(x-b_o)^2} - \frac{c}{(x-c_o)^2}\right] y=0, \tag{38}$$

$$y = (x-a_o)^{-a}(x-b_o)^{-b}(x-c_o)^{-c}\, e^{\pm i\beta\int (x-a_o)^a (x-b_o)^b (x-c_o)^c dx} \ , \tag{38a}$$

$$Y'' + \left\{\beta^2 (x-a_o)^{2a}(x-b_o)^{2b}(x-c_o)^{2c} - \frac{1}{2}\left[\frac{a}{(x-a_o)^2} + \frac{b}{(x-b_o)^2} + \frac{c}{(x-c_o)^2}\right]\right.$$

$$\left. - \frac{1}{4}\left[\frac{a}{x-a_o} + \frac{b}{x-b_o} + \frac{c}{x-c_o}\right]^2\right\} y = 0, \tag{38b}$$

Equation (38) appears to be a differential equation of great generality and in fact for a=b=c=1 it is in a form similar to the

Bôcher* equation.[8] If we set $\beta=0$, eq. (38) is in a form similar to the Riemann or Popperitz equation,[9] from which the hypergeometric equation can be derived. When $\beta=0$, the solution of eq. (38) becomes

$$\left.\begin{aligned} y &= (x-a_o)^{-a}(x-b_o)^{-b}(x-c_o)^{-c} \\ &\left[C_1 + C_2\int (x-a_o)^{a}(x-b_o)^{b}(x-c_o)^{c}dx\right] \end{aligned}\right\} \qquad (39)$$

(III) (A) (5) For $\mu = \alpha_1 + \frac{\alpha_2}{x}$:

$$y'' + \left(\alpha_1 + \frac{\alpha_2}{x}\right) y' + \left(\beta^2 x^{2\alpha_2} e^{2\alpha_1 x} - \frac{\alpha_2}{x^2}\right) y = 0 \quad (40)$$

$$y = \frac{1}{x^{\alpha_2} e^{\alpha_1 x}} \; e^{\pm i\beta\int x^{\alpha_2} e^{\alpha_1 x}dx} , \qquad (40a)$$

$$Y'' + \left[\beta^2 x^{2\alpha_2} e^{2\alpha_1 x} - \frac{1}{4}\left(\alpha_1+\frac{\alpha_2}{x}\right)^2 - \frac{\alpha_2}{2x^2}\right] Y = 0 \quad (40b)$$

It is obvious that various other differential equations and their respective solutions can be obtained by expressing μ as other functions of x but we do not attempt here to exhaust all possible cases.

Now we return to the general case of eq. (30) and eq. (30a) when k is any arbitrary nonzero constant.

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*The Bôcher equation is $y''+P(x)y'+Q(x)y=0$, where

$$P(x)=\frac{1}{2}\left[\frac{m_1}{x-a_1}+\frac{m_2}{x-a_2}+\dots\frac{m_{n-1}}{x-a_{n-1}}\right],\quad Q(x)=\frac{1}{4}\left[\frac{A_o+A_1x+\dots+A_\ell x^\ell}{(x-a_1)^{m_1}(x-a_2)^{m_2}\dots(x-a_{n-1})^{m_{n-1}}}\right]$$

and m_i, n and ℓ are non-negative integers.

(III) (B) (1) For μ= constant α:

$$y'' + \alpha y' + \left\{\frac{\beta^2 e^{2\alpha x}}{\left[1-\frac{k}{2\alpha}(\gamma e^{\alpha x}-1)\right]^4}\right\} y = 0, \quad \gamma = e^{-\alpha x_o}, \quad (41)$$

$$y = \left[1-\frac{k}{2\alpha}(\gamma e^{\alpha x}-1)\right] \exp\left\{-\alpha x \pm i \frac{2\beta}{k\gamma\left[1-\frac{k}{2\alpha}(\alpha e^{\alpha x}-1)\right]}\right\} \quad (41a)$$

$$Y'' + \left\{\frac{\beta^2 e^{2\alpha x}}{\left[1-\frac{k}{2\alpha}(\gamma e^{\alpha x}-1)\right]^4} - \frac{\alpha^2}{4}\right\} Y = 0 \quad (41b)$$

The degenerate case of eq. (41) can be obtained by letting α=0, and it becomes

$$y'' + \frac{\beta^2}{\left[(1-\frac{kx}{2})\right]^4} y = 0 = Y'' + \frac{\beta^2}{\left[(1-\frac{kx}{2})\right]^4} Y, \quad (42)$$

with

$$y = \left(1 - \frac{kx}{2}\right) e^{\pm i \frac{4\beta}{k(2-kx)}} = Y. \quad (42a)$$

(III) (B) (2) For $\mu = \frac{\alpha}{x}$, $\alpha \neq -1$:

$$y'' + \frac{\alpha}{x}y' + \left[\frac{\beta^2 x^{2\alpha}}{\left(1-\frac{k\gamma}{2}\frac{x^{\alpha+1}}{\alpha+1}\right)^4} - \frac{\alpha}{x^2}\right] y = 0, \quad \gamma = x_o^{-\alpha} \quad (43)$$

$$y = \left(1 - \frac{k\alpha}{2}\frac{x^{\alpha+1}}{\alpha+1}\right) x^{-\alpha} e^{\pm i \frac{2\beta}{k\gamma\left(1-\frac{k\gamma}{2}\frac{x^{\alpha+1}}{\alpha+1}\right)}} \quad (43a)$$

$$Y'' + \left[\frac{\beta^2 x^2}{\left(1 - \frac{k\gamma}{2} \frac{x^{\alpha+1}}{\alpha+1}\right)^4} - \frac{\alpha(\alpha+2)}{4x^2} \right] Y = 0 \quad . \qquad (43b)$$

Note eq. (43) also reduces to eq. (42) when $\alpha=0$. When $\alpha=1$, eqs. (30) and (30a) have to be used directly.

(III) (B) (3) For μ equal to other function of x.

(IV) Let $Q' =$ constant ξ, $\xi \neq 0$: (44)

Equation (7) becomes

$$(Q')^2 = \omega^2 - \frac{\mu'}{2} - \frac{\mu^2}{4} = \xi^2 \quad . \qquad (45)$$

(IV) (A) For constant μ_c:

Equation (45) becomes

$$Q' = \omega^2 - \frac{\mu_c^2}{4} = \xi^2 \quad . \qquad (46)$$

Equation (46) indicates that ω^2 must be a constant also, and therefore this case reduces identically to case (I).

(IV) (B) For constant ω_c^2 :

Equation (45) becomes

$$\mu' + \frac{1}{2}\mu^2 = 2\left[\omega_c^2 - (Q')^2\right] = 2\left(\omega_c^2 - \xi^2\right) = 2\lambda^2 , \qquad (47)$$

where

$$\lambda^2 = \omega_c^2 - \xi^2 = \omega_c^2 - (Q')^2$$

Equation (47) is similar to eq. (15) and its general solution is

$$\mu = 2\lambda \frac{e^{\lambda x} - ke^{-\lambda x}}{e^{\lambda x} + ke^{-\lambda x}} \quad . \tag{48}$$

From eq. (6), we have

$$y = \frac{C_1 e^{i\xi x} + C_2 e^{-i\xi x}}{e^{\lambda x} + ke^{-\lambda x}} \quad , \tag{49}$$

as the solution of

$$y'' + 2\lambda \frac{e^{\lambda x} - ke^{-\lambda x}}{e^{\lambda x} + ke^{-\lambda x}} \; y' + \omega_c^2 \, y = 0 \; , \tag{49a}$$

which has the normal form

$$Y'' + \;(\omega_c^2 - \lambda^2)\; Y = Y'' + \xi^2 \; Y = 0 \quad . \tag{49b}$$

(IV) (B) (1) For k=0 this case reduces to case (IV) (A).

(IV) (B) (2) For k>0 :

Equation (49a) reduces to

$$y'' + 2\lambda \; \tan h(\lambda x + \phi) \; y' + \omega_c^2 y = 0 \; , \tag{49c}$$

where ϕ is a phase constant and is equal to zero for k=1.

(IV) (B) (3) For k< 0 :

$$y'' + 2\lambda \; \cot h \; (\lambda x + \phi) \; y' + \omega_c^2 y = 0 \; , \tag{49d}$$

where ϕ=0 for k= -1 .

(V) Let $\frac{Q''}{Q'} = -\mu$: (50)

Equation (7) then reduces to the condition that

$$\omega = Q' = Ke^{-\int \mu dx}. \tag{51}$$

Then

$$y = e^{\pm iK\int e^{-\int \mu dx} dx}, \tag{51a}$$

and

$$y'' + \mu y' + \left(K^2 e^{-2\int \mu dx}\right) y = 0 \tag{51b}$$

$$Y'' + \left(K^2 e^{-2\int \mu dx} - \frac{\mu'}{2} - \frac{\mu^2}{4}\right) Y = 0 \tag{51c}$$

(V) (A) For μ = constant α:

$$y'' + \alpha y' + \beta^2 e^{-2\alpha x} y = 0, \tag{52}$$

$$y = e^{\pm i\frac{\beta}{\alpha}e^{-\alpha x}} \tag{52a}$$

$$Y'' + \left(\beta^2 e^{-2\alpha x} - \frac{\alpha^2}{4}\right) Y = 0. \tag{52b}$$

(V) (B) For $\mu = \alpha_1 + \alpha_2 x$:

$$y'' + (\alpha_1 + \alpha_2 x) y + \beta^2 e^{-(2\alpha_1 x + \alpha_2 x^2)} y = 0 \tag{53}$$

$$Y = e^{\pm i\beta \int e^{-\left(\alpha_1 x + \frac{1}{2}\alpha_2 x^2\right)} dx} \tag{53a}$$

$$Y'' + \left[\beta^2 e^{-(2\alpha_1 x + \alpha_2 x^2)} - \frac{(\alpha_1 + \alpha_2 x)^2}{4} - \frac{\alpha_2}{2}\right] Y = 0 \tag{53b}$$

(V) (C) For $\mu = \frac{\alpha_1}{\alpha_2+x}$:

$$y'' + \frac{\alpha_1}{\alpha_2+x} y' + \frac{\beta^2}{(\alpha_2+x)^{2\alpha_1}} y = 0, \tag{54}$$

$$y = e^{\pm i \frac{\beta(\alpha_2+x)^{1-\alpha_1}}{1-\alpha_1}}, \tag{54a}$$

$$Y'' + \left[\frac{\beta^2}{(\alpha_2+x)^{2\alpha_1}} - \frac{1}{4}\frac{\alpha_1(\alpha_1-2)}{(\alpha_2+x)^2}\right] Y \quad 0. \tag{54b}$$

For $\alpha_1=1$ eq. (54) is reduced to a special form of Cauchy's equation,[10] which can be converted into an equation with constant coefficients by the transformation $x=e^t$. Equation (54a) is not valid for $\alpha_1=1$ and eq. (51a) must be used directly to obtain the solution $y = (\alpha_2 + x)^{\pm\beta}$. In an effort to show that eq. (54) can be cast in a specific known form[11]

$$y'' + \frac{1-a-b}{x} y' + \frac{ab}{x^2} y = 0,$$

we let $\alpha_1 = 1-a-b$, $\alpha_2=0$, and $\beta^2=ab$ under the constraint $a=-b$.

(V) (D) For $\mu = \frac{c}{x} - \frac{a+b-c+1}{1-x} = \frac{c}{x(1-x)} - \frac{a+b+1}{1-x}$:

$$y'' + \left[\frac{c}{x(1-x)} - \frac{a+b+1}{1-x}\right] y' + \frac{\beta^2}{x^{2c}(1-x)^{2(a+b-c+1)}} y = 0, \tag{55}$$

$$y = e^{\pm i\beta/x} x^{-c} (1-x)^{-(a+b-c+1)} \tag{55a}$$

$$\left.\begin{aligned} Y'' + \Bigg[&\frac{\beta^2}{x^{2c}(1-x)^{2(a+b-c+1)}} - \frac{c(c-2)}{4x^2} + \frac{c(a+b-c+1)}{2x(1-x)} - \\ &\frac{(a+b-c+1)(a+b-c-1)}{4(1-x)^2} \Bigg] Y = 0 \end{aligned}\right\} \tag{55b}$$

If we set c=1 and a+b=1, these equations reduce, respectively, to the following simple forms:

$$y'' + \left(\frac{1}{x} - \frac{1}{1-x}\right) y' + \frac{\beta^2}{x^2(1-x)^2} y = 0 \quad , \tag{55c}$$

$$y = C_1\left(\frac{x}{1-x}\right)^{i\beta} + C_2\left(\frac{1-x}{x}\right)^{i\beta} \quad , \tag{55d}$$

$$Y'' + \frac{1 + 4\beta^2}{4x^2(1-x)^2} Y = 0 \quad . \tag{55e}$$

Note the simple forms of these equations in this case are due to the constraint that the numerator of μ is the derivative of the denominator of μ. This constraint[12] always appears in separation equations obtained by separating the partial differential equation $\nabla^2\psi + k^2\psi = 0$ in various separable coordinates, where ∇^2 is the Laplace operator.

If we set, in eq. (55),

$$b= -a, \quad c=\frac{1}{2} , \text{ and } \beta^2= -ab \quad , \tag{56}$$

eq. (55) degenerates into a special case of Gauss' hypergeometric equation

$$y'' + \left[\frac{\frac{1}{2}}{x(1-x)} - \frac{1}{1-x}\right] y' + \frac{a^2}{x(1-x)} y = 0 \qquad (57)$$

If we keep the constraints of eq. (56) we find that the following equations will be satisfied for $m = -\frac{1}{2}$:

$$a=m-n, \quad b=m+n+1, \text{ and } c=m+1. \qquad (58)$$

Substituting eq. (58) in eq. (55), we obtain a special case ($m= -\frac{1}{2}$) of a transformed Legendre equation[13] of order m and degree n

$$y'' + \left[\frac{m+1}{x(1-x)} - \frac{1}{1-x}\right] y' - \frac{(m-n)(m+n+1)}{x(1-x)} y=0$$

$$m = -\frac{1}{2}, \qquad (59)$$

$$y = e^{\pm i(n+\frac{1}{2})\cos^{-1}(1-2x)} \sim P_n^{-\frac{1}{2}}(x), \qquad (59a)$$

$$Y'' + \left[\frac{(n+\frac{1}{2})^2}{x(1-x)} + \frac{3-4x+4x^2}{(4x)^2(1-x)^2}\right] Y = 0. \qquad (59b)$$

In fact eqs. (57) and (59) are identical for $m= -\frac{1}{2}$ and $a= -(n+\frac{1}{2})$ and belong to the special case when the solution of the hypergeometric equation can be expressed in finite terms of elementary functions.

(V) (E) For $\mu = \frac{c}{-x} - \frac{a+b-c+1}{1-(-x)} = -\frac{c}{x(1+x)} - \frac{a+b+1}{1+x}$:

$$y'' - \left[\frac{c}{x(1+x)} + \frac{a+b+1}{1+x}\right] y' + \frac{\beta^2}{x^{-2c}(1+x)^{-2(a+b-c+1)}} y=0, \qquad (60)$$

$$y = e^{\pm i\beta \int x^{c} (1+x)^{(a+b-c+1)} dx}, \qquad (60a)$$

$$Y'' + \left[\frac{\beta^2}{x^{-2c}(1+x)^{-2(a+b-c+1)}} - \frac{c(c+2)}{4x^2} - \frac{c(a+b-c+1)}{2x(1+x)} - \frac{(a+b-c+1)(a+b-c+3)}{4(1+x)^2}\right] Y=0 \qquad (60b)$$

If we set $a = -\frac{3}{2}$, $b=c= -\frac{1}{2}$, and $\beta^2= -ab$ in the above eqs., we have

$$y'' + \left[\frac{1}{2}\,\frac{1}{x(1+x)} + \frac{1}{1+x}\right] y' - \frac{3}{4x(1+x)} y = 0 \quad , \qquad (61)$$

$$y = (\sqrt{x} + \sqrt{1+x})^{\pm\sqrt{3}} \quad , \qquad (61a)$$

$$Y'' + \left[\frac{3+4x+4x^2}{(4x)^2(1+x)^2} - \frac{3}{4x(1+x)}\right] Y = 0 \quad . \qquad (61b)$$

(V) (F) For $\mu= \frac{a}{x-a_o} + \frac{b}{x-b_o} + \frac{c}{x-c_o}$:

$$y'' + \left(\frac{a}{x-a_o} + \frac{b}{x-b_o} + \frac{c}{x-c_o}\right) y' + \frac{\beta^2}{(x-a_o)^{2a}(x-b_o)^{2b}(x-c_o)^{2c}} y = 0 \quad , \qquad (62)$$

$$y = e^{\pm i\beta\int (x-a_o)^{-a}(x-b_o)^{-b}(x-c_o)^{-c}dx} , \tag{62a}$$

$$Y'' + \left\{\frac{\beta^2}{(x-a_o)^{2a}(x-b_o)^{2b}(x-c_o)^{2c}} - \frac{1}{4}\left[\frac{a(a-2)}{(x-a_o)^2} + \frac{b(b-2)}{(x-b_o)^2} + \frac{c(c-2)}{(x-c_o)^2}\right] - \frac{1}{2}\left[\frac{ab(x-c_o)+bc(x-a_o)\ \ ac(x-b_o)}{(x-a_o)(x-b_o)(x-c_o)}\right]\right\} Y=0 \tag{62b}$$

Note eq. (62) is clearly in a form of Bocher equation[8]. If we set a=b=c=1, eq. (62) becomes Fuchsian type with four regular singular points at the points $x=a_o$, b_o, c_o, and ∞, and thus can be recognized as in a form of Riemann or Papperitz equation[9].

(VI) Let $\left(\frac{Q''}{Q'}\right)' - \frac{1}{2}\left(\frac{Q''}{Q'}\right)^2 = -2(Q')^2+2\omega^2-\mu'-\frac{\mu^2}{2} = f(x)$: (63)

It will be seen that the philosophy of approach in this case is different from that in cases (III) through (V). For a particular choice of f(x), we may obtain a solution of the equation

$$\left(\frac{Q''}{Q'}\right)' - \frac{1}{2}\left(\frac{Q''}{Q'}\right)^2 = f(x) \quad . \tag{63a}$$

For any arbitrary μ, we have

$$\omega^2 = \frac{1}{2}\left(f(x) + \mu' + \frac{\mu^2}{2}\right) + (Q')^2 \quad . \tag{63b}$$

Let the solution of eq. (63a) be S(x), then

$$\frac{Q''}{Q'} = S(x), \qquad Q' = Ke^{\int S(x)dx}, \tag{64}$$

$$y'' + \mu y' + \left[\frac{1}{2}\left(f(x)+\mu'+\frac{\mu^2}{2}\right) + K^2 e^{2\int S(x)dx}\right]y=0 \tag{65}$$

$$y = e^{-\frac{1}{2}\int(\mu+S)dx}\; e^{\pm iK\int e^{\int Sdx}dx}, \tag{65a}$$

$$Y'' + \left[\frac{1}{2}f(x) + K^2 e^{2\int S(x)dx}\right] Y = 0 \tag{65b}$$

Note that eq. (65b) is independent of μ and solely dependent of f(x). Accordingly for a prescribed f(x) the form of eq. (65b) will be the same for any chosen $\mu(x)$.

(VI) (A) Let f(x) = 0 :

The general solution of eq. (63) is, from eq. (18) ,

$$S(x) = \frac{Q''}{Q'} = \frac{2k}{1-kx}\;; \quad Q' = \frac{K}{(1-kx)^2},$$

$$Q = \frac{K}{k(1-kx)} + K_1,$$

where both k and K are arbitrary constants and $k \neq 0$. When $k=0$, this case reduces to case (IV).

(VI) (A) (1) For $\mu = \alpha_1 + \alpha_2 x$:

$$y'' + (\alpha_1+\alpha_2 x)y' + \left[\frac{\alpha_2}{2} + \frac{1}{4}(\alpha_1+\alpha_2 x)^2 + \frac{K^2}{(1-kx)^4}\right] y = 0 \quad , \qquad (66)$$

$$y=(1-kx)\ e^{-\frac{x}{4}(2\alpha_1+\alpha_2 x)}\ e^{\pm i\frac{K}{k(1-kx)}} \ , \ k\neq 0 \qquad (66a)$$

$$Y'' + \frac{K^2}{(1-kx)^4} \ Y = 0 \quad . \qquad (66b)$$

Note that eq. (66b) is identical to eq. (42).

(VI) (A) (2) For $\mu = \frac{\alpha_1}{\alpha_2+x}$:

$$y' + \frac{\alpha_1}{\alpha_2+x}\ y' + \left[\frac{\alpha_1(\alpha_1-2)}{4(\alpha_2+x)^2} + \frac{K^2}{(1-kx)^4}\right] \ y = 0 \qquad (67)$$

$$y = \frac{(1-kx)}{(\alpha_2+x)^{\frac{\alpha_1}{2}}}\ e^{\pm i\frac{K}{k(1-kx)}} \quad , \quad k\neq 0 \qquad (67a)$$

Note the particular simple form of eq. (67) when $\alpha_1=2$ (or the trivial case of $\alpha_1=0$). This is the consequence of the fact that $\mu = \frac{\mu_2}{\alpha_2+x}$ is the general solution of $\mu' + \frac{\mu^2}{2} = 0$. When $\alpha_1=0$, eq. (67) reduces to eq. (42).

(VI) (A) (3) For $\mu = 2\alpha_1 \tan h\ (\alpha_1 x+\phi)$ or $-2\alpha_1 \cot h\ (\alpha_1 x+\phi)$:

$$y'' + 2\alpha_1 \tan h(\alpha_1 x+\phi) + \left[2\alpha_1^2 + \frac{K^2}{(1-kx)^4}\right] \ y=0 \ , \qquad (68)$$

$$y = (1-kx)\, e^{\pm i \frac{K}{k(1-kx)}} \left[\operatorname{sec\,h}^{-1} (\alpha_1 x + \phi)\right] \quad (68a)$$

$$y'' - 2\alpha_1 \cot h\, (\alpha_1 x+\phi) + \left[2\alpha_1^2 + \frac{K^2}{(1-kx)^4}\right] y=0 \quad (68b)$$

$$y = (1-kx)\, e^{\pm i \frac{K}{k(1-kx)}} \left[\operatorname{csc\,h}^{-1} (\alpha_1 x + \phi)\right] . \quad (68c)$$

(VI) (A) (4) For $\mu = \frac{1}{(\alpha_2+x)} \Bigg[(1 \pm \sqrt{1+2\alpha_1}$

$$\mp \frac{2\sqrt{1+2\alpha_1}}{1 \mp 2\alpha_3\sqrt{1+2\alpha_1}\,(\alpha_2+x)^{\pm\sqrt{1+2\alpha_1}}} \Bigg] \quad (69)$$

It can be shown by direct substitution (use either upper signs or lower signs only) that eq. (69) is the general solution of $\mu' + \frac{\mu^2}{2} = \frac{\alpha_1}{(\alpha_2+x)^2}$. For $\alpha_1=\alpha_3=0$ we obtain from eq. (69) $\mu=0$ or $\frac{2}{\alpha_2+x}$ and case (VI) (A) (2) therefore can be derived from this case.

$$y'' + \frac{1 \mp \sqrt{1+2\alpha_1} \mp 2\alpha_3\sqrt{1+2\alpha_1}\,(1 \pm \sqrt{1+2\alpha_1})\,(\alpha_2+x)^{\pm\sqrt{1+2\alpha_1}}}{(\alpha_2+x)\left[1 \mp 2\alpha_3\sqrt{1+2\alpha_1}\,(\alpha_2+x)^{\pm\sqrt{1+2\alpha_1}}\right]}\, y'$$

$$+ \left[\frac{\alpha_1}{2(\alpha_2+x)^2} + \frac{K^2}{(1-kx)^4}\right] y = 0 \quad (70)$$

$$y = (1-kx)\, e^{\pm i \frac{K}{k(1-kx)}}\; e^{-\frac{1}{2}\int \mu dx} \quad , \quad (70a)$$

(VI) (A) (5) For $\mu = \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2}$:

$$y'' + \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{x^2}\right) y' + \left[\left(\alpha_1 + \frac{2\alpha_2}{x}\right)\frac{(\alpha_1 - 2)}{4x^2} + \frac{\alpha_2^2}{4x^4} + \frac{K^2}{(1-kx)^4}\right] y = 0 \qquad (71)$$

$$y = (1-kx)\ x^{-\frac{1}{2}\alpha_1} \exp\left[\frac{\alpha_2}{2x} \pm i\ \frac{K}{k(1-kx)}\right] \qquad (71a)$$

If $\alpha_1=2$, eq. (71) has a simpler form because of the fact $\mu = \frac{2}{x} + \frac{\alpha_2}{x^2}$ is a particular solution of $\mu' + \frac{\mu^2}{2} = \frac{\mu_2^2}{2x^4}$.

$$\text{(VI) (A) (6) For } \mu = \frac{2}{x} + \left(\frac{\alpha_2}{x^2}\right)\frac{e^{-\frac{1}{2}\alpha_2} - \alpha_3 e^{\frac{1}{2}\alpha_2}}{e^{-\frac{1}{2}\alpha_2} + \alpha_3 e^{\frac{1}{2}\alpha_2}} \qquad (72)$$

Equation (72) is the general solution of $\mu' + \frac{\mu^2}{2} = \frac{\alpha_2^2}{2x^4}$ and case (VI) (A) (5) can be derived from it by letting $\alpha_3=0$ and $\alpha_1=2$.

$$y'' + \left(\frac{2}{x} + \frac{\alpha_2}{x^2} \cdot \frac{e^{-\frac{\alpha_2}{2}} - \alpha_3\ e^{\frac{\alpha_2}{2}}}{e^{-\frac{\alpha_2}{2}} + \alpha_3\ e^{\frac{\alpha_2}{2}}}\right) y' + \left[\frac{\alpha_2^2}{4x^4} + \frac{K^2}{(1-kx)^4}\right] y = 0 \qquad (73)$$

$$y = (1-kx)\ e^{\pm i \frac{K}{k(1-kx)}}\ e^{-\frac{1}{2}\int \mu dx} \ . \qquad (73a)$$

(VI) (A) (7) For μ equal to other function of x.

(VI) (B) Let $f(x) = 2\xi^2$, where ξ is a constant.

The general solution of eq. (63a) then is, from eq. (16a),

$$S(x) = \frac{Q''}{Q'} = i2\xi \ \frac{e^{i\xi x} - ke^{-i\xi x}}{e^{i\xi x} + ke^{-\xi x}} \ ,$$

$$Q' = K\ (e^{i\xi x} + ke^{-i\xi x})^{-2} \ , \quad Q = i\frac{K}{2\xi}\ (e^{i2\xi x}+k)^{-1} + K_1 \ ,$$

(VI) (B) (1) For $\mu = \alpha_1 + \alpha_2 x$:

$$Y'' + (\alpha_1 + \alpha_2 x)\ y' + \left[\left(\xi^2 + \frac{\alpha_2}{2}\right) + \frac{1}{4}\left(\alpha_1 + \alpha_2 x\right)^2 + K^2 (e^{i\xi x} + ke^{-i\xi x})^{-4}\right] Y = 0 \qquad (74)$$

$$y = (e^{i\xi x} + ke^{-i\xi x})\ e^{\pm\frac{K}{2\xi}(e^{i2\xi x}+k)^{-1}}\ e^{-\frac{x}{4}(2\alpha_1+\alpha_2 x)} , \qquad (74a)$$

$$Y'' + \left[\xi^2 + K^2 (e^{i\xi x}+ke^{-i\xi x})^{-4}\right] Y = 0 \qquad (74b)$$

If we set $\alpha_1 = -2\xi^2$ and $\alpha_2 = 0$ in eq. (74), it has the following simple form:

$$y'' - 2\xi^2 y' + K^2 (e^{i\xi x}+ke^{-i\xi x})^{-4}\ y = 0 \qquad (74c)$$

(VI) (B) (2) For $\mu = \frac{a}{x-a_o} + \frac{b}{x-b_o}$:

$$y'' + \left(\frac{a}{x-a_o} + \frac{b}{x-b_o}\right) y' + \left[\xi^2 + \frac{a(a-2)}{4(x-a_o)^2} + \frac{b(b-2)}{4(x-b_o)^2} + \frac{ab}{2(x-a_o)(x-b_o)} + \frac{K^2}{(e^{i\xi x}+ke^{-i\xi x})^4}\right] Y=0 \qquad (75)$$

$$y = (e^{i\xi x}+ke^{-i\xi x})\ e^{\pm\frac{K}{2}(e^{i2\xi x}+k)^{-1}} (x-a_o)^{-\frac{a}{2}} (x-b_o)^{-\frac{b}{2}} \qquad (75a)$$

If we set a=1, a_o=b=K=k=0, eq. (75) becomes the Bessel equation of order $n=\pm\frac{1}{2}$:

$$y'' + \frac{1}{x} y' + \left(\xi^2 - \frac{n^2}{x^2}\right) y = 0 , \quad n=\pm\frac{1}{2} \qquad (75b)$$

$$y = x^{-\frac{1}{2}} e^{\pm i\xi x} = Z_{\pm\frac{1}{2}} (\xi x) \quad . \qquad (75c)$$

Accordingly, we see that Bessel's equation of order $n=\pm\frac{1}{2}$ is expressible in elementary functions as shown in eq. (75c). If we set a=2, a_o=b=k=K=0, eq. (75) becomes the spherical Bessel equation[14] of order m=0 or -1:

$$y'' + \frac{2}{x} y' + \left[\xi^2 - \frac{m(m+1)}{x^2}\right] y = 0, \quad m=0 \text{ or } -1, \qquad (75d)$$

$$y = x^{-1} \left(C_1 e^{i\xi x} + C_2 e^{-i\xi x}\right) = C_3 j_o (\xi x) + C_4 j_{-1} (\xi x) \qquad (75e)$$

where $j_o(\xi x)$ and $j_{-1}(\xi x)$ are the spherical Bessel functions of the first kind of order 0 and -1 respectively. Note the connection between eqs. (75c) and (75e), i.e., $y_c = \sqrt{x}\, y_e$ We shall use this opportunity to show briefly a simple and direct method for expressing the solution of the Bessel eq. (75b) for $n = m + \frac{1}{2}$, where m is any integer, in finite terms of elementary functions. Using the transformation $y = \frac{1}{\sqrt{x}} e^{\pm i\xi x} z$, we can convert the Bessel equation into

$$z'' \pm i2\xi z' - \frac{m(m+1)}{x^2} z = 0 \quad . \qquad (75f)$$

If m is allowed to be any value, eq. (75f) can only be solved by an infinite series which involves Gamma functions. When m, however, is an integer the series terminates as a polynomial. The solutions of eq. (75b) therefore are:

For $n=\pm\frac{3}{2}$ (m=0 or -1), $y=\frac{1}{\sqrt{x}} e^{\pm i\xi x} z = \sqrt{x} \quad e^{\pm i\xi x}$,

$n=\pm\frac{3}{2}$ (m=1 or -2), $y=\frac{1}{\sqrt{x}} e^{\pm i\xi x} (\xi \pm \frac{i}{x}) \left(C_1 + C_2 \int (\xi \pm \frac{i}{x})^{-2} e^{\mp i2\xi x} dx\right)$

$n=\pm\frac{5}{2}$ (m=3 or -3) $\quad y=\frac{1}{\sqrt{x}}e^{\pm i\xi x}\left[\left(\xi^2-\frac{3}{x^2}\right)\pm i\frac{3}{x}\,\xi\right].$

$$\left[C_1+C_2\int\left(\xi^2-\frac{3}{x^2}\pm i\frac{3\xi}{x}\right)^{-2}e^{\mp i2\xi x}\,dx\right]$$

$n=\pm\frac{7}{2}$ $\quad y=\frac{1}{\sqrt{x}}e^{\pm i\xi x}\left[\pm\xi\left(\xi^2-\frac{15}{x^2}\right)-i\frac{3}{x}\left(\frac{5}{x^2}\pm 2\xi\right)\right].$

$$\left[C_1+C_2\int\left(\pm\xi^3\mp\frac{15\xi}{x^2}\pm i\frac{6\xi}{x}-i\frac{15}{x^3}\right)^{-2}e^{\mp i2\xi x}\,dx\right]$$

$n=\pm\frac{9}{2}$

(VI) (B) (3) For $\mu=\dfrac{\alpha_1}{(\alpha_2+x)^2}$:

$$y''+\frac{\alpha_1}{(\alpha_2+x)^2}+\left[\xi^2+\frac{\alpha_1(\alpha_1-4\alpha_2-4x)}{4(\alpha_2+x)^4}+\frac{K^2}{(e^{i\xi x}+ke^{-i\xi x})^4}\right]y=0 \qquad (76)$$

$$y=(e^{i\xi x}+ke^{-i\xi x})\,e^{\pm\frac{K}{2\xi}(e^{i2\xi x}+k)^{-1}}\,e^{\frac{\alpha_1}{2(\alpha_2+x)}} \qquad (76a)$$

If we set $\alpha_2=0$, $K=k=0$, then

$$y''+\frac{\alpha_1}{x^2}y'+\left(\xi^2+\frac{\alpha_1^2}{4x^4}-\frac{\alpha_1}{x^3}\right)y=0\;, \qquad (76b)$$

$$y=e^{\left(\frac{\alpha_1}{2x}\pm i\xi x\right)} \qquad (76c)$$

(VI) (B) (4) For $\mu=ae^{bx}$:

$$y'' + ae^{bx}y' + \left[\xi^2 + \frac{ab}{2} e^{bx} + \frac{a^2}{4} e^{2bx} + K^2 \left(e^{i\xi x} + ke^{-i\xi x}\right)^{-4}\right] y = 0 \qquad (77)$$

$$y = \left(e^{i\xi x} + ke^{-i\xi x}\right) e^{\frac{\pm K}{2}\left(e^{i2\xi x} + k\right)^{-1}} e^{-\frac{a}{2b} e^{bx}} \qquad (77a)$$

For $k=0$, $K= \pm i\frac{a}{2}$, and $\xi= i\frac{b}{2}$, eqs. (77) and (77a) reduce respectively to

$$y'' + ae^{bx}y' + \left(\frac{ab}{2}e^{bx} - \frac{b^2}{4}\right) y = 0 \quad , \qquad (77b)$$

$$y = C_1e^{-\frac{b}{2}x} + C_2e^{-\left(\frac{b}{2} + \frac{a}{b} e^{bx}\right)} \quad . \qquad (77c)$$

(VI) (B) (5) For μ equal to other function of x

(VI) (C) Let $f(x) = \frac{\xi}{(\lambda+x)^2}$:

From case (VI) (A) (4) we know that a particular solution of eq. (63 a) for the above f(x) is:

$$S(x) = \frac{Q''}{Q'} = \frac{\beta}{\lambda+x} \quad , \qquad \beta = -1\pm \sqrt{1-2\xi} \quad ;$$

$$Q' = K(\lambda\pm x)^{\beta} \qquad Q = \frac{K}{\beta+1} (\lambda+x)^{\beta+1} + K_1 \ .$$

The general solution of eq. (63a), of course, can be used, but the resulting differential equation is rather complicated.

(VI) (C) (1) For $\mu = \alpha_1 + \alpha_2 x$:

$$y'' + (\alpha_1 + \alpha_2 x)\, y' + \left[\frac{\xi}{2(\lambda+x)^2} + \frac{\alpha_2}{2} + \frac{1}{4}(\alpha_1 + \alpha_2 x)^2 + K^2 (\lambda+x)^{2\beta}\right] y = 0 \tag{78}$$

$$y = (\lambda+x)^{-\frac{\beta}{2}}\, e^{\pm i \frac{K}{\beta+1}(\lambda+x)^{\beta+1}}\, e^{-\frac{x}{4}(2\alpha_1+\alpha_2 x)}, \tag{78a}$$

$$Y'' + \left[\frac{\xi}{2(\lambda+x)^2} + K^2 (\lambda+x)^{2\beta}\right] Y = 0 \tag{78b}$$

where

$$\beta = -1 \pm \sqrt{1-2\xi}$$

If we set $\alpha_2 = \lambda = K = 0$, then

$$y'' + \alpha_1 y' + \left(\frac{\xi}{2x^2} + \frac{\alpha_1^2}{4}\right) y = 0 \quad , \tag{78c}$$

$$y = x^{-\frac{\beta}{2}}\, e^{-\frac{\alpha_1}{2}x} \quad . \tag{78d}$$

Note the close resemblance between eqs. (78c) and (75f).

(VI) (C) (2) For $\mu = \frac{\alpha_1}{\alpha_2 + x}$:

$$y'' + \frac{\alpha_1}{\alpha_2+x} y + \left[\frac{\xi}{2(\lambda+x)^2} + \frac{\alpha_1(\alpha_1-2)}{4(\alpha_2+x)^2} + K^2(\lambda+x)^{2\beta} \right] y = 0 \quad , \tag{79}$$

$$y = (\lambda+x)^{-\frac{\beta}{2}} (\alpha_2+x)^{-\frac{\alpha_1}{2}} e^{\underline{+} i \frac{K}{\beta+1} (\lambda+1)^{\beta+1}} \quad , \tag{79a}$$

where

$$\beta = -1 \pm \sqrt{1-2\xi} \quad .$$

If we let $\alpha_2 = \lambda = 0$, eqs. (79) and (79a) become, respectively,

$$y'' + \frac{\alpha_1}{x} y' + \left[\frac{\alpha_1(\alpha_1-2)+2\xi}{4x^2} + K^2 x^{2(-1\underline{+}\sqrt{1-2\xi})} \right] y = 0 \tag{79b}$$

$$y = x^{-\frac{1}{2}(\alpha_1 - 1 \underline{+} \sqrt{1-2\xi})} \left[C_1 e^{i \frac{K}{\sqrt{1-2\xi}} x^{\underline{+}\sqrt{1-2\xi}}} + C_2 e^{-i \frac{K}{\sqrt{1-2\xi}} x^{\underline{+}\sqrt{1-2\xi}}} \right] . \tag{79c}$$

Equation (79b) can be identified as eq. (37), the Malmstén[3] equation, if

$$a = \alpha_1, \qquad b = K^2, \qquad c = \frac{1}{4} (\alpha_1^2 - 2\alpha_1 + 2\xi) \ ,$$

$$m = 2(-1 \underline{+} \sqrt{1-2\xi}) \quad . \tag{79d}$$

where α_1, K, and ξ are three arbitrary constants.

Eliminating ξ in eq. (79d), we have

$$m = 2 \left[-1 \pm \sqrt{(1-a)^2+4C}\right] \tag{79e}$$

Accordingly the condition of eq. (79e) is required for the integrability of the Malmstén equation in finite terms of elementary functions. For c=0 we have, from eq. (79d), $\pm \sqrt{1-2\xi} = (\alpha_1-1)$, and eq. (79b) and (79c), respectively, become

$$y'' + \frac{\alpha_1}{x} y' + K^2 x^{2(\alpha_1-2)} y = 0, \tag{79f}$$

$$y = x^{1-\alpha_1} e^{\pm i \frac{K}{1-\alpha_1} x^{\alpha_1-1}}. \tag{79g}$$

For $\alpha_1=2$ eqs. (79c) and (79g) reduce, respectively, to eqs. (75d) and (75e). Note that eq. (65) will have a very simple form, if we set

$$\mu' + \frac{1}{2} \mu^2 = -f(x). \tag{80}$$

Since $f(x) = \frac{\xi}{(\lambda+x)^2}$ in this case, we have a particular solution of eq. (80) as $\mu = \frac{\beta}{\lambda+x}$, where $\beta=-1\pm\sqrt{1-2\beta}$. This is equivalent to setting $\alpha_2=\lambda$ and $\alpha_1= -\beta$ in eq. (79) and the latter becomes

$$y'' - \frac{\beta}{\lambda+x} y' + K^2(\lambda+x)^{2\beta} y = 0 \quad , \tag{81}$$

$$y = e^{\pm i\frac{K}{\beta+1}(\lambda+x)^{\beta+1}} \tag{81a}$$

Note eq. (81) is identical to eq. (54).

If we set K=0, eq. (79b) becomes the general Cauchy equation[7]:

$$y'' + \frac{\alpha_1}{x} y' + \frac{\alpha_1(\alpha_1-2) + 2\xi}{4x^2} y = 0 \quad , \tag{82}$$

$$y = x^{-\frac{1}{2}(\alpha_1 - 1\pm\sqrt{1-2\xi})} \tag{82a}$$

When $\alpha_1=1$ in eq. (54), it reduces to the form of eq. (82).

Let $\frac{1}{4}[\alpha_1(\alpha_1-2) + 2\xi] = \alpha_2$ in eq. (82) then eq. (82a) becomes $y=x^{-\frac{1}{2}\left(\alpha_1-1\pm i\sqrt{4\alpha_2-(\alpha_1-1)^2}\right)}$. If we set $\alpha_1=1$ and $\xi = -\frac{3}{2}$, eq. (79b) is reduced to eq. (35) for $n = \pm\frac{1}{2}$ or

$$y'' + \frac{1}{x} y' + \left(\frac{K^2}{x^6} - \frac{1}{x^2}\right) y = 0 \quad , \tag{82b}$$

$$y = xe^{\pm i\frac{K}{2}x^{-2}} \quad . \tag{82c}$$

If we set $\alpha_1=1$ and $\xi=0$, eq. (79b) is reduced to eq. (75b) or

$$y'' + \frac{1}{x} y' + \left(\frac{K^2}{x^4} - \frac{1}{4x^2}\right) y = 0 \quad , \tag{82d}$$

$$y = x^{\frac{1}{2}} e^{\pm ikx^{-1}} \quad . \tag{82e}$$

If we set $\alpha_1=2$ and $\xi=0$, eq. (79b) is reduced to eqs. (75d) and (79e) or

$$y'' + \frac{2}{x} y' + \frac{K^2}{x^4} y = 0 \quad , \tag{82f}$$

$$y = e^{\pm ikx^{-1}} \quad . \tag{82g}$$

Note eq. (82d) and (82f) are also the special cases of eq. (67). We can see accordingly that eq. (79) is a very general differential equation with a simple pole at $x = -\alpha_2$ and another singularity at $x = -\lambda$.

(VI) (C) (3) For $\mu=\frac{2}{x} + \frac{\alpha}{x^2}$:

$$y'' + \left(\frac{2}{x} + \frac{\alpha}{x^2}\right) y' + \left[\frac{\xi}{2(\lambda+x)^2} + \frac{\alpha^2}{4x^4} + K^2(\lambda+x)^{2\beta}\right] y=0 \tag{83}$$

$$y = (\lambda+x)^{-\frac{\beta}{2}} x^{-1} e^{\frac{\alpha}{2x}} e^{\pm i\frac{K}{\beta+1}(\lambda+x)^{\beta+1}} \quad , \tag{83a}$$

where

$$\beta= -1 \pm\sqrt{1-2\xi} \quad .$$

For $\alpha=0$ in eq. (83) and for $\alpha_1=2$ in eq. (79b), these two equations become identical.

(IV) (C) (4) For μ equal to other function of x.

(VI) (D) Let $f(x) = \frac{\xi^2}{2(\lambda+x)^4}$

From case (VI) (A) (6) we know that a particular solution of eq. (63a) for the above f(x) is:

$$S(x) = \frac{Q''}{Q'} = \frac{-2}{(\lambda+x)} + i \frac{\xi}{(\lambda+x)^2} ,$$

$$Q' = K (\lambda+x)^{-2} e^{-i\xi(\lambda+x)^{-1}}$$

$$Q = Ke^{-i\xi(\lambda+x)^{-1}} + K_1 ,$$

(VI) (D) (1) For $\mu = \alpha_1 + \alpha_2 x$:

$$y' + (\alpha_1+\alpha_2 x)y' + \left[\frac{\xi^2}{4(\lambda+x)^4} + \frac{\alpha_2}{2} + \frac{1}{4}(\alpha_1+\alpha_2 x)^2 + K^2 \frac{e^{-i2\xi(\lambda+x)^{-1}}}{(\lambda+x)^4}\right] y = 0 , \tag{84}$$

$$y = (\lambda+x) e^{i\frac{\xi}{2}(\lambda+x)^{-1}} e^{\pm ike^{-i\xi(\lambda+x)^{-1}}} e^{-\frac{x}{4}(2\alpha_1+\alpha_2 x)} , \tag{84a}$$

$$Y'' + \left[\frac{\xi^2}{4(\lambda+x)^4} + K^2 \frac{e^{-i2\xi(\lambda+x)^{-1}}}{(\lambda+x)^4}\right] Y = 0 \tag{84b}$$

(VI) (D) (2) For $\mu = \frac{\alpha_1}{\alpha_2+x}$:

$$y'' + \frac{\alpha_1}{\alpha_2+x}\,y + \left[\frac{\xi^2}{4(\lambda+x)^4} + \frac{\alpha_1(\alpha_1-2)}{4(\alpha_2+x)^2} + K^2\,\frac{e^{-i2\xi(\lambda+x)^{-1}}}{(\lambda+x)^4}\right]y=0 \quad (85)$$

$$y = (\lambda+x)\; e^{i\frac{\xi}{2}(\lambda+x)^{-1}}\; e^{\pm ike^{-i\xi(\lambda+x)^{-1}}}\;(\alpha_2+x)^{-\frac{\alpha_1}{2}} \quad (85b)$$

(VI) (D) (3) For $\mu = \frac{2}{\lambda+x} + \frac{\alpha}{(\lambda+x)^2}$:

$$y'' + \left[\frac{2}{\lambda+x} + \frac{\alpha}{(\lambda+x)^2}\right]y' + \left[\frac{\xi^2+\alpha^2}{4(\lambda+x)^4} + K^2\,\frac{e^{-i2\xi(\lambda+x)^{-1}}}{(\lambda+x)^4}\right]y = 0 \quad , \quad (86)$$

$$y = e^{(i\frac{\xi}{2} + \frac{\alpha}{2})(\lambda+x)^{-1}}\; e^{\pm iKe^{-i\xi(\lambda+x)^{-1}}} \quad (86a)$$

If we set $\alpha= -i\xi$, eqs. (86) and (86a) reduce to, respectively,

$$y'' + \left[\frac{2}{\lambda+x} - i\,\frac{\xi}{(\lambda+x)^2}\right] y' + K^2\,\frac{e^{-i2\xi(\lambda+x)^{-1}}}{(\lambda+x)^4}\; y=0 \quad (86b)$$

$$y = e^{\pm iKe^{-i\xi(\lambda+x)^{-1}}} \quad (86c)$$

(VI) (D) (4) For μ equal to other function of x.

(VI) (E) Let $f(x) = \xi-\frac{1}{2}\,\xi^2x^2$:

$$S(x) = \frac{Q''}{Q'} = \xi x \;, \qquad Q' = Ke^{\frac{\xi}{2}x^2} \;.$$

(VI) (E) (1) For $\mu = \alpha_1 + \alpha_2 x$:

$$y'' + (\alpha_1+\alpha_2 x)\, y' + \left[\frac{1}{2}(\xi+\alpha_2) - \frac{1}{4}\xi^2 x^2 + \frac{1}{4}(\alpha_1+\alpha_2 x)^2 + K^2 e^{\xi x^2}\right] y = 0 \qquad (87)$$

$$y = e^{-\frac{1}{4}\xi x^2}\, e^{\pm iK\int e^{\frac{1}{2}\xi x^2} dx}\, e^{-\frac{1}{4}x(2\alpha_1+\alpha_2 x)} , \qquad (87a)$$

$$Y'' + \left[\frac{1}{4}\xi(2-\xi x^2) + K^2 e^{\xi x^2}\right] Y = 0 . \qquad (87b)$$

If we set $\alpha_1=0$ and $\alpha_2= -\xi$, eqs. (87) and (87a) become, respectively,

$$y'' - \xi x y' + K^2 e^{\xi x^2} y = 0 , \qquad (87c)$$

$$y = e^{\pm ik\int e^{\frac{1}{2}\xi x^2} dx} . \qquad (87d)$$

Note the identity between eqs. (87c) and (53).

If we set $\alpha_1=0$ and $\alpha_2= \xi$, eqs. (87) and (87a) become, respectively,

$$y'' + \xi x y' + (\xi+K^2 e^{\xi x^2})\, y = 0 , \qquad (87e)$$

$$y = e^{-\frac{1}{2}\xi x^2}\, e^{\pm iK\int e^{\frac{1}{2}\xi x^2} dx} . \qquad (87f)$$

Equations (87c) and (87e) show clearly the effect of changing the sign of α_2 in eq. (87).

If we set K = 0 in eq. (87e), it has the simple form

$$y'' + \xi x y' + \xi y = 0 \quad , \qquad (87g)$$

$$y = e^{-\frac{1}{2}\xi x^2} \left(C_1 + C_2 \int e^{\frac{1}{2}\xi x^2} dx \right) . \qquad (87h)$$

Equation (87g) is in the form of eq. (24), the solution of which is eq. (23). Now note the similarity between eq. (87g) and the differential equation associated with Hermite polynomial for n = 1:

$$y'' - \xi x y' + \xi n y = 0, \; n=1 \qquad (88)$$

$$y = x \left(C_1 + C_2 \int x^{-2} e^{\frac{1}{2}\xi x^2} dx \right) , \qquad (88a)$$

For n=0

$$y = C_1 + C_2 \int e^{\frac{1}{2}\xi x^2} dx \quad , \qquad (88b)$$

For n=2

$$y = (1-\xi x^2) \left[C_1 + C_2 \int (1-\xi x^2)^{-2} e^{\frac{1}{2}\xi x^2} dx \right] , \qquad (88c)$$

For n=3

$$y = \cdot \cdot \cdot$$

The difference between the solution eq. (87h) and the solution eq. (88a) is certainly quite surprising due to a simple sign change in the coefficient of y' in the differential equations (87g) and (88). If we set K= 0 in eq. (87b), it becomes the Weber equation[14] for n = 0:

$$Y'' + \xi \left[+ \frac{1}{2} (1-\frac{\xi}{2} x^2)\right] Y = 0, \quad n = 0 , \tag{89}$$

$$Y = e^{-\frac{1}{4}\xi x^2} \left(C_1 + C_2 \int e^{\frac{1}{2}\xi x^2} dx \right) , \tag{89a}$$

Note that eq. (89) is the normal form of eq. (88).

(VI) (E) (2) Let $\mu = \frac{\alpha_1}{\alpha_2 + x}$:

$$y + \frac{\alpha_1}{\alpha_2+x} y' + \left[\frac{\xi}{4} (2-\xi x^2) + \frac{\alpha_1(\alpha_1-2)}{4(\alpha_2+x)^2} + K^2 e^{\xi x^2}\right] y = 0 ,$$

$$y = e^{-\frac{\xi}{4}x^2} e^{\pm ik \int e^{\frac{\xi}{2}x^2} dx} (\alpha_2+x)^{-\frac{\alpha_1}{2}} .$$

(VI) (E) (3) Let $\mu = \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2}$:

$$y'' + \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{x^2}\right) y' + \left[\frac{\xi}{4}(2-\xi x^2) + (\alpha_1 + \frac{2\alpha_2}{x}) \frac{(\alpha_1-2)}{4x^2} + \frac{\alpha_2^2}{4x^4} + K^2 e^{\xi x^2}\right] y = 0 , \tag{90}$$

(VI) (E) (4) μ equal to other function of x.

(VI) (F) Choose f(x) equal to other function of x.

such that $(\frac{Q''}{Q'})' - \frac{1}{2}(\frac{Q''}{Q'})^2 = f(x)$ can be solved in finite terms of elementary functions. For example, if $f(x) = \xi x^{-\frac{4}{3}}$, a particular solution[16] is $\frac{Q''}{Q'} = -3\xi x^{-\frac{1}{3}} (1+\frac{3}{2}\sqrt{2\xi}\, x^{\frac{1}{3}})^{-1}$ and

$$Q' = \left(1 + \frac{3}{2} \sqrt{2\xi}\, x^{\frac{1}{3}}\right) e^{-\frac{3}{2} \sqrt{2\xi}\, x^{\frac{1}{3}}} .$$

4.0 EQUATIONS WITH ARBITRARY COEFFICIENTS

In this section we will show first that Sharpe's equation[17]

$$y'' + \frac{1}{x}y' + (\xi^2 + \frac{A}{x})\, y = 0 \quad , \qquad (91)$$

can be solved in finite terms for a particular choice of the coefficient A in eq. (91). Equation (91) is a generalization of Bessel equation of order zero and was investigated by Sharpe during 1881-1900. He showed that the solution of eq. (91) for $\xi=1$ can be expressed in a definite integral form

$$y = K \int_0^{\frac{1}{2}\pi} \cos\, (x \cos\theta + A \log \cot \tfrac{1}{2}\theta)\, d\theta \qquad (91a)$$

Since eq. (91) can not be fitted into any of the forms of differential equations discussed in the previous section, we may consider the coefficients as arbitrary and shall try to solve it in finite terms by using eq. (9)

$$\left(\frac{z''}{z'}\right)' - \frac{1}{2}\left(\frac{z''}{z'}\right)^2 = 2\left(\xi^2 + \frac{A}{x} + \frac{1}{4x^2}\right) \quad (91b)$$

Equation (91b) is soluble in finite terms, if $A=i\xi$, and has the solution

$$\frac{z''}{z'} = i2\xi - \frac{1}{x} \quad , \quad z' = kx^{-1}\, e^{i2\xi x} \quad (91c)$$

The solution of eq. (91) for $A=i\xi$ is, therefore, from eq. (10),

$$y = e^{-i\xi x}\left(C_1 + C_2 \int \frac{e^{-i2\xi x}}{x}\, dx\right) \quad . \quad (91d)$$

We will show next that the generalized Laguere equation

$$y'' + \left(\frac{\alpha+1}{x} - 1\right) y' + \frac{n}{x}\, y = 0 \quad , \quad (92)$$

can be solved in finite terms by using eq. (9)

$$\left(\frac{z''}{z'}\right)' - \frac{1}{2}\left(\frac{z''}{z'}\right)^2 = \frac{1-\alpha^2}{2x^2} + \frac{2n+1+\alpha}{x} - \frac{1}{2} \quad . \quad (92a)$$

Equation (92a) is soluble in finite terms, if

$$(a)\ \alpha = -n, \quad \text{then } \frac{z''}{z'} = 1 - \frac{n+1}{x}, \quad z' = K_a x^{-(n+1)} e^x, \quad (92b)$$

$$\text{(b)}\quad \alpha = -(n+1),\ \text{then}\ \frac{z''}{z'} = -1+\frac{n}{x}\ ,\quad z' = K_b x^n e^{-x} \qquad (92c)$$

The solutions of eq. (92) are, from eq. (10), the Laguere polynomials L_n^m.

(a) for $\alpha = -n$,

$$y_\alpha = x^n\left(C_1 + C_2\int x^{-(n+1)} e^x dx\right) = L_n^{-n}(x)\ , \qquad (92d)$$

(b) for $\alpha = -(n+1)$,

$$y_b = e^x\left(C_3 + C_4\int x^n\ e^{-x} dx\right) = L_n^{-(n+1)}(x)\ . \qquad (92e)$$

Lastly we mention in brief that the following equation, obtained by linearizing the nonlinear differential equation of a restricted three-body problem,

$$y'' + \frac{\varepsilon}{x^2}\, y' + \omega_c^2 y = 0\ , \qquad (93)$$

can be solved only approximately in finite terms. Note the similarity between eqs. (93) and (76b). We are interested in an approximate solution of eq. (93) in the region $\varepsilon^2 < x < \varepsilon^1$, where ε is a very small perturbing constant. ω_c is also a constant. It is well-known that in the region $\varepsilon^1 < x < \varepsilon^0$, an approximate solution by Poincaré's small parameter perturbation method should be useful and it can be shown that the error (1st order solution) is the order $\frac{\varepsilon^2}{x}$, which becomes significant in the region $x<\varepsilon$. An approximate solution of eq. (93) by our method has the following form[18]

$$y = e^{i\left(\omega x - \frac{\varepsilon^2}{2x}\right)}\left[C_1 + C_2\int_{x_i}^{x} e^{-i\left(2\omega x - \frac{\varepsilon^2}{x}\right)} e^{\frac{\varepsilon}{x}}\, dx\right] \quad (93a)$$

and its error can be shown to be of the order ε^2 and almost independent of x in the range $\varepsilon^2<x<\sqrt{\varepsilon}$. The improvement of accuracy amounts to a few orders of magnitude if $\varepsilon<<1$.

5.0 CONCLUSIONS AND SUMMARY

We have shown that classes of second order ordinary linear differential equations with coefficients $\mu(x)$ and $\omega^2(x)$ related in a specific fashion can be solved in finite terms by means of elementary functions. We also have shown explicitly the solutions for several specific relations between coefficients $\mu(x)$ and $\omega^2(x)$, but have, of course, not exhausted all possible cases. Other equations with "arbitrary" coefficients may be solved in terms of higher transcendental functions, and Moon and Spencer[19] have listed various separation equations according to specific types of Bôcher's equation and to the number of singularities, together with their solutions in terms of appropriate higher transcendental functions. An approximate solution of certain equations with "arbitrary" coefficients usually can be obtained by solving approximately its equivalent equation in Riccati form [eq. (9)].

A perusal of the elementary-functions solutions of the differential equations shown in the previous sections also indicates that when a differential equation has (A) a regular singular point or (B) an irregular singular point, its general solution has, respectively, (A) a pole or branch point or (B) an essential singularity.[20][21] We also note that the elementary-functions solutions of the differential equations are all in agreement with Liouville's two theorems concerning linear differential equations.[4]

Ince[22] and Kambe[23] have classified differential equations according to the number and the nature of their singular points. Our main classification here is based on various specific relations between the coefficients $\mu(x)$ and $\omega^2(x)$ and our sub-classification on various arbitrary specifications of $\mu(x)$, which can be any analytic or singular function. A concise classification is shown in the attached table. Those second order homogeneous linear differential equations solved in terms of elementary functions and listed by Kambe[22] belong to some of our classifications.

6.0 DIFFERENTIAL EQUATION TABLE

$y'' + \mu(x)y' + \omega^2(x)\ y = 0$

Class (1): $\mu(x)$ and $\omega^2(x)$ are related

(I)

ω_c^2 and μ_c both constant:

$y'' + \mu_c y' + \omega_c^2\ y = 0,$ $\qquad Y'' + \left(\omega_c^2 - \frac{1}{4}\ \mu_c^2\right)\ Y = 0$

$$y = e^{\frac{1}{2}\mu_c x}\ e^{\pm i\left(\omega_c^2 - \frac{1}{4}\mu_c^2\right)^{\frac{1}{2}}\ x}$$

when $\omega_c^2 - \frac{1}{4}\ \mu_c^2 = 0,$ $\qquad y = \left(C_1 + C_2\ x\right)\ e^{-\frac{1}{2}\mu_c x}$

(II)

$\omega^2 = \mu'$ or 0: μ any function of x

$y'' + \mu y' + \mu' y = 0$, or $y'' + \mu y' = 0$

$Y'' + \frac{1}{2}\ (\mu' - \frac{1}{2}\ \mu^2)\ Y = 0$, or $Y'' + \frac{1}{2}\left(\mu' - \frac{1}{2}\ \mu^2\right)\ Y = 0$

$y = e^{-\int\mu dx}\ \left(C_1 + C_2\int\ e^{\int\mu dx}dx\right)$, or $Y = C_3 + C_4\int e^{-\int\mu dx}dx$

(III)

$$\omega^2 = (Q')^2 + \mu' = K^2 e^{2\int\mu dx}\ \left(1 - \frac{k}{2}\int\ e^{\int\mu dx}dx\right)^{-4} + \mu':$$

μ any function of x

$$y'' + \mu y' + \left[K^2 e^{2\int \mu dx}\left(1 - \tfrac{1}{2}k \int e^{\int \mu dx}\, dx\right)^{-4} + \mu'\right] y = 0$$

$$Y'' + \left[K^2 e^{2\int \mu dx}\left(1 - \tfrac{1}{2}k \int e^{\int \mu dx}\, dx\right)^{-4} + \frac{\mu'}{2_2} - \frac{\mu^2}{4}\right] Y = 0$$

$$y = \left(1 - \frac{k}{2}\int e^{\int \mu dx}\right) e^{-\int \mu dx}\, e^{\pm i \frac{2K}{k}\left(1 - \frac{k}{2}\int e^{\int \mu dx}\, dx\right)^{-1}}$$

when $k = 0$, $\quad y = e^{-\int \mu dx}\, e^{\pm i\, K \int e^{\int \mu dx}\, dx}$

<u>(IV)</u>

ω_c^2 is constant (Q' = constant ξ)

$$y'' + 2\lambda \frac{e^{\lambda x} - ke^{-\lambda x}}{e^{\lambda x} + ke^{-\lambda x}} y' + \omega_c^2\, y = 0, \qquad Y'' + \xi^2\, Y = 0$$

$$y = \frac{C_1 e^{i\xi x} + C_2 e^{-i\xi x}}{e^{\lambda x} + ke^{-\lambda x}}, \quad \text{where } \lambda^2 = \omega_c^2 - \xi^2$$

<u>(V)</u>

$\omega = Q' = Ke^{-\int \mu dx}$: $\qquad$ μ any function of x

$$y'' + \mu y' + \left(K^2 e^{-2\int \mu dx}\right) y = 0, \qquad Y'' + \left(K^2 e^{-2\int \mu dx} - \frac{\mu'}{2} - \frac{\mu^2}{4}\right) Y = 0$$

$$y = e^{\pm iK \int e^{-\int \mu dx} dx}$$

<u>(VI)</u>

$$\left(\frac{Q'}{Q'}\right)' - \frac{1}{2}\left(\frac{Q''}{Q'}\right)^2 = 2\omega^2 - 2(Q')^2 - \mu' - \frac{\mu^2}{2} = f(x) : \mu \text{ any function of } x$$

$$\left(\frac{Q''}{Q'}\right)' - \frac{1}{2}\left(\frac{Q''}{Q'}\right)^2 = f(x), \quad \frac{Q''}{Q'} = S(x), \quad Q' = ke^{\int S(x)dx}$$

$$\omega^2 = \frac{1}{2}\left(f(x) + \mu' + \frac{\mu^2}{2}\right) + (Q')^2$$

$$y'' + \mu y' + \left[\frac{1}{2}\left(f(x) + \mu' + \frac{\mu^2}{2}\right) + K^2 e^{2\int S(x)dx}\right] y = 0$$

$$y = e^{-\frac{1}{2}\int (\mu+S)\,dx}\; e^{\pm iK\int e^{\int Sdx}dx}$$

$$Y'' + \left[\frac{1}{2} f(x) + K^2 e^{2\int Sdx}\right] Y = 0$$

(VI) (A) $f(x) = 0,$ $\qquad Q' = K\,(1-kx)^{-2}$

(VI) (B) $f(x) = \text{constant } 2\xi^2,$ $\qquad Q' = K\left(e^{i\xi x} + ke^{-i\xi x}\right)^{-2}$

(VI) (C) $f(x) = \xi(\lambda+x)^{-2}$ $\qquad Q' = K(\lambda+x)^{(-1\pm\sqrt{1-2\xi})}$

(VI) (D) $f(x) = \frac{1}{2}\xi^2(\lambda+x)^{-4},$ $\qquad Q' = k(\lambda+x)^{-2}e^{-i\xi(\lambda+x)^{-1}}$

(VI) (E) $f(x) = \xi(1-\frac{1}{2}\xi x^2),$ $\qquad Q' = Ke^{\frac{1}{2}\xi x^2}$

(VI) (F) $f(x) = \xi x^{-\frac{4}{3}}$ $$Q' = \left(1+\frac{3}{2}\sqrt{2\xi}\ x^{\frac{1}{3}}\right)e^{-\frac{3}{2}\sqrt{2\xi}\ x^{\frac{1}{3}}}$$

(VI) (G) $f(x)$ = other function of x

Class (2): $\mu(x)$ and $\omega^2(x)$ are arbitrary

(I)

Differential equations soluble by known higher transcendental functions

(II)

Differential equations soluble in definite integral form or continued fraction form

(III)

Differential equations which can be solved only by approximate methods or other (unknown yet) higher transcendental functions.

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Attachment
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